Mathematics II, Spring 2020
Complex Analysis

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Complex functions

A function from a subset of complex numbers into the set of complex numbers is called a complex function.

$S \subset \mathbb{C}$ a mapping $f : S \to \mathbb{C}$ is called a complex function if it is well defined.

The set S is called the domain of the function. Sometime if the domain is not mentioned then the largest possible set where f is well defined is taken to be the domain.

The set $\{f(z) : z \in S\} \subset \mathbb{C}$ is called the range of the function.

Examples

1. $f : \mathbb{C} \longrightarrow \mathbb{C}$ defined by $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ where $a_i \in \mathbb{C}$ is called a polynomial function.
2. $f = \frac{P(z)}{Q(z)}$ where P, Q are polynomial functions, is called a rational function, note that the domain of the function is all of complex plane except for the roots of Q.
3. $f(z) = e^z$ is the exponential function which is defined by
$$e^z = e^x e^{iy} = e^x(\cos y + i \sin y)$$
for $z = x + iy$. Note that the domain is all of the complex plane.

Examples. cont

1. If $z = x + iy$ then define $f(z) = x^2 - y^2$ is a real valued function since the range is contained in the real line.
2. The function $f(z) = z + z_0$ for some fixed complex number z_0 is called a translation.
3. $f(z) = e^{i\theta_0} z$ is called a rotation since the image is z rotated by the angle θ_0.
4. $f(z) = az + b$ for some complex numbers a, b is an affine transformation of the real vector space $\mathbb{C}$ to itself.
5. $f(z) = \overline{z}$ is a reflection about the X axis.

Limit

Definition

Let the function f be defined on a deleted neighborhood of a point z_0 then we say

$$\lim_{z \to z_o} f(z) = w_0,$$

if for each positive real number ϵ there is a positive real number δ such that

$$|f(z) - w_0| < \epsilon \quad \text{whenever} \quad |z - z_0| < \delta$$

Uniqueness

As we "move" closer to the point z_0 the value of the function moves closer to the value w_o. Note that a limit may or may not exist. But if it exists then one can show as in the next theorem that it is unique.

Theorem 1

If a limit $\lim_{z \to z_0} f(z)$ exists then it is unique.

Uniqueness

To prove the assertion let us assume if possible there are more than one limits, let us say w_1, w_2 are two limits.

So for each positive $\epsilon > 0$ there are δ_1, δ_2 positive such that

$$|f(z) - w_1| < \epsilon \quad \text{whenever} \quad |z - z_0| < \delta_1 \quad \text{and}$$

$$|f(z) - w_2| < \epsilon \quad \text{whenever} \quad |z - z_0| < \delta_2.$$

So whenever $|z - z_0| < \delta = \min\{\delta_1, \delta_2\}$, we have

$$|w_1 - w_2| = |(f(z) - w_2) - (f(z) - w_1)| \leq |f(z) - w_1| + |f(z) - w_2|$$

$$< 2\epsilon$$

Since ϵ in the above was arbitrary, we must have $w_1 = w_2$.

Theorem 2

Let us suppose that $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy$ and u, v are real valued functions of two real variables. And let us say $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0$ then :

$$\lim_{z \to z_0} f(z) = w_0$$

if and only if

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0.$$

Algebra of Limits

Theorem 3

Let us assume $\lim\limits_{z \to z_0} f(z) = w_0$ and $\lim\limits_{z \to z_0} g(z) = p_0$ Then:

1. $\lim\limits_{z \to z_0} (f(z) + g(z)) = w_0 + p_0$
2. $\lim\limits_{z \to z_0} f(z)g(z) = w_0 p_0$
3. If $p_0 \neq 0$ then $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{w_0}{p_0}$

Problems

Show that the limit of the function $f(z) = \left(\frac{z}{\bar{z}}\right)^2$ as z tends to 0 does not exist.

Find which of the following limits exists:

1. $\lim\limits_{z\to 1} \dfrac{1-\bar{z}}{1-z}$.
2. $\lim\limits_{z\to 0} \dfrac{z^2-\bar{z}^2}{z}$.
3. $\lim\limits_{z\to 0} \dfrac{z}{\operatorname{Re} z}$.

Limit to infinity

Theorem 4

1. $\lim_{z\to z_0} f(z) = \infty$ if and only if $\lim_{z\to z_0} \frac{1}{f(z)} = 0$
2. $\lim_{z\to\infty} f(z) = w_0$ if and only if $\lim_{z\to 0} f(\frac{1}{z}) = w_0$.
3. $\lim_{z\to\infty} f(z) = \infty$ if and only if $\frac{1}{\lim_{z\to 0} f(1/z)} = 0$

Use the above theorem to find the following limits

1. $$\lim_{z\to\infty} \frac{4z^2}{(z-1)^2}$$

2. $$\lim_{z\to 1} \frac{1}{(z-1)^3}$$

3. $$\lim_{z\to\infty} \frac{z^2+1}{z-1}$$