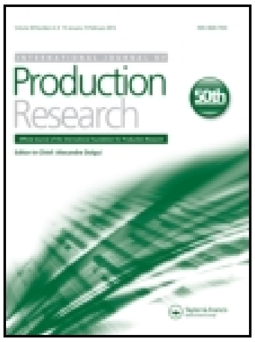

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# Integrated maintenance planning and production scheduling with Markovian deteriorating machine conditions

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In many industries, production capacity diminishes as machine conditions deteriorate. Maintenance operations improve machine conditions, but also occupy potential production time, possibly delaying the customer orders. Therefore, one challenge is to determine the joint maintenance and production schedule to minimize the combined costs of maintenance and lost production over the long term. In this paper, we address the problem of integrated maintenance and production scheduling in a deteriorating multi-machine production system over multiple periods. Assuming that at the beginning of each period the demand becomes known and machine conditions are observable, we formulate a Markov decision process model to determine the maintenance plan and develop sufficient conditions guaranteeing its monotonicity in both machine condition and demand. We then formulate an integer programming model to find the maintenance and the production schedule in each period. Our computational results show that exploiting online condition monitoring information in maintenance and production decisions leads to 21% cost savings on average compared to a greedy heuristic and that the benefit of incorporating long-term information in making short-term decisions is highest in industries with medium failure rates.

**Keywords:** maintenance planning; maintenance scheduling; scheduling; flow shop scheduling; optimization; operations management

## 1. Introduction

Machine deterioration is one of the main causes of production capacity reduction and consequently of delay in customer orders in many manufacturing industries (Kaufman and Lewis 2007; Sloan 2008). For example, a dull drill bit, a contaminated cooling system or a worn-out crankshaft sensor in manufacturing can significantly slow down production. Improving machine conditions by maintenance is a strategy to restore production capacity, increasing on-time delivery of customer orders. However, maintenance results in periods of machine unavailability that could be otherwise used for processing customer orders. Therefore, utilizing machine condition information to simultaneously schedule maintenance and production activities, maximizing the number of customer orders delivered by their due dates is a challenging problem.

The problem of integrated maintenance and production scheduling has been addressed in the scheduling community but the modelling of maintenance concepts is typically not as rich as in the maintenance literature. More specifically, there is no model of the effect of maintenance on machine condition (Ma, Chu, and Zuo 2010), there is no explicit connection between machine deterioration and the processing times of the customer orders and there are no decisions about maintenance planning as the time windows for maintenance are typically given (e.g. Kuo and Yang 2008; Mosheiov and Sidney 2010; Kellerer, Rustogi, and Strusevich 2013).

In this paper, we address the problem of integrated maintenance and production scheduling in a multi-machine production environment over multiple periods where different customer orders have to be scheduled on machines in sequence and are due at different times. The goal is to determine the assignment of maintenance to machines and to find the schedule of the orders in each time period. Unlike the work in the scheduling literature, we adopt a rich model of maintenance concepts as have been defined in the maintenance community (Cho and Parlar 1991; Dekker, Wildeman, and van der Duyn Schouten 1997; McCall 1965; Nicolai and Dekker 2008; Wang 2002). We explicitly model the effect of maintenance and machine deterioration, respectively, on machine conditions and the processing times of the customer orders, and consider maintenance as a long-term decision.

We design two different modules to solve the problem. The first module uses a Markov decision process (MDP) model to determine the maintenance plan over multiple periods, abstracting the combinatorics of the production scheduling problem: the customer orders are considered similar, requiring the same production capacity, and are due at the same time. The maintenance duration is considered to be negligible. The maintenance plan is a decision rule that determines which machines

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should be maintained based on their states and the number of customer orders. We derive sufficient conditions to guarantee that the optimal maintenance plan has a switching curve structure which is monotone in both machine state and the number of customer orders. The set of customer orders and the maintenance activities, if any, then constitute the set of operations in the second module. We solve a mixed integer programming (MIP) model to assign a start-time to each operation within the time period. The planned maintenance and production schedule is then executed, the cost of the period is realized, the new states of machines and the number of customer orders are observed, and the procedure repeats to find the schedule for the next time period.

To gain insight into situations where exploiting online condition monitoring information is beneficial, we compare the designed algorithm to a heuristic approach where both maintenance planning and production scheduling decisions are made by dispatching policies. Our computational results demonstrate that incorporating accurate information about machine deterioration decreases the total discounted cost of maintenance and lost production on average 21%. It is further shown that the benefit of using online condition monitoring information increases when the failure rate is medium and when the discount factor is high. In the former, the maintenance planning decision has a larger effect on the short-term production scheduling decision, and in the latter the long-term impact of short-term decisions has a more significant weight on the total discounted cost.

This paper is organized as follows. Section 2 provides an overview of the relevant literature. Our problem is formally defined in Section 3. Our solution approaches are then explained and the sufficient conditions for the switching curve optimal maintenance policy are derived in Sections 4 and 5, respectively. Computational results are then presented, followed by a discussion in Sections 6 and 7. We complete the paper with a conclusion in Section 8. Some of the proofs, the execution of the planned schedules given by different algorithms and the experimental set-up are discussed in the appendices.

## 2. Literature review

The problem of integrated maintenance and production scheduling is studied in both maintenance and scheduling communities adopting a long-term and a short-term decision horizon, respectively.[1]

Maintenance literature typically addresses the relationship between maintenance and production scheduling on a tactical level where the production process is abstracted as a single machine and there is an explicit model representing machine deterioration. For example, the condition of the machine may be represented by its age where the time to failure follows a probability distribution (Chakraborty, Giri, and Chaudhuri 2009; Tseng 1996). As another example, the state of the machine is represented using a finite discrete set, i.e. $\{0, 1, \ldots, N\}$ where 0 and $N$ indicate the new and the failed conditions of the machine, the middle states represent intermediate levels of machine health, and the machine changes its state based on a probability distribution (Kaufman and Lewis 2007; Sloan 2004). This literature has developed in two directions, both with the goal of the minimization of the total discounted costs over an infinite time horizon. The first direction assumes that all customer orders are similar and addresses the joint problem of maintenance and lot-sizing, *how much* to produce (Groenevelt, Pintelon, and Seidmann 1992a, 1992b; Rosenblatt and Lee 1986; Sloan 2004; Kaufman and Lewis 2007; Yao et al. 2005). The second research stream assumes different customer orders and addresses the joint problem of maintenance and product dispatching, *which* product to produce next (Batun and Maillart 2012; Kazaz and Sloan 2013; Sloan 2008). Our work in this paper is different as we address the problem of integrated maintenance and production scheduling on the operational level, assuming a multi-machine production process where different customer orders have to be scheduled on machines in sequence. The goal is to determine *when* each order on each machine is started.

The problem of integrated maintenance and production scheduling on the operational level is also studied in the scheduling literature. It is assumed that there are a number of planned maintenance activities that must be inserted into the schedule among the regular production jobs such that a given operational performance measure is optimized. This problem is studied from two perspectives. The first assumes that a machine undergoing maintenance is unavailable for production jobs and packs the jobs into the gaps created between unavailability intervals (Chen 2006; Kovacs and Beck 2007; Kuo and Chang 2007; Lee 2004; Ma, Chu, and Zuo 2010; Schmidt 2000). This body of work assumes that there is no correlation between machine conditions and the processing times of the production jobs, neglecting the practical effect of maintenance on machine conditions (Kellerer, Rustogi, and Strusevich 2013; Rustogi and Strusevich 2012). The second perspective models different processing times for production jobs depending on only whether they come before or after maintenance (Lee and Leon 2001; Mosheiov and Sidney 2010) or on the assigned group and their positions within the group, where the number of groups indicates the number of maintenance activities (Kellerer, Rustogi, and Strusevich 2013; Kuo and Yang 2008; Rustogi and Strusevich 2012). The distinctiveness of our work compared to the scheduling literature is twofold. First, we do not assume that maintenance is already planned. We consider maintenance planning as a long-term decision integrated with short-term maintenance and production scheduling. Second, we model the effect of maintenance and machine deterioration on machine conditions and the processing times of the customer orders.

## 3. Problem definition

In this section, we formally define the problem and argue that it is intractable to solve directly. In the next section, we, therefore, define a decomposition approach.

Figure 1 is a snapshot of the problem at time 0, where rectangles represent machines. Machines deteriorate as they are used for production; the filled colours in Figure 1 illustrate machine conditions where darker colours indicate higher levels of deterioration. Maintenance improves machine conditions but it takes production time and delays the delivery of the customer orders. Each order has a specific processing requirement and a due date and should be processed on all machines in sequence. The goal is to simultaneously schedule maintenance and customer orders to minimize the total cost of maintenance and lost production in the long term.

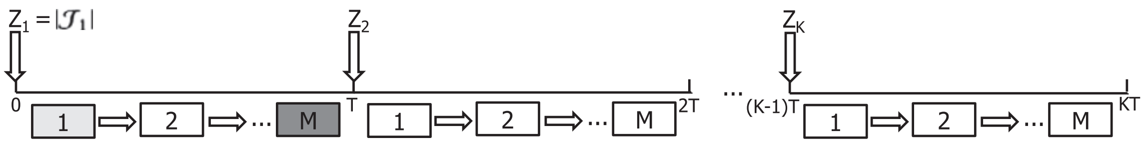


Figure 1. Snapshot of the problem at time 0.

Formally, we consider a series (flow shop) manufacturing facility with $M$ machines producing products to meet demand due at different time points over $K$ discrete periods, each with length $T$. All machines deteriorate as they are used for production. The deteriorating machine $m \in \{1, \ldots, M\}$ can be in one of $N_m$ operational states $\{0, \ldots, N_m - 1\}$ or in a failure state $N_m$. The state process $(X_t^m : t \in \mathbb{R}^+)$, the state of machine $m$ at time $t$, follows a continuous time homogeneous Markov chain with state space $\mathbb{S}_m = \{0, \ldots, N_m\}$. The state transition rate matrix of machine $m$ is defined as $Q^m = [q_{ik}^m]_{(N_m+1)\times(N_m+1)}$ where $-q_{ii}^m$ is the rate at which the machine changes its state when in state $i$ and $q_{ik}^m$ is the rate of transition to state $k$ leaving state $i$.

$$q_{ik}^m = \lim_{h\to 0} \frac{\Pr(X_h^m = k | X_0^m = i)}{h}, \quad i, k \in \mathbb{S}_m, \ i \neq k,$$
$$q_{ii}^m = -\sum_{k\neq i} q_{ik}^m.$$

As a machine deteriorates, its production rate decreases. The production rate of machine $m$ depends on its state and is denoted as $r^m(i)$, $i \in \mathbb{S}_m$. In each state of machine $m$ at time $t$, two actions can be performed. Therefore, the action space of machine $m$ is $\mathbb{A}_m = \{0, 1\}$ where $a_t^m \in \mathbb{A}_m$ is the action taken on machine $m$ at time $t$ being equal to 1 if maintained and 0 otherwise. Performing maintenance on machine $m$ at state $i$ takes $t_p^m$ units of time, costs $\tau^m(i)$ and transitions it to state $k$ with probability $R_{ik}^m$. The deterioration process and maintenance operations behaviours of machine $m$ are summarized below where "B" stands for behaviour.

- B1: Each state represents a level of machine deterioration. Higher states indicate higher levels of deterioration or worse machine conditions, that is, state $i$ is worse than state $k$ if $i > k$.
- B2: Machine conditions deteriorate as a result of production without performing maintenance. Machine states do not therefore improve, i.e. $q_{ik}^m = 0, \forall k < i$.
- B3: Maintenance improves machine conditions, thus, machine states do not worsen, i.e. $R_{ik}^m = \Pr(X_{t+t_p^m}^m = k | X_t^m = i, a_t^m = 1) = 0, \ \forall k > i$. It is worth mentioning that $R_{ik}^m$ only depends on machine states, not on time $t$ or maintenance duration, $t_p^m$.
- B4: Production rate does not increase as the machine deteriorates, it is non-increasing in the machine state, i.e. $r^m(i) \geq r^m(i+1)$. Furthermore, the production rate at failure state equals 0, $r^m(N_m) = 0$.
- B5: Maintenance cost does not decrease as the machine deteriorates, it is non-decreasing in the machine state, i.e. $\tau^m(i) \leq \tau^m(i+1)$.
- B6: As the machine deteriorates, the rate of transition to worse states increases. In other words, the transition rate matrix is monotone (Keilson and Kester 1977), more specifically, $\sum_{k\geq l} q_{ik}^m < \sum_{k\geq l} q_{(i+1)k}^m, \forall l \in \mathbb{S}_m, \ l \geq (i+2)$.
- B7: As the machine deteriorates, the probability of going to a better state after maintenance does not increase, i.e. $\Pr(X_{t+t_p^m}^m \leq l | X_t^m = i, a_t^m = 1) \geq \Pr(X_{t+t_p^m}^m \leq l | X_t^m = i+1, a_t^m = 1)$ (or $\sum_{k\leq l} R_{ik}^m \geq \sum_{k\leq l} R_{(i+1)k}^m, \forall l \in \mathbb{S}_m$).

Further let $Z_k$ be the random demand (the number of customer orders) of period $k$. We assume that $Z_1, \ldots, Z_K$ are independent and identically distributed (i.i.d) with probability mass function $g(z)$. At the beginning of each time period, the demand corresponding to a set of production jobs becomes known. The set of production jobs in time period $k$ denoted as $\mathcal{J}_k$ is the realization of the random variable $Z_k$, i.e. $Z_k = |\mathcal{J}_k|$. In Figure 1, $|\mathcal{J}_1|$ represents the known demand of the first time period and $Z_k$ denotes the random demand for the future period $k$. Each production job $j$ in time period $k$, $j \in \mathcal{J}_k$, can only be processed in time period $k$, has to be processed on all machines in sequence and has a due date of $d_j$. The due date of each job is a time point within period $k$. The processing time of a job on machine $m$ at state $i$ is a random variable denoted as $Y^m(i)$ having the expected value $\frac{1}{r^m(i)}$. We assume that $Y^m(i) = Y^m(0) + (\frac{1}{r^m(i)} - \frac{1}{r^m(0)})$ where $Y^m(0)$ is the random variable representing the processing time of a job on machine $m$ at its best state (or the nominal processing time) and $(\frac{1}{r^m(i)} - \frac{1}{r^m(0)})$ is the increase in the processing time as the machine deteriorates to state $i$. Since the demand becomes known at the beginning of each time period, the nominal processing times of the jobs, $n_{jm}$, $j \in \mathcal{J}_k$, which are the realization of the random variable $Y^m(0)$, also become known. Therefore, the processing time of job $j$ on machine $m$ at state $i$ denoted as $\mathcal{P}_{jm}(i)$, which is the realization of the random variable $Y^m(i)$, is then known being equal to $\mathcal{P}_{jm}(i) = n_{jm} + (\frac{1}{r^m(i)} - \frac{1}{r^m(0)})$. If the processing of production job $j$ on the last machine in sequence, $M$, is not completed before its due date, the production job is lost at cost of $h$.

We denote the state of the system at the beginning of time period $k$ as $\mathcal{X}_k = (i_k^1, \ldots, i_k^M, |\mathcal{J}_k|)$ which consists of machine states, $(i_k^1, \ldots, i_k^M)$, and the number of customer orders, $|\mathcal{J}_k|$. We further define $\mathcal{Y}_k = (y_k^1, \ldots, y_k^M, st_{jm}, st_{pm})$ to represent the maintenance and production scheduling decisions in period $k$ where $y_k^m$ determines if machine $m$ is maintained at period $k$ or not, $st_{jm}$ is the start-time of job $j \in \mathcal{J}_k$ on machine $m \in \{1, \ldots, M\}$, and $st_{pm}$ is the start-time of maintenance operation on machine $m$, if maintained in period $k$, i.e. $y_k^m = 1$.[2] Given the initial system state, $\mathcal{X}_1$, the goal of the problem is to find the maintenance and production scheduling decisions in each period, $\mathcal{Y}_k$, $\forall k \in K$, such that the total expected discounted cost is minimized over an infinite horizon, i.e. when $K \to \infty$. The optimization problem is shown in Figure 2.

$$\min_{\mathcal{Y}_1, \mathcal{Y}_2, \ldots} E_{\mathbb{X}}[\sum_{k=1}^{\infty} \rho^{k-1} C(\mathcal{X}_k, \mathcal{Y}_k) | \mathcal{X}_1] \tag{1}$$

$$\text{s.t. } C(\mathcal{X}_k, \mathcal{Y}_k) = \sum_{m=1}^{M} E_{X^m_{st_{pm}}}[\tau^m(X^m_{st_{pm}})] y_k^m + h \sum_{j \in \mathcal{J}_k} u_j, \qquad \forall k \tag{2}$$

$$\sum_{m=1}^{M} y_k^m \leq C, \qquad \forall k \tag{3}$$

$$st_{jm} + \mathcal{P}^e_{jm}(st_{jm}, n_{jm}) \leq st_{j(m+1)}, \qquad \forall j \in \mathcal{J}_k,\ \forall m\ (m \neq M),\ \forall k \tag{4}$$

$$st_{pm} + t_p^m + \mathcal{B}(y_k^m - 1) \leq T, \qquad \forall m,\ \forall k \tag{5}$$

$$st_{jm} + \mathcal{P}^e_{jm}(st_{jm}, n_{jm}) \leq st_{pm} + \mathcal{B}(1 - b_{jm}), \qquad \forall j \in \mathcal{J}_k,\ \forall m,\ \forall k \tag{6}$$

$$st_{pm} + t_p^m \leq st_{jm} + \mathcal{B} b_{jm}, \qquad \forall j \in \mathcal{J}_k,\ \forall m,\ \forall k \tag{7}$$

$$1 - b_{jm} \leq y_k^m \qquad \forall j \in \mathcal{J}_k,\ \forall m,\ \forall k \tag{8}$$

$$st_{jM} + \mathcal{P}^e_{jM}(st_{jM}, n_{jM}) \leq d_j + \mathcal{B} u_j, \qquad \forall j \in \mathcal{J}_k,\ \forall k \tag{9}$$

$$st_{jm} + \mathcal{P}^e_{jm}(st_{jm}, n_{jm}) \leq st_{im} + \mathcal{B}(1 - x_{jim}), \qquad \forall j, i \in \mathcal{J}_k\ (j > i),\ \forall m,\ \forall k \tag{10}$$

$$st_{im} + \mathcal{P}^e_{jm}(st_{jm}, n_{jm}) \leq st_{jm} + \mathcal{B} x_{jim}, \qquad \forall j, i \in \mathcal{J}_k\ (j > i),\ \forall m,\ \forall k \tag{11}$$

$$y_k^m, u_j, x_{jim}, b_{jm} \in \{0, 1\}, \qquad \forall j, i \in \mathcal{J}_k,\ \forall m,\ \forall k$$

$$st_{jm}, p_{jm}, st_{pm} \in \mathbb{Z}^+ \cup \{0\}, \qquad \forall j \in \mathcal{J}_k,\ \forall m,\ \forall k$$

Figure 2. The optimization model.

In the objective function (1), $\rho$ is the discount factor and $C(\mathcal{X}_k, \mathcal{Y}_k)$ is the total maintenance and lost production cost of period $k$ given state $\mathcal{X}_k$ and the decision $\mathcal{Y}_k$ taken. The expected value is calculated over the system state space $\mathbb{X} = \{0, \ldots, N_1\} \times \ldots \times \{0, \ldots, N_M\} \times \mathbb{Z}^+$ where $\mathbb{Z}^+$ is the set of non-negative integers. To define the total maintenance and lost production cost of period $k$, extra decision variables are defined in Table 1.

Equation (2) shows the total maintenance and lost production cost in period $k$ where the first term is the expected maintenance cost and the second term is the lost production cost of period $k$. Since the state of machine $m$ at time of performing maintenance, i.e. $st_{pm}$, is random, the expected value is calculated over $X^m_{st_{pm}}$.

The problem at period $k$ is subject to maintenance planning and maintenance/production scheduling constraints as defined below.

Table 1. Extra decision variables for maintenance/production scheduling in period $k$.

| | |
|---|---|
| $u_j$ | $u_j = 1$ iff job $j$ is lost. |
| $x_{jim}$ | $x_{jim} = 1$ iff job $j$ is processed before job $i$ on machine $m$. |
| $b_{jm}$ | $b_{jm} = 1$ iff job $j$ is processed before preventive maintenance on machine $m$. |

*Maintenance planning constraint* Constraints (3) enforce the maintenance capacity limit, $\mathcal{C}$, denoting the maximum number of machines that can be maintained in period $k$.

*Maintenance/production scheduling constraints* As already mentioned, the processing time of job $j$ on machine $m$ is dependent on machine $m$'s state. Since the state of machine $m$ at time $st_{jm}$ is random and several transitions might also happen within the processing of the job, the processing time of job $j$ on machine $m$ is random. Therefore, the expected processing time of job $j$ on machine $m$ is defined as $\mathcal{P}^e_{jm}(st_{jm}, n_{jm})$ which is a function of its start-time and its nominal processing time. The details of calculating the expected processing times are provided in Section 5.2.1.

The detailed descriptions of the maintenance/production scheduling constraints in period $k$ for each realization of the demand, $\mathcal{J}_k$, are provided below.

- Constraints (4) enforce the precedence constraints: the job should be finished on an upstream machine before its processing starts on downstream machines.
- Constraints (5) ensure that maintenance activities on machines requiring maintenance at time period $k$, $y^m_k = 1$, are scheduled within the length of the time period where $\mathcal{B}$ is a big value.
- Constraints (6)–(8) define the relationships between the binary decision variables $b_{jm}$ and the maintenance decisions. Respectively, the constraints guarantee that: if a job is processed before maintenance ($b_{jm} = 1$), then its processing is finished before maintenance is started; if a job is processed after maintenance ($b_{jm} = 0$), then maintenance is performed before processing the job is started; if a machine does not require maintenance, $y^m_k = 0$, all jobs are processed before maintenance, $b_{jm} = 1$.
- Since $M$ is the last machine, Constraints (9) define whether job $j$ in time period $k$ is lost or not. If a job is not finished before or at its due date, it is then lost.
- Constraints (10) and (11) are disjunctive constraints ensuring that all jobs on a machine form a total ordering, meaning that no two jobs execute at the same time.

Since the deterioration of each machine independently follows a continuous time Markov chain and the demand is also an independent and identically distributed random variable, the above optimization problem is a constrained MDP model with infinite countable state and action spaces. Both state and action spaces are prohibitively large and there is no close-form expression for the cost of single period given the state and the action taken. Solving the model in Figure 2 as a single MDP is therefore computationally intractable. In the next section, we first decompose the problem and then present two different solution approaches to approximately solve the decomposed problem.

## 4. Decomposing the problem

At the beginning of each time period, there are two different decisions: assigning maintenance to machines and scheduling maintenance and production jobs. Therefore, we decompose the global problem in Figure 2 into two sub-problems: a maintenance planning problem (MPP) and a production scheduling problem (PSP).

The decomposition approach is shown in Figure 3. The MPP is solved once to determine the maintenance policy. Then, at the beginning of each period, the sequence of the events is as follows: the system state is observed, the maintenance policy is used to determine the machines for maintenance, and the PSP problem is solved to find maintenance and production scheduling decisions for the current period. The planned maintenance and production schedule is then executed and the cost of the period is observed.

In this section, we define each sub-problem.

### 4.1 *The maintenance planning problem*

In the MPP, the maintenance/production scheduling problem (PSP) is abstracted. More specifically, the following is assumed:

(1) All production jobs on machine $m \in \{1, \ldots, M\}$ have the same processing time, equal to $\frac{1}{r^m(i_m)}$ where $i_m$ is the state of machine $m$ at the beginning of the period.

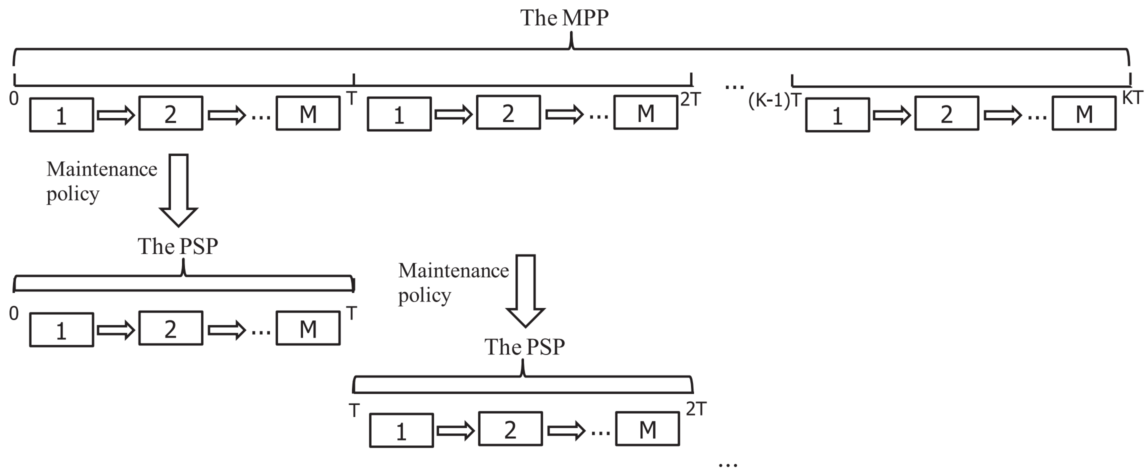


Figure 3. Schematic representation of the decomposition approach.

(2) All production jobs are due at the end of the time period ($d_j = T$).
(3) Maintenance, if any, is performed at the beginning of the time period and it has a negligible duration ($t_p^m = 0$). More specifically, the action performed on machine $m$ at time 0, $a_0^m$, equals 1 if maintained ($y_k^m = 1$) and equals 0, otherwise ($y_k^m = 0$). Therefore, $y_k^m = a_0^m$.

As noted in Section 3, there is a limit on the number of machines maintained in each time period implying that machines are negatively and economically dependent (Nicolai and Dekker 2008). However, in the MPP, machines are considered independently. A MDP model is used for each machine independently to find the optimal maintenance policy over an infinite horizon such that the total expected discounted cost of maintenance and a lower bound on the lost production is minimized. In the MDP model of each machine, one decision is made at the beginning of each period: whether to maintain the machine or not. To consider the maintenance capacity limit in the MDP, the conditions of the machines should be represented as a vector of size $M$ with its elements being the state of machines which results in $\prod_{m=1}^{M}(N_m + 1)$ different levels of system deterioration. Therefore, solving one MDP to make the maintenance decisions of all machines is computationally intractable due to the size of the state space. In this section, the index of machine, i.e. $m$, is excluded from the notation as the MDP is solved for each machine independently.

In the MDP, the following information is required: the production rate of a machine at each state, $r(i)$; the maintenance cost of a machine at each state, $\tau(i)$; the transition probability that a machine changes its state in a time interval with length $T$; and the demand distribution.

The machine transition probability, $p_{ik}^{a_0}$, represents the probability that the machine is in state $k$ at the beginning of next time period given its current state is $i$ and action $a_0$ is taken at the beginning of the current period. Since the deterioration process of the machine follows a homogeneous continuous time Markov chain, the transition probabilities do not vary in time. Taking the behaviour of deterioration process into account, we have:

$$p_{ik}^{0} = \Pr(X_T = k | X_0 = i, a_0 = 0) = \begin{cases} p_{ik} & 0 \le i \le k \le N, \\ 1 & i = N, k = N, \\ 0 & 0 \le k < i \le N, \end{cases}$$

$$\begin{aligned} p_{ik}^{1} = \Pr(X_T = k | X_0 = i, a_0 = 1) &= \sum_{l=0}^{N} \Pr(X_{0^+} = l | X_0 = i, a_0 = 1) \cdot \Pr(X_T = k | X_{0^+} = l, a_{0^+} = 0) \\ &= \sum_{l=0}^{N} R_{il} p_{lk}^{0}, \end{aligned}$$

where $p_{ik}$ is the probability of changing the state from $i$ to $k$ as a result of production within a period of $T$ time units. Using matrix notation, $P^0 = [p_{ik}^0] = e^{QT}$ and $P^1 = [p_{ik}^1] = R \times P^0$ where $R = [R_{ik}]$.

We define the maintenance cost, $\tau(i, a_0)$, and the production rate, $r(i, a_0)$, when machine is in state $i$ and action $a_0$ is taken as follows.

$$\tau(i, 0) = 0,\ \ \tau(i, 1) = \tau(i),$$
$$r(i, 0) = r(i),\ \ r(i, 1) = \sum_{l=0}^{N} R_{il} r(l).$$

Assuming the machine is in state $i$, demand is $z$, and action $a_0$ is taken, the total maintenance and lost production cost of a single period, $C(i, z, a_0)$, is defined below where the first and the second terms denote maintenance cost and lost production cost, respectively. The number of products produced by the machine equals $Tr(i, a_0)$, and since in a series manufacturing all demand needs to be produced by each machine, $(z - Tr(i, a_0))^+ = \max(0, z - Tr(i, a_0))$ denotes the number of lost products.

$$C(i, z, a_0) = \tau(i, a_0) + h(z - Tr(i, a_0))^+.$$

Let $V_n(i, z, a_0)$ be the total discounted expected maintenance and lost production cost when the machine state is $i$, the demand is $z$, action $a_0$ is taken at the beginning of the period and $n$ time periods are remaining. The minimal discounted cost, $V_n(i, z) = \min_{a_0 \in \{0,1\}}(V_n(i, z, a_0))$, can then be found by solving the following recursive equation.

$$V_n(i, z) = \min_{a_0 \in \{0,1\}} \left[ C(i, z, a_0) + \rho \sum_{k=0}^{N} p_{ik}^{a_0} \sum_{\delta=0}^{\infty} g(\delta) V_{n-1}(k, \delta) \right], \tag{12}$$

where $V_0(i, z) = 0,\ \forall i, z$. Recall that $\rho$ is the discount factor and $g(\delta)$ is the probability that the demand in the next period equals $\delta$. For the optimality Equation (12), Theorem 6.2.10 in Puterman (1994, 154) guarantees that there exists an optimal stationary policy because the state space is countable, the costs are bounded and stationary, i.e. they do not change from one decision point to another, the transition probabilities are stationary, and the action space for each state is finite.

The infinite-horizon equivalent of Equation (12) can be written as:

$$V(i, z) = \min_{a_0 \in \{0,1\}} \left[ C(i, z, a_0) + \rho \sum_{k=0}^{N} p_{ik}^{a_0} \sum_{\delta=0}^{\infty} g(\delta) V(k, \delta) \right]. \tag{13}$$

The following Lemma shows that there is a solution to Equation (13).

LEMMA 1 *$V(i, z) = \lim_{n\to\infty} V_n(i, z)$ exists for $\forall i,\ \forall z$.*

*Proof* Consider $\pi$ as a policy and let $W_\pi(i, z)$ denote the expected value of this policy when the initial machine state is $i$ and the demand is $z$. Based on Theorems 8–13 in Heyman and Sobel (1984), if $W_\pi(i, z) < \infty$ for $\forall i,\ \forall z$, then $V(i, z) = \lim_{n\to\infty} V_n(i, z)$ exists for $\forall i,\ \forall z$. We consider $\pi$ as a policy where we always maintain the machine. Since single period value function, $V_1(i, z) = C(i, z, 1)$, is bounded and the discount factor, $\rho$, is less than 1, $W_\pi(i, z) < \infty$ for $\forall i,\ \forall z$ which completes the proof. □

The solution to Equation (13) finds the maintenance decision for a given machine state and the demand (see Section 5.1).

### 4.2 *The production scheduling problem*

As already mentioned, at the beginning of time period $k$, the state of each machine and the demand are observed. Let assume that $i_m$ is the state of machine $m$ and $|\mathcal{J}_k|$ is the number of customer orders (demand). The decision rule identified in the MPP is then used for each machine, identifying the set of machines requiring maintenance denoted as $\mathcal{Q}$. Since the maintenance capacity limit is not considered in the MPP, the number of machines requiring maintenance might be more than the maintenance limit, i.e. $|\mathcal{Q}| > \mathcal{C}$. To adjust the maintenance plan, $\mathcal{C}$ machines need to be selected for maintenance. We define the penalty cost $\varphi_m = V(i_m, |\mathcal{J}_k|, 0) - V(i_m, |\mathcal{J}_k|, 1),\ \forall m \in \mathcal{Q}$ denoting the cost of deviating from the optimal long-term maintenance plan for machine $m \in \mathcal{Q}$. The MIP model for the PSP problem in time period $k$ is shown in Figure 4 assigning start-times to both maintenance and production activities such that the sum of the actual lost production cost and the deviation cost from the optimal long-term maintenance plan is minimized.

The first term in the objective function (14) is the lost production cost for unsatisfied demand where each late job corresponds to an unsatisfied customer order. The second term represents the penalty cost for deviating from the long-term optimal maintenance plan. Constraints (4)–(11) are detailed above in Section 3. Constraint (15) ensures the maintenance capacity limit.

$$\min h \sum_{j=1}^{|\mathcal{J}_k|} u_j + \sum_{m \in Q} \varphi_m (1 - y_k^m) \tag{14}$$

s.t. Constraints (4) to (11)

$$\sum_{m \in Q} y_k^m = \min(C, |Q|) \tag{15}$$

$$b_{jm}, y_k^m \in \{0,1\}, \quad st_{pm} \in \mathbb{Z}^+ \cup \{0\}, \qquad \forall j \in \mathcal{J}_k, \ \forall m \in Q$$

$$u_j, x_{jim} \in \{0,1\}, \quad st_{jm} \in \mathbb{Z}^+ \cup \{0\}, \qquad \forall j, i \in \mathcal{J}_k, \ \forall m$$

Figure 4. The PSP model for time period $k$.

If we relax the PSP by assuming there is no deterioration and that $M = 2$, then the PSP problem corresponds to a two machine flow shop with the objective of minimizing the number of tardy jobs, an NP-complete problem (Lenstra, Rinnooy Kan, and Brucker 1977). Therefore, the PSP problem which generalizes the two machine flow shop is also NP-complete.

## 5. Solving the decomposed problem

We experiment with two approaches called *MDP-MIP* and *Myopic-EDD* to solve the decomposed problem. In the first approach, the policy improvement algorithm and the MIP are used to solve the MPP and the PSP. In the second approach, the MPP and the PSP are both solved heuristically.

In this section, we discuss solution approaches for solving the MPP and the PSP.

### 5.1 *Maintenance planning problem*

Many solution approaches are available to solve the dynamic programming models (Heyman and Sobel 1984; Puterman 1994), each with specific advantages and disadvantages. To solve the recursive Equation (13), we use the policy improvement algorithm (Heyman and Sobel 1984, 145), which is also applied by Sloan (2004). The structure of the optimal maintenance policy in the MPP is discussed in this section. It is worth noting that we do not utilize the developed structural properties to design an algorithm for solving the MDP faster because we aim at presenting a framework which is applicable to any Markovian deterioration process. Furthermore, the number of machine states in our experimental study is small (see Section 6.1). However, developing a crafted algorithm based on the structural properties is a promising direction for decreasing the computational time especially as the size of the state space increases.

In this section, we also present a heuristic approach for finding the maintenance decisions when the information on machine conditions is not available.

#### 5.1.1 *MDP approach*

The main result on the structural property of the optimal maintenance policy is stated in Theorem 1. The index of machine, $m$, is omitted from the notation in this section as the MDP is solved for each machine independently.

THEOREM 1 *B4, B5, B6, B7 (see Section 3) and the following two conditions guarantee that for each $z \in \mathbb{Z}^+$, there exists a threshold state, $\hat{i}_z$, such that the optimal maintenance policy maintains the machine in state $(i, z)$ if $i \geq \hat{i}_z$ and does not maintain if $i < \hat{i}_z$. Furthermore, the threshold state $\hat{i}_z$ is non-increasing in $z$.*

- *A1: $C(i,z,0) - C(i,z,1)$ is non-decreasing in $i$, $\forall z$. It means that as the machine condition gets worse, the difference between the single period total cost of not-maintaining and maintaining decreases more for a given demand.*
- *A2: $\sum_{l \leq k \leq i} R_{ik} \geq \sum_{l \leq k \leq i+1} R_{(i+1)k}$, $\forall l \leq i$. It means that the probability of being worse than a specific state after performing maintenance does not increase as the machine gets worse.*

Theorem 1 guarantees a switching curve optimal maintenance policy for a machine. Conditions A2 and B7 together imply that $R_{ii} = R_{(i+1)i} + R_{(i+1)(i+1)}$ and $R_{ij} = R_{(i+1)j}$, $\forall j \leq (i-1)$ in maintenance probability matrix, $R = [R_{ik}]$.

While the previously studied conditions on these type of problems (Sloan and Shanthikumar 2000; Sloan 2004; Sloan 2008) are on transition probability matrix, $P^{a_0} = [p_{ik}^{a_0}]$, we derive the sufficient conditions on the transition rate matrix,

$Q = [q_{ik}]$, and the maintenance probability matrix, $R = [R_{ik}]$, to guarantee the optimal switching curve policy. The conditions on transition rate matrix which guarantee a monotone Markov model are similar to those found in the literature (Keilson and Kester 1977; Lindqvist 1987). The conditions on the maintenance probability matrix are however novel where the effect of maintenance on the production is considered uncertain such that maintenance does not make the machine new with probability 1. Moreover, in our problem, the demand is a state variable since it is known at the beginning of each period. In the literature, the demand becomes known at the end of the period.

The conditions stated in Theorem 1 are sufficient conditions to guarantee that an optimal threshold maintenance policy exists. There might be situations where the optimal maintenance policy has a threshold type though some of the conditions of Theorem 1 do not hold. For example, assume that the condition A1 does not hold true, meaning that for a given increase in machine deterioration, the decrease in lost production cost is less than the increase in maintenance cost. If the probability of transitioning to a better state after maintenance also increases as the machine deteriorates, i.e. $R_{(i+1)j} > R_{ij},\ \forall j \leq (i-1)$, the extra spending on maintenance might trade off with the benefit of probabilistic improvement in machine conditions in the long term. However, the precise characterization of the situations where the trade-off occurs is hard.

The steps that we take to prove Theorem 1 are as follows: first, we prove that B6, B7 and A2 result in three conditions on the transition probability matrix denoted as C1, C2 and C3 which are stated below. These conditions along with B4, B5 and A1 are then used to prove that $V(i,z)$ is non-decreasing in $i$ and in $z$, and finally we prove the theorem.

- C1: $\Pr(X_t \geq l | X_0 = i, a_0 = 0) \leq \Pr(X_t \geq l | X_0 = i+1, a_0 = 0)$.
- C2: $\Pr(X_t \geq l | X_0 = i, a_0 = 1) \leq \Pr(X_t \geq l | X_0 = i+1, a_0 = 1)$.
- C3: $\Pr(X_t \geq l | X_0 = i, a_0 = 0) - \Pr(X_t \geq l | X_0 = i, a_0 = 1)$
  $\leq \Pr(X_t \geq l | X_0 = i+1, a_0 = 0) - \Pr(X_t \geq l | X_0 = i+1, a_0 = 1)$.

Conditions C1 and C2 indicate that as the machine gets worse, it is more likely to be in a worse state after $t$ units of time regardless of the action taken in the current time 0. Condition C3 means that the likelihood of going to a worse state after performing maintenance decreases more when the machine gets worse.

The three conditions are represented using $p_{ij}^{a_0}$ as below where $t = T$.

- C1: $\sum_{k=l}^{N} p_{ik}^{0} \leq \sum_{k=l}^{N} p_{(i+1)k}^{0}$.
- C2: $\sum_{k=l}^{N} p_{ik}^{1} \leq \sum_{k=l}^{N} p_{(i+1)k}^{1}$.
- C3: $\sum_{k=l}^{N} p_{ik}^{0} - \sum_{k=l}^{N} p_{ik}^{1} \leq \sum_{k=l}^{N} p_{(i+1)k}^{0} - \sum_{k=l}^{N} p_{(i+1)k}^{1}$.

Lemma 2 shows that the conditions on the transition rate matrix and on the maintenance probability matrix guarantee that the conditions on the transition probability matrix hold true.

LEMMA 2 *B6, B7 and A2 guarantee that C1, C2 and C3 hold true.*

*Proof* We prove Lemma 2 in three parts. First, we show that B6 guarantees C1. Second, we show that B6 and B7 guarantee C2. Finally, we prove that B6 and A2 guarantee C3 which completes the proof. The full proof is presented in the Appendix 1. □

The following lemma shows that the value function $V(i,z)$ is non-decreasing in $i$ and in $z$.

LEMMA 3 *$V(i,z)$ is non-decreasing in $i$ for $\forall z \in \mathbb{Z}^+$ and in $z$ for $\forall i \in \{0,\ldots,N\}$.*

*Proof* See Appendix 1 □

Finally, we prove Theorem 1.

*Proof of Theorem 1* Assume that the optimal action in state $(\hat{i}_z, z)$ is $a_0 = 1$; therefore, we have

$$V(\hat{i}_z, z, 0) \geq V(\hat{i}_z, z, 1) \Leftrightarrow C(\hat{i}_z, z, 0) + \rho \sum_{k=0}^{N} p_{\hat{i}_z k}^{0} \sum_{\delta=0}^{\infty} g(\delta) V(k,\delta) - C(\hat{i}_z, z, 1) - \rho \sum_{k=0}^{N} p_{\hat{i}_z k}^{1} \sum_{\delta=0}^{\infty} g(\delta) V(k,\delta) \geq 0. \tag{16}$$

Based on A1: $C(\hat{i}_z, z, 0) - C(\hat{i}_z, z, 1) \leq C(\hat{i}_z + 1, z, 0) - C(\hat{i}_z + 1, z, 1)$. (17)

Based on C3: $\sum_{k=0}^{N} p_{\hat{i}_z k}^{0} - \sum_{k=0}^{N} p_{\hat{i}_z k}^{1} \leq \sum_{k=0}^{N} p_{(\hat{i}_z+1)k}^{0} - \sum_{k=0}^{N} p_{(\hat{i}_z+1)k}^{1}$.

Using the same reasoning as in the proof of Lemma 2 (step 2) where $R_{\hat{i}_z k} = p^0_{\hat{i}_z k} - p^1_{\hat{i}_z k}$ and $f(k) = \sum_{\delta=0}^{\infty} g(\delta)V(k,\delta)$, we have

$$\sum_{k=0}^{N}\left(p^0_{\hat{i}_z k} - p^1_{\hat{i}_z k}\right)\sum_{\delta=0}^{\infty} g(\delta)V(k,\delta) \leq \sum_{k=0}^{N}\left(p^0_{(\hat{i}_z+1)k} - p^1_{(\hat{i}_z+1)k}\right)\sum_{\delta=0}^{\infty} g(\delta)V(k,\delta).$$

By summing the above inequality and inequality (17), we have

$$\begin{aligned}
&V(\hat{i}_z+1, z, 0) - V(\hat{i}_z+1, z, 1)\\
&= C(\hat{i}_z+1, z, 0) - C(\hat{i}_z+1, z, 1) + \rho\sum_{k=0}^{N} p^0_{(\hat{i}_z+1)k}\sum_{\delta=0}^{\infty} g(\delta)V(k,\delta) - \rho\sum_{k=0}^{N} p^1_{(\hat{i}_z+1)k}\sum_{\delta=0}^{\infty} g(\delta)V(k,\delta)\\
&\geq C(\hat{i}_z, z, 0) - C(\hat{i}_z, z, 1) + \rho\sum_{k=0}^{N} p^0_{\hat{i}_z k}\sum_{\delta=0}^{\infty} g(\delta)V(k,\delta) - \rho\sum_{k=0}^{N} p^1_{\hat{i}_z k}\sum_{\delta=0}^{\infty} V(k,\delta) \geq 0.
\end{aligned}$$

The last inequality follows from (16). Since $V(\hat{i}_z+1, z, 0) - V(\hat{i}_z+1, z, 1) \geq 0$, the optimal action in state $(\hat{i}_z+1, z)$ is $a_0 = 1$. Therefore, we proved that for $\forall z$, there is a threshold state $\hat{i}_z$ such that in all states $(i, z)$ where $i \geq \hat{i}_z$, the optimal action is $a_0 = 1$.

Similarly, we can show that for $\forall i$, there is a threshold demand $\hat{z}_i$ where it is optimal to maintain the machine in state $(i, z)$ if $z \geq \hat{z}_i$ and not to maintain if $z < \hat{z}_i$. Let assume that the optimal action in state $(i, \hat{z}_i)$ is $a_0 = 1$, we therefore have:

$$V(i, \hat{z}_i, 0) - V(i, \hat{z}_i, 1) = C(i, \hat{z}_i, 0) - C(i, \hat{z}_i, 1) = h(\hat{z}_i - Tr(i, 0))^+ - h(\hat{z}_i - Tr(i, 1))^+ - \tau(i, 1) \geq 0.$$

By discussion on all possibilities, we can show that $C(i, \hat{z}_i+1, 0) - C(i, \hat{z}_i+1, 1) \geq C(i, \hat{z}_i, 0) - C(i, \hat{z}_i, 1)$. We then have:

$$V(i, \hat{z}_i+1, 0) - V(i, \hat{z}_i+1, 1) = C(i, \hat{z}_i+1, 0) - C(i, \hat{z}_i+1, 1) \geq C(i, \hat{z}_i, 0) - C(i, \hat{z}_i, 1) \geq 0,$$

therefore, the optimal action in state $(i, \hat{z}_i+1)$ is $a_0 = 1$ proving the existence of the threshold demand $\hat{z}_i$ for $\forall i$.

We showed that: (i) for $\forall z$, there is $\hat{i}_z$ where the optimal action in state $(i, z)$ with $i \geq \hat{i}_z$ is to maintain the machine and (ii) for $\forall i$, there is $\hat{z}_i$ where the optimal action in state $(i, z)$ with $z \geq \hat{z}_i$ is to maintain the machine.

Assume that $\hat{i}_z$ and $\hat{i}_{z+1}$ are the threshold states for $z$ and $z+1$, respectively. If $\hat{i}_{z+1} \geq \hat{i}_z$, then the optimal action in state $(\hat{i}_{z+1}, z)$ is to maintain the machine which contradicts our assumption that $\hat{i}_{z+1}$ is the threshold state for $z+1$. Therefore, we can conclude that $\hat{i}_z$ is non-increasing in the demand $z$. □

#### 5.1.2 *Heuristic approach*

In the heuristic approach to solving the MPP, the idea is to postpone incurring maintenance cost as long as possible. In other words, the maintenance plan is "run-to-failure" where machine $m$ is maintained only if it is in the failed state $N_m$ and there is, as a result, no production at the beginning of the period.

### 5.2 ***PSP***

We present two approaches to solve the PSP: a MIP model and a dispatch rule. We have also developed a constraint programming (CP) model to solve the PSP. However, since our early experimentation showed that the MIP model outperforms CP in terms of the run time, we did not pursue the CP model further. The details on the preliminary experiment can be found in Aramon Bajestani (2014, Chapter 6).

#### 5.2.1 *MIP approach*

As noted in Section 3, the processing time of job $j \in \mathcal{J}_k$ on machine $m$ is random with the expected value $\mathcal{P}^e_{jm}(st_{jm}, n_{jm})$ which is a function of its start-time and its nominal processing time. Calculating this expected value is analytically hard because (i) the probability that the machine is in a specific state depends on the time of performing maintenance and the job's start-time which are both decision variables, and (ii) several transitions might happen during the processing of the job with their probabilities dependent on time. We therefore approximate $\mathcal{P}^e_{jm} = n_{jm} + \frac{1}{A^m(i_m)} - \frac{1}{r^m(0)}$ where $A^m(i_m)$ is the average production rate of machine $m$ during the time period given its state is $i_m$ at the beginning of the period. To approximate the average production rate of a machine, we simply assume that it is the average between the expected production rate of machine $m$ at the beginning and at the end of the period. We distinguish between the following two cases:

- $m \notin Q$: If machine $m$ does not need maintenance, the expected production rate of machine $m$ at the beginning of the time period equals $r^m(i_m)$. Since the state of machine $m$ is known at the beginning of the time period, its production rate is not random and is known. Therefore, its expected value equals $r^m(i_m)$. The expected production rate at the end of the time period equals $\sum_{k=0}^{N_m} p_{i_m k}^{m0} r^m(k)$ where $p_{i_m k}^{m0}$ is the transition probability that machine $m$ changes its state from $i_m$ to $k$ within $T$ time units given it has not been maintained. Therefore, the average production rate of machine $m$ can be calculated as follows:

$$A^m(i_m) = \frac{1}{2}\left[r^m(i_m) + \sum_{k=0}^{N_m} p_{i_m k}^{m0} r^m(k)\right], \quad \text{if } m \notin Q. \tag{18}$$

- $m \in Q$: If machine $m$ needs maintenance, the probability that the machine is in a given state in the start-time of job $j$ is dependent on both the state of the machine at the beginning of the period and on the time of performing maintenance. Although the start-time of maintenance is a decision variable, to approximate the expected value, we need to make an assumption on maintenance start-time. We make the same assumption as in the MPP, that maintenance is performed at the beginning of the period with a negligible time. Assuming that machine $m$ is instantaneously maintained at the beginning of the time period, its state changes from $i_m$ to $k$ with probability $R_{i_m k}^m$. We then have the same problem as when machine $m$ does not need maintenance with the only difference that the machine's initial state is $k$. The average production rate can therefore be approximated as follows:

$$A^m(i) = \sum_{k=0}^{N_m} R_{i_m k}^m \frac{1}{2}\left[r^m(k) + \sum_{j=0}^{N_m} p_{kj}^{m0} r^m(j)\right], \quad \text{if } m \in Q. \tag{19}$$

  As already mentioned, $R_{i_m k}^m$ is the probability that machine $m$ changes its state from $i_m$ to $k$ as a result of maintenance.

We have also developed a method for calculating the exact average production rate of machine $m$ using more rigorous probability analysis in both cases of $m \notin Q$ and $m \in Q$ using the same assumption that maintenance is performed at the beginning of the period with a negligible duration. However, we do not use the exact method in our experimental study because the approximation model's error is very small and the exact method is not computationally efficient as the number of machine states and the length of the scheduling horizon increase (Aramon Bajestani, Banjevic, and Beck 2013).

Replacing $\mathcal{P}_{jm}^e(st_{jm}, n_{jm})$ with its approximation, we rely on the default branch-and-bound search in the IBM ILOG CPLEX 12.3 solver, a state-of-the-art commercial MIP solver, to solve the MIP model in Figure 4.

#### 5.2.2 *Heuristic approach*

To solve the PSP heuristically, the idea is to avoid paying the lost cost in the short-term. We use the dispatching policy Earliest Due Date (EDD) where the customer orders are processed in non-decreasing order of their due dates. The EDD dispatching rule minimizes the maximum lateness in a single machine problem when all jobs are available at time zero (Pinedo 2005).

### 5.3 ***Computational complexity and performance analysis***

In this section, we discuss the performance of two solution approaches in terms of computation time and solution quality.

*Computation time* In the MDP-MIP approach, the policy improvement algorithm is used once to determine the maintenance plan and the MIP is used at the beginning of each period to make the scheduling decisions. The policy improvement algorithm for solving the discounted MDP with a constant discount factor runs in strongly polynomial time (Ye 2011; Hansen, Miltersen, and Zwick 2013). However, since the PSP problem in Figure 4 is a NP-complete problem, the MIP model runs in exponential time. Therefore, the MDP-MIP approach has exponential complexity.

In the Myopic-EDD approach, both maintenance and scheduling decisions are made heuristically. The heuristic approach for solving the MPP runs in constant time and the EDD rule for solving the PSP runs in $O(n\log(n))$ time where $n$ is the number of jobs. As a result, the Myopic-EDD approach has polynomial complexity.

*Solution quality* To evaluate the solution quality of both approaches, our goal is to determine how much the total discounted cost increases by not knowing the complete information on machines deterioration in the worst case. However, analyzing the worst-case performance of both solution approaches is technically very difficult since the stochastic deterioration of

Table 2. The range of mean time to failure (MTTF) for different deterioration factors.

| Deterioration factor | MTTF |
|---|---|
| 1 | $[10^5, 10^7]$ |
| 2 | $[10^4, 10^6]$ |
| 3 | $[10^3, 10^5]$ |
| 4 | $[10^2, 10^4]$ |
| 5 | $[10, 10^3]$ |

machines is dependent on maintenance and scheduling decisions. We, therefore, empirically compare the performance of the MDP-MIP and the Myopic-EDD approaches in the next section.

## 6. Computational study

In this section we discuss the results of our computational experiments. The next sub-section describes the problem instances and the experimental details. We then compare the performance of the solution approaches.

### 6.1 *Experimental set-up*

In our problem instances, the number of machines is set at {3, 4, 5} where each machine has five states. The demand of each period is generated from the integer uniform distributions $U[4, 6]$ and $U[8, 12]$. Five different deterioration factors are considered numbered from 1 to 5. As the deterioration factor increases, the mean time to failure for machines decreases. Table 2 shows the range of the mean time to failure (MTTF) for different deterioration factors. These ranges are chosen to reflect the range of the real mean time to failure of different machines used in real industrial applications (see Aramon Bajestani (2014, Chapter 6) for more details.). Two instances for each combination of the parameters are generated yielding 60 problem instances.

The length of the time period is set at 50 and at 100 in problem instances with the number of customer orders generated from $U[4, 6]$ and $U[8, 12]$, respectively. The details of the other parameters are provided in Appendix 3.

The policy improvement algorithm, heuristic policies and the simulation are implemented in C++. The MIP formulation of the PSP in the MDP-MIP approach is solved using CPLEX 12.3. In the MDP-MIP approach, the time limit for solving the problem in each time period is 600 seconds. If the optimal solution is not found within the time limit, the best feasible schedule found by the time limit is executed.

### 6.2 *Experimental results*

We solve the problem online since we need to observe the state of the system at the beginning of each time interval. Therefore, to compare the performance of the two solution approaches, we estimate the total discounted expected cost through simulation. After the maintenance plan and the production/maintenance schedule are determined at the beginning of each period, the schedule is executed and the real cost of the period is observed. The details on simulating the execution of each period are given in Appendix 2.

For each problem instance, we compute the following quantities:

(1) $C_{\text{MDP-MIP}}$, estimated total discounted cost of the MDP-MIP approach, and
(2) $C_{\text{Myopic-EDD}}$, estimated total discounted cost of the Myopic-EDD approach.

The number of time periods, $K$, is chosen such that $\rho^K > 10^{-4}$ where $\rho$ is the discount factor.[3] The total cost of each run equals the discounted sum of the costs over $K$ periods. The total number of simulation runs is set at 20. Finally, $C_{\text{MDP-MIP}}$ ($C_{\text{Myopic-EDD}}$) equals the average of the discounted costs over the simulation runs.

The difference between the normalized total discounted costs for each instance is calculated as $\frac{C_{\text{MDP-MIP}} - C_{\text{Myopic-EDD}}}{C_{\text{Myopic-EDD}}}$. Table 3 and Figure 5 show the mean and the standard error of the difference between the normalized total discounted costs for five deterioration factors and four discount factors.[4]

Tables 4 and 5 show the mean and the standard error of the run-time of the PSP problem per period, and the percentage of the periods where the PSP times out in the MDP-MIP approach. The run-times of the MDP and the simulation are not included in the time reported in Tables 4 and 5 since they are very short, less than a second, and are the same in all periods. It is worth mentioning that the average run-time to find $C_{\text{MDP-MIP}}$ approximately equals the multiplication of the times in

Table 3. The mean and the standard error (SE) of the difference between the normalized total discounted costs.

| Deterioration factor | $\rho = 0.2$ Mean | SE | $\rho = 0.5$ Mean | SE | $\rho = 0.8$ Mean | SE | $\rho = 0.95$ Mean | SE |
|---|---|---|---|---|---|---|---|---|
| 1 | −0.03 | 0.10 | 0.05 | 0.07 | 0.14 | 0.12 | 0.23 | 0.18 |
| 2 | −0.02 | 0.11 | 0.05 | 0.07 | 0.16 | 0.12 | 0.30 | 0.16 |
| 3 | 0.02 | 0.14 | 0.09 | 0.08 | 0.24 | 0.09 | 0.42 | 0.13 |
| 4 | 0.20 | 0.23 | 0.26 | 0.16 | 0.32 | 0.07 | 0.32 | 0.06 |
| 5 | 0.33 | 0.15 | 0.22 | 0.14 | 0.1 | 0.02 | 0.08 | 0.04 |
| {1,2,3,4,5} | 0.14 | 0.32 | 0.22 | 0.19 | 0.24 | 0.14 | 0.27 | 0.17 |

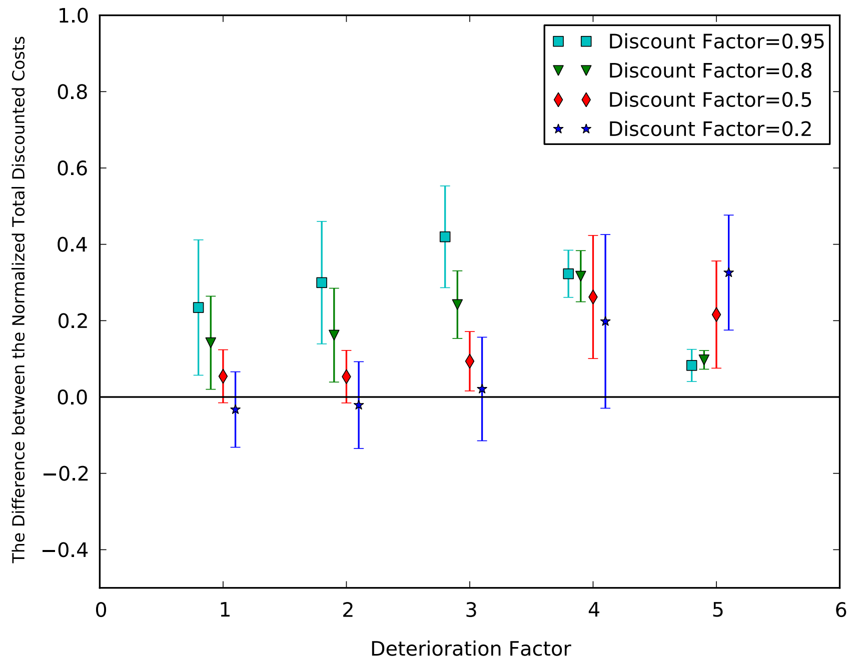


Figure 5. The mean and the standard error of the difference between the normalized total discounted costs for different deterioration factors and discount factors.

Tables 4 and 5 by $K$ and by the number of simulation runs. The Myopic-EDD heuristic approach finds a solution per-period almost instantaneously.

Figure 5 indicates the clear superiority of the MDP-MIP approach over the Myopic-EDD approach since the difference between the normalized total discounted costs is positive for the majority of problem instances. As shown in Table 3, the MDP-MIP approach decreases the mean of the total discounted cost by 21% over all deterioration factors and discount factors.

We further make the following observations:

- The performance of the MDP-MIP approach increases as the discount factor increases for all deterioration factors except 5. The long-term impact of the per-period decision significantly increases as the discount factor approaches 1. Since the MDP-MIP approach incorporates the long-term effect of the current decisions in the model for determining the optimal maintenance policy, its performance, as expected, improves for higher discount factors.
- At deterioration factor 5 when machines deteriorate very quickly, the mean time between failures in our experimental set-up equals 148. Recall that the length of the scheduling horizon is 50 or 100. Therefore, we expect that at the beginning of many periods, machines are in the failed state and both MDP-MIP and Myopic-EDD almost always make the same maintenance decisions, i.e. maintaining the failed machines. A closer look to the data shows that the average per-period maintenance costs for the deterioration factor 5 in the MDP-MIP and in the Myopic-EDD approaches, shown in Tables 6 and 7, are very close. The performance difference between the algorithms mainly results from using an optimization model in the MDP-MIP for solving the PSP rather than a heuristic dispatch rule.

Table 4. The mean and the standard error (SE) of the PSP run-time (sec) per period and the percentage of timed-out periods in the MDP-MIP approach for $\rho = 0.2$ and $\rho = 0.5$.

| | $\rho = 0.2$ | | | | | | $\rho = 0.5$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $U[4, 6]$ | | | $U[8, 12]$ | | | $U[4, 6]$ | | | $U[8, 12]$ | | |
| | MDP-MIP | | | MDP-MIP | | | MDP-MIP | | | MDP-MIP | | |
| Deterioration factor | Mean | SE | Timed-out | Mean | SE | Timed-out | Mean | SE | Timed-out | Mean | SE | Timed-out |
| 1 | 0.04 | 0.08 | 0 | 223.99 | 262.63 | 19 | 0.02 | 0.01 | 0 | 305.18 | 252.19 | 26 |
| 2 | 0.04 | 0.08 | 0 | 218.72 | 259.81 | 21 | 0.02 | 0.01 | 0 | 304.88 | 253.90 | 27 |
| 3 | 0.04 | 0.08 | 0 | 220.91 | 262.48 | 25 | 0.02 | 0.02 | 0 | 296.69 | 253.23 | 27 |
| 4 | 0.04 | 0.07 | 0 | 211.72 | 260.11 | 20 | 0.02 | 0.02 | 0 | 207.76 | 244.86 | 19 |
| 5 | 0.02 | 0.03 | 0 | 10.18 | 58.30 | 0 | 0.01 | 0.01 | 0 | 16.45 | 83.48 | 2 |

Table 5. The mean and the standard error (SE) of the PSP run-time (sec) per period and the percentage of timed-out periods in the MDP-MIP approach for $\rho = 0.8$ and $\rho = 0.95$.

| | $\rho = 0.8$ | | | | | | $\rho = 0.95$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | $U[4, 6]$ | | | $U[8, 12]$ | | | $U[4, 6]$ | | |
| | MDP-MIP | | | MDP-MIP | | | MDP-MIP | | |
| Deterioration factor | Mean | SE | Timed-out | Mean | SE | Timed-out | Mean | SE | Timed-out |
| 1 | 0.03 | 0.05 | 0 | 204.06 | 235.62 | 16 | 0.05 | 0.08 | 0 |
| 2 | 0.03 | 0.05 | 0 | 196.39 | 232.49 | 14 | 0.05 | 0.08 | 0 |
| 3 | 0.03 | 0.04 | 0 | 163.52 | 224.01 | 14 | 0.03 | 0.06 | 0 |
| 4 | 0.02 | 0.02 | 0 | 59.58 | 155.22 | 5 | 0.02 | 0.02 | 0 |
| 5 | 0.01 | 0.01 | 0 | 0.22 | 2.65 | 0 | 0.01 | 0.01 | 0 |

Table 6. The mean per-period maintenance cost (maintenance) and the mean per-period lost production cost (lost) for different approaches, different deterioration factors, and discount factors 0.2 and 0.5.

| | $\rho = 0.2$ | | | | $\rho = 0.5$ | | | |
|---|---|---|---|---|---|---|---|---|
| | MDP-MIP | | Myopic-EDD | | MDP-MIP | | Myopic-EDD | |
| Deterioration factor | Maintenance | Lost | Maintenance | Lost | Maintenance | Lost | Maintenance | Lost |
| 1 | 10.87 | 2246.86 | 0.00 | 3557.09 | 3.16 | 2533.12 | 0.00 | 3526.06 |
| 2 | 11.68 | 2285.50 | 0.15 | 3548.72 | 3.40 | 2558.53 | 0.04 | 3593.70 |
| 3 | 11.68 | 2339.26 | 0.97 | 3783.91 | 5.79 | 2724.17 | 1.08 | 4185.52 |
| 4 | 20.58 | 3159.57 | 12.08 | 4947.67 | 23.81 | 3936.07 | 26.78 | 6171.17 |
| 5 | 79.11 | 5241.55 | 87.53 | 5871.12 | 95.43 | 6098.96 | 105.86 | 6466.76 |

Table 7. The mean per-period maintenance cost (maintenance) and the mean per-period lost production cost (lost) for different approaches, different deterioration factors, and discount factors 0.8 and 0.95.

| | $\rho = 0.8$ | | | | $\rho = 0.95$ | | | |
|---|---|---|---|---|---|---|---|---|
| | MDP-MIP | | Myopic-EDD | | MDP-MIP | | Myopic-EDD | |
| Deterioration factor | Maintenance | Lost | Maintenance | Lost | Maintenance | Lost | Maintenance | Lost |
| 1 | 2.55 | 3385.35 | 0.02 | 5171.26 | 1.15 | 1838.68 | 0.01 | 2726.71 |
| 2 | 2.80 | 3473.41 | 0.22 | 5424.99 | 1.27 | 1884.99 | 0.12 | 2966.59 |
| 3 | 5.33 | 3884.93 | 2.84 | 6269.57 | 3.91 | 2310.99 | 2.87 | 3705.36 |
| 4 | 29.41 | 5385.31 | 68.57 | 7017.75 | 26.27 | 3059.35 | 104.99 | 3908.54 |
| 5 | 113.92 | 6923.48 | 118.35 | 7072.82 | 100.55 | 3843.79 | 126.20 | 3934.89 |

It is worth mentioning that as shown in Figure 5, their performance difference due to different production scheduling decisions is significant for lower discount factors.

- Tables 6 and 7 show that as the deterioration factor increases, the average per-period maintenance cost and lost production cost, respectively, increases and decreases for the MDP-MIP approach compared to the Myopic-EDD. The only exception is that both per-period costs decrease for the deterioration factor 4 and the discount factors $\rho = 0.8$ and $\rho = 0.95$. We expect that the savings on the lost production cost would be higher than the spending on maintenance cost for medium deterioration factors, i.e. 3 and 4, resulting in a better performance for the MDP-MIP approach. In the extreme low or high deterioration factors, the machines are frequently either in very good or in very bad conditions. Therefore, both approaches make similar maintenance decisions. Furthermore, because the processing times of the production jobs is either very short or very long in extreme cases, all customer orders are met or lost regardless of the approach used for the PSP. Figure 5 shows that the performance of the MDP-MIP approach, as expected, improves for medium deterioration factors especially as the discount factor increases. The MDP-MIP has the highest performance at the deterioration factor 3 and the discount factor 0.95.

## 7. Discussion

The experimental results demonstrate that utilizing machine condition information is beneficial particularly for high discount factors and medium deterioration factors. In this section, we discuss the practical relevance of our results.

(1) In low-failure industries, those that have MTTF of $10^4$ to $10^7$ time units as seen more frequently in machinery[5] based on, for example, safety, semi-conductor, circuit breakers, distribution transformers, boilers and condensers equipments (Bently 1999; Carter 1986; Green 1969; Smith 1985; Wright 1984), our results demonstrate that utilizing the online machine condition information in maintenance and production scheduling decisions decreases the mean total discounted cost 13% compared to a greedy heuristic. It is also shown that in low-failure industries where mean time to failure of machines is significantly longer than the length of the scheduling horizon, the benefit of using machine condition information increases as the discount factor increases.

(2) For industries with medium MTTF of $10^2$–$10^5$ time units, those that are mostly based on mechanical, electrical, transistors, turbines, pumps or circulators equipments (Bently 1999; Carter 1986; Green 1969; Smith 1985; Wright 1984), our results demonstrate the highest decrease in the mean total discounted cost, 30% on average, compared to low- and high-failure industries. In these industries, the frequency that machine conditions change is not very low or very high, the maintenance decisions therefore have a more significant impact on the production scheduling decisions within each period yielding a higher benefit. The superiority of utilizing machine condition information also improves as discount factor approaches 1 since the long-term weight of short-term decisions increases.

(3) In high-failure industries with MTTF of 10–$10^3$ units of time which are perhaps using some kind of electronic or pneumatic equipment (Bently 1999; Carter 1986; Green 1969; Smith 1985; Wright 1984), our results show the decrease of 19% in the total discounted cost. In such industries where machine conditions change very quickly, the current decisions have a higher impact in the short term and as shown in Figure 5, the benefit of incorporating information on machine conditions decreases as the discount factor increases.

## 8. Conclusion

In this paper, we addressed the interdependency between maintenance and production scheduling in a multi-machine production system where each machine deterioration process is modeled using a continuous time Markov chain. Machine conditions are characterised by a discrete set of states and can be partially controlled, that is, performing maintenance on machines stochastically improves their conditions. At the beginning of a period, the state of each machine is observed and the customer orders (demand) become known. The machines that need maintenance are then determined and a start-time is assigned to each production and maintenance activity within the period. The goal is to minimize the total discounted cost of maintenance and lost production in the long term.

To solve the problem, we decompose the global problem into maintenance planning and production scheduling sub-problems. A MDP model is developed in the maintenance planning sub-problem to determine the maintenance plan for each machine individually where the scheduling combinatorics are abstracted. More specifically, all customer orders are assumed similar and due at the end of the period. After the machines for maintenance are determined using the maintenance plan, a mixed-integer programming model is solved in the production scheduling sub-problem to find the schedule of maintenance and production activities within the period incorporating all scheduling combinatorics. The planned schedule is then executed, the real cost of the period is realized, the new machine states and the customer orders are observed and the same procedure repeats.

We have derived sufficient conditions in the maintenance planning sub-problem guaranteeing that the optimal maintenance plan has a switching curve structure which is monotone in both machine state and the demand.

The computational results demonstrate that utilizing online machine condition information in maintenance and scheduling decisions decreases the total discounted cost on average 21% compared to a greedy heuristic approach. It is also shown that the benefit of incorporating long-term information in making short-term decisions increases for high discount factors and medium-failure industries where the long-term impact of short-term decisions is higher and where the maintenance decisions effect on short-term production scheduling decisions is more significant.

## Notes

1. Comprehensive reviews of both literatures addressing the relationship between maintenance and production are provided by Aramon Bajestani (2014, Chapter 2).
2. If machine $m$ is maintained at time period $k$ ($y_k^m = 1$) and $st_{pm}$ is the start-time of maintenance operation, it means that $a^m_{st_{pm}} = 1$ and $a_t^m = 0,\ \forall t\ (0 \leq t \leq T, t \neq st_{pm})$. In case $y_k^m = 0$, then $a_t^m = 0,\ \forall t\ (0 \leq t \leq T)$.
3. The number of time periods , $K$, equals 6, 14, 42 and 180 for discount factors of 0.2, 0.5, 0.8 and 0.95, respectively.
4. Because of the high computational time to find $C_{\text{MDP-MIP}}$, there are no results for the case where the number of customer orders is generated from $U[8, 12]$ and the discount factor is 0.95. Therefore, the mean and the standard deviation for $\rho = 0.95$ in Table 3 and Figure 5 are calculated over the demand situation of $U[4, 6]$.
5. See Aramon Bajestani (2014, Chapter 6) for more details.

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## Appendix 1. Proofs of Lemmas 2 and 3

This appendix presents the proofs for several lemmas discussed in Section 5.1.1.

LEMMA 2 *B6, B7, and A2 guarantee that C1, C2, and C3 hold true.*

*Proof* We prove Lemma 2 in three parts. Step 1: we show that B6 guarantees C1. Step 2: we show that B6 and B7 guarantee C2. Step 3: we prove that B6 and A2 guarantee C3 which completes the proof.

*Step 1* B6 guarantees C1. That is, if B6: $\sum_{k\geq l} q_{ik} < \sum_{k\geq l} q_{(i+1)k}$, $\forall l \geq (i+2)$ is true, then C1: $\sum_{k=l}^{N} p_{ik}^{0} \leq \sum_{k=l}^{N} p_{(i+1)k}^{0}$, $\forall l$ holds true.

The proof of this part is based on induction. The *base case* is to show $\Pr(X_\Delta \geq l | X_0 = i, a_0 = 0) \leq \Pr(X_\Delta \geq l | X_0 = i+1, a_0 = 0)$, $\forall l$ for a small $\Delta \geq 0$. We discuss the following two cases:

(1) If $l \leq i+1$, then $\Pr(X_\Delta \geq l | X_0 = i+1, a_0 = 0) = 1$ and the inequality is obvious.
(2) If $l \geq i+2$, then

$$\begin{aligned}
&\Pr(X_\Delta \geq l | X_0 = i+1, a_0 = 0) - \Pr(X_\Delta \geq l | X_0 = i, a_0 = 0) \\
&= \sum_{k\geq l} (q_{(i+1)k}\Delta + o_{(i+1)k}(\Delta)) - \sum_{k\geq l} (q_{ik}\Delta + o_{ik}(\Delta)) \\
&= \Delta \left[ \sum_{k\geq l} q_{(i+1)k} - \sum_{k\geq l} q_{ik} + \frac{o_{i+1}(\Delta) - o_i(\Delta)}{\Delta} \right],
\end{aligned}$$

where $\sum_{k\geq l} o_{(i+1)k}(\Delta) = o_{i+1}(\Delta)$ and $\sum_{k\geq l} o_{ik}(\Delta) = o_i(\Delta)$. Since as $\Delta \to 0$, $\frac{o_{i+1}(\Delta)-o_i(\Delta)}{\Delta} \to 0$ and $\sum_{k\geq l} q_{(i+1)k} - \sum_{k\geq l} q_{ik} > 0$, there exists a small $\Delta_0$ such that $\forall\Delta \leq \Delta_0$, $\Pr(X_\Delta \geq l|X_0 = i+1, a_0 = 0) - \Pr(X_\Delta \geq l|X_0 = i, a_0 = 0) \geq 0$.

We choose $j > 0$ and big such that $\Delta = \frac{t}{j}$ is very small. Therefore, given 1 and 2 we have shown the base case which is

$$\Pr(X_\Delta \geq l|X_0 = i, a_0 = 0) \leq \Pr(X_\Delta \geq l|X_0 = i+1, a_0 = 0), \quad \forall l.$$

The *induction assumption* is $\Pr(X_{(j-1)\Delta} \geq l|X_0 = i, a_0 = 0) \leq \Pr(X_{(j-1)\Delta} \geq l|X_0 = i+1, a_0 = 0),\ \forall l$. The *last step* is to show $\Pr(X_{j\Delta} \geq l|X_0 = i, a_0 = 0) \leq \Pr(X_{j\Delta} \geq l|X_0 = i+1, a_0 = 0),\ \forall l$.

We have

$$\Pr(X_{j\Delta} \geq l|X_0 = i, a_0 = 0) = \sum_{k=0}^{N} \Pr(X_\Delta = k|X_0 = i, a_0 = 0)\cdot\Pr(X_{j\Delta} \geq l|X_\Delta = k, a_\Delta = 0),$$

$$\Pr(X_{j\Delta} \geq l|X_0 = i+1, a_0 = 0) = \sum_{k=0}^{N} \Pr(X_\Delta = k|X_0 = i+1, a_0 = 0)\cdot\Pr(X_{j\Delta} \geq l|X_\Delta = k, a_\Delta = 0).$$

Defining $f(k) = \Pr(X_{j\Delta} \geq l|X_\Delta = k, a_\Delta = 0)$ and $\Pr(Y_i = k) = \Pr(X_\Delta = k|X_0 = i, a_0 = 0)$, we have

$$\Pr(X_{j\Delta} \geq l|X_0 = i, a_0 = 0) = E[f(Y_i)],$$
$$\Pr(X_{j\Delta} \geq l|X_0 = i+1, a_0 = 0) = E[f(Y_{i+1})].$$

In the base case, we showed that

$$\Pr(X_\Delta \geq l|X_0 = i, a_0 = 0) \leq \Pr(X_\Delta \geq l|X_0 = i+1, a_0 = 0), \quad \forall l.$$

Using the definition of $\Pr(Y_i = k)$, we have

$$\Pr(Y_i \geq l) \leq \Pr(Y_{i+1} \geq l),$$

meaning that $Y_i$ is stochastically smaller than $Y_{i+1}$, $Y_i \leq_{st} Y_{i+1}$. Since $f(k)$ is non-decreasing in $k$ because of the induction assumption, $E[f(Y_i)] \leq E[f(Y_{i+1})]$. Letting $t = T$ where $j\Delta = T$, the proof of step 1 is complete.

*Step 2* B6 and B7 guarantee C2. That is, if

(i) B6: $\sum_{k\geq l} q_{ik} < \sum_{k\geq l} q_{(i+1)k},\ \forall l \geq i+2$, and
(ii) B7: $\sum_{k\leq l} R_{ik} \geq \sum_{k\leq l} R_{(i+1)k},\ \forall l$ are true,

then C2: $\sum_{k=l}^{N} p^1_{ik} \leq \sum_{k=l}^{N} p^1_{(i+1)k},\ \forall l$ holds true.

We have

$$\begin{aligned}\Pr(X_t \geq l|X_0 = i, a_0 = 1) &= \sum_{k=0}^{N} \Pr(X_{0^+} = k|X_0 = i, a_0 = 1)\cdot\Pr(X_t \geq l|X_{0^+} = k, a_{0^+} = 0)\\ &= \sum_{k=0}^{N} R_{ik}\Pr(X_t \geq l|X_{0^+} = k, a_{0^+} = 0) = E[f(Y_i)],\\ \Pr(X_t \geq l|X_0 = i+1, a_0 = 1) &= \sum_{k=0}^{N} \Pr(X_{0^+} = k|X_0 = i+1, a_0 = 1)\cdot\Pr(X_t \geq l|X_{0^+} = k, a_{0^+} = 0)\\ &= \sum_{k=0}^{N} R_{(i+1)k}\Pr(X_t \geq l|X_{0^+} = k, a_{0^+} = 0) = E[f(Y_{i+1})],\end{aligned}$$

where $R_{ik} = \Pr(Y_i = k) = \Pr(X_{0^+} = k|X_0 = i, a_0 = 1)$ and $f(k) = \Pr(X_t \geq l|X_{0^+} = k, a_{0^+} = 0)$. Since B7 indicates that $\Pr(Y_i \leq l) \geq \Pr(Y_{i+1} \leq l)$, we conclude that $Y_i \leq_{st} Y_{i+1}$. Furthermore, in the proof of step 1, it is shown that $f(k)$ is non-decreasing in $k$. Therefore, $E[f(Y_i)] \leq E[f(Y_{i+1})]$. Letting $t = T$, the proof of the step 2 is complete.

*Step 3* B6 and A2 guarantee C3. That is, if

(i) B6: $\sum_{k\geq l} q_{ik} < \sum_{k\geq l} q_{(i+1)k},\ \forall l \geq i+2$, and
(ii) A2: $\sum_{l\leq k\leq i} R_{ik} \geq \sum_{l\leq k\leq i+1} R_{(i+1)k},\ \forall l \leq i$ are true,

then C3: $\sum_{k=l}^{N} p^0_{ik} - \sum_{k=l}^{N} p^1_{ik} \leq \sum_{k=l}^{N} p^0_{(i+1)k} - \sum_{k=l}^{N} p^1_{(i+1)k},\ \forall l$ holds true.

The proof of this part is based on induction. The *base case* is to show that:

$$\Pr(X_\Delta \geq l|X_0 = i, a_0 = 0) - \Pr(X_\Delta \geq l|X_0 = i, a_0 = 1)$$
$$\leq \Pr(X_\Delta \geq l|X_0 = i+1, a_0 = 0) - \Pr(X_\Delta \geq l|X_0 = i+1, a_0 = 1), \quad \forall l.$$

To do so, we first need to prove

$$\Pr(X_{0^+} \geq l|X_0 = i, a_0 = 0) - \Pr(X_{0^+} \geq l|X_0 = i, a_0 = 1)$$
$$\leq \Pr(X_{0^+} \geq l|X_0 = i+1, a_0 = 0) - \Pr(X_{0^+} \geq l|X_0 = i+1, a_0 = 1), \quad \forall l. \tag{20}$$

Let discuss the following two cases:

(1) if $l \geq i+1$, then the inequality (20) is obvious as $0 - 0 \leq 1 - c$ where $0 \leq c \leq 1$.
(2) if $l \leq i$, based on A2, we have

$$\Pr(X_{0^+} \geq l|X_0 = i, a_0 = 1) \geq \Pr(X_{0^+} \geq l|X_0 = i+1, a_0 = 1).$$

By adding $\Pr(X_{0^+} \geq l|X_0 = i, a_0 = 0)$ to the above inequality, we have:

$$\Pr(X_{0^+} \geq l|X_0 = i, a_0 = 0) - \Pr(X_{0^+} \geq l|X_0 = i, a_0 = 1)$$
$$\leq \Pr(X_{0^+} \geq l|X_0 = i, a_0 = 0) - \Pr(X_{0^+} \geq l|X_0 = i+1, a_0 = 1). \tag{21}$$

In the proof of step 1, we have shown $\Pr(X_{0^+} \geq l|X_0 = i, a_0 = 0) \leq \Pr(X_{0^+} \geq l|X_0 = i+1, a_0 = 0)$; therefore, we can write (21) as:

$$\Pr(X_{0^+} \geq l|X_0 = i, a = 0) - \Pr(X_{0^+} \geq l|X_0 = i, a = 1)$$
$$\leq \Pr(X_{0^+} \geq l|X_0 = i+1, a = 0) - \Pr(X_{0^+} \geq l|X_0 = i+1, a = 1).$$

Given 1 and 2, we have shown that the inequality (20) holds true. Then, we have

$$\begin{aligned}
&\Pr(X_\Delta \geq l|X_0 = i, a_0 = 0) - \Pr(X_\Delta \geq l|X_0 = i, a_0 = 1) \\
&\quad = \sum_{k=0}^{N} \Pr(X_{0^+} = k|X_0 = i, a_0 = 0) \cdot \Pr(X_\Delta \geq l|X_{0^+} = k, a_{0^+} = 0) \\
&\quad\quad - \sum_{k=0}^{N} \Pr(X_{0^+} = k|X_0 = i, a_0 = 1) \cdot \Pr(X_\Delta \geq l|X_{0^+} = k, a_{0^+} = 0).
\end{aligned}$$

Defining $f(k) = \Pr(X_\Delta \geq l|X_{0^+} = k, a_{0^+} = 0)$, $\Pr(Y_i = k) = \Pr(X_{0^+} = k|X_0 = i, a_0 = 0)$, and $\Pr(Z_i = k) = \Pr(X_{0^+} = k|X_0 = i, a_0 = 1)$, we have

$$\Pr(X_\Delta \geq l|X_0 = i, a_0 = 0) - \Pr(X_\Delta \geq l|X_0 = i, a_0 = 1) = E[f(Y_i)] - E[f(Z_i)].$$

Following the same as above, we have

$$\Pr(X_\Delta \geq l|X_0 = i+1, a_0 = 0) - \Pr(X_\Delta \geq l|X_0 = i+1, a_0 = 1) = E[f(Y_{i+1})] - E[f(Z_{i+1})].$$

Knowing that $E[f(Y)] = f(0) + \sum_{j\geq 1}[f(j) - f(j-1)] \cdot \Pr(Y \geq j)$, we then have

$$E[f(Y_i)] - E[f(Z_i)] = \sum_{j\geq 1}[f(j) - f(j-1)] \cdot [\Pr(Y_i \geq j) - \Pr(Z_i \geq j)],$$
$$E[f(Y_{i+1})] - E[f(Z_{i+1})] = \sum_{j\geq 1}[f(j) - f(j-1)] \cdot [\Pr(Y_{i+1} \geq j) - \Pr(Z_{i+1} \geq j)].$$

Based on inequality (20), we have

$$\Pr(Y_i \geq j) - \Pr(Z_i \geq j) \leq \Pr(Y_{i+1} \geq j) - \Pr(Z_{i+1} \geq j).$$

Considering that $f$ is non-decreasing based on the proof of step 1, we have

$$E[f(Y_i)] - E[f(Z_i)] \leq E[f(Y_{i+1})] - E[f(Z_{i+1})].$$

which completes the proof of the base case. Let us choose $j > 0$ and big such that $\Delta = \frac{T}{j}$ is small enough. The *induction assumption* is $\Pr(X_{(j-1)\Delta} \geq l|X_0 = i, a_0 = 0) - \Pr(X_{(j-1)\Delta} \geq l|X_0 = i, a_0 = 1) \leq \Pr(X_{(j-1)\Delta} \geq l|X_0 = i+1, a_0 = 0) - \Pr(X_{(j-1)\Delta} \geq$

$l|X_0 = i+1, a_0 = 1), \ \forall l$. The *last step* is to show $\Pr(X_{j\Delta} \geq l|X_0 = i, a_0 = 0) - \Pr(X_{j\Delta} \geq l|X_0 = i, a_0 = 1) \leq \Pr(X_{j\Delta} \geq l|X_0 = i+1, a_0 = 0) - \Pr(X_{j\Delta} \geq l|X_0 = i+1, a_0 = 1), \ \forall l$ where $T = j\Delta$.

The proof of the last step is similar to the base case where $\Delta$ and $j\Delta$ replace $0^+$ and $\Delta$, respectively. □

LEMMA 3 *$V(i, z)$ is non-decreasing in $i$ for $\forall z \in \mathbb{Z}^+$ and in $z$ for $\forall i \in \{0, \dots, N\}$.*

*Proof* We use the induction to show the first part of the lemma. We first show that the single period value function, $V_1(i, z)$ is non-decreasing in $i$ for $\forall z$.

Based on B5, the maintenance cost $\tau(i, a_0)$ is non-decreasing in $i$ for $\forall a_0 \in \{0, 1\}$. B4 ensures that $r(i, 0) = r(i)$ is non-increasing in $i$. B4 and B7 also guarantee that $r(i, 1) = \sum_{k=0}^{N} R_{ik} r(k)$ is non-increasing in $i$ (the proof is exactly the same as step 2 of Lemma 2 with the only difference that $f(k) = r(k)$ is non-increasing in $k$, consequently $r(i, 1)$ is non-increasing). Therefore, lost production cost, $h(z - Tr(i, a_0))^+$ is non-decreasing in $i$ for $\forall a_0, \ \forall z$. Finally, $C(i, z, a_0)$ as the sum of maintenance and lost production cost is non-decreasing in $i$ for $\forall a_0, \ \forall z$ which concludes that the single period value function $V_1(i, z)$ is non-decreasing in $i$ for $\forall z$.

The induction assumption is that $V_{(n-1)}(i, z)$ is non-decreasing in $i$. Based on C1, C2 and the induction assumption, $\sum_{k=0}^{N} p_{ik}^{a_0} \sum_{\delta=0}^{\infty} g(\delta) V_{(n-1)}(k, \delta)$ is non-decreasing in $i$ (the proof is exactly the same as step 2 of Lemma 2 where $R_{ik} = p_{ik}^{a_0}$ and $f(k) = \sum_{\delta=0}^{\infty} g(\delta) V_{(n-1)}(k, \delta)$.). Since we already showed that $C(i, z, a_0)$ is non-decreasing in $i$, $V_n(i, z) = \min_{a_0 \in \{0,1\}} [C(i, z, a_0) + \sum_{k=0}^{N} p_{ik}^{a_0} \sum_{\delta=0}^{\infty} g(\delta) V_{(n-1)}(k, \delta)]$ is therefore non-decreasing in $i$. By Lemma 1, we have $V(i, z) = \lim_{n\to\infty} V_n(i, z)$. Thus, $V(i, z)$ is non-decreasing in $i$ which completes the proof of the first part.

For the second part, we have $V(i, z) = \min_{a_0 \in \{0,1\}} (\tau(i, a_0) + h(z - Tr(i, a_0))^+ + \sum_{k=0}^{N} p_{ik}^{a_0} \sum_{\delta=0}^{\infty} g(\delta) V(k, \delta))$. Since the single period cost is only a function of the current demand $z$, it is obvious that $V(i, z)$ is non-decreasing in $z$ for $\forall i$. □

## Appendix 2. The execution of the schedules

Given the start-times assigned to both production jobs and the maintenance job on each machine in the PSP, we first determine the sequence of the jobs on each machine. We then start from the first machine, iterate through the jobs processing each at the earliest available time.

The production rates of machines, the processing times of the jobs, the maintenance cost and the effect of maintenance on machines are dependent on machine states. We therefore simulate the state of each machine at each time point during a period, i.e. $X_t^m, \ \forall m, \ \forall t \ (0 \leq t \leq T)$. To simulate the states of machines at every time point, we need to simulate the next time that the machine leaves its current state, called *transition time*, and the new state that the machine transitions into. The state of machine $m$ at the beginning of the period is $i_m$. Since each machine deterioration process follows a continuous time Markov chain, the time that machine $m$ leaves its state, $t_n$, has an exponential distribution with parameter $-q_{i_m i_m}^m$ and with probability density function $h(t|i_m)$. Furthermore, the probability that the machine transitions into state $j$ after leaving state $i_m$ equals $\frac{q_{i_m j}}{-q_{i_m i_m}}$ (Ross 2010, 384). Therefore, the state that machine $m$ transitions into after leaving its current state is a random variable, $X_{t_n}^m$, with probability mass function, $\varphi(j|i_m) = \Pr(X_{t_n} = j|i_m) = \frac{q_{i_m j}}{-q_{i_m i_m}}$. The pseudocode for simulating the state of machine $m$ during a period is given in Algorithm 1.

**Algorithm 1** Simulating the state of machine $m$ at each time point within a time period.

```
current state ← i_m
current time ← 0
next transition time (t_n) ← current time + generate a random time with
                             probability density function h(t|current state)
X_t^m = i_m, ∀ 0 ≤ t ≤ t_n
while t_n < T do
   current time ← t_n
   X_{t_n}^m ← generate a random state with probability mass function φ(j|current state)
   current state ← X_{t_n}^m
   next transition time (t_n) ← current time + generate a random time with
                                probability density function h(t|current state)
   X_t^m = X_{t_n}^m, ∀ current time < t ≤ t_n
end while
```

Knowing the state of each machine at each time point, we can simulate the processing times of the production jobs to find the completion time of each job on each machine. Algorithm 2 shows the pseudocode for simulating the execution of production job $j$ on machine $m$ started at time $t$ given the next transition time of machine $m$ is $t_n$. Recall that $n_{jm}$ denotes the processing time of job $j$ on machine $m$ in its best state. We further define $inc_{jm}$ to denote the increase in the processing time which is dependent on the machine states. Note that $\{x|y\}$ in Algorithm 2 denotes that the remaining processing time of the job is $x$ if the machine is in state $y$.

If the completion time of job $j$ on machine $m$ returned by Algorithm 2 exceeds its due date, $d_j$, it is not executed on the downstream machines and is added to the list of late jobs.

Algorithm 3 defines the pseudocode for simulating the execution of a maintenance job at time $t$ on machine $m$ where the maintenance cost up to time $t$ is denoted as $C$. When maintenance is performed, we need to simulate the state that the machine transitions into. Recall that $R_{ij}^m$ is the probability that machine $m$ changes its state from $i$ to $j$ after maintenance, therefore the state that machine $m$ transitions into is a random variable with probability mass function $\xi(j|i) = R_{ij}^m$.

**Algorithm 2** Simulating the execution of production job $j$ on machine $m$ started at time $t$.

current time $\leftarrow t$
current state $\leftarrow X_t^m$
$inc_{jm} = \frac{1}{r^m(X_t^m)} - \frac{1}{r^m(0)}$
remaining processing time $\leftarrow \{n_{jm}|0\} + inc_{jm}$
completion time $\leftarrow$ current time + remaining processing time
next transition time $(t_n) \leftarrow$ given by Algorithm 1
**while** completion time $> t_n$ **do**

  the state of machine at next transition time $(X_{t_n}^m) \leftarrow$ given by Algorithm 1
  $inc_{jm} = \frac{1}{r^m(X_{t_n}^m)} - \frac{1}{r^m(\text{current state})}$
  remaining processing time $\leftarrow \{(\text{completion time} - t_n)|\text{ current state}\} + inc_{jm}$
  current time $\leftarrow t_n$
  current state $\leftarrow X_{t_n}^m$
  completion time $\leftarrow$ current time + remaining processing time
  next transition time $(t_n) \leftarrow$ given by Algorithm 1

**end while**
completion time of job $j$ on machine $m \leftarrow$ completion time

After the execution of the jobs on the last machine, $M$, is finished, the size of the late job list is multiplied by $h$ to determine the lost production cost. The sum of the maintenance cost and the lost production cost defines the total cost of the executed schedule in the period.

In simulating the achieved schedule, the idleness of machine $m$, waiting for the jobs to be finished on upstream machines, is not included in the remaining time to next transition from the current state. Assuming that the current time is $t$ and the next transition time of machine $m$ is $t_n$, this means that machine $m$ leaves its current state after $(t_n - t)$ units of time processing the production jobs.

**Algorithm 3** Simulating the execution of maintenance job $j$ on machine $m$ started at time $t$.

current time $\leftarrow t$
current state $\leftarrow X_t^m$
maintenance cost $\leftarrow C$
maintenance cost $\leftarrow$ maintenance cost $+\tau^m(X_t^m)$
maintenance time $\leftarrow t_p^m$
completion time $\leftarrow$ current time + maintenance time
current time $\leftarrow t + t_p^m$
$X_{t+t_p^m}^m \leftarrow$ generate a random state with probability mass function $\xi(j|\text{current state})$
current state $\leftarrow X_{t+t_p^m}^m$

## Appendix 3. Experimental set-up

We explain the simulation of the data related to time periods, machines and jobs in the next three sections.

### 3.1 *Time periods*

Table 8. The range of the data related to time periods where $M$ is the number of machines.

| | | |
|---|---|---|
| Discount factor $(\rho)$ | | $\{0.2, 0.5, 0.8, 0.95\}$ |
| Number of time periods $(K)$ | 6 | if $\rho = 0.2$ |
| | 14 | if $\rho = 0.5$ |
| | 42 | if $\rho = 0.8$ |
| | 180 | if $\rho = 0.95$ |
| Lost cost $(h)$ | | $U[50, 100]$ |
| Demand $(Z_k)$ | | $U[4, 6], U[8, 12]$ |
| Length of time period $(T)$ | 50 | if $Z_k \sim U[4, 6]$ |
| | 100 | if $Z_k \sim U[8, 12]$ |
| Maintenance capacity $(\mathcal{C})$ | | $\lfloor \frac{M}{2} \rfloor$ |

### 3.2 *Machines*

*Number of Machines* The number of machines varies between three and five, i.e. $M \in \{3, 4, 5\}$.
*Number of States* The number of states for each machine, $N_m + 1$, equals 5.
*Initial State* The initial state of each machine, $i_m$, equals $(X - 1)$ if $X < 2$ and $(X - 2)$, otherwise where $X$ is generated from the discrete uniform distribution $U[1, N_m]$.
*Maintenance Time* The maintenance duration for machine $m$, $t_p^m$, is drawn from discrete uniform distribution $U[0.05 \times T, 0.15 \times T]$ where $T$ is the length of the time period.
*Transition Rate* The state transition rate matrix of machine $m$ is defined as $Q^m = [q_{ik}^m]_{(N_m+1)\times(N_m+1)}$ where $\sum_{k \geq l} q_{ik}^m < \sum_{k \geq l} q_{(i+1)k}^m$, $\forall l \geq (i + 2)$. To generate such a matrix for machine $m$ and for each deterioration factor, a value is first assigned to $q_{00}^m$ such that $q_{00}^m = -V(\mathcal{DF})$, $\forall \mathcal{DF} \in \{1, 2, 3, 4, 5\}$ where $V = [\frac{1}{5\times10^5}, \frac{1}{5\times10^4}, \frac{1}{5\times10^3}, \frac{1}{5\times10^2}, \frac{1}{5\times10^1}]$. As the deterioration factor increases, the mean time that the machine spends in its best state, $\frac{-1}{q_{00}^m}$, becomes shorter.

After generating $q_{00}^m$, the other elements of the first row of the matrix are generated following Algorithm 4. Since the sum of the elements of each row equals 0, we simulate how $-q_{00}^m$ is divided between the other elements. As shown in Algorithm 4, we first generate a random number from $U[1, N_m]$ representing the number of states that the machine transitions into leaving its best state. We then generate a vector containing a random permutation of the integers from 1 to the generated number to find the ratio based on which $-q_{00}^m$ is divided.

**Algorithm 4** Simulating $q_{0i}^m, \forall i > 0$

$q_{0i}^m \leftarrow 0, \forall i > 0$
number of states to go, $n \leftarrow U[1, N_m]$
$\mathbb{D} \leftarrow$ *randperm*(n)
**for** $i = 1 : n$ **do**
  $q_{0i}^m \leftarrow \frac{-q_{00}^m \times \mathbb{D}(i)}{\sum \mathbb{D}}$
**end for**

We then need to generate the other elements of the matrix such that $\sum_{k \geq l} q_{ik}^m < \sum_{k \geq l} q_{(i+1)k}^m$, $\forall l \geq (i + 2)$. Algorithm 5 shows the procedure. As shown in Figure 6, the stated condition requires that the sum of elements in the $i$-th row from the $l$th column to the last column should be less than the sum of corresponding elements in the $(i + 1)$-th row, i.e. $s_1 < s_2 + q_{(i+1)l}^m$. Line 10 guarantees that the stated condition holds.

**Algorithm 5** Simulating $q_{(i+1)l}^m, \forall i \geq 0, \ \forall l$

1: $q_{(i+1)l}^m \leftarrow 0, \forall i \geq 0, \ \forall l$
2: **for** $i = 0 : N_m - 2$ **do**
3:   **for** $l = N_m : -1 : i + 2$ **do**
4:     $s_1 \leftarrow \sum_{k=l}^{N_m} q_{ik}^m$
5:     **if** $l = N_m$ **then**
6:       $s_2 \leftarrow 0$
7:     **else**
8:       $s_2 \leftarrow \sum_{k=l+1}^{N_m} q_{(i+1)k}^m$
9:     **end if**
10:     $q_{(i+1)l}^m \leftarrow \max(0, s_1 - s_2) + \frac{U[0, \mathbb{V}(\mathcal{DF})]}{2}$
11:   **end for**
12:   $q_{(i+1)(i+1)}^m \leftarrow -\sum_{j=i+2}^{N_m} q_{(i+1)j}^m$
13: **end for**

*Maintenance Probability* The maintenance probability matrix $R^m = [R_{ik}^m]$ should be generated such that

$$R_{ii}^m = R_{(i+1)i}^m + R_{(i+1)(i+1)}^m, \tag{22}$$

$$R_{ij}^m = R_{(i+1)j}^m, \quad \forall j \leq (i - 1). \tag{23}$$

Algorithm 6 shows the procedure for generating maintenance probability matrix.

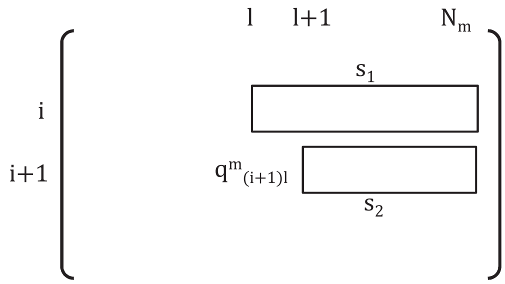


Figure 6. Transition rate matrix.

First, we assign 0 to $R^m_{N_m N_m}$, the probability of not leaving the failure state after maintenance. Second, we generate the other elements of the last row of the maintenance probability matrix. Since the sum of the elements in each row equals 1, we need to randomly divide $(1 - R^m_{N_m N_m})$ among the other elements. The idea is the same as the one used in generating the first row of the transition rate matrix. As shown in Algorithm 6, we generate a random number from $U[1, N_m]$ representing the number of states that the machine transitions into leaving its worst state after maintenance. We then generate a vector containing a random permutation of the integers from 1 to the generated number to find the ratio based on which $(1 - R^m_{N_m N_m})$ is divided. The other rows of the matrix are generated following both conditions (22) and (23) as shown in lines 10 and 12.

**Algorithm 6** Simulating $R^m$

1: $R^m_{ij} \leftarrow 0, \ \forall i, j$
2: number of states to go, $n$, $\leftarrow U[1, N_m]$
3: $\mathbb{D} \leftarrow randperm(n)$
4: index $\leftarrow 0$
5: **for** $j = N_m - 1 : -1 : N_m - n$ **do**
6: $\quad R^m_{N_m j} \leftarrow \frac{(1-R^m_{N_m N_m}) \times \mathbb{D}(n - \text{index})}{\sum \mathbb{D}}$
7: $\quad$ index $\leftarrow$ index + 1
8: **end for**
9: **for** $i = N_m - 1 : -1 : 1$ **do**
10: $\quad R^m_{ii} \leftarrow R^m_{(i+1)(i+1)} + R^m_{(i+1)i}$
11: $\quad$ **for** $j = 0 : i - 1$ **do**
12: $\quad\quad R^m_{ij} \leftarrow R^m_{(i+1)j}$
13: $\quad$ **end for**
14: **end for**
15: $R^m_{00} \leftarrow 1$

*Transition Probability* The transition probability matrix of machine $m$, $P^{ma} = [p^{ma}_{ik}]$, is the probability of changing the state from $i$ to $k$ within $T$ units of time given action $a$. Since $P^{m0} = e^{Q^m T}$, the matrix exponential function, *expm*, in MATLAB is used to calculate $P^{m0}$, i.e. $P^{m0} = expm(Q^m T)$. Then we have $P^{m1} = R^m \times P^{m0}$ where $R^m$ is the maintenance probability matrix.

*Production Rate* The production rate of machine $m$ at state $i \in \{0, \dots, N_m\}$ is $W(i+1)$ where we set $N_m = 4$ and $W = [0.2, 0.15, 0.1, 0.05, 0]$. The production rate of machine $m$ at state $i$ given action $a$, $r^m(i, a)$, is calculated as below where $R^m_{il}$ is the maintenance probability.

$$r^m(i, 0) = W(i+1), \ r^m(i, 1) = \sum_{l=0}^{N_m} R^m_{il} W(l+1).$$

*Maintenance Cost* We need to generate maintenance cost of machine $m$ at state $i$, $\tau^m(i)$, such that condition (24) holds true.

$$\begin{aligned} \tau^m(i+1) - \tau^m(i) \le\ & h(z - Tr^m(i+1, 0))^+ - h(z - Tr^m(i+1, 1))^+ \\ & - h(z - Tr^m(i, 0))^+ + h(z - Tr^m(i, 1))^+, \quad \forall i, \ \forall z. \end{aligned} \tag{24}$$

Moreover, $\tau^m(i+1) \geq \tau^m(i)$ implying that the left-hand side of condition (24) is greater than 0. Algorithm 7 shows the procedure that we use to generate the maintenance cost at each state for each machine where $a$ and $b$ are the lower and the upper bounds for demand. First maintenance cost at the worst state is generated from the uniform distribution $U[50, 100]$. As shown in Line 6 of Algorithm 7, if the right-hand side of condition (24) is less than 0, the required condition does not hold and we re-initiate the procedure of generating the data from the beginning. Otherwise, as line 9 shows in Algorithm 7, $\tau^m(i)$ is generated such that condition (24) holds true.

**Algorithm 7** Simulating $\tau^m(i), \ \forall i$

1: $\tau^m(N_m) \leftarrow U[50, 100]$
2: **for** $i = N_m - 1 : -1 : 0$ **do**
3:     **for** $z = a : b$ **do**
4:         $c \leftarrow h(z - Tr^m(i+1,0))^+ - h(z - Tr^m(i+1,1))^+ - h(z - Tr^m(i,0))^+ + h(z - Tr^m(i,1))^+$
5:         **if** $c < 0$ **then**
6:             re-initiate simulating the data
7:         **end if**
8:     **end for**
9:     $\tau^m(i) \leftarrow \tau^m(i+1)$
10: **end for**

### 3.3 *Jobs*

*Nominal Processing Time* The nominal processing time of the production activity $j$ on machine $m$, $n_{jm}$, is drawn from the discrete uniform distribution $U[1, 9]$. It is worth mentioning that the nominal processing time distribution is chosen such that its mean equals $\frac{1}{r^m(0)} = 5$.

*Due date* The due date of production activity $j$ is set at $\min(T, f^d \times \sum_{m=1}^{M} n_{jm})$ where $f^d$, the due-date factor, is 2.5 and 3 for $T = 50$ and $T = 100$, respectively.