Hybrid heuristic for a 3-level integrated lot-sizing and cutting stock problem with multiple suppliers and distribution decisions

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Abstract

In this paper, a generalized 3-level integrated lot-sizing and cutting stock problem (*G3ILSCS*) proposed in the literature is extended to take into account other relevant decisions of the supply chain. The extensions consist of the selection of suppliers for the raw materials used in the cutting process and the distribution of the final products from the production plant to a warehouse. To solve the integrated problem, a hybrid heuristic is proposed aiming to overcome the difficulties present in the integrated problem, i.e., the high number of variables, the multi-level structure, and the integrality of the decisions variables. The hybrid method embeds two decomposition approaches in each iteration of the algorithm: column generation and a relax-and-fix procedure. Due to the features of the problem, an innovative column generation procedure is proposed to manage the cutting patterns in the cutting stock problem, and the cargo configurations in the distribution problem. The models and solution approaches are analyzed with numerical experiments to assess the impact of incorporating other decisions of the supply chain into the integrated problem, as well as the performance of the hybrid heuristic when seeking a solution to the integrated problem.

Keywords: Integrated Lot-Sizing and Cutting Stock Problems, Supplier Selection, Distribution, Decomposition Approaches, Hybrid Heuristic.

1. Introduction

Given the current competitive pressures, companies are being forced to reexamine the processes and planning of their supply chain. For this, the development of new technologies, using mathematical models, is being employed to enhance the decision-making in different industrial and logistics settings. In this paper, we are interested in the tactical/operational decision planning in manufacturing industries that have their production processes linked to the cutting of raw materials (objects) and the production planning of end products (final products). These problems are known in the literature as the cutting stock problem (*CSP*) (Wang and Wäscher, 2002; Wäscher et al., 2007; Morabito et al., 2009; Gomes et al., 2016) and lot-sizing problem (*LSP*) (Brahimi et al., 2006; Buschkühl et al., 2010; Jans and Degraeve, 2008, 2007; Glock et al., 2014; Brahimi et al., 2017), respectively.

In these manufacturing settings, objects of large sizes, which are purchased and/or produced, are kept in stock to be cut later into smaller pieces of different sizes, using cutting patterns, in order to meet internal

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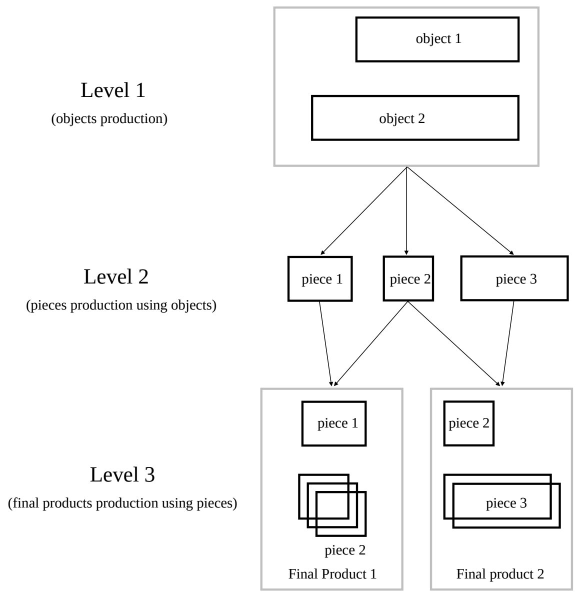
demand. These pieces then go to downstream levels of the production plant to produce the final products, which are then used to meet the clients' demand. Thus, the manufacturing processes involve planning the acquisition, production, and cutting of the objects, along with the production of the final products. Although in such production plants these processes affect each other, the decisions are usually taken separately, as has been pointed out in many studies in the literature (Thomas and Griffin, 1996; Drexl and Kimms, 1997; Jans and Degraeve, 2008; Melega et al., 2018). This tendency is, however, changing because practical cases have shown the benefits of dealing with an integrated approach (Poltroniere et al., 2008; Gramani et al., 2009; Vanzela et al., 2017).

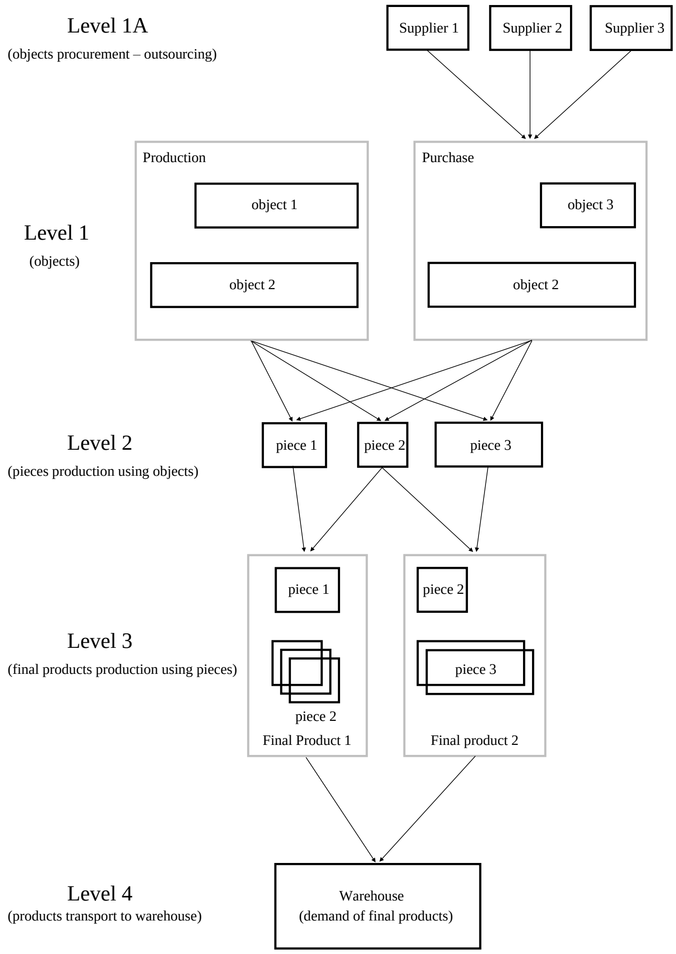
According to a recent literature review (Melega et al., 2018), a problem is classified as an integrated lot-sizing and cutting stock problem if it has two types of integration: integration across time periods and integration between production levels. In the review paper, the authors presented a 3-level integrated lot-sizing and cutting stock problem (*G3ILSCS*), which is exclusively used as a tool in the classification of the literature, i.e., there is no computational assessment regarding the proposed mathematical model. In this problem (see Part (a) in Figure 1), level 1 corresponds to the production planning of the objects used to fulfill the downstream level. Level 2 corresponds to the cutting process, in which the produced objects are cut into pieces by a cutting machine, according to cutting patterns. These pieces are then used to produce the final products at level 3. Therefore, the 3-level integrated problem has a multi-level production structure with limited resources in the production lines and a bill-of-material relationship, where a dependency of requirements exists among final products, pieces (final products components), and objects.

In this paper, the *G3ILSCS* problem is extended in order to consider other relevant decisions of the supply chain (see Part (b) in Figure 1). The first extension, (level 1A), consists of an alternative means to the acquisition of objects, as multiple suppliers are available (Ho et al., 2010; Chai et al., 2013). In addition to the production of the objects at the production plant (level 1), the objects can also be purchased from different suppliers that offer different objects with various fixed and variable costs. In the problem, the purchasing decisions define the number of objects to be purchased from each supplier, taking into account the inventory of these objects, in order to meet the internal demand for objects in the cutting process. In this way, the management of objects is related to the decisions on the number of produced and purchased objects, and it influences the cutting process at the production plant. Therefore, an integration between the decisions involving the production of objects, supplier selection, and the cutting operations will enable more efficient resource utilization with the objective of minimizing the overall expenses associated with the objects and the cutting process.

The second extension, (level 4), is associated with the loading and distribution configuration of the vehicles used in the transportation of the final products from the production plant to a warehouse. More specifically, it concerns the loading of the final products into vehicles, with the goal of transporting them from the production plant to a warehouse. In the problem, the loading and distribution decisions are associated with how to arrange loads of final products inside the vehicles in order to make better use of the volumetric and weight capacity of the vehicle, aiming to reduce the number of used vehicles. In this study, we allow the mixing of different final products as cargo configurations into the vehicles. As the loading decisions for the transport of final products are directly related to the lots of final products, it is appropriate to consider these decisions jointly, resulting in an integrated lot-sizing and loading problem. Note, however, that when considering the distribution decisions, we are not interested in the route of the vehicles or the transportation of the final products to the customers, since we only consider the transportation between the plant and the warehouse. Literature reviews on integrated lot-sizing and distribution problems can be found in Thomas and Griffin (1996), Bertazzi and Speranza (1999), Erengüç et al. (1999), and Darvish et al. (2021).



(a) 3-level integrated problem (*G3ILSCS*). Source: Melega et al. (2018).



(b) 4-level integrated problem with multiple supplier and distribution (*G4ILSCS*)

Figure 1: Multi-level structures for the integrated problem and extensions.

Considering the problems addressed in our study are classified as NP-hard (Maes et al., 1991; McDiarmid, 1999), we propose a hybrid heuristic with the purpose of overcoming the difficulties present in the integrated problem, which are, more specifically, the high number of decision variables, the multi-level structure, and the integrality of the decisions variables. The goal of the hybrid heuristic is to provide a good trade-off between solution quality and computational effort when solving the integrated problem. The hybrid heuristic consists of a decomposition solution approach that uses, in each iteration of the algorithm, column generation and relax-and-fix procedures. Firstly, the hybrid heuristic starts with the column generation procedure, which is used as a first step in order to generate an initial matrix of columns to the integrated problem. Note that we have two types of columns in this problem: cutting patterns for the cutting of objects into pieces at level 2 and cargo configurations for the loading of final products into the vehicles at level 4. After that, the relax-and-fix procedure decomposes the integrated problem into smaller subproblems in order to obtain a feasible solution. Between each iteration of the relax-and-fix procedure, the column generation procedure is also applied in order to search for more attractive columns for the integrated problem (cutting patterns and cargo configurations). We consider three selection strategies to decompose the integrated problem based on the time period, final product, and production level. The approaches are analyzed in computational experiments, in which the main objective is to evaluate the impact of incorporating other decisions of the supply chain into the integrated problem, as well as to assess the performance of the hybrid heuristic when seeking a solution to the 3-level integrated lot-sizing and cutting stock problem with multiple suppliers for the objects and distribution of the final products.

The contributions of this paper are three-fold: (i) we present mathematical formulations to the 4-level integrated lot-sizing and cutting stock problem with supplier selection and distribution decisions (*G4ILSCS*). Although such extensions are relevant to the supply chain, to the best of our knowledge, they have been explored separately in the literature; (ii) we propose a hybrid heuristic that combines two decomposition approaches iteratively in order to enhance the search for a feasible solution to the integrated problem; (iii) we present, for the first time, a computational study of the 3-level integrated lot-sizing and cutting stock problem, *G3ILSCS*, as well as to the *G4ILSCS*. Our objective is to evaluate the impact of incorporating these additional supply chain decisions and to assess the performance of the hybrid heuristic when solving the integrated problem.

This paper is organized as follows. Section 2 provides a literature review of the papers discussing various levels of integration among lot-sizing, cutting stock, supplier selection, and distribution decisions. In Section 3, the mathematical models *G3ILSCS* and *G4ILSCS* are presented. The hybrid heuristic proposed in this study for the integrated problem is described in Section 4, followed by a computational study in Section 5. Finally, conclusions and future research ideas are presented in Section 6.

2. Literature review

In this literature review, we present a brief overview of the papers addressing an integration among lot-sizing, cutting stock, supplier selection, and distribution decisions. As we have mentioned before, we are not aware of any studies extending the integrated lot-sizing and cutting stock problem with supplier selection or cargo configuration decisions. Therefore, we will discuss these extensions separately.

Firstly, supplier selection is considered an important decision that the managers of a manufacturing plant have to perform. Due to its significance in supply chain management, it has attracted considerable attention of researchers, who have looked at it from several angles (Ho et al., 2010; Chai et al., 2013). The recent tendency is towards approaches that consider simultaneously the decisions related to supplier selection and other problems. In this study, supplier selection consists of the procurement of raw materials (objects) from different suppliers. According to our problem structure, such decisions take into account the inventory management of these objects in the production plant and the cutting planning, i.e., it is linked to levels 1 and 2 of the integrated problem, which address the lot-sizing problem (for the objects) and cutting stock problem, respectively. Therefore, our interest is in studies that approach supplier selection with these decisions in an integrated way. However, we have only found a vast literature on supplier selection with lot-sizing problems.

In the integrated lot-sizing and supplier selection problem, the main goal is to decide which supplier(s) should be selected and how much product(s) should be ordered from the selected supplier(s) in order to meet the client's demand and minimize purchasing, ordering, and holding cost. Most of these studies manage the supply of final products, i.e., there is not a multi-level structure and no process is performed after the acquisition of the product(s) from the supplier(s). Over the years other features from practical applications have been incorporated into the integrated problem in order to get closer to the complex reality, such as lot-sizing and supplier selection problems with quantity discounts (all-unit and incremental discounts), multiple suppliers for a single product (Tempelmeier, 2002; Choudhary and Shankar, 2013, 2014; Mazdeh et al., 2015), multi-product without any type of discounts (Basnet and Leung, 2005; Rezaei and Davoodi, 2011), and multi-objective lot-sizing problems with supplier selection for a single and multiple products (Ustun and Demirtas, 2008; Rezaei and Davoodi, 2011). A comprehensive literature review on lot-sizing problems with supplier selection is provided by Aissaoui et al. (2007), and a classification under different quantity discounts can be found in Benton and Park (1996).

The problem structure presented in these studies differs from the one addressed in our integrated problem due to the multi-level structure, in which the supplies (objects) are the inputs for the downstream levels, i.e., there are operations (cutting process) performed using the products provided by the suppliers in order to obtain the client's orders. In the literature, this type of structure is seen in just a few papers. Senyigit and Soylemez (2012) devised a multi-level multi-product mixed-integer linear programming model with stochastic demand for the integrated lot-sizing and supplier selection problem, in which the supplies consist of components used to produce the final product. The aim is to minimize the costs incurred from purchasing the components (purchasing and transportation cost), as well as the costs related to the production of final products (production, operation, transportation and lost sales costs). There is also a capacity limitation at the suppliers. More related to our study, Cunha et al. (2018) and Mohammadi et al. (2020) proposed mathematical models for the integrated lot-sizing and supplier selection problem for raw materials purchasing, considering different suppliers and quantity discounts. Cunha et al. (2018) presented the integrated problem observed in a chemical industry, which minimizes the costs incurred from the acquisition and inventory of raw materials, and production of final products. Mohammadi et al. (2020) address a multi-level integrated lot-sizing and scheduling problem with multiple production lines, sequence-dependent setups, and supplier selection for raw material purchasing, which are used to manufacture products on the production lines. The authors consider two different types of lot-sizing models (aggregated and disaggregated) under deterministic and stochastic demands of the final products.

Despite many advantages seen when integrating supplier selection decisions into the management of a production plant, the approaches proposed for the integrated lot-sizing and cutting stock problems have not sufficiently addressed the acquisition process of raw materials (objects), as it has been pointed out in a recent literature review on integrated problems (Melega et al., 2018). This literature review shows that only a few studies consider the management of objects, along with the integrated problem. In these studies, the management of objects consists of a lot-sizing problem related to the production of objects, in practical applications found in the paper, copper, wood, furniture, and steel industries (Reinders, 1992; Hendry et al., 1996; Poltroniere et al., 2008; Silva et al., 2014; Viegas et al., 2016). In fact, the papers in the literature on integrated lot-sizing and cutting stock problems model either the production process of objects or consider an unlimited number of objects available in the cutting process, which is not realistic for most of the practical applications. Therefore, the consideration of supplier selection for raw material purchasing with different suppliers, like the one addressed in our study, has not been explored in the literature of integrated problems and can be seen as an strategy to accurately estimate the impact of these costs in the production planning.

Secondly, the distribution decisions considered in the present study take into account the arrangements and the distribution of the final products from the production plant to a warehouse, which are directly related to the decisions on the lots of final products. In the literature, several studies treat distribution decisions together with the lot-sizing decisions. However, the focus of these papers is mainly on the transportation costs incurred from the routing problem between the production plant and the customers (Bard and Nananukul, 2009; Armentano et al., 2011; Adulyasak et al., 2015; Miranda et al., 2018a,b). Our study considers a different perspective: we are interested in the loadings/arrangements of final products into vehicles to determine the costs incurred with the number of vehicles needed to transport the final products from the production plant to the warehouse, in the so-called integrated lot-sizing and loading problem.

There are only a few papers in the literature addressing such decisions simultaneously. A related study is seen in van Norden and van de Velde (2005), where the authors study a large European manufacturing, with heavily fluctuating monthly shipments between its manufacturing plant and its central warehouse. The cargo costs depend on the type of contract established with the logistics operator. The proposed mathematical models consist of an extension of the Wagner-Whitin model (Wagner and Whitin, 1958) by considering multiple products and transportation costs. The aim is to minimize the production and distribution costs considering different costs related to the number of used vehicles according to the freight rates established in the contract. Molina et al. (2009, 2016) extended this problem by considering backlogging, capacity limitation in the production process, and different approaches to the transportation of items. Firstly, the items are loaded onto pallets without being mixed (i.e., single item pallets) and the problem consists of determining the number of pallets used in transportation. Next, the problem is extended to consider the arrangement of these single-item pallets (of the same size) into vehicles of a homogeneous and heterogeneous fleet. Lastly, the items are loaded directly into the vehicles, without intermediate devices. The proposed models are solved with a commercial optimization package and show the potential of these models to address different practical situations.

Finally, it is worth mentioning that over the last decades, with the approach of integrating problems, several papers in the literature present computational results showing the benefits of integrating lot-sizing and cutting stock problems instead of taking these decisions separately. Gramani et al. (2009) observed an improvement of the integrated problem (levels 2 and 3), when compared to the actions taken in practice by the sequencing these decisions. The main difference between the approaches occurred in the number of setups, with a reduction of around 18%, while the number of cut objects remained the same. Dealing with the same integration (levels 2 and 3), as well as specific settings found in the furniture industry, Vanzela et al. (2017) noticed that the integrated approach can handle more effectively the decisions when costs for managing the inventory of pieces are taken into account, with a reduction of the total costs up to 15%. The gains are even greater for the instances with high demands, reaching a gain of 29.4% compared to the sequential method used by the studied company. Considering an integration between levels 1 and 2 arising in a company in the paper industry, Poldi and de Araujo (2016) compared the multi-period integrated problem with the sequential lot-by-lot policy. The gains of the integrated strategy was, on average, 54.38%, reaching up to 77% of improvements in the total costs, while reducing the number of cut objects.

3. Mathematical formulations

This section presents the mathematical models addressed in our study. The first one is the generalized 3-level integrated lot-sizing and cutting stock problem (*G3ILSCS*) proposed in the literature solely and exclusively as a tool to classify the literature of integrated problems (Melega et al., 2018). The second model is the extension of *G3ILSCS* that takes into account the decisions related to the supplier selection for the objects. The last model is the *G3ILSCS* with the supplier selection decisions and the additional decisions related to the distribution of final products from the production plant to the warehouse, i.e., the *G4ILSCS*.

In order to define the mathematical models, we present the following sets, parameters, and decision variables.

Sets:
T: set of time periods (index t);
O: set of different types of objects (index o);
P: set of pieces (index p);
F: set of final products (index f);
J_o: set of cutting patterns for object type o (index j).

Level 1 parameters:
sc^1_{ot}: setup cost of object type o in time period t;
vc^1_{ot}: unit production cost of object type o in time period t;
hc^1_{ot}: unit holding cost of object type o in time period t;
st^1_{ot}: setup time of object type o in time period t;
vt^1_{ot}: unit production time of object type o in time period t;
Cap^1_t: production capacity (in time units) available to produce the objects in time period t;
M^1_{ot}: large number.

Level 2 parameters:
hc^2_{pt}: unit holding cost of piece p in time period t;
sc^2_{ojt}: setup cost for cutting object type o according to cutting pattern j in time period t;
vc^2_{ojt}: cost of cutting object type o according to cutting pattern j in time period t;
st^2_{ojt}: setup time of object type o cut according to cutting pattern j in time period t;
vt^2_{ojt}: production time to cut object type o according to cutting pattern j in time period t;
a^2_{pojt}: number of pieces p cut from object type o using cutting pattern j in time period t;
r_{fp}: number of pieces p required to produce the final product f;
Cap^2_t: cutting capacity (in time units) available in time period t;
M^2_{ojt}: large number.

Level 3 parameters:
sc^3_{ft}: setup cost of final product f in time period t;
vc^3_{ft}: unit production cost of final product f in time period t;
hc^3_{ft}: unit holding cost of final product f in time period t;
d^3_{ft}: demand of final product f in time period t;
$sd^3_{ft\tau}$: sum of demand of final product f from time period t until time period τ;
st^3_{ft}: setup time of final product f in time period t;
vt^3_{ft}: unit production time of final product f in time period t;
Cap^3_t: production capacity (in time units) available to produce the final products in time period t;
M^3_{ft}: large number.

Level 1 decision variables:
Y^1_{ot}: binary variable indicating the production or not of object type o in time period t;
X^1_{ot}: production quantity of object type o in time period t;
S^1_{ot}: inventory of object type o at the end of time period t.

Level 2 decision variables:
Y^2_{ojt}: binary variable indicating the setup or not for cutting object type o according to cutting pattern j in time period t;
Z^2_{ojt}: number of objects type o cut according to cutting pattern j in time period t;
S^2_{pt}: inventory of piece p at the end of time period t.

Level 3 decision variables:
Y^3_{ft}: binary variable indicating the setup or not of final product f in time period t;
X^3_{ft}: production quantity of final product f in time period t;
S^3_{ft}: inventory of final product f at the end of time period t.

3.1. 3-level integrated problem

The *G3ILSCS* model, as defined in Melega et al. (2018), consists of a production environment composed of three levels and multiple periods, in a deterministic setting. Level 1 corresponds to the production planning of objects, that have to be produced considering a capacitated environment in order to be cut in the downstream level (level 2). These objects might be of several types, each with a different length, cost, and production time. Level 2 is associated with the cutting process, in which the produced objects are cut into pieces according to cutting patterns (Gilmore and Gomory, 1961, 1963) by a cutting machine with limited resources. In the cutting process only one dimension, i.e., the length, is taken into account when cutting the objects. The cut pieces can be used as components to assemble the final products or directly as final products. In the last case, the pieces still need to undergo some finishing process before being ready as final products. It is at level 3 that the production of the final products occurs and the independent demand for final products has to be met in each time period.

For simplicity, we assume that the lead-time between all the levels in the *G3ILSCS* is zero, i.e., it is possible to produce an object in the same time period in which the object is cut into pieces, and in addition, still in the same time period, these cut pieces can be used to produce the demanded final products. However, a lead-time offset of at least one period can be incorporated throughout the levels of the *G3ILSCS* model to allow the disaggregation into a machine schedule (Almeder, 2010; Pochet and Wolsey, 2006). It is also assumed, for simplicity, that the capacities in the supply chain are dedicated, exclusive, and independent for each level. The link between the different time periods is provided by the inventory at each level. There is a bill-of-material relationship, for which the dependent demand of final products triggers a dependent demand for pieces, and indirectly, for objects. Therefore, the decisions of the *G3ILSCS* problem determine simultaneously a production plan that defines for every time period: setups and production quantities for products and objects, and the cutting patterns with their corresponding frequencies, considering limited resources at each level, while searching for a global optimal solution that minimizes the overall costs of the production planning.

Model *G3ILSCS*

$$\min \sum_{t\in T}\sum_{o\in O}\left(sc^1_{ot}Y^1_{ot} + vc^1_{ot}X^1_{ot} + hc^1_{ot}S^1_{ot}\right) + \sum_{t\in T}\left(\sum_{o\in O}\sum_{j\in J_o}\left(sc^2_{ojt}Y^2_{ojt} + vc^2_{ojt}Z^2_{ojt}\right) + \sum_{p\in P} hc^2_t S^2_{pt}\right) + \sum_{t\in T}\sum_{f\in F}\left(sc^3_{ft}Y^3_{ft} + vc^3_{ft}X^3_{ft} + hc^3_{ft}S^3_{ft}\right) \quad (1)$$

Subject to:

Level 3

$$S^3_{f,t-1} + X^3_{ft} = d^3_{ft} + S^3_{ft} \qquad \forall f,\ \forall t \quad (2)$$

$$X^3_{ft} \le M^3_{ft}Y^3_{ft} \qquad \forall f,\ \forall t \quad (3)$$

$$\sum_{f\in F}\left(st^3_{ft}Y^3_{ft} + vt^3_{ft}X^3_{ft}\right) \le Cap^3_t \qquad \forall t \quad (4)$$

$$X^3_{ft},\ S^3_{ft} \ge 0,\ Y^3_{ft} \in \{0,1\} \qquad \forall f,\ \forall t \quad (5)$$

Link levels 2 and 3

$$S^2_{p,t-1} + \sum_{o\in O}\sum_{j\in J_o} a^2_{pojt}Z^2_{ojt} = \sum_{f\in F} r_{fp}X^3_{ft} + S^2_{pt} \qquad \forall p,\ \forall t \quad (6)$$

Level 2

$$Z^2_{ojt} \le M^2_{ojt}Y^2_{ojt} \qquad \forall j,\ \forall o,\ \forall t \quad (7)$$

$$\sum_{o\in O}\sum_{j\in J_o}\left(st^2_{ojt}Y^2_{ojt} + vt^2_{ojt}Z^2_{ojt}\right) \le Cap^2_t \qquad \forall t \quad (8)$$

$$S^2_{pt} \ge 0,\ Z^2_{ojt} \in \mathbb{Z}_+,\ Y^2_{ojt} \in \{0,1\} \qquad \forall p,\ \forall o,\ \forall j,\ \forall t \quad (9)$$

Link levels 1 and 2

$$S^1_{o,t-1} + X^1_{ot} = \sum_{j\in J_o} Z^2_{ojt} + S^1_{ot} \qquad \forall o,\ \forall t \quad (10)$$

Level 1

$$X^1_{ot} \le M^1_{ot}Y^1_{ot} \qquad \forall o,\ \forall t \quad (11)$$

$$\sum_{o\in O}\left(st^1_{ot}Y^1_{ot} + vt^1_{ot}X^1_{ot}\right) \le Cap^1_t \qquad \forall t \quad (12)$$

$$X^1_{ot},\ S^1_{ot} \ge 0,\ Y^1_{ot} \in \{0,1\} \qquad \forall o,\ \forall t \quad (13)$$

The objective function (1) minimizes the overall costs at each level. At level 1, the costs are related to the production of the objects and include a fixed setup cost, the production cost, and the inventory holding cost. At level 2, the costs refer to the cutting process. At this level, we take into account the setup cost of setting up the cutting machine according to a cutting pattern, the cost of cutting the object, and the cost of holding the pieces in inventory. The last terms in the objective function corresponds to the setup, production, and inventory costs of final products at level 3.

Constraints (2) - (5) refer to the production of final products and can be seen as the lot-sizing problem at level 3. Constraints (2) are the inventory balance constraints for the final products. Constraints (3) force the setup variable to one if any production takes place in a time period for a final product. Constraints (4) impose the capacity limitation in the production process for the final products, taking into account production and setup time. Constraints (5) are the non-negativity and integrality constraints of the decision variables at level 3.

Constraints (6) - (9) are related to the process of cutting objects into pieces at level 2 and can be seen as a multi-period cutting stock problem with setup and capacity constraints. Constraints (6) ensure that the dependent demand for pieces ($\sum_{f \in F} r_{fp} X^3_{ft}$) is satisfied through the cutting of objects using cutting patterns. These constraints model the interdependency between the decisions of level 2 and level 3, i.e., an integration between the *CSP* and *LSP*. Constraints (7) force a cutting pattern setup in the cutting machine, whenever an object is cut according to a new cutting pattern. The setup addressed in this paper is independent of the preceding cutting pattern. The capacity constraints (8) consider the use of only one machine in the cutting process and take into account the time consumed for setting up each new cutting pattern, and the time for cutting objects. Constraints (9) are the non-negativity and integrality constraints of the decision variables at level 2.

Constraints (10) - (13) are related to the production of objects and can be seen as the lot-sizing problem at level 1. Constraints (10) are the inventory balance constraints that ensure the production of a sufficient amount of objects needed to supply the cutting process ($\sum_{j \in J_o} Z^2_{ojt}$). These constraints also correspond to the integration between level 1 and level 2. Constraints (11) are the setup forcing constraints associated with the production of objects. The capacity limitation in the production of objects is modeled by constraints (12), which take into account the time spent to set up the machine for a specific object type and the time needed to produce the objects. Finally, constraints (13) are the non-negativity and integrality constraints for the decision variables at level 1.

In the *G3ILSCS* model, there are the parameters in the setup constraints regarding the production of objects (M^1_{ot}), the number of objects cut according to a cutting pattern (M^2_{ojt}), and the production of final products (M^3_{ft}). Considering that each level has a capacity limitation, the production and cutting quantities can be limited by the maximum production capacity available at their respective level. In addition, the remaining demand of final products is taken into account to limit the production of final products in each time period, as there is no need to produce more than the demanded quantity. To limit the number of objects cut in constraint (7), we calculate M^2_{ojt} to not only take into account the capacity limits, but also the remaining demand of pieces (see equation (15)). Therefore, the values to these parameters are taken so that no solution is eliminated when solving the mathematical model and tight values are obtained by strengthening such constraints. We present these calculations for each one of these parameters as follows:

$$M^1_{ot} = \left(\frac{Cap^1_t - st^1_{ot}}{vt^1_{ot}}\right) \qquad \forall o,\ \forall t \tag{14}$$

$$M^2_{ojt} = \min\left\{\begin{array}{l} \left\lfloor \dfrac{Cap^2_t - st^2_{ojt}}{vt^2_{ojt}} \right\rfloor \\ \max\limits_{p,\ a^2_{pojt}>0} \left\{ \left\lceil \dfrac{\sum_{f\in F} r_{fp} sd^3_{ftT}}{a^2_{pojt}} \right\rceil \right\} \end{array}\right. \qquad \forall j,\ \forall o,\ \forall t \tag{15}$$

$$M^3_{ft} = \min\left\{ sd^3_{ftT}, \frac{Cap^3_t - st^3_{ft}}{vt^3_{ft}} \right\} \qquad \forall f,\ \forall t \tag{16}$$

3.2. 3-level integrated problem with supplier selection

In this section, the *G3ILSCS* is extended to consider an alternative means to the acquisition of objects demanded in the cutting process. The objects can be produced at level 1 of the production plant, as well as be purchased from external suppliers. The supplier selection (level 1A) takes into account a set of suppliers offering different types of objects, which can be purchased considering fixed and variable costs, and the decisions define the number of objects to be purchased from each supplier. For simplicity, we assume that the supplier's capacity is unlimited and the order lead-time for objects to arrive after the purchase is zero. In addition, the price of objects does not vary according to the number of objects ordered, i.e., there is no discount rate. Thus, the integrated problem optimizes an integrated production planning for the production of final products, cutting process, and the production planing and procurement of objects, with the objective of minimizing the total costs of the integrated problem.

To model the case with multiple suppliers for the objects in the *G3ILSCS* problem, we consider the additional parameters and decision variables as follows.

Set:
S: set of suppliers (index s).

Level 1A parameters:
sc^{1A}_{st}: ordering cost from supplier s in time period t;
pc^{1A}_{ost}: purchasing cost of object type o from supplier s in time period t;
M^{1A}_{st}: large number.

Level 1A decision variables:
Y^{1A}_{st}: binary variable indicating if any object is purchased from supplier s in time period t;
X^{1A}_{ost}: quantity purchased of object type o from supplier s in time period t.

$$\min \sum_{t\in T}\sum_{s\in S}\left(sc^{1A}_{st}Y^{1A}_{st} + \sum_{o\in O} pc^{1A}_{ost}X^{1A}_{ost}\right) + \sum_{t\in T}\sum_{o\in O}\left(sc^1_{ot}Y^1_{ot} + vc^1_{ot}X^1_{ot} + hc^1_{ot}S^1_{ot}\right) +$$
$$\sum_{t\in T}\left(\sum_{o\in O}\sum_{j\in J_o}\left(sc^2_{ojt}Y^2_{ojt} + vc^2_{ojt}Z^2_{ojt}\right) + \sum_{p\in P} hc^2_t S^2_{pt}\right) + \sum_{t\in T}\sum_{f\in F}\left(sc^3_{ft}Y^3_{ft} + vc^3_{ft}X^3_{ft} + hc^3_{ft}S^3_{ft}\right) \tag{17}$$

Subject to:

$$(2) - (9), (11) - (13)$$

Link levels 1, 1A and 2

$$S^{1}_{o,t-1} + X^{1}_{ot} + \sum_{s \in S} X^{1A}_{ost} = \sum_{j \in J_o} Z^{2}_{ojt} + S^{1}_{ot} \qquad \forall o,\ \forall t \tag{18}$$

Level 1A

$$\sum_{o \in O} X^{1A}_{ost} \leq M^{1A}_{st} Y^{1A}_{st} \qquad \forall s,\ \forall t \tag{19}$$

$$X^{1A}_{ost} \geq 0,\ Y^{1A}_{st} \in \{0, 1\} \qquad \forall o, \forall s,\ \forall t \tag{20}$$

The objective function (17) minimizes the overall costs at each level of the integrated problem. At level 1A, related to the procurement of objects, the costs are composed of a fixed ordering cost from each external supplier, regardless of the size of the order, and a variable cost according to the total quantity of objects purchased. The other terms in the objective function are the same as the ones in (1), which are related to the production of objects, the cutting process, and the production of final products.

Constraints (18) - (20) are responsible for the supplier selection of the objects at level 1A of the integrated problem. Constraints (18) control the inventory balancing of objects in the planning horizon, and replace constraints (10). In the left-hand side, the availability of objects purchased from the various suppliers ($\sum_{s \in S} X^{1A}_{ost}$) is taken into account. These constraints also correspond to the integration between level 1, level 1A, and level 2 of the problem, i.e., the integration between the lot-sizing and cutting stock decisions, along with supplier selection decisions. Constraint (19) is the order setup constraint with respect to the objects purchased. Constraints (20) are the non-negativity and integrality constraints for the decision variables in level 1A.

To model the order setup constraint (19), we use parameter M^{1A}_{st}, which provides an upper limit on the number of objects that can be purchased from supplier s in period t. As the suppliers do not have capacity issues, this quantity cannot be limited by the cutting capacity. Instead, we consider it as the number of objects used to meet the remaining dependent demand of pieces, given by equation (21). In case there is a supplier with a supply limitation, M^{1A}_{st} can be adjusted to take into account the supply capacity limit of the supplier.

$$M^{1A}_{st} = \sum_{p \in P} \left(\max_{o,j,\ a^{2}_{pojt} > 0} \left\{ \left\lceil \frac{\sum_{f \in F} r_{fp} sd^{3}_{ftT}}{a^{2}_{pojt}} \right\rceil \right\} \right) \qquad \forall s,\ \forall t \tag{21}$$

3.3. 4-level integrated problem

In this section, the extension of the *G3ILSCS* model with supplier selection and distribution decisions, called *G4ILSCS*, is presented. In the *G4ILSCS*, level 4 is responsible for the distribution between the production plant and the warehouse, more specifically, the loading/arrangements of the final products into the vehicles. The distribution decisions define the number of vehicles needed to transport the final products and are associated with the distribution configuration of the vehicles used in the transportation. In this way, such problem can be directly related to the production lot-sizing decisions of final products (Molina et al., 2009, 2016). Therefore, the *G4ILSCS* problem combines the lot-sizing and cutting stock decisions with loading decisions.

We approach the distribution decisions at level 4 of the *G4ILSCS* by allowing the mixing of final products in the load arrangement of the vehicles, taking into account limited volume and weight capacity of the trucks. In the *G4ILSCS*, the demand of final products is shifted from the production plant to the warehouse and in both sites, there is the possibility of inventory. We assume a zero distribution lead time. Thus, ordered products can be used to satisfy demand in the same time period in which they have been produced. The loading of the final products takes into account only one dimension related to the physical arrangement of the final products in the vehicles. However, two capacity limitations are addressed, which are in terms of the weight and volume of the final products in the vehicles. The distribution decisions are modeled using cargo configurations known a priori to load the vehicles. A cargo configuration is a feasible proposal for loading the truck. It indicated for all final products how many units are loaded in a vehicle, taking into account the weight and volume limits. These cargo configurations are obtained via an extended formulation (Gilmore and Gomory, 1961, 1963).

To define the mathematical model *G4ILSCS*, we present the following sets, parameters, and decision variables.

Additional sets:
K: set of different types of vehicles (index k);
G_k: set of cargo configurations for vehicle k (index g).

Level 4 parameters:
d^4_{ft}: demand (at the warehouse) of final product f in time period t;
hc^4_{ft}: unit holding cost of final product f at the warehouse in time period t;
b^4_{fkgt}: number of final products f loaded in vehicle k using cargo configuration g in time period t;
uc^4_k: fixed cost of using one truck of vehicle type k.

Level 4 decision variables:
H^4_{kgt}: number of vehicles type k loaded using cargo configuration g in time period t;
S^4_{ft}: inventory of final product f in the warehouse at the end of time period t.

Model *G4ILSCS*

$$\min \sum_{t\in T}\sum_{s\in S}\left(sc^{1A}_{st}Y^{1A}_{st} + \sum_{o\in O} pc^{1A}_{ost}X^{1A}_{ost}\right) + \sum_{t\in T}\sum_{o\in O}\left(sc^1_{ot}Y^1_{ot} + vc^1_{ot}X^1_{ot} + hc^1_{ot}S^1_{ot}\right) +$$
$$\sum_{t\in T}\left(\sum_{o\in O}\sum_{j\in J_o}\left(sc^2_{ojt}Y^2_{ojt} + vc^2_{ojt}Z^2_{ojt}\right) + \sum_{p\in P} hc^2_t S^2_{pt}\right) + \sum_{t\in T}\sum_{f\in F}\left(sc^3_{ft}Y^3_{ft} + vc^3_{ft}X^3_{ft} + hc^3_{ft}S^3_{ft}\right) +$$
$$\sum_{t\in T}\left(\sum_{k\in K}\sum_{g\in G_k} uc^4_k H^4_{kgt} + \sum_{f\in F} hc^4_{ft}S^4_{ft}\right) \tag{22}$$

Subject to:

$$(3) - (9), (11) - (13), (18) - (20)$$

Link levels 3 and 4

$$S^3_{f,t-1} + X^3_{ft} = \sum_{k \in K} \sum_{g \in G_k} b^4_{fkgt} H^4_{kgt} + S^3_{ft} \qquad \forall f,\ \forall t \qquad (23)$$

Level 4

$$S^4_{f,t-1} + \sum_{k \in K} \sum_{g \in G_k} b^4_{fkgt} H^4_{kgt} = d^4_{ft} + S^4_{ft} \qquad \forall f,\ \forall t \qquad (24)$$

$$S^4_{ft} \geq 0, H^4_{kgt} \in \mathbb{Z}_+ \qquad \forall f,\ \forall k,\ \forall g,\ \forall t \qquad (25)$$

The objective function (22) minimizes the overall costs at each level of the integrated problem. In addition to the previously discussed costs, we add the costs at level 4: the distribution costs defined by the number of vehicles used in the transportation of the final products from the production plant to the warehouse, and the inventory of final products at the warehouse.

Constraints (23) ensure that all final products produced in the manufacturing plant, and needed to meet the clients' demand, are transported to the warehouse by the loading of the final products into vehicles using a known set of cargo configurations. These constraints also model the interdependence between levels 3 and 4, which consists of the linking constraints between the lot-sizing and loading decisions. Constraints (24) are the inventory balance constraints of the final products, at the warehouse, that takes into account the shipments from the production plant. Note that, the final products can be kept in stock either at the production plant in order to consolidate the cargo in the vehicles or at the warehouse to meet the external demand. In each one of these sites, the inventory quantity can differ (S^3_{ft} and S^4_{ft}) and the associated costs (hc^3_{ft} and hc^4_{ft}), as well. Constraints (25) are the non-negativity and integrality constraints for the decision variables at level 4.

4. Hybrid heuristic for the integrated problems

In this section, the hybrid heuristic proposed to solve *G3ILSCS* and *G4ILSCS* is presented. This solution method is strongly related to two well-known decomposition procedures from the literature: the column generation and the relax-and-fix procedures. These approaches have been successfully used to solve these problems separately. In fact, the column generation procedure has been largely used either as a solution method or embedded into other approaches to solve cutting stock problems in many manufacturing settings (Reinders, 1992; Gramani et al., 2009; Melega et al., 2016; Wu et al., 2017; Lemos et al., 2021; Signorini et al., 2021), while heuristics based on the relax-and-fix procedure have been used in solving multi-level lot-sizing problems (Stadtler, 2003; Akartunali and Miller, 2009; Ferreira et al., 2010; Mohammadi et al., 2010; Fiorotto et al., 2020; Toscano et al., 2020).

In the literature, some studies (Liang et al., 2015; Melega et al., 2020; Nascimento et al., 2020; Andrade et al., 2021) have combined column generation and relax-and-fix procedures sequentially: first the column generation solves the linear relaxation of the problem, and afterwards, considering the set of generated columns, the relax-and fix tries to find a feasible solution to the problem. In this paper, we are proposing a Hybrid Heuristic (*HH*) that consists of an iterative approach where the column generation and the relax-and fix procedures are used in each iteration of the algorithm in order to generate new attractive columns to the problem, using the column generation procedure, while finding a feasible solution to the problem, using the relax-and-fix procedure.

The hybrid heuristic starts applying the column generation procedure as a first step in order to generate the matrix of columns at levels 2 and 4 of the integrated problem (cutting patterns for the cutting process and cargo configurations for the vehicle loading). Next, the relax-and-fix procedure starts searching for a feasible solution to the integrated problem. Between each iteration of the relax-and-fix procedure, the column generation procedure is also applied. For this, after fixing the binary decision variables according to a selected freezing strategy in the relax-and-fix, column generation is applied to the linear relaxation of the resulting mixed-integer problem, in order to search for more attractive columns. The motivation behind the interaction between these approaches comes from the fact that when applying heuristics based on relax-and-fix procedures, several different mixed-integer problems are solved sequentially until a feasible solution to the original problem is found. In this case, each mixed-integer problem obtained in a step of the relax-and-fix procedure is different from the previous one due to the recently fixed variables. Thus, when applying the column generation in each step of the relax-and-fix procedure, new columns might be attractive to the current linear problem, and may improve the search for a solution to the original problem during the relax-and-fix procedure in the hybrid heuristic. In the following sections, each step of the hybrid heuristic is presented in more details (see a flowchart of the hybrid heuristic in Figure A.2).

4.1. Column generation procedure

The column generation procedure, embedded in the hybrid heuristic, is used to overcome the difficulties present in the cutting stock and distribution problems, which consist of the high number of possible cutting patterns and cargo configurations, i.e., the high number of Z^2_{ojt} and H^4_{kgt} decision variables at level 2 and 4 of the integrated problem, respectively.

To present the column generation procedure, some parameters are necessary.

L_o:	length of object type o;
l_p:	length of pieces type p;
w_f:	weight of the final product f;
v_f:	volume of the final product f;
$CapW^4_k$:	capacity (weight) of vehicle type k;
$CapV^4_k$:	capacity (volume) of vehicle type k.

Each type of object and each type of cut piece is characterized by its length. The final product is characterized by its weight and volume and each vehicle type has a specific capacity limit with respect to the weight and volume.

In the column generation, the initial integrated problem is composed of columns which are the homogeneous cutting patterns and cargo configurations, in the so-called restricted master problem (RMP). The (homogenous) columns for the Z^2_{ojt} and H^4_{kgt} are given as follows.

- $(0, \cdots, a^2_{pot}, \cdots, 0)$ where $a^2_{pot} = \left\lfloor \frac{L_o}{l_p} \right\rfloor, \ \forall p \in P, \ \forall o \in O, \ \forall t \in T;$
- $(0, \cdots, b^4_{fkt}, \cdots, 0)$ where $b^4_{fkt} = \min\left\{ \left\lfloor \frac{CapW^4_k}{w_f} \right\rfloor; \left\lfloor \frac{CapV^4_k}{v_f} \right\rfloor \right\}, \ \forall f \in F, \ \forall k \in K, \ \forall t \in T.$

Note that, as the column generation procedure evolves and the cutting patterns and cargo configurations are generated, the corresponding Z^2_{ojt} and H^4_{kgt} variables are generated. The columns related to the other decision variables are already present in the restricted master problem, except the setup decision variables Y^2_{ojt}. When a new cutting pattern is generated, the corresponding Z^2_{ojt} decision variables are created, along with the corresponding the setup decision variable Y^2_{ojt}, which are added to constraints (7) and (8), and to the objective function of the problem (see Melega et al. (2020)).

When the column generation starts, the integrality constraints of the variables in the initial *RMP* are relaxed, and the current restricted master problem is solved. The dual values associated with constraints (6), (8), (10) or (18), (23) and (24) are recovered. It is important to mention that, in the linear relaxation of the integrated problem, constraints (7) become an equality constraints and the variables Y^2_{ojt} can be replaced by Z^2_{ojt}/M^2_{ojt} in constraint (8) and in the objective function; consequently constraints (7) can be eliminated from the mathematical model, i.e., the dual variables associated to this constraints are not used in the column generation procedure (for more details see Melega et al. (2020)).

Let π_{pt}, γ_t, τ_{ot}, λ_{ft}, and δ_{ft} be the dual values recovered from constraints (6), (8), (10) or (18), (23) and (24), respectively. For each time period t, and object type o, a subproblem of the type (26) - (28) is solved in order to find attractive cutting patterns for the cutting stock problem in the restricted master problem. In the same way, for each time period t and vehicle k, a subproblem of the type (29) - (32) is solved to find attractive cargo configurations for the distribution problem in the restricted master problem. The subproblem for the cutting patterns consists of a knapsack problem, whereas for the cargo configurations it is a generalized assignment problem with two resource constraints related to the volume and weight limitations of the vehicle.

- *Cutting patterns subproblem*

$$OF_{SUBJ_{ot}} = \min \ vc^2_{ojt} - \sum_{p \in P} \pi_{pt} \alpha_{pojt} - vt^2_{ojt} \gamma_t + \tau_{ot} \tag{26}$$

Subject to:

$$\sum_{p \in P} l_p \alpha_{pojt} \leq L_o \tag{27}$$

$$\alpha_{pojt} \in \mathbb{Z}_+ \qquad \forall p \tag{28}$$

- *Cargo configurations subproblem*

$$OF_{SUBK_{kt}} = \min \ uc^4_k + \sum_{f \in F} (\lambda_{ft} - \delta_{ft}) \beta_{fkgt} \tag{29}$$

Subject to:

$$\sum_{f \in F} v_f \beta_{fkgt} \leq CapV^4_k \tag{30}$$

$$\sum_{f \in F} w_f \beta_{fkgt} \leq CapW^4_k \tag{31}$$

$$\beta_{fkgt} \in \mathbb{Z}_+ \qquad \forall f \tag{32}$$

For each time period t and object type o, if $OF_{SUBJ_{ot}} < 0$, the optimal solution α^*_{pojt} consists of a new cutting pattern, which is included in the restricted master problem, with $a^2_{pojt} = \alpha^*_{pojt}$. In the same way, for each time period t and vehicle k, if $OF_{SUBK_{kt}} < 0$, the optimal solution β^*_{fkgt} consists of a new cargo configuration, which is added to the restricted master problem, with $b^2_{fkgt} = \beta^*_{fkgt}$. After the analysis of both subproblems, the resulting restricted master problem, with the added columns, is solved. When the generated columns in both subproblems no longer price out attractively, that is, $OF_{SUBJ_{ot}} \geq 0$, $\forall o$, $\forall t$, $OF_{SUBK_{kt}} \geq 0$, $\forall k$, $\forall t$, the column generation procedure stops by optimality and a linear problem (*LP*) with all the generated columns is obtained. We also consider a maximum number of iterations as a stopping criteria in order to interrupt the column generation procedure in case of a tailing off effect with the procedure. We have applied the stabilization technique, called *dual-optimal inequalities* (Valério de Carvalho, 2005; Ben Amor et al., 2006), in an attempt to overcome the tailing off effect in the column generation procedure, however, this strategy did not help to improve the results.

At each iteration of the column generation, we calculate a lower bound on the objective function of the restricted master problem, which is a lower bound to the original problem (Lasdon, 2002; Degraeve and Jans, 2007). Such a lower bound is calculated only during the first time the column generation procedure is applied to the restricted master problem, i.e., when no variables have been fixed yet in the relax-and-fix heuristic (see in Figure A.2, until $CG = 0$). The lower bound, LB, is obtained using the formula (33).

$$LB = OF^r_{RMP} + \sum_{t \in T} \sum_{o \in O} OF^r_{SUBJ_{ot}} + \sum_{t \in T} \sum_{k \in K} OF^r_{SUBK_{kt}} \tag{33}$$

where, OF^r_{RMP}, $OF^r_{SUBJ_{ot}}$ and $OF^r_{SUBK_{kt}}$ are the objective function values of the restricted master problem, cutting pattern subproblem (o, t), and cargo configuration subproblem (k, t) at iteration r, respectively.

When the column generation stops, either by optimality or after a maximum number of iterations, the hybrid heuristic continues. In the first iteration of the hybrid heuristic ($CG = 0$), after applying the column generation procedure, we consider a selection strategy to manage the generated columns in the heuristic. For this, we fix the binary setup decision variable related to the setup of the cutting patterns to zero ($Y^2_{ojt} = 0$) when their value is equal to zero in the solution of the linear relaxation. After that, the relax-and-fix procedure is applied in order to find a feasible solution to the integrated problem.

4.2. *Relax-and-fix procedure*

The relax-and-fix procedure embedded in the hybrid heuristic is used to manage the integrality of the decision variables and take advantage of the multi-level structure, while searching for a feasible solution to the integrated problem. The relax-and-fix procedure is applied to the resulting *LP* obtained after the application of the column generation procedure. The set of binary decision variables (Y^3_{ft}, Y^2_{ojt}, Y^1_{ot}, and Y^{1A}_{st}), considering three selection strategies (time period, final product, and production level), are used to decompose the problem. The decompositions are described in more detail as follows.

- **Time period decomposition: *HH_T***
  In the time period decomposition, the entire planning horizon is divided into three parts according to the binary decision variables, which are: fixed, integral, and relaxed decisions. In the planning horizon composed of Δ periods, called window, the integrality constraints of the binary decision variables are added to the problem. Inside of each window, there is a number of periods (ψ), in which the binary decision variables are fixed at their binary values. For periods preceding the window, binary decision variables have already been made (fixed) in previous steps and for later periods, binary decision variables are relaxed. The hybrid heuristic starts with the addition of the integrality constraints for the decision variables in the window. After that, the resulting mixed-integer problem is solved. The resulting problem can be infeasible or no solution can be found during the available time. In either case, the hybrid heuristic stops without finding a feasible solution to the integrated problem.

When the solution of the resulting mixed-integer problem is found, and there are still relaxed binary decision variables, the binary decision variables for the ψ periods are fixed and the integrality constraints for all the decision variables are removed from the problem, resulting in a linear problem. The column generation procedure is then applied. At the end of the column generation procedure, the windows moves by ψ periods to the next window, Δ periods, for which the integrality constraints for the binary decision variables are added. The resulting mixed-integer problem is then processed in the same manner until all the integrality constraints for the relaxed binary decision variables have been added. In the last iteration, the integrality constraints of the remaining decision variables are added to the problem, as well. A solution from this last iteration, if it exists, corresponds to a feasible solution to the integrated problem solved by the hybrid heuristic with the time period decomposition. For an algorithm of the relax-and-fix procedure with time decomposition see Algorithm 1 in Appendix A.

- **Final product decomposition: *HH_F***
  In the final product decomposition, the set of final products is split into three parts according to binary decision variables, which are: fixed, integral and relaxed decisions. In the set of final products composed of Δ products, the entire planning horizon is considered, as well as, the set of cutting patterns related to these final products, that is, for those cutting patterns which have a piece belonging to a final product in the window. For the Δ products, the integrality constraint of the binary decision variables are added to the resulting mixed-integer problem, if it has not been added in previous steps. The remaining of the decomposition proceeds as in the time period decomposition.

- **Level decomposition: *HH_L***
  In the level decomposition, the production levels are split into three parts according to binary decision variables, which are: fixed, integral and relaxed decisions. In each iteration, the integrality constraints of binary decision variables belonging to a production level are added to the problem, while the remaining decision variables are either relaxed or have already been fixed in previous iterations (see Table 1). After the solution of the resulting mixed-integer problem, these binary decision variables are fixed. The column generation procedure is then applied. At the end of the column generation procedure, the integrality constraints of binary decision variables belonging to another production level are added to the problem according to the order of the decomposition (see Table 1). The resulting mixed-integer problem is then processed in the same manner until all the integrality constraints for the relaxed binary decision variables have been added. In the last iteration, the integrality constraints of the remaining decision variables are added to the problem, as well. Therefore, the solution of the resulting problem, if it exists, corresponds to a feasible solution to the integrated problem with level decomposition.

The heuristic used as a benchmark for the other approaches proposed in this paper is a Column Generation-based Heuristic (*CGH*). In the *CGH*, the integrality constraints of all the variables are imposed to the resulting LP with all the generated columns, which is obtained after the first application of the column generation procedure. This resulting mixed-integer problem is then solved by an optimization package. Such an approach can be seen as basically solving the mixed-integer mathematical model, obtained after the column generation procedure, by an optimization package without any decomposition strategy.

5. Computational experiments

This section presents the data generation and the computational results obtained by applying the hybrid heuristic to the *G3ILSCS* and *G4ILSCS* models (see Section 3). The proposed models and solution approach

Variations	Iteration 1	Iteration 2	Iteration 3
HH_L123	$Y^1_{ot},\ Y^{1A}_{st} \in \{0,1\}$	$Y^2_{ojt} \in \{0,1\}$ $Y^1_{ot} = \tilde{Y}^1_{ot}\ \ \forall o,\ \forall t$ $Y^{1A}_{st} = \tilde{Y}^{1A}_{st}\ \ \forall s,\ \forall t$	$Y^3_{ft} \in \{0,1\}$ $Z^2_{ojt},\ H^4_{kgt} \in \mathbb{Z}_+$ $Y^2_{ojt} = \tilde{Y}^2_{ojt}\ \ \forall o,\ \forall j,\ \forall t$
HH_L213	$Y^2_{ojt} \in \{0,1\}$	$Y^1_{ot},\ Y^{1A}_{st} \in \{0,1\}$ $Y^2_{ojt} = \tilde{Y}^2_{ojt}\ \ \forall o,\ \forall j,\ \forall t$	$Y^3_{ft} \in \{0,1\}$ $Z^2_{ojt},\ H^4_{kgt} \in \mathbb{Z}_+$ $Y^1_{ot} = \tilde{Y}^1_{ot}\ \ \forall o,\ \forall t$ $Y^{1A}_{st} = \tilde{Y}^{1A}_{st}\ \ \forall s,\ \forall t$
HH_L231	$Y^2_{ojt} \in \{0,1\}$	$Y^3_{ft} \in \{0,1\}$ $Y^2_{ojt} = \tilde{Y}^2_{ojt}\ \ \forall o,\ \forall j,\ \forall t$	$Y^1_{ot},\ Y^{1A}_{st} \in \{0,1\}$ $Z^2_{ojt},\ H^4_{kgt} \in \mathbb{Z}_+$ $Y^3_{ft} = \tilde{Y}^3_{ft}\ \ \forall f,\ \forall t$
HH_L321	$Y^3_{ft} \in \{0,1\}$	$Y^2_{ojt} \in \{0,1\}$ $Y^3_{ft} = \tilde{Y}^3_{ft}\ \ \forall f,\ \forall t$	$Y^1_{ot},\ Y^{1A}_{st} \in \{0,1\}$ $Z^2_{ojt},\ H^4_{kgt} \in \mathbb{Z}_+$ $Y^2_{ojt} = \tilde{Y}^2_{ojt}\ \ \forall o,\ \forall j,\ \forall t$

Table 1: Level decomposition variations

were implemented in C++ using the C callable library of the IBM ILOG Cplex 22.1.1. All the computational tests were conducted using a computer with 2 processors Intel(R) 3.07GHz (96 of RAM). We impose the use of a single processor (threads = 1), a time limit of 1800 seconds and an optimality gap of 0.1% in Cplex, while the remaining parameters are set to their default values.

The column generation procedure stops by optimality, i.e., the procedure cannot generate columns anymore, or heuristically, when the procedure is interrupted after a maximum number of 20 iterations without improvements in the objective function value of the restricted master problem. In either case, a lower bound is obtained using formula (33). The computational time is split equally among all iterations of the hybrid heuristic and the time left in an iteration is added to the next iteration. At the end of the hybrid heuristic, if a solution is found, it consists of a feasible solution to the integrated problem.

The gap is calculated according to the equation (34), where Z_H is the objective function value from the heuristic and Z_{LB} is the lower bound obtained using formula (33).

$$GAP = \frac{100(Z_{HH} - Z_{LB})}{Z_{LB}} \tag{34}$$

The tested instances and computational experiments are discussed in the following sections.

5.1. Computational study: G3ILSCS model

In this section, we analyze the behavior of the solution approaches when solving the generalized 3-level integrated problem (model (1) - (13)). It is worth mentioning that, although the *G3ILSCS* has been proposed in the literature, no computational analysis regarding the behavior of the 3-level integrated problem or solution methods were previously proposed to the problem.

5.1.1. Data set: G3ILSCS model

The data set used to generate instances for the *G3ILSCS* is based on well-known data from the literature of the lot-sizing and cutting stock problems, which are the instances from Trigeiro et al. (1989) and the CUTGEN1 generator proposed by Gau and Wäscher (1995). We generate the instances following the parameter settings as displayed in Table 2. When data is generated from a given interval $[a, b]$, we use a uniform distribution.

Parameters	Data variation	**Parameters**	Data variation	**Parameters**	Data variation
$\|F\|$	10, 20, 30	$\|P\|$	$\|F\|$	$\|T\|$	20
$\sum_{p \in P} r_{fp}$	1, 2	L_o	10,000	$\|O\|$	1
d^3_{ft}	[0,200]	l_p	$\begin{cases} [0.01, 0.8]\ L_o, & \text{Variate;} \\ [0.01, 0.8]\ L_o, & \text{Small;} \\ [0.2, 0.8]\ L_o, & \text{Large.} \end{cases}$	Cap^1_t	$\begin{cases} Cap^3_t, & \text{if } \sum_{p \in P} r_{fp} = 1; \\ 2\ Cap^3_t, & \text{if } \sum_{p \in P} r_{fp} = 2. \end{cases}$
sc^3_{ft}	[400,1200]	sc^2_{ojt}	10 [400, 1200]	sc^1_{ot}	10 [400, 1200]
vc^3_{ft}	0	vc^2_{ojt}	0.5 L_o	vc^1_{ot}	0
hc^3_{ft}	[0.8, 1.2]	hc^2_{pt}	$\begin{cases} [0.8, 1.2], & \text{if } \sum_{p \in P} r_{fp} = 1; \\ 0.5\ [0.8, 1.2], & \text{if } \sum_{p \in P} r_{fp} = 2. \end{cases}$	hc^1_{ot}	0.02 vc^2_{ojt}
st^3_{ft}	[21,65]	st^2_{ojt}	[21,65]	st^1_{ot}	[21,65]
vt^3_{ft}	1	vt^2_{ojt}	1	vt^o_t	1
Cap^3_t	$\frac{\sum_{t \in T} \sum_{f \in F} (vt^3_{ft} d^3_{ft} + st^3_{ft})}{T}$	Cap^2_t	$\begin{cases} Cap^3_t, & \text{if } \sum_{p \in P} r_{fp} = 1; \\ 2\ Cap^3_t, & \text{if } \sum_{p \in P} r_{fp} = 2. \end{cases}$		

Table 2: Data sets

Combining these scenarios, we obtain 18 sets of parameters (three variations to the number of final products ($|F|$) and length of pieces (l_p) and two variations to the number of pieces required in a final product ($\sum_{p \in P} r_{fp}$). For each variation of these parameters, 10 different instances are generated, totaling 180 instances in this computational analysis.

5.1.2. Parameter settings

In the hybrid heuristic, it is require the initialization of specific parameters for its execution. To identify the most suitable parameter values, a range of variations is tested for each decomposition strategy (time period, final product, and production level). The evaluation process considers the trade-off between solution quality and computational time. The variation that achieves a favorable trade-off is selected for further analysis and investigation within the paper. This selection ensures an exploration of the hybrid heuristic's performance with respect to the identified parameters.

- **Time period decomposition: *HH_T***
  We tested different sizes for the window in terms of the number of periods for which the integrality constraints of the binary decision variables are added to the problem (see Table 3). The variations mainly differ in terms of the number of feasible instances, for which variation (*vi*) is able to find a feasible solution to all the instances and presented a good tradeoff between gap and computational time, and hence, it is used in the further analysis of the hybrid heuristic with time period decomposition.

Tested variations *HH_T*	Δ	ψ	Tested variations *HH_T*	Δ	ψ
(*i*)	3	1	(*iv*)	10	3
(*ii*)	5	1	(*v*)	12	3
(*iii*)	7	2	**(*vi*)**	14	4

Table 3: Tested values in the Hybrid Heuristic with time period decomposition

- **Final product decomposition: *HH_F***
  We tested different sizes for the window in terms of the number of final products for which the integrality constraints of the binary decision variables are added to the problem (see Table 4). We

also consider 3 variations to the ordering of the final products in a window, which are according to the data file, the demand of the final products, and the value for the *EOQ* - Economic Order Quantity Andriolo et al. (2014)). The results showed the choice to the final products based on their demand with variation (*iii*) showed the best results in terms of number of feasible instances and the best tradeoff between gap and computational time, and hence, it is used in the further analysis of the hybrid heuristic with final product decomposition.

	$\|F\| = 10$		$\|F\| = 20$		$\|F\| = 30$	
Tested variations *HH_F*	Δ	ψ	Δ	ψ	Δ	ψ
(*i*)	7	2	12	3	17	5
(*ii*)	8	3	15	5	20	7
(*iii*)	9	3	17	5	25	10

Table 4: Tested values in the Hybrid Heuristic with final product decomposition

- **Level decomposition: *HH_L***
  Considering the variations for the level decomposition according to Table 1, the results showed that de level decompositions *HH_L123* and *HH_L231* presented the best tradeoffs in terms of gap and computational time with the highest number of feasible instances, and hence, they are used in the further analysis of the hybrid heuristic with level decomposition.

5.1.3. Computational analysis: G3ILSCS model

In this first analysis, we evaluate the behavior of the column generation procedure in different points of the hybrid heuristic. For this, Table 5 shows the average results, for each data variation, with the column generation when obtaining a lower bound for the integrated problem (columns Initial columns) and during its application in the relax-and-fix heuristic (columns ♯Additional Columns). Note that the lower bound values are the same for all the solution approaches and are obtained using equation (33). The last column, Total time, corresponds to the total time used by the column generation in the hybrid heuristic.

The computational time used by the column generation is quite small, less than 9 seconds on average, ranging from 0.54 up to 30.52 seconds, on average. For the classes where the length of pieces is small, the procedure is the most time-consuming and generates the highest number of columns. Due to this hight quantity of columns, the percentage of fixed columns reaches 84.5% on average, while when the length of pieces is large, the average percentage of fixed columns is 77.9%. There is also an interesting difference when comparing the number of additional columns generated with the several variations for the hybrid heuristic. The *HH_F* generates the smallest number of additional columns, while the *HH_L231* generates two times more columns. This can be explained by the fact that the *HH_F* has the smallest number of iterations of the hybrid heuristic, and hence, the column generation procedure. On the other hand, the *HH_L231* has the cutting patterns setup variables fixed in the first iteration of the heuristic, which can impact the generation of more additional columns, when solving the other levels of the problem in the following iterations of the method.

In the remaining analysis of this section, we evaluate the behavior of the different variations of the hybrid heuristic, in terms of the gap (percentage) and the computational time (in seconds), and compared the results with the benchmark approach (*CGH*). The average is calculated using only those instances solved by all the heuristics. The number of instances used to calculate the average in each set of parameters corresponds to the column Number instances. For more information about the status of the solutions see Table B.14 in Appendix B.

$\|F\|$	r_{fp}	l_p	Initial columns		♯Additional columns				Total time
			♯Columns	% Columns fixed	*HH_T*	*HH_F*	*HH_L123*	*HH_L231*	
10	1	Variated	536.00	81.59	395.00	228.00	300.70	524.90	0.74
		Small	872.00	87.07	613.20	488.00	394.10	1027.90	10.37
		Large	460.00	80.07	369.56	197.78	329.00	385.44	0.57
	2	Variated	522.00	78.92	384.20	264.00	393.70	519.90	0.62
		Small	844.00	85.55	652.70	514.00	336.90	1060.30	10.85
		Large	472.00	78.16	330.00	198.00	327.60	384.90	0.54
20	1	Variated	954.00	81.39	514.00	300.00	535.50	780.00	2.21
		Small	1678.00	85.45	842.70	784.00	759.40	1805.70	24.88
		Large	758.00	78.70	408.40	220.00	436.30	532.00	1.28
	2	Variated	968.00	80.03	495.10	336.00	502.90	732.80	2.25
		Small	1616.00	84.38	1128.50	977.50	616.63	1925.38	30.52
		Large	820.00	77.27	400.50	190.00	395.10	540.00	1.37
30	1	Variated	1362.00	80.39	597.22	417.78	712.22	975.56	5.11
		Small	1960.00	82.59	1299.30	1192.00	1242.80	2367.60	27.27
		Large	1072.00	76.87	442.50	266.00	499.50	684.00	2.03
	2	Variated	1354.00	78.67	651.00	448.00	678.00	1006.00	4.52
		Small	2010.00	82.36	1597.10	1318.00	920.20	2391.70	29.87
		Large	1148.00	76.61	429.90	254.00	468.00	608.00	2.15
Average			1078.11	80.89	637.98	473.64	546.65	1007.44	8.73

Table 5: Analysis of the solution approaches: column generation procedure

Table 6 summarizes the average results, for each number of final products ($|F|$). We can see that all the approaches present quite small gaps, smaller than 0.85% on average (from 0.79% to 0.87%). Some of the hybrid heuristics are able to find gaps that are on average at least as good as the benchmark *CGH*, having *HH_L123* reaching the smallest ones, with an average of 0.80%, followed by *HH_T* with a gap of 81% on average. Furthermore, *HH_L123* finds the smallest values to the gap in all variations, spending on average 47% less computational time than the *CGH*. The fastest approach is the *HH_L231*, which may be due to the fact that the bottleneck level 2, is solved in the first iteration of the hybrid heuristic with a smaller time limit, when compared with the *HH_L123*, where the time limit left after solving level 1 (previous iteration) is added to the time limit of the next iteration to solve level 2. This may also explain why *HH_L231* has the worse average gaps using the hybrid heuristic. Although there is a slight increase in the gap as the number of final products increases, there is no clear impact of the data set. The impact of the data set occur in the time, which increases as the number of final products increase. This can be explained by the fact that as the size of the problem increases, the approaches take more time to find a solution to the problem, and hence, the time increases.

Table 7 shows the average results for each number of pieces required to compose a final product (r_{fp}). Note that the number of different pieces and final products are the same, however, the demand of pieces is doubled when the final products require two pieces ($r_{fp} = 2$). When the required number of pieces in a final products is doubled, the gap is able to reduce 49.81% on average, among all the approaches. This can be explained by the fact that with more pieces, better cutting patterns can be generated, and hence, better solutions can be found, reducing the total cost. The *HH_L123* is able to find the best values for the gap in both variations of the parameter, while the *HH_L231* is the fastest approach. Surprisingly, the high demand of pieces does not increase the computational time, but in fact, it helps decreasing 27.79% on average (from 11.19% to 36.05%).

In Table 8, we present the average results according to the variations for the length of pieces (l_p). The results show that the consideration of only small pieces led to a solution with less quality, i.e., large gaps, whereas the existence of either large items or variate items had the opposite effect. The benchmark *CGH* finds the smallest gaps when the length of pieces is variate and large. However, when the length of pieces is small, the heuristics are able to reduce the gap in comparison with the *CGH*, for which *HH_L123* finds the smallest value, 1.77% on average. Comparing the computational time, the approaches have quite the same behavior, for which the small length of pieces is very time-consuming for all the approaches, using almost all the available time to solve the problem, whereas the large variation uses only 14.84% of this time to be solved. Such an analysis reinforces the impact of data set on the results.

	Number	Gap (%)						Computational Time (sec.)					
$\|F\|$	instances	*CGH*	*RFH_T*	*HH_T*	*HH_F*	*HH_L123*	*HH_L231*	*CGH*	*RFH_T*	*HH_T*	*HH_F*	*HH_L123*	*HH_L231*
10	59	0.82	0.84	0.82	0.82	0.82	0.87	1198.3	746.1	865.8	810.9	601.9	475.5
20	58	0.83	0.81	0.79	0.83	0.79	0.83	1262.7	820.9	871.7	918.8	689.9	613.8
30	59	0.89	0.86	0.83	0.86	0.81	0.86	1443.5	930.3	974.9	1021.1	781.0	690.2
Average	176	0.85	0.84	0.81	0.83	0.80	0.85	1301.7	832.5	904.3	916.9	690.9	593.1

Table 6: Analysis according to the number of final products ($|F|$)

	Number	Gap (%)						Computational Time (sec.)					
r_{fp}	Instances	*CGH*	*RFH_T*	*HH_T*	*HH_F*	*HH_L123*	*HH_L231*	*CGH*	*RFH_T*	*HH_T*	*HH_F*	*HH_L123*	*HH_L231*
1	88	1.16	1.12	1.09	1.11	1.07	1.12	1570.7	980.1	1103.2	1112.1	777.9	628.2
2	88	0.54	0.56	0.54	0.56	0.54	0.59	1032.8	684.9	705.4	721.7	604.0	557.9
Average	176	0.85	0.84	0.81	0.83	0.80	0.85	1301.7	832.5	904.3	916.9	690.9	593.1

Table 7: Analysis according to the composition of final products (r_{fp})

	Number	Gap (%)						Computational Time (sec.)					
l_p	Instances	*CGH*	*RFH_T*	*HH_T*	*HH_F*	*HH_L123*	*HH_L231*	*CGH*	*RFH_T*	*HH_T*	*HH_F*	*HH_L123*	*HH_L231*
Variate	59	0.29	0.35	0.32	0.31	0.37	0.37	1360.4	567.0	732.6	731.5	321.4	161.1
Small	58	2.04	1.92	1.89	1.96	1.77	1.92	1800.0	1788.1	1800.0	1746.3	1731.1	1618.9
Large	59	0.22	0.28	0.25	0.25	0.28	0.29	753.2	158.5	195.5	287.0	38.0	16.6
Average	176	0.85	0.84	0.81	0.83	0.80	0.85	1301.7	832.5	904.3	916.9	690.9	593.1

Table 8: Analysis according to the length of pieces (l_p)

In conclusion, with these analysis of the results for the different solution approaches to solve the *G3ILSCS*, we can say that the *HH_L123* and *HH_T* have shown a good tradeoff between solution quality, by finding the smallest gaps, and computational effort, whereas *HH_L231* has the shortest computational effort. In the next section, we evaluate such approaches in a more difficult problem and compare their behavior.

5.2. Computational Study: G4ILSCS Model

The other study presented in this paper comprehends an analysis of the solution approaches when solving the 4-level integrated problem (model (3) - (9), (11) - (13), (18) - (20), and (22) - (25)). The different variations for the hybrid heuristic are compared to the benchmark approach *CGH*. For all the computational tests, we impose the use of a single processor (threads = 1), a time limit of 3600 seconds and an optimality gap of 0.1% in Cplex, while the remaining parameters are set to their default values.

5.2.1. Data Set G4ILSCS Model

Additionally to the previous data sets, the values to the new parameters present in the *G4ILSCS* are based on the studies of Basnet and Leung (2005) and Molina et al. (2016) for the supplier and distribution decisions (see Table 9). We use a uniform distribution for data generated from a given interval $[a, b]$.

Parameters	Data variation	**Parameters**	Data variation
$\|S\|$	5, 10, 15	$\|K\|$	1
sc_{st}^{1A}	[10.000, 12.000]	uc_k^4	100, 500, 3000
pc_{ost}^{1A}	[1, 5]	d_{ft}^4	[0,200]
		hc_{ft}^4	hc_{ft}^3
$CapV_k^4,\ CapW_k^4$	100	$v_f,\ w_f$	$\begin{cases} [2,5], & \text{Low;} \\ [2, 10], & \text{Medium;} \\ [5, 15], & \text{Large.} \end{cases}$

Table 9: Data sets

Some variations of the previous data set are fixed as follows:

- number of final products: $|F| = 10$;
- number of pieces required in a final product: $\sum_{p \in P} r_{fp} = 1$ (each final product corresponds directly to a piece);
- production capacity of final products: $Cap_t^3 = cap_t/0.7$;
  The capacity cap_t is generated using $\frac{\sum_{t \in T} \sum_{f \in F}(vt_{ft}^3 d_{ft}^3 + st_{ft}^3)}{T}$.
- capacity of the cutting machine: $Cap_t^2 = Cap_t^3$;
- production capacity of objects: $Cap_t^1 = Cap_t^3$;
- length of pieces: $l_p \in [0.01, 0.8]\ L_o$ (Variate);

In the data variation, the values to the capacity of vehicles, in terms of volume and weight, are expressed in percentage, that is, the vehicles have a capacity of 100%, and the utilization of the final products in the vehicles, in terms of volume and weight, varies between 2% and 10% according to each vehicle type. The calculation of the volume of each final product is considered as dependent on its length. The reason for this assumption is that in certain cargo scenarios, weight and volume serve as concurrent capacity constraints and should be taken into account when dealing with distribution decisions.

We combine these scenarios and 27 sets of parameters are obtained (three variations to the number of suppliers, unit utilization cost of vehicles, and the weight and volume of the final product). For each variation, 10 different instances are generated, totalling 270 instances in this computational analysis.

5.2.2. Computational Analysis: G4ILSCS model

In this first analysis for *G4ILSCS* model, Table 10 presents the status of the solutions (number of feasible, unknown, or infeasible instances) obtained with each heuristic method. We can see that the solution approaches have varying degrees of success in finding feasible solutions. Among the different decomposition strategies for the hybrid heuristics, the *HH_L123* has the highest number of instances with a feasible solution for all values of $|S|$, solving almost 95% of the instances in the analysis, followed by *HH_T*. On the other hand, the benchmark *CGH* has relatively lower numbers of instances with a feasible solution, solving only 71% of the instances. The majority of instances without a feasible solution using the benchmark *CGH* is due to the constraint on the time limit, whereas using the hybrid heuristic is due to infeasibility reached during the heuristic, which can be explained by the fact that depending on the decomposing strategy the constraints of capacity and dependency among the decisions of the problem might generate infeasibilities. Overall, this analysis suggests that the *HH_L123* and *HH_T* approaches perform the best in finding feasible instances, while the benchmark *CGH* encounters a significant number of instances without a feasible solution.

			Feasible		
$\lvert S \rvert$	*CGH*	*HH_T*	*HH_F*	*HH_L123*	*HH_L231*
10	72	82	78	86	75
20	67	80	75	84	73
30	54	73	68	84	82
Sum	193	235	221	254	230

			Infeasible		
$\lvert S \rvert$	*CGH*	*HH_T*	*HH_F*	*HH_L123*	*HH_L231*
10	0	5	1	2	14
20	1	8	1	4	17
30	0	9	0	2	8
Sum	1	22	2	8	39

			Unknown		
$\lvert S \rvert$	*CGH*	*HH_T*	*HH_F*	*HH_L123*	*HH_L231*
10	18	3	11	2	1
20	22	2	14	2	0
30	36	8	22	4	0
Sum	76	13	47	8	1

Table 10: Analysis of the solution approaches for *G4ILSCS*: status of the solution

In the remaining of the analysis in this section, we consider only the *HH_L123* and *HH_T* approaches, which are compared with the benchmark *CGH*, aiming to better investigate and determine the most effective solution approach for the *G4ILSCS* problem, as well as the impact of some of these parameters on the solution of the integrated problem. The averages are calculated using only those instances solved by all the heuristics. The number of instances used to calculate the average in each set of parameters corresponds to the column Number instances.

Table 11, 12 and 13 present the average results obtained with the *CGH*, *HH_T* and *HH_L123* approaches for the number of suppliers ($|S|$), utilization cost of vehicles (uc^4k) and weight and volume utilization of the final product in the vehicles (v_f and w_f), respectively. We can see that all the approaches present quite small gaps, smaller than 0.62% on average (from 0.48% to 0.70%) for all the different analysis. The results indicate that *HH_T* consistently achieves the lowest gap across all the approaches for all data variation. It is in terms of the computational time that the differences between the approaches become more evident with the data variation. In fact, the benchmark uses on average all the computational time available to solve the instances, followed very closely by *HH_T*, while *HH_L123* uses only 35% of the time available. When the weight and volume utilization of the final product in the vehicles is low, the instances are very time-consuming, increasing up to 50% the computational time, when compared with the large variation for the *HH_L123*, which states the impact of this data on the results. These findings suggest that *HH_L123* consistently outperforms *CGH* and *HH_T* in terms of computational time, while *HH_T* offers the lowest gaps using competitive computational effort, when compared with the benchmark *CGH*. Therefore. the results in this paper highlights the benefit of using more tailored heuristics to solve integrated problems.

$\|S\|$	Number Instances	Gap (%)			Computational Time (sec.)		
		CGH	*HH_T*	*HH_L123*	*CGH*	*HH_T*	*HH_L123*
5	63	0.59	0.49	0.57	3600.00	3320.18	1389.84
10	54	0.70	0.50	0.58	3600.00	3463.48	1266.07
15	38	0.55	0.48	0.56	3600.00	3366.55	1091.52
Average	155	0.62	0.49	0.57	3600.00	3381.47	1273.59

Table 11: Analysis according to the number of suppliers ($|S|$)

uc_k^4	Number Instances	Gap (%)			Computational Time (sec.)		
		CGH	*HH_T*	*HH_L123*	*CGH*	*HH_T*	*HH_L123*
100	55	0.65	0.50	0.59	3600.00	3340.25	1199.37
500	44	0.65	0.49	0.57	3600.00	3448.08	1213.42
3000	56	0.57	0.48	0.56	3600.00	3369.62	1393.75
Average	155	0.62	0.49	0.57	3600.00	3381.47	1273.59

Table 12: Analysis according to the utilization cost of vehicles (uc_k^4)

v_f and w_f	Number Instances	Gap (%)			Computational Time (sec.)		
		CGH	*HH_T*	*HH_L123*	*CGH*	*HH_T*	*HH_L123*
Low	44	0.64	0.52	0.59	3600.00	3525.37	1857.84
Medium	56	0.60	0.50	0.57	3600.00	3402.25	1126.91
Large	55	0.62	0.47	0.56	3600.00	3245.20	955.52
Average	155	0.62	0.49	0.57	3600.00	3381.47	1273.59

Table 13: Analysis according to the weight and volume utilization of the final product in the vehicles (v_f and w_f)

6. Conclusions and Future Perspective

In conclusion, this paper addresses the challenges faced by manufacturing industries in optimizing their supply chain decision-making processes through the integration of mathematical models. Specifically, the focus is on the tactical and operational planning of production processes that involve both the cutting of raw materials and the production of final products. The main contribution of this research lies in the extension of the existing 3-level integrated lot-sizing and cutting stock problem (*G3ILSCS*) to incorporate two additional decisions of the supply chain: supplier selection for raw materials and distribution of final products from the production plant to a warehouse. These extensions lead to a novel problem with the formulation of the 4-level integrated lot-sizing and cutting stock problem with supplier selection and distribution decisions (*G4ILSCS*).

To tackle the complexity of the integrated problem, a hybrid heuristic is proposed, which combines two decomposition approaches: column generation and the relax-and-fix procedures. The hybrid heuristic consists of an iterative approach where the column generation and the relax-and fix procedures are used in each iteration of the algorithm in order to generate new attractive columns to the problem (cutting patterns for the cutting process and cargo configurations for the vehicle loading), using the column generation procedure, while finding a feasible solution to the problem, using the relax-and-fix procedure with three selection strategies (time period, final product, and production level), to decompose the problem.

In the analysis for the *G3ILSCS* model, the main results indicate that the *HH_L123* and *HH_T* approaches show promising performance in terms of both solution quality, as measured by the smallest gaps, and computational efficiency, when compared to the traditional *CGH* benchmark. Moreover, the *HH_L231* approach achieves the same gap, on average, but with a significantly reduced computational time, using over 50% less time than the *CGH* method. For the *G4ILSCS* model, it is more evident that, when it comes to identifying feasible instances for the problem, the *HH_L123* and *HH_T* approaches stand out as the most effective performers. Their remarkable capabilities in locating viable solutions are clearly demonstrated, showcasing their potential as methodologies for the integrated problem. Furthermore, the findings consistently demonstrate that the *HH_L123* approach outshines both *CGH* and *HH_T* in the aspect of computational time, whereas the *HH_T* showcases the lowest gaps with competitive time usage. This comprehensive analyses underscores the immense potential and distinct advantages offered by using specific approaches when solving integrated problems.

In summary, this paper contributes to the field of supply chain optimization by proposing innovative formulations for integrated lot-sizing and cutting stock problems, incorporating supplier selection and distribution decisions, along with the development of a hybrid heuristic that offers a powerful tool for solving such complex problems. The results emphasize the benefits of integrating decision-making processes within the supply chain, leading to more efficient resource utilization and cost reduction. The insights gained from this research can guide manufacturing industries in their efforts to enhance production planning. Future research in this area could explore additional real-world case studies and further refine the proposed heuristic to address more intricate supply chain scenarios.

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Appendix A. Additional information: hybrid heuristic

Figure A.2 present a flowchart of the hybrid heuristic proposed that consists of an iterative approach where the column generation and the relax-and fix procedures are used in each iteration of the algorithm in order to generate new attractive columns to the integrated problem, using the column generation procedure, while finding a feasible solution to the problem, using the relax-and-fix procedure.

The Algorithm 1 displays the steps of the relax-and-fix heuristic used in the hybrid heuristic that consists of an approach to iteratively relax constraints, solve a problem and fix variables in order to find a feasible solution to the problem using a decomposition based on the time periods. The same idea is used for the decomposition based on the final products.

Algorithm 1 Relax-and-fix with time period decomposition - *HH_T*

Input: Δ: number of periods in the time window.
  ψ: number of periods with no overlapping.
  Set the time window to the ordered set $\Omega = \{1, \dots, \Delta\}$.
Step 1: Relax the integrality of all variables.
while $\Delta \leq |T|$ **do**
  Step 2: Add the integrality constraints:
    $Y^1_{ot} \in \{0,1\}, \ \forall o, \ \forall t \in \Omega$
    $Y^{1A}_{st} \in \{0,1\}, \ \forall s, \ \forall t \in \Omega$
    $Y^2_{ojt} \in \{0,1\}, \ \forall o, \ \forall j, \ \forall t \in \Omega$
    $Y^3_{ft} \in \{0,1\}, \ \forall f, \ \forall t \in \Omega$
  if Last iteration is reached **then** add the integrality constraints:
    $Z^2_{ojt} \in \mathbb{Z}_+, \ \forall o, \ \forall j, \ \forall t$
    $H^4_{kgt} \in \mathbb{Z}_+, \ \forall k, \ \forall g, \ \forall t$
  end if
  Step 3: Solve the resulting mixed-integer problem.
  if Resulting mixed-integer integrated problem is infeasible **then**
    No solution is found by the Hybrid Heuristic.
    break
  end if
  Step 4: Fix the binary decision variables:
    $Y^1_{ot} = \widetilde{Y}^1_{ot}, \ \forall o,$ first ψ periods in Ω
    $Y^{1A}_{st} = \widetilde{Y}^{1A}_{st}, \ \forall i,$ first ψ periods in Ω
    $Y^2_{ojt} = \widetilde{Y}^2_{ojt}, \ \forall o, \ \forall j,$ first ψ periods in Ω
    $Y^3_{ft} = \widetilde{Y}^3_{ft}, \ \forall i,$ first ψ periods in Ω
  Step 5: Move forward the set Ω by ψ periods.
end while
Output: Feasible solution is found by Hybrid Heuristic.

Appendix B. Additional computational results

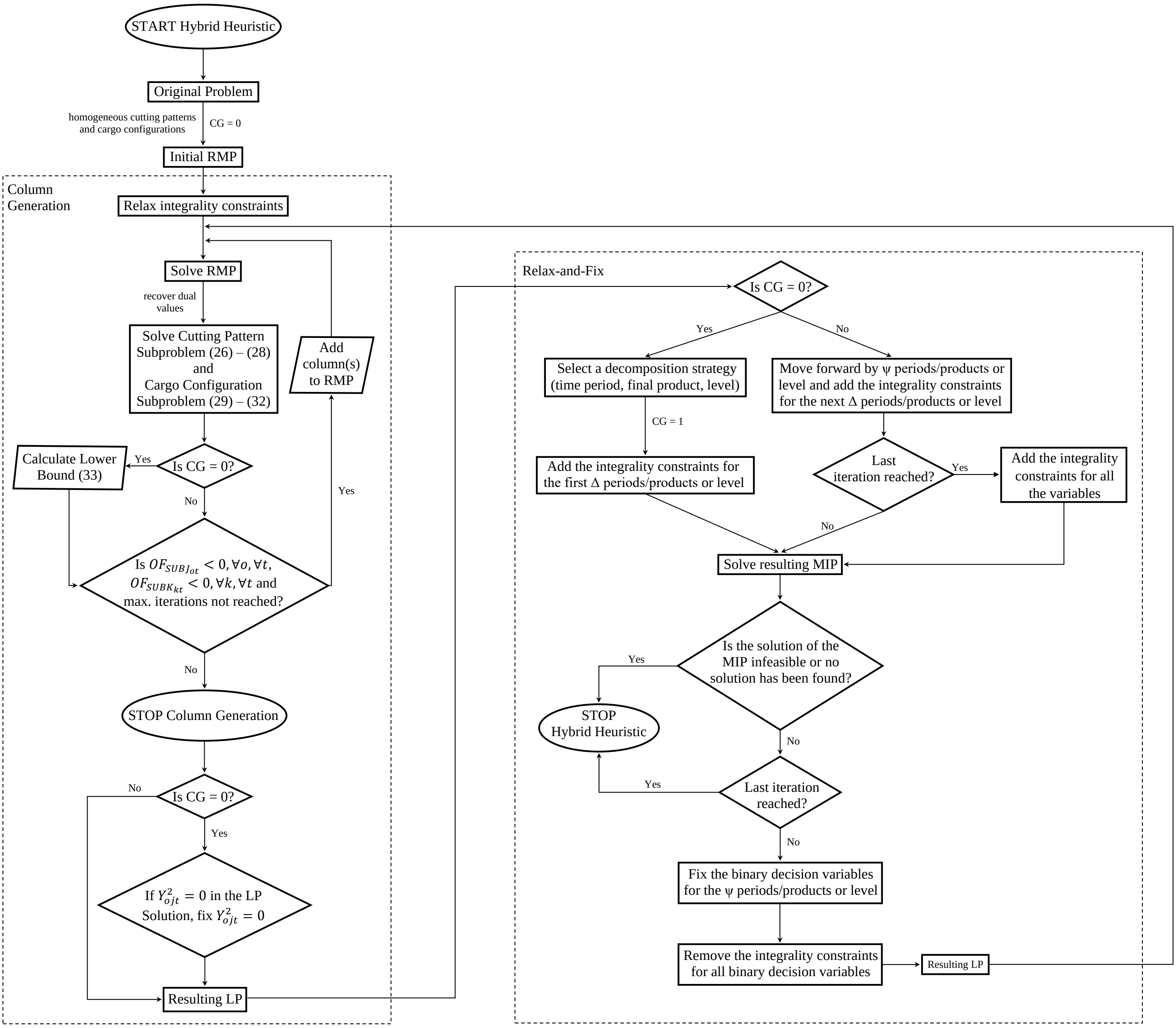

Figure A.2: Flowchart of the Hybrid Heuristic.

$\lvert F\rvert$	r_{fp}	l_p	*CGH**	Number of feasible instances				
				CGH	*HH_T*	*HH_F*	*HH_L123*	*HH_L231*
10	1	Variated	10	10	10	10	10	10
		Small	8[b]	10	10	10	10	10
		Large	10	10	10	10	9[a]	10
	2	Variated	10	10	10	10	10	10
		Small	9[b]	10	10	10	10	10
		Large	10	10	10	10	10	10
20	1	Variated	9[b]	10	10	10	10	10
		Small	9[b]	10	10	10	10	10
		Large	9[b]	10	10	10	10	10
	2	Variated	10	10	10	10	10	10
		Small	10	10	10	8[b]	10	10
		Large	9[b]	10	10	10	10	10
30	1	Variated	10	10	10	10	10	9[a]
		Small	7[b]	10	10	10	10	10
		Large	10	10	10	10	10	10
	2	Variated	10	10	10	10	10	10
		Small	10	10	10	10	10	10
		Large	10	10	10	10	10	10
Sum			170	180	180	178	179	179

[a] Infeasible instances.

[b] Unknown instances.

* Without fixing strategy.

Table B.14: Analysis of the solution approaches for *G3ILSCS*: status of the solution