Qubits

Notationally, vectors are represented in Dirac's "bra"-"ket" form: $|u\rangle$ and $\langle u|$ for adjoint. Operators can be expressed in terms of the outer product between two vectors: $|\psi\rangle\langle\psi|$.

A qubit is in a *pure* state if it's measurement is completely determined. For individual qubits, in Dirac notation, these are $|0\rangle$ and $|1\rangle$. As vectors:

$$|0\rangle = \begin{bmatrix}1\\0\end{bmatrix} \qquad |1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$$

The states $\{|0\rangle, |1\rangle\}$ are referred to as the standard basis.

A qubit is in a superposition if it's components have two or more distinct states. Most common is the Hadamard, or diagonal, basis $\{|+\rangle, |-\rangle\}$ where

$$|+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$
$$|-\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\end{bmatrix} - \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Basis

(Revise) The fundamental bases for measurement is the standard basis $\{|0\rangle, |1\rangle\}$ and the Hadamard basis $\{|+\rangle, |-\rangle\}$

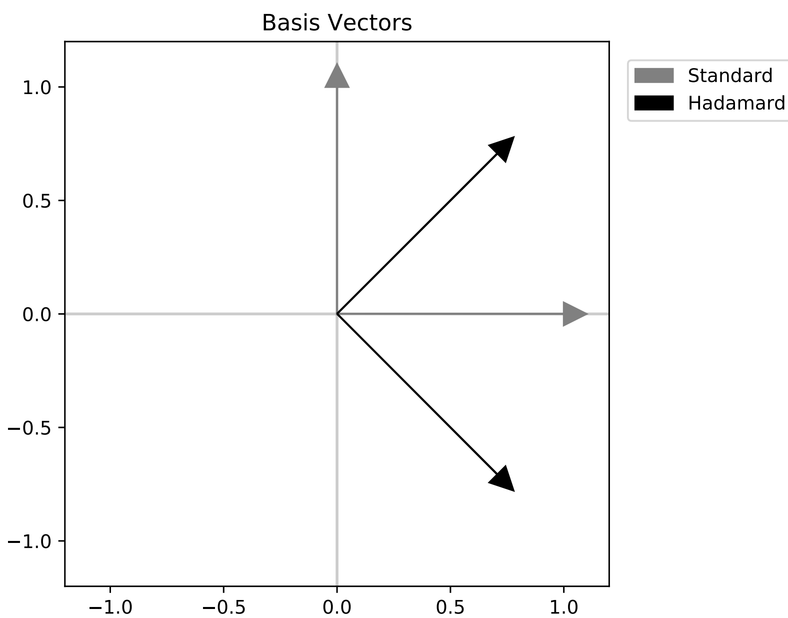

Figure 1

General unitary operators: Arbitrary unitary operators can be effectively implemented in IBMQX with parametric gates:

$$U_1(\theta) = \begin{bmatrix}1 & 0\\0 & e^{i\theta}\end{bmatrix} \qquad U_2(\varphi,\theta) = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -e^{(i\theta)}\\e^{i\varphi} & e^{i(\theta+\varphi)}\end{bmatrix} \qquad U_3(\rho,\varphi,\theta) = \begin{bmatrix}\cos(\theta/2) & -e^{i\rho}\sin(\theta/2)\\e^{i\varphi}\sin(\theta/2) & e^{i(\rho+\varphi)}\cos(\theta/2)\end{bmatrix}$$

where ρ, φ, and θ are angles corresponding to rotations in $\mathbb{C}^3$.

Standard Quantum Gates

$X = U_3(\pi, 0, \pi)$
$Y = U_3(\pi, \pi/2, \pi/2)$
$Z = U_1(\pi)$
$H = U_2(0, \pi)$

$S = U_1(\pi/2)$
$S^\dagger = U_1(-\pi/2)$
$T = U_1(\pi/3)$
$T^\dagger = U_1(\pi/4)$

Standard Rotation Gates

$R_x(\theta) = U_3(\theta, -\pi/2, \pi/2)$
$R_y(\theta) = U_3(\theta, 0, 0)$
$R_z(\theta) = U_1(\varphi)$

Controlled Phase Gates

$C_Z = HC_XH$
$C_Y = S^\dagger C_X S$
$C_H = HS^\dagger C_X HTC_X THSXS$

Pauli Operators

X gate (Bit flip) :

$$X = R_x(\pi) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Z gate (Phase flip):

$$Z = R_z(\pi) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Y gate(Bit and phase flip):

$$Y = R_y(\pi) = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$
$$Y = XZ$$

S gate($\pi/4$ phase rotation):

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$
$$S = \sqrt{Z}$$
$$\begin{aligned} S : X &\mapsto Y \\ Z &\mapsto Z \end{aligned}$$

T gate(Bit and phase flip):

$$T = R_z(\pi/4) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$$
$$T = \sqrt{S}$$

Controlled-Not (CNOT):

$$\bigwedge X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = |0\rangle\langle 1| \otimes I + |1\rangle\langle 1| \otimes X$$

Scratch

Hermitian Formalism

A linear system of equations is typically represented as $A\vec{x} = \vec{b}$, where A is the matrix containing coefficients of some unknown vector $\vec{x}$. Given $\vec{b}$, we want to find a $\vec{x} = A^{-1}\vec{b}$. Consider representing the matrix A as an observable acting on a set of qubits. Since A is Hermitian, a measurement of $|x\rangle$ can be specified by an observable decomposed as

$$A = \sum \lambda_i |e_i\rangle\langle e_i|$$

Applying A to some quantum state $|\psi\rangle$ followed by a measurement, the resulting state $|\psi'\rangle = A|\psi\rangle$ is an eigenstate of A. For example, if A has orthonormal eigenvectors $\{|e_1\rangle,\ |e_2\rangle\}$ with associated eigenvalues $\{\lambda_1, \lambda_2\}$. Then for some $|\psi\rangle$,

$$\begin{aligned} A|\psi\rangle &= \sum \lambda_i |e_i\rangle\langle e_i| = \big(\lambda_1|e_1\rangle\langle e_1| + \lambda_2|e_2\rangle\langle e_2|\big)|\psi\rangle \\ &= \lambda_1|e_1\rangle\langle e_1||\psi\rangle + \lambda_2|e_2\rangle\langle e_2||\psi\rangle \end{aligned}$$

More concretely, A linear system whose coefficient matrix has eigenspace in $\{|+\rangle, |-\rangle\}$ can be effectively solved. For example, consider the system $A|x\rangle = |b\rangle$ where

$$A = \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{bmatrix} \quad |x\rangle = \frac{1}{2\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad |b\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Figure 2 illustrates that both $|b\rangle$ and $|x\rangle$ lie on one of A's eigenvectors which turn out to be vectors in the Hadamard basis as seen in Figure 1.

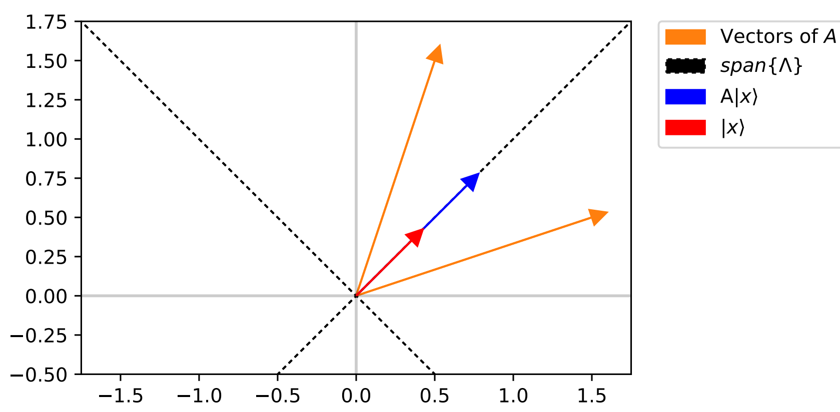

Figure 2

$$A = \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}$$

with eigenvalues $\lambda_i = \{1, 2\}$ with associated eigenvectors

$$|u_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \tag{1}$$
$$= |+\rangle \tag{2}$$
$$|u_2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix} \tag{3}$$
$$= \frac{-1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} \tag{4}$$
$$= (-1)|-\rangle \tag{5}$$

Pulling out the negative sign in (4) can be done because it is just a global phase; it is an equivalent state. From (2) and (5), the eigspace decomposition of A is:

$$\begin{aligned} A &= \sum \lambda_i |u_i\rangle\langle u_i| \\ &= 2|+\rangle\langle +| + (-1)|-\rangle\langle -| \end{aligned}$$

and solving for $|x\rangle$

$$\begin{aligned} A^{-1}|b\rangle &= \big(2^{-1}|+\rangle\langle +| - 1^{-1}|-\rangle\langle -|\ \big)|+\rangle \\ &= \frac{1}{2}|+\rangle\langle +|+\rangle - |-\rangle\langle -|+\rangle \\ &= \frac{1}{2}|+\rangle(1) - |-\rangle(0) \\ &= \frac{1}{2}|+\rangle \\ &= \left(\frac{1}{2}\right)\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ &= \frac{1}{2\sqrt{2}}(|0\rangle + |1\rangle) \\ &= |x\rangle \end{aligned}$$