MATHEMATICS II MATH F112

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Matrix inverse

An $n \times n$ matrix A is invertible (nonsingular) if there exist an $n \times n$ matrix B such that

$$AB = BA = I_n$$

The matrix B is called an inverse of A. If there exist no such B, then A is called noninvertible (or singular).

Example

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 3/2 \\ 1 & -1 \end{bmatrix}$$

$$AB = BA = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Important Result

If a square matrix A has an inverse, then the inverse is unique.

How will it help us solve system of equations?

Because if we can express a system of equations in the form AX = h

Then we can multiply both sides by the inverse matrix

$$A^{-1}AX = A^{-1}b$$

And we can then know the values of *X* because

$$A^{-1}A = I$$

From now onwards we will denote the inverse of A, if exist, by A^{-1} .

Example 2: Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

The problem of finding inverse lead us to the problem of solving linear systems (???).

Properties of Inverse

(i) If A is nonsingular matrix, then its inverse is also nonsingular and

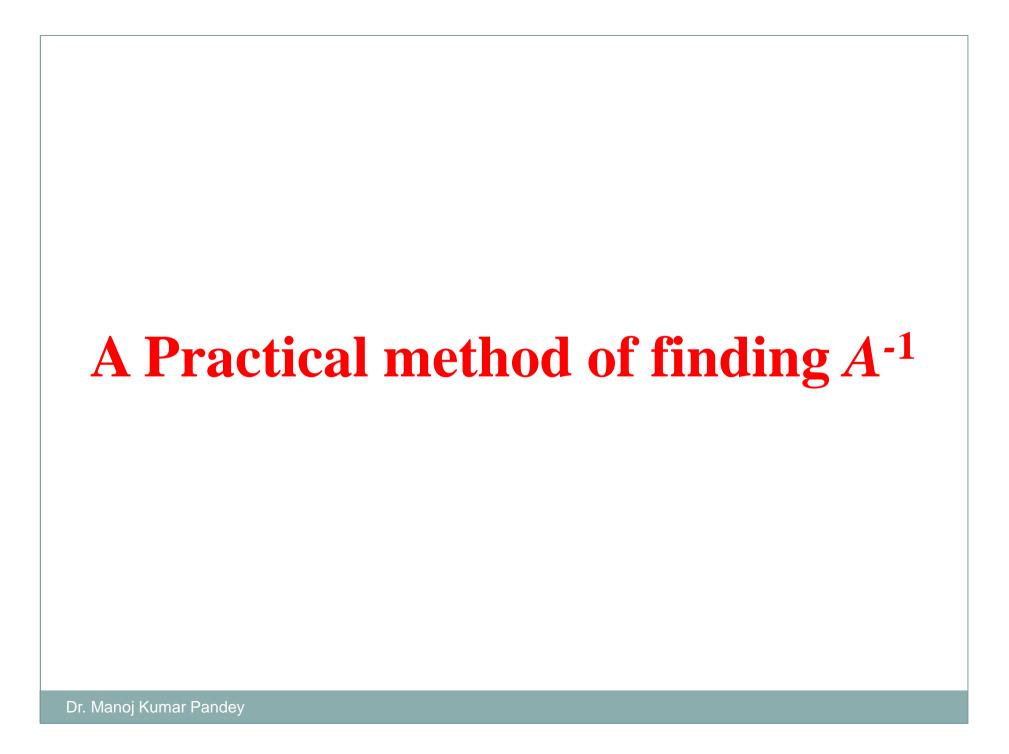
$$(A^{-1})^{-1} = A$$

(ii) If A and B are nonsingular matrices of same size, then AB is nonsingular and

$$(AB)^{-1} = B^{-1}A^{-1}$$

(iii) If A is a nonsingular matrix, then

$$(A^T)^{-1} = (A^{-1})^T$$



A Practical method of finding A⁻¹

• If A is an $n \times n$ matrix, we first construct the $n \times 2n$ matrix that has the entries of A on the left and of the identity matrix I_n on the right:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 1 \end{bmatrix}$$

- We then use the elementary row operations on this new large matrix to change the *left side* into the identity matrix.
 - This means that we are changing the large matrix to reduced row-echelon form.
- The right side is transformed automatically into A^{-1}
- In case the left side is not reduced to an identity matrix then ?????

Example 1:



$$A = \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix}$$

We begin with

We then transform the left half of this new matrix into the identity matrix—by performing a sequence of elementary row operations on the entire new matrix.

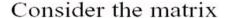
$$A^{-1} = \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix}$$

• We calculate AA^{-1} and $A^{-1}A$, and verify that both products give the identity matrix I_3 .

$$AA^{-1} = \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix} \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 2:



$$A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$$

Applying the procedure of Example 4 yields

$$\begin{bmatrix} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{bmatrix}$$

We added -2 times the first row to the second and added the first row to the third.

We added the second row to the third.

Since we have obtained a row of zeros on the left side, A is not invertible.

Problems:

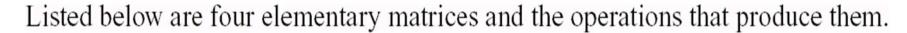
Find the inverse of the following matrices

$$(i) A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{bmatrix} \quad (ii) B = \begin{bmatrix} 10 & 41 & -5 \\ -1 & -12 & 1 \\ 3 & 20 & -2 \end{bmatrix}$$

Elementary Matrices

A square matrix *E* is said to be *elementary matrix* if it can be obtained from an identity matrix by performing a *single* elementary row operation.

Examples:



$$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
Multiply the second row of I_2 by -3 .

Interchange the second and fourth rows of I_3 to the first row.

Add 3 times the third row of I_3 to the first row.

Interchange the first row of I_3 by 1.

Theorem: Row operation by matrix multiplication

If the elementary matrix E results from performing a certain row operation on I_m and if A is an $m \times n$ matrix, then the product EA is the matrix that results when this same row operation is performed on A.

Example:

Let
$$E_{1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \quad E_{2} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$E_{3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Compute E_1A , E_2A , and E_3A , and describe how these products can be obtained by elementary row operations on A.

Verify that

$$E_{1}A = \begin{bmatrix} a & b & c \\ d & e & f \\ g - 4a & h - 4b & i - 4c \end{bmatrix}, E_{2}A = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}$$

$$E_{3}A = \begin{bmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{bmatrix}$$

• Addition of -4 times row 1 of A to row 3 produces E_1A .

• An interchange of rows 1 and 2 of A produces E_2A , and multiplication of row 3 of A by 5 produces E_3A

Theorem

Every elementary matrix is invertible, and the inverse is also an elementary matrix.

Important Result

- (1) An $n \times n$ matrix A is non-singular if and only if it is row equivalent to I_n .
- (2) If A is an $n \times n$ matrix, the homogeneous system AX=0, has nontrivial solution if and only if A is singular.
 - (3) If A is an $n \times n$ matrix, then A is nonsingular if and only if AX=b has a unique solution for every b.

Theorem

If A is invertible then A is expressible as a product of elementary matrices.

Problems:

Find the inverse of the following matrices, if it exist

$$(i) A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ 5 & -2 & -3 \end{bmatrix} \quad (ii) B = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 7 & 6 \end{bmatrix}$$

Problems:

Show that square matrix A is invertible if and only if det(A) is not zero.