Electrical Sciences
Lectures Topic- AC Analysis

Varun Sharma

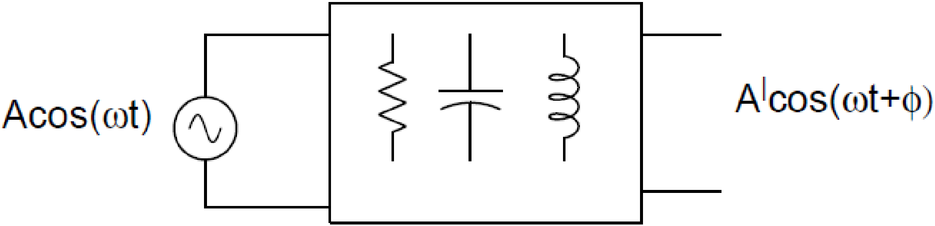
BITS Pilani,
K. K. Birla Goa Campus

February 28, 2020

Outline

Limitations of time-domain analysis

- Most common form of input (source of excitation) is a sinusoid. When a sinusoidal is input to a linear circuit the output is also a sinusoid with a same frequency.
- To analyse these circuits in time domain, we need to solve a complex differential equation and then find the solution.
- Is there a simpler way of analysing circuits with sinusoidal inputs?.



Limitations of time-domain analysis

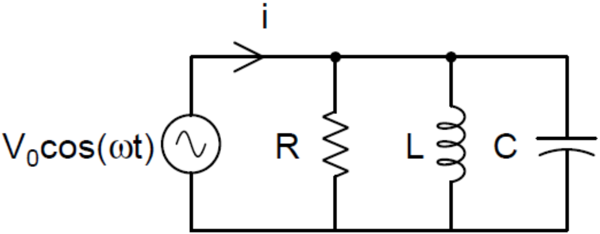
- When excited by a voltage source, the currents in each element is given by.

$$i_R(t) = \frac{V_o}{R} cos(\omega t)$$

$$i_C(t) = C\frac{dV}{dt} = V_o\omega Csin(\omega t) = V_o\omega Ccos(\omega t + \frac{\pi}{2})$$

Similarly

$$i_L(t) = \frac{V_o}{\omega L} cos(\omega t - \frac{\pi}{2})$$



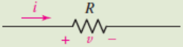
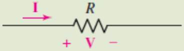
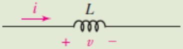
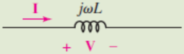
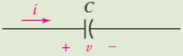
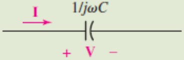
Introduction to Phasors

- Since only amplitude and phase of the sinusoid are changing then it is convenient to represent the quantities as phasors.
- A sinusoid of the form $Acos(\omega t + \phi)$ can be expressed as .

$$A\angle\phi$$

- Thus, a phasor is essentially a complex number.

Comparisons between Time-Domain & Frequency Domain

Time Domain		Frequency Domain	
	$v = Ri$	$\mathbf{V} = R\mathbf{I}$	
	$v = L\dfrac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$	
	$v = \dfrac{1}{C}\int i\,dt$	$\mathbf{V} = \dfrac{1}{j\omega C}\mathbf{I}$	

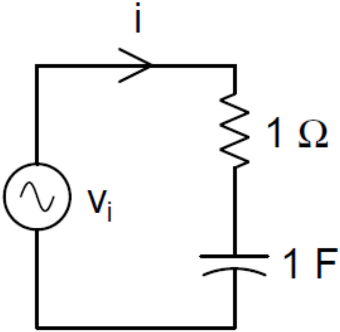
- The complex nature of the V/I of inductors and capacitors leads us to a new term called impedance denoted by Z.

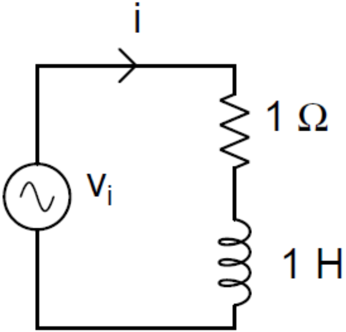
$$Z = R \pm jX$$

Where, R (the real part) represents the resistance and the X (the imaginary part) represents the reactance.

Solve using Phasor Notations

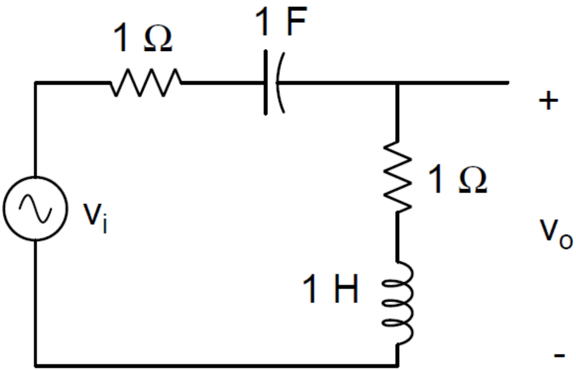
- Find and draw the phasors for the current and voltage in the circuits shown below. It is given $v_i = 4\cos(2t + \pi/4)$





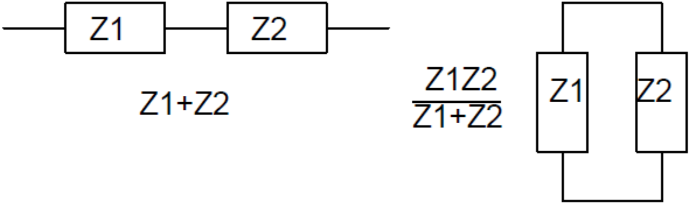
Solve using Phasor Notations

- Find the voltage v_o in the circuit shown below. It is given $v_i = 4\cos(2t + \pi/4)$



Impedances in Series-Parallel

- Impedances in series and parallel

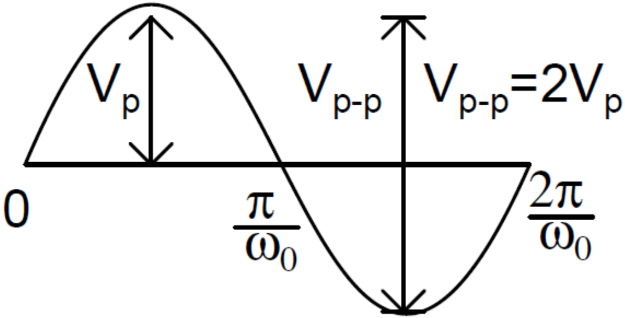


- Impedance of a series Rl circuit is $Z = R + j\omega L$
- Imepdance of a series RC circuit is $Z = R - j/\omega C$

Some Details About Sinusoidal Function

- Properties of a Sinusoid. A sinusiodal voltage signal of the form

$$v(t) = V_p \sin(\omega_0 t)$$



Some Details About Sinusoidal Function

- Peak Value–The maximum value of the sinusoid above a reference voltage or current (zero Volts/Amperes). The peak value of the sinusoidal signal $V_p \sin(\omega_0 t)$ is V_p.
- Peak-Peak Value- The difference between the maximum and minimum value of the sinusoid w.r.t a reference level (refernce level is taken as zero usually)

$$V_{pp} = 2V_p$$

Some Details About Sinusoidal Function

- The average value of a periodic signal is the mean value (area) of the waveform over a single cycle

$$V_{av} = \frac{1}{T} \int_{t_0}^{t_0+T} v(t)\, dt$$

- The average value of a sinusoid ($V_p \sin(\omega_0 t)$) ix zero

$$v_{av} = \frac{1}{T} \int_{t_0}^{t_0+T} V_0 \sin(\omega_0 t)\, dt = 0$$

Some Details About Sinusoidal Function

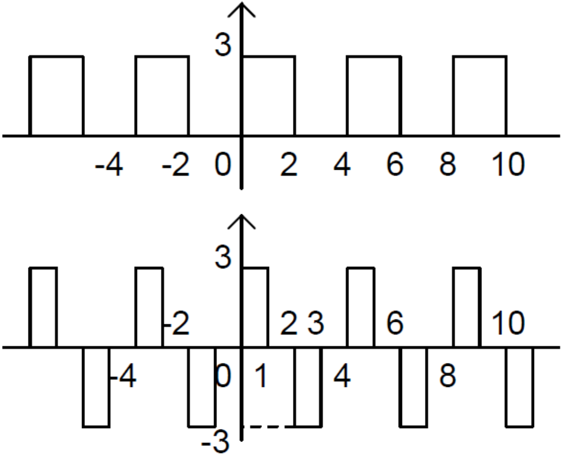
- The root-mean-square value of a sinusiod is of value in AC circuits and it has special significance. For a periodic signal $v(t)$ of time period T. The rms value is given by

$$v_{rms} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} v^2(t)\,dt}$$

- The rms value of a sinusiod $V_p \sin(\omega_0 t)$ is $V_p/\sqrt{2}$. It is independent of the frequency or time period of the sine wave

Some Details About Sinusoidal Function

- Find the average, peak, peak to peak and rms values of the signals shown below



Some Details About Sinusoidal Function

- a) The peak value is $V_p = 3$ and $V_{p-p} = 3$. The average value is

$$V_{avg} = \frac{1}{T}\int_0^T v(t)\,dt = \frac{1}{4}\int_0^2 3\,dt = 1.5$$

- The rms value of the waveform is

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T (v(t))^2\,dt} = \sqrt{\frac{1}{4}\int_0^2 (3)^2\,dt}$$

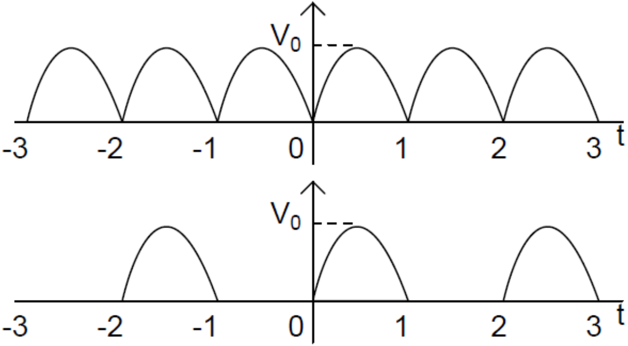
$$= \frac{3}{\sqrt{2}}$$

- b) SImilarly it can be shown that $V_p = 3$ $V_{p-p} = 3-(-3) = 6$.

$$V_{avg} = 0 \;\;\&\;\; V_{rms} = \frac{3}{\sqrt{2}}$$

Some Details About Sinusoidal Function

- Find the average, peak, peak to peak and rms values of the signals shown below



Some Details About Sinusoidal Function

- a) The peak value is $V_p = V_0$ and $V_{p-p} = V_0$. The average value is

$$V_{avg} = \frac{1}{T}\int_0^T v(t)\,dt = \frac{1}{1}\int_0^1 V_0 sin(\pi t)\,dt = \frac{2V_0}{\pi}$$

- The rms value of the waveform is

$$v_{rms} = \sqrt{\frac{1}{T}\int_0^T (v(t))^2\,dt} = \sqrt{\frac{1}{1}\int_0^1 (V_0 sin(\pi t))^2\,dt}$$

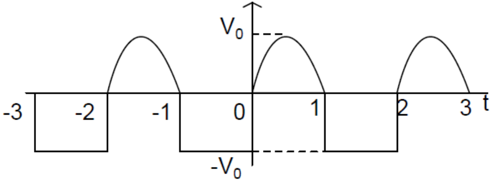
$$= \frac{V_0}{\sqrt{2}}$$

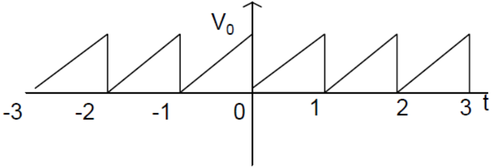
- b) SImilarly for a half wave rectified waveform it can be shown that $V_p = V_0$, $V_{p-p} = V_p$.

$$V_{avg} = \frac{V_0}{\pi} \;\&\; V_{rms} = \frac{V_0}{2}$$

Some Details About Sinusoidal Function

- Find the average, peak, peak to peak and rms values of the signals shown below





Instantaneous Power

- The instaneous power absorbed by an element with a voltage $v(t)$ across it and a current $i(t)$ passing through it is given by
$$p(t) = v(t)i(t)$$
- For an AC circuit the voltage and current need not be in phase like a dc circuit. Let $v(t) = V_0 \cos(\omega t)$ and $i(t) = I_0 \cos(\omega t + \phi)$, then the power absorbed by the element is given by

$$p(t) = V_0 I_0 \cos(\omega t) \cos(\omega t + \phi)$$
$$= \frac{V_0 I_0}{2}(\cos(\phi) + \cos(2\omega t + \phi))$$

- Thus the instaneous power oscillates a frequency of 2ω.

Average Power

- The instantaneous power has two components. An oscillating component and a dc (constant) component

$$p(t) == \frac{V_0 I_0}{2}(\cos(\phi) + \cos(2\omega t + \phi))$$

- The average power is given by

$$P = \frac{1}{T}\int_0^T p(t)\,dt = \frac{V_0 I_0}{2}\cos(\phi)$$

- The average power is the actual power dissipated as heat in a circuit

Average Power dissipated in Resistor

- The average power absorbed by a resistor is given by

$$P = \frac{1}{T}\int_0^T p(t)\,dt = \frac{V_0 I_0}{2}\cos(\phi)$$

- Since voltage and current are in phase for a resistor $V_0\cos(\omega t) = IR \implies I = (V_0/R)\cos(\omega t)$, $\phi = 0$
- The average power absorbed by a resistor is

$$\therefore P = \frac{V_0 I_0}{2} = \frac{V_0{}^2}{2R} = \frac{I_0{}^2 R}{2}$$

- The average power can also be expressed in terms of the rms values $V_e = V_0/\sqrt{2}$ and $I_e = I_0/\sqrt{2}$

$$P = \frac{V_e{}^2}{R} = I_e{}^2 R$$

Average Power dissipated in Inductor and Capacitor

- The average power absorbed by a capacitor is given by

$$P = \frac{1}{T}\int_0^T p(t)\,dt = \frac{V_0 I_0}{2}\cos(\phi)$$

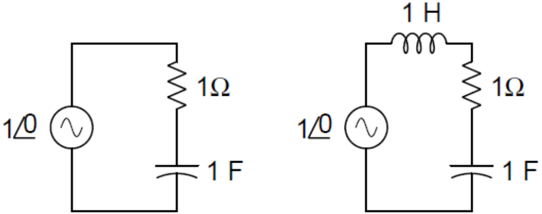
- For a capacitor, voltage and current are out of phase by 90^0. If $v(t) = V_0\cos(\omega t)$, then $i(t) = \omega C V_0 \cos(\omega t + \pi/2)$
  $\phi = \pi/2$

$$\therefore P = \frac{V_0 I_0}{2}\cos(\pi/2) = 0$$

- Thus the average power absorbed by a capacitor is Zero
- Similarly it can be shown that the averge power absorbed by an inductor is also zero

Average Power- Problems

- Find the average power absorbed by each element in the circuit given below. Given $\omega = 1\ rad/s$



Power Factor & Apparent Power

- In an AC circuit, the current and voltage across an impedance are related by

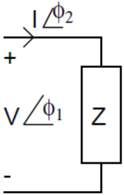
$$Z = \frac{\mathbf{V}}{\mathbf{I}} = \frac{V\angle\phi_1}{I\angle\phi_2} = \frac{|V|}{|I|}\angle(\phi_1 - \phi_2) = \frac{|V|}{|I|}\angle\theta$$

The power absorbed in the load is given by

$$P = \frac{1}{2}|V||I|\cos(\theta) = V_{rms}I_{rms}\cos(\theta)$$

The product $V_{rms}I_{rms}$ is called as Apparent power and it is always greater or equal to the Average power

$$P_{app} \geq P_{AV}$$



Power Factor

- The ratio of the average power to apprent power is called power factor

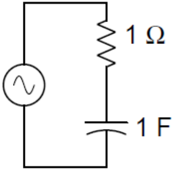
$$pf = \cos(\theta) = \frac{P}{V_{rms} I_{rms}} = \frac{P_{AV}}{P_{app}}$$

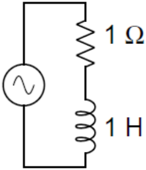
  Power factor is a metric used by electrical engineers for designing low cost electrical distribution systems

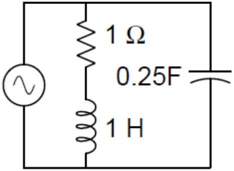
- if the current lags the voltage, then the pf is lagging pf. Pf is lagging for an inductive load. Since for an inductive load the current lags the voltage
- If the current leads the voltage, then the pf is leading pf. pf is leading for a capacitive load
- The impedance $Z = R + jX$ can be inductive $X > 0$ or capacitive $X < 0$

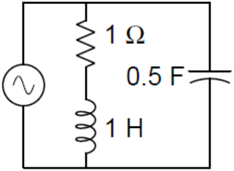
Power Factor- Problem

- Find the average power, apparent power and power factor (lagging or leading) for the following circuits. Also find the current drawn from the voltage source in each case. Given that the voltage is $1\angle 0$ and $\omega = 1\ rad/sec$.









Power Factor- Problem

- a) The current

$$I = \frac{1}{1-j} = \frac{1+j}{2} = \frac{1}{\sqrt{2}}\angle\pi/4$$

The rms vlaues of current and voltage are $I_{rms} = 1/2$ and $V_{tms} = 1/\sqrt{2}$.

$$\therefore P_{app} = V_{rms}I_{rms} = \frac{1}{2\sqrt{2}} = 0.353\ VA$$

$$P_{AV} = P_{app}\cos(\theta) = \frac{1}{4} = 0.25\ W$$

- The pf is leading since the current is leading the volatage

$$pf = \cos(\theta) = \frac{1}{\sqrt{2}}$$

Power Factor- Problem

- b) The current

$$I = \frac{1}{1+j} = \frac{1-j}{2} = \frac{1}{\sqrt{2}}\angle -\pi/4$$

The rms vlaues of current and voltage are
$I_{rms} = 1/2 = 0.5\,A$ and $V_{tms} = 1/\sqrt{2} = 0.707\,A$.

$$\therefore P_{app} = V_{rms} I_{rms} = \frac{1}{2\sqrt{2}} = 0.353\ VA$$

$$P_{AV} = P_{app}\cos(\theta) = \frac{1}{4} = 0.25\ W$$

- The pf is lagging since the current is lagging the volatage

$$pf = \cos(\theta) = \frac{1}{\sqrt{2}} = 0.707$$

Power Factor- Problem

- c) The current in the circuit is given by

$$I = \frac{1}{1+j} + \frac{j}{4} = \frac{1-j}{2} + \frac{j}{4}$$

$$= \frac{1}{2} - \frac{j}{4} = \frac{\sqrt{5}}{4}\angle{-26.565} = 0.56\angle{-26.656^0}$$

The rms vlaues of current and voltage are $I_{rms} = 0.395\,A$ and $V_{tms} = 1/\sqrt{2}\,V$.

$$\therefore P_{app} = V_{rms}I_{rms} = \frac{0.395}{\sqrt{2}} = 0.2795\,VA$$

$$P_{AV} = P_{app}\cos(\theta) = \frac{1}{4} = 0.25\,W$$

- The pf is lagging since the current is lagging the volatage

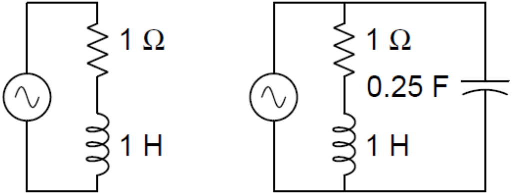
$$pf = \cos(26.565) = 0.894$$

Power Factor- Problem

- What happened to the power factor in the two circuits

$$pf1 = \cos(45) = 0.707 \;\;\&\;\; pf2 = \cos(26.565) = 0.894$$

- By adding a capacitor in parallel to the load the power factor increases. The average power remains the same but the rms current delivered by the load reduces from 0.5 A to 0.395 A



Power Factor- Problem

- d) The current in the circuit is given by

$$I = \frac{1}{1+j} + \frac{j}{2} = \frac{1-j}{2} + \frac{j}{2}$$

$$= \frac{1}{2} = 0.5\angle 0^0$$

The rms vlaues of current and voltage are $I_{rms} = 0.353\ A$ and $V_{tms} = 1/\sqrt{2}\ V$.

$$\therefore P_{app} = V_{rms} I_{rms} = \frac{1}{2\sqrt{2}\cdot\sqrt{2}} = 0.25\ VA$$

$$P_{AV} = P_{app}\cos(\theta) = \frac{1}{4} = 0.25\ W$$

- The pf in this case is unity

$$pf = \cos(0) = 1$$

Complex Power

- After having defined complex voltages and currents, the next logical step is to introduce complex power. Again the average power can be seen as the real part of a complex number.
- If $V_e\angle\phi_1$ is a voltage phasor and $I_e\angle\phi_2$ is the current phase across a load. Here V_e and I_e represent the rms values of the current and voltage across the load. Then the average power is given by

$$P_{AV} = P = V_e I_e \cos(\phi_1 - \phi_2) = Re\left[V_e I_e e^{j(\phi_1-\phi_2)}\right]$$

$$= Re\left[V_e e^{j\phi_1} I_e e^{-j\phi_2}\right]$$

- Therefore we can define the complex power S

$$S = V_{rms} {I_{rms}}^*$$

where $V_{rms} = V_e e^{j\phi_1}$ and $I_{rms} = I_e e^{j\phi_2}$ are the rms [hasors of the voltage and current

Complex Power

- The complex power is given by

$$S = V_{rms} {I_{rms}}^{*} = V_e I_e \cos(\theta) + j V_e I_e \sin(\theta)$$
$$= P + jQ$$

where $\theta = \phi_1 - \phi_2$ is the phase difference between voltage and current phasors

- The term P is the average power or the real power and Q is called as the reactive power

$$P = V_e I_e \cos(\theta)$$

- The term Q is called as the reactive power

$$Q = V_e I_e \sin(\theta)$$

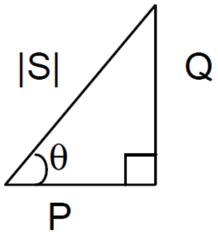
Complex Power and Power Triangle

- The magnitude of the complex power S is the apparent power

$$P_{app} = |S| = V_e I_e = \sqrt{P^2 + Q^2}$$

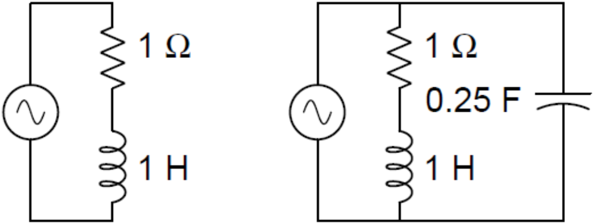
- The pf aggle is given by $\theta = tan^{-1}(Q/P)$
- The pf is given by

$$\cos(\theta) = \frac{P_{AV}}{P_{app}} = \frac{P}{\sqrt{P^2 + Q^2}} = \frac{P}{|S|}$$



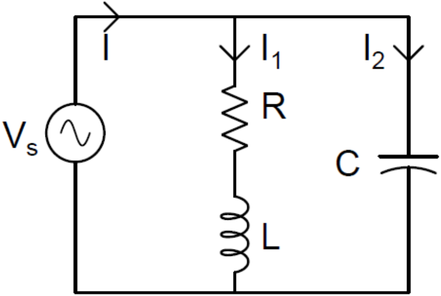
Complex Power - Problem

- Find the complex power absorbed by each circuit element in the circuit given below. given that the rms voltage phasor is $1\angle 0$ and $\omega = 1\, rad/sec$



Complex Power - Problem

- In the circuit given below, the resistance, capacitance and inductance values are chosen such that $L = 2CR^2$. Now the frequency of the circuit is varied from $0 \rightarrow \infty$. Find the frequency at which the power factor becomes unity and What is the magnitude of the current I and the complex power at that frequency in terms of R and C. Given $V_s = V_0\angle 0$



Complex Power - Problem

- The current in the circuit is given by

$$I = I_1 + I_2 = \frac{V_0}{R + j\omega L} + V_0(j\omega C)$$

$$= \frac{V_0(R - j\omega L)}{R^2 + (\omega L)^2} + jV_0\omega C$$

$$= \frac{V_0 R}{R^2 + (\omega L)^2} + jV_0\left(\omega C - \frac{\omega L}{R^2 + (\omega L)^2}\right)$$

- Since the pf is unity at the frequency of interest (ω_0), the complex part of the current is zero

$$\therefore \omega_0 C = \frac{\omega_0 L}{R^2 + (\omega_0 L)^2}$$

$$\implies \omega_0 = \sqrt{\frac{1}{LC} - \left(\frac{R}{L}\right)^2}$$

Complex Power - Problem

- Since it is given that $L = 2CR^2$,

$$\omega_0 = \sqrt{\frac{1}{LC} - \left(\frac{R}{L}\right)^2} = \frac{1}{2RC}$$

- The current at this frequency is

$$I = \frac{V_0 R}{R^2 + (\omega_0 L)^2} = \frac{V_0}{2R}$$

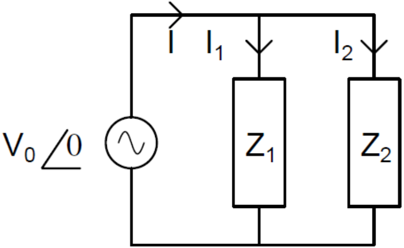
- The complex power delivered by the source is

$$S = \frac{V_s}{\sqrt{2}} \cdot \frac{I^*}{\sqrt{2}} P = \frac{1}{2} V_0 \cdot \frac{V_0}{2R} = \frac{{V_0}^2}{4R}$$

The complex power in this case is purely real and the reactive power delivered by the source is zero at this frequency

Power Factor Correction - Problem & Extension

- In the circuit given below, A voltage source with a rms phasor $V_0\angle 0$ is driving two loads Z_1 and Z_2. The power absorbed by the load Z_1 is P Watts with a pf of 0^0 and the power absorbed by the load Z_2 is P with a pf of $\cos(\theta)$ lagging. Find the value of the capacitor in terms of $P, \omega \& V_0$ to be added in parallel such the pf of the overall load is unity. Now if $P = 1\ kW$, $\omega = 100\ Hz$ $\theta = 45^0$ and $V_0 = 100\ V$. Find the value of the capacitance in F

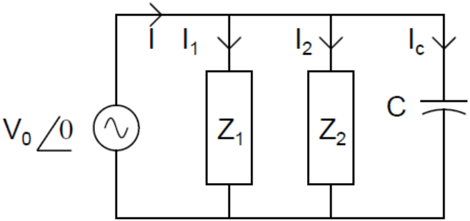


Power Factor Correction - Problem & Extension

- After adding the capactior The current in the circuit is given by

$$I = I_1 + I_2 + I_c = \frac{P}{V_0}\angle 0 + \frac{P}{V_0\cos(\theta)}\angle{-\theta} + j\omega C V_0$$

$$= \frac{2P}{V_0} - j\frac{P}{V_0}\tan(\theta) + j\omega C V_0$$



Power Factor Correction - Problem & Extension

- The total current is

$$= \frac{2P}{V_0} + j(\omega C V_0 - \frac{P}{V_0}\tan(\theta))$$

For unifty pf condition, the complex part of the resulting current has to be zero, thus we have

$$\frac{P}{V_0}\tan(\theta) = \omega C V_0$$

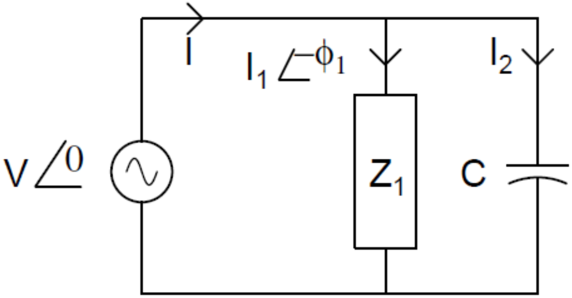
$$\implies C = \frac{P}{\omega {V_0}^2}\tan(\theta)$$

For a $P = 1000\ W$, $\theta = 45^0$, $\omega = 100\ Hz$ & $V_0 = 100\ V$ we have

$$C = 1\ mF$$

Power Factor Correction - Problem & Extension

- Let $\cos(\phi_1)$ be the pf before the pf correction. The pf correction involves adding a capacitor to reduce the magnitude of the current supplied by the voltage source. Let $\cos(\theta)$ be the pf after pf correction



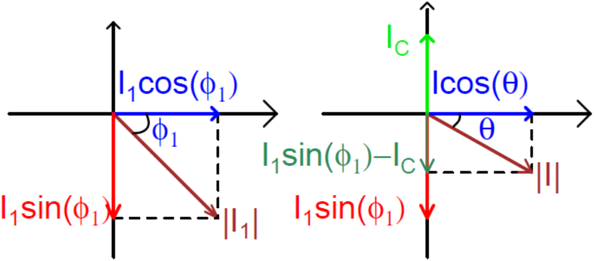
Power Factor Correction - Problem & Extension

- The supply current I, with a lagging pf of $\cos(\theta)$ is given by

  $$I = I_1 + I_2 = I_1\cos(\phi_1) - j(I_1\sin(\phi_1) - I_C) = I\cos(\theta) - jI\sin(\theta)$$

  Since the real part of the current is unchanged we have

  $$I\cos(\theta) = I_1\cos(\phi_1)$$



Power Factor Correction - Problem & Extension

- The complex power delivered by the source before pf correction is

$$S_1 = V I_1^* = V I_1 \cos(\phi_1) + jV I_1 \sin(\phi_1) = P + jQ_1$$

- The complex power delivered by the source after pf correction is

$$S = VI^* = VI \cos(\theta) + jVI \sin(\theta)$$

Since $I = I_1 + I_2 = I_1 \cos(\phi_1) - j(I_1 \sin(\phi_1) - \omega CV)$

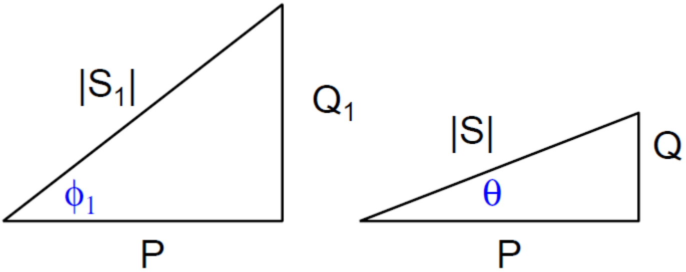
$$S = VI^* = VI_1 \cos(\phi_1) + jV(I_1 \sin(\phi_1) - \omega CV) = P + jQ$$

- Thus the real power remains the same in both cases, but the reactive power changes

Power Factor Correction - Problem & Extension

- The complex power before and after correction has the same real part. The apparent power reduces after pf correction.

$$\therefore VI_1\cos(\phi_1) = VI\cos(\theta) \implies I_1\cos(\phi_1) = I\cos(\theta)$$



Power Factor Correction - Problem & Extension

- The complex power before pf correction is $S_1 = VI_1 \cos(\phi_1) + jVI_1 \sin(\phi_1) = P + jQ_1$
- The complex power after pf correction is

$$S = VI_1 \cos(\phi_1) + jV(I_1 \sin(\phi_1) - \omega CV) = P + jQ$$

- Tha value of the capacitor can also be found from the reactive power

$$Q = VI \sin(\theta) = VI_1 \sin(\phi_1) - \omega CV^2 = Q_1 - \omega CV^2$$

$$C = \frac{Q_1 - Q}{\omega V^2}$$