INFORMS

DYNAMIC VERSION OF THE ECONOMIC LOT SIZE MODEL*†

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A forward algorithm for a solution to the following dynamic version of the economic lot size model is given: allowing the possibility of demands for a single item, inventory holding charges, and setup costs to vary over N periods, we desire a minimum total cost inventory management scheme which satisfies known demand in every period. Disjoint planning horizons are shown to be possible which eliminate the necessity of having data for the full N periods.

1. Introduction

By now the "square root formula" [7] (equation 8 below) for an economic lot size under the assumption of a steady-state demand rate is well known. The calculation is predicated upon a balancing of the costs of holding inventory against the costs of placing an order. When the assumption of a steady-state demand rate is dropped—i.e., when the amounts demanded in each period are known but are different—and furthermore, when inventory costs vary from period to period, the square root formula (applied to the overall average demand and costs) no longer assures a minimum cost solution. We shall present a simple algorithm for solving the dynamic version of the model.[1]

The mathematical model may be viewed as a "one-way temporal feasibility" problem, in that it is feasible to order inventory in period t for demand in period $t + k$ but not vice versa. This suggests that the same model also permits an alternative interpretation as the following "one-way technological feasibility" problem [1]. Suppose a manufacturer produces an item having N possible values for a certain critical dimension; for example, the item may be steel beams of various strengths.[2] He anticipates a known demand schedule for the N types of steel beams, and it is feasible to substitute a beam of strength g_1 for a beam of strength g_2 if and only if $g_1 > g_2$. Producing each kind of a beam requires a setup cost, and using a beam in excess of the required strength incurs a charge in terms of wasted steel. The operator of the steel mill wishes to know how many beams of each type to produce in order to minimize total costs.

2. Mathematical Model

As in the standard lot size formulation, we assume that the buying (or manufacturing) costs and selling price of the item are constant throughout all time

periods, and consequently only the costs of inventory management are of concern. In the t-th period, $t = 1, 2, \cdots, N$, we let

d_t = amount demanded

i_t = interest charge per unit of inventory carried forward to period $t + 1$

s_t = ordering (or setup) cost

x_t = amount ordered (or manufactured).[3]

We assume that all period demands and costs are non-negative. The problem is to find a program $x_t \geqq 0$, $t = 1, 2, \cdots, N$, such that all demands are met at a minimum total cost; any such program, which need not be unique, will be termed *optimal*.

Of course one method of solving the optimization problem is to enumerate 2^{N-1} combinations of either ordering or not ordering in each period (we assume an order is placed in the first period).[4] A more efficient algorithm evolves from a dynamic programming characterization of an optimal policy [2, 3, 4].

Let I denote the inventory entering a period and I_0 initial inventory; for period t

$$I = I_0 + \sum_{j=1}^{t-1} x_j - \sum_{j=1}^{t-1} d_j \geqq 0. \tag{1}$$

We may write the functional equation [2, 4] representing the minimal cost policy for periods t through N, given incoming inventory I, as

$$f_t(I) = \min_{\substack{x_t \geqq 0 \\ I + x_t \geqq d_t}} [i_{t-1} I + \delta(x_t)s_t + f_{t+1}(I + x_t - d_t)] \tag{2}$$

where

$$\delta(x_t) = \begin{cases} 0 \text{ if } x_t = 0 \\ 1 \text{ if } x_t > 0. \end{cases} \tag{3}$$

In period N we have

$$f_N(I) = \min_{\substack{x_N \geqq 0 \\ I + x_N = d_N}} [i_{N-1} I + \delta(x_N)s_N]. \tag{4}$$

Consequently we compute f_t, starting at $t = N$, as a function of I; ultimately we derive f_1, thereby obtaining an optimal solution as I for period 1 is specified. Theorem 2 below establishes that it is permissible to confine consideration to only $N + 2 - t$, $t > 1$, values of I at period t.

By taking cognizance of the special properties of our model, we may formulate an alternative functional equation which has the advantage of potentially requiring less than N periods' data to obtain an optimal program; that is, it may

[3] We confine ourselves, as one does in the static model, to situations in which (nearly) constant lead or delivery time is a workable approximation to reality.

[4] Formally the model may also be posed as a fixed charge linear programming problem; see W. M. Hirsch and G. B. Dantzig, "The Fixed Charge Problem," RAND Corporation RM-1383, December 1954.

be possible without any loss of optimality to narrow our program commitment to a shorter "planning horizon" than N periods on the sole basis of data for this horizon. Just as one may prove that in a linear programming model it suffices to investigate only *basic* sets of variables in search of an optimal solution, we shall demonstrate that in our model an optimal solution exists among a very simple class of policies.

It is necessary to postulate that $d_1 \geqq 0$ is demand in period 1 *net* of starting inventory.[5] Then the fundamental proposition underlying our approach asserts that it is sufficient to consider programs in which at period t one does not *both* place an order and bring in inventory.

Theorem 1. There exists an optimal program such that $Ix_t = 0$ for all t (where I is inventory entering period t).

Proof: Suppose an optimal program suggests both to place an order in period t and to bring in I (i.e., $Ix_t > 0$). Then it is no more costly to reschedule the purchase of I by including the quantity in x_t, for this alteration does not incur any *additional* ordering cost and does save the cost $i_{t-1}I \geqq 0$.

Note that the theorem does not hold if our model includes buying or production costs which are not constant and identical for all periods. In the latter case, economies of scale might very well call for the carrying of inventory into period t even when an order or setup takes place in t [6].

Two corollaries follow from the theorem.

Theorem 2. There exists an optimal program such that for all t

$$x_t = 0 \quad \text{or} \quad \sum_{j=t}^{k} d_j \quad \text{for some} \quad k, t \leqq k \leqq N.$$

Proof: Since all demands must be met, any other value for x_t implies there exists a period $t^* \geqq t$ such that $Ix_{t^*} > 0$; but Theorem 1 assures that it is sufficient to consider programs in which such a condition does not arise.

The implication of Theorem 2 is that we can limit the values of I in (2) for period t to zero and the cumulative sums of demand for periods t up to N. If initial inventory is zero, then only $N(N + 1)/2$ different values of I in toto over the entire N periods need be examined.

Theorem 3. There exists an optimal program such that if d_{t^*} is satisfied by some $x_{t^{**}}$, $t^{**} < t^*$, then d_t, $t = t^{**} + 1, \cdots, t^* - 1$, is also satisfied by $x_{t^{**}}$.

Proof: In a program not satisfying the theorem, either I for period t^{**} is positive or I for period t^* is brought into some period t', $t^{**} < t' < t^*$, where $x_{t'} > 0$; but again by Theorem 1, it is sufficient to consider programs in which such conditions do not arise.

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[5] In other words, we let $I_0 = 0$ by netting out starting inventory from demand in period 1. If the level of starting inventory in fact exceeds the total demand in period 1, then the "forward" algorithm to be suggested may not be correct. In particular, Theorem 1 below may not hold for period 1; in such a case (2) still remains applicable. A sufficient condition for the existence of a forward solution is that s_t is monotonically non-increasing. An optimal solution is found then by using up initial inventory period by period until, at some t, the inventory remaining does not meet the demand; at this point, our suggested algorithm is commenced.

We next investigate a condition under which we may divide our problem into two smaller subproblems.

Theorem 4. Given that $I = 0$ for period t, it is optimal to consider periods 1 through $t - 1$ by themselves.

Proof: By hypothesis, (2) in period $t - 1$ for the N period model is

$$f_{t-1}(I) = \min_{\substack{x_{t-1} \geqq 0 \\ I + x_{t-1} = d_{t-1}}} [i_{t-2}I + \delta(x_{t-1})s_{t-1} + f_t(0)], \qquad (5)$$

and for the $t - 1$ period model is correspondingly

$$g_{t-1}(I) = \min_{\substack{x_{t-1} \geqq 0 \\ I + x_{t-1} = d_{t-1}}} [i_{t-2}I + \delta(x_{t-1})s_{t-1}]. \qquad (6)$$

But the functional relations (5) and (6) differ only by a constant $f_t(0)$. Consequently what is optimal for (6) is optimal for (5), and by the recursive structure of the model, the latter conclusion continues to hold for all the earlier periods.

We may now offer an alternative formulation to (2). Let $F(t)$ denote the minimal cost program for periods 1 through t. Then

$$F(t) = \min \left[\begin{array}{c} \min_{1 \leqq j < t} \left[s_j + \sum_{h=j}^{t-1} \sum_{k=h+1}^{t} i_h d_k + F(j - 1) \right] \\ s_t + F(t - 1) \end{array} \right] \qquad (7)$$

where $F(1) = s_1$ and $F(0) = 0$. That is, the minimum cost for the first t periods comprises a setup cost in period j, plus charges for filling demand d_k, $k = j + 1, \cdots, t$, by carrying inventory from period j, plus the cost of adopting an optimal policy in periods 1 through $j - 1$ taken by themselves. Theorems 2, 3, and 4 guarantee that at period t we shall find an optimum program of this type. With the present formulation, (7) is computed, starting at $t = 1$. At any period t, (7) implies that only t policies need to be considered. The minimum in (7) need not be unique, so that there may be alternative optimal solutions. When we derive $F(N)$, we shall have solved the problem for N is the last period to be considered.

Finally we come to what is perhaps the most interesting property of our model.

The Planning Horizon Theorem.[6] If at period t^* the minimum in (7) occurs for $j = t^{**} \leqq t^*$, then in periods $t > t^*$ it is sufficient to consider only $t^{**} \leqq j \leqq t$. In particular, if $t^* = t^{**}$, then it is sufficient to consider programs such that $x_{t^*} > 0$.

Proof: Without loss of optimality we restrict our attention to programs of the form specified in Theorems 1–4. Suppose a program suggests that d_t is satisfied by $x_{t^{***}}$, where $t^{***} < t^{**} \leqq t^* < t$. Then by Theorem 3 d_{t^*} is also satisfied by $x_{t^{***}}$. But by hypothesis we know that costs are not increased by rescheduling the program to let d_{t^*} be satisfied by $x_{t^{**}} > 0$.

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[6] The reader may wish to prove the corresponding theorem for (2): Let I^{**} be the value of incoming inventory associated with $\min_I f_{t^*}(I)$; then in period $t < t^*$ it is sufficient to consider only $0 \leqq I \leqq I^{**} + \sum_{j=t}^{t^*-1} d_j$. In particular, if $I^{**} = 0$, then it is sufficient to consider programs such that $I = 0$ at period t^*.

The planning horizon theorem states in part that if it is optimal to incur a setup cost in period t^* when periods 1 through t^* are considered by themselves, then we may let $x_{t^*} > 0$ in the N period model without foregoing optimality. By Theorems 1 and 4 it follows further that we may adopt an optimal program for periods 1 through $t^* - 1$ considered separately.

3. The Algorithm

The algorithm at period t^*, $t^* = 1, 2, \cdots, N$, may be generally stated as

1. Consider the policies of ordering at period t^{**}, $t^{**} = 1, 2, \cdots, t^*$, and filling demands d_t, $t = t^{**}, t^{**} + 1, \cdots, t^*$, by this order.

2. Determine the total cost of these t^* different policies by adding the ordering and holding costs associated with placing an order at period t^{**}, and the cost of acting optimally for periods 1 through $t^{**} - 1$ considered by themselves. The latter cost has been determined previously in the computations for periods $t = 1, 2, \cdots, t^* - 1$.

3. From these t^* alternatives, select the minimum cost policy for periods 1 through t^* considered independently.

4. Proceed to period $t^* + 1$ (or stop if $t^* = N$).

Table 1 portrays the symbolic scheme for the algorithm. The notation $(1, 2, \cdots, t^{**}) \underline{t^{**} + 1, t^{**} + 2, \cdots, t^*}$ in Table 1 indicates that an order is placed in period $t^{**} + 1$ to cover the demands of d_t, $t = t^{**} + 1, t^{**} + 2, \cdots, t^*$, and the optimal policy is adopted for periods 1 through t^{**} considered separately. At the bottom of the table we record the minimum cost plan for periods 1 through t^*.

In general, it may be necessary to test N policies at the N-th period, implying a table of $N(N + 1)/2$ entries (versus 2^{N-1} for all possibilities). Thus the forward algorithm (7) is at least as efficient as (2). As we shall see, the number of entries usually is much smaller than this number if we make full use of the planning horizon theorem.

TABLE 1

Month t	1	2	3	4	$\cdots$	N
Ordering cost	s_1	s_2	s_3	s_4	$\cdots$	s_N
Demand	d_1	d_2	d_3	d_4	$\cdots$	d_N
$(1, 2, \cdots, t-1)\underline{t}$	1	$(1)\underline{2}$	$(1, 2)\underline{3}$	$(1, 2, 3)\underline{4}$	$\cdots$	$(1, 2, \cdots, N-1)\underline{N}$
$(1, 2, \cdots, t-2)$		$\underline{12}$	$(1)\underline{23}$	$(1, 2)\underline{34}$	$\cdots$	$(1, 2, \cdots, N-2)$
$\underline{t-1, t}$						$\underline{N-1, N}$
$(1, 2, \cdots, t-3)$			$\underline{123}$	$(1)\underline{234}$	$\cdots$	$(1, 2, \cdots, N-3)$
$\underline{t-2, t-1, t}$						$\underline{N-2, N-1, N}$
$(1, 2, \cdots, t-4)$				$\underline{1243}$	$\cdots$	$(1, 2, \cdots, N-4)$
$\underline{t-3, \cdots, t}$						$\underline{N-3, N-2, N-1, N}$
$\vdots$						$\vdots$
Minimum cost						
Optimal policy						
$(1, 2, \cdots, t)$	(1)	$(1, 2)$	$(1, 2, 3)$	$(1, 2, 3, 4)$		$(1, 2, \cdots, N)$

4. An Example

Table 2 presents a sample set of data for a 12 month period; to simplify computations we have let $i_t = 1$ for all t; Table 3 contains the specific calculations.

To illustrate, the optimal plan for period 1 alone is to order (entailing an ordering cost of 85), Table 3. Two possibilities must be evaluated for period 2: order in period 2, and use the best policy for period 1 considered alone (at a cost of $102 + 85 = 187$); or order in period 1 for both periods, and carry inventory into period 2 (at a cost of $85 + 29 = 114$). The better policy is the latter one. In period 3 there are three alternatives: order in period 3, and use the best policy for periods 1 and 2 considered alone (at a cost of $102 + 114 = 216$); or order in period 2 for the latter two periods and use the best policy for period 1

TABLE 2

Month t	d_t	s_t	i_t
1	69	85	1
2	29	102	1
3	36	102	1
4	61	101	1
5	61	98	1
6	26	114	1
7	34	105	1
8	67	86	1
9	45	119	1
10	67	110	1
11	79	98	1
12	56	114	1
Average..................	52.5	102.8	1

TABLE 3

Month t	1	2	3	4	5	6	7	8	9	10	11	12
Ordering cost	85	102	102	101	98	114	105	86	119	110	98	114
Demand	69	29	36	61	61	26	34	67	45	67	79	56
	85	187	216	287	375	462	505	555	674	710	808	903
		114	223	277	348	401	496	572	600	741	789	864
			186			400	469			734		901
							502					
Minimum cost	85	114	186	277	348	400	469	555	600	710	789	864
Optimal policy*	1	12	123	34	45	456	567	8	89	10	10, 11	11, 12

* Only the last order period is shown; 567 indicates that the optimal policy for periods 1 through 7 is to order in period 5 to satisfy d_5, d_6, and d_7, and adopt an optimal policy for periods 1 through 4 considered separately.

considered alone (at a cost of $102 + 36 + 85 = 223$); or order in period 1 for the entire three periods (at a cost of $85 + 29 + 36 + 36 = 186$).

In our example, it is clear that it would never pay to carry inventory from periods 1 or 2 to meet d_4, since the carrying charges would exceed the ordering cost in period 4. A fortiori it would never pay to carry inventory from periods 1 or 2 to meet d_5, d_6, $\cdots$, d_N, because to do so would also imply that inventory was being carried to period 4 (Theorem 3).

Note that periods 1 through 8, and 8 through 10 comprise planning horizons. Whenever a time horizon (or a simplification of the type mentioned in the previous paragraph) arises, the entries in the table can be truncated below the southeast diagonal through the entry for $(1, 2, \cdots, t^* - 1)\underline{t^*}$, as we have done in Table 3.

For our set of data the optimal policy is

1. Order at period 11, $x_{11} = 79 + 56 = 135$, and use the optimal policy for periods 1 through 10, implying
2. Order at period 10, $x_{10} = 67$, and use the optimal policy for periods 1 through 9, implying
3. Order at period 8, $x_8 = 67 + 45 = 112$, and use the optimal policy for periods 1 through 7, implying
4. Order at period 5, $x_5 = 61 + 26 + 34 = 121$, and use the optimal policy for periods 1 through 4, implying
5. Order at period 3, $x_3 = 36 + 61 = 97$, and use the optimal policy for periods 1 through 2, implying
6. Order at period 1, $x_1 = 69 + 29 = 98$.

The total cost of the optimal policy is 864.

By use of the suggested tabular form, it is also relatively easy to make sensitivity analyses of the solution. For example, the ordering cost in period 2 would have to decrease by more than 73 in order to make it less costly to setup in period 2 than carry inventory from period 1; ordering cost in period 11 would have to increase by more than 37 in order to make it less costly to order in period 10 for the last three periods.

5. A Steady State Example

In the case of steady state demand and constant ordering and holding costs, our algorithm yields the same result as the standard "square root formula." Assume that throughout the entire year monthly demand $d = 52.5$, ordering (setup) cost $s = 102.80$, and interest charge $i = 1$. The square root formula for the order quantity gives

$$Q = \sqrt{2\,ds/i} = \sqrt{2 \times 52.5 \times 102.8/1} = 104. \tag{8}$$

Since this is approximately two months demand, we round Q to 105 units for comparison purposes.

Applying our algorithm yields that for the first two and three periods, the optimal policies are $\underline{12}$ and $(1, 2)\underline{3}$, indicating that the first two periods comprise

a planning horizon. In the steady state case, all planning horizons are the same, i.e., orders will be placed every two months. Therefore annual costs are easily obtained as six times the costs for one planning horizon, amounting to 931.80.

Annual total variable costs may be calculated with the standard lot size model as[7]

$$12[(Q - d)i/2 + ds/Q] = 12[(105 - 52.5)1/2 + 52.5(102.80)/105]$$
$$= 931.80.$$

Thus the two models are equally as costly. If the square root formula had not resulted in the ordering of an integral number of months' supply, the costs under the two methods would have been different due to the discrete division of time in our model. However this difference vanishes once the length of our time period is reduced.

References

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[7] To effect comparability with our model, the inventory carrying charge term has been reduced from $Qi/2$ to $(Q - d)i/2$. This change is required because our model, for the sake of simplicity, applies carrying charges to ending inventory only. As the reader may convince himself, adding carrying charges to that inventory used within a period would change total costs by a fixed amount and would not affect the solution of an optimal policy.