MATHEMATICS II

MATH F112

Department of Mathematics

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Linear Transformations

A function *T* that maps a vector space *V* into a vector space *W*:

$$T: V \xrightarrow{\text{mapping}} W, \qquad V, W: \text{vector spaces}$$

V: the domain of T $\qquad$ W: the codomain of T

Image of v under *T*:

If $\mathbf{v}$ is a vector in V and $\mathbf{w}$ is a vector in W such that

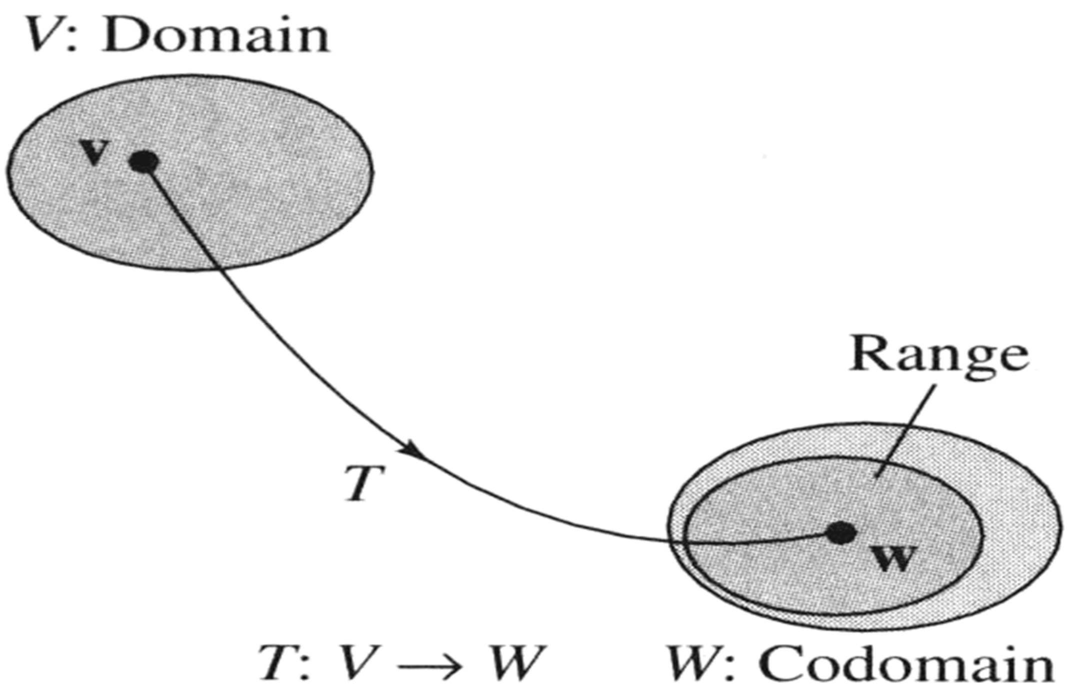
$$T(\mathbf{v}) = \mathbf{w},$$

then $\mathbf{w}$ is called the image of $\mathbf{v}$ under T

The range of *T*:

The set of all images of vectors in V

The graphical representations of the domain, codomain, and range



Linear Transformations (LT)

Let V, W be vector spaces, a function

$$L : V \to W$$

is called a linear transformation from V into W if the following two properties are true

$$(1)\ L(\mathbf{u}+\mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v}), \quad \forall \mathbf{u}, \mathbf{v} \in V$$

$$(2)\ L(c\mathbf{u}) = cL(\mathbf{u}), \quad \forall c \in R$$

In some special case if $V=W$ then the LT is called a linear operator on V.

$$L : V \to V$$

Remark:

A linear transformation is said to be operation preserving

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \qquad\qquad T(c\mathbf{u}) = cT(\mathbf{u})$$

Addition in V

Addition in W

Scalar multiplication in V

Scalar multiplication in W

Examples

Let

$$L : R^2 \to R^2$$

be defined as

$$(a)\ L(x, y) = (x - y, x + 2y)$$
$$(b)\ L(x, y) = (x, -y)$$

Show that L is a LT.

Verifying a linear transformation L from R^2 into R^2

$\mathbf{u}=(u_1,u_2),\ \ \mathbf{v}=(v_1,v_2)$: vector in R^2, c : any real number

(1) Vector addition :

$$\mathbf{u}+\mathbf{v}=(u_1,u_2)+(v_1,v_2)=(u_1+v_1,u_2+v_2)$$

$$\begin{aligned}
L(\mathbf{u}+\mathbf{v}) &= L(u_1+v_1,u_2+v_2) \\
&=((u_1+v_1)-(u_2+v_2),(u_1+v_1)+2(u_2+v_2)) \\
&=((u_1-u_2)+(v_1-v_2),(u_1+2u_2)+(v_1+2v_2)) \\
&=(u_1-u_2,u_1+2u_2)+(v_1-v_2,v_1+2v_2) \\
&=L(\mathbf{u})+L(\mathbf{v})
\end{aligned}$$

(2) Scalar multiplication

$$c\mathbf{u} = c(u_1, u_2) = (cu_1, cu_2)$$

$$L(c\mathbf{u}) = L(cu_1, cu_2) = (cu_1 - cu_2, cu_1 + 2cu_2)$$

$$= c(u_1 - u_2, u_1 + 2u_2)$$

$$= cL(\mathbf{u})$$

Therefore, L is a linear transformation

Two uses of the term "linear".

(1) $f(x)=x+1$ is called a linear function because its graph is a line

(2) $f(x)=x+1$ is not a linear transformation from a vector space R into R because it preserves neither vector addition nor scalar multiplication

Zero transformation:

$$L : V \to W \qquad L(\mathbf{v}) = \mathbf{0}, \ \forall \mathbf{v} \in V$$

Identity transformation:

$$L : V \to V \qquad L(\mathbf{v}) = \mathbf{v}, \ \forall \mathbf{v} \in V$$

Projection transformation:

$$L : R^3 \to R^3 \qquad L(x, y, z) = (x, y, 0)$$

Reflection transformation:

$$L : R^2 \to R^2 \qquad L(x, y) = (x, -y)$$

Properties of linear transformations

$$L: V \to W, \quad \mathbf{u}, \mathbf{v} \in V$$

(1) $L(\mathbf{0}_v) = \mathbf{0}_w$ ($L(c\mathbf{v}) = cL(\mathbf{v})$ for c=0)

(2) $L(-\mathbf{v}) = -L(\mathbf{v})$ ($L(c\mathbf{v}) = cL(\mathbf{v})$ for c=-1)

(3) $L(\mathbf{u}-\mathbf{v}) = L(\mathbf{u}) - L(\mathbf{v})$ (L($\mathbf{u}$+(-$\mathbf{v}$))=$L(\mathbf{u})+L(-\mathbf{v})$ and property (2))

(4) If $\mathbf{v} = c_1v_1 + c_2v_2 + \cdots + c_nv_n,$

then $L(\mathbf{v}) = L(c_1v_1 + c_2v_2 + \cdots + c_nv_n)$

$$= c_1L(v_1) + c_2L(v_2) + \cdots + c_nL(v_n)$$

(Iteratively using $L(\mathbf{u}+\mathbf{v})=L(\mathbf{u})+L(\mathbf{v})$ and $L(c\mathbf{v}) = cL(\mathbf{v})$)

Problems

Which of the following are LT's

$$(a)\ \ L: R^3 \to R^2,\quad L(x, y, z) = (0, 0)$$

$$(b)\ \ L: R^3 \to R^3,\quad L(x, y, z) = (1, 2, -1)$$

$$(c)\ \ L: R^3 \to R^2,\quad L(x, y, z) = (x^2 + y, y - z)$$

$$(d)\ L: P_2 \to P_1,\quad L(ax^2 + bx + c) = 2ax + b + c$$

$$(e)\ \ L: P_2 \to P_1,\quad L(ax^2 + bx + c) = (a + 2)x + b - c$$

The transpose function is a linear transformation from $M_{m\times n}$ into $M_{n\times m}$

$$T(A) = A^T \quad (T : M_{m\times n} \to M_{n\times m})$$

Show that T is a linear transformation

Sol:

$$A, B \in M_{m\times n}$$

$$T(A+B) = (A+B)^T = A^T + B^T = T(A) + T(B)$$

$$T(cA) = (cA)^T = cA^T = cT(A)$$

Therefore, T (the transpose function) is a linear transformation from $M_{m\times n}$ into $M_{n\times m}$

Theorem

Let $L: V \rightarrow W$ be a LT of an n-dim. vector space V into a vector space W. Also let

$$S = \{v_1, v_2, \cdots v_n\}$$

be a basis for V. If $u \in V$, then $L(u)$ is completely determined by the set

$$\{L(v_1), L(v_2), \cdots, L(v_n)\}.$$

Example

Let $L : P_1 \rightarrow P_2$ be a LT for which

$$L(x+1) = x^2 - 1 \text{ and } L(x-1) = x^2 + x$$

then find the LT and what is $L(7x+3)$

The Kernel and Range of a Linear Transformation

Kernel of a linear transformation *L*

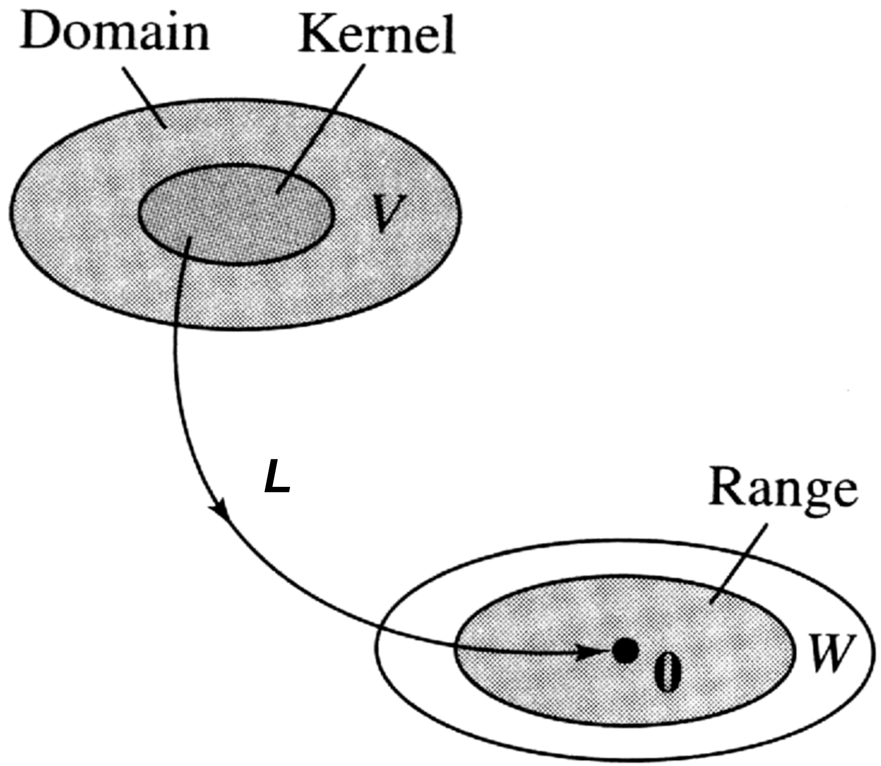
Let $L:V \to W$ be a linear transformation. Then the set of all vectors $\mathbf{v}$ in V that satisfy $L(\mathbf{v}) = \mathbf{0}$ is called the kernel of L and is denoted by $\ker(L)$

$$\ker(L) = \{\mathbf{v} \mid L(\mathbf{v}) = \mathbf{0},\ \mathbf{v} \in V\}$$

Range of a linear transformation *L*

Let $L:V \to W$ be a linear transformation. Then the set of all vectors $\mathbf{w}$ in W that are images of any vectors in V is called the range of L and is denoted by $\text{range}(L)$

$$\text{range}(L) = \{L(\mathbf{v}) \mid \forall \mathbf{v} \in V\}$$


Domain
Kernel
V
L
Range
0
W

Important Results

Let $L : V \rightarrow W$ be a linear transformation then

(1) The kernel of L is a subspace of V.

(2) The range of L is a subspace of W.

One-One LT

A LT $L: V \rightarrow W$ is said to be one to one if for all v_1, v_2 in V $v_1 \neq v_2$ implies that $\mathrm{L}(v_1) \neq L(v_2)$. An equivalent statement is that L is one to one if for all v_1, v_2 in V, $\mathrm{L}(v_1) = L(v_2)$ implies that $v_1 = v_2$.

$L: V \rightarrow W$ is said to be onto if $\mathrm{range}(L) = W$.

Theorem:

A LT $L:V \to W$ is one to one if and only if $\ker(L) = \{0_v\}$

Example

Let $L : \mathrm{R}^3 \to R^3$ be defined by

$$L(x, y, z) = (x - y, x + 2y, z)$$

(a) Show that L is a LT.

(b) Find a basis for ker(L).

(c) Find a basis for range(L).

(d) Is L one one.

(e) Is L onto.

Problem

Let $L : \mathrm{R}^3 \to R^3$ be defined by

$$L\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

(a) Show that L is a LT.

(b) Find a basis for ker(L).

(c) Find a basis for range(L).

(d) Is L one one.

(e) Is L onto.

Matrices for Linear Transformations

Two representations of the linear transformation $T: R^3 \to R^3$:

$$(1)\ T(x_1, x_2, x_3) = (2x_1 + x_2 - x_3, -x_1 + 3x_2 - 2x_3, 3x_2 + 4x_3)$$

$$(2)\ T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 3 & -2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Three reasons for matrix representation of a linear transformation:

- It is simpler to write
- It is simpler to read
- It is more easily adapted for computer use

Matrix of a LT

Let $L: V \to W$: be a LT, let us consider the **ordered** bases

$$S = \{v_1, v_2, \cdots v_n\} \text{ be a basis for } V,$$

$$T = \{w_1, w_2, \cdots w_m\} \text{ be a basis for } W,$$

$$\begin{array}{l}
L(v_1) = a_{11} w_1 + a_{21} w_2 + \cdots + a_{m1} w_m \\
L(v_2) = a_{12} w_1 + a_{22} w_2 + \cdots + a_{m2} w_m \\
\quad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
L(v_n) = a_{1n} w_1 + a_{2n} w_2 + \cdots + a_{mn} w_m
\end{array}$$

Matrix of a LT

The coordinate vectors of

$$L(v_1),\ L(v_2),\cdots L(v_n)\ \text{ w.r.t basis } T \text{ are :}$$

$$[L(v_1)]_T = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix},[L(v_2)]_T = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \cdots,[L(v_n)]_T = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Matrix of a LT

Then the matrix of L w.r.t bases S and T is

$$L_{ST} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{m2} \end{bmatrix}$$

Example

Let $L : \mathrm{R}^3 \to R^2$ be defined by

$$L(x, y, z) = (x + y, y - z)$$

find the matrix of L w.r.t

$\mathrm{S} = \{(1,0,1), (0,1,1), (1,1,1)\}$ and $T = \{(1,2), (-1,1)\}$.

Use the matrix of L to compute the image of (2,3,1).

Let A be the matrix of L w.r.to bases S and T :

$$\begin{tikzcd}
X \arrow[r, "L"] \arrow[d] & L(X) \arrow[d] \\
{[X]_S} \arrow[r, "A"'] & {[L(X)]_T = A[X]_S}
\end{tikzcd}$$

Quiz-1 5-marks

Let $L : P_1 \to P_2$ be defined by

$$L(p(x)) = xp(x)$$

find the matrix of L w.r.t

$S = \{x+1, x-1\}$ and $T = \{1, x, x^2\}$.

Rank of a linear transformation $T:V\rightarrow W$:

$$\text{rank}(T) = \text{the dimension of the range of } T = \dim(\text{range}(T))$$

Nullity of a linear transformation $T:V\rightarrow W$:

$$\text{nullity}(T) = \text{the dimension of the kernel of } T = \dim(\ker(T))$$

Note:

If $T: R^n \rightarrow R^m$ is a linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$, then

$$\text{rank}(T) = \dim(\text{range}(T)) = \dim(CS(A)) = \text{rank}(A)$$

$$\text{nullity}(T) = \dim(\ker(T)) = \dim(NS(A)) = \text{nullity}(A)$$

Dimension Theorem (Rank Nullity Theorem)

Let T: $V \longrightarrow W$ be a linear transformation from an n-dimensional vector space V (i.e. the dim(domain of T) is n) into a vector space W. Then

$$\text{rank}(T) + \text{nullity}(T) = n$$

$$\text{(i.e. } \dim(\text{range of } T) + \dim(\text{kernel of } T) = \dim(\text{domain of } T))$$

Note

If we change the order of the vectors in the bases S and T, then the matrix of L may change.

Change of basis lead to a different matrix of L.

Theorem

Let $T:V \to V$ be a linear operator, where V is an n-dimensional vector space. Let $B = \{v_1, v_2, \cdots v_n\}$ and $B' = \{w_1, w_2, \cdots w_n\}$ be a bases for V, and let P be the transition matrix from B' *to* B. If A is the matrix of T w.r.to B then $P^{-1}AP$ is the matrix of T w.r.to the basis B'.

$$P = \begin{bmatrix} [w_1]_B & [w_2]_B & [w_3]_B & \cdots & [w_n]_B \end{bmatrix}$$

Transition Matrices

$T : V \rightarrow V$ (a linear transformation)

$B = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ (a basis of V)

$B' = \{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_n\}$ (a basis of V)

$A = \left[[T(\mathbf{v}_1)]_B \;\; [T(\mathbf{v}_2)]_B \;\; \cdots \;\; [T(\mathbf{v}_n)]_B \right]$ (matrix of T relative to B)

$A' = \left[[T(\mathbf{w}_1)]_{B'} \;\; [T(\mathbf{w}_2)]_{B'} \;\; \cdots \;\; [T(\mathbf{w}_n)]_{B'} \right]$ (matrix of T relative to B')

$$\therefore [T(\mathbf{v})]_B = A[\mathbf{v}]_B, \text{ and } [T(\mathbf{v})]_{B'} = A'[\mathbf{v}]_{B'}$$

$P = \left[[\mathbf{w}_1]_B \;\; [\mathbf{w}_2]_B \;\; \cdots \;\; [\mathbf{w}_n]_B \right]$ (transition matrix from B' to B)

$P^{-1} = \left[[\mathbf{v}_1]_{B'} \;\; [\mathbf{v}_2]_{B'} \;\; \cdots \;\; [\mathbf{v}_n]_{B'} \right]$ (transition matrix from B to B')

$$\therefore [\mathbf{v}]_B = P[\mathbf{v}]_{B'}, \text{ and } [\mathbf{v}]_{B'} = P^{-1}[\mathbf{v}]_B$$

Example

Let $L : \mathrm{R}^2 \rightarrow R^2$ be defined by

$$L(x, y) = (x + y, x - 2y)$$

find the matrix of L w.r.t

$S = \{(1,0), (0,1)\}$ and use it to find the matrix of L w.r.t $T = \{(1,-1), (2,1)\}$.

Isomorphism

A linear transformation $T: V \rightarrow W$ that is one to one and onto is called an isomorphism, and the vector spaces V and W are are said to be isomorphic to each other

Theorem:

Every real n-dimensional vector space is isomorphic to R^n

Isomorphic vector spaces

The following vector spaces are isomorphic to each other

(a) $R^4 = 4\text{-space}$

(b) $M_{4\times1} =$ space of all 4×1 matrices

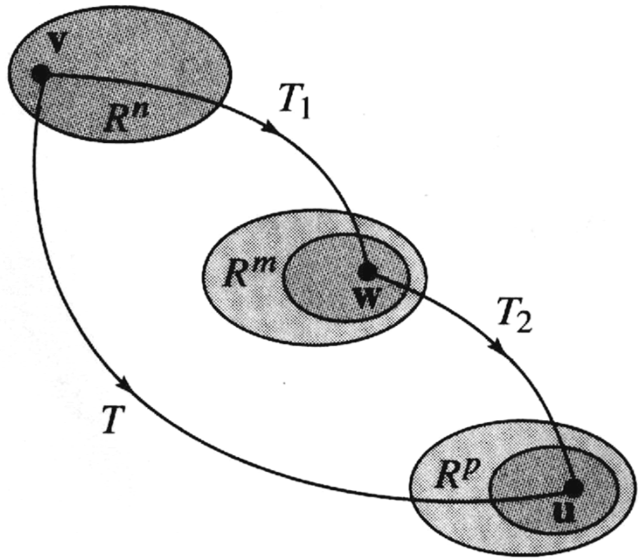
(c) $M_{2\times2} =$ space of all 2×2 matrices

(d) $P_3(x) =$ space of all polynomials of degree 3 or less

composition of T_1:$R^n \rightarrow R^m$ with T_2:$R^m \rightarrow R^p$:

$$T(\mathbf{v}) = T_2(T_1(\mathbf{v})), \quad \mathbf{v} \in R^n$$

This composition is denoted by $T = T_2 \circ T_1$



Composition of Transformations

Inverse linear transformation:

If $T_1 : R^n \to R^n$ and $T_2 : R^n \to R^n$ are L.T. s.t. for every $\mathbf{v}$ in R^n

$$T_2(T_1(\mathbf{v})) = \mathbf{v} \quad \text{and} \quad T_1(T_2(\mathbf{v})) = \mathbf{v}$$

Then T_2 is called the inverse of T_1 and T_1 is said to be invertible

Finding the inverse of a linear transformation

The linear transformation $T: R^3 \rightarrow R^3$ is defined by

$$T(x_1, x_2, x_3) = (2x_1 + 3x_2 + x_3, 3x_1 + 3x_2 + x_3, 2x_1 + 4x_2 + x_3)$$

Show that T is invertible, and find its inverse

Sol:

The standard matrix for T (with respect to standard basis)

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}$$

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\text{G.-J. E.}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 6 & -2 & -3 \end{array}\right] = \left[I \mid A^{-1} \right]$$

Therefore T is invertible and the standard matrix for T^{-1} is A^{-1}

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

$$T^{-1}(\mathbf{v}) = A^{-1}\mathbf{v} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_1 + x_2 \\ -x_1 + x_3 \\ 6x_1 - 2x_2 - 3x_3 \end{bmatrix}$$

In other words,

$$T^{-1}(x_1, x_2, x_3) = (-x_1 + x_2, -x_1 + x_3, 6x_1 - 2x_2 - 3x_3)$$