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# Predictive maintenance: The one-unit replacement model

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**Abstract**

Predictive maintenance consists in deciding whether or not to maintain a system according to its state. In this paper, we address the case of one-unit systems, i.e. systems whose state is given by the value taken by one variable. The system is replaced in the case of breakdowns, but also when a maintenance is decided, at a lowest cost than in the breakdowns case.

We first propose a global approach to the problem and provide a characterisation of the optimal maintenance. We then focus on the case when the state transition between two successive periods follows a negative exponential distribution. © 1998 Elsevier Science B.V. All rights reserved.

*Keywords:* Predictive maintenance; Dynamic programming; Markov process; Optimal replacement policy

## 1. Introduction

The maintenance cost ranges between 15% (for manufacturing companies) and 40% (for iron and steel industry) of the cost of the manufactured goods [1]. In the United States, this corresponds to more than 200 billion dollars every year. This shows the importance of maintenance from an economical point of view.

Usually, three different types of maintenance are considered, that is:

1. the corrective maintenance, which consists in repairing a system only after a breakdown occurred;
2. the preventive maintenance, which consists in maintaining the system periodically to prevent a breakdown. Statistics of failures are used to define the period (for instance, every 100 working hours, or every 10 000 kms for a car);
3. the predictive maintenance, which consists in starting a maintenance operation only when required by the state of the system, i.e. when a potential failure is detected.

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We are interested in studying systems whose evolution is represented by a random variable, the value of which represents the state of deterioration of the system. Such a model is referred to as a one-unit model. The goal is to minimise the maintenance policy using the information about the stochastic process which governs the evolution of the random variable and its current value.

One of the first research work was done by Taylor [2]. In the model studied in this work, the system is subject to shocks, the occurrence of which follows a Poisson distribution. The magnitude of the damage resulting from a shock, i.e. the deterioration of the system, follows a negative exponential distribution, and the magnitudes of the damages are independent from each other. The maintenance and repair costs are assumed to be constant. Under these hypotheses, Taylor showed that the optimal policy is a control limit policy (CLP), i.e. that there exists a limit $x^*$ such that if the state of deterioration of the system is greater than $x^*$, but less than a breakdown limit, then a maintenance is required, while no action is required if the state of deterioration is less than $x^*$.

Bergman [3] introduced a model more general than the previous one: the value of the random variable (state of deterioration of the system) is non-decreasing, and the maintenance and repair costs are constant. In this case, Bergman proves that the optimal policy is still a CLP.

In both the previous models, the state of deterioration of the system evolves continuously.

Aven et al. [4] considered the case when the shocks occur at predefined instants. Assuming that the costs are constant, they showed that the optimal policy is a CLP.

Zuckerman [5] provided the conditions under which the optimal policy is a CLP, that is the properties of the physical systems to which a CLP applies.

We are interested in studying systems whose deterioration is continuous but known periodically, since this situation corresponds to the practical case of systems supervised numerically. This paper addresses some theoretical aspects of the predictive maintenance in this case. It should be noticed that the main goal of this paper is to provide a mathematical tool to manage predictive maintenance for systems the behaviour of which is characterised by probability rules that are as general as possible. The results provided hereafter extend and generalise the ones provided by the references presented at the end of the publication.

In Section 2, we propose a global approach of the problem defined as a discrete-time infinite state space model. In this section, we provide a characterisation of the optimal maintenance policy for one-unit systems. A good review of the work done for one-unit systems, i.e. systems whose state is given by a scalar, is given in the chapter “shock models” of Ref. [6].

Section 3 is devoted to the particular case when a state transition between two successive periods follows a negative exponential distribution. In this case, it is possible to provide some simple results about the optimal policy, i.e. the least expensive maintenance policy.

Section 4 is the conclusion.

## 2. Global approach to the discrete-time infinite state space problem

The models already developed in the literature have two main drawbacks, that is (i) the state of the system is supposed to be known continuously along the time and to be defined by parameters taking discrete values, and (ii) the only possible maintenance action consists in replacing a unit at a constant cost.

The model used in this paper seems to be closer to real-life situations than the models previously proposed in the literature, since it is less constrained by the properties of the probability rules.

### 2.1. Problem formulation

The state of the system is represented by the value taken by a random variable $X$. This random variable is observed at times $t_i$ ($i = 1, 2, \dots$), and $x_i$ is the value of $X$ at time $t_i$. In other words, $x_i$ is the state of the system at time $t_i$. This state is equal to 0 if the system is new. The greater $x_i$, the closer the system to a breakdown. We assume that

$x_{i+1}$ depends only on $x_i$; thus, the stochastic process ($x_i$, $i = 0, 1, \dots$) is a discrete-time infinite state space Markov process. We denote by $f(x_{i+1}, x_i)$ the probability density function (PDF) which provides the probability for $X$ to take the value $x_{i+1}$ at time $t_{i+1}$ given that its value at time $t_i$ is $x_i$.

The system breaks down if the value of $X$ exceeds a given value $L$. In this case, the system has to be replaced by a new one at cost $R(x)$, where $x$ is the value of $X$ at the breakdown state.

We further assume that the following properties hold for the PDF function $f(x_{i+1}, x_i)$ (these properties are referred to as ($P_1$) in the remaining of the paper):

1. $f(x_{i+1}, x_i)$ is integrable in $x_{i+1}$ on $[h(x_i), +\infty)$, $h(x_i) \in [0, +\infty)$: note that $h(x_i)$ may be, in some cases, less than $x_i$;
2. if $a$ and $b$ are two real values such that $a < b$, there exists one real value $v$ which verifies

   $f(v, a) = f(v, b)$,

   $f(v^+, a) < f(v^+, b)$,

   $f(x, a) \geqslant f(x, b), \quad \forall x < v,$

   and

   $f(x, a) \leqslant f(x, b), \quad \forall x > v.$

Indeed, $h(x_i)$ is usually greater than or equal to $x_i$ since the state of the system becomes closer to breakdown as the time increases.

Condition (2) is given to introduce a general shape for the probability distribution: as the parameter increases, the maximum value of the density decreases and the standard deviation increases.

An example of such a PDF is represented in Fig. 1.

Note that any set of PDFs such that $f(u, x) = f(u - x, 0)$ and such that $f(\bullet, x)$ is either unimodal or decreasing with respect to $x$ verifies ($P_1$).

These properties are realistic. They mean that the lower $x_i$, the more likely $x_{i+1}$ is low. In other words, the farther the system from the breakdown at time $t_i$, the more likely the system is far from breakdown at time $t_{i+1}$.

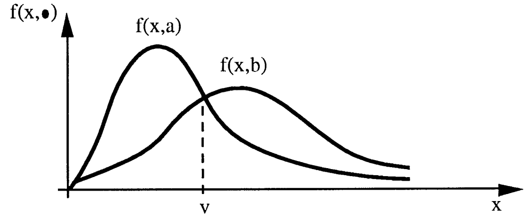


Fig. 1. Example of PDF.

There are two options when the system is in working order at time $t_i$, i.e. when the value $x_i$ of the random variable $X$ is less than $L$:

1. either no intervention is decided; in this case $c(x_i)$ has to be paid to guarantee a standard working mode of the system on $[t_i, t_{i+1})$ ($c(x)$ is a non-decreasing function of $x$),
2. or a preventive replacement of the system is decided; in this case, $r(x_i)$ has to be paid and the value of the random variable is reinitialised at 0; in this case, $c(0)$ has to be paid to guarantee a standard working mode of the system on $[t_i, t_{i+1})$.

To simplify the presentation, we assume that $R(\bullet)$ and $r(\bullet)$ are constant, i.e. $R(x) = R, \quad \forall x > L$ and $r(x) = r, \quad \forall x \in [0, L]$, and that $R > r$. The last assumption means that it is always cheaper to anticipate a replacement of the system than to wait until a breakdown occurs.

We introduce a function $d(\bullet)$ which indicates the decision to be made according to the value $x$ of variable $X$. This function is defined as follows:

$$d(x) = \begin{cases} x & \text{if no replacement is done,} \\ 0 & \text{if the system is replaced.} \end{cases}$$

Function $d(x)$ indicates the next state of the system according to the decision made when the current state is $x$.

As mentioned before, $d(\bullet) = 0$ brings the value of $X$ back to 0. We assume that this change is instantaneous. In other words, when it is decided to replace at time $t_i$, the replacement is supposed to be completed at time $t_i$. This assumption is only to simplify the notations and does not modify the results: introducing replacement times would only result in some translations of the state of the system along the time axis.

The goal is to minimise the expected cumulative cost of the maintenance policy at horizon $t_{N-1}$, i.e. on $[0, t_{N-1}]$, knowing that the initial state is $x_0$.

The maintenance policy $d(\bullet)$ being known, the mathematical expectation of the cost is

$$J_d^N(x_0) = E\left[\sum_{k=0}^{N-1} g_d(x_k)\right], \tag{1}$$

where $E$ denotes the mathematical expectation and $g_d(x_k)$ is the cost corresponding to policy $d(\bullet)$ for period $[t_k, t_{k+1})$,

$$g_d(x_k) = \begin{cases} c(x_k) & \text{if } \delta(x_k) = 1 \text{ and } d(x_k) = x_k, \\ c(0) + r & \text{if } \delta(x_k) = 1 \text{ and } d(x_k) = 0, \\ c(0) + R & \text{if } \delta(x_k) = 0. \end{cases}$$

In this formulation

$$\delta(x_k) = \begin{cases} 1 & \text{if } x_k \in [0, L] \text{ (the system is in working order)}, \\ 0 & \text{otherwise.} \end{cases}$$

If $D$ is the set of possible policies, the goal is to find the optimal policy $d^* \in D$ such that

$$J_{d^*}^N(x_0) = \operatorname*{Min}_{d \in D} J_d^N(x_0) = J^N(x_0).$$

The fact that $D$ is the set of possible policies can be expressed by saying that $D$ is the set of functions $d$: $[0, +\infty) \to [0, +\infty)$ such that $d(x) = 0$ *or* $x$, depending on the decision made.

### 2.2. Optimal policy on a finite horizon

In this section, we first formulate the problem as a dynamic programming problem. We then derive some properties of the optimal solution from the dynamic programming formulation.

#### 2.2.1. Dynamic programming formulation

The state $x_0$ of the system is known at time $t_0$. If the system is in working order ($\delta(x_0) = 1$) we may or may not replace the system. In the first case, the cost incurred on period $[t_0, t_1)$ is $r + c(0)$, which corresponds to the replacement cost plus the running cost on $[t_0, t_1)$ when the state is 0 at time $t_0$. If no replacement is decided, the cost corresponding to the same period is $c(x_0)$. If the system is out of order ($\delta(x_0) = 0$) then we have to replace the system, and the cost is $c(0) + R$.

Finally, the following equalities hold:

If $\delta(x_0) = 1$,

$$J^N(x_0) = \operatorname{Min}\left\{c(x_0) + \operatorname*{Min}_{d \in D}\left(E\left[\sum_{k=1}^{N-1} g_d(x_k)/x_0\right]\right),\right.$$
$$\left. r + c(0) + \operatorname*{Min}_{d \in D}\left(E\left[\sum_{k=1}^{N-1} g_d(x_k)/x_0 = 0\right]\right)\right\}. \tag{2}$$

If $\delta(x_0) = 0$,

$$J^N(x_0) = R + c(0) + \operatorname*{Min}_{d \in D}\left(E\left[\sum_{k=1}^{N-1} g_d(x_k)/x_0 = 0\right]\right). \tag{3}$$

The conditional expectation

$$E\left[\sum_{k=1}^{N-1} g_d(x_k)/x_0\right]$$

is the expected cumulative cost of the maintenance policy $d$ on $[t_1, t_{N-1}]$, that is on a horizon of length $N-1$, knowing that $x_0$ was the state at time $t_0$.

This conditional expectation can be expressed using PDF $f(x_1, x_0)$, assuming that this function is integrable on $[h(x_0), +\infty)$. We obtain

$$E\left[\sum_{k=1}^{N-1} g_d(x_k)/x_0\right] = \int_{h(x_0)}^{+\infty} E\left[\sum_{k=1}^{N-1} g_d(x_k)\right] f(x_1, x_0)\, \mathrm{d}x_1.$$

Considering relation (1), this can be rewritten as

$$E\left[\sum_{k=1}^{N-1} g_d(x_k)/x_0\right] = \int_{h(x_0)}^{+\infty} J_d^{N-1}(x_1) f(x_1, x_0)\, \mathrm{d}x_1.$$

Since $J^{N-1}(x_1) \leqslant J_d^{N-1}(x_1)$, $\forall x_1 \in [h(x_0), +\infty)$, $\forall d \in D$, it is possible to write

$$\operatorname*{Min}_{d \in D}\left(\int_{h(x_0)}^{+\infty} J_d^{N-1}(x_1) f(x_1, x_0)\, \mathrm{d}x_1\right)$$

$$= \int_{h(x_0)}^{+\infty} \min_{d \in D} (J_d^{N-1}(x_1)) f(x_1, x_0)\, \mathrm{d}x_1$$

$$= \int_{h(x_0)}^{+\infty} J^{N-1}(x_1) f(x_1, x_0)\, \mathrm{d}x_1.$$

Eqs. (2) and (3) can be simplified as

If $\delta(x_0) = 1$,

$$J^N(x_0) = \mathrm{Min}\left\{ c(x_0) + \int_{h(x_0)}^{+\infty} J^{N-1}(x_1) f(x_1, x_0)\, \mathrm{d}x_1, \right.$$
$$\left. r + c(0) + \int_{h(x_0)}^{+\infty} J^{N-1}(x_1) f(x_1, 0)\, \mathrm{d}x_1 \right\}.$$

If $\delta(x_0) = 0$,

$$J^N(x_0) = R + c(0) + \int_{h(x_0)}^{+\infty} J^{N-1}(x_1) f(x_1, x_0)\, \mathrm{d}x_1.$$

Assuming that a steady-state exists, we finally obtain the following dynamic programming formulation:

$$J^N(x) = \mathrm{Min}\left\{ c(x) + \int_{h(x)}^{+\infty} J^{N-1}(s) f(s, x)\, \mathrm{d}s, r + J^N(0) \right\} \quad \text{if } 0 < x \leqslant L, \tag{4}$$

$$J^N(x) = R + J^N(0) \quad \text{if } x > L. \tag{5}$$

Furthermore, since no maintenance occurs when the state of the system (i.e. the random variable) is equal to 0, we have

$$J^N(0) = c(0) + \int_{h(0)}^{+\infty} J^{N-1}(v) f(v, 0)\, \mathrm{d}v. \tag{4'}$$

The border conditions are, indeed: $J^N(x) = 0$, $\forall x$ and $N \leqslant 0$.

### 2.2.2. Properties of the optimal solution

We introduce Lemma 1 which will be useful to characterise the optimal policy.

**Lemma 1.** *Let $f(u, x)$ be a PDF which verifies properties $(P_1)$ and, in addition, let us assume that $h(x)$ is a non-decreasing function of $x$. Then, if $t(x)$ is a non-decreasing function, integrable and positive on $E_x = [h(x), +\infty)$ for any $x \in [0, +\infty)$, the following property holds*:

$$B(x) = \int_{h(x)}^{+\infty} f(u, x) t(u)\, du \quad \textit{is non-decreasing in } x.$$

**Proof.** Let $x_0$ and $x_1$ be two real-positive values such that $x_1 > x_0$. We want to prove that $B(x_1) - B(x_0) \geqslant 0$.

$$B(x_1) - B(x_0) = \int_{h(x_1)}^{+\infty} f(u, x_1) t(u)\, \mathrm{d}u - \int_{h(x_0)}^{+\infty} f(u, x_0) t(u)\, \mathrm{d}u,$$

$h(x)$ being a non-decreasing function, we can write

$$B(x_1) - B(x_0) = \int_{h(x_1)}^{+\infty} [f(u, x_1) - f(u, x_0)] t(u)\, \mathrm{d}u - \int_{h(x_0)}^{h(x_1)} f(u, x_0) t(u)\, \mathrm{d}u$$

and

$$\int_{h(x_0)}^{h(x_1)} f(u, x_0) t(u)\, \mathrm{d}u = C(x_0, x_1) \geqslant 0.$$

Let $A(x_0, x_1)$ be the point such that $f(A(x_0, x_1), x_0) = f(A(x_0, x_1), x_1)$. Such a point exists according to property $(P_1)$ – (2). Thus, $B(x_1) - B(x_0)$ can be rewritten as

$$B(x_1) - B(x_0) = \int_{h(x_1)}^{A(x_0, x_1)} [f(u, x_1) - f(u, x_0)] t(u)\, \mathrm{d}u + \int_{A(x_0, x_1)}^{+\infty} [f(u, x_1) - f(u, x_0)] t(u)\, \mathrm{d}u - C(x_0, x_1). \tag{6}$$

We know that $t(u)$ is a non-decreasing function of $u$. As a consequence:

$t(A(x_0, x_1)) \leqslant t(u), \quad \forall u \geqslant A(x_0, x_1)$.

But, setting $t(A(x_0, x_1)) = K$, equality (6) and property $(P)$ – (2) leads to

$$B(x_1) - B(x_0) \geqslant \int_{h(x_1)}^{A(x_0, x_1)} [f(u, x_1) - f(u, x_0)]t(u)\,\mathrm{d}u + K \int_{A(x_0, x_1)}^{+\infty} [f(u, x_1) - f(u, x_0)]\,\mathrm{d}u - C(x_0, x_1). \tag{7}$$

According to the generalised mean-value theorem, there exists $\alpha \in [t(h(x_1)), t(A(x_0, x_1))]$ such that

$$\int_{h(x_1)}^{A(x_0, x_1)} [f(u, x_1) - f(u, x_0)]t(u)\,\mathrm{d}u = \alpha \int_{h(x_1)}^{A(x_0, x_1)} [f(u, x_1) - f(u, x_0)]\,\mathrm{d}u.$$

Thus, inequality (7) can be rewritten as

$$B(x_1) - B(x_0) \geqslant \alpha \int_{h(x_1)}^{A(x_0, x_1)} [f(u, x_1) - f(u, x_0)]\,\mathrm{d}u + K \int_{A(x_0, x_1)}^{+\infty} [f(u, x_1) - f(u, x_0)]\,\mathrm{d}u - C(x_0, x_1).$$

This inequality leads to

$$B(x_1) - B(x_0) \geqslant \alpha \int_{h(x_1)}^{A(x_0, x_1)} [f(u, x_1) - f(u, x_0)]\,\mathrm{d}u + K\left\{1 - \int_{h(x_1)}^{A(x_0, x_1)} f(u, x_1)\,\mathrm{d}u - 1 + \int_{h(x_0)}^{A(x_0, x_1)} f(u,x_0)\,\mathrm{d}u\right\} - C(x_0, x_1).$$

Thus,

$$B(x_1) - B(x_0) \geqslant (\alpha - K) \int_{h(x_1)}^{A(x_0, x_1)} [f(u, x_1) - f(u, x_0)] \times \mathrm{d}u + K \int_{h(x_0)}^{h(x_1)} f(u, x_0)\,\mathrm{d}u - C(x_0, x_1). \tag{8}$$

Since $\alpha \leqslant K$ and $f(u, x_1) \leqslant f(u, x_0)$ on $[h(x_1), A(x_0, x_1)]$, the first term of the second member of inequality (8) is non-negative.

Furthermore, using the generalised mean-value theorem

$$C(x_0, x_1) = \int_{h(x_0)}^{h(x_1)} f(u, x_0)t(u)\,\mathrm{d}u = \beta \int_{h(x_0)}^{h(x_1)} f(u, x_0)\,\mathrm{d}u$$

with $t(h(x_0)) \leqslant \beta \leqslant t(h(x_1))$,

and

$$K \int_{h(x_0)}^{h(x_1)} f(u, x_0)\,\mathrm{d}u - C(x_0, x_1) = (K - \beta) \int_{h(x_0)}^{h(x_1)} f(u, x_0)\,\mathrm{d}u \geqslant 0,$$

since $\beta \leqslant t(h(x_1)) \leqslant t(A(x_0, x_1)) = K$.

Finally, the second member of inequality (8) is positive, and $B(x_1) - B(x_0) \geqslant 0$ if $x_1 > x_0$: $B(x)$ is non-decreasing. □

We use Lemma 1 to prove that the expected cumulative cost is a non-decreasing function of the value of variable $X$ which models the state of the system.

**Theorem 1.** *When the replacement costs $r$ and $R$ are constant, the optimal expected cumulative cost on horizon $N$, i.e. $J^N(x)$, is a non-decreasing function of $x$.*

**Proof.** According to relation (5), $J^N(x)$ is constant for $x > L$, and Theorem 1 holds in this case. We now assume that $x \in [0, L]$. Since $c(x)$ is a non-decreasing function of $x$,

$$c(x) + \int_{h(x)}^{+\infty} J^{N-1}(s) f(s, x)\,\mathrm{d}s$$

is a non-decreasing function of $x$ if $J^{N-1}(x)$ is a non-decreasing function of $s$ (see Lemma 1). Furthermore, $r + J^N(0)$ is constant. Thus, according to relation (4), $J^N(x)$ is a non-decreasing function of $x$ if $J^{N-1}(s)$ is a non-decreasing function of $s$.

As a consequence (induction reasoning), $J^N(x)$ is a non-decreasing function of $x$ if $J^1(x)$ is a non-decreasing function of $x$.

$J^1(x)$ is the optimal maintenance cost on one period. Thus,

$$J^1(x) = g_{d^*}(x) = \begin{cases} \mathrm{Min}\{c(x), c(0) + r\} & \text{if } x \leqslant L, \\ c(0) + R & \text{if } x > L, \end{cases}$$

which shows that $J^1(x)$ is not decreasing. □

Using Theorem 1, we established Theorem 2 which claims that an optimal replacement policy is a "control limit" policy.

**Theorem 2.** *When the replacement costs r and R are constant, there exists a value* $\lambda_N^*$ *such that the following policy is optimal*:

(i) *for* $0 \leqslant x \leqslant \lambda_N^*$, $d^*(x) = x$, *i.e. the decision is to let the system evolve as it is*,

(ii) *for* $\lambda_N^* \leqslant x \leqslant L$, $d^*(x) = 0$, *i.e. the decision is to replace the system.*

**Proof.** According to Theorem 1;

$$c(x) + \int_{h(x)}^{+\infty} J^{N-1}(s) f(s, x)\,\mathrm{d}s$$

is a non-decreasing function of $x$ since $c(x)$ and $J^{N-1}(x)$ are non-decreasing functions of $x$.

Furthermore, according to equality (4), the optimal decision is to replace the system as soon as

$$c(x) + \int_{h(x)}^{+\infty} J^{N-1}(s) f(s, x)\,\mathrm{d}s \geqslant r + J^N(0).$$

Thus,

$$\lambda_N^* = \mathop{\mathrm{Min}}_{\lambda \in [0,\ \infty)} \left\{ \lambda; c(\lambda) + \int_{h(\lambda)}^{+\infty} J^{N-1}(v) f(v, \lambda)\,\mathrm{d}v \geqslant r + J^N(0) \right\}. \quad \square \tag{9}$$

**Remark.** $\lambda_N^*$ may be greater than $L$. In this case, the optimal policy consists in replacing the system only in case of breakdown.

The evolution of the state of the system is similar to the one represented in Fig. 2.

*2.2.3. A numerical example*

In this example, the breakdown limit is $L = 10$. The PDF is a negative exponential function

$$f(u, x) = \begin{cases} \mu \mathrm{e}^{-\mu(u-x)} & \text{if } u > x, \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

with $\mu = 1$.

The costs are defined as follows:

1. standard working cost

$$c(x) = \begin{cases} x/8 & \text{if } x < 7.5, \\ x/4 & \text{otherwise,} \end{cases}$$

2. preventive maintenance cost

$r = 1.5$,

3. repair cost

$R = 5$.

By applying the dynamic programming Eqs. (4) and (5) and Eq. (9), we obtain the maintenance limit $\lambda_N^*$ and the optimal mathematical expectation of the average cost as a function of horizon $N$. The results are grouped in Table 1.

As we can see, the maintenance limit $\lambda_N^*$ is small compared to the breakdown limit. This is due to the facts that the standard working cost is much smaller if $x < 7.5$ than if $7.5 \leqslant x \leqslant 10$, and that $R$ is much bigger than $r$, compared to the mean value of the difference between two successive values of $X$, which is 1. Furthermore, the maintenance limit $\lambda_N^*$ converges much faster than the average cost.

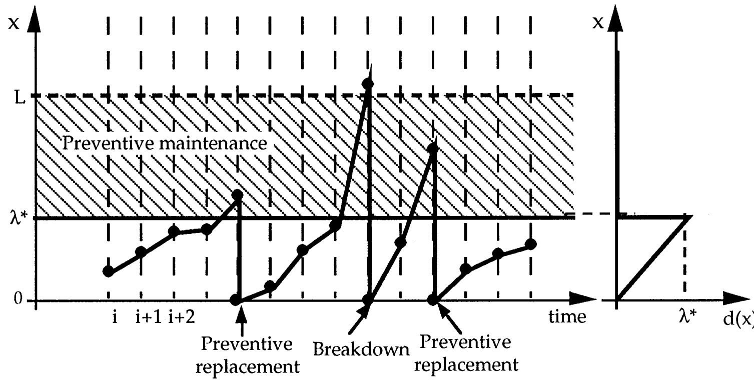


Fig. 2. Evolution of the state of the system.

Table 1
Maintenance limit and optimal cost

| | $N = 10$ | $N = 20$ | $N = 50$ | $N = 100$ |
|---|---|---|---|---|
| $\lambda_N^*$ | 3.92 | 3.95 | 3.95 | 3.95 |
| $I^N(0)$ | 0.37 | 0.44 | 0.48 | 0.49 |

## 3. The negative exponential distribution

The case when the PDF is negative exponential can be solved using the approach introduced in Section 2, assuming that the repair cost $R$ and the replacement cost are constant, and assuming also that $c(x)$ is a non-decreasing function of $x$.

There is also a more immediate approach when the PDF is a negative exponential function. According to the results of Section 2, we know that the control policy is a CLP. Thus, the only remaining computational effort is the one which leads to the control limit: it is the goal of this section.

To simplify the presentation, we assume that the standard working cost is constant. As a consequence, we can ignore this cost.

### 3.1. Problem formulation

Let $\lambda$ be the parameter such that

1. when the state of the system is less than or equal to $\lambda$ ($0 \leqslant x \leqslant \lambda$), no maintenance (replacement) occurs, and the cost is equal to zero until the next checking time,
2. when the state of the system is greater than $\lambda$ and less than or equal to $L$, a preventive maintenance is performed at cost $r$, and the state of the system becomes instantaneously equal to zero.

Furthermore, when the system breaks down, i.e. when $x > L$, then it is instantaneously replaced by a new system at cost $R$.

The average cost to be minimised is thus

$$C(\lambda) = r\,\Pr\{\lambda < X \leqslant L\} + R\,\Pr\{X > L\}.$$

By setting $\alpha = R/r$, the function to be minimised becomes

$$\varphi(\lambda) = \frac{1}{r}C(\lambda) = \Pr\{\lambda < X \leqslant L\} + \alpha\,\Pr\{X > L\}$$

and $\alpha \geqslant 1$. This function is the average normalised cost.

The goal is to define $\lambda^*$ such that

$$\varphi(\lambda^*) = \min_{\lambda \in (0,\, L]} \varphi(\lambda).$$

In the particular case considered in this section, it is possible to find an analytical formulation of the cost according to $\lambda$ and the parameter $\mu$ of the negative exponential PDF.

### 3.2. Density of the random variable $X$

In Theorem 3, we give the probability density of $X$ in steady state. We denote this probability density by $p(x)$.

**Theorem 3.** *In steady state, if the maintenance limit is* $\lambda$
(i) $p(x)$ *is constant if* $0 \leqslant x \leqslant \lambda$,
(ii) $p(x)$ *decreases exponentially if* $x > \lambda$.
*More precisely*

$$p(x) = \begin{cases} \dfrac{\mu}{1+\mu\lambda} & \text{for } x \leqslant \lambda, \\ \mu\dfrac{e^{-\mu(x-\lambda)}}{1+\mu\lambda} & \text{for } x > \lambda. \end{cases}$$

**Proof.** Using $\vee$ as the logical operator "or" and $\wedge$ as the logical operator "and", the probability density can be written as

$$\begin{aligned} p(x)\,dx &= \Pr\{x \leqslant X_i < x + dx\} \\ &= \Pr\{(x \leqslant X_i < x + dx) \\ &\quad \wedge (X_{i-1} \leqslant \lambda \vee X_{i-1} > \lambda)\} \\ &= \Pr\{(x \leqslant X_i < x + dx) \wedge (X_{i-1} \leqslant \lambda)\} \\ &\quad + \Pr\{(x \leqslant X_i < x + dx) \wedge (X_{i-1} \leqslant \lambda)\}. \end{aligned} \tag{11}$$

The first term of the right-hand side of Eq. (11) can be rewritten as

$$\begin{aligned} &\Pr\{(x \leqslant X_i < x + dx) \wedge (X_{i-1} \leqslant \lambda)\} \\ &= \int_{0+}^{\mathrm{Min}(x,\lambda)} \Pr\{(x \leqslant X_i < x + dx) \wedge (z \leqslant X_{i-1} \\ &\leqslant z + dz)\} = \int_{0+}^{\mathrm{Min}(x,\lambda)} \Pr\{(x \leqslant X_i < x + dx)/ \\ &(X_{i-1} = z)\}\Pr\{z \leqslant X_{i-1} < z + dz\}. \end{aligned} \tag{12}$$

The second term of the right-hand side of Eq. (11) can be rewritten as

$$\begin{aligned} &\Pr\{(x \leqslant X_i < x + dx) \wedge (X_{i-1} > \lambda)\} \\ &= \int_{\lambda}^{+\infty} \Pr\{(x \leqslant X_i < x + dx) \wedge (z \leqslant X_{i-1} \\ &\leqslant z + dz)\} \\ &= \int_{\lambda}^{+\infty} \Pr\{(x \leqslant X_i < x + dx)/ \\ &(X_{i-1} = 0)\}\Pr\{z \leqslant X_{i-1} < z + dz\}. \end{aligned} \tag{13}$$

Using Eqs. (12) and (13), relation (11) becomes

$$\begin{aligned} p(x)\,dx &= \Pr\{(x \leqslant X_i < x + dx)\} \\ &= \int_{0+}^{\mathrm{Min}(x,\lambda)} \Pr\{(x \leqslant X_i < x + dx)/(X_{i-1} = z)\} \\ &\quad \times p(z)\,dz + \int_{\lambda}^{+\infty} \Pr\{(x \leqslant X_i < x + dx)/ \\ &\quad (X_{i-1} = 0)\}p(z)\,dz. \end{aligned}$$

Or, by introducing $f(\bullet, \bullet)$ (see Eq. (10)), we obtain

$$p(x) = \int_{0+}^{\mathrm{Min}(x,\lambda)} \mu e^{-\mu(x-z)} p(z)\,dz + \int_{\lambda}^{+\infty} \mu e^{-\mu x} p(z)\,dz. \tag{14}$$

If $x \leqslant \lambda$, we obtain the derivate

$$\begin{aligned} \frac{dp(x)}{dx} &= \frac{d}{dx}\left[\int_{0+}^{x} \mu e^{-\mu(x-z)} p(z)\,dz + \int_{\lambda}^{+\infty} \mu e^{-\mu x} p(z)\,dz\right] \\ &= \mu \int_{0+}^{x} \mu e^{-\mu(x-z)} p(z)\,dz + \mu p(x) \\ &\quad - \mu \int_{\lambda}^{+\infty} \mu e^{-\mu x} p(z)\,dz \\ &= \mu p(x) - \mu p(x) = 0. \end{aligned}$$

The derivate being equal to zero, $p(x)$ is a constant $c$ for $x \leqslant \lambda$.

We further consider the case when $x > \lambda$. Eq. (14) becomes

$$p(x) = \int_{0+}^{\lambda} \mu e^{-\mu(x-z)} c\,dz + \mu e^{-\mu x} \int_{\lambda}^{+\infty} p(z)\,dz. \tag{15}$$

We set

$$P_0 = \int_{\lambda}^{+\infty} p(z)\,\mathrm{d}z,$$

which is the probability to make a preventive maintenance or to replace the system on account of breakdowns. Using $P_0$, we obtain

$$p(x) = \int_{0+}^{\lambda} \mu \mathrm{e}^{-\mu(x-z)} c\,\mathrm{d}z + \mu \mathrm{e}^{-\mu x} P_0$$

$$= \mu \mathrm{e}^{-\mu x}\left(\frac{c}{\mu}(\mathrm{e}^{\mu\lambda} - 1) + P_0\right).$$

Furthermore,

$$\Pr\{0 < X < \lambda\} + \Pr\{X \geqslant \lambda\} = 1$$

or

$$\int_{0+}^{\lambda} c\,\mathrm{d}z + P_0 = 1,$$

which leads to

$$c = \frac{\mu}{1+\mu\lambda} \quad \text{and} \quad P_0 = \frac{1}{1+\mu\lambda}.$$

The formulation of the density is straightforward. □

### 3.3. Optimal policy

We first calculate the cost according to $\lambda$, parameters $\alpha$ and $\mu$ being known.

**Theorem 4.** *In steady state, the average normalised cost, that is the cost per time unit, is*

$$\varphi(\lambda) = \frac{(\alpha - 1)\mathrm{e}^{-\mu(L-\lambda)} + 1}{1 + \mu\lambda}.$$

**Proof.** *Using the formulation of the PDF $f(\bullet,\bullet)$, we obtain*

$$\Pr\{\lambda < X_t \leqslant L\} = P\{(X_t > \lambda) \wedge (X_{t-1} \leqslant \lambda)\} - P\{(X_t > L) \wedge (X_{t-1} \leqslant \lambda)\}$$

and

$$\Pr\{X > L\} = \int_{L}^{+\infty} \left(\mu\,\mathrm{e}^{-\mu x_i} P_0 + \int_{0+}^{\lambda} c\mu\,\mathrm{e}^{-\mu(x_i - x_{i-1})}\,\mathrm{d}x_{i-1}\right)\mathrm{d}x_i$$

$$= \frac{\mathrm{e}^{-\mu(L-\lambda)}}{1+\mu\lambda}.$$

Since $\varphi(\lambda) = \Pr\{\lambda < X \leqslant L\} + \alpha \Pr\{X > L\}$, Theorem 4 holds. □

**Theorem 5.** *$\varphi$ is a convex function. As a consequence, $\lambda^*$ such that*

$$\varphi(\lambda^*) = \underset{\lambda \in [0, +\infty)}{\mathrm{Min}} \varphi(\lambda)$$

*is unique, and it verifies*

$$\lambda^* \mathrm{e}^{\mu\lambda^*} = \frac{1}{\mu \mathrm{A}} \quad \text{with } A = (\alpha - 1)\mathrm{e}^{-\mu L}.$$

*Indeed, $\lambda^*$ may or not be less than L.*

**Proof.** The average normalised cost (see Theorem 4) can be rewritten as

$$\varphi(\lambda) = \frac{A\,\mathrm{e}^{\mu\lambda} + 1}{1+\mu\lambda} \quad \text{with } A = (\alpha - 1)\mathrm{e}^{-\mu L}.$$

Thus,

$$\frac{\mathrm{d}\varphi(\lambda)}{\mathrm{d}\lambda} = \frac{\mu^2 A \lambda\,\mathrm{e}^{\mu\lambda} - \mu}{(1+\mu\lambda)^2}$$

and

$$\frac{\mathrm{d}^2\varphi(\lambda)}{\mathrm{d}\lambda^2} = \frac{\mu^2(2 + A\,\mathrm{e}^{\mu\lambda} + A\mu^2\lambda^2\,\mathrm{e}^{\mu\lambda})}{(1+\mu\lambda)^3} \geqslant 0.$$

As a consequence, $\varphi(\lambda)$ is convex and the minimum of $\varphi(\lambda)$ is obtained for $\mathrm{d}\varphi(\lambda)/\mathrm{d}x = 0$, that is for $\lambda*$ such that

$$\lambda^* \mathrm{e}^{\mu\lambda^*} = \frac{1}{\mu A}. \quad \square \tag{16}$$

**Remarks.**

1. If $\lambda^* > L$, then it is optimal to never decide a preventive maintenance: the optimal behaviour

is to wait for a breakdown before replacing the system.

2. $\lambda^*$ is greater than $L$ when $\mu L\, e^{2\mu L} < 1/(\alpha - 1)$. When $\alpha$ tends towards 1 that is when $r$ tends towards $R$, then $1/(\alpha - 1)$ tends to $+\infty$ and the above inequality always holds. In other words, when the cost of a preventive replacement tends towards the cost of a replacement due to a breakdown, the optimal behaviour is to react only in case of breakdown.
3. According to Theorem 5, $\lambda^*$ exists and is unique. It can be obtained, for instance, by applying Newton's iterative approach.

## 4. Conclusions

In this paper, a general predictive replacement model has been presented for one-unit systems. This model is based on dynamics programming. It leads to an optimal predictive maintenance policy. Under some hypotheses which fit with most of the practical situations, it has been proven that the policy is a control limit policy.

We also provided an analytical approach if the transition probability between states is a negative exponential function.

The next step will consist in studying the model where various levels of maintenance are possible, each level having different costs. We will also extend the study to two-unit models.

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