Exercise 4.3

If two routes separate that means that at a common kth bit they chose different ways. This again means that they must have different goals, differing on the kth bit. Since bit correction is going from left to right, these routes will never rejoin, since they will differ on this kth bit until their goals are reached.

Exercise 4.7

Given a solution $\hat{x}$ to the linear program relaxation we use the following simple rounding procedure.

$$\overline{x_{i0}} = 1 \ \& \ \overline{x_{i1}} = 0 \quad \iff \quad \widehat{x_{i0}} > \frac{1}{2}$$

There are two cases:

If $\overline{x_{i0}} = 1$ and $\overline{x_{i1}} = 0$ we have $\widehat{x_{i0}} > \frac{1}{2}$ and hence $\overline{x_{i0}} < 2\widehat{x_{i0}}$. Also we trivially have $\overline{x_{i1}} \leq 2\widehat{x_{i1}}$.

If $\overline{x_{i0}} = 0$ and $\overline{x_{i1}} = 1$ then $\widehat{x_{i0}} \leq 1/2$ and hence $\widehat{x_{i1}} \geq 1/2$ and hence $\overline{x_{i1}} \leq 2\widehat{x_{i1}}$ and $\overline{x_{i0}} \leq 2\widehat{x_{i0}}$ trivially holds.

In any case we have $\overline{x_{ij}} \leq 2\widehat{x_{ij}}$.
We then have for any boundary b

$$\begin{aligned}\sum_{i \in T_{b0}} \overline{x_{i0}} + \sum_{i \in T_{b1}} \overline{x_{i1}} &\leq \sum_{i \in T_{b0}} 2\widehat{x_{i0}} + \sum_{i \in T_{b1}} 2\widehat{x_{i1}} \\ &\leq 2\left(\sum_{i \in T_{b0}} \widehat{x_{i0}} + \sum_{i \in T_{b1}} \widehat{x_{i1}}\right) \leq 2\widehat{w} \leq w_o\end{aligned}$$

by definition of w_S this yields $w_S \leq w_o$ as desired.

Problem 4.13

The $0-1$ linear program solving this problem is the following:

- Minimize $||c||_1$
- $c_i \in \{0, 1\}$ for $0 \leq i \leq m = \#U$

- $Mc \geq 1$

We will consider the linear program relaxation where $c_i \in [0,1]$, which can be solved in time polynomial in the size of M.

Now given a solution to the linear program relaxation, $\widehat{c}$, we sample a $0-1$ vector c such that $P(c_i = 1) = \min\{1, 8\widehat{c}_i \log n\}$.

For $1 \leq j \leq n$ let M_j denote the jth row of M, and for $1 \leq i \leq m$ define

$$X_i = \min\{M_{ji}, c_i\}$$

Then the X_i are poison trials and $M_j \cdot c = \sum_{i=1}^m X_i$. Furthermore

$$E(M_j \cdot c) = \sum_{i=1}^m EX_i = 8 \log n M_j \cdot \widehat{c} \geq 8 \log n$$

since the fact that $\widehat{c}$ is a solution to the linear program relaxation yields $M_j \cdot \widehat{c} \geq 1$.

Now the generalized Chernoff bound yields

$$P\left(\sum_{i=1}^m X_i < (1-\delta) 8 \log n\right) < \exp\left(-8 \log n \frac{\delta^2}{2}\right) = \frac{1}{n^{4\delta^2}}$$

for $0 < \delta \leq 1$.
Now choosing δ sufficiently close to 1 we can ensure that $(1-\delta)\, 8 \log n \leq 1$ and that $\delta^2 \geq \frac{1}{2}$, and then the above yields

$$P\left(\sum_{i=1}^m X_i = 0\right) = P\left(\sum_{i=1}^m X_i < (1-\delta) 8 \log n\right) < \frac{1}{n^{4\delta^2}} \leq \frac{1}{n^2}$$

By the above observations we have proven that $P(M_j \cdot c = 0) \leq \frac{1}{n^2}$.
Hence we can now bound the probability that the vector c obtained by the randomized rounding procedure described above is not a set-cover by

$$\begin{aligned}
&P(c \text{ is not a set-cover}) \\
=&P\left(\cup_{j=1}^n (M_j \cdot c = 0)\right) \\
\leq&\sum_{j=1}^n P(M_j \cdot c = 0) \\
\leq&\sum_{j=1}^n \frac{1}{n^2} = \frac{1}{n}
\end{aligned}$$

To check whether c is in fact a set-cover we just have to compute the matrix product Mc and check whether any entry is 0, which is certainly polynomial in the size of M.

Thus we have deviced a Monte Carlo algorithm with expected polynomial running time such that a given output can be verified in polynomial time and the probability of a correct output is at least $1 - \frac{1}{n}$.
Then by exercise 1.3 this can be used to construct a Las Vegas algorithm whose expected running time is certainly polynomial.

To compute bounds on the expected size of the set-cover observe that since $\widehat{c}$ is a solution to the linear program relaxation we certainly have $||\widehat{c}||_1 \leq C(M)$ and hence

$$E||c||_1 \leq \sum_{i=1}^{m} \widehat{c}_i 8 \log n = ||\widehat{c}||_1 8 \log n \leq C(M) 8 \log n$$

as desired.