MATHEMATICS II

MATH F112

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Text Book (Linear Algebra)

- Elementary Linear Algebra with Supplemental Applications

by Howard Anton and Chris Rorres,
Wiley & Sons, 11th Ed., 2017

Text Book (Complex Variables)

- Complex Variables and applications,

by R.V. Churchill and J.W. Brown,
McGraw-Hill, 8th edition, 2014.

Interesting Books: Linear Algebra

- Gilbert Strang, Linear Algebra and its Applications.
- B. Kolman and D. R. Hill, Introductory Linear Algebra, An Applied First Course.
- K. Hoffman and R. Kunge, Linear Algebra

Evaluation Components

Component	Duration	Marks (300)	Remarks
Mid-Sem. Exam	1 hour 30 min.	90	Closed Book
Announced Quizzes		80	Open Book
Comprehensive Exam	3 hours	120	Closed Book

About the Course

- **Linear Algebra**
- **Complex Analysis (theory of complex valued functions)**

Linear Algebra

- Solving system of linear equations
  - Gaussian Elimination
  - Elementary Matrices and method of finding inverse
- Vector Spaces
- Linear Transformations
- Eigenvalues and Eigenvectors

Complex Variables

- Functions of complex variables
  - Limit, Continuity, Differentiability.
  - Analytic functions
- Elementary Functions
- Integrations
- Power Series and its extension
- Applications of Complex valued functions
  - Evaluating improper integrals

I/C Instructions

- Follow the course page in LMS regularly for any updates (particularly for the course handout).

- Chamber Consultation Hours (section 1-3)
  - Chamber Number – CC207
  - Friday- 3:00 – 5:30 PM

Introduction to Linear Algebra

Linear Algebra has become as basic and as applicable as calculus and fortunately it is easier

- Gilbert Strang

System of Linear Equations

Let us consider a system of linear equations

$$x + y = 2$$

$$2x + 3y = 5$$

Let us take the following questions.

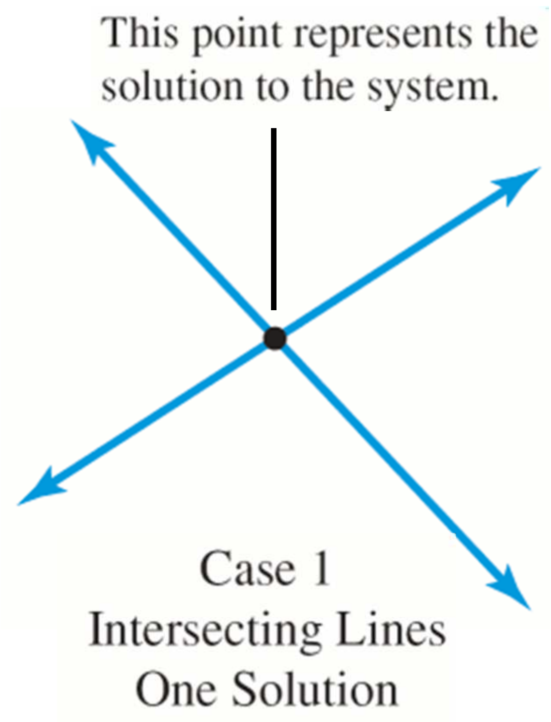
What is a solution ?

How to solve?

How these system of equations appear in applications?

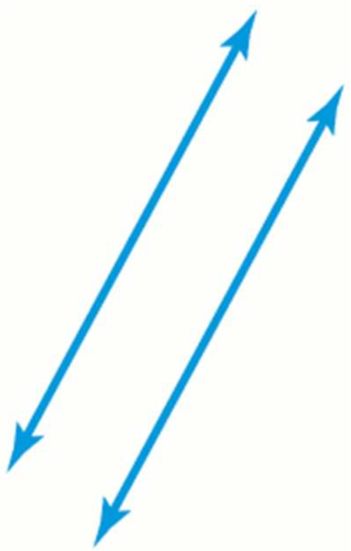
Systems of Linear Equations

Graphing a system of two linear equations in two unknowns gives one of three possible situations:



Case 1: Lines intersecting in a single point. The ordered pair that represents this point is the *unique solution* for the system.

Systems of Linear Equations

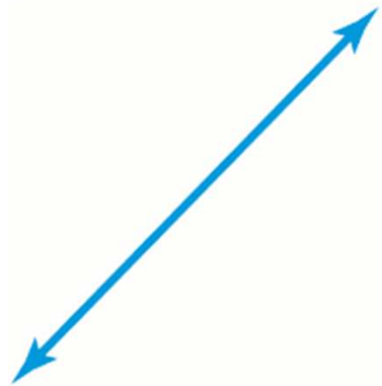


Case 2
Parallel Lines
No Solutions

Case 2: Lines that are distinct parallel lines and therefore don't intersect at all. Because the lines have no common points, this means that the system has *no solutions*.

Systems of Linear Equations

Case 3: Two lines that are the same line. The lines have an infinite number of points in common, so the system will have *an infinite number of solutions.*



Case 3
Same Line
Infinite Number
of Solutions

Elementary Row Operations (ERO)

Elementary row operations on an m x n matrix **A**:

- Interchange rows of **A**
- Multiply a row by a non-zero constant
- Add a constant times row r to row s.

Solving Linear System

Let us consider the linear system:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad\qquad \vdots \qquad\qquad\quad \vdots \qquad &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_{m1}
\end{aligned}
$$

We can abbreviate the system by writing only the rectangular array of numbers

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} & b_1 \\ a_{21} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{bmatrix} = [\, A \mid b \,]$$

This is called the **augmented matrix** for the system.

Example

Let us consider the linear system:

$$x_2 - 4x_3 = 8$$
$$2x_1 - 3x_2 + 2x_3 = 1$$
$$5x_1 - 8x_2 + 7x_3 = 1$$

The augmented matrix is

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

Elementary Row Operations (ERO)

Symbol	Description
$R_i \rightarrow R_i + kR_j$	Change the *i*th row by adding *k* times row *j* to it. Then, put the result back in row *i*.
$R_i \rightarrow kR_i$	Multiply the *i*th row by *k*.
$R_i \leftrightarrow R_j$	Interchange the *i*th and *j*th rows.

Solving Linear System

Solve the system using Elementary Row operations:

$$x + 2y + 3z = 6$$

$$2x - 3y + 2z = 14$$

$$3x + y - z = -2$$

Problem

Solve:

$$x + y + 2z = 9$$

$$2x + 4y - 3z = 1$$

$$3x + 6y - 5z = 0$$

Row Echelon and Reduced Row Echelon Form

An m x n matrix $\boldsymbol{A}$ is said to be in **row echelon** form if it satisfies the following properties:

1. The first nonzero number in each row (reading from left to right) is 1. This is called the leading entry (**leading 1**).

2. The leading entry in each row is to the right of the leading entry in the row immediately above it.

3. All rows consisting entirely of zeros are at the bottom of the matrix.

Row Echelon and Reduced Row Echelon Form

A matrix is in **reduced row echelon** form if it is in row echelon form and also satisfies the following condition:

4. Every column that contains a leading 1 has zeros everywhere else in that column.

Examples:

Row-Echelon Form	Reduced Row-Echelon Form
$$\begin{bmatrix} 1 & 3 & -6 & 10 & 0 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ Leading 1's shift to the right in successive rows.	$$\begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ Leading 1's have 0's above and below them.

Problems

Check whether the following matrices are in row echelon form or in reduced row echelon form:

$$(i)\ \begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix} \qquad (ii)\ \begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(iii)\ \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition

An m x n matrix ***A*** is said to be row equivalent to an m x n matrix ***B*** if ***B*** can be obtained by applying a finite sequence of elementary row operations to the matrix ***A*.**

Theorem

Let $\boldsymbol{A}X=\boldsymbol{b}$ and $\boldsymbol{C}X=\boldsymbol{d}$ be two linear systems each of m equations in n unknowns. If the augmented matrices $[\boldsymbol{A}|\boldsymbol{b}]$ and $[\boldsymbol{C}|\boldsymbol{d}]$ of these systems are row equivalent, then both linear systems have exactly the same solutions .

Methods of Solving Linear Systems

The results established provide us with two methods for solving linear systems:

- Gauss Elimination Method (based on row echelon form)
- Gauss-Jordan Elimination Method (based on reduced row echelon form)

Example

Solve the linear system

$$x + 2y + 3z = 9$$
$$2x - y + z = 8$$
$$3x - z = 3$$

by

(i) Gauss Elimination Method

(ii) Gauss-Jordan Elimination Method

Problems

Solve the linear system of equations by using both Gauss Elimination and Gauss-Jordan Elimination methods

$$(i)\quad \begin{aligned} x+y+2z-5w&=3\\ 2x+5y-z-9w&=-3\\ 2x+y-z+3w&=-11\\ x-3y+2z+7w&=-5 \end{aligned}$$

$$(ii)\quad \begin{aligned} x+2y+3z+4w&=5\\ x+3y+5z+7w&=11\\ x-z-2w&=-6 \end{aligned}$$

Linear Systems

Let us consider the linear system $\boldsymbol{AX=b}$:

(a) The linear system is said to be consistent if it has at least one solution.

(b) The linear system with no solution is said to be inconsistent system.

Homogeneous Linear Systems

The linear system $\boldsymbol{AX}=\mathbf{0}$ is said to be homogeneous system. A homogeneous system is always consistent (??).

Solve the homogeneous system by Gauss elimination method

$$
\begin{aligned}
x+y+z+w&=0\\
x+\qquad\quad w&=0\\
x+2y+z&=0
\end{aligned}
$$

Rank of a Matrix

Let ***A*** be an m x n matrix, then the rank of ***A***, denoted by rank(***A***), is equal to the number of non-zero rows in its reduced row echelon form.

Problems

Find the ranks of the following matrices:

$$(i)\ A=\begin{bmatrix}1 & 2 & 3\\ 2 & -1 & 1\\ 3 & 0 & -1\end{bmatrix}\quad (ii)\ B=\begin{bmatrix}1 & 1 & 2 & -5\\ 2 & 5 & -1 & -9\\ 2 & 1 & -1 & 3\\ 1 & -3 & 2 & 7\end{bmatrix}$$

$$(iii)\ C=\begin{bmatrix}1 & 2 & 3 & 4\\ 1 & 3 & 5 & 7\\ 1 & 0 & -1 & -2\end{bmatrix}$$

Important Result

Let $\boldsymbol{AX}=\boldsymbol{b}$ be a linear system with m equations and n unknowns then :

(a) The linear system is consistent if rank($\boldsymbol{A}$) = rank($\boldsymbol{A}|\boldsymbol{b}$).

- It has unique solution is rank($\boldsymbol{A}$) = rank($\boldsymbol{A}|\boldsymbol{b}$)=$n$.
- It has infinitely many solutions if rank($\boldsymbol{A}$) = rank($\boldsymbol{A}|\boldsymbol{b}$) < n.

(b) The linear system is inconsistent if rank($\boldsymbol{A}$) is not equal to rank($\boldsymbol{A}|\boldsymbol{b}$).

Example

For what values of a the linear system

$$x+2y-3z=4$$
$$3x-y+5z=2$$
$$4x+y+(a^2-14)z=a+2$$

has

(a) Unique solution
(b) Infinitely many solutions
(c) No solution.

Example

For what values of a the linear system

$$x+2y-3z=4$$

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$$4x+y+(a^2-14)z=a+2$$

has

(a) Unique solution

(b) Infinitely many solutions

(c) No solution.

Problems

(1) Solve the following linear systems using both Gauss elimination and Gauss-Jordan elimination methods:

$$(i)\quad \begin{aligned} x+y+z&=1 \\ x+y-2z&=3 \\ 2x+y+z&=2 \end{aligned} \qquad\qquad (ii)\quad \begin{aligned} x+y+2z+3w&=13 \\ x-2y+z+w&=8 \\ 3x+y+z-w&=1 \end{aligned}$$

Problems

(2) Find an equation relating a, b and c so that the linear system :

$$x + 2y - 3z = a$$
$$2x + 3y + 3z = b$$
$$5x + 9y - 6z = c$$

is consistent for any values of a, b and c that satisfy that equation.

Ans: $-3a-b+c=0$