Randomized Algorithms assignment 6

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Proof of corollary 5.12

By assumption we have that any vertex of any dependency graph can have degree at most d.
Furthermore the assumption that $ep(d+1) \leq 1$ implies

$$p(d+1) \leq \frac{1}{e} \leq \left(1 - \frac{1}{d-1}\right)^d$$

which implies

$$p \leq \frac{1}{d+1}\left(1 - \frac{1}{d-1}\right)^d \leq \frac{1}{d-1}\left(1 - \frac{1}{d-1}\right)^d$$

Hence for $d > 2$ we let $x_i = \frac{1}{d-1}$ for $i = 1, \cdots, n$ and then we obtain for any $i = 1, \cdots, n$ that

$$x_i \prod_{(i,j)\in E} (1 - x_i) \geq \frac{1}{d-1}\left(1 - \frac{1}{d-1}\right)^d \geq p = P(\mathcal{E}_i)$$

so by the Lovasz local lemma we obtain that

$$P\left(\bigcap_{i=1}^{n} \overline{\mathcal{E}_i}\right) \geq \prod_{i=1}^{n} (1 - x_i) = \left(1 - \frac{1}{d-1}\right)^n > 0$$

as desired.

If $d = 1$ we have

$$\left(1 - \frac{1}{d+1}\right)^d = \frac{1}{2} \geq \frac{1}{e}$$

and if $d = 2$ we have

$$\left(1 - \frac{1}{d+1}\right)^d = \left(\frac{2}{3}\right)^2 = \frac{4}{9} \geq \frac{1}{e}$$

so in either case we can apply the same argument with $x_i = \frac{1}{d+1}$.

Problem 5.13

Let $v_1, \cdots, v_n$ denote the vertices of the graph.
As in section 5.6 of Motwani & Raghavan we consider the decision tree of the experiment.
Each node is labelled by two sets $[A, B]$ corresponding to the assignments made so far, i.e. the root r is labelled $[\emptyset, \emptyset]$ and if a is any node at level i labelled by $[A, B]$ we have $\#(A \cup B) = i$, and if c and d are its left and right child respectively, they will be labelled $[A \cup \{v_{i+1}\}, B]$ and $[A, B \cup \{v_{i+1}\}]$ respectively.