MATHEMATICS II

MATH F112

Department of Mathematics

BITS Pilani K K Birla Goa Campus

Inverse of a Matrix

Matrix inverse

An n x n matrix ***A*** is invertible (nonsingular) if there exist an n x n matrix ***B*** such that

$$AB = BA = I_n$$

The matrix ***B*** is called an inverse of ***A*.** If there exist no such ***B***, then ***A*** is called noninvertible (or singular).

Example

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 3/2 \\ 1 & -1 \end{bmatrix}$$

$$AB = BA = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Important Result

If a square matrix ***A*** has an inverse, then the inverse is unique.

How will it help us solve system of equations?

Because if we can express a system of equations in the form

$$AX = b$$

Then we can multiply both sides by the inverse matrix

$$A^{-1}AX = A^{-1}b$$

And we can then know the values of X because

$$\boxed{A^{-1}A = I}$$

$$\boxed{X = A^{-1}b}$$

From now onwards we will denote the inverse of ***A***, if exist, by ***A*$^{-1}$**.

Example 2: Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

The problem of finding inverse lead us to the problem of solving linear systems (???).

Properties of Inverse

(i) If $\boldsymbol{A}$ is nonsingular matrix , then its inverse is also nonsingular and

$$(A^{-1})^{-1} = A$$

(ii) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are nonsingular matrices of same size, then $\boldsymbol{AB}$ is nonsingular and

$$(AB)^{-1} = B^{-1}A^{-1}$$

(iii) If A is a nonsingular matrix , then

$$(A^T)^{-1} = (A^{-1})^T$$

A Practical method of finding A^{-1}

A Practical method of finding A^{-1}

- If $\boldsymbol{A}$ is an n x n matrix, we first construct the $\boldsymbol{n}$ **x** $\boldsymbol{2n}$ matrix that has the entries of $\boldsymbol{A}$ on the left and of the identity matrix $\boldsymbol{I_n}$ on the right:

$$\left[\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 1 \end{array}\right]$$

- We then use the elementary row operations on this new large matrix to change the ***left side*** into the identity matrix.
  - This means that we are changing the large matrix to reduced row-echelon form.
- The right side is transformed automatically into A^{-1}
- ***In case the left side is not reduced to an identity matrix then ?????***

Example 1:

Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix}$$

We begin with

$$\left[\begin{array}{ccc:ccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 2 & -3 & -6 & 0 & 1 & 0 \\ -3 & 6 & 15 & 0 & 0 & 1 \end{array}\right]$$

We then transform the left half of this new matrix into the identity matrix—by performing a sequence of elementary row operations on the entire new matrix.

$$A^{-1} = \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix}$$

- We calculate AA^{-1} and $A^{-1}A$, and verify that both products give the identity matrix I_3.

$$AA^{-1} = \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix} \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 2:

Consider the matrix

$$A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$$

Applying the procedure of Example 4 yields

$$\left[\begin{array}{rrr|rrr} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{rrr|rrr} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right]$$

We added -2 times the first row to the second and added the first row to the third.

$$\left[\begin{array}{rrr|rrr} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

We added the second row to the third.

Since we have obtained a row of zeros on the left side, A is not invertible.

Problems :

Find the inverse of the following matrices

$$(i)\ A=\begin{bmatrix}1 & 1 & 1\\ 0 & 2 & 3\\ 5 & 5 & 1\end{bmatrix}\quad (ii)\ B=\begin{bmatrix}10 & 41 & -5\\ -1 & -12 & 1\\ 3 & 20 & -2\end{bmatrix}$$

Elementary Matrices

A square matrix ***E*** is said to be ***elementary matrix*** if it can be obtained from an identity matrix by performing a *single* elementary row operation.

Examples:

Listed below are four elementary matrices and the operations that produce them.

$$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiply the second row of I_2 by -3.

Interchange the second and fourth rows of I_4.

Add 3 times the third row of I_3 to the first row.

Multiply the first row of I_3 by 1.

Theorem: Row operation by matrix multiplication

If the elementary matrix $\boldsymbol{E}$ results from performing a certain row operation on $\boldsymbol{I_m}$ and if $\boldsymbol{A}$ is an m x n matrix, then the product $\boldsymbol{EA}$ is the matrix that results when this same row operation is performed on $\boldsymbol{A}$.

Example:

Let

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Compute E_1A, E_2A, and E_3A, and describe how these products can be obtained by elementary row operations on A.

Verify that

$$E_1A=\begin{bmatrix} a & b & c \\ d & e & f \\ g-4a & h-4b & i-4c \end{bmatrix}, \quad E_2A=\begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix},$$

$$E_3A=\begin{bmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{bmatrix}$$

- Addition of -4 times row 1 of A to row 3 produces E_1A.

- An interchange of rows 1 and 2 of A produces E_2A, and multiplication of row 3 of A by 5 produces E_3A

Theorem

Every elementary matrix is invertible, and the inverse is also an elementary matrix.

Important Result

(1) An n x n matrix $\boldsymbol{A}$ is non-singular if and only if it is row equivalent to $\boldsymbol{I_n}$.

(2) If A is an n x n matrix, the homogeneous system $\boldsymbol{AX=0}$, has nontrivial solution if and only if $\boldsymbol{A}$ is singular.

(3) If A is an n x n matrix, then A is nonsingular if and only if $\boldsymbol{AX=b}$ has a unique solution for every $\boldsymbol{b}$.

Theorem

If $\boldsymbol{A}$ is invertible then $\boldsymbol{A}$ is expressible as a product of elementary matrices.

Problems :

Find the inverse of the following matrices, if it exist

$$(i)\ A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ 5 & -2 & -3 \end{bmatrix} \quad (ii)\ B = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 7 & 6 \end{bmatrix}$$

Problems:

Show that square matrix $\boldsymbol{A}$ is invertible if and only if $\det(\boldsymbol{A})$ is not zero.