Management Science

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

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MANAGEMENT SCIENCE
Vol. 30, No. 10, October 1984
Printed in U.S.A.

STRONG FORMULATIONS FOR MULTI-ITEM CAPACITATED LOT SIZING*

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Multi-item capacitated lot-sizing problems are reformulated using a class of valid inequalities, which are facets for the single-item uncapacitated problem. Computational results using this reformulation are reported, and problems with up to 20 items and 13 periods have been solved to optimality using a commercial mixed integer code. We also show how the valid inequalities can easily be generated as part of a cutting plane algorithm, and suggest a further class of inequalities that is useful for single-item capacitated problems.
(INVENTORY/PRODUCTION—LOT SIZING)

The aim of this paper is to present a new approach to the solution of capacitated lot-sizing problems:

$$\min \sum_{i=1}^{I} \sum_{t=1}^{T} (p_{it}s_{it} + c_{it}x_{it} + f_{it}y_{it}), \quad s_{i,t-1} + x_{it} = d_{it} + s_{it} \qquad \forall i, t,$$

$$\sum_{i=1}^{I} x_{it} \leq L_t \qquad \forall t, \qquad x_{it} \leq L_t y_{it} \qquad \forall i, t, \tag{$\mathscr{P}$}$$

$$x_{it}, s_{it} \geq 0, \qquad y_{it} \in \{0, 1\} \qquad \forall i, t,$$

where x_{it}, s_{it} represent production level and end stock of item i in period t, $y_{it} \in \{0, 1\}$ indicates whether a set-up cost must be incurred for item i in period t (i.e. $x_{it} > 0$ implies $y_{it} = 1$), d_{it}, p_{it}, c_{it}, f_{it} are the demand, storage, production and set-up costs respectively, and L_t is the machine capacity in period t.

The approach we take is to reformulate ($\mathscr{P}$) by the addition of strong valid inequalities, with the aim of obtaining a good approximation of the convex hull of solutions of ($\mathscr{P}$). The reformulated problem is then tackled using a branch and bound code. This approach has the practical advantage that generally available (mixed integer) linear programming software can be used, and it remains a valid approach when ($\mathscr{P}$), or variants of ($\mathscr{P}$), form part of a more complex production and inventory model.

It is well known that ($\mathscr{P}$) is a well-solved problem in the constant capacity, single-item case, i.e. when $I = 1$ and $L_t = L$ for all t. In particular in the uncapacitated case when $L_t = +\infty$ for all t, the Wagner–Whitin algorithm is a well-known and efficient solution procedure. The fact that there is an efficient algorithm suggests on theoretical grounds, see Grötschel, Lovasz and Schrijver (1981), that one can probably find a useful description of the convex hull of solutions in this special case.

We now briefly describe the contents of this paper. In §1 we describe a class of valid inequalities for the single-item uncapacitated model. In §2 almost all of the inequalities are then shown to be facets, which means that any individual inequality cannot be strengthened. What is more, we state a result, proved in a companion paper (Barany et al. 1983), that the class of inequalities completely describes the convex hull. This means essentially that no other valid inequalities are needed for this special problem, and it can be solved by LP.

In §3 we show how inequalities of the class can be very easily generated in a cutting plane procedure. This leaves one with the choice of either reformulating the problem initially by adding some or all of the inequalities, or just generating those that are needed as cutting planes.

In §4 we describe how the valid inequalities for the single-item uncapacitated problem were used to reformulate the multi-item capacitated problem ($\mathscr{P}$). In particular we solve to optimality a variety of problems with up to 20 items and 13 time periods.

*Accepted by Marshall L. Fisher, former Departmental Editor; received April 5, 1983. This paper has been with the authors 3 weeks for 1 revision.

0025-1909/84/3010/1255$01.25

1. Valid Inequalities for Lot-Sizing Problems

Here we consider the set of feasible solutions to the uncapacitated problem:

$$
\begin{aligned}
X_T = \{(x, y, s) \in R^{3T}: \quad & x_1 = d_1 + s_1 & \\
& s_{t-1} + x_t = d_t + s_t, & t = 2, \dots, T, \\
& s_T = 0, & \\
& x_t \leqslant d_{tT} y_t, & t = 1, \dots, T, \\
& x_t, s_t \geqslant 0, \quad y_t \in \{0, 1\}, & t = 1, \dots, T\}
\end{aligned}
$$

where d_{il} denotes $\sum_{t=i}^{l} d_t$. Note that X_T appears as a substructure of most (capacitated, hierarchical, etc.) lot-sizing problems.

Below we shall describe a family of valid inequalities for X_T. It is first worth noticing that it is possible to eliminate the "stock" variables s_t from the description of X_T, giving the set $X_T^* \subset R^{2T}$ defined by the following inequalities:

$$\sum_{t=1}^{i} x_t \geqslant d_{1i}, \qquad i = 1, \dots, T-1, \tag{1}$$

$$\sum_{t=1}^{T} x_t = d_{1T}, \tag{2}$$

$$x_t \leqslant d_{tT} y_t, \qquad i = 1, \dots, T, \tag{3}$$

$$x_t \geqslant 0, \qquad i = 1, \dots, T, \tag{4}$$

$$0 \leqslant y_t \leqslant 1, \qquad i = 1, \dots, T, \tag{5}$$

$$y_t \text{ integer}, \qquad i = 1, \dots, T. \tag{6}$$

THEOREM 1. *For any* $1 \leqslant l \leqslant T$, $L = \{1, \dots, l\}$, *and* $S \subseteq L$, *the inequality* $\sum_{t \in S} x_t + \sum_{i \in L \setminus S} d_{il} y_i \geqslant d_{1l}$ *is a valid inequality for* X_T, *or* X_T^*.

PROOF. Given a point $(x, y) \in X_T^*$, suppose that $y_i = 0 \ \forall i \in L \setminus S$. Then

$$\sum_{i \in S} x_i + \sum_{i \in L \setminus S} d_{il} y_i = \sum_{i=1}^{l} x_i \geqslant d_{1l} \qquad \text{as} \quad x_i = 0 \quad \forall i \in L \setminus S.$$

Suppose on the contrary that $k = \arg\min_i \{i \in L \setminus S, y_i = 1\}$. Then as before $y_i = 0$ and hence $x_i = 0 \ \forall i \in (L \setminus S) \cap \{1, \dots, k-1\}$. Hence

$$\sum_{i \in S} x_i + \sum_{i \in L \setminus S} d_{il} y_i \geqslant \sum_{i=1}^{k-1} x_i + d_{kl} \geqslant d_{1k-1} + d_{kl} = d_{1l}. \quad \text{Q.E.D.}$$

An alternative way to write the above inequality that is useful computationally is given by:

COROLLARY. *The valid inequality of Theorem* 1 *can be written as*: $\sum_{i \in L \setminus S} x_i \leqslant \sum_{i \in L \setminus S} d_{il} y_i + s_l$.

PROOF. Substitute $s_l = \sum_{i=1}^{l} x_i - d_{1l}$. Q.E.D.

2. Facets for the Single-Item Uncapacitated Lot-Sizing Problem

Here we show that almost all the (l, S) inequalities are facets of conv(X_T^*), which means they are necessary if we wish to describe conv(X_T^*) by a system of linear inequalities. First we need to consider the dimension of the solution set X_T^*.

PROPOSITION 2. *If* $d_t > 0$, $t = 1, \ldots, T$, $\dim(X_T) = \dim(X_T^*) = 2T - 2$.

PROOF. As $d_1 > 0$, all points in X_T^* satisfy both $y_1 = 1$ and $\sum_{i=1}^{T} x_i = d_{1T}$ and therefore $\dim(X_T^*) \leqslant 2T - 2$. We now exhibit $2T - 1$ affinely independent points in X_T^*. For $j = 1, \ldots, T$, set $x_t = d_t, y_t = 1$, $t < j$; $x_j = d_{jT}, y_j = 1$; $x_j = y_j = 0, j > t$. For $j = 2, \ldots, T$, set $x_1 = d_{1T}, y_1 = 1, y_j = 1, x_j = 0, x_t = y_t = 0$ otherwise. Q.E.D.

THEOREM 3. *If $d_t > 0$, $t = 1, \ldots, T$, the (l, S) inequality defines a facet of X_T whenever $l < T$, $1 \in S$ and $L \backslash S \neq \emptyset$. All these facets are distinct.*

PROOF. Let $k = \arg\min\{i \in L \backslash S\} > 1$. Consider first the points on the (l, S) inequality with $x_i = y_i = 0$ $\forall i \in \{k, \ldots, l\}$. Consider the problem for periods $1, \ldots, k-1$ with $d_i' = d_i$, $i = 1, \ldots, k-2$, $d_{k-1}' = d_{k-1,l}$. By Proposition 2 we obtain $2(k-1) - 1$ affinely independent solutions, $(x_A^p, y_A^p) \in R^{2(k-1)}$. Similarly considering the problem for periods $l+1, \ldots, T$, we obtain $2(T-l) - 1$ affinely independent solutions $(x_B^q, y_B^q) \in R^{2(T-l)}$. Combining these vectors and inserting $x_t = y_t = 0$, $i \in \{k, \ldots, l\}$ gives $2T - 2(l-k) - 5$ affinely independent solutions $(x_A^1, y_A^1, \underline{0}, \underline{0}, x_B^q, y_B^q)_{q=1}^{2(T-l)-1}$ and $(x_A^p, y_A^p, \underline{0}, \underline{0}, x_B^1, y_B^1)_{p=2}^{2(k-1)-1}$ with $s_l = 0$.

We now exhibit two new affinely independent solutions for each $j \in \{k, \ldots, l\}$. For given j, we take $x_i = y_i = 0$ $\forall i \in \{k, \ldots, j-1\}$.

Case 1. $j \in L \backslash S$. Take the solution with $x_j = d_{jl}$, $y_j = 1$, $x_i = y_i = 0$, $i = j+1, \ldots, l$ and $s_l = 0$, and the solution with $x_j = d_{j,l+1}$, $y_j = 1$, $x_i = y_i = 0$, $i = j+1, \ldots, l$ and $s_l = d_{l+1}$. It is clear how each of these vectors can be extended to a vector $(x, y) \in X_T^*$.

Case 2. $j \in \{k, \ldots, l\} \cap S$. Take the first solution in the same way as in Case 1. To get a second solution, take $x_1 = d_{1l}$, $y_2 = 1$, $x_j = 0$, $y_j = 1$, $x_t = y_t = 0$, $t \in L - \{1, j\}$, $x_t = d_t$. $y_t = 1$, $t \geqslant l+1$. A final solution is obtained by modifying the second solution of Case 1 with $j = k$ by setting $y_{l+1} = 1$.

We have exhibited $(2T - 2(l-k) - 5) + 2(l - k + 1) + 1 = 2T - 2$ affinely independent solutions, and hence the inequality is a facet.

It is readily seen that none of the inequalities differs just by a multiple of $y_1 = 1$ and $\sum_{t=1}^{T} x_t = d_{1T}$, and hence the facets are distinct. Q.E.D.

PROPOSITION 4. *The inequalities $x_t \leqslant d_{tT} y_t$, $t = 2, \ldots, T$, define distinct facets of X_T if $d_t > 0$, $t = 1, \ldots, T$.*

PROOF. $\dim\{(x, y) \in X_T^* : x_t = y_t = 0\} = \dim(X_T^*) - 2$ if $t > 1$. This gives $2T - 3$ affinely independent solutions with $x_t = y_t = 0$. Any point with $x_t = d_{tT}$, $y_t = 1$, $y_j = x_j = 0, j > t$, is independent of these, and hence these $2T - 2$ points define a facet.
Q.E.D.

PROPOSITION 5. *The inequalities $y_t \leqslant 1$, $t = 2, \ldots, T$, define facets.*

Even though it is not strictly necessary for the computational work described below, the fact that the (l, S) inequalities are facets is something of a guarantee of their value as cutting planes. It is particularly reassuring if one knows that these are all the facets, so one can be sure not to be missing some cuts that might be even more effective. The following result, proved in a companion paper, provides this guarantee.

Let P_T be the polyhedron defined as: $\{(x, y) \in R^{2T}$ satisfying (2), (4), (5) and $\sum_{i \in S} x_i + \sum_{i \in L \backslash S} d_{il} y_i \geqslant d_{1l}$ $\forall l, S\}$.

THEOREM (Barany et al. 1983). $P_T = \mathrm{conv}(X_T^*)$.

This also means that the linear program: $\max\{cx + fy : (x, y) \in P_T\}$ always gives optimal extreme point solutions with y integer, and therefore solves the uncapacitated lot-sizing problem.

3. The Separation Problem for X_T^*

Given the class of (l, S) inequalities that we have obtained, there appear to be two obvious ways in which they might be used.

The first is to reformulate the problem a priori by adding some or all of the (l, S) inequalities in the description of X_T or X_T^*. This is essentially the approach we take in §4. If we choose to add all the inequalities, then we have reformulated the problem as: $\max\{cx + fy : (x, y) \in P_T, y \text{ integer}\}$.

The second approach is to introduce the (l, S) inequalities as cutting planes. To implement this approach we need to solve the "*Separation Problem*" for P_T, namely given a point $(x^*, y^*) \in R^{2T}$ satisfying (1)–(5), find an (l, S) inequality cutting it off, or decide that $(x^*, y^*) \in P_T$.

The Separation Algorithm

Given (x^*, y^*) satisfying (1) – (5), for $l = 1, \dots, T$, find

$$S_l \subseteq L = \{1, \dots, l\} \qquad \text{where} \quad i \in S_l \quad \text{if} \quad x_i^* \leqslant d_{il} y_i^*$$
$$\text{and} \quad i \in L \setminus S_l \quad \text{if} \quad x_i^* > d_{il} y_i^* .$$

Check if $\sum_{i \in L \setminus S_l} d_{il} y_i^* < d_{1l}$. If so, the (l, S_l) inequality is violated. Q.E.D.

If no violation has been found, each of the (l, S) inequalities is satisfied, because for each l,

$$\min_{S \subseteq L} \left\{ \sum_{i \in S} x_i^* + \sum_{i \in L \setminus S} d_{il} y_i^* - d_{1l} \right\} = \sum_{i \in S_l} x_i^* + \sum_{i \in L \setminus S_l} d_{il} y_i^* - d_{1l} \geqslant 0.$$

Hence $(x^*, y^*) \in P_T$ by Theorem 6.

This algorithm can obviously be used as part of a very simple cutting plane algorithm, and note that if one keeps adding cuts, one terminates with an optimal solution to the linear program: $\max\{cx + fy : (x, y) \in P_T\}$.

4. Practical Solutions to Lot-Sizing Problems

Consider now the multi-item capacitated lot-sizing problem, which was formulated in the introduction as:

$$Z = \min \sum_i \sum_t (p_{it} s_{it} + c_{it} x_{it} + f_{it} y_{it}) \quad \text{s.t.}$$
$$(x_{it}, y_{it}, s_{it}) \in X_T^i, \quad i = 1, \dots, I, \quad \sum_i x_{it} \leqslant L_t, \quad t = 1, \dots, T, \qquad (\mathscr{P})$$

where X_T^i denotes the set of feasible solutions to the uncapacitated problem for item i.

Our earlier results tell us that $(\mathscr{P})$ can be reformulated as:

$$Z = \min \sum_i \sum_t (p_{it} s_{it} + c_{it} x_{it} + f_{it} y_{it}) \quad \text{s.t.}$$
$$(x_{it}, y_{it}, s_{it}) \in P_T^i, \quad i = 1, \dots, I, \quad \sum_i x_{it} \leqslant L_t, \quad t = 1, \dots, T, \quad y_{it} \text{ integer} \quad \forall i, t, \qquad (\mathscr{P}')$$

where $P_T^i = \operatorname{conv}(X_T^i)$.

Let $(\mathscr{L}\mathscr{P}')$ with optimal value $Z'_{\mathscr{L}\mathscr{P}}$ be the linear programming relaxation of $(\mathscr{P}')$.

$(\mathscr{L}\mathscr{P}')$ can be solved in various ways. The Separation Algorithm of the previous section provides a cutting plane algorithm. Lagrangean relaxation, dualising the capacity constraints, also leads to the optimal value $Z'_{\mathscr{L}\mathscr{P}}$. However, the main points to emphasise are:

(i) we obtain a strong lower bound $Z'_{\mathscr{L}\mathscr{P}}$ by solving $(\mathscr{L}\mathscr{P}')$ which is impossible with most of the heuristics used to date;

(ii) the reformulation $(\mathscr{P}')$ permits us to solve to optimally some problems that had previously appeared insolvable.

The approach we have tested computationally is of adding inequalities to the initial formulation, and then solving the reformulated problem using commercial MIP software. This avoids the development of any special purpose code.

To illustrate the above, the choice of inequalities to add was made on the following grounds:

(a) The number of facets is exponential in T, and hence in practice a subset must be selected.

(b) The relative importance of the facets (in terms of cutting strength) appears to decrease as $k = l - \alpha$ increases, where $\alpha = \arg\min\{i \in L \setminus S\}$. Therefore the most important are those with $k = 0$.

$k = 0$, $x_l \leqslant d_l y_l + s_l$, or written differently $\sum_{i=1}^{l-1} x_i + d_l y_l \geqslant d_{1l}$,

$k = 1$, $x_{l-1} + x_l \leqslant d_{l-1,l} y_{l-1} + d_l y_l + s_l$, and $x_{l-1} \leqslant d_{l-1,l} y_{l-1} + s_l$.

(c) For the problems tested, only inequalities with $S = \{1, \ldots, l-k-1\}$, $L \setminus S = \{l-k, \ldots, l\}$ were generated, so that the 2nd inequality for $k = 1$ (above) is not used.

(d) It follows that adding inequalities for $k \leqslant k^*$, $I[T + (T-1) + \cdots + (T-k^*)] = O(k^* IT)$ inequalities are added.

Both multiple and single item test problems are considered. The multiple item problems were a set of four 8-item, 8-period problems from Thizy and Van Wassenhove (1982), a 20-item, 13-period problem from Dixon and Silver (1981), and some 20-item, 12-period problems from Graves (1982). Both in Thizy and Van Wassenhove (1982) and Graves (1982) the authors used Lagrangean relaxation. The single item problems are variations on a problem from Peterson and Silver (1979) with differing capacities.

All problems were solved on a Data General MV8000 using the SCICONIC mixed integer programming software. This MV8000 is roughly 6 times slower than an IBM 3033U for this kind of calculation.

The strategy adopted for the 8×8 and the 20×12 problems was to add inequalities of the type described above for $k \leqslant k^*$, solve the linear program, drop the inactive rows, and then carry out branch and bound. For the 20×13 problem we demonstrate the effect of adding the (l, S) inequalities for different values of k^*. The computational results are given in Tables 1, 2, and 3. The number of simplex pivots, the CPU time and the number of nodes in the branch and bound tree are displayed at four states: at the LP optimum, at the first integer solution, at the optimal integer solution and at termination. Both the LP and the Branch and Bound were run using the default options of SCICONIC.

For the single item uncapacitated problems we know from Theorem 6 that it suffices to add all the (l, S) inequalities. For these problems we examined how large $k = l - \alpha$ needed to be to obtain an integer LP solution. For the single item capacitated problems, we also added a priori the inequalities described below:

PROPOSITION 8 (Van Roy and Wolsey 1983). *If* $\lambda = \sum_{i=t}^{l} L_i - d_{tl} > 0$, $\sum_{i=t}^{l} x_i + \sum_{i=t}^{l} (L_i - \lambda)^+ (1 - y_i) \leqslant d_{tl} + s_l$ *is a Valid Inequality for* $X_T \cap \{(x_t, y_t) : x_t \leqslant L_t \ \forall t\}$, *where* $(Z)^+$ *denotes* $\max\{Z, 0\}$.

As Table 4 shows, we always obtained integer solutions to the linear program for these few examples. For some comparative computational results, see Baker *et al.* (1978).

TABLE 1

8 × 8 *Problems* (*Thizy and Van Wassenhove* 1982)[3]

	Capacity	Rows[1]	Columns	LP value	pivots	secs	1st IP value	pivots[2]	secs[2]	nodes	Optimal IP value	pivots[2]	secs[2]	nodes	search completed pivots[2]	secs[2]	nodes
A	[350, 500]	288	184	7996.7	337	37	8500	156	40	24	8430	3921	940	707	8658	2314	1934
B	400	270	184	7722.3	282	23	7910	76	24	15	= 1st IP				1809	446	421
C	500	279	184	7534.2	228	19	7800	68	32	23	7610	223	91	106	223	94	112
D	600	280	184	7464.2	214	17	7570	33	17	11	7520	38	22	15	115	43	39

[1]Rows = total number of rows for $k^* = 3$ (= 345) less the number of (l, S) inequalities inactive in the LP.
[2]Total number of pivots/secs are obtained by adding these values to the pivots/secs for the LP.
[3]In Thizy and Van Wassenhove (1982) no solutions were given for A, B and C.

TABLE 2

20 × 13 *Problem* (*Dixon and Silver* 1981)

Solution Strategy	Rows	Columns	LP value	pivots	secs	1st IP value	pivots[1]	secs[1]	nodes	Optimal IP value	pivots[1]	secs[1]	nodes	search completed pivots[1]	secs[1]	nodes
No (l, S) in-equalities	534	760	2526.4	356	35	6233.0	1095	355	257	Not attempted						
$k^* = 0$	794	760	5364.5	502	66	6016.9	126	120	52	Not attempted						
$k^* = 1$	1034	760	5656.6	578	97	5840.8	90	93	26	Not attempted						
$k^* = 2$	1254	760	5661.6	650	134	5807.6	48	102	24	= 1st IP				487	798	213

[1]Total number of pivots/secs are obtained by adding these values to the pivots/secs for the LP.

TABLE 3

20 × 12 *Problems* (*Graves* 1982)

	Resource Coverage	Set up Cost	Rows	Columns	LP value	pivots	secs	1st IP value	pivots[3]	secs[3]	nodes	Optimal IP value	pivots[3]	secs[3]	nodes	Search completed pivots[3]	secs[3]	nodes
1	80%	low	1153[1]	712	15867	1096	236	15870	12	26	10	= 1st IP				19	51	21
2	120%	low	1153[1]	712	6484	812	167	6484	0	3	1	= 1st IP				0	3	1
3	80%	high	1088[2]	712	59148	1238	681	59684	585	853	60	59657[4]	1288	1789	142	4864	5848	490[4]
4	120%	high	1137[2]	712	54518	2017	1141	54673	109	589	41	54538	343	1228	94	555	2020	212

[1]Total number of rows for $k^* = 2$.
[2]Total number of rows for $k^* = 12$ (= 2053) less the number of (l, S) inequalities inactive in the LP.
[3]Total number of pivots/secs are obtained by adding these values to the pivots/secs for the LP.
[4]Best solution found before search was truncated at node 490.

TABLE 4

1 × 12 *Problems* (*Peterson and Silver* 1979)

	Capacity	k^*	Rows	Cols.	LP value	pivots	secs	1st IP value	total pivots	total secs	nodes
U	∞	No	37	35	140.2	24	0.5	501.2	39	1.9	17
		0	49	35	479.8	36	0.7	501.2	38	1.2	3
		1	60	35	501.2	38	0.8	501.2	38	0.8	1
A	150	No	37	35	576.0	28	0.5	703.6	36	1.5	10
		0	49	35	665.1	35	0.7	679.2	38	1.3	4
		1	66	35	670.4	41	0.9	679.2	43	1.4	2
		2	80	35	670.4	40	1.1	679.2	41	1.5	2
		4	102	35	679.2	42	1.5	679.2	42	1.5	1
B	180	3	90	35	579.6	55	1.7	579.6	55	1.7	1
C	220	1	65	35	527.2	40	0.9	527.2	40	0.9	1
D	[150, 220]	3	91	35	605.2	52	1.6	605.2	52	1.6	1
E	[80, 220]	3	102	35	643.2	58	1.9	643.2	58	1.9	1

5. Conclusions

It appears that the (l, S) inequalities provide a valuable computational tool in the formulation and resolution of lot-sizing problems, and this should also hold for more complicated models with embedded lot-sizing problems. However it is clearly important to obtain even stronger valid inequalities for the multi-item capacitated problem that take into account the capacity constraints. Other extensions to include models with backlogging and multiple stages are under investigation. We are also planning to test an alternative formulation based on a simple plant location model due to Krarup and Bilde (1977), which is used in one of our proofs of Theorem 6. This formulation leads to a model with $O(IT^2)$ constraints and variables for the problem $(\mathscr{P})$.[1]

[1] Research supported by the Projet d'Action Concertée.

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