Mathematics II, Spring 2020

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Part II Complex Variables

R.V. Churchill and J.W. Brown, *Complex Variables and Applications*, McGraw-Hill, 8th edition, 2014.

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Consulting Hours: Thursday 4PM to 5PM.

Definition

A complex number is a pair (x, y) of real numbers x, y.

It is customary to denote a complex number by z. Here the real numbers x and y are known as *real* and *imaginary* parts of z, respectively and written as

$$x = \operatorname{Re} z \qquad y = \operatorname{Im} z$$

Addition and Multiplication

The complex numbers $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ are “added” and “multiplied” as follows:

$$z_1 + z_2 = (x_1, y_2) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$$z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$$

Notations

We can write a complex number $z = (x, y)$ as a sum

$$(x, 0) + (0, 1)(y, 0)$$

Let us identify the complex number $(x, 0)$ with the real number x, we see that:

$$(x, y) = x + (0, 1)y$$

Note that $(0, 1)(0, 1) = (-1, 0) = -1$, so denoting $(0, 1)$ by the symbol i we can write a complex number $(x, y) = x + \mathrm{i}y$, where $\mathrm{i}^2 = -1$.

Algebraic Properties

Additive identity $0 = (0, 0)$.

Additive inverse of $z = (x, y)$ is $-z = (-x, -y)$.

Multiplicative identity $1 = (1, 0)$.

Multiplicative inverse of $z = (x, y) \neq (0, 0)$ is

$$z^{-1} = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right)$$

Algebraic Properties

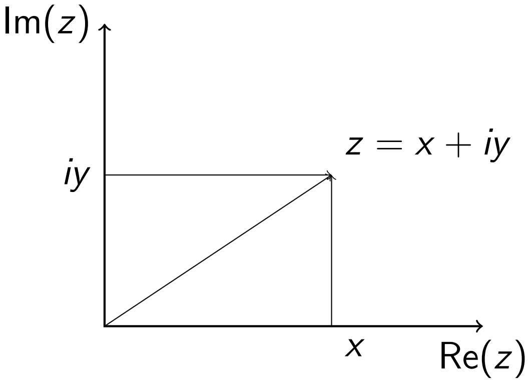
1. Two complex numbers $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ are equal if and only if $x_1 = x_2$ and $y_1 = y_2$.
2. If product of two complex numbers z_1, z_2 is equal to zero then at least one of the two is equal to zero.
3. The binomial theorem holds for the complex numbers, i.e.,

$$(z_1 + z_2)^n = \sum_{k=0}^{k=n} \binom{n}{k} z_1^{n-k} z_2^k$$

Argand plane

The complex number $x + iy$ may be visualized as a point in a plane with coordinated (x, y), this plane is thought to be the plane where the complex numbers reside, as a result it is called “the complex plane” wiki. Sometime this plane is also called “the Argand plane” after the mathematician Argand who popularized this visualization method. It is a very good visualization method that brings out the geometric properties and applications of the complex numbers quite effectively as we will see soon.

Argand plane



Absolute value

Absolute value

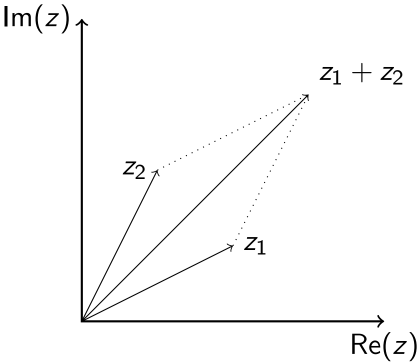
The distance from the origin to the point (x, y) is called the absolute value of the complex number $z = x + \mathrm{i}y$. Which is denoted by $|z|$.

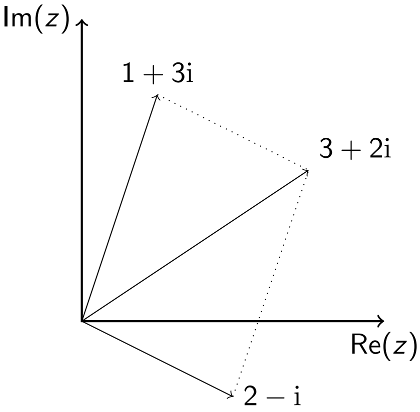
So for a complex number $z = x + \mathrm{i}y$ we can write:

$$|z| = \sqrt{x^2 + y^2} = \sqrt{(\text{ Re } z)^2 + (\text{ Im } z)^2}$$

Note: $\text{Re } z \leq |z|$ and $\text{Im } z \leq |z|$,
$z \neq 0$ if and only if $|z| \neq 0$.

Vectors and Moduli



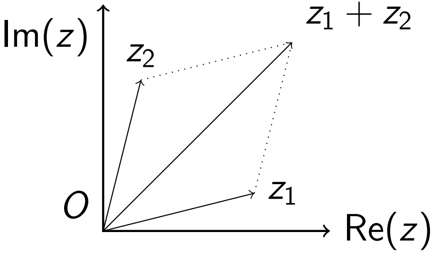

Im(z)
1 + 3i
3 + 2i
Re(z)
2 − i

Triangle inequality

Triangle inequality

For any two complex numbers z_1, z_2, $|z_1 + z_2| \leq |z_1| + |z_2|$.

The above inequality follows from the diagram:



Corollaries of triangle inequality

1. $|z_1 + z_2| \geq ||z_1| - |z_2||$

Conjugate

Conjugate

For a complex number $z = x + \mathrm{i}y$ the complex number $x - \mathrm{i}y$ is called the conjugate of z and it is denoted by $\bar{z}$.

1. Using the conjugate one can write $|z|^2 = z\bar{z}$.
2. The conjugate satisfies $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$ and $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$ for $z_2 \neq 0$.

Parallelogram identity and other geometric properties

Parallelogram Law

z_1, z_2 are complex numbers then

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$$

Polar form

A complex number $z = (x + \mathrm{i}y)$ on the complex plane have polar coordinates (r, θ), where $r = |z|$ and θ has an infinite number of possible values, including negative ones, that deffer by integral multiple of 2π.

Those values can be determined from the equation $\tan\theta = y/x$, where the quadrant containing the point corresponding to z must be specified.

Each value of θ is called the argument of z and the set of all such values is denoted by $\arg z$.

Polar form

The value of the argument in the interval $(-\pi, \pi]$ is called the **principal value** of the argument, denoted by Arg z. So,

$$arg\ z = \{Arg\ z + 2n\pi\ :\ n = 0, \pm 1, \pm 2, \pm 3, \ldots\}.$$

So we can write a complex number

$$z = |z|\cos(arg\,(z)) + \mathrm{i}|z|\sin(arg\,(z))$$

If z_1, z_2 are two complex numbers with arguments θ_1, θ_2 respectively then we have

$$z_1 z_2 = |z_1||z_2|(\cos(\theta_1 + \theta_2) + \mathrm{i}\sin(\theta_1 + \theta_2))$$

De Moivre's identity

Euler's formula

The expression

$$e^{i\theta} = \cos\theta + i\sin\theta$$

is called Euler's formula, it will be put on a sound footing when we introduce complex exponential by it's power series, but for now let us take the exponential as a symbol.

From the previous slides we can write a complex number z as $z = |z|e^{i\theta}$ where $\theta = arg(z)$.

Cont.

1. Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ be two complex numbers then
$$z_1 z_2 = r_1 r_2 e^{\mathrm{i}(\theta_1+\theta_2)}$$
2. If $r_2 \neq 0$ then we can also write the division as
$$z_1 z_2^{-1} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$
3. $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ and $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$

Cont.

De Moivre's Identity

For any complex number $re^{i\theta}$ and an integer n we have:

$$(re^{i\theta})^n = r^n e^{in\theta}$$

Application

A good number of application of De Moivre's identity may be found in trigonometry.

Example Let us find a formula for $\sin 3\theta$ in terms of $\sin\theta$. Consider the complex number $e^{i\theta} = \cos\theta + i\sin\theta$, let us raise both sides to the power 3.

$$(e^{i\theta})^3 = e^{i3\theta} = \cos 3\theta + i\sin 3\theta$$

Now the left hand side is equal to $(\cos\theta + i\sin\theta)^3$

Which after a simple calculation

$$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$$

Roots and arguments

So equating the imaginary part of both the sides we get:

$$\sin 3\theta = 3\cos^2\theta \sin\theta - \sin^3\theta.$$

One application of the polar form of a complex number is the ease of finding roots.
Let us consider the complex number $z_0 = r_0 e^{i\theta_0}$ and let us have a natural number $n \in \mathbb{N}$, we want to find a nth root of the complex number z_0.
So we are looking for a complex number $z = re^{i\theta}$ such that

$$z^n = z_0 \qquad \text{or} \qquad r^n e^{in\theta} = r_o e^{i\theta_0}$$

Cont.

Now equating the modulus and the arguments we get

$$r^n = r_0, \text{and} \qquad n\theta = \theta_0 + 2\pi k \quad \text{for} k \in \mathbb{Z}$$

Or we get

$$r = \sqrt[n]{r_0} \quad \text{and} \quad \theta = \frac{\theta_0}{n} + \frac{2\pi k}{n}$$

for $k \in \mathbb{Z}$.
So the distinct roots are obtained for the values $k = 0, 1, 2, \ldots, n-1$ and the distinct roots are given by

$$c_k = \sqrt[n]{r_0} e^{i(\frac{\theta}{n} + \frac{2\pi k}{n})}$$

Quetions

1. Find the fifth roots of $(1+i)$.

2. Show that three distinct points in the complex plane z_1, z_2, z_3 are collinear if and only if

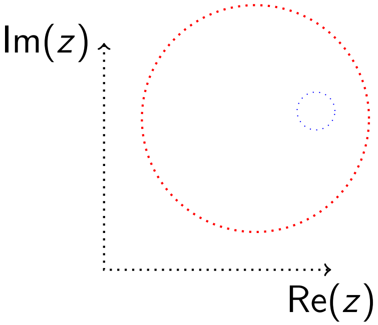
$$z_1\overline{(z_2 - z_3)} + z_2\overline{(z_3 - z_1)} + z_3\overline{(z_1 - z_2)} = 0$$

Neighborhoods

1. $|z_1 - z_2|$ defines a "distance" between z_1 and z_2 on the complex plane.
2. Given z_0 a complex number and ϵ a real positive number, we have a ϵ neighborhood of z_0 defined as the set $\{z \in \mathbb{C} : |z - z_0| < \epsilon\}$.
3. A deleted neighborhood of z_0 is the set of points $\{z \in \mathbb{C} : 0 < |z - z_0| < \epsilon\}$.

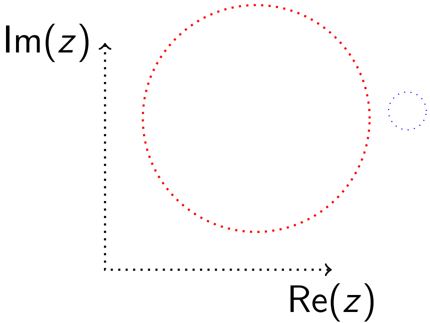
Interior Point

1. A point z is called an interior point of a set $S \subset \mathbb{C}$ if there is an ϵ neighborhood of z for some ϵ which is completely contained in S.



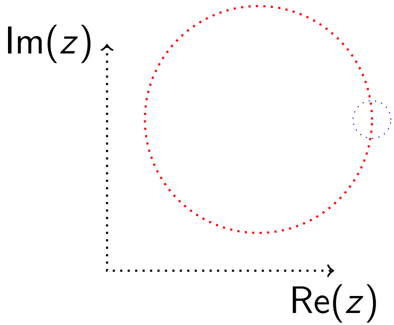
Exterior Point

1. It is called an exterior point if there is a neighborhood which is completely not contained in S. Or completely contained in the complement of the set S.



Boundary Point and Boundary

1. It is called a boundary point if it is neither an interior point nor an exterior point of the set S.

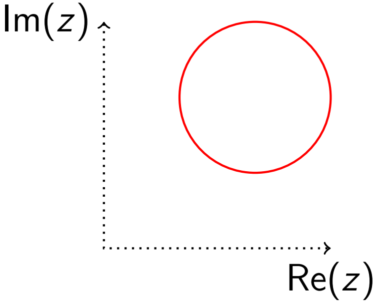


2. The set of boundary points is called the boundary of the set.

Open and Closed Sets

1. A set is called open if it contains none of it's boundary points.

2. A set is closed if it contains all it's boundary points.



Connected Set, Domain and Region

1. A set is called connected if given any two points in the set there is a polygonal line connecting the two points inside the set.
2. A nonempty open connected set in the complex plane is called a domain.
3. A domain with some (none, all, some) of it's boundary points is called a region.

Bounded Regions and Accumulation Point

1. A region is called bounded if it is contained in a circle of radius R for some R.
   $H = \{z \in \mathbb{C} : Re(z) > 0\}$ is an example of an unbounded region.

2. A point z is called an accumulation point (or a limit point) of S if every deleted neighborhood of the point contains a point of the set S.

$$\{\frac{1}{n} : n \in \mathbb{N}\}$$