A SURVEY OF MAINTENANCE MODELS: THE CONTROL AND SURVEILLANCE OF DETERIORATING SYSTEMS*

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ABSTRACT

The literature on maintenance models is surveyed. The focus is on work appearing since the 1965 survey, "Maintenance Policies for Stochastically Failing Equipment: A Survey" by John McCall and the 1965 book, *The Mathematical Theory of Reliability*, by Richard Barlow and Frank Proschan. The survey includes models which involve an optimal decision to procure, inspect, and repair and/or replace a unit subject to deterioration in service.

INTRODUCTION

For nearly two decades, there has been a large and continuing interest in the study of maintenance models for items with stochastic failure. This interest has its roots in many military and industrial applications. Lately, however, new applications have arisen in such areas as health, ecology, and the environment. Although it is not possible to detail these many applications of maintenance models, some of them are: the maintenance of complex electronic and/or mechanical equipment, maintenance of the human body, inspection and control of pollutants in the environment, and maintenance of ecological balance in populations of plants and animals.

Just as the interest in maintainability has grown and changed, so has the sophistication of the models and control policies for solving the maintenance problems. Many of the important early maintenance and/or inspection models possessed an elegance and simplicity which led to easily implementable policies. These models were later generalized; and although much of the elegance and simplicity remains, some of the new results are complex and require the use of large computers for implementation.

In two earlier and excellent works, the area of maintainability was rather thoroughly researched and surveyed up to 1965. These works are: *The Mathematical Theory of Reliability*, by Richard Barlow and Frank Proschan [1965] and the paper, "Maintenance Policies for Stochastically Failing Equipment: A survey," by John McCall [1965]. Here, the presentations of these earlier works will not be repeated, but rather the coverage is primarily the results which have appeared since 1965. There are a few exceptions to this rule, however, when it was necessary to review some key models which appeared prior

to 1965 and which formed the foundation of many of the later studies. Furthermore, in this survey it is assumed the reader is familiar with such concepts as Markov processes, Poisson processes, dynamic programming, linear programming, Lagrange multipliers, and other concepts and techniques with which the holder of a master's degree or an experienced practitioner in statistics or operations research would be familiar.

Since there are many threads of activity which contribute to the total fabric of maintainability and since it is not possible to include all of them in a reasonably sized survey paper, this study includes only those models which involve an optimal *decision* to procure, inspect, repair and/or replace a unit subject to deterioration in service. Not included are models which describe the operating characteristics of a system, such as repairmen or machine interference models, unless these models involve some optimization as well. Also not included are models which deal with the controlled reliability of a system such as the design of redundant systems unless they also include aspects of optimal inspection, repair, and/or replacement decisions. Each of these excluded areas is large, important, and worthy of a survey in its own right. Furthermore, each lies slightly outside of what many people would call maintainability. Even with these restrictions, however, the reader will see that the many topics covered in this survey are rich with theoretical significance and practical application.

There are many possible ways to classify the works in maintainability. One could establish a multi-dimensional grid whose coordinates would be (i) states of the system, such as deterioration level, age, number of spares, number of units in service, number of state variables, etc., (ii) actions available, such as repair, replacement, opportunistic replacement, replacement of spares, continuous monitoring, discrete inspections, destructive inspections, etc., (ii) the time horizon involved, such as finite or infinite and discrete or continuous, (iv) knowledge of the system, such as complete knowledge or partial knowledge involving such things as noisy observation of the states, unknown costs, unknown failure distributions, etc., (v) stochastic or deterministic models, (vi) objectives of the system, such as minimize long run expected average costs-per-unit time, minimize expected total discounted costs, minimize total costs, etc., and (vii) methods of solution, such as linear programming. dynamic programming, generalized Lagrange multipliers, etc.

Then in each cell of the grid, one could conceivably place every paper written on maintainability. The empty cells would mean those research areas were irrelevant or not yet explored. To some extent, McCall [1965] uses such a classification scheme (on a modest scale) by considering the primary categories of "preparedness" and "preventive maintenance" models with and without complete information.

Although classification schemes based on some breakdown involving two or more of the seven categories mentioned above are useful for establishing an underlying general theory of maintainability and also for exposing some unexplored areas, such a scheme has not been rigidly followed here. Rather the papers have been classified in such a manner that a practitioner might be able to find the models relevant to his maintenance problem (to a large extent this approach parallels that taken by Barlow and Proschan [1965]. Thus, there are two major sections with several subsections in each.

The first section surveys discrete time maintenance models. That is, at discrete points in time, a unit (or units) is monitored and a decision is made to repair, replace, and/or restock the unit(s). The second section surveys continuous time maintenance models. Actions and events are not *a priori* restricted to occur only within a discrete subset of the time axis.

In the discrete time section, most of the models are Markov decision models in which the state of

the system is described by the level of deterioration and/or the number of spare units available in inventory. This section is subdivided by segregating models with no restocking from those which involve an inventory (restocking) decision. The subsection on models with no restocking is further subdivided on the basis of kinds of information available.

The continuous time models are subdivided into several topic categories also. The first of these is concerned with applications of control theory to maintainability. The control function $m(t)$ acquires the interpretation "rate of maintenance expenditure at time t." By the use of control theory, $m(t)$ is selected to maximize discounted return. The second subsection deals with age replacement models thereby giving new enhancements to the classical model, such as age dependent operating cost and the optimal provision of spares. Also described in this subsection are certain policies that regulate time of replacement according to the occurrence of the kth failure. The third subsection describes shock models, i.e., systems where the unit is subject to external shocks according to some stochastic process, and these shocks affect its failure characteristics. Subsection four deals with interacting repair activities. Here the decisionmaker must control a system composed of more than one unit. Specifically discussed are such "system-wide" activities as opportunistic replacement, cannibalization, multistage replacement, and variable repair rate. Also discussed are models involving incomplete information. The decisionmaker's information may be incomplete only in regard to the current state of the system, in which case inspection models arise. On the other hand, he may not know completely the probability law governing the system or the actual cost implication of various actions. In the former case, max-min and Bayesian strategies have been devised. In the latter case, the statistical implications of a related sampling procedure are presented. The concluding section indicates some areas for future research.

Although the authors have tried to give a reasonably complete survey, the reader will note that untranslated Eastern European and Asiatic papers are missing. Any other papers that are not included were either inadvertently overlooked, not directly bearing on the topics of this survey, or no longer in print. The authors apologize to both the readers and the researchers if we have omitted any relevant papers.

1. DISCRETE TIME MAINTENANCE MODELS

In this section, models are reviewed which utilize information regarding the degree of deterioration of the unit or units in order to select the best action at certain discrete points in time. The deterioration may be described by such factors as "wear and tear," "fatigue," etc. In some cases, inspection decisions must be made in order to ascertain the current state of the unit before a repair or replacement action is taken. In others, it may be assumed that the current state is always known at the beginning of a period and the available actions are to replace the unit or to choose one of several repair activities which will tend to decrease the degree of deterioration. Often the decisionmaker must take these actions under conditions of incomplete information about costs, underlying failure laws, or noisy observations of the unit's state. Models involving complete knowledge are treated first, then the models with incomplete information are covered. In these latter models, it is always assumed that there are new replacements available (i.e., an unlimited supply of new spares). The section concludes with models involving the periodic restocking of inventories of spare parts.

Because of the Markovian nature of the state and action transition processes, Markov decision theory and inventory theory are the primary approaches used in the model formulations. Consequently,

linear programming and dynamic programming are the primary techniques used to solve these models.

COMPLETE INFORMATION

A unit (or several units) is inspected every period and a decision is made to repair or replace the unit whenever it is found to be in a certain set of states. In the absence of a decision to repair or replace, it is assumed that the unit deteriorates stochastically through a finite set of states denoted by the set of integers $\{0, 1, \ldots, L\}$ according to a Markov chain. The state 0 denotes a new or completely renovated unit and the state L an inoperative or failed unit. After inspecting the unit a decision is made either to repair, replace, or do nothing to the units. In most of the models there are only two decisions every period: decision 1 means do nothing and decision 2 means replace.

Depending upon the assumptions concerning the time horizon, the amount of information available, the nature of the cost functions, the objectives of the models, system constraints, and numbers of units, different authors have produced many interesting and significant results for variations on this basic model. The basic model was originally introduced by Derman [1962] and extended by Klein [1962]. Although these two important papers are clearly presented in Barlow and Proschan [1965] and Derman [1970], the basic model is briefly given here prior to surveying later generalizations.

The unit is observed at times $t = 0, 1, 2, \ldots$ to be in one of the states $X_t \epsilon \{0, 1, \ldots, L\}$. If no action (decision 1) is taken, then p_{ij} denotes the probability of moving from state i to state j in one period. If the unit is replaced (decision 2), then the unit moves immediately into state 0, and the transition during the period is governed by the probabilities $\{p_{0j}\}$. It is assumed that

$$(1) \qquad p_{i0} = 0, \qquad\qquad i = 0, \ldots, L-1,$$

$$(2) \qquad p_{L0} = 1, \qquad \text{and}$$

$$(3) \qquad p_{iL}^{(t)} > 0 \qquad\qquad \text{for some } t \text{ and each } i = 0, \ldots, L-1.$$

Condition (1) implies the unit is never as good as "new" after its first period of service; condition (2) implies the unit must be replaced on failure; and condition (3) implies the unit will eventually fail and that the underlying Markov chain, $\{X_t\}$, has a single ergodic class and the steady state probabilities exist.

Upon inspecting the unit at any time, it is possible to replace the item before failure. In this way it may be possible to avoid the consequences of failure or further deterioration of the unit. These decisions to replace the unit or do nothing are summarized by a set of decision rules (or replacement rules) based on the entire history of the process up to time t. Although the most generality is achieved by considering these rules as elements of the class of nonstationary, randomized rules, conditions have been established by Derman [1970] which enable one to restrict attention to stationary nonrandomized rules. These latter rules are of the type that the set $\{0, 1, \ldots, L\}$ is partitioned into two subsets $\mathscr{R}$ and $\mathscr{D}$. If $X_t \epsilon \mathscr{R}$ then replace the unit and if $X_t \epsilon \mathscr{D}$, do nothing. Most of the models surveyed here satisfy the conditions given by Derman or else restrict their attention only to the class of stationary nonrandomized rules which do not utilize all prior history and can be denoted by R_i for $i = 0, \ldots, L$, where R_i is the action taken when $X_t = i$.

By making an intervening decision before observing state L, the behavior of the system is modified and the evolution of the system under a replacement rule results in a modified Markov chain. The

only costs consist of c_1 to replace a unit that has not yet failed, and a higher cost c_2 to replace a failed unit.

The objective is to minimize the expected long run average cost-per-unit time.

By making an additional assumption on the original transition probabilities

(4)
$$\sum_{j=k}^{L} p_{ij} \text{ is nondecreasing in } i = 0, \ldots, L-1$$

$$\text{for each fixed } k = 0, \ldots, L.$$

Derman has shown that the optimal replacement rule R^* is a "control limit" rule; that is, there is a state $i^* \, \epsilon \{0, 1, \ldots, L\}$ such that if the observed state k satisfies $k \geqq i^*$ then replace the unit and if $k < i^*$ do nothing. This key result reduces the size of the set of rules R in the minimization from 2^{L-1} rules to at most $L+1$ rules. The same key result holds when the objective is changed to minimize the total long run discounted costs.

Assumption (4) implies that if no replacements are made, the probability of deterioration increases as the initial state increases. Barlow and Proschan [1965] point out that this conditional probability is equivalent to assuming the original Markov chain is IFR.

The mathematical programming problems resulting from the above objectives have a natural formulation as dynamic programs, and in this context, successive approximations and policy iteration techniques may be used (see Bellman [1957], Bellman and Dreyfus [1962], and Howard [1971]). Derman has provided an interesting formulation in the long run average cost case leading to a linear programming problem. On the surface it may appear that the linear programming problem is very difficult to handle for most important problems because of its large size; however, recent work in large-scale linear programming (cf. Lasdon [1974]) gives hope for solving these very large Markov decision problems.

Kolesar [1966] considers the same model with the cost function generalized to allow an "occupancy" cost, A_i, associated with being in each state i. With the added condition $0 \leqq A_1 \leqq \ldots \leqq A_L$, he demonstrates that a control limit rule is optimal in the long run average cost case.

Now the $L+1$ control limit rules can be represented by the integers $\{i | i = 0, \ldots, L\}$. In his case where a control limit rule (or rules, if uniqueness is not present) is optimal, Kolesar shows that the cost function is integer quasi-convex in i, i.e., if i^* is the smallest optimal control limit then φ_{R_i} is nonincreasing for $i < i^*$ and nondecreasing for $i \geqq i^*$ (see Greenberg and Pierskalla [1971] for results on quasi-convexity). This observation enables him to develop more efficient algorithms to solve the average cost problem. He further observes that in the linear programming formulation of the model the restriction to control limit rules allows one to modify the simplex algorithm so that at most L changes of feasible bases are needed before the optimal rule is obtained.

The results of Derman and Kolesar were later extended by Ross [1969a] to the case where the state of the system can be represented by an element of some nonempty Borel subset S of the nonnegative real line. Thus, continuous or denumerable state spaces are now allowed. Let $x \, \epsilon \, S$ denote the state of the system upon inspection at the beginning of a period, and F_x the cumulative distribution function describing the next state given that the decision is made not to replace defined for $x > 0$. The failed state is denoted by 0. Thus $p(x) \equiv F_x(0)$ is the probability the unit currently in state x will fail before the next inspection.

In parallel with the finite state models, a control limit policy, R_y, replaces the unit at time t if $X_t \geqq y$ or $X_t > y$ for some control limit $y \epsilon (0, \infty]$ and does nothing otherwise. Ross then proves the following. If $g(x)$, the cost of being in state x, is a nondecreasing bounded function of x for $x > 0$, p_x is a nondecreasing function of x for $x > 0$, and for each $y > 0$ the function $(1 - F_x(y))/(1 - p_x)$ is a nondecreasing function of x for $x > 0$, then there is a control limit policy R_y that is optimal in the long run discounted case. Under some additional conditions on $g(x)$ and $F_x(y)$, he then shows that a control limit policy is optimal for the long run average cost case as well.

In a recent paper Kao [1973] considers a different type of generalization of the basic Derman model. He considers the problem of a deterioration process which moves from state 0 to state L, but the time spent in each state before a transition is a random variable depending on the transition. This problem is modelled as a discrete time finite state semi-Markov process. Since the underlying process is one of deterioration it is assumed that $p_{ij} = 0$ for $j \leqq i$ and that the holding time in state i before going to j is a positive, finite, integer-valued random variable. As in Kolesar, Kao assumes an occupancy cost A_i per unit time, a fixed cost c_i of replacement and a variable cost v_i of replacement per unit time if a decision to replace is made when the unit is in state i. Under reasonable conditions on the cost functions and the expected sojourn time in each state, Kao proves that a control limit policy is optimal over the class of stationary nonrandomized policies for the long run average cost case. He then generalizes the ' model to include the time the unit has spent in the state as well as the state itself in determining when to replace the item.

Until now, it has been assumed that the decisionmaker is continuously aware of the state of the unit. That assumption will now be dropped. To ascertain the true state of the system, the decisionmaker must inspect it, which is an action entailing a particular cost. Rules for scheduling inspections now enter into the policy.

In an early paper, Klein [1962] was concerned with the different levels of decisions which could be made upon the inspection of a unit. These decisions ranged from replace with a new unit to various types and degrees of repair. In addition, he considered the decision as to when the next inspection should be taken. He showed that the problem could be formulated as a linear fractional program and then reduced to a linear program to be solved.

More specifically, there are costs associated with inspections and repairs. A repair moves the unit from state j to state i at cost $r(j, i)$; replacement is equivalent to moving it from state j to state 0. The cost of inspection while in state j is $c(j)$. After an inspection, the decisionmaker can choose to skip the next m time periods before making another inspection. It is assumed that m is bounded by some number, say M. If the unit is discovered to have failed at some time before inspection, it is assumed that its failure time can be determined during the inspection.

The states are $\{0, 1, \ldots, L, L(1), \ldots, L(M)\}$, where $L(m)$ denotes that the unit failed m time periods ago. The action (j, m) denotes placing the unit in state j by repair or replacement and skipping m periods until the next inspection. The transition probabilities are the same as before. For this model it is assumed $p_{ij} = 0$ if $i > j$, $p_{L, L(1)} = p_{L(1), L(2)} = \ldots = p_{L(M), L(M)} = 1$, and $p_{iL}^{(t)} > 0$ for some t and all $i = 0, \ldots, L-1$.

A linear programming formulation for the long run average cost case can be found in Klein. Derman [1970] shows that a stationary policy is optimal.

Much in the spirit of Derman's and Klein's models, but by using the functional equation of dynamic programming, Eppen [1965] gives sufficient conditions on the cost functions and transition matrices

to show that the total discounted expected costs are minimized from period n to the end of the finite horizon by the following policy: there exists a sequence $\{i_n^*\}$ of critical numbers such that in period n if the state of the deteriorating unit exceeds i_n^*, then return the system to state i_n^* (by repair); otherwise do not repair. This policy parallels the critical number policy in the single product stochastic inventory model without setup cost.

In an interesting paper, Hinomoto [1971] considers the sequential control of N homogeneous units by generalizing the approach developed by Klein. At each decision point the decisionmaker determines the level of the repair activity and the time of the next inspection. Hinomoto considers two different plans for the maintenance of the N units. The first plan is a fixed rotating sequence for the units, i.e., find an optimal permutation schedule from the set of all permutations of the numbers $1, \ldots, N$ and then inspect (and repair, if needed) the units sequentially in the order given by the permutation. The second plan is a priority plan which alters the fixed sequencing by queueing those units for repair or replacement which have fallen below a certain critical level of performance. (In the first plan, such a unit would just have to wait its turn in the permutation sequence.) He formulates both problems as linear programs and finds the optimal decisions over the class of fixed cyclic inspection patterns (except for the temporary displacing priorities in the second plan).

Other authors Derman [1970], Eckles [1968], Sondik [1971], and Smallwood and Sondik [1973] also allow for this greater variety in the decision space by allowing repair or replacement decisions $j \epsilon \{0, 1, \ldots, L\}$ (more will be said about their results in the next section).

Incomplete Information

A problem which often arises in the study of maintenance is the lack of information concerning many aspects of the model. This lack of information may take many different forms. In his earlier survey, McCall [1965] treated the case of lack of knowledge of the underlying failure distribution. In addition to this case, there can be lack of information due to (i) random costs or unknown costs and (ii) noisy observations on the state of the system. A paper treating the costs as random variables with known distribution is given by Kalymon [1972]. The papers concerning unknown costs are generalizations by Kolesar [1967] and Beja [1969] of another basic model due to Derman [1963a]. The papers on noisy observations are by Eckles [1968], Sondik [1971], and Smallwood and Sondik [1973]. Concluding this section, the papers by Satia [1968] and Satia and Lave [1973] on the lack of knowledge of the underlying probability distribution are mentioned.

Kalymon [1972] generalizes the Derman model of the previous section by considering a stochastic replacement cost determined by a Markov chain. This chain is assumed to be conditionally independent of the Markov chain defining the deterioration of the unit from period to period. The cost, C, of a new unit in period t is a random variable which takes on a finite set of values. There is a separable salvage value of $-[r(c_t) + s(x_t)]$ and an occupancy cost $A(x_t)$ when the machine is in state x_t and the realized actual cost of a new machine is c_t in period t.

When the cost functions $c + r(c)$, $s(x)$, and $A(x)$ are nondecreasing in their arguments, the Markov chains $\{X_t | t = 0, 1, \ldots\}$ and $\{C_t | t = 0, 1, \ldots\}$ have distribution functions which are IFR, i.e., they satisfy (4) and its analog, and the conditional expected costs satisfy a certain condition, then the optimal policy is a control limit policy for the nonstationary finite horizon discounted cost function. This result is then generalized to the infinite horizon ergodic chains case for both the long run dis-

counted and the long run average cost cases. In this context, there can be a different control limit policy for each realized replacement cost c.

In another interesting variation on his earlier basic model, Derman [1963a] considered the problem where the unit is in one of several operative states $0, 1, \ldots, n$ and several inoperative states $n + 1, \ldots, L$. A decision $k \in \{1, \ldots, K\}$ is made to replace or partially repair the unit at some level k once it is observed in states $1, \ldots, n$ and only to replace if in states $n+1, \ldots, L$. There is no explicit cost structure available nor can one be easily inferred other than that the cost of failure is much greater than the cost of replacement. The objective is to maximize the expected length of time between replacements, called the cycle time, subject to the constraint that the probability a replacement is made when the process is in state j is no greater than some preassigned number α_j for $j = n + 1, \ldots, L$. Derman develops a linear programming formulation of this model over the class of rules which repeat every time the unit is renewed.

In a subsequent paper, Kolesar [1967] restricts the set of decisions to only two: replace or do not replace. From the class of stationary randomized rules, a generalized control limit rule is defined by

$$\text{do not replace} \qquad\qquad\qquad\qquad\qquad\qquad \text{when } i < m,$$

$$\text{replace with probability } \frac{\beta \pi_i(R_m)}{\beta \pi_i(R_m) + (1-\beta)\pi_i(R_{m+1})} \quad \text{when } i = m,$$

$$\text{replace} \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{when } i > m,$$

where $0 \leq \beta \leq 1$ and R_m and R_{m+1} are control limit rules (as defined previously). The coefficient β which forms the convex combination of the steady state probabilities for the two nonrandomized control limit rules R_m and R_{m+1} is obtained as the coefficient which satisfies $\pi_j^* = \beta \pi_j(R_m) + (1 - \beta)\pi_j(R_{m+1})$ and the π_j^* minimize π_0 on the set $\sum_{j=0}^{L} \pi_j = 1$, $\sum_{i=0}^{L} \pi_i p_{ij} = \pi_j$, $\pi_j \geq 0$ and $\frac{\pi_L}{\pi_0} \leq \tau$, where τ is the predetermined maximum tolerable probability of failure in a cycle. Under the IFR assumption (4), Kolesar establishes that the optimal stationary policy is either do not replace until state L, or else there are control limit rules R_m and R_{m+1} such that a generalized control limit rule is optimal when the maximum tolerable probability of failure in a cycle is τ.

Beja [1969] does not make the IFR type of assumption (4) on the transition probabilities. In so doing he is able to consider more general transition matrices, e.g., the "bath-tub" variety. Let y_i and x_i denote the probability of failure before replacement and the expected time before replacement given that the present state is i, respectively. For every state i the set of constraints

$$y_i/x_i \leq \tau, \qquad \text{for a given } \tau,$$

impose restrictions on the potential hazard encountered while in state i. With these additional constraints on the unit, Beja shows that from the class of stationary randomized policies, one of the 2^{L-1} nonrandomized policies (i.e., replace or keep whenever the process is in state $1, 2, \ldots, L-1$) is optimal. He then demonstrates what the "implicit" replacement and failure costs are, given the optimal solution which maximizes the cycle length.

The problems of uncertainty in studying a maintenance problem are treated somewhat differently by Eckles [1968], Sondik [1971], and Smallwood and Sondik [1973]. The underlying process for the unit is still the finite state, discrete time Markov chain as before and there are K possible decisions from which a selection k is to be made. For example, these decisions could be to replace, repair or do nothing. Let $p_{ij}^k(t)$ be the true conditional probability that the unit goes to state j in the next period given that it is in state i at period t and decision $k \equiv k(t)$ was taken. An inspection of the unit after taking decision k in state i during period t yields the observation x where $q_{xi}^k(t)$ is the conditional probability of outcome x given state i and decision k. In this manner, it is possible to characterize maintenance-inspection problems ranging from those whose observation of the state reveal knowledge of the actual state with certainty (as in the previous models) to those which provide no information of the new state of the system. It is assumed (i) that the actual age of the unit in use in period t is always known with certainty given the sample history of observations and decisions $\mathscr{H}_t$, and (ii) there is an underlying cost $c_{ij}^k(t)$ which represents the cost of going to state j, given state i and decision k in period t. Letting

$$c_i^k(t) = \sum_{j=0}^{L} p_{ij}^k(t) c_{ij}^k(t)$$

be the expected one period cost given (i, k, t), and assuming that for any sample history $\mathscr{H}_t$ the one-step transition matrix $P(\mathscr{H}_t)$ and the current age of the unit $t(\mathscr{H}_t)$ are jointly a sufficient statistic for $\mathscr{H}_t$, then there exists an optimal solution which minimizes the expected total discounted costs where the only information needed are the one-step transition matrix $P(\mathscr{H}_t)$ and the current age of the unit $t(\mathscr{H}_t)$ and not the entire prior history $\mathscr{H}_t$. Bayes Theorem is then used to update $P(\mathscr{H}_t)$ to $P(\mathscr{H}_{t+1})$ (or $P(\mathscr{H}_{t-1})$ if one numbers backward as is often done in finite time dynamic programming). Eckles formulates the problem as a dynamic program and under the standard assumption that whenever a unit is replaced it is completely renewed (its age is 0) and its transition probabilities are then independent of the past history of the process, he presents an algorithm for finding an optimal nonrandomized age replacement policy for this renewal process.

Sondik [1971] and Smallwood and Sondik [1973] treat the same problem as Eckles; however, they demonstrate that the entire history of the process is contained in the information vector

$$\pi(t) = (\pi_0(t), \; . \; . \; ., \pi_L(t)),$$

where

$\pi_j(t) \equiv$ conditional probability that the actual state at time t is j given observation x, history $\mathscr{H}_{t-1}$ and decision $k(t)$,

and that by Bayes Theorem

$$\pi_j(t+1) = \frac{\sum_i \pi_i(t) p_{ij}^k(t) q_{xj}^k(t)}{\sum_i \sum_j \pi_i(t) p_{ij}^k(t) q_{xj}^k(t)}.$$

Thus, if $\hat{\pi}(t)$ is the statistic generated by the outcomes $\mathscr{H}_t$ then $\hat{\pi}(t+1)$ is a sufficient statistic for $\mathscr{H}_{t+1}$. This approach differs slightly from the Eckles model in that

$q_{xj}^k(t) \equiv$ conditional probability that the outcome x is observed given that the true outcome
is j and that decision k was taken just prior to the inspection.

Thus, $\pi(t)$ behaves as a discrete time, continuous state Markov process. They then demonstrate that the expected total discounted cost function is piecewise linear and convex. Using this structure they develop an efficient algorithm which makes the rather difficult large state space problem relatively easy to solve.

In conclusion of this subsection the case of incomplete knowledge of the probability law governing the system's evolution is briefly considered. As was mentioned earlier McCall [1965] surveyed this case extensively.

The optimization of a maintenance problem modelled as a Markov chain with unknown transition probabilities is usually approached from either a game theoretic (i.e., max-min or max-max) or from a Bayesian point of view. The max-min approach essentially seeks the maximum overall possible policies of the minimum of the total expected long run discounted return with respect to all possible transition matrices (within some set). The max-max approach is defined analogously. Satia [1968] proves there exists a pure, stationary policy for this latter decision process that is optimal.

Satia and Lave [1973] discuss this earlier Satia result and present an algorithm similar to Howard's Policy Improvement Algorithm. Their algorithm will converge to within any predetermined ϵ-interval about the true max-min solution in a finite number of iterations. Satia and Lave also formulate a Bayesian approach to the problem and present an implicit enumeration algorithm for its solution.

Maintenance and Inventory Models

Most maintainability models which provide for the replacement of a unit assume that the replacement items are drawn from an infinite stock. However, for some models, this stock is not infinite; indeed, its management becomes a control variable.

There are many inventory papers which treat stock replenishment problems for stochastically failing equipment: Falkner [1969], Prawda and Wright [1972], Sherbrooke [1968, 1971], Sobel [1967], Porteus and Lansdowne [1974], Silver [1972], Miller [1973], Moore, et al. [1970], Demmy [1974], and Drinkwater and Hastings [1967]. These papers do not consider problems where a decision must be made to repair, replace or inspect a unit or units. Rather they assume that once a unit has failed, it must be repaired or replaced and the decision process is how much inventory to stock either initially, periodically, or continuously.

To be more specific, Prawda and Wright [1972] examine a system where there may be many identical units in operation. These units fail in one of two ways. The unit fails and is repairable with probability distribution $F_i(\cdot)$ or nonrepairable with probability distribution $\mathscr{R}_i(\cdot)$ in period $i = 1, 2, \ldots$. Repairable units go to a repair facility and after $\mathscr{N}$ periods are renewed and returned to inventory. Nonrepairable units are discarded. If the available inventory is insufficient to meet replacement needs, these needs are backlogged until inventory is available. An order for new units is placed at the beginning of each period and delivery is received λ periods later. They consider the two problems of how much to order every period to minimize the expected total discounted costs and to minimize the ex-

pected long run average cost-per-unit time. In their model, there are four costs: ordering, holding, shortage, and salvage. Following some earlier work in inventory theory by Veinott [1965], Prawda and Wright show that for an ordering cost of c per unit and explicit quasi-convexity of the single period cost function, the optimal policy is a stationary single critical number in each period. That is, in period t and before ordering in this period if the state of the system determined by the stock on hand, in repair, and on backorder is denoted by x_t, then there is a number $\bar{y}$ such that

$$\text{if } x_t < \bar{y} \text{ order } \bar{y} - x_t,$$

otherwise do not order. They also consider the case where there is a setup cost K every time an order is placed and give results on the optimal order quantities.

Taking a different approach, Sherbrooke [1968, 1971] considers the problem of determining the stock levels at each echelon of a multiechelon multiunit inventory system of repairable units. The idea is that at various sites $j = 1, \ldots, J$ there are repairable units of types $i = 1, \ldots, I$ in use and in inventory, y_{ij}; and there is a central facility, 0, which maintains a buffer inventory, y_{i0}, for use at the sites when needed. Each site including the central facility has repair capabilities. The problem is to determine the y_{ij} for $i = 1, \ldots, I$ and $j = 0, 1, \ldots, J$ which minimize the total weighted expected number of units backlogged at any point in time subject to a budget constraint on repair and operating costs. The failure of units are independent and identically distributed (by unit type) according to a logarithmic Poisson distribution with constant variance to mean ratios. There is no transshipment among sites $1, \ldots, J$ and the repair decisions are not explicitly entered in the model (only implicitly through the logarithmic Poisson failure rates). It is also assumed that there are an infinite number of repair activities, so that the repair times are independent of the number of units being repaired. Sherbrooke shows that the generalized Lagrange multiplier approach of Everett [1963] and Greenberg and Robbins [1972] can be used to obtain near optimal solutions to the problem.

Porteus and Lansdowne [1974] consider the same model, but assume the failure process is Poisson, that $y_{i0} = 0$ for all $i = 1, \ldots, I$ and that the mean repair times for the repair of unit type i at site j undergoing repair work of level k can be controlled. Using a generalized Lagrange multiplier algorithm, they obtain the spare stock quantities and mean repair times which minimize either the long run average cost-per-unit time or the long run total discounted cost.

In a related paper also based on Sherbrooke's work, Silver [1972] determines the inventory levels, y_i, for repairable subassemblies ($i = 1, \ldots, I$) of a major assembly. The failure process is Poisson for each subassembly, the failure of a single subassembly makes the entire unit inoperative, and cannibalization of major assemblies awaiting repair is assumed. Under the additional assumption that there is no inventory of the major assembly (i.e., $y_0 = 0$), Silver shows that the entire optimization problem is separable and easily solved. In the case $y_0 > 0$, he gives a near-optimal algorithm to obtain y_i for $i = 0$, $1, \ldots, I$.

Miller [1971] considers the optimal stocking problem also; however, in [1973] he asks the question to which site should a repaired item be sent after it is repaired at a central facility given that transportation times differ and that only the central facility can repair items. He formulates a single-item multi-location model and shows by simulation that the policy of shipping the repaired item to the site which realizes the greatest marginal decrease in the expected backorders from this one additional unit (computed over the time required to transport the unit) is better than the current Air Force shipping policy.

Furthermore, under the assumption that repairs are instantaneous after a failure, he proves that this policy is optimal.

Along different lines, Derman and Lieberman [1967] consider a joint replacement and stocking problem. Inspections are made every time period. An initial stock of N identical units are on hand. At the end of each period, a decision is made to replace the currently working unit or not to replace it. If a replacement is made, the new unit works at a level s with probability f_s and continues to work at the same level until it either fails or is replaced. Its life-length is a random variable with a geometric distribution. If at the end of the period the unit in service has failed, it is replaced provided there are units still in stock. If there are none in stock, the system is down for one period of time while N units are reordered.

The state space is described by $\{(n, s) : n = 1, \ldots, N; s = 1, 2, \ldots\} \cup \{0\}$, where n denotes the number of (identical) units on hand including the one in service, and s denotes the performance level of the unit in service. Note that the levels of service are a denumerable set. The element 0 denotes no units in stock and the system is down. The available actions are $\{1, 2\}$, where 1 denotes no replacement and 2 denotes replacement (or reorder).

Using the resulting transition probabilities and the cost functions given by

$$g_2(0) = C$$

$$g_1(n, s) = g_2(n, s) \qquad \text{for } n = 1, \ldots, N; s = 1, 2, \ldots;$$

where $g_1(n, s)$ is nondecreasing in s for each fixed n, Derman and Lieberman show that for a fixed N, there exists a sequence of numbers $s_1, s_2, \ldots, s_N$ such that the optimal solution for minimizing the expected average cost-per-unit time is a stationary policy of the form

$$R_{(n, s)} = \begin{cases} 1 \text{ if } s < s_n, \\ \\ 2 \text{ if } s \geq s_n. \end{cases}$$

Under additional assumptions on the cost functions, in order to determine the optimal N, only a finite number of possible choices for N need to be investigated.

This joint maintenance-inventory model is generalized by Ross [1969a] to allow a continuous state space and deterioration of the component from period to period. He establishes optimal policies of the same form as Derman and Lieberman and shows that when the inventory is 0, it is optimal to order N^* units.

2. CONTINUOUS TIME MAINTENANCE MODELS

In this section maintenance models are considered which do not contain the assumption that maintenance or inspection activity is *a priori* restricted to a particular discrete set of points in time. That is to say, in these models the actions of the decisionmaker may potentially take place anywhere on the continuous time axis (although there may be only a discrete number of such actions).

In certain of these models, the maintenance activity is permitted to occur as a continuous stream. That is, the decisionmaker must optimize over functions $m(\cdot)$ where $m(t)$ is the maintenance expenditure rate at time t. These models are considered first.

Control Theory Models

Determination of the optimal maintenance schedule and the sale date for a unit in a deterministic environment has been under study by a number of different authors. Early solutions to the problem are to be found in Masse [1962]. Naslund [1966] was the first to solve the problem by making use of the maximum principle.

In more recent work, Thompson [1968] presents a simple maintenance model which illustrates the application of the maximum principle technique to maintenance problems. The model contains the following factors: the (unknown) sale date of the unit T, the present value $V(T)$ of the unit if its sale date is T, the salvage value $S(t)$ of the unit at time t, the net operating receipts $Q(t)$ at time t, the rate of interest r, the number of dollars spent on maintenance $m(t)$ at time t where maintenance refers to money spent over and above necessary repairs, the maintenance effectiveness function $f(t)$ at time t (in units of dollars added to $S(t)$ per dollars spent on maintenance), the obsolescence function $d(t)$ at time t (dollars subtracted from $S(t)$), and the production rate p at time t.

It is assumed that $d, f,$ and m are piecewise continuous, d is nondecreasing and f is nonincreasing, p is constant over time, and, for some given constant M, $0 \leqq m(t) \leqq M$.

The object of this model is to choose a maintenance policy $m(t)$ and the sale date T^* to maximize $V(T)$. Thompson begins by solving for the optimal maintenance policy for a fixed sale date T.

He shows that the Hamiltonian is linear in control m, hence the optimal maintenance policy will be of the bang-bang type (piecewise constant). The optimal maintenance policy is obtained by solving $f(t) = r/(p - (p-r)e^{-r(T-t)})$ for the unique point T'. Then

$$m(t) = \begin{cases} M, & t < T' \\ \text{arbitrary}, & t = T' \\ 0, & t > T'. \end{cases}$$

It is clear that $m(t)$, the optimal control, is a function of T. In this respect, it is piecewise constant; hence, the solution of $\partial V/\partial T = 0$ for T is simplified. Thompson presents the details of this procedure, illustrates the model with a few examples and extends the model to the case of a variable production rate.

Arora and Lele [1970] extend Thompson's model by considering the effect of technological progress. This was accomplished by including a term for obsolesence, due to such progress, in the state equation for the salvage value of the unit.

Kamien and Schwartz [1971] consider a maintenance model which represents another extension of Thompson's work. In their model, it is assumed that the value of the unit's output is independent of its age while the probability of failure increases with its age. Furthermore, it is assumed that revenue and scrap value are both independent of age. The probability of failure is influenced by the amount of money spent on maintenance according to the following differential equation:

$$\frac{dF(t)}{dt} = (1 - u(t))h(t)(1 - F(t)).$$

The function $u(t) \in [0, 1]$ is the level of maintenance at time t; $F(t)$ is the time-to-failure distribution provided the unit is given maintenance according to the schedule $u(\cdot)$; $\hat{F}(t)$ is the corresponding dis-

tribution provided the unit receives no maintenance; and $h(t) = \hat{F}'(t)/(1 - \hat{F}(t))$ is the "natural" failure rate of the unit. The object of maintenance is to reduce the probability of failure. Expected revenue from the unit's output is maximized by selecting the appropriate control function, and the optimal sale date T^* is determined.

Kamien and Schwartz characterize the solution to this problem by deriving necessary conditions for the optimal sale date. In addition, they use the maximum principle to characterize necessary conditions for optimal control and in particular show that it is not of the bang-bang type. They also prove sufficiency of their necessary conditions. These results are of interest since the underlying process in this case is controlled by influencing the failure rate and not the salvage value as was the case in Thompson's paper.

In two recent papers by Sethi and Morton [1972] and Sethi [1973], the basic one-unit model is extended to the situation of maintaining a chain of units. In addition, conditions for a changing technological environment affecting both production and maintenance requirements of future units is assumed. In this dynamic model, prices of future units are also allowed to vary. In the first paper, a finite horizon problem is considered and a solution procedure for determination of the optimal maintenance schedule for each unit in the chain is derived. In addition, conditions for bang-bang control are discussed. In the second paper, the problem is solved where the maintenance policy is assumed to be stationary in the sense that the same maintenance policy is applied to each unit in the chain. The optimal maintenance schedule is characterized and again conditions for bang-bang control are presented. Finally, the computation of the optimal replacement period is posed in terms of a nonlinear programming problem.

In a related paper by Tapiero [1973], a sequence of n units is also considered. This paper is a direct generalization of Thompson's model. Characterization of optimal maintenance schedules and a discussion of replacement times are presented. Tapiero demonstrates that the decision to replace a unit depends only on the relative value of the current unit and the subsequent unit. Thus a replacement is effected when the subsequent unit becomes more profitable to operate. Tapiero refers to this condition as "technical obsolescence."

The above maintenance policies permitted a (piecewise) continuous rate of maintenance expenditure through time. Consider now policies that apply a maintenance action only at discrete instants of time; for example, replacing an item upon its reaching a certain age or upon its third breakdown. Such policies can nonetheless be distinguished from those in Part 1 because the event precipitating the maintenance action is, in general, permitted to occur anywhere on a continuous time axis.

Age Dependent Replacement Models

In the earlier models of age replacement (cf., Barlow and Proschan, 1965) the replacement of the unit at failure costs c_2 while replacement before failure costs $c_1 < c_2$. It was shown that if F, the distribution of time-to-failure, had a strictly increasing failure rate then there existed a unique T^* such that expected cost-per-unit time was minimized if the unit was replaced at age T^* or at failure, whichever occurred first.

Glasser [1967] has obtained solutions to the age replacement problem for three specific distributions, the truncated normal, the gamma, and the Weibull.

Fox [1966b] showed the optimality of an age replacement policy under a total discounted cost criterion. For a continuous and strictly increasing hazard rate, he derives an integral equation which can be solved for the optimal T^*.

Schaefer [1971] extends the standard age replacement model by including an age-dependent cost. Such a cost may reflect the increasing burden of routine maintenance as the unit ages, its diminishing productivity, or reduced salvage value for the unit at replacement due to depreciation. Specifically, he expresses the total cost up to time t as

$$C(t) = c_1 N_1(t) + c_2 N_2(t) + c_3 \left[\sum_{i=1}^{N(t)} Z_i^\alpha + (t - S_{N(t)})^\alpha \right],$$

where $N_1(t)$ is the number of replacements due to reaching the age replacement level T which have occurred by time t, $N_2(t)$ is the number of replacements due to failure by time t, $N(t) = N_1(t) + N_2(t)$, $c_3 > 0$, Z_i is min (X_i, T) where X_i would be the uninhibited life of the ith unit in the sequence of replacements and T is the fixed age at which replacement is to be made, S_k is the time of kth replacement, and $0 < \alpha < 1$. The goal is to minimize the long run average cost-per-unit time. By an argument similar to the one given in Chapter IV of Barlow and Proschan [1965], it can be shown that the optimum policy is nonrandom if the failure distribution F is continuous. The case of an exponential life distribution is analyzed in more detail.

R. Cleroux and M. Hanscom [1974] considered a very similar model. One of the differences is that the age-dependent cost $c_3(ik)$ is incurred only at discrete times, the multiples of the positive constant k. Moreover, $c_3(ik)$ need not be increasing in i. For F continuous and IFR, they show that the optimal age replacement policy is nonrandom for an infinite time span. They then develop sufficient conditions for the optimal replacement interval T^* to be a finite number and to be restricted to a certain finite set.

The notion of an age-dependent cost structure is further generalized by M. R. Wolfe and R. Subramanian [1974]. If the nth unit is replaced at age T the total cost incurred over the life of that unit is

$$W_n = \int_0^T [Y_n + r(s)] ds + K_n,$$

where Y_n and K_n are independent random variables forming renewal processes and $r(\cdot)$ is a differentiable and strictly increasing function. K_n is the cost of replacement and $Y_n + r(s)$ is the cost rate at time s after the installation of the nth unit. There are no failures. In order to minimize the expected total cost-per-unit time, the decisionmaker determines a critical threshold value c^* such that when the cost rate exceeds c^* he replaces the unit. For a particular realization of Y_n, this occurs at age $r^{-1}(c^* - Y_n)$. Procedures for determining c^* are given, and for $r(t)$ linear, they derive an explicit solution.

An example where an optimum age replacement policy is found for a two-unit redundant system is provided by T. Nakagawa and S. Osaki [1974]. While one of the identical units is in operation, the other is in standby status, immune to all failure or aging effects. When the operating unit is sent to repair, either for preventive maintenance (which renews the unit) or because of failure, the standby unit takes over. If an operating unit should fail when the other unit is still at the repair facility, the system becomes inoperative and the most recent failure must wait its turn for repair. Under the assumption that a preventive maintenance activity for a unit entails less mean time in repair than does the repair of a unit's failure, it may be advisable to routinely schedule such preventive maintenance after a unit has been in operation for time T. (If at that point in time the operating unit lacks a standby, preventive maintenance is delayed until the current repair work is completed.) Nakagawa and Osaki derive the optimal

value for this T under the assumptions of increasing failure rate and with regard to maximizing the long run proportion of time the system is operating. Their arguments utilize the regenerative properties of the system at those epochs when the operating unit enters the repair facility and a standby unit is available to take over.

With very expensive and complicated systems, the failure of a single component unit would not very reasonably call for replacing the entire system. Instead the system could be restored to operation by replacing the single failed unit. Because the great bulk of that system's components were not renewed, the probability distribution for the system's remaining life remains essentially what it was at the instant before failure. That is, the failure and subsequent repair activity often do not affect the system's failure rate. The action of restoring a failed system to operation without affecting its failure rate is called minimal repair. Barlow and Hunter [1960] incorporated this notion in their Policy Type II replacement Model (Policy Type I was the simple age replacement model—replace at age T or failure, whichever comes first). A Type II Policy assumes that the unit is replaced after functioning for T units (downtime is not included in the T). Any failures before that time would be dealt with by minimal repair. The optimization of a Type II policy with instantaneous minimal repair is equivalent to the age replacement model which incorporated an age-dependent cost of operating the unit. This age-dependent cost is the expected cost of incurring the expense of minimal repair which depends on the unit's age via the failure rate. Bellman [1955] and Descamp [1965] applied dynamic programming to this problem. Sivazlian [1973] generalized their work by permitting a positive downtime for the minimal repair following a failure. This downtime is random with an arbitrary distribution. Using the functional equation technique he derives an explicit expression for the long term total expected cost. Further, he derives necessary and sufficient conditions for the optimal policy to be of the "Type II" form described above.

Makabe and Morimura [1963a, 1963b, 1965] and Morimura [1970] introduce a Policy of Type III and Type III'. Under Policy Type III, the unit is replaced at the kth failure. The $k–1$ previous failures are corrected with minimal repair. A Policy Type III, Morimura feels, would be easier to implement in many practical situations than a Type II policy.

An optimal policy of Type III is shown to exist for strictly IFR distributions, both with respect to the criterion of expected fraction of time operating in $[0, \infty)$, called limiting efficiency; and with respect to what Makabe and Morimura [1963a, 1963b] call the maintenance cost rate which is defined to be

[cost-per-unit downtime] × [expected fraction of downtime]
+ [expected cost of all repairs and replacements during a unit time].

Also for distributions of strictly increasing failure rate, Morimura [1970] has shown the existence of an optimal policy (with respect to limiting efficiency) for a larger class of policies, Type III' policies. This class of policies is specified by two critical numbers t^* and k. All failures before the kth failure are corrected only with minimal repair. If the kth failure occurs before an accumulated operating time of t^*, it is corrected by minimal repair and the next failure induces replacement. But if the kth failure occurs after t^*, it induces replacement of the unit. Clearly, if $t^* = 0$, this reduces to a policy of Type III.

Even if the replacement rule is purely according to age, it cannot always be assumed that the provision of replacement units is outside the purview of the decisionmaker. Falkner [1968] examines a maintenance-inventory problem where there is a single unit which is operating and it fails according to an IFR distribution $F(\cdot)$. When the unit reaches a certain age or when it fails, it is replaced by another

new unit. The problem is to find both the initial number of spares, N, to produce and the age replacement policies $\tau_j(t)$, $j=1, \ldots, N+1$, for the original new unit and the spares in order to minimize the expected total operating cost of the system over a finite time interval $[0, T]$. This problem is formulated as a dynamic program with nondecreasing costs for holding stock (h per unit), stockout penalty (p per unit) and unscheduled replacement (r per unit). Under the assumption $F(0) = 0$ Falkner shows that the optimal number of spares is bounded above by the greatest integer less than or equal to p/h, where $h > 0$. With additional assumptions on $F(\cdot)$ he is able to characterize the cost structure and the structure of the optimal age replacement policies.

In a later paper, Falkner [1969] specializes this model by removing the age replacement decision process and is able to obtain stronger results on the optimal initial number of spares. He gives an application using the negative exponential failure distribution.

Shock Models

In most maintenance models the time-to-failure random variable of a unit is considered intrinsic to that unit. But it is possible to take a different view—regarding the unit as being subject to exterior shocks, each of which damages (or causes wear) in such a way that the damage accumulated up to a particular time defines the unit's probability of failure at that time.

For example, A-Hameed and Proschan [1973] set up the shock process as a nonhomogeneous Poisson process. This process, combined with the probabilities P_k that the unit will survive k shocks, induces a time-to-failure distribution for the unit. Various properties of the distribution $\{P_k\}$ are related to corresponding properties in the induced time-to-failure distribution. For example, if the former distribution is IFR, so is the latter.

The question of optimal replacement rules in the context of a shock model has received attention only very recently. Taylor [1973] considers a unit subject to a shock process where the decisionmaker knows at all times the level of accumulated damage from the shocks. At the occurrence of a shock he has the option of replacing the item at cost c_1. If the item should fail, it must be replaced at cost $c_2 > c_1$. The shocks occur according to a Poisson process and each shock causes a random amount of damage which accumulates additively. The device may fail only at the occurrence of a shock and then with a probability which depends on the accumulated damage. If the probability of the device failing is an increasing function of the accumulated damage, Taylor proves that the optimal replacement rule is of the following control limit form: replace the device at failure or when accumulated damage first exceeds a critical control level Z^*. He gives an equation which implicitly defines Z^* in terms of the replacement costs and other system parameters.

Richard Feldman [1974] and [1975] has permitted a more general stochastic process to represent the incidence of damage to the unit. In particular, both the time to the next shock and the degree of damage inflicted by that shock may depend on the current level of accumulated damage. He then derives the optimal control limit rule for replacement.

Interacting Repair Activities Models

For a system composed of many units, the repair or replacement of one unit should sometimes be considered in conjunction with what happens to the other units. Below are discussed four ways in which

a maintenance policy either gives rise to, or exploits, interactions among the units of a system; namely, opportunistic policies, cannibalization policies, multistage replacement policies, and variable repair rate policies.

Opportunistic policies exploit economies of scale in the repair or replacement activity. That is, two or more repair activities done concurrently may cost less than if they are done separately. So the necessity of performing at least one repair might provide the economic justification to do several others at the same time.

In cannibalization and multistage replacement models, units of the same type are utilized at different locations in the system. In response to a failure at one location, an identical unit may be transferred there from another location. In the multistage model, a new item must enter the system at some location so that all units are restored to operation. Also in multistage models, the purpose of transferring units among locations is to locate those units, which because of their age are less likely to fail, at those locations where failure is most costly. In the cannibalization model, on the other hand, no new item enters the system when a transfer is made. Since the performance of the system depends on which items are functioning, the purpose of the transfer is to provide the system with the best possible configuration of functioning units.

The last type of interacting activities model to be discussed is a variable repair rate model; that is, when the repair capacity of a system is limited and under the decisionmaker's control, he may wish to modify that capacity according to the number of items in a down state.

Some recent work has examined opportunistic repair in the context of the following system structure: There are two classes of components, 1 and 2. Class 1 contains M standby redundant components so that upon the failure of the currently operating class-1 component, a standby takes over. When all the class-1 standbys have failed, the system suffers catastrophic failure. The class-2 components, on the other hand, form a series system; if one of them should fail, the system suffers a minor breakdown. The operator always knows the state of the system.

When a minor breakdown occurs, there is the opportunity for opportunistic repair of those class-1 items which have failed. Kulshrestha [1968a] examined such a policy under the assumption that the class-1 units fail according to a general distribution and the ith component of class-2 fails with the constant rate λ_i $(i = 1, 2, \ldots, N)$. The time to complete repairs follows a general distribution. He then compares this policy to the corresponding nonopportunistic policy upon assuming the failure distribution of the class-1 units to be exponential.

Nakamichi, et al. [1974] examine the same system, but they require that any class-1 unit enter repair upon its own failure — not waiting for either a minor or major system failure. They further assume that in a major breakdown, as soon as the class-1 unit being serviced is repaired, it is reinstalled permitting the system to operate again. And in a minor breakdown, as soon as the failed class-2 unit is repaired the system operates again. In the case where a class-2 unit fails when a class-1 unit is under repair, two alternatives are examined. (1) It is repaired immediately, interrupting the repair of the class-1 unit and (2) it's repair awaits the completion of the class-1 unit's repair.

The notion of cannibalization stems from the situation where a working (operative) part in a system may be removed from one location to replace a failed part in another location. This "cannibalization" would be done in order to improve the operation of the system in some sense.

The foundations for the study of cannibalization were established by Hirsch, Meisner, and Boll [1968]. Each part in the system is classified by type $(1, \ldots, N)$ and location $(1, \ldots, n)$. The set of

parts is then partitioned into subsets such that any two parts in a subset are interchangeable, but parts in different subsets are not. At any point in time, the status of all parts in their locations is given by a binary n-tuple $v=(v_1, \ldots, v_n)$,

where

$$v_i = \begin{cases} 0 & \text{implies part } i \text{ has failed} \\ \\ 1 & \text{implies part } i \text{ is operating.} \end{cases}$$

It is assumed that failures are detected immediately. The state of the system is given by a monotonic increasing "structure" function $\varphi(v)$ which takes values from $\{0, 1, \ldots, M\}$, where 0 is total system failure and M is the best performance. In the event there are spare parts available and a failure occurs, the failed part is immediately replaced by a spare part and the structure function is unchanged. However, if there are no spares, then part shortages exist and the problem is to find cannibalizations (transformations that make feasible interchanges of parts) which maximize φ over all feasible interchanges. Any cannibalization T which maximizes φ is called admissible. Under an assumption, known as the "minimum condition," on the composite transformation $\varphi \cdot T \equiv \varphi T$, Hirsch, Meisner, and Boll explicitly characterize the state of the system for any admissible cannibalization T as a function of the number of working parts of each type. The minimum condition asserts that $\varphi T(v)$ is equal to the minimum value of $\varphi T \pi_i(v)$ over $i = 1, \ldots, N$, where $\pi_i(v)$ is the operation of making all parts of type j operable (for all $j \neq i$) while the status of parts of type i are held constant. Thus, in a sense, a single part type determines the value of φT for any v. Using the minimum condition and the explicit characterization of the state of the system, φT, for any admissible T, they demonstrate the probability laws of the state of the system under the additional assumptions that (i) the failure distributions of parts of type i are identically distributed and do not depend on the location of the part for each $i = 1, \ldots, N$, and (ii) the lifetimes of all parts are independent random variables.

Simon [1970, 1972] generalizes the results of Hirsch, Meisner, and Boll by relaxing the restrictions on the interchangeability of parts. He classifies their interchangeability as closed, isolated, and/or communicating classes or parts. When the closure, isolation, or communication aspects are relaxed, then the result of Hirsch, Meisner, and Boll that all admissible cannibalizations are equivalent no longer holds. The question becomes: from the set of admissible policies for a given v, are there policies which are "better" than others? With the additional objective of maintaining the most flexibility for future cannibalization, Simon [1970] demonstrates that certain admissible policies are uniformly better than others. In another paper [1972] Simon establishes upper and lower bounds on φT under his more general interchangeability rules and from these bounds he also develops bounds on the probability laws of the state of the system.

In a later work, Hochberg [1973] returns to the interchangeability classes of Hirsch, Meisner, and Boll; however, rather than allowing each part at its location to occupy only one of two states—failed or operating (i.e., 0-1), he assumes it can be in any one of k states

$$\{a_i\}_{i=1}^k \text{ where } 0 = a_k < \ldots < a_2 < a_1 = 1.$$

Each successively smaller state represents a decreasing level of performance. With this generalized description of the status of each part at any point in time and the minimum condition, Hochberg (paral-

leling the work of Hirsch, et al.) obtains a characterization of the state of the system φT (i.e., the performance level over admissible cannibalizations) as a function of the number of working parts at each level a_k. He then develops the probability distribution of φT.

Implicit in the Hirsch, Meisner, and Boll, Simon, and Hochberg works is the assumption that all parts start operating from time zero and are continuously subject to failure as long as the system operates. For many systems when a working part is in a major assembly that has failed for other reasons, this working part does not experience any further deterioration or stress until it is cannibalized and commences to operate again. Rolfe [1970] looked at this aspect of cannibalization. He considered a group of S identical major assemblies which operate independently of each other. A major assembly contains N distinct subassemblies (or parts). Each part is interchangeable with its corresponding part in the other major assemblies, but with no other parts. All working assemblies operate continuously for a period of time T before they are inspected. During this period of operation, parts may fail but the major assemblies continue to function. Upon inspection, failed parts are immediately replaced from an initial stock of spares until the stock is exhausted after which they are replaced by cannibalized parts from other nonworking major assemblies. Assemblies containing any failed parts are not used in the next period of operation. It is assumed that all parts fail independently while operating and that parts of type i have failure distribution $1 - e^{\lambda_i t}$. With these assumptions the state vector of the stochastic process is described by the number of working parts of each type $n_i(t)$ available at the end of each operating period t. This stochastic process forms an $S^N + 1$ state, absorbing Markov chain where the absorbing state is reached when $n_i(t) = 0$ for some $i = 1, 2, \ldots, N$. Rolfe develops the expected number of good major assemblies at the end of any period and because of obvious computational difficulties when S and/or N are large, he gives approximations and lower bounds for this expectation.

As mentioned before, multistage models differ from cannibalization models in that new items enter the system to replace items transferred to other locations. Bartholomew [1963] examined the following model: Suppose a system contains N units with independent but identically distributed times-to-failure. These N units, although stochastically identical, are partitioned into two classes, class I and class II (according, perhaps, to their function or location within the system). Upon the failure of any unit, it must be replaced — at a cost k_1 for items in the first class and at cost k_2 for items in the second class.

The procedure Bartholomew analyzes, the so-called two-stage replacement strategy, is to replace all failures that occur amongst items in class II with items in class I and to replace items in class I, which either failed or were transferred to class II, with new items. There is a cost β for transferring one item from class I to class II. Also, there is a purchase cost per item (which turns out to be irrelevant). It is assumed that replacing and transferring items takes no time.

The total number of unit failures remains unaffected by the above procedure. But the proportion of failures in class I vis-a-vis class II may change. Bartholomew determines the following condition under which a two-stage replacement strategy is better than simple replacement at failure.

If ρ is the transfer rate from class I to class II, n_i is the number of units of class i, and μ is the mean time-to-failure of a unit; then the two-stage strategy is preferable if

$$n_2 (k_2 - k_1) \mu^{-1} - n_1 \rho (k_2 - k_1 + \beta) > 0.$$

The determination of ρ depends on whether an item from class I is selected for transfer to class II on a random basis or by an oldest-first rule. Upon assuming $\beta = 0$, he derives an approximation for ρ for each of these cases.

One implication of his results is that for IFR time-to-failure distributions, $k_1 > k_2$ makes the two-stage scheme preferable to simple replacement at failure. In the DFR case with $k_1 < k_2$, the two-stage scheme is also better.

Naik and Nair [1965a, 1965b] have generalized the above scheme to multistage replacement strategies. Marathe and Nair [1966b] investigate multistage block replacement strategies. This latter model requires the assumption that there is a reserve of units, the "interstage inventory," attached to each class.

In his model of a variable repair rate problem, Crabill [1974] considers a collection of $M+R$ units. The state of the system, $i (i = 1, \ldots, M+R)$, designates the number of these units in operating condition. A cost-per-unit time of $C(i)$ is charged when in state i. Min(i, M) of the i units in operating condition are actually in production, and only these are subject to failure—at the constant hazard rate λ. When a unit fails, it enters the single-server repair facility (or its queue). On the basis of the current state of the system, the decisionmaker selects action $k (k = 1, \ldots, K)$ which provides the repair facility with an exponential repair rate u_k at per-unit-time cost n_k. The object is to minimize long run cost-per-unit time. Using Markov decision theory, Crabill presents sufficient conditions for particular service rates to be eliminated from consideration regardless of state. Furthermore, he provides sufficient conditions ensuring that the optimal service rate is a nonincreasing function of the system's state.

Incomplete Information

As with discrete time models, the decisionmaker may lack complete information about any of the following: the current state of the system (unless he performs an inspection), the probability law governing the system's stochastic behavior (e.g., the time-to-failure distribution of a single unit), and the cost implications of particular operating policies.

When the current state of the system is not known, the problem arises of jointly determining a replacement and inspection policy. In a model by Savage [1962] the state of the unit moves from $x = 0$ (a new unit) to $x = k$, $k = 1, 2, \ldots$, according to a Poisson process. In state x, income is earned at the rate $i(x)$. At a cost L, the decisionmaker can inspect the unit and thereby learn its true state. After each inspection, he elects either to schedule another inspection $T(x)$ time units into the future and not replace the unit at present, or to replace the unit (thus returning to $x = 0$), with the next inspection $T(0)$ time units into the future. Replacement requires m units of time at a cost of c per unit.

The objective of the decisionmaker is to minimize the long run average cost-per-unit time by specifying the "how-long-to-next-inspection" function $T(x)$ and the set W of all states which will call for another inspection rather than replacement. Savage shows that if $i(x)$ is nonincreasing, then $T(x)$ is strictly decreasing for x in W and bounds on $T(x)$ are derived. More explicit results are derived for two special cases of $i(x)$.

Antelman and Savage [1965] consider a parallel problem, namely, the process governing the change of states is Brownian Motion rather than a Poisson process. In particular, it is then possible to move to an improved state without a replacement.

These models were generalized by Chitgopekar [1974]. He considers a larger (but still finite) action space which includes the class of all random time-to-inspection policies while permitting a more general stochastic process to govern the change of states. He shows that the optimal policy is nonrandom.

Keller [1974] utilized optimal control theory for an approximate method of selecting that inspection schedule, for a unit subject to failure, which will minimize cost until the detection of the first failure. Each inspection has cost L and the cost $H(t)$ is incurred when detection of the failure by an inspection occurs time t after failure. This problem is placed in the control theory framework by assuming that the tests are so frequent that they can be described by a smooth density $n(t)$ which denotes the number of checks per unit time. In other words, at time t, the tests are scheduled $1/n(t)$ units of time apart. He then derives an integral equation for $n(t)$ and uses this solution to minimize the expected cost up to detection of the first failure.

Several authors consider the problem of incomplete information concerning the time-to-failure distribution governing the system. A single unit is subject to random failure which can be detected with probability p by an inspection. Each inspection has cost L and $v \cdot t$ is the cost of a failure which remains undetected for a duration t. The problem is to schedule inspections so as to minimize these two costs over the period the unit remains installed which can be no longer than the time when failure is first discovered by an inspection.

Derman [1961] has found a minimax optimal solution to this problem when the time-to-failure distribution is totally unknown. However, it was necessary to assume a finite time horizon T; i.e., the cost accounting stops at either the first inspection to detect failure or at time T, whichever happens first. The reason for this is that for any possible inspection schedule there exists a distribution which would induce an arbitrarily high expected cost during an infinite time horizon. Hence, a minimax solution would not then exist. McCall [1965] gives Derman's results.

Roeloffs [1963, 1967] (letting Derman's probability of detection, p, equal one), found the minimax inspection schedule when a single percentile is all that is known of the time-to-failure distribution. That is, y and π are known where $F(y) = \pi$. He finds the minimax schedule $(x_1, x_2, \ldots, x_{m+n})$, where $0 \leq x_1 \leq \ldots \leq x_m \leq y \leq x_{m+1} \leq \ldots \leq x_{m+n} \leq T$, and the corresponding expected cost for a cost structure identical to Derman's. Naturally, the expected costs are less than Derman's due to the additional information. It is interesting to note that the form of his solution for selecting the inspection points after y, is identical to that of Derman. That is to say, the information contained in the percentile has no bearing in carrying out a minimax surveillance after that percentile is crossed. Roeloffs also finds the optimal inspection schedule to minimize expected cost-per-unit time (rather than per period of installation). However, in doing this, he sets $T = y$. Kander and Raviv [1974] utilize dynamic programming to model this problem for an arbitrary T.

Combining the unknown time-to-failure distribution with the traditional age replacement problem (cost c_2 for replacement at failure, cost $c_1 < c_2$ for scheduled replacement), Fox [1967] used a Bayesian approach. As the realizations of times-to-failure or scheduled replacements are progressively observed, the decisionmaker learns more about the underlying time-to-failure distribution. Using the Weibull time-to-failure distribution with the gamma *a priori* distribution for the parameter, his objective is to minimize the long run expected discounted cost. He derives certain asymptotic optimality conditions on the stationary policy for an arbitrary fixed number of replacements.

In actual maintainability applications, the exact form of the function relating expected cost to the control variable is often not known. In such cases it may be appropriate to precede any attempt at optimization by an experiment which "samples" the resultant costs for various values of the control variable. Then, using least-square techniques, estimate the shape of the actual cost function—thereby permitting an inference as to the optimal value for the control variable.

Such a model was proposed by Dean and Marks [1965] and analyzed by Elandt-Johnson [1967]. If routine maintenance is provided a machine (or vehicle fleet) with frequency x per year, the average resulting cost is $C(x) = bx + D(x)$ where b is the average cost of a scheduled maintenance and $D(x)$ is the unknown expected cost-per-unit time of providing emergency nonscheduled maintenance. Presumably, $D(x)$ is decreasing. $C(x)$ is assumed to possess an absolute minimum, say at x_0. For the specific form

$$D(x) = E\left[\sum_{k=0}^{n} A_k x^k + \Theta \right],$$

where Θ is a random variable with distribution $N(0, \sigma^2)$ and the A_k's are not known, least-square estimates (a_k) are obtained for the coefficients (A_k). By those estimated coefficients, the estimated total cost function can be minimized for $\hat{x}_0$.

Elandt-Johnson [1967] provides a normal distribution which approximates the true distribution of the statistic $\hat{x}_0$. She further shows that $E[C(x_0) - C(\hat{x}_0)]$ depends on the degree and coefficients of the polynomial

$$\sum_{k=1}^{n} A_k x^k.$$

3. REMARKS

The foregoing survey has been primarily expository rather than critical. This approach was taken in order to be able to include most available works on maintainability in a reasonably sized monograph. It was felt that the more papers included, the more useful it would be to a general reader, practitioner, or research scholar or teacher interested in entering the area.

Now that the reader has seen what has been accomplished in maintainability, the question becomes, "what new work needs to be undertaken?"

In the case of deteriorating single-unit inspection, repair and/or replacement models, there have been many generalizations of the basic Derman models. For example, the state space has been allowed to become nondiscrete, the costs a nonstationary random variable and the sojourn time in a state a random variable. It would appear that most future research on these types of models would be to add more system constraints and to develop efficient algorithms to solve the large linear, nonlinear or dynamic programs involved, rather than to provide additional refinements to the assumptions of the basic model. Such efforts will tend to be directed toward particular problems in the sense that the models will become more specialized and tailored to particular situations. Similar comments hold for the area of age-dependent replacement models and the maintenance-salvage models.

The area requiring the most future work is the study of multiechelon multipart interaction maintenance models. This area is perhaps the most difficult to handle mathematically especially when the interactions occur because of stochastic dependence among the parts. In many such cases, very little can be said about the operating characteristics or optimal solutions of the models. Often the only way these problems can be handled is by simulation; consequently, few general results will be obtained in the near future. In the long run, however, work will be initiated to handle more complex kinds of stochastic dependence, where the dependence is restrictively proscribed.

Multipart, multiechelon models which involve economic dependencies are often less difficult to model and solve than the stochastic dependencies. Frequently, as in the inventory-maintenance models, large mathematical programs result and the question becomes how to solve them rather than how to model or describe the problem. Future research will see an increase in the general applicability of these models. Of course, making them more general and/or realistic usually removes the optimality of simple, elegant policies and one is dependent on large computers to solve the complicated mathematical programs involved. Fortunately, many decisionmakers in the military branches, other governmental agencies, industry, and private nonprofit organizations now possess the sophistication to use more complicated techniques and approaches. The phenomenal progress of space age technology is to be thanked for this development, as well as the need to maintain more increasingly complex devices.

Perhaps the most future research will go into the development of new models with different constraints that are needed to handle maintenance problems in health, ecology, and the environment. With the advent of a national health insurance program, there will be an increased effort nationwide on preventive medicine in order to cut costs and to extend the lifetimes of previously disadvantaged groups. Similarly, with the depletion of many of our plant and animal resources, new concepts of maintainability must be developed to restore balances for future populations. Finally, there are many environmental problems that require inspection and corrective action. Models for all of these problems will be built. Some general theory for these models will be established and large scale versions of the problems will be solved.

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