Exercise 4.7

Given a solution $\hat{x}$ to the linear program relaxation we use the following simple rounding procedure.

$$\overline{x_{i0}} = 1 \ \& \ \overline{x_{i1}} = 0 \iff \widehat{x_{i0}} > \frac{1}{2}$$

There are two cases:

If $\overline{x_{i0}} = 1$ and $\overline{x_{i1}} = 0$ we have $\widehat{x_{i0}} > \frac{1}{2}$ and hence $\overline{x_{i0}} < 2\widehat{x_{i0}}$. Also we trivially have $\overline{x_{i1}} \leq 2\widehat{x_{i1}}$.

If $\overline{x_{i0}} = 0$ and $\overline{x_{i1}} = 1$ then $\widehat{x_{i0}} \leq 1/2$ and hence $\widehat{x_{i1}} \geq 1/2$ and hence $\overline{x_{i1}} \leq 2\widehat{x_{i1}}$ and $\overline{x_{i0}} \leq 2\widehat{x_{i0}}$ trivially holds.

In any case we have $\overline{x_{ij}} \leq 2\widehat{x_{ij}}$.
We then have for any boundary b

$$\begin{aligned}\sum_{i \in T_{b0}} \overline{x_{i0}} + \sum_{i \in T_{b1}} \overline{x_{i1}} &\leq \sum_{i \in T_{b0}} 2\widehat{x_{i0}} + \sum_{i \in T_{b1}} 2\widehat{x_{i1}} \\ &\leq 2\left(\sum_{i \in T_{b0}} \widehat{x_{i0}} + \sum_{i \in T_{b1}} \widehat{x_{i1}}\right) \leq 2\widehat{w} \leq w_o\end{aligned}$$

by definition of w_S this yields $w_S \leq w_o$ as desired.

Problem 4.13

The $0-1$ linear program solving this problem is the following:

- Minimize $||c||_1$
- $c_i \in \{0, 1\}$ for $0 \leq i \leq m = \#U$
- $Mc \geq 1$

We will consider the linear program relaxation where $c_i \in [0, 1]$, which can be solved in time polynomial in the size of M.

Now given a solution to the linear program relaxation, $\widehat{c}$, we sample a $0-1$ vector c such that $P(c_i = 1) = \min\{1, 8\widehat{c_i}\log n\}$.

For $1 \leq j \leq n$ let M_j denote the jth row of M, and for $1 \leq i \leq m$ define

$$X_i = \min\{M_{ji}, c_i\}$$

Then the X_i are poison trials and $M_j \cdot c = \sum_{i=1}^m X_i$. Furthermore

$$E(M_j \cdot c) = \sum_{i=1}^{m} EX_i = 8 \log n M_j \cdot \widehat{c} \geq 8 \log n$$

since the fact that $\widehat{c}$ is a solution to the linear program relaxation yields $M_j \cdot \widehat{c} \geq 1$.

Now the generalized Chernoff bound yields

$$P\left(\sum_{i=1}^{m} X_i < (1-\delta) 8 \log n\right) < \exp\left(-8 \log n \frac{\delta^2}{2}\right) = \frac{1}{n^{4\delta^2}}$$

for $0 < \delta \leq 1$.
Now choosing δ sufficiently close to 1 we can ensure that $(1-\delta)\, 8 \log n \leq 1$ and that $\delta^2 \geq \frac{1}{2}$, and then the above yields

$$P\left(\sum_{i=1}^{m} X_i = 0\right) = P\left(\sum_{i=1}^{m} X_i < (1-\delta) 8 \log n\right) < \frac{1}{n^{4\delta^2}} \leq \frac{1}{n^2}$$

By the above observations we have proven that $P(M_j \cdot c = 0) \leq \frac{1}{n^2}$.
Hence we can now bound the probability that the vector c obtained by the randomized rounding procedure described above is not a set-cover by

$$\begin{aligned}
&P\left(c \text{ is not a set-cover}\right) \\
=&P\left(\cup_{j=1}^{n} \left(M_j \cdot c = 0\right)\right) \\
\leq&\sum_{j=1}^{n} P\left(M_j \cdot c = 0\right) \\
\leq&\sum_{j=1}^{n} \frac{1}{n^2} = \frac{1}{n}
\end{aligned}$$

To check whether c is in fact a set-cover we just have to compute the matrix product Mc and check whether any entry is 0, which is certainly polynomial in the size of M.

Thus we have deviced a Monte Carlo algorithm with expected polynomial running time such that a given output can be verified in polynomial time and the probability of a correct output is at least $1 - \frac{1}{n}$.
Then by exercise 1.3 this can be used to construct a Las Vegas algorithm whose expected running time is certainly polynomial.

To compute bounds on the expected size of the set-cover observe that since $\widehat{c}$ is a solution to the linear program relaxation we certainly have $||\widehat{c}||_1 \leq C(M)$ and hence

$$E||c||_1 \leq \sum_{i=1}^{m} \widehat{c}_i 8 \log n = ||\widehat{c}||_1 8 \log n \leq C(M) 8 \log n$$

as desired.