Monads and other abstractions

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4 Jul 2014

Mathematics

Meaning

There isn’t any.

Abstraction

Structures, and relationships between structures.

Sets

“Stuff”

A set is a collection of elements.

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A set is a collection of elements.

```
type Set a = a -> Bool
```

Extensional

Can be defined by stating its elements.

$$\{ \mathit{True}, \mathit{False} \}$$

Intensional

Or by describing them.

$$\{\, x \mid x \in \mathbb{N}, \mathit{even}(x) \,\}$$

Distinction

Values can be extensionally equal, but intensionally distinct.

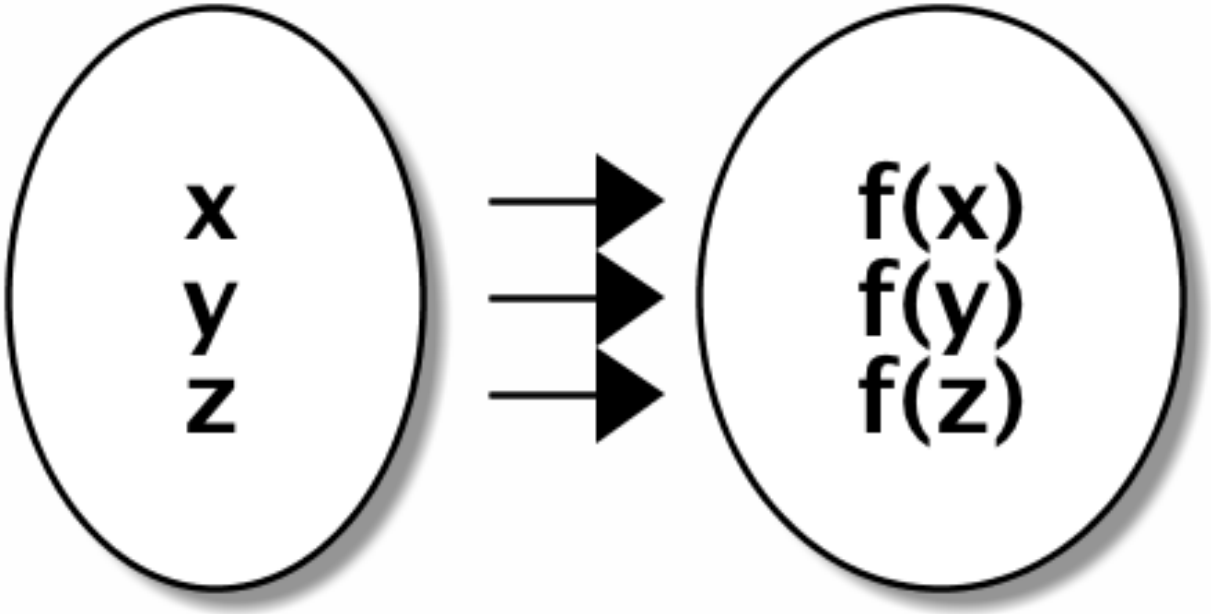
$n \mapsto 2(n + 5)$
$n \mapsto 2n + 10$

Deceptively simple

With just a basic definition, and seven axioms (we've already seen two!), you can generate a good deal of mathematics.

Functions

As maps



Higher-order functions

$$id\ x = x$$

$$(f \circ g)\ x = f(g(x))$$

Properties of functions

$$f : cod \rightarrow dom$$

Properties of functions

$$f : cod \to dom$$

Definition (Idempotent)

$f \circ f = f$

Properties of functions

$$f : cod \to dom$$

Definition (Idempotent)

$f \circ f = f$

Definition (Involutive)

$f \circ f = id$

Homomorphism

“Structure preserving.”

Isomorphism

An isomorphism is a pair of functions satisfying two equations:

$$f \circ g = id_{dom(f)}$$

$$g \circ f = id_{dom(g)}$$

Isomorphism

In terms of the types involved:

$$A \cong B$$

$$g : A \to B$$

$$f : B \to A$$

Laws

Imposing structure

In the absence of meaning, laws create structure.

Principled restriction

Laws restrict how functions and values relate to each other.

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```
class Monoid a where
    mempty  :: a
    mappend :: a -> a -> a
```

Algebras

Algebraic Structures

Magmas

Semigroups

Monoids

Groups

Type Algebras

Equational Reasoning

Quantification

Existential

$$\exists p, P(p)$$

Universal

$$\forall p, P(p)$$

Universal

True?

$\forall\ x, \exists\ y \rightarrow x = y$

Universal

True?

$\forall\, x, \exists\, y \rightarrow x \neq y$

Parametricity

Curry-Howard Isomorphism

Free objects

Category Theory

Functors

Applicatives

Monads

Free Monads