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Economic production lot-sizing for an unreliable machine under imperfect age-based maintenance policy

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Abstract

This paper is concerned with the joint determination of both economic production quantity and preventive maintenance (PM) schedules under the realistic assumption that the production facility is subject to random failure and the maintenance is imperfect. The manufacturing system is assumed to deteriorate while in operation, with an increasing failure rate. The system undergoes PM either upon failure or after having reached a predetermined age, whichever of them occurs first. As is often the case in real manufacturing applications, maintenance activities are imperfect and unable to restore the system to its original healthy state. In this work, we propose a model that could be used to determine the optimal number of production runs and the sequence of PM schedules that minimizes the long-term average cost. Some useful properties of the cost function are developed to characterize the optimal policy. An algorithm is also proposed to find the optimal solutions to the problem at hand. Numerical results are provided to illustrate both the use of the algorithm in the study of the optimal cost function and the latter's sensitivity to different changes in cost factors.

Keywords: Preventive maintenance; Imperfect maintenance; Lot-sizing; Economic production quantity; Integrated models

1. Introduction

The field of inventory management has recently witnessed an intense research activity in adding more realism to the available economic manufacturing quantity (EMQ) models so as to address real-world applications. The objective of such an activity is to provide a modeling tool that closely represents real manufacturing applications. The prime motivation for seeking more realistic models stems from the fact that by including in the EMQ model, the cost and resource factors that may dominate a particular application, these models will then greatly help in both reducing the operation, production, and inventory costs, and helping companies survive in a global competitive market. Real world applications are not as perfect as assumed in the classical EMQ model where production systems are considered free from defects, deterioration, and failure (Lee and Rosenblatt, 1987; Nahamias, 1989). Such systems are indeed rare, if not totally inexistent

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in practice. In a real industrial environment, system deterioration and random failures are highly likely to occur. Dohi et al. (2001) recognize that, in such conditions, the classical EMQ model loses its usefulness. These authors affirm, "...effects of machine breakdown and corrective maintenance in economic production lot sizing decisions should be examined exactly in uncertain environment without reliable manufacturing facilities" (p. 1).

Production models with unreliable machines have been studied in many papers. Rosenblatt and Lee (1986) and Porteus (1986) studied the effect of the presence of defective items on economic quantities. In both studies, the deterioration of the production process was assumed to be a random process, where no possible operator interventions or maintenance activities were considered. They found that the optimal EMQ was smaller than the one produced by the classical EMQ model. Lee and Rosenblatt (1987) considered the simultaneous determination of production cycle and inspection schedules in a production system. They proposed a simple relationship to determine whether maintenance by inspection is necessary. They also showed that when maintenance by inspection is adopted, the optimal inspection schedules are equally spaced throughout the production cycle. Lee and Rosenblatt (1989) also extended their work to consider the case where no immediate knowledge of the system's state is available unless inspection of produced items is performed. Maintenance activities are subsequently initiated based on the inspection results. The random deterioration process considered in their model follows an exponential failure distribution. Lee and Rosenblatt succeeded in determining the optimal lot size as well as the optimal inspection schedule. Groenevelt et al. (1992) addressed the EMQ problem of a manufacturing process with constant failure rate and negligible repair times. In a similar approach to that of (Rosenblatt and Lee, 1986), they observed that the optimal EMQ is smaller than that provided by the classical EMQ model. Later on, Lin et al. (1991) and Hariga and Ben Daya (1998) extended the work of Groenevelt et al. (1992) by allowing the random deterioration process to follow a general probability distribution. Recently, Lee and Srinivason (2001) considered an integrated production/inventory and preventive maintenance (PM) model. The policy adopted therein specifies both the maximum inventory level (S) and the number of times (N) preventive maintenance activities are carried out. Their objective was to determine the optimal control policy, (S, N), that minimizes the long-term average cost of operating the facility per unit of time. They established the properties of the various cost components. In their model, Lee and Srinivason assumed that, during production run, the unreliable production facility is minimally repaired in the case of failure. They also assumed that the facility undergoes a PM activity that brings it to an "as-good-as-new" state at the end of every production run. However, these models failed to address real industrial situations where maintenance is not always perfect.

Maintenance could be classified into four categories according to the degree to which the operating conditions of an item are restored after maintenance activities (Pham and Wang, 1996): Perfect repair or perfect maintenance, minimal repair or minimal maintenance, imperfect repair or imperfect maintenance, and worse repair or maintenance. In particular, Pham and Wang (1996) define imperfect repair or imperfect maintenance as maintenance actions that improve the effective age and the health of the system compared to what they were just before maintenance. After PM, the condition of the system is somewhere between as-good-as-new and as-bad-as-old. Several models for imperfect PM have been proposed in the literature. Nguyen and Murthy (1981) studied the problem of optimal preventive maintenance policies for repairable systems. The authors considered two PM policies: Policy I and Policy II as explained below:

PM policy I: PM is age-based. The system undergoes the ith PM at failure or at age T_{p_i} (i.e., T_{p_i} hours from the last PM or replacement), whichever of them occurs first. After $(N-1)$ PMs, the system is replaced. In the case of failure, an additional cost is incurred (El-Ferik and Ben-Daya, 2006).

PM policy II: The ith PM is always carried out after a predetermined time T_{p_i}. The system is replaced after $(N-1)$ PMs. In the case of failure, a minimal repair is carried out (Nakagawa, 1988; Lin et al., 2000, 2001).

Ben-Daya and Khursheed (2002) extended Lee and Srinivason's (2001) model by considering imperfect PM activities according to maintenance policy II. They considered a system subject to random failures with increasing failure rates as in (Nakagawa, 1988). In their proposed model, the PM activities reduce the rate of occurrence of failures. Since the failure rates were strictly increasing with the age of the system, the post-PM improvement was expressed in terms of reduction of the effective age of the production facility. Later on, Ben-Daya (2002) addressed the problem of lot size determination with imperfect production process and imperfect maintenance. Here too, the maintenance was performed according to policy II.

Aghezzaf et al. (2006) proposed a combined production and maintenance model in a batch production system context. The model they developed allows the determination of an integrated production and maintenance plan that satisfies the demand over the entire horizon without backlogging, and which minimizes the expected total production and maintenance costs. These authors considered the case of a production system that is subject to random failure. They adopted a periodic maintenance policy where a minimal repair is performed at failure as in (Barlow and Hunter, 1960). In such an integrated model, the maintenance activities at any period reduce the system's available production capacity.

In many industrial applications, PM can restore only part of the system's performance (Zhao et al., 2006). Examples of such systems are abundant and include aircraft, conveyor belts, photocopy equipment, stamping presses (Yano, 2006), gas turbine engines (Zhao et al., 2006), bus engines (Lai et al., 2000), and many others. Imperfect preventive maintenance models are particularly useful for such systems where repeated repairs improve the system's condition in the short term. However, in the long term, the system's failure rate tends to increase due to aging.

In this paper, we consider the joint determination of the economic lot size for an unreliable manufacturing facility maintained according to maintenance policy I, i.e., the maintenance is considered to be age-based with an increasing failure rate. Preventive maintenance activities are performed on the facility as soon as the system fails or reaches an age T_p, whichever of them occurs first. Also, PM is assumed to be imperfect as it could restore the system's effect age back to zero with a higher failure rate (Nguyen and Murthy, 1981). The demand and production rates are constant. A production run restarts after a failure has occurred or a PM action has taken place as soon as the inventory level drops to zero. After N production cycles, the system is replaced. The control policy considered in this paper aims at identifying the optimal policy (T_p, N) that minimizes a general long-term average cost that includes the setup cost, holding cost, cost of lost sales, cost of replacement, breakdown cost, and preventive maintenance cost. A formulation of the problem is presented in Section 2. The mathematical cost model is developed in Section 3. In Section 4, an analysis of the model is carried out and some useful properties of the optimal solution derived. In addition, an algorithm to compute the optimal solution will be suggested. Numerical results and a sensitivity analysis, for the special case where the failure time follows a Weibull distribution, are presented in Section 5.

2. Problem formulation

First, the following notation will be adopted to develop the model proposed in this paper.

$\mathrm{AVC}(\{T_{p_i}\}, N)$ long term average cost
$\mathrm{TC}(\{T_{p_i}\}, N)$ total cost per cycle
$\mathrm{CL}(\{T_{p_i}\}, N)$ cycle length
$\mathrm{SC}(\{T_{p_i}\}, N)$ setup cost
$\mathrm{LS}(\{T_{p_i}\}, N)$ lost sales cost
$\mathrm{HC}(\{T_{p_i}\}, N)$ inventory holding cost
$\mathrm{PM}(\{T_{p_i}\}, N)$ preventive maintenance cost
$\mathrm{BC}(\{T_{p_i}\}, N)$ breakdown cost
d demand rate
u production rate
K_1 setup cost
K_2 setup cost after replacement
C_{h} holding cost per unit held in inventory per unit of time
$r_i(t)$ failure rate for the ith production run
C_ℓ cost of lost sales per unit
C_{B} cost of additional repair in case of failure (used with Policy I)
C_{pm} cost of PM
C_{r} cost of replacement per unit of time
N number of production runs per cycle
T_{p_i} system age during the ith production run

$F_i(t)$ the probability distribution of the facility age
$f_i(t)$ the probability density function $\frac{dF_i(t)}{dt}$
$R_i(t)$ the reliability function $\overline{F}_i = 1 - F_i(t)$
$F_r(t)$ the probability distribution of the lead time
$f_r(t)$ the probability density function $\frac{dF_r(t)}{dt}$
T_r the random variable representing the replacement time
$E[.]$ the expected value
LHS left hand side
RHS right hand side

As in the classical economic production lot-sizing model, the demand rate, d, and the production rate, u, are both constant with $u > d$. The manufacturing system is supposed to deteriorate while in operation with an increasing failure rate. The PM activities are imperfect and are unable to restore the facility to its original condition. Thus, as illustrated in Fig. 1, after the PM activity is performed, the facility is between an as-good-as-new state and an as-bad-as-old one. Indeed, while the age of the system is restored back to zero, the facility is more likely to fail with a failure rate, $r_i(t) = A_i r(t)$, higher than what it was in the previous production run.

There are two types of costs involved here: a setup cost (K_1), incurred for each production run, and a holding cost (C_h) per unit held in inventory for each unit of time. The setup, PM, and repair times are presumed negligible. The system undergoes a certain number of such PM activities before initiating either a replacement or a major system overhaul. Furthermore, the adopted maintenance policy is age-based as per policy I. Thus, after the ith PM, a new production run is initiated as soon as the inventory level drops to zero, and continues to do so, until the facility fails or its age reaches a predetermined value T_{p_i}, $i = 1, 2, \ldots, N-1$, whichever occurs first. Every PM activity costs C_{PM} and an additional cost C_B is considered if the PM is performed after occurrence of a failure. After N production runs, either a system replacement or a major system overhaul, each lasting a random amount of time, brings the system back to an "as-good-as-new" state. During this activity, if the inventory decreases to zero, then the operation will resume as soon as this activity ends. No backlog is permitted, and any demand received during the period where the stock is zero is lost with a cost C_ℓ for each unit produced per unit of time. The replacement time has an average value T_r and follows a general distribution $f_r(t)$. The replacement cost is given by $C_R(t) = K_2 + C_r t$. The problem here is then to determine the optimal sequence $\{T^*_{p_i}\}, i = 1, 2, \ldots, N^*$, and the optimal number of production runs, N^*, that minimize the

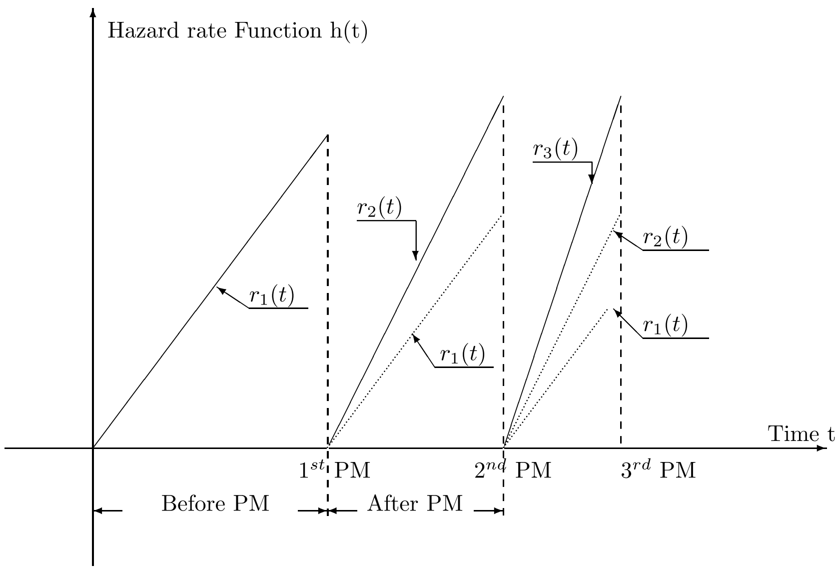

Fig. 1. Imperfect PM: The age is restored back to zero after each PM but the system has higher hazard rate.

long-term average cost when maintenance policy I is adopted. The following are the assumptions adopted in this paper.

(1) The demand and production rates are assumed to be constant.
(2) The setup time and repair time are negligible.
(3) The ith PM is assumed to be imperfect.
(4) $r_{i+1}(0) = r_i(0) = \cdots = r_1(0) = 0$.
(5) $r_{i+1}(t) \geqslant r_i(t)$ for all t and $i = 1, 2, \ldots, N$.

In the remainder of the paper, *the explicit dependence of the different costs on* $(\{T_{p_i}\}, N)$ *will not be highlighted, unless it is necessary to do so for clarity*. The expected total cost (E[TC]) comprises the expected setup cost (E[SC]), the expected inventory holding cost (E[HC]), the expected lost sales cost (E[LS]), the expected preventive maintenance cost (E[PM]), and the expected breakdown cost (E[BC]). The expressions of these various cost components will be derived in the next section.

3. Model formulation

Before starting the model formulation, the following proposition (El-Ferik, 2003), needed to simplify the different cost expressions, will be re-established.

Proposition 1. *Let $h(t)$ be an L_2-function and $f(t)$ be a continuous probability density function of a positive random variable, then*:

$$\int_a^x h(t)f(t)\,\mathrm{d}t + h(x)\overline{F}(x) = h(a)\overline{F}(a) + \int_a^x h'(t)\overline{F}(t)\,\mathrm{d}t.$$

Proof. The proof is reproduced here from (El-Ferik, 2003) in order to make the paper self-contained.

Using integration by part:

$$\begin{aligned}\int_a^x h(t)f(t)\,\mathrm{d}t + h(x)\overline{F}(x) &= [h(t)F(t)]_a^x - \int_a^x h'(t)F(t)\,\mathrm{d}t + h(x) - h(x)F(X)\\ &= -h(a)F(a) - \int_a^x h'(t)F(t)\,\mathrm{d}t + h(x) - h(a) + h(a)\\ &= h(a)(1 - F(a)) - \int_a^x h'(t)F(t)\,\mathrm{d}t + \int_a^x h'(t)\,\mathrm{d}t\\ &= h(a)\overline{F}(a) + \int_a^x h'(t)\overline{F}(t)\,\mathrm{d}t. \qquad \square\end{aligned}$$

In the remainder of this section, we present the derivation of the different terms used in our model.

Expected cycle length:
The expected cycle length, (E[CL]), is given by

$$\begin{aligned}E[\mathrm{CL}] = \frac{u}{d}\left\{\sum_{i=1}^N \int_0^{T_{p_i}} \tau f_i(\tau)\,\mathrm{d}\tau + T_{p_i}R_i(T_{p_i})\right\} + \int_0^{T_{p_N}}\left\{\int_{\frac{u-d}{d}\tau}^{\infty}\left(t - \frac{u-d}{d}\tau\right)f_{\mathrm{r}}(t)\,\mathrm{d}t\right\}f_N(\tau)\,\mathrm{d}\tau\\ + \left\{\int_{\frac{u-d}{d}T_{p_N}}^{\infty}\left(t - \frac{u-d}{d}T_{p_N}\right)f_{\mathrm{r}}(t)\,\mathrm{d}t\right\}R_N(T_{p_N}). \qquad (1)\end{aligned}$$

Using Proposition 1, (1) becomes:

$$E[\mathrm{CL}] = \frac{u}{d}\left\{\sum_{i=1}^N \int_0^{T_{p_i}} R_i(\tau)\,\mathrm{d}\tau\right\} + \overline{T}_{\mathrm{r}} - \frac{u-d}{d}\left\{\int_0^{T_{p_N}} R_N(\tau)\,\mathrm{d}\tau - \int_0^{T_{p_N}} F_{\mathrm{r}}\left(\frac{u-d}{d}\tau\right)R_N(\tau)\,\mathrm{d}\tau\right\}. \qquad (2)$$

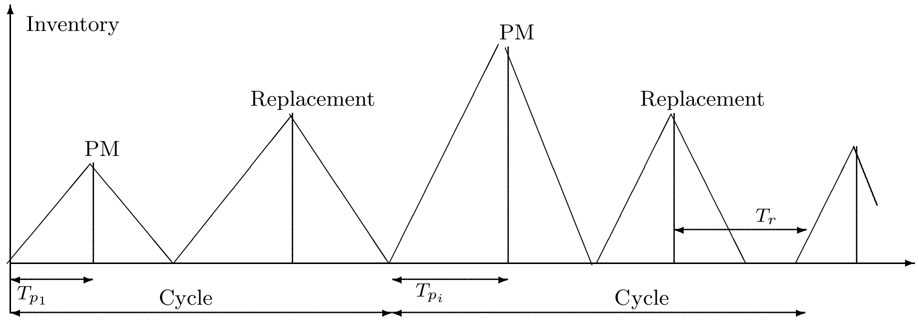

Fig. 2. Sample path realization of the production.

Expected inventory holding cost:

The expected holding cost, (E[HC]), is defined as the area below any sample path of the inventory as seen in Fig. 2.

This leads to:

$$E[\mathrm{HC}] = \frac{C_h(u-d)u}{2d}\sum_{i=1}^{N}\left\{\int_0^{T_{p_i}}\tau^2 f_i(\tau)d\tau + T_{p_i}^2 R_i(T_{p_i})\right\}. \tag{3}$$

Applying Proposition 1, Eq. (3) reduces to:

$$E[\mathrm{HC}] = \frac{C_h(u-d)u}{d}\sum_{i=1}^{N}\int_0^{T_{p_i}}\tau R_i(\tau)\,\mathrm{d}\tau. \tag{4}$$

Expected lost sales cost:

Since times for setups, repair, and preventive maintenance actions are assumed here to be negligible, lost sales can only occur during the replacement activity which is planned after the Nth production run, when the inventory is completely depleted. When the inventory reaches zero, the demand received by the facility triggers the process of lost sales. This can be expressed as:

$$\begin{aligned} E[\mathrm{LS}] = C_\ell d\Bigg\{&\int_0^{T_{p_N}}\left\{\int_{\frac{u-d}{\mathrm{d}}\tau}^{\infty}\left(t-\frac{u-d}{\mathrm{d}}\tau\right)f_\mathrm{r}(t)\,\mathrm{d}t\right\}f_N(\tau)\,\mathrm{d}\tau \\ &+\int_{\frac{u-d}{d}T_{p_N}}^{\infty}\left(t-\frac{u-d}{d}T_{p_N}\right)f_\mathrm{r}(t)\,\mathrm{d}t\Bigg\}R_N(T_{p_N}). \end{aligned} \tag{5}$$

Using Proposition 1, Eq. (5) can be rearranged as follows:

$$\begin{aligned} E[\mathrm{LS}] &= C_\ell d\left\{\overline{T}_\mathrm{r}-\frac{u-d}{d}\left\{\int_0^{T_{p_N}}\left\{\int_{\frac{u-d}{\mathrm{d}}\tau}^{\infty}\left(t-\frac{u-d}{\mathrm{d}}\tau\right)f_\mathrm{r}(t)\,\mathrm{d}t\right\}R_N(\tau)\mathrm{d}\tau\right\}\right\} \\ &= C_\ell d\left\{\overline{T}_\mathrm{r}-\frac{u-d}{d}\left\{\int_0^{T_{p_N}}R_N(\tau)\,\mathrm{d}\tau-\int_0^{T_{p_N}}F_\mathrm{r}\left(\frac{u-d}{d}\tau\right)R_N(\tau)\,\mathrm{d}\tau\right\}\right\}. \end{aligned} \tag{6}$$

Expected preventive maintenance cost:

The expected preventive maintenance cost, (E[PM]), is the total incurred cost due to PM activities. Thus,

$$E[\mathrm{PM}] = (N-1)C_\mathrm{pm}. \tag{7}$$

Expected failure cost:

The additional cost due to possible failures, (E[BC]), is given by:

$$E[\mathrm{BC}] = C_\mathrm{B}\sum_{i=0}^{N-1}F_i(T_{p_i}). \tag{8}$$

Expected replacement cost:

The expected replacement cost, (E[RC]), is a linear function of both the average replacement time, $\overline{T}_{\mathrm{r}}$, and the cost K_2. Thus,

$$E[\mathrm{RC}] = K_2 + C_{\mathrm{r}}E[T_{\mathrm{r}}] = K_2 + C_{\mathrm{r}}\overline{T}_{\mathrm{r}}. \tag{9}$$

Expected set up cost:

The expected set-up cost per production run, (E[SC]), is assumed to be constant and equal to K_1. Therefore, for N production runs, we have:

$$E[\mathrm{SC}] = NK_1. \tag{10}$$

The total expected cost per cycle is the sum of all the previously-listed costs:

$$E[\mathrm{TC}(\{T_{p_i}\}, N)] = E[\mathrm{HC}] + E[\mathrm{LS}] + E[\mathrm{PM}] + E[\mathrm{BC}] + E[\mathrm{SC}] + E[\mathrm{RC}]. \tag{11}$$

Using renewal process theory, we ca state that the long-term average cost is the ratio of the total cost per cycle to the expected cycle length:

$$E[\mathrm{AVC}(\{T_{p_i}\}, N)] = \frac{E[\mathrm{TC}(\{T_{p_i}\}, N)]}{E[\mathrm{CL}(\{T_{p_i}\}, N)]}. \tag{12}$$

In the following section, we develop some useful properties of the cost function that are needed to characterize the optimal policy $(\{T^*_{p_i}\}, N^*), i = 1, 2, \ldots, N^*$.

4. Model analysis and optimality conditions

To find the optimal sequence $\{T^*_{p_i}\}$ and the optimal N^*, the average cost $E[\mathrm{AVC}(\{T_{p_i}\}, N)]$, $i = 1, 2, \ldots, N$, must be minimized over an $(N+1)$-parameter space. Thus, the necessary optimality conditions are obtained as follows:

$$\frac{\partial E[\mathrm{AVC}(\{T^*_{p_i}\}, N^*)]}{\partial T_{p_k}} = \left.\frac{\frac{\partial E[\mathrm{TC}]}{\partial T_{p_k}} E[\mathrm{CL}] - E[\mathrm{TC}]\frac{\partial E[\mathrm{CL}]}{\partial T_{p_k}}}{E[\mathrm{CL}]^2}\right|_{T^*_{p_k}, N^*} = 0 \quad \text{for } k = 1, 2, \ldots, N^*. \tag{13}$$

will be derived in two steps:

(1) Compute the partial derivatives of $E[\mathrm{AVC}(\{T_{p_i}\}, N)]$, with respect to T_{p_k}, for a particular k, where $i = 1, 2, \ldots, k, \ldots, N$.
(2) Equate the partial derivatives to zero.

4.1. *Obtaining the partial derivatives of* $E[\mathrm{AVC}(T_{p_i} N)]$ *with respect to* T_{p_k}

Let

$$q(k, T_{p_k}) = \frac{\partial E[\mathrm{TC}]}{\partial T_{p_k}} E[\mathrm{CL}] - E[\mathrm{TC}]\frac{\partial E[\mathrm{CL}]}{\partial T_{p_k}} = \left\{\frac{\partial E[\mathrm{TC}]}{\partial T_{p_k}} - \frac{\partial E[\mathrm{CL}]}{\partial T_{p_k}} E[\mathrm{AVC}]\right\} E[\mathrm{CL}]. \tag{14}$$

First, for $k < N$, $\frac{\partial \mathrm{ETC}}{\partial T_{p_k}}$ can be written using Eqs. (6), (2) and (4), as:

$$\frac{\partial E[\mathrm{TC}]}{\partial T_{p_k}} = C_{\mathrm{B}} f_k(T_{p_k}) + \frac{C_h(u-d)u}{d} T_{p_k} R_k(T_{p_k}). \tag{15}$$

Partial derivatives of the cycle's length with respect to T_{p_k}, for $k = 1, 2, \ldots, N-1$, are given by:

$$\frac{\partial E[\mathrm{CL}]}{\partial T_{p_k}} = \frac{u}{d} R_k(T_{p_k}). \tag{16}$$

Moreover, when $k = N$,

$$\frac{\partial E[\mathrm{TC}]}{\partial T_{p_N}} = C_{\mathrm{B}} f_N(T_{p_N}) + C_h \frac{(u-d)u}{d} T_{p_N} R_N(T_{p_N}) - C_\ell d \frac{u-d}{d} \overline{F}_{\mathrm{r}}\left(\frac{u-d}{d} T_{p_N}\right) R_N(T_{p_N}), \tag{17}$$

which leads to:

$$\frac{\partial E[\mathrm{CL}]}{\partial T_{p_N}} = \frac{u}{d} R_N(T_{p_N}) - \frac{u-d}{d} \overline{F}_{\mathrm{r}}\left(\frac{u-d}{d} T_{p_N}\right) R_N(T_{p_N}) = \left\{1 + \frac{u-d}{d} F_{\mathrm{r}}\left(\frac{u-d}{d} T_{p_N}\right)\right\} R_N(T_{p_N}). \tag{18}$$

4.2. Equating the partial derivatives to zero

Based on these results, and using Eqs. (15) and (16), the necessary optimality conditions ($q(k, T_{p_k}) = 0$) can be expressed for $k < N$ and for $k = N$ as follows:

for $k < N$,

$$C_{\mathrm{B}} f_k(T_{p_k}) + \frac{C_h(u-d)u}{d} T_{p_k} R_k(T_{p_k}) - \frac{u}{d} R_k(T_{p_k}) E[\mathrm{AVC}] = 0. \tag{19}$$

Re-arranging (19) yields:

$$C_{\mathrm{B}} r_k(T_{p_k}) + \frac{C_h(u-d)u}{d} T_{p_k} - \frac{u}{d} E[\mathrm{AVC}] = 0, \quad \text{for } k < N. \tag{20}$$

Similarly, for $k = N$,

$$\begin{aligned} &C_{\mathrm{B}} f_N(T_{p_N}) + C_h \frac{(u-d)u}{d} T_{p_N} R_N(T_{p_N}) - \left\{ C_\ell d \left(\frac{u-d}{d}\right) \overline{F}_{\mathrm{r}}\left(\frac{u-d}{d} T_{p_N}\right) \right. \\ &\quad \left. - \left\{1 + \frac{u-d}{d} F_{\mathrm{r}}\left(\frac{u-d}{d} T_{p_N}\right)\right\} E[\mathrm{AVC}] \right\} R_N(T_{p_N}) = 0. \end{aligned} \tag{21}$$

Therefore,

$$C_{\mathrm{B}} r_N(T_{p_N}) + \frac{(u-d)}{d}\left\{C_h u T_{p_N} - C_\ell d\right\} - E[\mathrm{AVC}] + \{C_\ell d - E[\mathrm{AVC}]\}\left(\frac{u-d}{d}\right) F_{\mathrm{r}}\left(\frac{u-d}{d} T_{p_N}\right) = 0. \tag{22}$$

It is important to study the characteristics and properties of the optimal solution based on the above-derived necessary optimality conditions. Indeed, the following lemma establishes the existence of possible solutions to (22).

Lemma 1. *There is, at least, one local minimum, $T^*_{p_N}$, of $E[AVC(\{T_{p_i}\}, N)]$ that can be a solution of equation (22).*

Proof. the proof is based on the asymptotic behavior of LHS term of (22). Indeed, if $T_{p_N} \to 0$, then $r_N(0) = 0$ and $F_{\mathrm{r}}(0) = 0$. Thus (22) reduces to a negative quantity as shown below:

$$-\left[\frac{(u-d)}{d}\{C_\ell d\} + E[\mathrm{AVC}]\right] < 0.$$

However, if T_{p_N} keeps increasing to infinity,i.e. $T_{p_N} \to \infty$

$$\lim_{T_{p_N} \to \infty} \left\{ C_{\mathrm{B}} r_N(T_{p_N}) + \frac{(u-d)}{d}\left\{C_h u T_{p_N} - C_\ell d\right\} - E[\mathrm{AVC}] + \{C_\ell d - E[\mathrm{AVC}]\}\left(\frac{u-d}{d}\right) F_{\mathrm{r}}\left(\frac{u-d}{d} T_{p_N}\right) \right\} > 0, \tag{23}$$

which completes the proof. □

Theorem 1. *For a given N, if the first local minimum of $E[AVC(\{T_{p_i}\}_{i=1,\ldots,N-1}, T_{p_N}, N)]$, expressed as a function of T_{p_N}, occurs at $T^*_{p_N}$, then:*

(1) *if* $E[AVC(\{T_{p_i}\}_{i=1,\dots,N-1}, T^*_{p_N}, N)] \geqslant C_\ell d$, *then*
$E[AVC(\{T_{p_i}\}_{i=1,\dots,N-1}, T_{p_N}, N)] \geqslant C_\ell d$ *for all* T_{p_N},

(2) *if* $E[AVC(\{T_{p_i}\}_{i=1,\dots,N-1}, T^*_{p_N}, N)] < C_\ell d$, *then* $T^*_{p_N}$ *is a global minimum.*

Proof. The proof of this theorem follows similar steps as in (Lee and Srinivason, 2001). Indeed, let us define the LHS term of Eq. (22) as $\phi(T_{p_N}, N)$.

$$\phi(T_{p_N}, N) = C_{\mathrm{B}} r_N(T_{p_N}) + \frac{(u-d)}{d}\{C_h u T_{p_N} - C_\ell d\} - E[\mathrm{AVC}] + \{C_\ell d - E[\mathrm{AVC}]\}\frac{u-d}{d}F_{\mathrm{r}}\left(\frac{u-d}{d}T_{p_N}\right). \tag{24}$$

In order to prove parts (1) and (2) of this theorem, we shall adopt a proof by contradiction. First let

$$E[\mathrm{AVC}(\{T_{p_i}\}_{i=1,\dots,N-1}, T^*_{p_N}, N)] \geqslant C_\ell d \quad \text{with} \quad \phi(T^*_{p_N}, N) = 0.$$

Assume the existence of another local minimum, $T_{p_{N2}}$, at which

$$E[\mathrm{AVC}(\{T_{p_i}\}_{i=1,\dots,N-1}, T_{p_N}2, N)] < C_\ell d.$$

Consequently, there exists a point $T_{p_{N1}}$, such that $T^*_{p_N} < T_{p_{N1}} < T_{p_{N2}}$ and $E[\mathrm{AVC}((\{T_{p_i}\}_{i=1,\dots,N-1}, T_{p_N}1, N), N)] \leqslant C_\ell d$ with $\phi(T^*_{p_N}, N) < 0$. There are three terms in the expression of $\phi(T_{p_N}, N)$. The first two terms are obviously increasing as a function of T_{p_N}. For the third term, we have:

$$-E[\mathrm{AVC}(\{T_{p_i}\}_{i=1,\dots,N-1}, T_{p_N}1, N)] > -E[\mathrm{AVC}(\{T_{p_i}\}_{i=1,\dots,N-1}, T^*_{p_N}, N)].$$

Since $F_{\mathrm{r}}(T_{p_N})$ is an increasing function of T_{p_N}, then $\phi(T_{p_{N1}}, N) > \phi(T^*_{p_N}, N)$. Therefore, $E[\mathrm{AVC}(\{T_{p_i}\}_{i=1,\dots,N-1}, T_{p_N}1\}, N)] \leqslant C_\ell d$ and $\phi(T_{p_{N1}}, N) > 0$ which leads to a contradiction. Therefore, we conclude that if

$$E[\mathrm{AVC}(\{T_{p_i}\}_{i=1,\dots,N-1}, T^*_{p_N}\}, N)] \geqslant C_\ell d,$$

then $E[\mathrm{AVC}(\{T_{p_i}\}_{i=1,\dots,N-1}, T_{p_N}, N)] \geqslant C_\ell d$ for all T_{p_N}.

To prove part (2) of this theorem, assume that

$$E\Big[\mathrm{AVC}(\{T_{p_i}\}_{i=1,\dots,N-1}, T^*_{p_N}, N)\Big] < C_\ell d$$

and suppose that there exists another local minimum, $T_{p_{N2}}$, at which

$$E\Big[\mathrm{AVC}(\{T_{p_i}\}_{i=1,\dots,N-1}, T_{p_N}2, N)\Big] < E\Big[\mathrm{AVC}(\{T_{p_i}\}_{i=1,\dots,N-1}, T^*_{p_N}, N)\Big].$$

Thus, there exists a point $T_{p_{N1}}$, such that $T^*_{p_N} < T_{p_{N1}} < T_{p_{N2}}$ with

$$E[\mathrm{AVC}(\{T_{p_i}\}_{i=1,\dots,N-1}, T_{p_N}1, N)] \leqslant E[\mathrm{AVC}(\{T_{p_i}\}_{i=1,\dots,N-1}, T^*_{p_N}, N)]$$

and $\phi(T_{p_{N1}}, N) < 0$. Considering that $T^*_{p_N} < T_{p_{N1}}$ and $\phi(T^*_{p_N}, N) = 0$, and following the same approach as in proof of part (1), we can show that $\phi(T_{p_{N1}}, N) > 0$, which again leads to a contradiction. Hence, if $E[\mathrm{AVC}(\{T_{p_i}\}_{i=1,\dots,N-1}, T^*_{p_N}, N)] < C_\ell d$, then $T^*_{p_N}$ is a global minimum. □

Furthermore,

Theorem 2. *For a fixed finite* T_{p_N}, *the solution,* T_{p_k}, $k = 1, 2, \dots, N-1$, *of Eq.* (20) *exists and is unique. Furthermore, for* $u > d$, *the sequence of PM intervals is a decreasing one, i.e.* $T_{p_1} > T_{p_2} > \cdots > T_{p_{N-1}}$.

Proof. Using (20) and (22),

$$
\begin{aligned}
&C_{\mathrm{B}} r_N(T_{p_N}) + \frac{(u-d)}{d}\left\{C_h u T_{p_N} - C_\ell d\right\} + C_\ell d \frac{u-d}{d} F_{\mathrm{r}}\left(\frac{u-d}{d} T_{p_N}\right) \\
&\quad = \left(1 + \frac{u-d}{d} F_{\mathrm{r}}\left(\frac{u-d}{d} T_{p_N}\right)\right)\left(C_{\mathrm{B}} r_k(T_{p_k}) + \frac{C_h (u-d) u}{d} T_{p_k}\right)\frac{d}{u}.
\end{aligned} \tag{25}
$$

Thus,

$$
C_{\mathrm{B}} r_k(T_{p_k}) + \frac{C_h (u-d) u}{d} T_{p_k} = \frac{C_{\mathrm{B}} r_N(T_{p_N}) + \frac{(u-d)}{d}\left\{C_h u T_{p_N} - C_\ell d\right\} + C_\ell d \frac{u-d}{d} F_{\mathrm{r}}(\frac{u-d}{d} T_{p_N})}{\left(1 + \frac{u-d}{d} F_{\mathrm{r}}(\frac{u-d}{d} T_{p_N})\right)\frac{d}{u}}. \tag{26}
$$

At $T_{p_k} = 0$, the LHS term of (26) is less than the RHS term. In addition, when $T_{p_k} \to \infty$, the LHS term tends to ∞. Furthermore, the derivative of the LHS term with respect to T_{p_k} is a strictly increasing function. Since, the RHS term of (26) is constant for a fixed T_{p_N}, $r_i(t) > r_{i-1}(t)$, for all t and i, and since $u > d$, the LHS of (26) will also increase as a function of k. Consequently, the sequence $\{T_{p_k}\}$, for $k = 1, 2, \ldots, N-1$, will be a decreasing one. □

The proof of the unimodality of the cost function, with respect to N, is not straightforward. Thus, the proof that the average cost function is convex on N is difficult to establish even for simple cases. Alternatively, the following algorithm can be used to determine the optimal solutions for the joint EPQ-maintenance problem.

Algorithm

Initialization Set $N = 1$ and let $T^0_{p_N}$ be an arbitrary initial value of T_{p_N}. Let ϵ be a termination criterion.

Step 1. Find the optimal T_{p_i}s, $i = 1, 2, \ldots, N-1$; verifying Eq. (22).

Step 2. Using T_{p_i}s, $i = 1, 2, \ldots, N-1$; obtained in Step 1, find a new value of T_{p_N}, T'_{p_N}, from Eq. (21).

Step 3. If $|T'_{p_N} - T_{p_N}| < \epsilon$, go to Step 4; otherwise return to Step 2 with $T'_{p_N} = T_{p_N}$.

Step 4. Find $E[\mathrm{AVC}(\{T_{p_i}\}, N)]$ for the current solution.

Step 5. If $E[\mathrm{AVC}(\{T_{p_i}\}, N)] > E[\mathrm{AVC}(\{T_{p_i}\}, N-1)]$, go to Step 6; otherwise return to Step 1 with $T_{p_N} = T'_{p_N}$ and $N = N + 1$.

Step 6. The optimal solution is $N^* = N - 1$ and the optimal T_{p_i}s, $i = 1, 2, \ldots, N$, corresponding to N^*.

5. Simulation results

In this example, the random failures follow a Weibull distribution with the following rate:

$$
r_k(t) = A_k \frac{t^\beta}{\theta}
$$

Table 1
PM schedule and long-term average cost as a function of the number of PM activities with $a_k = \frac{3k+1}{k+1}$

Production periods	$N = 1$	$N^* = 2$	$N = 3$	$N = 4$	$N = 5$	$N = 6$	$N = 7$
T_{p_1}	2.920	**2.664**	2.727	2.898	3.125	3.385	3.666
T_{p_2}		**2.531**	2.555	2.714	2.927	3.170	3.435
T_{p_3}			2.225	2.328	2.510	2.719	2.945
T_{p_4}				1.710	1.823	2.000	2.134
T_{p_5}					1.046	1.128	1.222
T_{p_6}						0.507	0.566
T_{p_7}							0.210
$E[\mathrm{AVC}(\{T_{p_i}\}, N)]$	45.923	**42.128**	43.129	45.822	49.405	53.513	57.951
$E[\mathrm{CL}(\{T_{p_i}\}, N)]$	3.826	**6.101**	7.633	8.638	9.279	9.680	9.926

with $\beta = 2$, $\theta = 4$, and $A_k = a_k A_{k-1}$, where $1 = a_0 < a_1 < a_2 \cdots < a_k$ and $A_0 = a_0 = 1$. The minimum mean-time-to-failure is defined for, $k = 0$, as $\mathrm{MTTF} = \theta^{1/\beta}\Gamma(1 + \frac{1}{\beta})$. Let $u = 25$; $d = 10$; $C_h = 1$; $C_{pm} = C_B = 2$; $Cr = 2$; $C_\ell = 5$; $K_1 = 50$; $K_2 = 50$; $\overline{T}_r = \mathrm{MTTF}$ using Eq. (26),

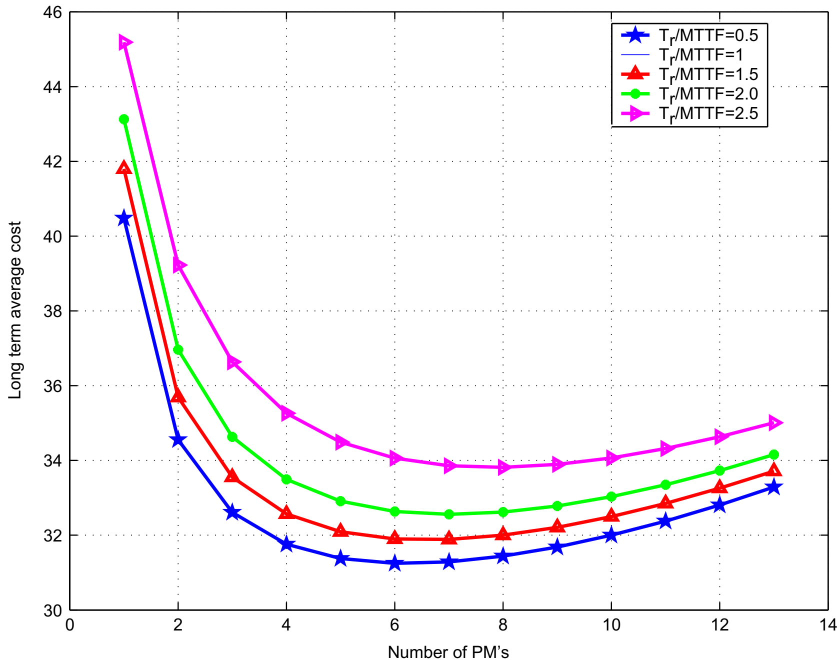

Fig. 3. Long-term average cost as a function of the number of PM activities for different $\overline{T}_r/\mathrm{MTTF}$.

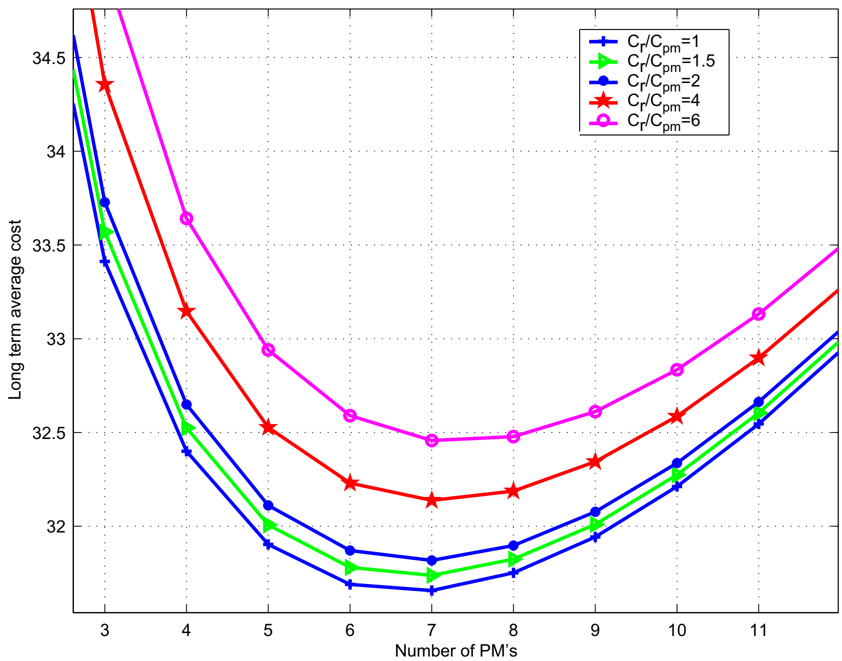

Fig. 4. Long-term average cost as a function of the number of PM activities for different $\frac{C_r}{C_{pm}}$.

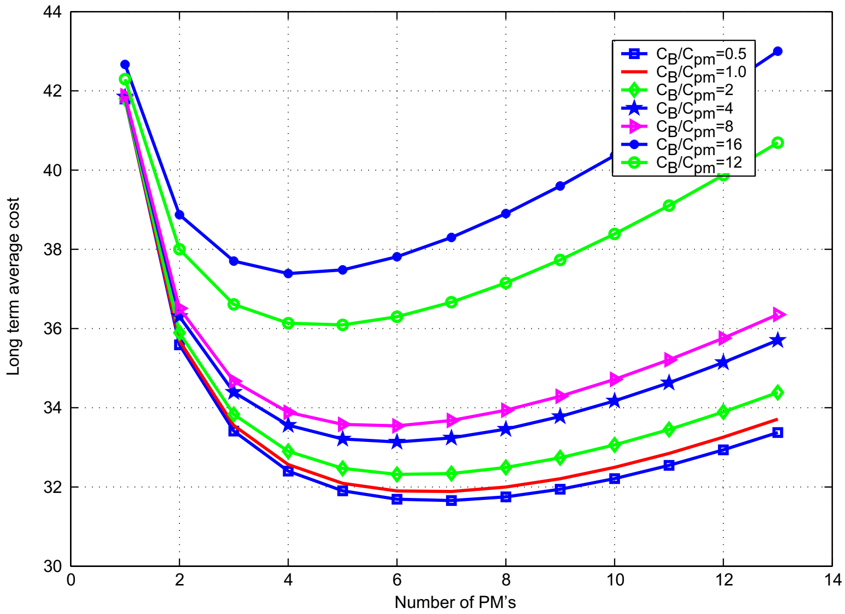

Fig. 5. Long-term average cost as a function of the number of PM activities for different $\frac{C_B}{C_{pm}}$.

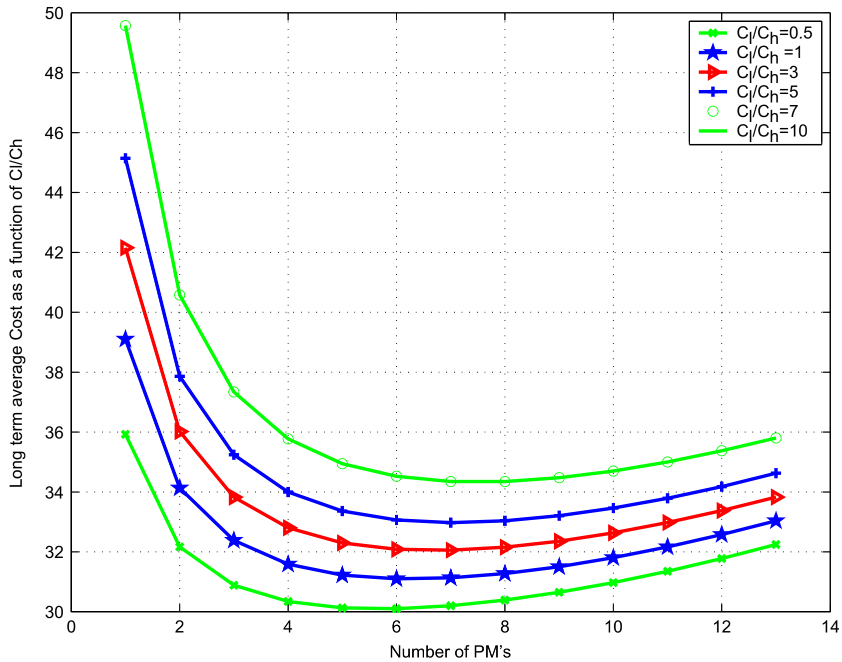

Fig. 6. Long-term average cost as a function of the number of PM activities for different $\frac{C_\ell}{C_h}$.

$$
\begin{aligned}
&C_B r_k(T_{p_k}) + \frac{C_h(u-d)u}{d} T_{p_k} = D(T_{p_N}),\\
&A_k \frac{1}{2} T_{p_k} + \frac{C_h(u-d)u}{d} T_{p_k} = D(T_{p_N}),\\
&T_{p_k} = \frac{D(T_{p_N})}{\frac{A_k}{2} + \frac{C_h(u-d)u}{d}}.
\end{aligned}
\tag{27}
$$

Using this expression of T_{p_k} (27) in Eq. (21) will lead to the determination of T_{p_N}. Table 1 summarizes the results of different simulation runs using the algorithm presented at the end of the previous section.

The numerical results in Table 1 indicate that the production periods are decreasing as the number of production runs increases. Since the production rate is constant, the economic production quantity, uT_{p_i}, at each run is also decreasing. For this numerical example, $N^* = 2$ is the optimal number of production runs. Furthermore, Figs. 3–6 demonstrate the sensitivity of the long-term average cost to different parameters of the model. It is worth noting here that the average cost is unimodal as a function of N. However, the analytical proof of this fact is not straightforward, even the simple cases, because of the variability of T_{p_k} in terms of T_{p_N} and N. Figs. 3 and 6 indicate that the higher T_r/MTTF or c_ℓ/C_h is, respectively, the higher the cost and N^* become. However, in terms of varying the parameter C_B/C_{pm}, Fig. 5 indicates that the higher the ratio is, the higher the cost and the smaller N^* become. From Figs. 3 and 5, we can deduce that it is preferable to delay the replacement and repair the system if the lost sales' cost is higher than the holding cost. However, if the cost of breakdown is much higher than the PM cost, it is preferable to replace the system early so as to reduce the risk of failure.

6. Conclusion

In this paper, the optimal control problem of a joint production-maintenance model has been addressed. The facility is assumed to deteriorate while in operation with an increasing failure rate. PM activities are initiated either in the case of a failure or when the facility reaches a certain age, whichever occurs first. After each PM, the age of the system is restored back to zero with a higher failure rate. At the end of the Nth production run, the system is replaced within a random period of time. An analysis of the cost function has also been carried out, and useful properties derived. our numerical simulation work illustrated the sensitivity of the long-term average cost to the ratios of key cost factors. Such sensitivity was also observed in the ratio of the mean-time-to-failure to the average time of replacement. Moreover, the optimal sequence of production runs was shown to be a decreasing one, which indicated that the production quantity would get smaller as the system deteriorates. This fact also suggests that using variable production rates could be a plausible solution to the problem at hand if a constant production quantity is to be secured at each production run.

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