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Lot-sizing problem with simultaneous backlogging and lost sales

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ABSTRACT
In the past decades, lost-sizing formulations have been adapted in order to represent more practical situations with the aim to propose realistic solutions to the problems. In this paper, we extend the lot-sizing formulations in order to incorporate simultaneously backlog and lost sales. We assume that if there is a stock-out, some proportion of the affected customers will be willing to accept a backlog, while others will not, resulting in lost sales. We propose formulations with four different assumptions for backlog: no backlog, unlimited backlog, restricted backlog, and multiple customer backlog, in combination with four possibilities of lost sales: no lost sales, fixed-proportion lost sales, variable-proportion lost sales, and unrestricted proportion of lost sales. The assumption of backlog with multiple customer types, which is when customers have different willingness to wait, is very little explored in the literature and extends the concept of the traditional single customer type. To model these problems, we use the facility location reformulation that provides more flexibility and structure to incorporate backlog and lost sales decisions simultaneously in the problem, as it allows us to specifically identify and track stock-outs. We present a fast relax-and-fix heuristic that takes advantage of the mathematical formulation provided by the reformulation and the backlog assumptions. An extensive computational experiments is performed, where the different formulations are compared with respect to structure of the solution and the performance of the heuristic. We also evaluate the impact that certain parameters have on the solution for the different formulations by conducting sensitivity analyses on various parameters, such as capacity, customer's willingness to wait, and costs.

KEYWORDS
Lot-sizing problems; backlog; lost sales

1. Introduction

One of the most researched issues in contemporary production planning is the lot-sizing problem, that considers the tradeoff between setup costs and holding costs in order to determine the optimal timing and level of production to satisfy deterministic or dynamic demand. Throughout the years, much has been done in terms of theories, models, and solution approaches for the problem (Karimi, Ghomi, and Wilson 2003; Jans and Degraeve 2008; Copil et al. 2017).

Commonly in the literature, it is assumed that the demand must be completely satisfied on time (Robinson, Narayanan, and Sahin 2009). In practice, however, companies face situations where this demand assumption does not hold, and this situation is commonly described as a stock-out. In this study, we consider this possibility. Once faced

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with a stock-out, two approaches can be used to manage the stock-outs: backlogging the unsatisfied demand and/or incurring lost sales.

The concept of backlogging is used to describe the situation in which clients with an unsatisfied demand in a specific period have their demand satisfied using production in later periods, within the planning horizon, at a given cost (Zangwill 1966; Pochet and Wolsey 1988). When industries are faced with stock-outs and clients are not willing to wait, the demand is lost (Sandbothe and Thompson 1990). This typically occurs in highly competitive and short seasonal industries like fast fashion. Companies within these industries have similar item offerings, so some customers prefer purchase their products from other companies if stock-outs occur, while others customers are willing to wait. As long as there are substitutes available, there can be lost sales.

Considering the increasingly effort of researchers to represent such realistic situations, we strive in that trend by studying the multi-item capacitated lot-sizing problem that takes into account, simultaneously, backlog and lost sales as means of dealing with stock-outs. To represent the negative impact of stock-outs, we consider penalty costs when demand is not met on-time either with backlog penalty cost (Gruson, Cordeau, and Jans 2018) or with lost sales penalty cost (Sandbothe and Thompson 1990). The backlog penalty cost is used to represent the costs associated with backlogging an item (administration cost, schedule changing cost, extra transportation cost, etc.), while the lost sales penalty cost will be used to represent the lost profit and the loss of customer goodwill. Therefore, our study applies to realistic situations where, in case of a stock-out, some customers are willing to wait (leading to backlog) and others are not willing to wait (leading to lost sales).

Although these two concepts have mostly been addressed in the literature separately, in this paper, we approach backlog and lost sales simultaneously. More specifically, this problem assumes multiple customers backlog and fixed lost sales, called *FLR-MB-FL*. The multiple customer backlog case describes a scenario where there are multiple customers types, each with a different behavior when faced with a backlog. The customer behavior is characterized by the maximum number of periods a customer is willing to wait for the backlogged items. This assumption of backlog with multiple customer types is very little explored in the literature and extends the concept of the traditional single customer type. For the fixed lost sales assumption, when a stock-out occurs, a minimum fixed percentage of the unsatisfied demand automatically results in lost sales, whereas the remaining unsatisfied demand must be backlogged and satisfied in later periods. This fixed percentage represents the customers who are not willing to wait in case of a stock-out. We have also considered other different assumptions for backlog in combination with other possibilities of lost sales (see Table 4).

The formulations are proposed and modeled using the facility location reformulation (Eppen and Martin 1987). This reformulation gives more flexibility and structure to incorporate both backlog and lost sales, as it allows us to specifically identify and track stock-outs, as well as properly account for the penalty costs associated to them. To optimize the *FLR-MB-FL* problem, we introduce a fast relax-and-fix heuristic that leverages the structure provided by the reformulation and the backlog assumptions. We conduct extensive computational experiments in order to evaluate the behavior of the solution provided with the various formulations and the performance of the heuristic when solving the problem. Additionally, we assess the impact of certain parameters of the problem by conducting sensitivity analyses on important factors such as capacity availability, customer's willingness to wait, and the cost of backlog and lost sales. To the best of our knowledge, this study is one of the earliest in the lot-sizing literature to address a simultaneous combination of backlog and lost sales strategies to deal with stock-outs.

The contribution of this study is threefold: (1) we propose new lot-sizing problems that incorporate backlog and lost sales simultaneously, including the backlog concept of multiple customers with different willingness to wait; (2) we develop a rapid relax-and-fix heuristic to solve the problem; (3) we conduct a sensitivity analysis of the impact of key parameters in tradeoff between backlog and lost sales.

Next, in Section 2, we present a literature review with the major studies addressing lot-sizing problems with backlog and lost sales. Section 3 describes the mathematical model for the multi-item capacitated lot-sizing problem with multiple customers backlog and fixed lost sales, and how to derive the other proposed models. In Section 4, the solution strategy is described and the computational results are presented in Section 5. Finally, the conclusions are presented in Section 6.

2. Literature review

In this section, we provide a literature review on lot-sizing problems addressing backlog and lost sales. We also present a brief overview of the reformulations that have been developed to similar lot-sizing problems.

In general, lot-sizing problems aim to determine the optimal timing (when) and level of production (how much), while meeting the production restrictions and satisfying the demand of clients. The problem can be classified using different angles in terms of timescale, demand behaviour, and time horizon (Jans and Degraeve 2008). The classical Economic Order Quantity model (EOQ), developed by Harris (1990), is seen as the base concept of several lot-sizing models in the literature. The EOQ model assumes a continuous timescale, no production capacity, and a deterministic and constant demand rate with only one type of item. One of the contributions of the EOQ model is that it highlights the tradeoff between setup and holding costs. The lot-sizing research has moved towards using dynamic demand in order to incorporate realistic conditions, so that a more applicable production plan can be obtained. The goal of the classical dynamic lot-sizing problem is to satisfy demand that varies over time, through production during several [time periods (Wagner and Whitin 1958; Robinson, Narayanan, and Sahin 2009; Buschkühl et al. 2010). The literature on the dynamic lot-sizing problem has been extended to take into account several other industrial settings, one which is of particular interest in this paper is when stock-outs occur. A review of different models and approaches for lot-sizing problems can be found in Karimi, Ghomi, and Wilson (2003), Brahimi et al. (2006), Jans and Degraeve (2008), Glock, Grosse, and Ries (2014), and Copil et al. (2017).

Practical settings suggest that stock-outs may be planned when demand is highly dynamic or setup costs are considerably high, and hence, a designed production plan might not fulfill the demand of items on time. When faced with stock-outs, backlogging the demand can be a possible option, which means that the demand of items in a specific period will be satisfied by production in later periods. Zangwill (1966) can be seen as the first study to introduce backlog by allowing inventory levels to be negative. The author adapts the deterministic single-item lot-sizing model from Wagner and Whitin (1958) and proposed a dynamic programming algorithm with piecewise concave inventory cost structure. This model is extended in order to be represented by a single source network in Zangwill (1969). Swoveland (1975) proposes a model with piecewise concave cost functions for production, holding, and backlogging, where inventory capacities and maximal backorder levels must be respected. In the paper, the author state that such a strategy may also ensure that items are backlogged for no more than a prescribed maximum number of periods. Pochet and Wolsey (1988) present mixed-integer reformulations for the uncapacitated problem with backlogging, where all variables, including backlog, have an associated cost in the objective function to be minimized. The authors study the convex hull of solutions and develop valid inequalities to the problem, which are also tested on multi-item capacitated lot-sizing problems with backlogging. In Constantino (2000), the multi-item capacitated lot-sizing problem with backlog is addressed. To solve real life instances from chemical industries, cutting planes in a branch-and-cut algorithm is developed. Chand et al. (2007) address a dynamic lot-sizing problem with a single item and multiple customers, which consists of a generalization of the model presented in Wagner and Whitin (1958). The authors show that the problem with time-variant customer-dependent backlogging costs and shipping costs is NP-hard. A dynamic programming algorithm for the case when there is no speculative motive for backlogging is devised.

Researchers have also addressed backlog with service-level constraints. Gade and Küçükyavuz (2013) study a multi-item uncapacitated lot-sizing problem with service-level constraints. One of these constraints limit the number of periods with backlog, while the other ensures that at least a percentage of demand is met on time. The authors develop an algorithm based on the shortest path reformulation of the problem, which is also tested on multi-item capacitated lot-sizing problem with service-level constraints. Gruson, Cordeau, and Jans (2018) extend the studies in Gade and Küçükyavuz (2013), and propose various backlog and backorder related service-levels for the capacitated and uncapacitated lot-sizing problem. The authors use the facility location reformulation in order to distinguish between backorders and backlog, and study different service level constraints at an individual level, item-by-item, period-by-period, and globally. The results indicate that different constraints may lead to very different solutions. In a recent paper, Tomazella et al. (2023) proposed mathematical models for the integrated procurement and lot-sizing problem with multiple customers and backlogging. The study is motivated by an application in a manufacturing company specializing in the assembly of refrigeration equipment. In addition to the traditional cost and backlogging minimization models, the authors introduced service-level optimization models that consider cost control through budgetary constraints. Backlogging has also been combined into various extensions of lot-sizing problems, such as time-windows (Lee, Çetinkaya, and Wagelmans 2001; Absi, Kedad-Sidhoum, and Dauzère-Pérès 2011), multi-level (Wu et al. 2011), and two-level with cargo capacity (Solyali, Denizel, and Süral 2016).

Another relevant strategy that can be used to manage stock-outs is lost sales. Lost sales occur when demand is not met on time and the customer no longer wants the item, either because the item is not needed anymore, or because the customer decides to go to a competitor to purchase a similar item. When lost sales occur, a company may avoid paying the unit production cost, but faces a loss in revenue and customer goodwill. There is hence a cost associated to each unit of lost sales. Sandbothe and Thompson (1990) are some of the first to explore the lot-sizing problem with the concept of lost sales. The authors examine two versions of a capacitated problem with the possibility of lost sales, where production capacity either remains constant or vary over the time horizon. In the formulation, a single-item is considered. When the company is not able to meet demand on time, lost sales are incurred with a penalty cost per period. Aksen, Altinkemer, and Chand (2003) propose a profit maximization variant for the uncapacitated single-item lot-sizing problem (Wagner and Whitin 1958), where demand cannot be backlogged but does not have to be satisfied, becoming lost sales.

The paper adds a new perspective to the traditional tradeoff in lot-sizing problems, for which if the demand is not satisfied, the cost of a lost sale is equal to the lost revenue. The authors develop a recursive dynamic programming algorithm to solve the problem optimally and present several structural properties of the optimal solutions. Liu et al. (2007) explore an alternate scenario observed in process industries, where the demand is limited by inventory capacity rather than production capacity. The paper studies the single-item capacitated lot-sizing problem with time varying inventory restrictions and lost sales. Once inventory reaches its capacity for a given period, the unsatisfied demand is absorbed by the lost sale variable with no possibility of backlogging. The authors proposed several properties of the model at an optimal solution and based on these properties, developed a polynomial time algorithm.

The classic formulations for lot-sizing problems have limitations when backlog is considered (Gruson, Cordeau, and Jans 2018). In fact, such formulations are able to identify backlog but not backorders. Backorders represent unsatisfied demand on an individual level while backlog represents the sum of multiple backorders which have not been satisfied yet. In our study, we propose the development of formulations that require the information of backorders, and hence, the distinction between backlog and backorders is necessary, in order to properly account for the penalty costs involved in these approaches. Faced with this issue, we must turn to a different formulation that is capable of handling this matter in the problem. Krarup and Bilde (1977) proposed a facility location reformulation, which solves the uncapacitated lot-sizing problem as a linear program. Researchers have used this reformulation, or extensions of it, to include factors such as production capacity, backlog, service levels, and multi-processed items (Denizel et al. 2008; Gade and Küçükyavuz 2013; Gruson, Cordeau, and Jans 2018). The reformulation is shown to be able to provide better linear relaxations and improve computing times (Nemhauser and Wolsey 1988; Denizel et al. 2008). Krarup and Bilde (1977) reformulate the model specifically to address lot-sizing problems. The reformulation uses a production variable that is able to track separately in which period there is production and in which period demand is satisfied. This information facilitates the calculation of the total production cost, along with the associated penalty costs (backlog and lost sales).

In the lot-sizing literature (see Table 1), the idea of merging both backlog and lost sales, as strategies to deal with stock-outs, are generally not considered. In fact, to the best of our knowledge there are no papers that present mathematical formulations and further discussions about the impact of incorporating along backlog and lost sales in lot-sizing problems. In Absi, Kedad-Sidhoum, and Dauzère-Pérès (2011), the authors study a single-item uncapacitated lot-sizing problem with production time-windows, early production, backlog and lost sales. The study allows the evaluation of satisfying the demand of items before the given time window, at the expenses of an extra cost. For this, two formulations are proposed. The first one includes early production and lost sales, in which a cost is incurred for producing before the time window, and a lost sales cost for not satisfying demand before the due date. The second formulation involves the consideration of early production and backlogs, for which one there are costs associated. The authors mention about the consideration of backlog and lost sales simultaneously using their proposed models. However, there is no relation/proportionality considered when dealing with these two approaches and no formal mathematical formulation is presented.

We believe that considering both backlog and lost sales in an environment without time windows, can also be insightful. By removing the concept of production time windows, early production is no longer applicable. Thus, items that are produced

Table 1. Summary of the literature review.

Papers	Single item	Multi item	Capacity	Backlog	Multiple customers	Lost sales	Reformulation
Absi, Kedad-Sidhoum, and Dauzère-Pérès (2011)	✓			✓		✓	
Aksen, Altinkemer, and Chand (2003)	✓					✓	
Chand et al. (2007)	✓				✓		
Constantino (2000)		✓	✓	✓			
Gade and Küçükyavuz (2013)		✓		✓			✓
Gruson, Cordeau, and Jans (2018)		✓	✓	✓			✓
Liu et al. (2007)	✓		✓			✓	
Pochet and Wolsey (1988)	✓			✓			✓
Sandbothe and Thompson (1990)	✓		✓			✓	
Swoveland (1975)	✓		✓	✓			
Zangwill (1966, 1969)	✓			✓			✓
Our study		✓	✓	✓	✓	✓	✓

before the demanded period, simply become inventory, and a holding cost is charged in each period until they are used to meet the demand of clients. In addition, considering a scarce literature related to the multiple customers backlogging (Chand et al. 2007), the consideration of different willingness to wait can be very helpful in order to provide insights on the behavior of the related backlog, and hence, lost sales. This further motivates the study on such a problem in this paper.

3. Mathematical model

In this section, we present the formulations proposed for the multi-item capacitated lot-sizing problem with simultaneous backlogging and lost sales. More specifically, the formulation assumes multiple customers backlog and fixed lost sales, called *FLR-MB-FL* (Facility Location Reformulation with Multiple Backlog and Fixed Lost sales). The other proposed formulations, under different backlog and lost sales assumptions, are derived from *FLR-MB-FL* and are presented in Table 2.

The *FLR-MB-FL* considers a finite planning horizon with dynamic deterministic demand, under the assumption that every unit of demand is represented by a unique customer. The demand can be satisfied on time either from production in the current time period and/or from inventory carried over from the previous period. When the demand is not met on time, a stock-out occurs. A fixed percentage of the unsatisfied demand results in lost sales, while the remaining unsatisfied demand must be backlogged and satisfied in later periods. The multiple customers have different behaviors when faced with backlog, which is characterized by the different tolerance, in terms of the maximum number of periods a customer is willing to wait for the backlog. A penalty cost is incurred for every unit of demand not satisfied. We assume zero initial and ending inventory, and no backlog remaining at the end of the planning horizon.

To model this problem, we use the facility location reformulation proposed for the lot-sizing problem (Krarup and Bilde 1977) that provides the flexibility and structure to formulate the problem with both backlog and lost sales as strategies to deal with stock-outs. More specifically, the reformulation keeps track of at which period the stock-out occurred, how many units were backordered, and for how long it has been backlogged, in order to properly account for the penalty costs. In this way, we are able to specifically identify and track backlog and backorder levels. Finally, such a reformulation typically provides a better linear programming relaxation, and hence improves computational time.

Consider the following sets, parameters and decision variables for the *FLR-MB-FL* problem:

Sets and parameters:
T: set of periods $(\{1, 2, ..., m\})$;
I: set of items $(\{1, 2, ..., n\})$;
d_{it}: demand to be fulfilled of item i in period t;
sc_{it}: setup cost of production for item i during period t;
vc_{it}: unit production cost for item i during period t;
hc_{it}: unit holding cost for item i at the end of period t;
bc_{it}: unit backlog cost for item i at the end of period t;
lc_{it}: unit lost sale cost for item i in period t;
st_{it}: setup time for item i during period t;
vt_{it}: unit production time for item i during period t;
C_t: production capacity during period t;
τ: the maximum number of periods an item can be backlogged for;
α: percentage of stock-outs for a specific period that will remain as backlog;
$((1-\alpha)$: remainder of the stock-outs for a specific period that becomes lost sales);
β_k: percentage of customers willing to wait a maximum number of k periods $(k = \{1, \ldots, \tau\})$, with $\sum_k \beta_k = \alpha$;
c_{ikt}: cost of item i produced in period k to satisfy the demand in period t.

$$c_{ikt} = \begin{cases} vc_{ik} + \sum_{l=k}^{t-1} hc_{il}, & \text{if } k < t; \\ vc_{ik}, & \text{if } k = t; \\ vc_{ik} + \sum_{l=t}^{k-1} bc_{il}, & \text{if } k > t. \end{cases}$$

Decision Variables:
Y_{it}: equals to 1 if there is a setup for item i in period t; 0, otherwise;
L_{it}: amount of lost sales for item i in period t;

$$Z_{ikt}: \begin{cases} k \leq t, & \text{quantity of item } i \text{ produced in period } k \text{ to satisfy demand in period } t; \\ k > t, & \text{quantity of item } i \text{ produced in period } k \text{ that is backordered in period } t, \\ & \text{and have been backlogged for } (k-t) \text{ periods.} \end{cases}$$

S_{it}: quantity of item i stocked-out at the end of period t (amount of the demand of item i that is not satisfied on time).

Model FLR-MB-FL

$$\min \sum_{t \in T} \sum_{i \in I} \left(sc_{it} Y_{it} + lc_{it} L_{it} + \sum_{k=1}^{\min\{m, t+\tau\}} c_{ikt} Z_{ikt} \right) \tag{1}$$

Subject to:

$$\sum_{k=1}^{\min\{m,t+\tau\}} Z_{ikt} + L_{it} = d_{it} \qquad \forall i,\ \forall t \quad (2)$$

$$Z_{ikt} \le d_{it} Y_{ik} \qquad \forall i,\ \forall t,\ \forall k,\ 1 \le k \le \min\{m, t+\tau\} \quad (3)$$

$$\sum_{i \in I} \left(st_{it} Y_{it} + \sum_{k=\max\{1,t-\tau\}}^{m} vt_{it} Z_{itk} \right) \le C_t \qquad \forall t \quad (4)$$

$$S_{it} = \sum_{k=t+1}^{\min\{m,t+\tau\}} Z_{ikt} + L_{it} \qquad \forall i,\ \forall t \quad (5)$$

$$L_{it} = (1-\alpha) S_{it} \qquad \forall i,\ \forall t \quad (6)$$

$$\sum_{l=k}^{\min\{\tau,m-t\}} Z_{i,(t+l),t} \le \sum_{l=k}^{\min\{\tau,m-t\}} \beta_l S_{it} \qquad \forall i,\ \forall t,\ \forall k,\ 1 \le k \le \min\{\tau, m-t\} \quad (7)$$

$$Z_{ikt},\ L_{it},\ S_{it} \ge 0,\ Y_{it} \in \{0,1\} \qquad \forall i,\ \forall t,\ \forall k \quad (8)$$

The objective function (1) minimizes the sum of setup cost and lost sales costs, and production cost, holding cost and backlog cost of each item in each period. The demand balancing constraints (2) guarantee that the demand of each item i in each period t is satisfied by production in periods k or becomes lost sales. Constraints (3) are the setup constraints ensuring that there is a setup for item i in period k only when there is the production for item i in period t that is used to satisfied the demand in period k. Constraints (4) enforce the capacity restrictions in the production process by taking into account the setup time and production time needed to produce the items in each period t, including the backlogged items.

Constraints (5) calculate the amount of demand of item i in period t that is not satisfied on time, consisting in the amount backordered and lost sales, i.e., the amount stocked-out for item i in period t. The lost sales constraints (6) imposes the fixed percentage $(1-\alpha)$ of the stocked-out demand for item i in period t as lost sales. These constraints serve as a restriction on the number of stocked-out items that can be backlogged, given by a fixed percentage α of the stocked-out for item i in period t. The multiple customer type constraints (7) guarantee that each item i that is backlogged for at least k periods is restricted by the number of customers willing to wait for k periods. The customers willing to wait for k periods consists of all the customers willing to wait for a minimum of k periods or longer. Finally, constraints (8) impose the non-negativity and binary conditions of the decision variables.

We also propose several other formulations for the multi-item capacitated lot-sizing problem with simultaneous backlogging and lost sales by taking into account different assumptions for the backlog in combination with others for the lost sales. Table 2 shows a summary of the formulations addressed in our study and how each formulation can be derived from the *FLR-MB-FL* base formulation by either removing sets of constraints, restricting some parameters or fixing some decision variables. We describe these formulations in more detail as follows.

Firstly, lets clarify the meaning of each one of the studied cases for the backlog and lost sales. As in the base formulation, the multiple customer backlog case describes a scenario where there are multiple customers types (each represented by a unit of

Table 2. Variation of models

	Lost sales			
Backlog	**Fixed**	**Variable**	**Unrelated**	**No**
Multiple customer	*FLR-MB-FL* **Base Formulation** $\alpha = \sum_{k=1}^{tau} \beta_k \leq 100\%$	*FLR-MB-VL* $L_{it} \geq (1-\alpha)S_{it}$ in (6)	—	*FLR-MB* $L_{it} = 0$ $\alpha = \sum_{k=1}^{tau} \beta_k = 100\%$ Remove (6)
Restricted	*FLR-RB-FL* Remove (7)	*FLR-RB-VL* $L_{it} \geq (1-\alpha)S_{it}$ in (6) Remove (7)	—	*FLR-RB* $L_{it} = 0$ Remove (5) - (7)
Unrestricted	*FLR-UB-FL* $\tau = m - 1$ Remove (7)	*FLR-UB-VL* $\tau = m - 1$ $L_{it} \geq (1-\alpha)S_{it}$ in (6) Remove (7)	*FLR-UB-UL* $\tau = m - 1$ Remove (5) - (7)	*FLR-UB* $\tau = m - 1$ $L_{it} = 0$ Remove (5) - (7)
No	—	—	*FLR-UL* $Z_{ikt} = 0$, if $k > t$ Remove (5) - (7)	—

demand) and each customer has a different behavior when faced with a backlog, which is characterized by the maximum number of periods a customer is willing to wait. In the restricted backlog case, with only one type of customer, there is also a maximum number of periods for which the items can be backlogged for (τ), whereas, assuming the unlimited backlog case, the demand of the backlogged items can be satisfied during any period within the time horizon ($\tau = m - 1$).

About the lost sales assumption, with either fixed or variable lost sales strategies, when a stock-out occurs, a minimum fixed percentage of the unsatisfied demand automatically results in lost sales. This fixed percentage represents the customers who are not willing to wait in case of a stock-out. For the fixed-proportion lost sales case, the remaining unsatisfied demand must be backlogged and satisfied in later periods. As for the variable-proportion lost sales case, the company decides whether the remaining unsatisfied demand will be lost sales or become backlog, i.e., the company has the option to incur more lost sales if this is more advantageous for the company. This can occur when there is high demand and strict capacity levels, in which the additional lost sales for some items can also be beneficial if there are items with different characteristics in terms of backlog cost, lost sales cost, setup cost, capacity usage, etc. In the unrelated lost sales assumption, there is no fixed proportion of the unsatisfied demand that must result in lost sales, i.e., the model has the flexibility to determine how much of the demand, if any, is allocated to lost sales.

Now, let's describe how the models can be derived from the base formulation *FLR-MB-FL*. Starting by adding more flexibility to the lost sales assumption, it results in the formulation with multiple customer backlog and variable lost sales (*FLR-MB-VL*) that can be derived by changing constraints (6) to an inequality, where the amount of lost sales for item i in period t (L_{it}) can be bigger than the fixed percentage $(1 - \alpha)$ of the stocked-out for item i in period t. In essence, the fixed and variable lost sales formulations are similar with a minor adjustment to the lost sales constraints. However, this adjustment has an impact on the formulation's ability to provide feasible solutions. For instance, with fixed lost sales, some instances might be infeasible if the capacity is not sufficient, whereas with variable lost sales, the company always has the possibility to have all the demand as lost sale, and hence the problem can never be infeasible.

Considering the assumption that there is only one type of customers that has a certain level of willingness to wait for a maximum of τ periods, instead of multiple types, the formulation with restricted backlog and fixed lost sales (*FLR-RB-FL*) can be derived from the base formulation by simply removing constraints (7) that deal with the multiple customer backlog. From there, in order to add the variable lost sales (*FLR-RB-VL*), constraints (6) are altered to become inequalities, as described before.

The formulations under the unrestricted backlog assumption can be easily derived from their respective restricted version, (*FLR-RB-FL*, *FLR-RB-VL*), by no longer having a backlog restriction, i.e., assuming $\tau = m-1$, where m is the last period in the time horizon. By doing so, we can obtain the formulations with unrestricted backlog and fixed lost sales (*FLR-UB-FL*), or variable lost sales (*FLR-UB-VL*), respectively.

In the formulation with unrestricted backlog and unrelated lost sales (*FLR-UB-UL*), there are no backlog restrictions or proportionality among backlog and lost sales. Instead, both strategies are considered to deal with the stock-outs. In other words, the quantities assigned to backlog and lost sales are due to the search for an optimal solution with minimal cost. Such a formulation can be derived from the base formulation, by removing constraints (5) - (7), that impose relationship among backlog and lost sales, and removing the backlog restrictions, i.e., assuming $\tau = m - 1$. Note that this formulation has been mentioned in Absi, Kedad-Sidhoum, and Dauzère-Pérès (2011), but no formal mathematical model was presented by these authors.

Even if the goal of this paper is to propose formulations where backlog and lost sales are taken into account simultaneously, we have also formulate the problems addressing either backlog or lost sales, in order to show how the base formulation can be derived to those cases as well. The formulation with multiple customer backlog and no lost sales (*FLR-MB*) (Chand et al. 2007), can be obtained from the base formulation by setting the lost sales variables to zero ($L_{it} = 0$). From constraints (6) and due to $L_{it} = 0$, the percentage of stock-outs that remains as backlog corresponds to 100% ($\alpha = 100\%$), and hence, the sum of the percentage of customers willing to wait a maximum number of k periods is also 100%, i.e., $\alpha = \sum_{k=1}^{tau} \beta_k = 100\%$. Note that, due to this last affirmation, constraints (6) become redundant and can be removed. For the formulation with restricted backlog and no lost sales (*FLR-RB*), constraints (5) - (7) are removed due to the fixing of the lost sales variables ($L_{it} = 0$) and the consideration of only one type of customer. In the formulation with unrestricted backlog and no lost sales (*FLR-UB*) (Constantino 2000), it is assumed that there is no backlog restriction ($\tau = m - 1$) in the previous formulation. Finally, the formulation with no backlog and unrelated lost sales (*FLR-UL*) (Sandbothe and Thompson 1990) can be obtained from the base formulation by simply not allowing backlog, i.e., imposing $Z_{ikt} = 0$, if $k > t$ and removing constraints (5) - (7), that impose relationship among backlog and lost sales.

Amongst the formulations presented in this study, there is some a priori relationship between the optimal objective function value that can be identified. Under the same backlog assumption, a relationship holds between formulations with different types of lost sales: the variable lost sales formulation has an optimal objective function value that is smaller or equal to the value of the formulation with fixed lost sales. Considering formulations with the same lost sales assumption, given that formulations with unrestricted backlog provide more flexibility, the optimal objection function value is less than or equal to formulations with restricted backlog. Furthermore, the formulation with multiple customer types has an optimal objective function value larger than or equal to the formulations with restricted and unrestricted backlog assumption, assuming that for the restricted case, they have the same maximum restriction as the multiple customer type. Finally, the formulation with unrestricted assumption to both backlog and lost sales has an optimal objective function smaller than any other assumption with regards to backlog and lost sales.

4. Relax-and-fix heuristic

In this section, we describe the rapid relax-and-fix heuristic used to solve the multi-item capacitated lot-sizing problem with simultaneous multiple customer backlog and fixed lost sales. The relax-and-fix heuristic (Stadtler 2003; Akartunali and Miller 2009; Toledo et al. 2015) is an approach that has been successfully applied in various lot-sizing problems. The heuristic generates a feasible solution by decomposing the problem and solving a series of partially relaxed mixed-integer subproblems, each with a number of fixed, integer and relaxed variables. Such subproblems are quickly solved, when compared to the original problem, and provide, at the end of the heuristic, a solution that corresponds to a feasible solution to the original problem. The decomposition strategy takes advantage of the mathematical reformulation and the backlog assumptions. This is because, as in the restricted backlog case there is a maximum number of periods for which items can be backlogged, this restriction is used to decompose all the formulations.

The relax-and-fix heuristic (see Algorithm 1) starts by defining the number of periods the integrality constraints of the decision variables are added to the problem (Δ), and the number of periods the decision variables are fixed. In the first step, the integrality constraints of all the decision variables are relaxed. Next, the integrality constraints of the binary decision variables (Y_{it}) with periods belonging to the ordered set $\Omega = \{1, \ldots, \Delta\}$ are added to the problem and the resulting mixed-integer problem is solved. Following, the binary decision variables belonging to the first θ periods in Ω are fixed according to the current found solution. Then, the ordered set of periods Ω is moved forward by θ periods, for which the integrality constraints of the binary decision variables (Y_{it}) with periods belonging to the updated set Ω are added to the model. These steps are repeated until the integrality constraints of all the binary decision variables have been added to the problem or any resulting subproblem is infeasible. The final solution from the heuristic consists of a feasible solution to the problem.

Algorithm 1 Relax-and-fix heuristic

Input: Δ: number of periods to add integrality constraints.
    θ: number of periods to fix variables.
Step 1: Relax the integrality of all variables.
Step 2: Add the integrality constraints for periods in the ordered set $\Omega = \{1, \ldots, \Delta\}$:
    $Y_{it} \in \{0, 1\}, \quad \forall i, \ \forall t \in \Omega$
while $\Delta \leq m$ **do**
    Step 3: Solve the resulting mixed-integer subproblem
    if Resulting mixed-integer problem is infeasible **then**
        No solution is found
        break
    end if
    Step 4: Fix the binary decision variables:
        $Y_{it} = \widetilde{Y}_{it}, \quad \forall i,,$ first θ periods in Ω
    Step 5: Move forward set Ω by θ periods.
end while
Output: Feasible solution found.

We have tested different variations for Δ and θ parameters ($\Delta = \{4, 5, 6\}$ and $\theta = \{1, 2\}$), for which $\Delta = 6$ and $\theta = 1$ presented the best tradeoff among the comparisons and it is used in the further analysis of the heuristic.

5. Computational experiments

This section presents the dataset generation and the computational results for the multi-item capacitated lot-sizing problem with simultaneous backlogging and lost sales. The proposed models are implemented in C++ using the C callable library of the IBM ILOG Cplex 20.1.0. All the computational tests were conducted using a computer with 2 processors Intel(R) 3.07GHz (96 of RAM). We impose the use of a single processor (threads = 1) and a time limit of 1800 seconds, when solving the problem. The other CPLEX parameters were set to their default values. The tested instances and computational experiments are discussed as follows.

5.1. *Dataset: base case*

With the absence of an existing dataset having both backlog and lost sales parameters, the instances in the dataset, called G-dataset, from Trigeiro, Thomas, and McClain (1989), which were originally intended for the classical multi-item capacitated lot-sizing problem, are adapted to additionally take into account the parameters related to backlog and lost sales. The G-dataset, is a well-known set in the literature and has been used in other researches (Jans and Degraeve 2007; Süral, Denizel, and Wassenhove 2009; Fiorotto and de Araujo 2014).

The G-dataset presents different problems structures in terms of the number of items, periods and the number of instances in each subset, totalling 71 instances. The details about each subset is shown in Table 3 (for more information see Trigeiro, Thomas, and McClain (1989)).

Table 3. Dataset: problems structure

Name of instances	Items (n)	Periods (m)	Number of instances
G6-15	6	15	46
G12-15	12	15	5
G24-15	24	15	5
G6-30	6	30	5
G12-30	12	30	5
G24-30	24	30	5

The G-dataset uses a static production capacity level for all their instances, meaning that the capacity level for each period remains the same throughout the time horizon. The same static idea is used for other parameters, such as setup cost and inventory cost, by assuming that the value is fixed for all periods in the planning horizon. The instances used in our study consider the original values from the G-dataset for the parameters related to the multi-item capacitated lot-sizing problem (d_{it}, sc_i, vc_i, hc_{it}, st_i, vt_i, cap_t). As in the original G-dataset, there are no backlog or lost sales assumptions; all demand can be satisfied on-time. In order to create instances that have stock-outs, we decrease the original capacity down to 92.5% of its original level. By doing so, we can better expose the tradeoff between early production, on-time production, late production, or no production, due to a tight capacity limit. We have also tested other lower values to decrease the capacity, such as 90%, however, the number of infeasible instances considerably increase, and due to this fact we assumed a 92.5% down rate for the capacity.

For the parameters related to backlog and lost sales assumptions (bc_i, lc_i, τ, α, and β_k), we strived to keep the structure throughout the data variation to ensure comparability, and we ensured that the cost of lost sales is always greater than the cost if the item is backlogged and satisfied in a later period, i.e., we guarantee that $lc_i > \tau * bc_{it}$. Furthermore, we assume that $hc_{it} \leq bc_{it}$. The values for parameters generated in an interval of the type $[a, b]$ consider a uniform distribution.

- number of periods: $m = \{15, 30\}$;
- number of items: $n = \{6, 12, 24\}$;
- demand of items (average per period): $d_{it} = 100$;
  Original values from the instances in Trigeiro, Thomas, and McClain (1989).
- setup cost of an item: $sc_i \in [200, 1000]$;
- unit production cost of an item: $vc_i = 0$;
- unit holding cost of an item: $hc_i \in [1, 5]$;
- unit backlog cost of an item: $bc_i \in [6, 7]$;
- unit lost sale cost of an item: $lc_i \in [28, 33]$;
- setup time of an item: $st_i \in [10, 50]$;
- unit production time of an item: $vt_i = 1$;
- maximum number of periods for backlog: $\tau = \{1, 4\}$;
- percentage of stock-outs that will remain as backlog: $\alpha = 75\%$;
- percentage of customers willing to wait a maximum number of k periods ($k = \{1, \ldots, \tau\}$): $\beta_\tau = [25\%, 20\%, 15\%, 15\%]$;
- production capacity of items: $Cap_t = 92.5\% cap$;
  the capacity *cap* is the original value from the instances in Trigeiro, Thomas, and McClain (1989).

5.2. *Computational analysis: base case*

In this section, we analyze the behavior of the different mathematical formulations with various assumptions with regards to the backlog and lost sales. The performance measures include the objective function value, computational time (in seconds), and solution gap (in percentage). We also compare the structure of the solution related to the cost (in percentage) for each separated component in the objective function (setup, inventory, backlog and lost sale).

For this dataset, there were 5 instances in class $G6-15$ (with 6 item and 15 periods) that were infeasible using the *FLR-MB* formulation. Infeasible solutions are possible when using this formulation because it does not allow for lost sales and assumes a restricted backlog with $\tau = 4$, due to this fact, the *FLR-MB* can be seen as an inflexible formulation that can run into some infeasibility issues. In the remaining of this paper, these 5 instances are removed from the analysis. Note that we combine all the results for the different problems size of the G-dataset together in order to provide a better representation of the results.

Table 4 presents the average value obtained with the dataset for each model. We can see that, the worse objective function value is obtained with the *FLR-UL* formulation that allows only unrelated lost sales, whereas the best solution is obtained with the *FLR-UB-UL*, which accepts unrestricted quantities for both stocked-out options. This can be explained by the fact that the former is the most restricted model, while the later is the most flexible one, and hence, provide the best results in terms of solution quality. When only backlog is considered, there is more flexibility and less penalty costs, and hence, models that consider simultaneously backlog and lost sales, gave solutions that are higher than the respective formulations with only backlog but lower than the respective formulations with only lost sales. The objective function value slightly change when removing the restriction on backlog, for a certain assumption ($\tau = 4$), due to the result of a lenient restriction on backlog. In fact, by reducing the maximum number of periods an item can be backlogged for (from 4 periods to 1 period), we can create a larger restriction on backlog, producing a difference in the objective function value between formulations with unrestricted and restricted backlog.

In some cases, there is also an slight increase in the objective function when the formulation includes the assumption of multiple customer types, instead of a singular customer type. Under the same backlog assumption, the objective function values are very similar for both the fixed and variable lost sales versions, and we can conclude that there may be little benefit in incurring more lost sales than necessary. All formulations have a low average optimality gap, obtaining gaps not bigger than 0.28% (from 0.0661% to 0.2769%), which indicates that the formulations can find good solutions independently of the backlog and lost sales assumptions. Comparing the models in terms of computational time, it is possible to see that the time is impacted by the type of formulation, varying around 35% (from 814.18 seconds to 1257.49 seconds). Under the same backlog assumption and for a single customer, formulations that consider both backlog and lost sales have larger average computing times than those that only consider backlog. More specifically, for a single customer, relaxing the formulations by allowing variable lost sales increase the computational time up to 37%, when compared with no lost sales assumption (28% on average). For some cases, multiple customers increases the computational effort when compared to the same formulation with single customer. This result may be caused by the added complexity which the multiple customer types bring to the formulation in comparison with single customers, specially when no lost sales is allowed.

Table 4 also shows the structure of the solution among the various mathematical models, the column Setup indicates the percentage of the total cost that is made up of the cost to setup production in the objective function value. The following three rows do the same for inventory, backlog, and lost sales costs in that order. We can see that the *FLR-UL* formulation has the highest setup and lost sales percentages in the objective function, and when adding backlog, these percentages decrease. This is because, backlog can be used as a strategy to cut costs. In fact, this strategy is most effective when only backlog is allowed, when compared to both stock-out strategies. Another interesting fact is that allowing unrestricted backlog and lost sales, as in *FLR-UB-UL*, decreases the setup cost by 5.5% on average, while keeping the backlog percentage cost very similar to the formulations with only backlog, i.e., lost sales is only used in very little proportions. The multiple customer type formulations are less flexible and may be forced to incur more lost sales or additional production setups during periods of high demand. In summary, we see an apparent tradeoff between the cost of backlog and lost sales. The tradeoff, however may be sensitive to certain parameters. Changing certain parameters can affect the total number of stock-outs that are included in the optimal solution.

We perform a comparison of the various mathematical formulations considering two different strategies to solve the models, which are the optimization package and the heuristic. Table 5 shows the results in terms of objective function and computational time, along with the variation between these values when comparing the two approaches. We can see that the objective function values vary very slightly between the strategies, for which under the same backlog assumption, the model with variable lost sales presents the highest deviation in the objective function, up to 0.08%, among

Table 4. Results: base case

Backlog	Lost sales	Models	OF	Gap (%)	Time (sec.)	Structure of costs (%) Setup	Inventory	Backlog	Lost sales
Unrestricted	Unrestricted	*FLR-UB-UL*	124843.29	0.0950	998.25	59.73	25.59	14.49	0.19
Unrestricted	No	*FLR-UB*	124862.00	0.0992	1012.56	59.65	25.54	14.81	0.00
	Fixed	*FLR-UB-FL*	130818.88	0.2495	1239.81	62.61	27.31	4.26	5.81
	Variable	*FLR-UB-VL*	130808.13	0.2584	1257.49	62.52	27.48	3.93	6.07
Restricted ($\tau = 4$)	No	*FLR-RB*	124863.53	0.0892	954.89	59.70	25.50	14.81	0.00
	Fixed	*FLR-RB-FL*	130813.01	0.2287	1142.06	62.62	27.27	4.25	5.86
	Variable	*FLR-RB-VL*	130792.21	0.2259	1170.99	62.65	27.40	3.93	6.03
Restricted ($\tau = 1$)	No	*FLR-RB*	125132.22	0.0661	814.18	60.10	25.14	14.76	0.00
	Fixed	*FLR-RB-FL*	131227.04	0.1767	1019.64	62.99	26.71	4.00	6.30
	Variable	*FLR-RB-VL*	131085.18	0.1951	1117.55	62.93	26.99	3.41	6.67
No	Unrestricted	*FLR-UL*	135194.97	0.2769	1031.23	63.14	26.86	0.00	10.00
Multiple customers	No	*FLR-MB*	125547.78	0.1805	1135.63	60.29	26.11	13.60	0.00
	Fixed	*FLR-MB-FL*	131243.65	0.2100	1089.95	62.79	26.93	4.02	6.27
	Variable	*FLR-MB-VL*	130952.77	0.2200	1120.15	62.68	27.27	3.67	6.38

the methods. However, it is in the computational time that there is a huge variation between the approaches. In fact, while the optimization package takes 1104.82 seconds on average to solve the instances, the heuristic is able to reduce 95.55% on average by using only 49.08 seconds, on average to solve the instances. For the same backlog assumption, the highest reduction in the computation time occurs when no lost sales is allowed, whereas fixed and variable lost do not present a clear impact in the variation for the computational time. Therefore, we can state that the heuristic is able to considerably reduce the computational effort, while keeping the solution quality, independently of the mathematical models.

Table 5. Results base case: comparison of solution approaches

Backlog	Lost sales	Models	Mathematical model OF	Mathematical model Time (sec.)	Heuristic OF	Heuristic Time (sec.)	Variation (%) OF	Variation (%) Time
Unrestricted	Unrestricted	*FLR-UB-UL*	124843.29	998.25	124901.50	37.12	0.05	-96.28
Unrestricted	No	*FLR-UB*	124862.00	1012.56	124923.59	35.00	0.05	-96.54
	Fixed	*FLR-UB-FL*	130818.88	1239.81	130903.38	89.99	0.06	-92.74
	Variable	*FLR-UB-VL*	130808.13	1257.49	130889.12	65.15	0.06	-94.82
Restricted ($\tau = 4$)	No	*FLR-RB*	124863.53	954.89	124925.62	25.12	0.05	-97.37
	Fixed	*FLR-RB-FL*	130813.01	1142.06	130904.32	68.43	0.07	-94.01
	Variable	*FLR-RB-VL*	130792.21	1170.99	130891.85	61.16	0.08	-94.78
Restricted ($\tau = 1$)	No	*FLR-RB*	125132.22	814.18	125159.36	15.39	0.02	-98.11
	Fixed	*FLR-RB-FL*	131227.04	1019.64	131290.92	38.79	0.05	-96.20
	Variable	*FLR-RB-VL*	131085.18	1117.55	131152.00	33.51	0.05	-97.00
No	Unrestricted	*FLR-UL*	135194.97	1031.23	135251.69	83.40	0.04	-91.91
Multiple customers	No	*FLR-MB*	125547.78	1135.63	125616.32	20.43	0.05	-98.20
	Fixed	*FLR-MB-FL*	131243.65	1089.95	131305.96	56.17	0.05	-94.85
	Variable	*FLR-MB-VL*	130952.77	1120.15	131042.32	57.52	0.07	-94.87
	Average		129156.05	1078.88	129225.57	49.08	0.05	-95.55

5.3. *Sensitivity analysis*

In this section, we evaluate the impact that certain parameters have on the behavior of the formulations and structure of the solution. For this, we conduct a sensitivity analysis on the following parameters: percentage of available capacity, percentage of backlog (α), and backlog and lost sales costs.

The capacity can be seen as an important factor for the cause of stock-outs. Up to this point, we have tested the models with an adjusted capacity level of 92.5% from the original values in the dataset (Trigeiro, Thomas, and McClain 1989). For this sensitivity analysis, we perform tests using different capacity levels (95% and 100%)

in order to evaluate the impact of the capacity on the different formulations, totalling 213 instances. As we have already observed five infeasible instances in the *FLR-MB*, with a 92.5% capacity level, we did not further lower the capacity percentage to not obtain even more infeasible solutions.

The results presented in Table 6 show that the average objective function value decreases, 3.5% on average, as the capacity levels increases, for which the model *FLR-UL* is the most affected formulation, with an decrease of almost 5%, while *FLR-UB-UL* is the least, with a 2.2% decrease. This can be explained by the fact that *FLR-UL* has the smallest level of flexibility, and hence, when faced with lower capacity level, the formulation is forced into accepting more lost sales, which have a high cost. If we examine the formulations that consider both concepts simultaneously, we notice that the objective function value for formulations with multiple customers are more impacted than those with a single customer type. This is because in the formulation, the backlog constraints are more restrictive, and hence a lack of capacity will have bigger repercussions.

A change in capacity levels has a more significant impact on the computational time, decreasing 91.8% on average, when allowing 100% of the production capacity, compared to 92.5%. For a single customer, relaxing the formulations by allowing variable lost sales increases the standard deviation of the computational time among the different levels of capacity. Therefore, we can state that the capacity has a significant impact on the behavior of the formulations, since the computing time and the objective function value increase as capacity decreases.

The increase in total cost can be traced back to the change in the structure of the solution forced by the tightened capacity. In fact, the contribution of setup cost towards the total cost decreases when capacity is decreased, but not by a large amount, while the proportion of backlog and lost sales costs rise significantly, becoming more important. Under the same backlog assumption, the formulation that uses only backlog exhibit the smallest proportion of setup and inventory holding cost at all capacity levels compared to the other models that accept both stock-out strategies. The reason behind this is because for these formulations backlog constitutes a substantial proportion of the total costs. The stock-outs combined percentages (backlog and lost sales) increase from a decrease in capacity, which seems to be similar for all the formulations. Such a behavior appears to be logical as lower capacity levels will force some backlog and/or lost sales because of feasibility. Therefore, it might indicate that capacity has a similar impact on all formulations.

In the previous section, we have assumed that when faced with a stock-out, 75% of all customers are willing to wait for backlog. To explore the impact of having customers with different willingness to wait, we conduct a sensitivity analysis of the α parameter with six different values (percentages), as indicated in Table 7, totalling 426 instances. For problems with multiple customers, we adjust each customer type proportionally, while maintaining condition that $\sum_k \beta_k = \alpha$. Note that in this sensitivity analysis, we only consider the formulations with simultaneous backlog and lost sales.

Table 8 present an apparent impact on the objective function value when the percentage value for the backlog is altered. For all formulations, as the value of α is reduced, the variable lost sales version of the formulations provides a better total cost than the fixed lost sales version. In addition, the multiple customer types further increases the total costs, when compared to those that consider a single customer type, no matter the varied parameter. Looking exclusively for each formulation, the objective function value and gap decreases as a bigger percentage of customers are willing to wait. This is because that by allowing the possibility of more backlog, the formulation

Table 6. Sensitivity analysis: percentage of capacity

							Structure of costs (%)			
Backlog	**Lost sales**	**Models**	**Capacity (%)**	**OF**	**Gap (%)**	**Time (sec.)**	**Setup**	**Inventory**	**Backlog**	**Lost sales**
Unrestricted	Unrestricted	*FLR-UB-UL*	100%	122079.22	0.0090	53.07	63.61	27.76	8.63	0.00
			95%	123356.63	0.0255	517.38	62.06	26.43	11.47	0.04
			92.5% [1]	124842.82	0.0969	1023.63	59.72	25.59	14.50	0.19
Unrestricted	No	*FLR-UB*	100%	122079.62	0.0090	47.54	63.58	27.78	8.64	0.00
			95%	123360.01	0.0269	559.65	62.08	26.41	11.51	0.00
			92.5% [1]	124862.00	0.0992	1017.07	59.65	25.54	14.80	0.00
	Fixed	*FLR-UB-FL*	100%	125639.29	0.0093	164.36	67.26	28.53	1.72	2.49
			95%	128164.46	0.0691	758.95	65.13	27.75	2.96	4.16
			92.5% [1]	130813.84	0.2457	1241.28	62.58	27.33	4.27	5.83
	Variable	*FLR-UB-VL*	100%	125637.93	0.0092	169.84	67.26	28.54	1.69	2.52
			95%	128161.98	0.0716	756.54	65.15	27.79	2.86	4.19
			92.5% [1]	130805.79	0.2578	1258.26	62.53	27.50	3.93	6.05
Restricted ($\tau = 4$)	No	*FLR-RB*	100%	122079.82	0.0088	31.04	63.63	27.78	8.59	0.00
			95%	123360.51	0.0229	469.81	62.03	26.47	11.50	0.00
			92.5% [1]	124862.90	0.0901	966.84	59.70	25.49	14.81	0.00
	Fixed	*FLR-RB-FL*	100%	125639.62	0.0088	121.44	67.24	28.54	1.72	2.50
			95%	128164.63	0.0633	676.09	65.16	27.71	2.96	4.16
			92.5% [1]	130813.62	0.2304	1145.17	62.67	27.22	4.25	5.86
	Variable	*FLR-RB-VL*	100%	125638.09	0.0089	106.20	67.29	28.52	1.68	2.51
			95%	128160.11	0.0661	694.15	65.19	27.75	2.88	4.18
			92.5% [1]	130788.61	0.2259	1173.51	62.62	27.42	3.93	6.03
Restricted ($\tau = 1$)	No	*FLR-RB*	100%	122098.48	0.0087	18.86	63.80	27.61	8.59	0.00
			95%	123456.95	0.0189	476.80	62.28	26.23	11.49	0.00
			92.5% [1]	125132.09	0.0662	814.90	60.09	25.15	14.76	0.00
	Fixed	*FLR-RB-FL*	100%	125686.44	0.0089	77.99	67.63	28.31	1.58	2.48
			95%	128385.36	0.0573	682.17	65.49	27.45	2.74	4.32
			92.5% [1]	131226.85	0.1757	1021.76	63.04	26.67	3.99	6.29
	Variable	*FLR-RB-VL*	100%	125680.10	0.0090	98.87	67.64	28.35	1.52	2.49
			95%	128342.44	0.0619	677.12	65.42	27.52	2.62	4.44
			92.5% [1]	131082.25	0.1946	1119.18	62.89	27.02	3.40	6.68
No	Unrestricted	*FLR-UL*	100%	128559.11	0.0109	168.82	69.12	26.94	0.00	3.95
			95%	132186.17	0.0488	613.72	65.75	26.64	0.00	7.61
			92.5% [1]	135198.37	0.2796	1031.47	63.18	26.80	0.00	10.02
Multiple customers	No	*FLR-MB*	100%	122264.66	0.0090	37.71	64.17	28.05	7.78	0.00
			95%	123671.68	0.0241	509.27	62.75	26.85	10.40	0.00
			92.5% [1]	125545.53	0.1762	1132.14	60.20	26.18	13.61	0.00
	Fixed	*FLR-MB-FL*	100%	125683.52	0.0090	83.33	67.54	28.40	1.58	2.47
			95%	128382.30	0.0637	689.19	65.41	27.50	2.78	4.31
			92.5% [1]	131237.13	0.2056	1087.17	62.78	26.95	4.01	6.25
	Variable	*FLR-MB-VL*	100%	125676.56	0.0088	102.62	67.56	28.41	1.55	2.48
			95%	128271.50	0.0596	684.74	65.23	27.69	2.66	4.42
			92.5% [1]	130952.19	0.2189	1115.95	62.63	27.32	3.67	6.38

[1] Base case instances

has more flexibility and can therefore provide a better solution in the available running time. Furthermore, *FLR-RB-FL* exhibits a steeper standard deviation in the objective function value when α increases, while the formulations with multiple customer have more restrictions in place, which could explain the lowest standard deviation in the objective function values. The results do not provide enough evidences that the computational time of certain formulations are more affected than others or even the clear impact of the different values of α on the time.

To further analyze the impact of α on the models, we look into the changes in the structure of the solutions. As mentioned before, the reasoning for the better solutions is due to the added flexibility that backlogging provides. The results in Table 8 indicate that as the value of α increases up to 100%, the percentage of backlog cost increases. This slope supports the notion that the impact of backlog grows increasingly as the value of α increases as well. In a scenario where both backlog and lost sales are present, backlog seems to increase when customers are more patient, therefore allowing the added flexibility to structure a production schedule that reduces overall cost, since the percentage of backlog increases, while the lost sales decrease when the α increases. Furthermore, as the value of α increases, the percentage of setup cost decreases for all formulations.

In this final sensitivity analysis, we aim to determine the impact on the objective function value as there is more variability among items with respect to the backlog and lost sales cost. More specifically, we consider two additional scenarios: the backlog

Table 7. Relationship between α and β_τ

α (%)	β_τ (%)
55	$[18.33, 14.67, 11.00, 11.00]$
65	$[21.67, 17.33, 13.00, 13.00]$
75	$[25.00, 20.00, 15.00, 15.00]$
85	$[28.33, 22.67, 17.00, 17.00]$
95	$[31.67, 25.33, 19.00, 19.00]$
100	$[33.33, 26.67, 20.00, 20.00]$

Table 8. Sensitivity analysis: different percentages of backlog

							Structure of costs (%)			
Backlog	**Lost sales**	**Models**	**Backlog (α)**	**OF**	**Gap (%)**	**Time (sec.)**	**Setup**	**Inventory**	**Backlog**	**Lost sales**
Unrestricted	Fixed	*FLR-UB-FL*	55%	132509.92	0.2935	1148.98	62.86	27.25	2.37	7.52
			65%	131760.72	0.2672	1195.29	62.90	27.13	3.21	6.76
			75% [1]	130819.10	0.2521	1243.52	62.60	27.31	4.27	5.83
			85%	129408.06	0.2002	1161.28	62.23	26.93	6.10	4.74
			95%	127021.52	0.1536	1182.51	61.09	26.29	10.16	2.46
			100%	124866.23	0.1025	1055.04	59.65	25.60	14.75	0.00
	Variable	*FLR-UB-VL*	55%	132458.27	0.2848	1178.88	63.04	27.14	2.21	7.61
			65%	131735.03	0.2777	1207.06	62.77	27.29	2.93	7.01
			75% [1]	130807.93	0.2598	1258.76	62.56	27.47	3.90	6.06
			85%	129400.28	0.2080	1182.86	62.26	26.98	5.92	4.84
			95%	127005.43	0.1503	1147.94	60.97	26.45	9.99	2.59
			100%	124846.42	0.0991	1009.00	59.73	25.48	14.60	0.19
Restricted ($\tau = 4$)	Fixed	*FLR-RB-FL*	55%	132490.64	0.2547	1036.88	62.82	27.25	2.37	7.55
			65%	131764.43	0.2529	1169.74	62.64	27.30	3.25	6.81
			75% [1]	130814.90	0.2314	1149.38	62.62	27.27	4.25	5.85
			85%	129424.71	0.1950	1100.33	62.24	26.96	6.09	4.71
			95%	127013.76	0.1298	1111.65	61.02	26.35	10.17	2.47
			100%	124863.30	0.0893	942.15	59.69	25.50	14.81	0.00
	Variable	*FLR-RB-VL*	55%	132458.61	0.2574	1104.79	62.88	27.24	2.22	7.66
			65%	131734.93	0.2513	1175.91	62.53	27.50	2.95	7.02
			75% [1]	130790.68	0.2268	1175.74	62.59	27.44	3.93	6.04
			85%	129399.91	0.1906	1082.09	62.28	26.95	5.93	4.85
			95%	127003.70	0.1355	1129.69	61.06	26.33	10.01	2.59
			100%	124846.56	0.0911	975.34	59.85	25.51	14.45	0.19
Restricted ($\tau = 1$)	Fixed	*FLR-RB-FL*	55%	132955.54	0.2259	1034.02	63.23	26.79	2.06	7.92
			65%	132243.10	0.2246	991.22	63.01	26.81	2.88	7.30
			75% [1]	131226.97	0.1769	1019.37	62.99	26.72	3.99	6.30
			85%	129781.77	0.1592	893.04	62.74	26.48	5.86	4.92
			95%	127291.92	0.1111	914.42	61.33	25.91	10.19	2.57
			100%	125131.46	0.0674	815.84	60.03	25.27	14.70	0.00
	Variable	*FLR-RB-VL*	55%	132691.91	0.2562	1018.80	63.00	27.18	1.63	8.18
			65%	132006.04	0.2179	1048.86	62.94	27.20	2.36	7.49
			75% [1]	131088.79	0.1976	1119.21	62.92	26.96	3.42	6.70
			85%	129692.86	0.1534	982.76	62.60	26.91	5.30	5.19
			95%	127241.64	0.1229	890.19	61.23	26.19	9.84	2.74
			100%	125028.33	0.0662	866.66	60.13	25.21	14.36	0.30
Multiple customers	Fixed	*FLR-MB-FL*	55%	132948.25	0.2427	1048.76	63.14	26.91	2.11	7.85
			65%	132216.81	0.2175	1059.49	63.03	26.84	2.92	7.21
			75% [1]	131238.79	0.2057	1085.59	62.74	26.95	4.03	6.28
			85%	129813.77	0.1686	1021.84	62.60	26.65	5.87	4.88
			95%	127471.16	0.1393	1094.36	61.48	26.43	9.68	2.41
			100%	125540.48	0.1622	1112.49	60.24	26.14	13.62	0.00
	Variable	*FLR-MB-VL*	55%	132558.82	0.2574	1045.44	62.83	27.30	1.97	7.90
			65%	131863.22	0.2355	1087.53	62.75	27.39	2.59	7.27
			75% [1]	130953.60	0.2192	1112.26	62.68	27.27	3.67	6.38
			85%	129593.13	0.1754	1066.96	62.48	27.06	5.38	5.08
			95%	127305.34	0.1356	1076.14	61.39	26.73	9.08	2.81
			100%	125323.49	0.1075	1106.89	60.22	26.22	13.08	0.48

[1] Base case instances

and lost sales costs is multiplied by 5 for the half of the items; and the backlog and lost sales costs is multiplied by 5 for all the items (see Table 9). In total, 213 instances are analyzed.

Table 9. Backlog and lost sales costs variations

Parameter sensitivity analysis	Base case	Variations
Cost variations	$\text{Original} = \begin{cases} bc_i, & \forall i; \\ lc_i, & \forall i. \end{cases}$	$\text{Adjust1} = \begin{cases} 5bc_i, & i = 1, \ldots, n/2; \\ 5lc_i, & i = 1, \ldots, n/2. \end{cases}$
		$\text{Adjust2} = \begin{cases} 5bc_i, & \forall i; \\ 5lc_i, & \forall i. \end{cases}$

Table 10 shows the average results for each formulation considering the three scenarios. The results indicate that across all formulations, the objective function value experiences a significant increase when the costs are increased for certain items, and even further for all items, which is logical since some costs are increased. In fact, there is an increase of 28% on average, when the costs are altered for all items, for which *FLR-UB-UL* is the least affected, while *FLR-UL* is the most affected. The results indicate that an increase in the backlog and lost sales costs for certain items or even all the items do not seem to impact the computing time of the formulations in a consistent way. Indeed, some formulations have a longer computing time with the adjusted backlog and lost sales costs compared to the original costs, and others experience the opposite. However, the computing time remains fairly similar for all formulation. The formulations presented quite small gaps in this analysis, not bigger than 0.4555%, for which the impact is not clear when the costs are altered.

To explore in greater detail the impact of lost sales cost on the optimal solution, we review the structure of the solutions. For all cases, the proportion of backlog and lost sales cost change randomly when the costs are altered for only half of the items, however, it is when the costs are changed for all items that the percentage of these costs considerably increase in the objective function, sometimes more than doubling, while the setup and inventory cost decrease.

Table 10. Sensitivity analysis: variations on backlog and lost sales costs

			Cost				**Structure of costs (%)**			
Backlog	**Lost sales**	**Models**	**variations**	**OF**	**Gap (%)**	**Time (sec.)**	**Setup**	**Inventory**	**Backlog**	**Lost sales**
Unrestricted	Unrestricted	*FLR-UB-UL*	Original [1]	124841.89	0.0950	1000.55	59.72	25.60	14.49	0.19
			Adjust 1	127563.31	0.1349	992.89	61.49	25.77	12.01	0.72
			Adjust 2	158871.31	0.3542	1192.84	57.50	20.50	19.03	2.97
Unrestricted	No	*FLR-UB*	Original [1]	124862.00	0.0996	1014.50	59.65	42.95	25.54	20.20
			Adjust 1	127686.65	0.1466	1006.82	61.39	43.30	25.65	12.18
			Adjust 2	159337.62	0.3646	1218.17	57.30	39.17	20.23	20.70
	Fixed	*FLR-UB-FL*	Original [1]	130819.70	0.2515	1242.36	62.59	45.14	27.32	16.91
			Adjust 1	132204.58	0.2735	1161.55	62.76	44.75	27.03	14.15
			Adjust 2	167331.35	0.4556	1257.42	56.44	38.24	19.37	15.07
	Variable	*FLR-UB-VL*	Original [1]	130792.93	0.2502	1257.99	62.53	45.02	27.48	16.84
			Adjust 1	132111.17	0.2671	1187.98	62.71	44.50	27.28	14.47
			Adjust 2	166803.09	0.4452	1229.98	56.65	38.58	19.52	14.16
Restricted ($\tau = 4$)	No	*FLR-RB*	Original [1]	124862.96	0.0893	958.60	59.70	42.94	25.49	20.15
			Adjust 1	127684.54	0.1235	947.72	61.36	43.29	25.67	12.18
			Adjust 2	159348.65	0.3064	1180.59	57.26	39.14	20.24	20.87
	Fixed	*FLR-RB-FL*	Original [1]	130814.25	0.2298	1144.08	62.65	44.97	27.24	16.92
			Adjust 1	132222.49	0.2677	1121.20	62.82	44.49	26.97	14.24
			Adjust 2	167319.22	0.4080	1260.27	56.56	38.39	19.38	15.10
	Variable	*FLR-RB-VL*	Original [1]	130789.58	0.2254	1171.40	62.60	45.01	27.44	16.89
			Adjust 1	132122.60	0.2451	1115.07	62.75	44.51	27.21	14.44
			Adjust 2	166773.01	0.3379	1222.48	56.54	38.45	19.66	14.35
Restricted ($\tau = 1$)	No	*FLR-RB*	Original [1]	125132.42	0.0662	816.56	60.11	25.14	14.74	0.00
			Adjust 1	128986.83	0.1117	789.03	62.11	24.37	13.52	0.00
			Adjust 2	160874.49	0.2868	1005.14	56.88	19.95	23.16	0.00
	Fixed	*FLR-RB-FL*	Original [1]	131225.58	0.1762	1018.85	62.97	26.73	4.00	6.30
			Adjust 1	132887.46	0.2163	1051.29	63.44	26.16	4.05	6.35
			Adjust 2	169110.22	0.3942	1152.57	56.18	19.09	9.58	15.15
	Variable	*FLR-RB-VL*	Original [1]	131083.39	0.1958	1117.60	62.92	26.98	3.41	6.69
			Adjust 1	132442.03	0.2378	1044.59	63.15	26.89	3.05	6.92
			Adjust 2	167304.52	0.3679	1186.24	56.30	19.65	6.82	17.22
No	Unrestricted	*FLR-UL*	Original [1]	135184.54	0.2745	1034.57	63.14	45.21	26.85	14.71
			Adjust 1	135985.45	0.2435	1040.13	63.25	44.72	26.42	14.27
			Adjust 2	177369.49	0.3350	1077.16	53.88	36.80	18.88	10.45
Multiple customers	No	*FLR-MB*	Original [1]	125546.25	0.1771	1134.76	60.21	43.67	26.20	19.82
			Adjust 1	129272.86	0.2757	1006.26	62.60	44.24	24.88	12.29
			Adjust 2	161061.87	0.3608	1182.12	56.77	38.72	20.03	20.90
	Fixed	*FLR-MB-FL*	Original [1]	131238.42	0.2078	1087.78	62.74	45.08	26.95	16.41
			Adjust 1	132881.11	0.2345	1067.49	63.30	44.78	26.35	13.75
			Adjust 2	169053.51	0.4366	1183.73	56.07	38.11	19.27	14.68
	Variable	*FLR-MB-VL*	Original [1]	130952.25	0.2188	1116.64	62.69	44.99	27.29	16.61
			Adjust 1	132268.83	0.2437	1138.42	63.11	44.66	26.97	14.30
			Adjust 2	167025.86	0.3905	1215.92	56.44	38.46	19.72	14.22

[1] Base case instances

6. Conclusion and Remarks

In conclusion, this study addresses the lot-sizing problem in the context of stock-outs by considering both backlog and lost sales as strategies to manage unsatisfied demand. By incorporating these realistic scenarios, this research extends the existing literature and provides valuable insights into the impact of incorporating simultaneously backlog and lost sales in lot-sizing problems.

We introduces four different assumptions for backlog: no backlog, unlimited backlog, restricted backlog, and multiple customer backlog. These assumptions capture various scenarios and enhance the understanding of customer behavior when faced with backlogged items. Particularly, the exploration of backlog with multiple customer types is a contribution that expands upon the traditional single customer type concept. Furthermore, we investigates four possibilities of lost sales: no lost sales, fixed-proportion lost sales, variable-proportion lost sales, and unrestricted proportion of lost sales. These variations offer flexibility for decision-making in cases of stock-outs and enable companies to make informed choices regarding backlogging or incurring additional lost sales.

To model and optimize these problems, this study adopt the facility location reformulation, which provides a structured framework for incorporating backlog and lost sales decisions. The reformulation enables the identification and tracking of stock-outs, as well as the proper consideration of penalty costs associated with backlogging and lost sales. A fast heuristic is developed to efficiently solve the problem, taking advantage of the mathematical formulation provided by the reformulation and the backlog assumptions. The heuristic offers a practical approach for decision-makers to quickly obtain feasible solutions and make informed tradeoffs between backlog and lost sales strategies.

Computational experiments are conducted to evaluate the proposed formulations and the performance of the heuristic. The analysis indicates that removing the restriction on backlog, for a certain assumption ($\tau = 4$), results in slight changes in the cost and structure of solutions. Notably, the heuristic demonstrates its effectiveness by finding feasible solutions that are extremely close to the best ones, while significantly reducing computational time by an average of 95%. This highlights the efficiency and practicality of the heuristic approach in solving such problems. Additionally, performing the sensitivity analyses, we could observe that the capacity parameter has an impact on the performance of the formulations. As the capacity decreases, both the computing time and the objective function value increase, indicating the challenges and complexities associated with managing inventory under limited capacity conditions. It emphasizes the need for careful capacity planning and allocation to optimize production and satisfy customer demand effectively. Furthermore, analysis reveals that the objective function value and the gap between the solutions improve as a larger percentage of customers are willing to wait in case of stock-outs. This finding underscores the importance of understanding customer behavior and preferences when formulating strategies for backlog and lost sales. By considering the willingness to wait, companies can make informed decisions that minimize lost sales and enhance customer satisfaction. Finally, there is a slight increase in the objective function value when backlog and lost sales costs are higher for specific items, but the significance in this analysis is observed when the costs are increased for all items. This indicates the sensitivity of the objective function to cost changes and emphasizes the impact of cost on the optimization of the problem.

Overall, this study contributes to the lot-sizing literature by simultaneously addressing backlog and lost sales strategies in the context of stock-outs. The proposed formulations, heuristic approach, and comprehensive analyses provide valuable tools and knowledge for practitioners and researchers alike. The insights gained from this research can aid decision-makers in making more informed choices regarding inventory management and customer satisfaction when faced with stock-outs. Future research can build upon these findings to further refine and expand the understanding of lot-sizing problems in real-world scenarios.

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