

## **FACETS AND ALGORITHMS FOR CAPACITATED LOT SIZING**

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The dynamic economic lot sizing model, which lies at the core of numerous production planning applications, is one of the most highly studied models in all of operations research. And yet, capacitated multi-item versions of this problem remain computationally elusive. We study the polyhedral structure of an integer programming formulation of a single-item capacitated version of this problem, and use these results to develop solution methods for multi-item applications. In particular, we introduce a set of valid inequalities for the problem and show that they define facets of the underlying integer programming polyhedron. Computational results on several single and multiple product examples show that these inequalities can be used quite effectively to develop an efficient cutting plane/branch and bound procedure. Moreover, our results show that in many instances adding certain of these inequalities a priori to the problem formulation, and avoiding the generation of cutting planes, can be equally effective.

### **1. Introduction and motivation**

Allocating shared resources and determining lot sizes and schedules for multiple products is a central issue in operations management. Because of its practical importance, this problem has been the focus of extensive study and continues to receive considerable attention. Despite the rich variety of problem settings encountered in practice, a few prototypical models—and particularly the capacitated lot size problem—have become central in the production planning literature. The lot size model describes production operations for multiple products that incur a fixed cost (and/or time) whenever a facility is set up to manufacture any particular product. The capacitated version of the problem has numerous applications. For example, Gorenstein (1970) has used a variant of this model to support long range production decisions in a tire company, and Lasdon and Terjung (1971) have reported the use of a capacitated lot size model by a large manufacturing company. This paper studies the polyhedral structure of an integer programming formulation of the single item lot size problem and uses these results to develop efficient solution methods for multi-item applications.

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Two observations motivate this research. First, even though multiple product lot size problems have been the subject of extensive study, they can be solved optimally (or near optimally with performance guarantees) only for special cases. For example, although current methods can obtain good solutions to the capacitated lot size problem when the number of items is large compared to the number of time periods, no evidence in the literature suggests that these methods perform well for applications with a small or medium number of products. Second, as indicated in Section 2, increasing empirical evidence indicates that both pure and mixed integer programming problems can be solved to optimality in reasonable computation times by methods that use results about the underlying polyhedral structure of these problems.

One of the advantages of studying the polyhedral structure of the lot sizing model is that even if it occurs as a subproblem, its analysis may improve modeling and algorithm design for the larger problem that contains it. Several researchers have reported very effective computational results when using the analysis of substructures to improve problem formulations. For example, Crowder, Johnson and Padberg (1983) showed that even information from single constraints of a zero-one program (minimal cover cuts) can be effective in solving large scale models.

We organize the paper as follows. After briefly discussing previous contributions from the literature on both the lot size problem and the polyhedral structure of combinatorial problems, in Section 3, we describe the lot size problem and present a mixed integer programming formulation of it. In Section 4, we identify a class of valid inequalities for this model and derive the necessary and sufficient conditions for these inequalities to be facets. In Section 5, we show how to solve the separation problem efficiently for this class of inequalities and in Section 6, we present our computational results.

## **2. Previous contributions**

### *2.1. Lot size problem*

The papers of Manne (1958) and Wagner and Whitin (1958) are two of the seminal contributions in the area of lot sizing. Wagner and Whitin studied the uncapacitated problem and showed how to solve it efficiently via dynamic programming. Manne formulated the multiple item capacitated version of the problem as a mixed integer linear program and proposed solving a linear programming approximation of it. Subsequently, considerable research effort has refined and generalized the models and the solution approaches suggested by each of these two works. The book by Hax and Candea (1984) reviews a number of the mathematical programming models in the area.

Zangwill (1966), who extended the results of Wagner and Whitin to allow back-ordering of demand, also introduced a network representation of the problem (1969). Zabel (1964) and Eppen, Gould and Pashigian (1969) improved the planning horizon

results of Wagner and Whitin. Zangwill (1969) and Kalyon (1970) extended the results to include multiple facilities. Veinott (1969) used the properties of extreme points of Leontief systems to characterize the structure of solutions to single product concave cost problems and also proposed a method for an arborescent multiechelon structure. All these papers studied the uncapacitated version of the corresponding problems.

Adding capacity restrictions makes the problem much more difficult. In their extensive study of the complexity of the single item capacitated lot size problem, Bitran and Yanasse (1982) showed that most versions of the problem (i.e., with different assumptions regarding the variability of demand, costs and capacities) are NP-complete. Further, though the single item problem with constant capacities can be solved efficiently (Florian and Klein, 1971), the multiple item problem with constant capacities is NP-complete (Bitran and Yanasse, 1982).

Research on procedures to solve the single item capacitated problem has focused on methods to solve special cases (see, for example, Florian and Klein, 1971; Jagannathan and Rao, 1973; Love, 1973), or on approximation methods (see, for example, Bitran and Matsuo, 1986a). Baker et al. (1978) and Lambrecht and Vander Eecken (1978) provide exponential algorithms to solve the variable capacity version of the problem exactly.

The multiple item capacitated lot size problem can be formulated as a mixed integer linear program and several researchers have examined methods to solve this integer program. Manne (1958) formulated the problem to allow additional capacity to be bought at a cost and suggested solving a linear programming approximation of the problem. Dzielinski, Baker and Manne (1963), Dzielinski and Gomory (1965) and Lasdon and Terjung (1971) subsequently refined his approach. Kleindorfer and Newson (1975) showed that solving this linear programming relaxation is equivalent to solving the dual of the original problem. Kortanek, Sodaro and Soyster (1968), Gorenstein (1970) and Bahl (1983) also proposes similar approaches. Graves (1982) suggested using Lagrangian relaxation to solve a hierarchical production planning problem that includes the lot size model as a substructure. Researchers have also examined exact solution methods to solve the integer program. Murty (1968) and Gray (1971) proposed extreme point ranking procedures and Jones and Soland (1962) and Steinberg (1970) developed branch and bound procedures. As already mentioned, Manne's approximate formulation works well when the number of items is large compared to the number of time periods (see Manne, 1958; and Bitran and Matsuo, 1986b). In other situations, the gap between the linear programming approximation and the original integer program could be substantial and no evidence suggests that the existing methods would be able to obtain good solutions to the problem.

It is possible to reformulate the lot size model as a plant location problem and apply algorithms developed for the plant location problem to the lot size problem. Nemhauser and Wolsey (1988) discuss the relation between the two problems and provide references to the extensive plant location literature.

In applied contexts, practitioners continue to use heuristics even for problems that can be solved efficiently using optimization algorithms. Several researchers have proposed heuristics methods (Balinski, 1961; Cooper and Drebes, 1967; Denzler, 1969; Newson, 1975; Walker, 1976; Lambrecht and Vanderveken, 1979; Dixon and Siver, 1981; and Dogramaci, Panayiotopoulos and Adam, 1981). Maes and Van Wassenhove (1986a, 1986b) present computational results comparing the performance of some of these methods. Even simpler heuristics for the uncapacitated lot size problem are used in practice (see Orlicky, 1974). Axsater (1982, 1985), Bitran, Magnanti and Yanasse (1984) and Vachani (1984) have analyzed the worst case performance for these heuristics.

Recently, researchers have investigated the polyhedral structure of the mixed integer programming formulation of the problem. Barany, Van Roy and Wolsey (1984a, 1984b) characterized the convex hull of the solutions for the uncapacitated lot size problem and reported good computational results for the multiple item capacitated problem using facets of the single item problem. Eppen and Martin (1987) have proposed a reformulation of the single item uncapacitated problem as a shortest path problem that is equivalent to including a description of the convex hull of the problem. Pochet and Wolsey (1988) have described valid inequalities for the uncapacitated problem with backlogging.

## *2.2. Polyhedral structure of integer programming problems*

A considerable body of literature in the last fifteen years has studied the polyhedral structure of combinatorial optimization problems. More recently, several computational studies have used this theory for actual problem solving and have reported very impressive results in obtaining exact solutions to large integer programs. In this section, we briefly review several major contributions in the area, focusing on the computational work. For additional references and underlying concepts and notation, we refer the reader to the bibliography compiled by Grötschel (1985) and to the survey written by Hoffman and Padberg (1985).

The work of Crowder and Padberg (1980) and Padberg and Hong (1980) for the symmetric traveling salesman problem are landmark studies on the use of results from polyhedral theory for actual problem solving. These researchers were able to solve large traveling salesman problems to optimality using a combination of cutting planes, fixing variables using reduced cost information, and branch and bound strategies. More recently, Padberg and Grimaldi (1987) reported solving a larger (532-city) traveling salesman problem. The cutting planes used in these studies were derived from the facets of the traveling salesman polytope described by Grötschel and Padberg (1979a, 1979b).

Crowder, Johnson and Padberg (1983) also reported equally impressive results for a number of real world, large scale, zero one programming problems, again using a combination of preprocessing, facet-defining cutting planes, and branch and bound. More recently, Grötschel, Junger and Reinelt (1984, 1985) have obtained

very good results for the linear ordering problem, using facets of the problem in a cutting plane procedure. They state that the range of matrices considered in their study is representative of almost all input-output matrices that have been compiled to date in Europe. Barany, Von Roy and Wolsey (1984b) and Eppen and Martin (1985) also report considerable success in solving multiple item capacitated lot size problems using facets of the single item uncapacitated problem. Johnson, Kostreva and Suhl (1985) successfully used a strong cutting plane algorithm for solving a strategic planning problem.

The problems solved in all these studies are drawn from the class of NP-complete problems. A recent study also suggests that these methods might be successful in solving problems for which “good” algorithms are already available. Grötschel and Holland (1984) implemented a strong cutting plane algorithm for the problem of finding a maximum matching in a graph. This problem can be solved in polynomial time (Edmonds, 1965). The Grötschel and Holland algorithm uses a characterization of the matching polytope developed by Edmonds (1965) and a procedure of Padberg and Rao (1982) for choosing cuts to be added (i.e., solving the so called separation problem). Surprisingly, Grötschel and Holland’s algorithm solved large scale problems as well as or better (in terms of running time) than Edmonds’ polynomial time algorithm.

Though we have focused on problems for which computational studies have been reported, researchers have also investigated the facial structure of several other classes of integer programming problems, e.g., mixed zero-one problems (Padberg, Van Roy and Wolsey, 1985), the set packing problem (Padberg, 1973, 1979; Nemhauser and Trotter, 1974; Chvátal, 1975; Balas and Padberg, 1976; Trotter, 1976; Balas and Zemel, 1977), the uncapacitated plant location problem (Guignard, 1980; Cornuejols and Thizy, 1982; Cho et al., 1983a, 1983b), the capacitated plant location problem (Leung and Magnanti, 1989), the machine scheduling problem (Balas, 1984), the network design problem (Balakrishnan, 1987; Balakrishnan and Magnanti, 1985). All the results for these problems depend on the special structure of the problem class.

Chvátal (1973), Nemhauser and Wolsey (1985) and Martin and Schrage (1986) describe alternative approaches for generating valid inequalities and facets for integer programming and mixed integer programming problems. Their methods rely on constraint aggregation and coefficient reduction.

### 3. Problem description

We focus on the structure of the single item, single resource, constant capacity lot size problem. Since this problem can be solved in  $O(T^4)$  time (where  $T$  is the number of time periods in the planning horizon) via a dynamic programming algorithm (see Florian and Klein, 1971), our goal in identifying its facets is not to develop a competitive cutting plane procedure for this problem itself, but to obtain

“strong” valid inequalities for larger problems that contain it as a subproblem. In Section 7, we report on computational experience in using the inequalities in this way.

Our objective in studying the single item lot size problem is also to develop useful insights in modeling problems with similar structure. For example, the demand constraints in the lot size problem (or other problems that require demand in every period to be met without backordering) are usually formulated as

$$\sum_{i=1}^t x_i \geq \sum_{i=1}^t d_i \quad \forall t. \quad (3.1)$$

In this expression,  $x_i$  is the production and  $d_i$  is the demand in period  $i$ . Let  $y_i \in \{0, 1\}$  denote the state of the machine in period  $i$ , with  $y_i = 1$  if the machine is setup to produce the product and  $y_i = 0$  otherwise. Constraints (3.1) can be replaced by

$$\sum_{i=1}^t x_i \geq \sum_{i=1}^t d_i + d_{t+1}(1 - y_{t+1}). \quad (3.2)$$

This constraint is valid, since if  $y_{t+1} = 0$  then we require  $\sum_{i=1}^t x_i \geq \sum_{i=1}^{t+1} d_i$ , which is more stringent than (3.1), to meet demand in period  $t+1$ , whereas if  $y_{t+1} = 1$  then (3.2) and (3.1) are identical. Therefore, (3.2) is stronger than (3.1). Indeed, the demand constraints are not facets of the capacitated lot size problem (unless  $d_{t+1} = 0$ ), but constraints (3.2) are. (This result is true of the uncapacitated lot size problem as well—see Barany, Van Roy and Wolsey, 1984a). Thus, if we use any linear-programming-based method to solve a problem that includes constraints (3.1) and has corresponding setup variables, then replacing (3.1) by (3.2) should result in a tighter formulation.

The single item capacitated lot size problem, CLSP, that we study in this paper can be formulated as follows.

$$\text{CLSP} \quad \text{Minimize} \quad \sum_{t=1}^T p_t x_t + \sum_{t=1}^T s_t y_t \quad (3.3)$$

$$\text{subject to} \quad \sum_{i=1}^t x_i \geq \sum_{i=1}^t d_i \quad \forall t, \quad (3.4)$$

$$\sum_{i=1}^T x_i = \sum_{i=1}^T d_i, \quad (3.5)$$

$$x_t \leq \min\{d_{tT}, C\} y_t \quad \forall t, \quad (3.6)$$

$$x_t \geq 0, \quad 0 \leq y_t \leq 1, \quad \forall t, \quad (3.7)$$

$$y_t \text{ integer} \quad \forall t. \quad (3.8)$$

In this formulation,  $T$  is the finite horizon over which production is to be planned,  $x_t$  is the production in period  $t$ ,  $d_t$  is the demand in period  $t$  (that must be met without backordering) and  $y_t = 1$  if there is a setup in period  $t$  ( $y_t = 0$  otherwise).  $d_{tT}$  denotes the total demand in periods  $t$  through  $T$ , i.e.,  $d_{tT} = \sum_{i=t}^T d_i$ . Constraint (3.6) requires that  $y_t = 1$  if  $x_t > 0$ , and also ensures that the production in any period

does not exceed the capacity  $C$  or the demand remaining until the end of the horizon. We can assume without loss of generality that  $d_t \leq C$  for all  $t$ . (If  $d_j > C$  for some  $j$ , then we obtain an equivalent problem, with demands  $d'_t$ , by setting  $d'_t = d_t$  for  $t \neq j-1$  or  $j$ ,  $d'_{j-1} = d_{j-1} + (d_j - C)$  and  $d'_j = C$ . If  $d'_{j-1}$ , the resulting demand in period  $j-1$  exceeds the capacity  $C$ , we can repeat this procedure. Note that if  $d_1 > C$  then the problem is infeasible). We can also assume that  $d_T > 0$  since otherwise we could eliminate period  $T$  from the problem. We further assume that  $d_1 + d_2 \leq C$ , since if  $d_1 + d_2 > C$  (or  $\sum_{i=1}^t d_i > C(t-1)$  for some  $t > 1$ ) then all feasible solutions to CLSP have  $y_1 = y_2 = 1$  (or  $y_i = 1$  for  $i \leq t$ ), and we could include these conditions as constraints of CLSP. Instead of considering these special cases, we assume  $d_1 + d_2 \leq C$ . The objective function coefficient  $p_t$  includes both the inventory holding cost and the production cost per unit in period  $t$  ( $I_t$ , the ending inventory in period  $t$ , equals  $\sum_{i=1}^t x_i - \sum_{i=1}^t d_i$  and, hence, the holding cost  $h_t I_t$  can be written as a function of the  $x_i$  variables for  $1 \leq i \leq t$ ) and the coefficient  $s_t$  is the fixed cost of a setup in period  $t$ .

In the next section, we investigate the polyhedral structure of the convex hull of the feasible solutions to problem CLSP.

#### 4. Facets of the single item capacitated lot size problem

Let  $F_{LS}$  denote the set of feasible solutions to CLSP and let  $C_{LS}$  denote the convex hull of  $F_{LS}$ , i.e.,

$$C_{LS} = \text{conv}\{(x, y) \mid (x, y) \text{ satisfies constraints (3.4)–(3.8)}\}.$$

Let  $d_{hk}$  denote  $\sum_{t=h}^k d_t$  for all  $1 \leq h \leq k \leq T$ . Now consider  $u$  and  $v$  with  $1 \leq u \leq v \leq T$  and let  $r = d_{uv} \pmod{C}$ . (If  $d_{uv}$  is a multiple of  $C$ , we use the notation  $d_{uv} \pmod{C} = C$  rather than  $d_{uv} \pmod{C} = 0$ .) Further, let  $w$  be the first period between  $u$  and  $v$  that satisfies  $\lceil d_{uw}/C \rceil = \lceil d_{uv}/C \rceil$ .

The following inequality defines a facet of  $C_{LS}$ :

$$I_{u-1} + \sum_{t \in Q} x_t + \sum_{t \in R} r y_t + \sum_{t \in D} d_{tv} y_t \geq r \lceil d_{uv}/C \rceil \quad (f)$$

with

$$P = \{u, u+1, \dots, v\}, \quad Q \subseteq P,$$

$$R \cup D = P \setminus Q, \quad R = \{t \mid t \in P \setminus Q, t \leq w\}, \quad D = \{t \mid t \in P \setminus Q, t \geq w+1\},$$

and

$$I_t = \sum_{i=1}^t x_i - \sum_{i=1}^t d_i$$

as the ending inventory in period  $t$ ,  $\forall t$ . This inequality divides the set of periods  $\{u, u+1, \dots, v\}$  into the subsets  $Q$ ,  $R$  and  $D$ . The set  $Q$  comprises those periods  $t$  for which variable  $x_t$ , the Quantity produced in period  $t$ , is included in (f). The sets

$R$  and  $D$  comprise periods  $t$  for which  $y_t$  is included in (f);  $R$  contains those periods that have the Remainder  $r$  as the coefficient of  $y_t$ , and the set  $D$  those periods that have Demand parameters as the coefficients.

Note that these inequalities can be stated in terms of only the  $x$  and  $y$  variables to correspond to the variables in the formulation CLSP. However, we have included the inventory variable  $I_{u-1}$  in (f) for ease of exposition and will continue to do so subsequently. Our arguments can easily be translated for the inequalities stated exclusively in terms of the  $x$  and  $y$  variables only.

Also, notice that (f) can be restated as

$$I_{u-1} + \sum_{t \in Q} x_t + \sum_{t \in P \setminus Q} \min\{r, d_{tv}\} y_t \geq r \lceil d_{uv}/C \rceil. \quad (4.1)$$

Figure 4.1 illustrates that

$$r \leq d_{tv} \quad \text{for all } u \leq t \leq w$$

and

$$r > d_{tv} \quad \text{for all } w+1 \leq t \leq v.$$

Using this fact, we see that (f) is a more explicit statement of (4.1).

To illustrate inequality (f), consider a 9 period problem with capacity in each period equal to 6 and demand specified as follows:

$$d_1 = 1, d_2 = 3, d_3 = 4, d_4 = 5, d_5 = 1, d_6 = 2, d_7 = 3, d_8 = 1, d_9 = 2.$$

Let  $u = 3, v = 8$ , i.e.,  $P = \{3, 4, \dots, 8\}$ . For this set of periods,  $d_{38} = 16$ ,  $\lceil d_{38}/C \rceil = 3$ ,  $r = 4$  and  $w = 7$ . Choosing  $Q = \{5, 7\}$  results in the following valid inequality for this particular problem.

$$I_2 + 4y_3 + 4y_4 + x_5 + 4y_6 + x_7 + y_8 \geq 12. \quad (4.2)$$

This inequality is in fact a facet for the given problem instance as shown later in Theorem 4.2.

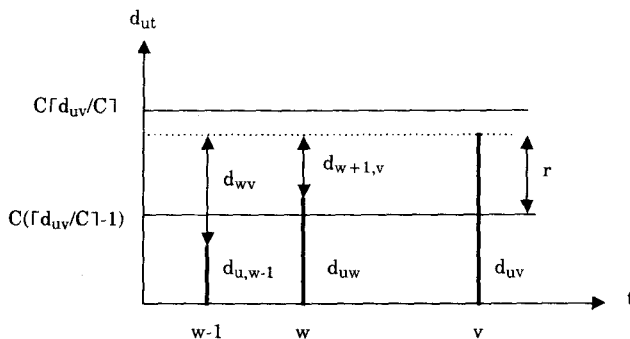


Fig. 4.1. Relation between  $d_{uv} \pmod C$  and  $d_{tv}$  for  $u \leq t \leq v$ .



Before proving that inequalities (f) define facets of  $\mathbb{C}_{LS}$ , we examine the relationship between these inequalities and other valid inequalities that have been proposed for the lot size model. The facet inequalities of the uncapacitated lot size problem derived by Barany, Van Roy and Wolsey (1984a) are

$$I_{u-1} + \sum_{t \in Q} x_t + \sum_{t \in P/Q} d_{tv} y_t \geq d_{uv} \quad (\ddagger)$$

with  $P = \{u, u+1, \dots, v\}$  and  $Q \subseteq P$ . If we set  $C = \infty$  in (f), we obtain  $r = d_{uv}$ ,  $w = u$ , and hence, the resulting inequality is the same as  $(\ddagger)$ . Therefore, the inequalities (f) generalize the facet inequalities  $(\ddagger)$  of the uncapacitated lot size problem.

Recently, Pochet (1988) has independently (and simultaneously) derived a class of valid inequalities for the CLSP that can be stated as follows:

$$I_{u-1} + \sum_{t \in Q} x_t + \sum_{t \in S} (\bar{d}_{tv} - \lambda)^+ y_t + \sum_{t \in L} \min\{\bar{d}_{uv} - \lambda, \bar{d}_{tv}\} y_t \geq \sum_{t \in S} (\bar{d}_{tv} - \lambda)^+ \quad (*)$$

with

$$P = \{u, u+1, \dots, v\}, \quad S \subseteq P \text{ and } u \in S,$$

$$Q \cup L = P \setminus S, \quad Q \cap L = \emptyset,$$

and

$$\bar{d}_{tv} = \min\{C, d_{tv}\}, \quad \lambda = \sum_{t \in S} \bar{d}_{tv} - d_{uv} \text{ and } S \text{ chosen so that } 0 \leq \lambda \leq \bar{d}_{uv}.$$

Pochet has derived sufficient conditions for a subset of inequalities in  $(*)$  obtained by setting  $L = \emptyset$  to be facets of  $\mathbb{C}_{LS}$ .

To illustrate Pochet's inequality, consider a 9 period problem with  $C = 6$  and

$$d_1 = 1, \quad d_2 = 3, \quad d_3 = 4, \quad d_4 = 4, \quad d_5 = 1, \quad d_6 = 1, \quad d_7 = 1, \quad d_8 = 5, \quad d_9 = 2.$$

This demand pattern differs slightly from that used to derive inequality (4.2).

Let  $u = 3, v = 8$ , i.e.,  $P = \{3, 4, \dots, 8\}$ . Choosing  $S = \{3, 4, 8\}$  gives  $d_{38} = d_{48} = 6, d_{88} = 5$  and  $\lambda = 1$ . The following inequality of the form  $(*)$ , with  $L = \emptyset$ , satisfies the sufficient conditions identified by Pochet and is a facet

$$I_2 + 5y_3 + 5y_4 + x_5 + x_6 + x_7 + 4y_8 \geq 14. \quad (4.3)$$

Note that the right side of inequality  $(*)$  with  $L = \emptyset$  is the sum of the coefficients of the  $y_t$  terms on the left side of the inequality. Since the facet given by (4.1) does not satisfy this condition, it cannot be generated by any of the facets  $(*)$  with  $L = \emptyset$ . Similarly, it is possible to show that the facet (4.3) cannot be generated by any of the inequalities (f). This example shows that neither the set of facets defined by Pochet nor the set of facets defined by us are subsets of each other. It is also possible to show that the inequalities  $(*)$  derived by Pochet subsume the facets defined by Theorem 4.2 but not all the valid inequalities (f).

**Theorem 4.1.** (f) is a valid inequality for CLSP.

**Proof.** Rewrite (f) as

$$I_{u-1} + \sum_{t \in Q} x_t + \sum_{t \in D} d_{tv} y_t \geq r \left( \lceil d_{uv}/C \rceil - \sum_{t \in R} y_t \right). \quad (4.4)$$

Let  $(x^*, y^*)$  be any feasible solution for CLSP. If  $\sum_{t \in R} y_t^*$ , the number of periods in  $R$  in which this solution incurs a setup, is greater than or equal to  $\lceil d_{uv}/C \rceil$ , then (4.4) is trivially satisfied. Next, suppose  $\sum_{t \in R} y_t^*$  is exactly one less than  $\lceil d_{uv}/C \rceil$ ; then production in these periods can supply at most  $d_{uv} - r$  units of the demand  $d_{uv}$ . If  $D = \emptyset$ , then the starting inventory in period  $u$  plus the production in the periods in the set  $Q$  must be at least  $r$  units and, hence, (4.4) is satisfied. If  $D \neq \emptyset$ , then let  $k = \min\{t \in D \mid y_t^* = 1\}$  and note that the inequality

$$I_{u-1}^* + \sum_{t \in Q, t \leq k-1} x_t^* \geq d_{u,k-1} - (d_{uv} - r)$$

is necessary to meet the demand in periods  $u$  to  $k-1$  since the maximum amount that can be produced in the periods in the set  $R$  is  $C \sum_{t \in R} y_t^* = d_{uv} - r$ . Thus

$$I_{u-1}^* + \sum_{t \in Q} x_t^* + \sum_{t \in D} d_{tv} y_t^* \geq I_{u-1}^* + \sum_{t \in Q, t \leq k-1} x_t^* + d_{kv} \geq d_{u,k-1} - (d_{uv} - r) + d_{kv} = r$$

and (4.2) is again satisfied. These observations account for the coefficient  $r$  of  $y_t$  in (f). If

$$\lceil d_{uv}/C \rceil - \sum_{t \in R} y_t^* = p$$

with  $p \geq 2$  and integer, then similar observations show that (4.4) is still valid. (In this case,  $d_{uv} - r - (p-1)C$  replaces  $d_{uv} - r$  in the previous argument.)  $\square$

**Remark 4.1.** (a) The last statement at the end of the proof shows that (f) is satisfied as a strict inequality whenever  $r < C$  and  $\sum_{t \in R} y_t^* \leq \lceil d_{uv}/C \rceil - 2$ .

(b) The proof also shows that (f) can be satisfied at equality only if  $\sum_{t \in R} y_t^*$  equals  $\lceil d_{uv}/C \rceil$  or  $\lceil d_{uv}/C \rceil - 1$ .

(c) Hence (f) cannot be a face of  $\mathbb{C}_{LS}$  if  $0 < |R| \leq \lceil d_{uv}/C \rceil - 2$ .

We next indicate how inequality (f) strengthens the linear programming relaxation of CLSP for the case  $D = \emptyset$ . The inequality

$$I_{u-1} + \sum_{t \in Q} x_t + C \sum_{t \in R} y_t \geq d_{uv} \quad (4.5)$$

is a linear combination of the constraints of CLSP and is satisfied by all solutions to the linear programming relaxation of CLSP. Figure 4.2 illustrates that inequality (f) cuts off some of the fractional solutions allowed by the LP-relaxation. Note that if  $r = C$  and  $D = \emptyset$ , then (f) is identical to (4.5) and hence is redundant.

We now establish some properties of solutions that satisfy (f) at equality and that will be useful in identifying conditions under which (f) is a facet of  $\mathbb{C}_{LS}$ . Let  $\mathbb{C}_{LS}^* = \{(x, y) \in \mathbb{C}_{LS} \mid (x, y) \text{ satisfies (f) at equality}\}$ .

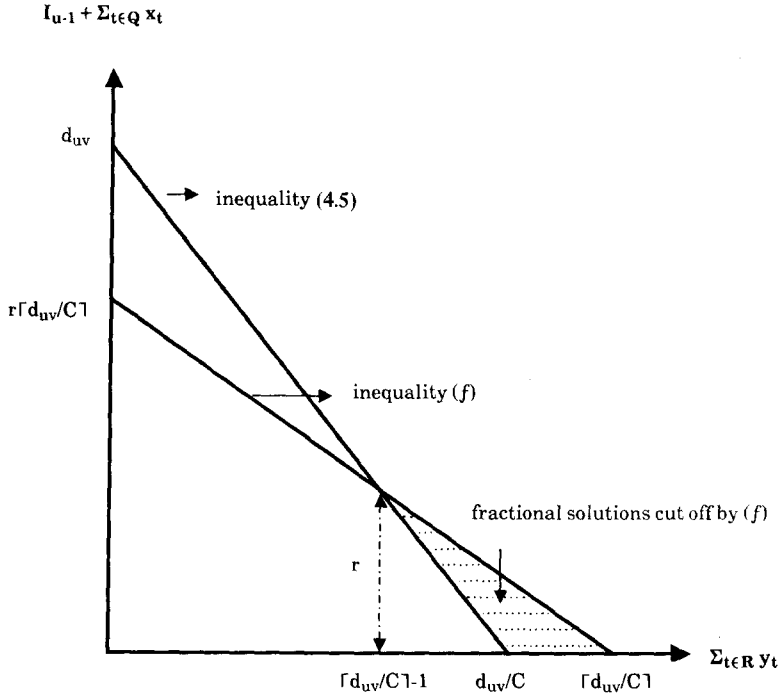


Fig. 4.2. Interpretation of inequality (f) for lot size problem.

**Lemma 4.1.** For any solution  $(x, y)$  that satisfies (f) at equality,  $I_v \leq \max\{C - r, C - d_{t'v}\}$  where  $t' = \max\{t \mid t \in R \cup D\}$ , i.e.,  $t'$  is the last period between  $u$  and  $v$  for which  $y_t$  is included in the inequality.

**Proof.** First, suppose that  $Q = P$ , i.e., there are no  $y_t$  terms in the inequality. Then, since  $I_{u-1} + \sum_{t=u}^v x_t \geq d_{uv}$ , (f) cannot be a face of  $\mathbb{C}_{LS}$  unless  $r = d_{uv}$  or  $r = C$ . If  $r$  equals  $d_{uv}$  or  $C$ , then  $r \lceil d_{uv}/C \rceil = d_{uv}$  and all  $(x, y) \in \mathbb{C}_{LS}^*$  satisfy  $I_{u-1} + \sum_{t=u}^v x_t = d_{uv}$  and, hence,  $I_v = 0$ .

Next, suppose that  $Q \neq P$  and  $R \neq \emptyset$ . Then, from Remark 4.1(b), for all  $(x, y) \in \mathbb{C}_{LS}^*$ ,

$$\sum_{t \in R} y_t \in \{ \lceil d_{uv}/C \rceil - 1, \lceil d_{uv}/C \rceil \}. \quad (4.6)$$

If  $\sum_{t \in R} y_t = \lceil d_{uv}/C \rceil$ , then  $I_{u-1} = \sum_{t \in Q} x_t = \sum_{t \in D} y_t = 0$  and the maximum amount that can be produced in the periods between  $u$  and  $v$  is  $C \lceil d_{uv}/C \rceil$ . Since  $d_{uv} = C(\lceil d_{uv}/C \rceil - 1) + r$ ,  $I_v \leq C - r$  in this case. If  $\sum_{t \in R} y_t = \lceil d_{uv}/C \rceil - 1$  and  $\sum_{t \in D} y_t = 0$ , then  $I_{u-1} + \sum_{t \in Q} x_t = r$  and  $I_v = 0$ . If  $y_t \neq 0$  for some  $t \in D$ , then let  $i$  be the first period in  $D$  for which  $y_i = 1$ . To satisfy demand up to period  $i-1$  requires that  $I_{u-1} + \sum_{t \in Q} x_t \geq d_{u,i-1} - C(\lceil d_{uv}/C \rceil - 1) = d_{uv} - d_{iv} - C(\lceil d_{uv}/C \rceil - 1) = r - d_{iv}$ . In fact, since the solution is in  $\mathbb{C}_{LS}^*$ ,  $I_{u-1} + \sum_{t \in Q} x_t = r - d_{iv}$  and  $y_t = 0$  for  $t > i$  and  $t \in D$ . Thus, the maximum inventory that can be available at the end of period  $v$  occurs

when period  $i$  produces to capacity and in this case  $I_v = C - d_{iv}$ . This conclusion establishes that  $I_v \leq \max\{C - r, C - d_{iv}\}$  for all  $(x, y) \in \mathbb{C}_{LS}^*$ .

A similar argument establishes this result if  $Q \neq P$  but  $R = \emptyset$ .  $\square$

The next result shows that (f) cannot be a face of  $\mathbb{C}_{LS}$  if the demands in the periods  $u, u+1, \dots, v$  grow too fast.

**Lemma 4.2.** *Inequality (f), with  $|R| \geq \lceil d_{uv}/C \rceil - 1$ , is a face of  $\mathbb{C}_{LS}$  if and only if for all  $q, u \leq q \leq v$ , the demand between periods  $u$  and  $q$ ,  $d_{uq} \leq r + C(|\{t: t \in R \text{ and } t \leq q\}|)$ .*

**Proof.** Note that if  $|R| \geq \lceil d_{uv}/C \rceil - 1$  and  $w+1 \leq q \leq v$ , then  $d_{uq} \leq r + C(|\{t: t \in R \text{ and } t \leq q\}|) = r + C|R|$  is trivially satisfied since  $d_{uq} \leq d_{uv} = r + C(\lceil d_{uv}/C \rceil - 1)$ . Consider  $u \leq q \leq w$ . All integer solutions in  $\mathbb{C}_{LS}^*$  satisfy (4.4). Thus, these solutions also satisfy

$$I_{u-1} + \sum_{i \in Q} x_i + \sum_{i \in D} d_{iv} y_i \in \{r, 0\}.$$

Consequently, the maximum quantity available to meet demand up to any period  $q, u \leq q \leq w$ , is  $r + C(|\{t: t \in R \text{ and } t \leq q\}|)$ . Thus, if  $d_{uq} > r + C(|\{t: t \in R \text{ and } t \leq q\}|)$ , then (f) cannot be a face of  $\mathbb{C}_{LS}$ , whereas if  $d_{uq} \leq r + C(|\{t: t \in R \text{ and } t \leq q\}|)$  then it is straightforward to construct  $(x, y) \in \mathbb{C}_{LS}^*$  and show that (f) is a face of  $\mathbb{C}_{LS}$ .  $\square$

Our objective now is to identify conditions under which inequality (f) is a facet of  $\mathbb{C}_{LS}$ . Several of these conditions impose restrictions on the demand between periods  $u$  and  $v$  and are related to the properties that we established in Lemmata 4.1 and 4.2. To simplify the exposition, we focus on the case  $Q \neq \emptyset, R \neq \emptyset, D \neq \emptyset$  and  $u \geq 3$  and then point out later how the discussion needs to be modified for the other cases. We first introduce some additional notation that we need for the subsequent discussion. Let  $N = \lceil d_{uv}/C \rceil = \lceil d_{uv}/C \rceil$ . Further, as in Lemma 4.1, define  $t'$  as the last period between  $u$  and  $v$  for which  $y$  is included in (f), i.e.,  $t' = \max\{t \mid t \in R \cup D\}$ .

**Theorem 4.2.** *Given  $Q \neq \emptyset, R \neq \emptyset, D \neq \emptyset$  and  $u \geq 3$ , (f) is a facet of  $\mathbb{C}_{LS}$  if and only if the following conditions are satisfied.*

- (i)  $r < C, v < T$  and if  $v \in Q$  then  $d_v > 0$ .
- (ii)  $d_{v+1} \leq \max\{C - r, C - d_{t'v}\}$ .
- (iii)  $|R| \geq \lceil d_{uv}/C \rceil$ .
- (iv)  $d_{uq} \leq r + C(|\{t: t \in R \text{ and } t \leq q\}| - 1)$  for all  $q$  for which  $d_{uq} > 0, u \leq q \leq v$ . (Note that if  $d_u > 0$ , this condition requires  $u \in R$  and  $d_u \leq r$ .)

**Proof.** We should first point out that though, for ease of exposition, we continue to use the inventory variable  $I_{u-1}$  in stating inequality (f) and in our discussion, the proof that (f) is a facet of  $\mathbb{C}_{LS}$  is actually for the equivalent inequality stated in terms of the  $x$  and  $y$  variables only, i.e., in terms of the variables used to define the

original problem CLSP. The proof of the theorem is fairly long and so we outline briefly the essential steps. If the conditions of the theorem are satisfied, then it is straightforward to show that (f) is a facet of  $\mathbb{C}_{LS}$  but not an improper face and we omit the details. We first discuss why all of the conditions (i)–(iv) are necessary for (f) to be a facet of  $\mathbb{C}_{LS}$  and then show that they are also sufficient.

*Necessity.* The proof that each of the conditions (i)–(iv) is necessary for (f) to be a facet of  $\mathbb{C}_{LS}$  uses the observations that if  $J$  is another valid inequality for CLSP that is not a multiple of (f) then (f) cannot be a facet of  $\mathbb{C}_{LS}$  if it is implied by  $J$  or if  $\mathbb{C}_{LS}^* \subset \mathbb{C}$ , where  $\mathbb{C} = \{(x, y) \in \mathbb{C}_{LS} \mid (x, y) \text{ satisfies } J \text{ at equality}\}$ .

(i) If  $r = C$ , then for some  $j$ ,  $u \leq j \leq w$ , replacing  $Cy_j$  by  $x_j$  in (f) results in another valid inequality that implies (f) since  $x_j \leq Cy_j$ . Moreover, by our blanket assumption that  $d_1 + d_2 \leq C$ ,  $x_j \neq Cy_j$  for some  $(x, y) \in \mathbb{C}_{LS}$ , and therefore, (f) cannot be a facet of  $\mathbb{C}_{LS}$ . Similarly, if  $v = T$ , and  $D = \emptyset$ , let  $j \in D$ . Replacing  $d_{jT}y_j$  by  $x_j$  results in a stronger inequality that is not equivalent to (f) and, hence, (f) cannot be a facet. If  $v \in Q$  and  $d_v = 0$  then (f) is equivalent to

$$I_{u-1} + \sum_{t \in Q, t \neq v} x_t + x_v + \sum_{t \in R} ry_t + \sum_{t \in D} d_{t,v-1}y_t \geq r \lceil d_{u,v-1}/C \rceil \quad (4.7)$$

since  $d_{tv} = d_{t,v-1}$  for all  $t \leq v-1$ . However, since  $x_v \geq 0$ , (4.7) is implied by

$$I_{u-1} + \sum_{t \in Q, t \neq v} x_t + \sum_{t \in R} ry_t + \sum_{t \in D} d_{t,v-1}y_t \geq r \lceil d_{u,v-1}/C \rceil.$$

Further, since  $x_v \neq 0$  for some feasible solution to LS, (f) cannot be a facet of  $\mathbb{C}_{LS}$ .

(ii) Lemma 4.1 shows that  $I_v \leq \max\{C - r, C - d_{rv}\}$  for all  $(x, y) \in \mathbb{C}_{LS}^*$ . Hence, if  $d_{v+1} > \max\{C - r, C - d_{rv}\}$ , then  $y_{v+1} = 1$  in all integer solutions in  $\mathbb{C}_{LS}^*$  and (f) cannot be a facet of  $\mathbb{C}_{LS}$ .

(iii) From Remark 4.1,  $|R| \geq N - 1$  for (f) to be a face of  $\mathbb{C}_{LS}$ . If  $|R| = N - 1$ , then (4.4) requires  $y_t = 1$  for all  $t \in R$  and, therefore, (f) cannot be a facet of  $\mathbb{C}_{LS}$ .

(iv) Lemma 4.2 shows that  $d_{uq} \leq r + C(|\{t: t \in R \text{ and } t \leq q\}| - 1)$  is necessary for (f) to be a face of  $\mathbb{C}_{LS}$ . If  $d_{uq} > r + C(|\{t: t \in R \text{ and } t \leq q\}| - 1)$ , then for all integer solutions in  $\mathbb{C}_{LS}^*$  we must have  $y_t = 1$  for all  $t \in R$  and  $t \leq q$  to meet demand up to  $q$ . This observation implies that (f) cannot be a facet of  $\mathbb{C}_{LS}$ . If  $d_u > 0$ , then this condition implies that  $u \in R$ . If  $d_u = 0$ , and  $u \in Q$ , then this inequality is identical to another one with  $P = \{u + 1, \dots, v\}$  since

$$\begin{aligned} I_{u-1} + x_u + \sum_{t \in Q, t \neq u} x_t + \sum_{t \in R} ry_t + \sum_{t \in D} d_{tv}y_t &\geq r \lceil d_{uv}/C \rceil \\ \Leftrightarrow I_u + \sum_{t \in Q, t \neq u} x_t + \sum_{t \in R} ry_t + \sum_{t \in D} d_{tv}y_t &\geq r \lceil d_{u+1,v}/C \rceil. \end{aligned}$$

Hence, we need not consider the case  $u \in Q$ .

Having established the necessity of conditions (i)–(iv), we now show that if the problem satisfies these conditions, then (f) is a facet of  $\mathbb{C}_{LS}$ . As mentioned earlier, it is easy to see that (f) is a face of  $\mathbb{C}_{LS}$ , but is not an improper face. Thus, we need to show only that  $\dim \mathbb{C}_{LS}^* = \dim \mathbb{C}_{LS} - 1$  to establish that (f) is a facet of  $\mathbb{C}_{LS}$ . We prove  $\dim \mathbb{C}_{LS}^* = \dim \mathbb{C}_{LS} - 1$  by showing that any arbitrary valid inequality that is

satisfied at equality by all  $(x, y) \in \mathbb{C}_{LS}^*$  is a linear combination of (f),  $\sum_{i=1}^T x_i = \sum_{i=1}^T d_i$  and  $y_1 = 1$ .

**Sufficiency.** Let  $\alpha x + \beta y = \delta$ , with  $\alpha \in \mathbb{R}^T$ ,  $\beta \in \mathbb{R}^T$  and  $\delta \in \mathbb{R}$ , represent an arbitrary equation that is satisfied by all  $(x, y) \in \mathbb{C}_{LS}^*$ . We will construct a sequence of points  $(x^1, y^1), (x^2, y^2), \dots$ , in  $\mathbb{C}_{LS}^*$  and use the fact that  $\alpha x^1 + \beta y^1 = \alpha x^2 + \beta y^2 = \dots = \delta$ , to show that

- (a)  $\beta_t = 0$ ,  $t \notin R \cup D$  and  $t \neq 1$ ,
- (b)  $a_t = a_1$ ,  $t \leq u-1$  or  $t \in Q$ ,
- (c)  $a_t = a_{v+1}$ ,  $t \in R \cup D$  or  $t \geq v+1$ ,
- (d)  $\beta_t = r(a_1 - a_{v+1})$ ,  $t \in R$ ,
- (e)  $\beta_t = d_{tv}(a_1 - a_{v+1})$ ,  $t \in D$ ,
- (f)  $\delta = a_{v+1} \sum_{i=1}^T d_i + \beta_1 + (a_1 - a_{v+1})\{r \lceil d_{uv}/C \rceil + d_{1,u-1}\}$ .

These conclusions establish the desired result that  $\alpha x + \beta y = \delta$  is a linear combination of  $\sum_{i=1}^T x_i = \sum_{i=1}^T d_i$ ,  $y_1 = 1$ , and

$$I_{u-1} + \sum_{t \in Q} x_t + \sum_{t \in R} r y_t + \sum_{t \in D} d_{tv} y_t = rN. \quad (4.8)$$

We first indicate how we will prove (a)–(f). If it is possible to construct two solutions in  $\mathbb{C}_{LS}^*$ , one with  $y_t = 0$ , the other with  $y_t = 1$ , the other with  $y_t = 1$ , and all other variables the same for both solutions, then  $\beta_t$  must be 0. We can ensure that the solutions are feasible by defining both solutions with  $x_t = 0$ . To prove that  $a_t = a_i$  for  $t \neq i$ , we need to be able to define two solutions in  $\mathbb{C}_{LS}^*$ , one with  $x_t = a+1$ ,  $x_i = b$ , the other with  $x_t = a$ ,  $x_i = b+1$ , for some nonnegative  $a$  and  $b$ , and all other variables the same in both solutions, i.e., we can shift a unit of production from period  $i$  to period  $t$ . Similarly, given  $a_t = a_i$ , to show that  $\beta_t = \beta_i$  for  $t \neq i$ , we construct a pair of solutions in  $\mathbb{C}_{LS}^*$ , one with  $y_t = 1$ ,  $x_t = a$ ,  $y_i = x_i = 0$ , the other with  $y_t = 1$ ,  $x_t = a$ ,  $y_i = x_i = 0$ , for some nonnegative  $a$ , and all other variables the same in both solutions.

We first show that  $\beta_t = 0$  for all  $t \notin R \cup D$ ,  $t \neq 1$ . Recall that to do so we need to construct a solution in  $\mathbb{C}_{LS}^*$  with  $x_t = y_t = 0$ . We consider the three cases  $2 \leq t \leq u-1$ ,  $t \in Q$  and  $t \geq v+1$  separately. Since  $d_1 + d_2 \leq C$  by assumption, it is easy to construct a solution with  $x_t = y_t = 0$  for  $2 \leq t \leq u-1$ . For future reference, we call this solution  $(x^1, y^1)$ .

We construct a solution with  $x_j = y_j = 0$  for a given  $j \in Q$  as follows. Let the production in each of the first  $u-1$  periods and in periods  $v+1$  to  $T$  to be equal to demand in that period. Let the production in periods  $t \in Q$  and  $t \in D$  and the corresponding setup variables be equal to 0. Finally, let the setup variables for the first  $N$  periods in the set  $R$  be equal to 1 and the other be equal to 0; production in the first  $N-1$  of these periods is at capacity and production in the  $N$ th period is equal to  $r$ . Condition (iv) of the theorem ensures that this plan meets demand in all periods between  $u$  and  $v$ . We refer to this solution as  $(x^2, y^2)$ .

For the third case, we ensure that  $x_j = y_j = 0$  for a given  $j \geq v+1$  by carrying sufficient inventory to meet demand in period  $j$ . Since  $d_{v+1} \leq C - d_{tv}$ , it is possible to produce  $d_{tv} + d_{v+1}$  in period  $t'$  to meet demand in periods  $t'$  to  $v+1$ . Thus, the capacity of period  $v+1$  can be used to produce for period  $j$ . To ensure that this

solution is in  $\mathbb{C}_{LS}^*$ , let the starting inventory  $I_{u-1}$  in period  $u$  equal  $r - d_{t'v}$ , the production in periods  $t \in Q$  and  $t \in D \setminus \{t'\}$  equal 0, and the production in each of the first  $N-1$  periods in the set  $R$  be equal to capacity  $C$ . (Note that since  $u \geq 3$ , it is possible to construct solutions with  $I_{u-1} = r$  (or less). However, if  $u = 2$ , this choice is not possible unless  $d_1 \leq C - r$  and thus, we need to consider the case  $u = 2$  (and  $u = 1$ ) separately.) We refer to this third solution as  $(x^3, y^3)$ .

We have shown that  $\beta_t = 0$  for all  $t \notin R \cup D$ ,  $t \neq 1$ . We now show that  $a_t = a_1$  for all  $2 \leq t \leq u-1$  or  $t \in Q$ . To do so, we need to construct a pair of solutions with the same setup variables equal to 1 in both the solutions,  $x_1 = a$ ,  $x_t = b+1$  in one and  $x_1 = a+1$ ,  $x_t = b$  in the other. We first construct a solution in  $\mathbb{C}_{LS}^*$  with  $x_1 = d_1$  and positive production in period  $j$  for some given  $2 \leq j \leq u-1$  or  $j \in Q$ . This construction is always possible. Now, we can obtain another solution by shifting a unit of production from period  $j$  to period 1 and keeping all other variables the same. Therefore,  $a_t = a_1$  for all  $2 \leq t \leq u-1$  or  $t \in Q$ .

Similar arguments show that  $a_t = a_{v+1}$  for all  $t \geq v+2$ . We now want to show that  $a_t = a_{v+1}$  for all  $t \in R \cup D$ . Consider  $i \in D$ . Construct a solution  $(x, y)$  that is similar to  $(x^3, y^3)$  with  $i$  playing the role of  $t'$ . In particular, let  $I_{u-1} = r - d_{iv}$ ,  $x_i = d_{iv} + 1$ ,  $x_{v+1} = d_{v+1} - 1$  and choose the other variables to obtain a feasible point. (If  $d_{v+1} = 0$ , the construction can be suitably modified.) Since this solution can be changed by shifting the one unit of production from period  $i$  to period  $v+1$  and keeping all other variables the same,  $a_t = a_{v+1}$  for all  $t \in D$ .

We now make some observations that will establish  $a_t = a_{v+1}$  for all  $t \in R$ . Let  $R = \{t_1, t_2, \dots, t_N\}$ . First, note that for any  $j \in R$  and  $j > t_N$ , period  $j$  can play the role of period  $t_N$  in the point  $(x^1, y^1)$ , i.e., we could have  $y_{t_N} = x_{t_N} = 0$  and  $y_j = 1$ ,  $x_j = r$  instead. Since  $r \leq C - 1$ , it is possible to modify these solutions by shifting one unit of production from period  $v+1$  (or later) to period  $j$ , and hence  $a_t = a_{v+1}$  for all  $t \in R$  and  $t \geq t_N$ . Similarly, for any  $j \in R$  and  $j \leq t_{N-1}$ , it is possible to modify  $(x^1, y^1)$  by shifting one unit of production from period  $j$  to period  $t_N$  and hence,  $a_j = a_{t_N}$ . Therefore,  $a_t = a_{v+1}$  for all  $t \in R$ .

The results so far establish that  $ax + \beta y = \delta$  is of the form

$$\begin{aligned} a_1 \left( \sum_{t=1}^{u-1} x_t + \sum_{t \in Q} x_t \right) + a_{v+1} \left( \sum_{t \in R \cup D} x_t + \sum_{t=v+1}^T x_t \right) + \beta_1 y_1 + \sum_{t \in R \cup D} \beta_t y_t &= \delta \\ \Leftrightarrow (a_1 - a_{v+1}) \left( \sum_{t=1}^{u-1} x_t + \sum_{t \in Q} x_t \right) + a_{v+1} \sum_{t=1}^T x_t + \beta_1 y_1 + \sum_{t \in R \cup D} \beta_t y_t &= \delta. \end{aligned} \quad (4.9)$$

We now show that  $\beta_t = r(a_1 - a_{v+1})$  for all  $t \in R$ . As mentioned earlier, for any  $j \in R$  and  $j > t_N$ , period  $j$  can play the role of period  $t_N$  in the solution  $(x^1, y^1)$ . Define  $(x^4, y^4)$  to be the same as  $(x^1, y^1)$  except that  $y_{t_N} = x_{t_N} = 0$  and  $y_j = 1$ ,  $x_j = r$  instead for some  $j > t_N$ . Construct  $(x^5, y^5) \in \mathbb{C}_{LS}^*$  with

$$I_{u-1}^5 = r, \quad x_j^5 = y_j^5 = 0 \quad \text{and} \quad x_t^5 = x_t^4, y_t^5 = y_t^4, \quad \text{for } t \geq u, t \neq j.$$

Comparing  $(x^4, y^4)$  with  $(x^5, y^5)$  gives  $\beta_t = r(a_1 - a_{v+1})$  for all  $t \in R$  and  $t \geq t_N$ . To show that  $\beta_t = \beta_{t_N}$  for all  $t \in R$  and  $t < t_N$ , note that if  $I_{u-1} = r$  in any feasible plan

for the first  $u-1$  periods, then setting production equal to capacity in any  $N-1$  of the first  $N$  periods in the set  $R$  and production equal to 0 in all the other periods between  $u$  and  $v$  will give a solution in  $\mathbb{C}_{LS}^*$ . Thus,  $\beta_t = \beta_{t_N}$  for all  $t \in R$  and  $t < t_N$ , and hence,  $\beta_t = r(a_1 - a_{v+1})$  for all  $t \in R$ .

Using similar arguments we can show that  $\beta_t = d_{uv}(a_1 - a_{v+1})$  for all  $t \in D$ . Thus, (4.9) is equivalent to

$$(a_1 - a_{v+1}) \left\{ \sum_{t=1}^{u-1} x_t + \sum_{t \in Q} x_t + \sum_{t \in R} r y_t + \sum_{t \in D} d_{uv} y_t \right\} + a_{v+1} \sum_{t=1}^T x_t + \beta_1 y_1 = \delta.$$

Since all  $(x, y) \in \mathbb{C}_{LS}^*$  satisfy  $y_1 = 1$ ,  $\sum_{t=1}^T x_t = \sum_{t=1}^T d_t$ , and (4.8),

$$\delta = a_{v+1} \sum_{i=1}^T d_i + \beta_1 + (a_1 - a_{v+1}) \{r \lceil d_{uv}/C \rceil + d_{1,u-1}\}.$$

Therefore,  $ax + \beta y = \delta$  is a linear combination of  $y_1 = 1$ ,  $\sum_{t=1}^T x_t = \sum_{t=1}^T d_t$ , and (4.8).  $\square$

We can now discuss why we need additional conditions if  $u < 3$  or if one of the sets  $Q, R$  or  $D$  is empty. Recall that the proof of Theorem 4.2 needs to be modified if  $u = 1$  or  $u = 2$  because some of the solutions  $(x, y) \in \mathbb{C}_{LS}^*$  that we constructed to prove the theorem required  $I_{u-1} = r$ . This choice is possible if  $u \geq 3$  (since  $d_1 + d_2 \leq C$ ), but may not be feasible for  $u = 2$  and obviously needs to be modified when  $u = 1$  since we assume  $I_0 = 0$ . If  $Q = P$  then (f) cannot be a face unless  $r = C$  or  $r = d_{uv}$ . Moreover, as mentioned in the proof of Lemma 4.2,  $I_v = 0$  for all  $(x, y) \in \mathbb{C}_{LS}$  in this case and, hence, (f) cannot be a facet unless  $d_{v+1} = 0$  if  $Q = P$ . Similarly, we need additional conditions for the other cases. For example, the general inequality (f) with  $u \geq 3$  is a facet of  $\mathbb{C}_{LS}$  if and only if the following conditions are satisfied.

- (i) If  $Q = P$ , then  $v < T$ ,  $d_{v+1} = 0$  and  $r = C$  or  $d_{uv}$ .
- (ii) If  $r = C$ , then  $t \in Q$  for all  $u \leq t \leq w$ .
- (iii) If  $v = T$ , then  $t \in Q$  for all  $w + 1 \leq t \leq T$ ; if, in addition,  $w = T$ , then  $T \in Q$ .
- (iv) If  $v \in Q$ , then  $d_v > 0$ .
- (v)  $d_{v+1} \leq \max\{C - r, C - d_{v'}\}$ .
- (vi) If  $\lceil d_{uv}/C \rceil \geq 2$  and  $r \leq C - 1$ , then  $|R| \geq \lceil d_{uv}/C \rceil$ .
- (vii) If  $R \neq \emptyset$ , then  $d_{uq} \leq R + C(|\{t: t \in R \text{ and } t \leq q\}| - 1)$  for all  $q$  for which  $d_{uq} > 0$ ,  $u \leq q \leq v$ . (Note that if  $d_u > 0$ , this condition requires  $u \in R$  and  $d_u \leq r$ .)

Note that most of these conditions are required for special cases of the inequality, for example, if  $Q = P$  or  $R = \emptyset$ . However, since we do not develop much further insight into the structure of CLSP from these special cases, we do not provide the details for the cases  $u = 1$  and  $u = 2$ .

### Trivial facets

We state the following results regarding the trivial facets of CLSP without including their proof, since they can be proved using arguments similar to those used to establish Theorem 4.2.



**Proposition 4.1.** *The demand constraint  $\sum_{i=1}^t x_i \geq d_t$  is a facet of  $\mathbb{C}_{LS}$  for all  $t, 1 \leq t \leq T-1$ , for which  $d_{t+1} = 0$ .  $\square$*

As discussed earlier, the demand constraint is implied by  $\sum_{i=1}^t x_i + y_{t+1}d_{t+1} \geq d_{t+1}$ , which is a special case of the facet inequalities (f). The two constraints are identical only if  $d_{t+1} = 0$ . Hence, the demand constraint cannot be a facet if  $d_{t+1} > 0$ .

**Proposition 4.2.** *The forcing constraint  $x_t \leq \min\{C, d_t\}y_t$  is a facet of  $\mathbb{C}_{LS}$  for all  $t$ .  $\square$*

**Proposition 4.3.** *The constraint  $x_t \geq 0$  is a facet of  $\mathbb{C}_{LS}$  for all  $t, 3 \leq t \leq T$ , for which  $d_{1t} \leq C(t-2)$ ;  $x_2 \geq 0$  is a facet if  $d_{13} \leq C$ .  $\square$*

We comment briefly on why the conditions are necessary. First, note that the constraint  $x_1 \geq 0$  is not a face of  $\mathbb{C}_{LS}$  since, by assumption,  $d_1 > 0$ . Suppose  $d_{1t} > C(t-2)$  for some  $t \geq 3$ . Then, all solutions that satisfy  $x_t \geq 0$  at equality, i.e., have  $x_t = 0$ , must have  $y_1 = y_2 = \dots = y_{t-1} = 1$  to satisfy demand up to period  $t$ . Thus,  $x_t \geq 0$  cannot be a facet in this case. Since we already required  $y_1 = 1$ , this observation does not hold for  $t = 2$ . However, if  $x_2 = 0$  and  $d_{13} > C$ , then in addition to  $y_1 = 1$  we require  $y_3 = 1$  and hence,  $x_2 \geq 0$  cannot be a facet in this case.

**Proposition 4.4.** *The constraint  $y_t \geq 0$  is a facet of  $\mathbb{C}_{LS}$  for all  $t, 3 \leq t \leq T$ , for which  $d_{1t} \leq C(t-2)$ .  $\square$*

**Proposition 4.5.** *The constraint  $y_t \leq 1$  is a facet for all  $t, 2 \leq t \leq T$ .  $\square$*

## 5. Separation problem

In this section, we show how to solve the separation problem for inequalities (f), the class of non-trivial facets for CLSP derived earlier. The separation problem can be described as follows. If we use inequalities (f) as part of a cutting plane procedure to solve CLSP (or any problem that contains CLSP as a subproblem), then given a feasible fractional point  $(x^*, y^*)$  for the linear programming relaxation of CLSP (or the larger problem), we need to identify an inequality (f) that cuts it off or determine that no such inequality exists. The following algorithm solves this problem:

Let  $(x^*, y^*)$  denote a given fractional solution for the problem. For each  $u$  and  $v$ , with  $u = 1, 2, \dots, T$  and  $u \leq v \leq T$ , find

- (1)  $r = d_{uv} \pmod{C}$ ;
- (2)  $w$  = the first period between  $u$  and  $v$  that satisfies  $\lceil d_{uw}/C \rceil = \lceil d_{uv}/C \rceil$ ;
- (3)  $R^* \subseteq \{u, u+1, \dots, w\}$ , defined by  $t \in R^*$  if  $ry_t^* \leq x_t^*$ ;
- (4)  $D^* \subseteq \{w+1, \dots, v\}$ , defined by  $t \in D^*$  if  $d_{tv}y_t^* \leq x_t^*$ ;
- (5)  $Q^* = P \setminus \{R^* \cup D^*\}$ , where  $P = \{u, u+1, \dots, v\}$ .
- (6) Check if

$$I_{u-1}^* + \sum_{t \in Q^*} x_t^* + \sum_{t \in R^*} ry_t^* + \sum_{t \in D^*} d_{tv}y_t^* < r \lceil d_{uv}/C \rceil. \quad (§)$$

If (§) is satisfied, then a violated inequality has been found. If (§) does not hold for any  $u$  and  $v$  then the solution  $(x^*, y^*)$  does not violate any of the inequalities (f). By our choice of  $Q^*$ ,  $R^*$  and  $D^*$ , this procedure will find a violated inequality (f) if there is one. It is straightforward to check that the running time of this algorithm is  $O(T^3)$ .

## 6. Computational results

In our computational studies, we use our characterization of the polyhedral structure of the single item lot size problem to develop and test a strong cutting plane algorithm for the multiple item problem. In this section, we describe the algorithm and present some computational results.

The multiple item problem (MLSP) can be formulated as follows:

$$\text{Minimize } \sum_{i=1}^I \sum_{t=1}^T p_{it} x_{it} + \sum_{i=1}^I \sum_{t=1}^T s_{it} y_{it} \quad (6.1)$$

$$\text{subject to } \sum_{j=1}^I x_{ij} \geq \sum_{j=1}^I d_{ij} \quad \forall i, t, \quad (6.2)$$

$$\sum_{j=1}^I x_{ij} = \sum_{j=1}^I d_{ij} \quad \forall i, \quad (6.3)$$

$$x_{it} \leq \min\{d_{i,T}, C\} y_{it} \quad \forall i, t, \quad (6.4)$$

$$\sum_{i=1}^I x_{it} \leq C \quad \forall t, \quad (6.5)$$

$$x_{it} \geq 0, \quad 0 \leq y_{it} \leq 1, \quad \forall i, t, \quad (6.6)$$

$$y_{it} \text{ integer} \quad \forall i, t. \quad (6.7)$$

In this formulation,  $x_{it}$  refers to the production of item  $i$  in period  $t$  and  $y_{it}$  is equal to 1 if the facility is setup to produce item  $i$  in period  $t$  and 0 otherwise. As in CLSP, the single product model,  $p_{it}$  is the production and inventory carrying cost for production of item  $i$  in period  $t$  and  $s_{it}$  is the cost of setting up the facility for item  $i$  in period  $t$ . The demand for item  $i$  in period  $t$  given by  $d_{it}$  and  $d_{i,T} = \sum_{j=1}^T d_{ij}$ . Constraints (6.2), (6.3) and (6.4) are analogous to constraints (3.4), (3.5) and (3.6) of the single item problem. Constraint (6.5) requires that the total production of all items in period  $t$  cannot exceed the capacity  $C$ .

The algorithm that we test for solving MLSP uses a combination of cutting planes and branch and bound as embodied in a system with the following features. The algorithm starts by solving a linear programming relaxation of the given problem. If the linear programming solution is fractional, then the system solves the separation problem for the family of facets described earlier to identify if this solution violates a valid inequality. If the algorithm finds such an inequality, it adds it to the current

linear programming relaxation of the problem and then solves the updated linear program. The system repeats this procedure until either the solution it generates is integral or violates none of the facet inequalities. On termination, if the linear programming solution is not integral, then the system uses branch and bound to obtain an optimal integral solution to the original problem.

Our computational experiments have two major objectives:

(i) To empirically estimate the reduction in the integrality gap, i.e., the gap between the optimal values of the original problem and its linear programming relaxation, caused by the addition of facet inequalities.

(ii) To determine if any specific subclass of the facet inequalities is more effective in reducing this gap so that we can develop insights into modeling these problems and also consider linear programming based solution methods that include these inequalities *a priori*.

Our goal is not to develop the most efficient cutting plane procedure that exploits our description of the problem's polyhedral structure; therefore, for each fractional solution encountered in the algorithm, we solve the separation problem, rather than use a faster heuristic, to identify a violated inequality. Moreover, instead of testing when it is better to add violated inequalities and when to invoke branch and bound, we use branch and bound only when no more valid inequalities can be added to tighten the linear program. The resolution of a number of such implementation issues is a prerequisite for developing a computationally efficient algorithm.

In the next two sections, we describe the data used to test the algorithm and present our computational results. The computations were all performed on a PRIME 850 computer using the LINDO mixed integer programming package for solving the linear programs as well as for the branch and bound computations. The matrix generators and the cut generation routines were coded in FORTRAN as part of the USER subroutine available with LINDO.

### 6.1. *Data sets*

Several researchers have reported computational experience for the capacitated lot size problem. However, most of the results are for problems with a large number of products as compared to the number of time periods. For these problems, the integrality gap tends to be small. We wanted to test the cutting plane procedure on problems that are expected to have a large integrality gap since it is for these problems that our procedure is most likely to offer advantages over other methods. Unfortunately, there are very few data sets in the literature for capacitated problems with the number of products comparable to the number of time periods. Thizy and Van Wassenhove (1986) present a set of 8 period, 8 product problems that they were unable to solve using Lagrangian relaxation. These problems have since been solved by Barany, Van Roy and Wolsey (1984b) and Eppen and Martin (1987). Barany, Van Roy and Wolsey use facets of the single item uncapacitated lot size problem to solve these problems and Eppen and Martin use a reformulation approach that is equivalent to including a description of the convex hull of the

single item uncapacitated lot size problem in the multiple item formulation. Thus, both these methods are based on a characterization of the polyhedral structure of the single item uncapacitated model.

To test our algorithm, we use variations of the Thizy and Van Wassenhove data, variations of a single item problem discussed by Peterson and Silver (1979) and some randomly generated problems. The four problems described by Thizy and Van Wassenhove are identical except for the available capacity. In all four problems, the total demand over the horizon for most of the 8 products is less than the capacity of a period. (The total demand over the horizon for the 8 products varies between 100 and 800 with the average being 365 and the capacity per period varies between 350 and 600. Note that for any 8 period 8 product problem, the average total demand for a product cannot exceed the average capacity per period since otherwise the total demand for all the products will exceed the total available capacity and the problem will be infeasible.) Therefore, the cuts that our algorithm would generate for these problems would be the same as those for the uncapacitated problem and we would not expect the performance of our method to be significantly different from that of Barany, Van Roy and Wolsey. To generate capacitated problems, we modified the Thizy and Van Wassenhove data sets to obtain 1, 2 and 4 product problems as follows. The demand for the 1 product problems was obtained by adding the demand of all 8 products in the corresponding time periods and the setup cost was set equal to 500. Four different 1 product problems were generated for this demand and setup cost by letting the capacity per period equal 400, 600, 1000 and 3000 respectively. Note that the 1 product problem with capacity per period equal to 3000 is an uncapacitated problem since the total demand over the horizon is 2920. Similarly, we generated 2 product problems by aggregating the demands of products 1–4 and 5–8 of the Thizy and Van Wassenhove data to obtain the demand for the two products. The setup costs for the products was 500 and 400, respectively, and two different problems were obtained by letting the capacity per period be 400 and 600, respectively. We also generated two 4 product problems in a similar manner by aggregating demands of products 1 and 2, 3 and 4, 5 and 6, and 7 and 8. The setup costs were equal to 500, 400, 300 and 200, respectively and the capacity per period was 400 and 600. We also used variations of a single item, 12 period, uncapacitated problem described by Peterson and Silver. For this data set, we let capacity equal 150, 180 and 220 in each period to obtain 3 different problems. The Peterson and Silver problems are the same as those used by Barany, Van Roy and Wolsey to test their algorithm.

We also tested the algorithm on some randomly generated 1 and 2 product problems. For all these problems, the capacity per period was equal to 100, the setup cost for each of the products was also equal to 100 and the holding cost per unit per period for each of the products was equal to 1. We then generated demand for each product in each period from a uniform distribution. The single product problems all had a planning horizon of 15 periods. The parameters of the distribution were determined to allow the ratio  $d_{1T}/C$  to vary between 3.7 and 11.2 (i.e., the

utilization of the facility's capacity varied between approximately 25% and 75%). Similarly, the two product problems all had a planning horizon of 12 periods and the ratio of average total demand/capacity per period was varied between 1.75 and 5.6 (i.e., the capacity utilization varied between approximately 29% and 93%).

## 6.2. Computational results

For a number of the problems tested, we found that most of the inequalities that were added to tighten the linear program were either of the type

$$\sum_{i=1}^t y_i \geq \lceil d_{1t}/C \rceil \quad (6.8)$$

or of the type

$$\sum_{i=1}^t x_i + d_{t+1} y_{t+1} \geq d_{1,t+1}. \quad (6.9)$$

Note that constraints (6.8) and (6.9) are both special cases of the facet inequality (f). If  $P = \{1, \dots, t\}$ ,  $Q = \emptyset$  and  $D = \emptyset$ , then (f) reduces to (6.8), whereas if  $P = \{1, \dots, t+1\}$ ,  $Q = \emptyset$ ,  $R = \emptyset$  and  $D = \{t+1\}$ , then (f) reduces to (6.9). Recall also that constraint (6.9) is a stronger formulation of the demand constraint of the lot size model. Therefore, to determine how much these inequalities tighten the linear program, we also solved the linear programming relaxation with both (6.8) and (6.9) included for all the products and all the time periods. In this case, we replaced the original demand constraints by (6.9). Tables 6.1 and 6.2 summarize the results for all the test problems. In these tables,  $v(\text{LP})$  refers to the optimal value of the linear programming relaxation,  $v(\text{LP1})$  to the optimal value of the linear program with both (6.8) and (6.9) included,  $v(\text{LP2})$  to the optimal value of the linear program after addition of all the facet inequalities that are generated by the cutting plane procedure, and  $v(\text{IP})$  to the optimal value of the original problem. For comparison purposes, we also implemented a strong cutting plane algorithm that uses facets of the uncapacitated lot size problem (inequality (‡)) derived by Barany, Van Roy and Wolsey.  $v(\text{LP3})$  refers to the optimal value of the linear program after addition of all facet inequalities of the uncapacitated problem that are generated by the cutting plane procedure. Recall that constraints (6.9) are also facets of the uncapacitated problem and we replaced the demand constraints by (6.9) when implementing the strong cutting plane procedure to obtain  $v(\text{LP3})$ .

The results in Table 6.1 show that the facet inequalities (f) are very effective in reducing the integrality gap for the multiple item capacitated lot size problem. In fact, note that the addition of only constraints (6.8) and (6.9) reduces the gap significantly. Moreover, when capacity is tight, an implementation using facets of the capacitated problem produces a much smaller integrality gap than does an implementation using facets of the uncapacitated problem. Indeed, for several of these problems (for example, LS1.1, LS1.2, LS1.8, LS2.1) inequalities (6.8) and (6.9) reduce the gap as much as all the facets of the uncapacitated problem. On the other

Table 6.1  
Reduction in Integrality gap for capacitated lot size problems with addition of facet inequalities

Problem name	No. of products	No. of periods	Utilization	LP value	LP value with (6.8) & (6.9)	LP value after capac. cuts (f)	LP value after uncapac. cuts (\$)	Optimal IP value	Integrality gap (%)	Reduction in gap by (6.8 & (6.9) (%)	Reduction in gap by ineq. (f) (%)	Reduction in gap by ineq. (\$) (%)
				$v(LP)$	$v(LP1)$	$v(LP2)$	$v(LP3)$	$v(IP)$	$\frac{v(IP) - v(LP)}{v(IP)}$	$\frac{v(LP1) - v(LP)}{v(IP) - v(LP)}$	$\frac{v(LP2) - v(LP)}{v(IP) - v(LP)}$	$\frac{v(LP3) - v(LP)}{v(IP) - v(LP)}$
LS1.1	1	8	7.30	12 970.0	13 320.0	13 320.0	13 206.3	13 320.0	2.6	100.0	100.0	67.6
LS1.2	1	8	4.87	11 423.3	12 612.7	12 642.7	12 603.4	12 730.0	10.3	91.0	93.3	90.3
LS1.3	1	8	2.92	10 651.6	12 365.1	12 490.0	12 490.0	12 490.0	14.7	93.2	100.0	100.0
LS1.4	1	8	0.97	10 165.2	12 267.9	12 490.0	12 490.0	12 490.0	18.6	90.4	100.0	100.0
LS1.5	1	12	8.00	2 699.2	2 788.3	2 802.4	2 789.1	2 802.4	3.7	86.5	100.0	87.1
LS1.6	1	12	6.67	2 563.3	2 690.2	2 702.8	2 700.2	2 702.8	5.2	91.0	100.0	98.1
LS1.7	1	12	5.45	2 457.7	2 635.4	2 650.4	2 649.2	2 650.4	7.3	92.2	100.0	99.4
LS1.8	1	15	11.18	8 861.0	9 168.9	9 208.6	9 149.8	9 226.0	4.0	84.4	95.2	79.1
LS1.9	1	15	7.39	5 307.3	5 602.4	5 641.3	5 605.3	5 649.0	6.0	86.4	97.7	87.2
LS1.10	1	15	3.69	3 195.5	3 454.8	3 514.0	3 514.0	3 514.0	9.1	81.4	100.0	100.0
LS2.1	2	8	3.65	12 283.1	13 657.7	13 681.1	13 657.0	13 710.0	10.4	96.3	98.0	96.3
LS2.2	2	8	2.43	11 363.3	13 304.4	13 630.0	13 630.0	13 630.0	16.6	85.6	100.0	100.0
LS2.3	2	12	5.57	6 960.6	7 254.4	7 330.4	7 273.5	7 427.0	6.3	63.0	79.3	67.1
LS2.4	2	12	4.08	5 518.9	6 002.0	6 088.5	6 082.1	6 136.0	10.1	78.3	92.3	91.3
LS2.5	2	12	3.20	3 975.9	4 394.8	4 511.5	4 511.5	4 519.0	12.0	77.1	98.6	98.6
LS2.6	2	12	1.74	2 661.1	3 010.8	3 098.0	3 098.0	3 098.0	14.1	80.0	100.0	100.0
LS4.1	4	8	1.83	12 663.5	14 997.0	15 732.5	15 732.5	16 030.0	21.0	69.3	91.2	91.2
LS4.2	4	8	1.22	12 203.3	14 726.3	15 448.3	15 448.3	15 500.0	21.3	76.5	98.6	98.6

Table 6.2  
Comparison of different solution methods for capacitated lot size problems

Problem name	No. of products	No. of periods	Cutting plane algorithm with facets (f) of capacitated model			Branch and bound with (6.8) and (6.9) included		Cutting plane algorithm with facets (‡) of uncapacitated model	
			No. of cuts added	No. of branches in B&B	CPU time (sec)	No. of branches in B&B	CPU time (sec)	No. of cuts added	
LS1.1	1	8	0	0	11	0	9	1	
LS1.2	1	8	8	2	28	4	9	1	
LS1.3	1	8	12	7	29	9	13	6	
LS1.4	1	8	5	9	18	6	10	6	
LS1.5	1	12	10	15	63	7	18	2	
LS1.6	1	12	11	6	49	8	17	6	
LS1.7	1	12	8	5	37	3	14	6	
LS1.8	1	15	29	32	271	23	31	9	
LS1.9	1	15	23	10	145	11	30	1	
LS1.10	1	15	60	24	1023	40	54	41	
LS2.1	2	8	12	5	99	5	40	9	
LS2.2	2	8	15	18	70	13	36	12	
LS2.3	2	12	45	160	2478	197	479	19	
LS2.4	2	12	43	30	738	91	181	23	
LS2.5	2	12	47	26	518	25	100	48	
LS2.6	2	12	28	25	290	45	77	41	
LS4.1	4	8	32	350	2427	852	2828	58	
LS4.2	4	8	24	102	486	548	977	67	

hand, for the four product problems the facets of the capacitated problem do not tighten the linear programming relaxation any more than the facets of the uncapacitated problem. This result probably reflects the fact that for both the 4 product problems, the ratio of average total demand to capacity per period is less than 2, i.e., the capacity restrictions are not tight.

Table 6.2 summarizes other performance measures: the number of cuts added to the initial linear program in the cutting plane procedures, and the number of branches generated in the branch and bound tree to obtain an optimal integer solution. The results in Table 6.1 suggest that adding constraints (6.8) and (6.9) a priori to the linear programming relaxation is very effective in reducing the integrality gap. To test whether it may be worthwhile to invoke branch and bound directly after solving the linear program with (6.8) and (6.9) included, we implemented this procedure as well; Table 6.2 compares the CPU times for the different methods. From this table, we see that for all the 1 and 2 product problems, it is faster to use branch and bound directly rather than first seeking the cutting plane procedure. However, the reverse is true for the 4 product problems. This result suggests that other classes of facet inequalities might be effective in tightening the linear programming relaxation in general and that we should consider adding these inequalities a priori and then using branch and bound directly instead of a cutting plane procedure. Tables 6.1 and 6.2 also show that for a majority of the 1 and 2 product problems, using facets (f) of the capacitated model adds more cuts than does an implementation using facets ( $\ddagger$ ) of the uncapacitated model; however, adding inequalities (f) reduces the integrality gap more than adding inequalities ( $\ddagger$ ). For the 4 product problems, the facets of the capacitated problem reduce the integrality gap as much as the facets of the uncapacitated problem; however, the cutting plane routine generates fewer of them. Thus, inequalities (f) seem to be more effective than the corresponding inequalities of the uncapacitated model in solving multi-item capacitated problems.

Table 6.3 summarizes information about the average integrality gaps for the problems we solved.

The computational results suggest that replacing the usual formulation of the demand constraints by (6.9) in lot size problems (and in more complex problems that contain the lot size model as a substructure) and including constraints (6.8)—which state that for every product  $i$  and a given time period  $t$ —the facility must incur at least  $\lceil d_{i,t}/C \rceil$  setups for product  $i$  by time period  $t$ , may yield significantly stronger formulations whenever capacity is tight. This experience also suggests that we might be able to avoid completely the more complex cutting plane approach and simply add constraints (6.8) and (6.9) to the model a priori and use a conventional branch and bound procedure. However, we need to explore in more detail whether any additional facet inequalities should also be added a priori.

An important issue that needs to be resolved is how to construct good feasible solutions from the fractional solutions obtained at the end of the cutting plane routine, since for these problems it takes a large number of branches to find an optimal integer solution even though the integrality gap after the addition of the



Table 6.3

Average integrality gaps for the capacitated lot size problems

No. of products	Average (%)	Average reduction in gap by inequalities (6.8) and (6.9) (%)	Average reduction in gap by inequalities (f) (%)	Average reduction in gap by inequalities ( $\ddagger$ ) (%)
	$\frac{v(\text{IP}) - v(\text{LP})}{v(\text{IP})}$	$\frac{v(\text{LP1}) - v(\text{LP})}{v(\text{IP}) - v(\text{LP})}$	$\frac{v(\text{LP2}) - v(\text{LP})}{v(\text{IP}) - v(\text{LP})}$	$\frac{v(\text{LP3}) - v(\text{LP})}{v(\text{IP}) - v(\text{LP})}$
1	8.2	89.7	98.6	90.9
2	11.6	80.1	94.7	92.2
4	21.2	72.9	94.8	94.8

facet inequalities is fairly small (see, for example, problem LS4.1). Unfortunately, the structure of the fractional solutions does not suggest a simple heuristic for this purpose. We could just round up all the fractional  $y$  variables to 1; however, in general this approach does not yield a good upper bound.

## 7. Conclusions

Our objective in this paper has been to study the polyhedral structure of the capacitated lot size model, a prototypical model in the production planning literature, and to subsequently use the results to develop efficient solution methods for problems that contain this model as a substructure. Our research was motivated by the following observations. First, even though this problem has been the focus of extensive study, available results from the literature are able to solve only special cases. Second, like many integer programming problems, the lot sizing problem can be formulated in several different ways and results about the structure and properties of alternative formulations should provide useful insights about modeling. Finally, the success of cutting plane procedures using facet inequalities in other problem domains indicates that these methods can be very effective in solving large integer programming problems to optimality in reasonable computation times.

Our study focused on examining the structure of the single item, single resource, constant capacity lot size model. We derived a class of non-trivial facets of the problem, which generalize the facet inequalities for the uncapacitated lot size problem, and showed how to solve efficiently the separation problem for this new class of inequalities. We then implemented and tested a strong cutting plane procedure for the multiple item problem. Our computational results for this problem, which show that these facet inequalities considerably reduce the integrality gap, have been very encouraging. Our results further indicate that adding even special cases—simple inequalities (6.8) concerning number of set-ups and an alternate

(stronger) formulation (6.9) of the demand constraints—of the facet inequalities a priori to the linear programming relaxation considerably reduces the integrality gap. This conclusion suggests that replacing the usual formulation of the demand constraints by the stronger formulation (6.9) and including the set-up inequalities (6.8) in the formulation of more complex problems that contain the lot size model as a substructure may yield stronger formulations in other settings as well.

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