A Theorical Analysis of Fitted Q-Iteration

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1 Abstract

2 Foreword

The main purpose of this master thesis for me, has been to uncover what (at present) it is possible to say (mathematically) about the convergence of Q-learning algorithms. In particular Q-learning algotihms using (deep) ANNs.

I came to realize during my reading of [TODO ref to YangXieWang] that it is quite error-prone with some errors not obviously fixable.

3 Disambiguation

- $[\phi] = 1$ when ϕ is true/holds and 0 otherwise, for a logical formula ϕ.
- $[q] = \{1, \ldots, q\}$ for $q \in \mathbb{N}$.
- $C_{\mathbb{K}}(X) = \{f : X \to \mathbb{K} \mid f \text{ continuous}\}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. $C(X) = C_{\mathbb{R}}(X)$
- ANN abrv. artificial neural network see definition 2.
- δ_a Dirac-measure of point a. I.e. $\delta_a(A) = [a \in A]$.
- $(\Omega, \mathcal{F}, \mathbb{P})$ the underlying measure space of all random variables and processes when not otherwise specified.
- $\mathbb{B}_n$ the n-dimensional Borel σ-algebra.

3.1 Notational deviations from [TODO ref YangXieWang]

Because σ is used ambigously in theorem 1 we denote the probability distribution 'σ' from [YangXieWang, thm. 6.2, p. 20] by ν instead.

I dislike the shorthand defined in [YangXieWang, p. 26 bottom]: $\|f\|_n^2 = 1/n \cdot \sum_{i=1}^n f(X_i)^2$. This is partially due to inconsistencies and abuse of this notation employed. For example $\|f\|_n$ is used as $1/n \sum_{i=1}^n f(X_i)$ as opposed another likely interpretation $\sqrt{\|f\|_n^2}$, whereas $\|f\|_n^{-1}$ is used to mean $1/(\|f\|_n)$. This is avoided by using finite dimensional p-norms instead. The conversion to my notation thus becomes $\|f\|_n \rightsquigarrow \|f\|_1/n$, $\|f\|_n^2 \rightsquigarrow \|f\|^2/n$, $\|f\|_n^{-1} \rightsquigarrow n\|f\|_1^{-1}$.

4 Introduction

4.1 Reinforcement Learning

In Reinforcement Learning (RL) we are concerned with finding an optimal policy for an agent in some environment. Typically (also in the case of Q-learning) this environment is a Markov decision process

Definition 1. A Markov decision process (MDP) $(\mathcal{S}, \mathcal{A}, P, R, \gamma)$ consists of

- $\mathcal{S}$ a set of states
- $\mathcal{A}$ a set of actions
- $P : \mathcal{S} \times \mathcal{A} \to \mathcal{P}(\mathcal{S})$ its Markov transition kernel
- $R : \mathcal{S} \times \mathcal{A} \to \mathcal{P}(\mathbb{R})$ its immediate reward distribution
- $\gamma \in (0, 1)$ the discount factor

A policy (for an MDP) is a function

$$\pi : \mathcal{S} \to \mathcal{P}(\mathcal{A})$$

With this we can define the state-value function $V^\pi : \mathcal{S} \to \mathbb{R}$

$$V^\pi(s) = \mathbb{E}\left(\sum_{t \geq 0} \gamma^t R_t \mid R_t \sim R(S_t, A_t), S_t \sim P(S_{t-1}, A_{t-1}), A_t \sim \pi(S_t), S_0 = s\right)$$

And the state-action-value (Q-) function $Q^\pi : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$

$$Q^\pi(s, a) = \mathbb{E}(R(s, a) + \gamma V^\pi(S_0) \mid S_0 \sim P(s, a))$$

The optimal Q-function is defined as

$$Q^*(s, a) = \sup_\pi Q^\pi(s, a)$$

One can show that there is a policy π^* such that $Q^* = Q^{\pi^*}$. This is the optimal policy - the goal of RL.

Note that V^π, Q^π and Q^* are usually infeasible to calculate to machine precision, unless $\mathcal{S} \times \mathcal{A}$ is finite and not very big.

4.2 Q-Learning

Let $\pi : \mathcal{S} \to \mathcal{P}(\mathcal{A})$ be a policy. We define the operator

$$(P^\pi Q)(s, a) = \mathbb{E}(Q(S', A') \mid S' \sim P(s, a), A' \sim \pi(S'))$$

Intuitively this operator yields the expected state-action-value function when looking *one step ahead* following the policy π and taking expectation of Q.

We define the operator T^π called the Bellman operator by

$$(T^\pi Q)(s, a) = \mathbb{E}R(s, a) + \gamma(P^\pi Q)(s, a)$$

This operator adjust the Q function to look more like Q^π making one "iteration" of "propagation of rewards" discounting with γ. Indeed it is easily seen that Q^π is a fixed point for T^π.

A *greedy* policy π with respect to a state-action value function Q is a policy which deterministically chooses an action with maximal value of Q for each state. That is $\pi(s) = \delta_a$ for some $a \in \operatorname{argmax}_a Q(s, a)$. We then write $\pi = \pi_Q$. With this we can define the operator T:

$$TQ = T^{\pi_Q} Q$$

called the Bellman *optimality* operator.

The Bellman optimality *equation* can then be written $Q^* = TQ^*$.

Proposition 1. Q^π is the unique fixed point of T^π.

Proof. Clearly $T^\pi Q^\pi = Q^\pi$. [TODO: rest of this proof] □

4.3 Artificial Neural Networks

Definition 2. An **ANN** (Artificial Neural Network) with structure $\{d_i\}_{i=0}^{L+1} \subseteq \mathbb{N}$, activation functions $\sigma_i = (\sigma_{ij} : \mathbb{R} \to \mathbb{R})_{j=1}^{d_i}$ and weights $\{W_i \in M^{d_i \times d_{i-1}}, v_i \in \mathbb{R}^{d_i}\}_{i=1}^{L+1}$ is the function $F : \mathbb{R}^{d_0} \to \mathbb{R}^{d_{L+1}}$

$$F = w_{L+1} \circ \sigma_L \circ w_L \circ \sigma_{L-1} \circ \cdots \circ w_1$$

where w_i is the affine function $x \mapsto W_i x + v_i$ for all i.

Here $\sigma_i(x_1, \ldots, x_{d_i}) = (\sigma_{i1}(x_1), \ldots, \sigma_{id_i}(x_{d_i}))$.

$L \in \mathbb{N}_0$ is called the number of hidden layers.

d_i is the number of neurons or nodes in layer i.

An ANN is called *deep* if there are two or more hidden layers.

4.4 Fitted Q-Iteration

We here present the algorithm which everything in this paper revolves around:

Algorithm 1: Fitted Q-Iteration Algorithm

Input: MDP $(\mathcal{S}, \mathcal{A}, P, R, \gamma)$, function class $\mathcal{F}$, sampling distribution ν, number of iterations K, number of samples n, initial estimator $\widetilde{Q}_0$

for $k = 0, 1, 2, \ldots, K-1$ **do**

Sample i.i.d. observations $\{(S_i, A_i), i \in [n]\}$ from ν obtain $R_i \sim R(S_i, A_i)$ and $S_i' \sim P(S_i, A_i)$

Let $Y_i = R_i + \gamma \cdot \max_{a \in \mathcal{A}} \widetilde{Q}_k(S_i', a)$

Update action-value function:

$$\widetilde{Q}_{k+1} \leftarrow \underset{f \in \mathcal{F}}{\operatorname{argmin}} \frac{1}{n} \sum_{i=1}^{n} (Y_i - f(S_i, A_i))^2$$

Define π_K as the greedy policy w.r.t. $\widetilde{Q}_K$

Output: An estimator $\widetilde{Q}_K$ of Q^* and policy π_K

5 Measure Theory

We are mostly concerned with a random process

$$(Z_i)_{i=1}^K = (S_i, A_i, R_i)_{i=1}^K \in (\mathcal{S} \times \mathcal{A} \times (0, R_{\max}))^K \tag{1}$$

where $\mathcal{S} \subseteq \mathbb{R}^d$ is compact and $\mathcal{A}$ is finite, so we can model this as a discrete (and finite) time random process in a compact subset of $\mathbb{R}^{d+1}$ having the Markov property, namely that

$$\mathbb{P}(Z_j \in A \mid Z_{j-1}, \ldots, Z_1) = \mathbb{P}(Z_j \in A \mid Z_{j-1}) \tag{2}$$

These random variables live on some background probability space, denote this $(\Omega, \mathcal{H}, \mathbb{P})$.

6 Assumptions

6.1 Assumption 1: Holder Smoothness

Definition 3. For $s, V \in \mathbb{R}$ a (s,V)-**Sparse ReLU Network** is an ANN f with any structure $\{d_i\}_{i \in [L+1]}$, all activation functions being *ReLU* i.e. $\sigma_{ij} = \max(\cdot, 0)$ and any weights (W_ℓ, v_ℓ) satisfying

- $\max_{\ell \in [L+1]} \left\| \widetilde{W}_\ell \right\|_\infty \leq 1$
- $\sum_{\ell=1}^{L+1} \left\| \widetilde{W}_\ell \right\|_0 \leq s$
- $\max_{j \in [d_{L+1}]} \left\| f_j \right\|_\infty \leq V$

Here $\widetilde{W}_\ell = (W_\ell, v_\ell)$.

The set of them we denote $\mathcal{F}(s, V)$.

Definition 4. Let $\mathcal{D} \subseteq \mathbb{R}^r$ be compact and $\beta, H > 0$. A function $f : \mathcal{D} \to \mathbb{R}$ we call Holder smooth if

$$\sum_{\alpha:|\alpha|<\beta} \|\partial^\alpha f\|_\infty + \sum_{\alpha:\|\alpha\|_1=\lfloor\beta\rfloor} \sup_{x\neq y} \frac{|\partial^\alpha(f(x)-f(y))|}{\|x-y\|_\infty^{\beta-\lfloor\beta\rfloor}} \leq H$$

Where $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{N}^r$. We write $f \in C_r(\mathcal{D}, \beta, H)$.

Definition 5. Let $t_j, p_j \in \mathbb{N}$, $t_j \leq p_j$ and $H_j, \beta_j > 0$ for $j \in [q]$. We say that f is a *Composition of Holder smooth Functions* when

$$f = g_q \circ \cdots \circ g_1$$

for some functions $g_j : [a_j, b_j]^{p_j} \to [a_{j+1}, b_{j+1}]^{p_{j+1}}$ that only depend on t_j of their inputs for each of their components g_{jk}, and satisfies $g_{jk} \in C_{t_j}([a_j, b_j]_j^t, \beta_j, H_j)$, i.e. they are Holder smooth. We denote the class of these functions

$$\mathcal{G}(\{p_j, t_j, \beta_j, H_j\}_{j\in[q]})$$

Definition 6. Define

$$\mathcal{F}_0 = \{f : \mathcal{S} \times \mathcal{A} \to \mathbb{R} \mid f(\cdot, a) \in \mathcal{F}(s, V)\ \forall a \in \mathcal{A}\}$$

and

$$\mathcal{G}_0 = \{f : \mathcal{S} \times \mathcal{A} \to \mathbb{R} \mid f(\cdot, a) = \mathcal{G}(\{p_j, t_j, \beta_t, H_j\}_{j\in[q]})\ \forall a \in \mathcal{A}\}$$

Assumption 1. It is assumed that $Tf \in \mathcal{G}_0$ for any $f \in \mathcal{F}_0$.

I.e. when using the Bellman optimality operator on our sparse ReLU networks, we should stay in the class of compositions of Holder smooth functions.

6.2 Assumption 2: Concentration Coefficients

Definition 7 (Concentration coefficients)**.** Let $\nu_1, \nu_2 \in \mathcal{P}(\mathcal{S} \times \mathcal{A})$ be probability measures, absolutely continuous w.r.t. m_λ Define

$$\kappa(m, \nu_1, \nu_2) = \sup_{\pi_1,\dots,\pi_m} \left[\mathbb{E}_{v_2}\left(\frac{\mathrm{d}(P^{\pi_m}\dots P^{\pi_1}\nu_1)}{\mathrm{d}\nu_2}\right)^2\right]^{1/2}$$

Assumption 2. Let ν be the sampling distribution from the algorithm, and μ the distribution over which we measure the error in the main theorem, then we assume

$$(1-\gamma)^2 \sum_{m\geq 1} \gamma^{m-1} m\kappa(m, \mu, \nu) = \phi_{\mu,\nu} < \infty$$

7 Main theorem

Theorem 1 (Yang, Xie, Wang)**.** For any $K \in \mathbb{N}$ let Q^{π_K} be the action-value function corresponding to policy π_K which is returned by Algorithm 1, when run with a sparse ReLU network on the form

$$\mathcal{F}_0 = \{f(\cdot, a) \in \mathcal{F}(L^*, \{d_j^*\}_{j=0}^{L^*+1}, s^*) \mid a \in \mathcal{A}\}$$

where

$$L^* \lesssim (\log n)^{\xi'}, d_0 = r, d_j^*, d_{L+1} = 1, \lesssim n^{\xi'}, s^* \asymp n^{\alpha^*} \cdot (\log n)^{\xi'}$$

Let μ be any distribution over $\mathcal{S} \times \mathcal{A}$. Under assumption 1 and assumption 2

$$\|Q^* - Q^{\pi_K}\|_{1,\mu} \leq C \cdot \frac{\phi_{\mu,\nu}\cdot\gamma}{(1-\gamma)^2} \cdot |\mathcal{A}| \cdot (\log n)^{\xi^*} \cdot n^{(\alpha^*-1)/2} + \frac{4\gamma^{K+1}}{(1-\gamma)^2} \cdot R_{\max}$$

Here $C, \xi', \xi^*, \phi_{\mu,\nu} \in \mathbb{R}_+$ and $\alpha^* \in (0, 1)$ are constants depending on the assumptions and $R_{\max}$ the maximum possible reward.

8 Proofs

Proof of main theorem. Using theorem 2 we get

$$\|Q^* - Q^{\pi_K}\|_{1,\mu} \leq \frac{2\phi_{\mu,\nu}\gamma}{(1-\gamma)^2} + \frac{4\gamma^{K+1}}{(1-\gamma)^2} R_{\max} \tag{3}$$

where $\varepsilon_{\max} = \max_{k\in[K]} \left\| T\widetilde{Q}_{k-1} - \widetilde{Q}_k \right\|_{2,\nu}$. Using theorem 3 with $Q = \widetilde{Q}_{k-1}$, $\mathcal{F} = \mathcal{F}_0$, $\epsilon = 1$ and $\delta = 1/n$, we get

$$\varepsilon_{\max} \leq 4n^{-1}V_{\max}^2 + 12\delta V_{\max} + 2\omega(\mathcal{F}_0) + 8\sqrt{2}V_{\max}n^{-1/2}\sqrt{\log N_0} + 16V_{\max}n^{-1} \tag{4}$$

where $N_0 = \left|\mathcal{N}(1/n, \mathcal{F}_0, \|\cdot\|_\infty)\right|$. To bound $\omega(\mathcal{F}_0)$ we want to employ the following lemma by

Lemma 1 (Approximation of Hölder Smooth Functions)**.** Let $m, M \in \mathbb{N}$ with $N \geq \max\{(\beta+1)^r, (H+1)\}$, $L = 8 + (m+5)(1 + \lceil \log_2 r \rceil)$, $d_0 = r, d_j = 12rN, d_{L+1} = 1$. Then for any $g \in \mathcal{C}_r([0,1]^r, \beta, H)$ there exists a ReLU network $f \in \mathcal{F}(L, \{d_j\}_{j=0}^{L+1}, s, V_{\max})$ with $s \leq 94r^2(\beta+1)^{2r}N(m+6)(1+\lceil \log_2 r \rceil)$ such that

$$\|f - g\|_\infty \leq (2H+1)3^{r+1}N2^{-m} + H2^\beta N^{-\beta/r}$$

Therefore the first step is to refit our Hölder Smooth compositions in $\mathcal{G}_0$ to be defined on a hyper-cube instead. This is a relatively simple procedure:

Let $f \in \mathcal{G}_0$ then $f(\cdot, a) \in \mathcal{G}(\{p_j, t_j, \beta_j, H_j\})$ so let $f(\cdot, a) = g_q \circ \cdots \circ g_1$ where $g_{jk} \in C_{t_j}([a_j, b_j]^{t_j}, \beta_j, H_j)$ for each $j \in [q]$ and $k \in [p_{j+1}]$. Define

$$h_1 = g_1/(2H_1) + 1/2 \tag{5}$$

$$h_j(u) = g_j(2H_{j-1}u - H_{j-1})/(2H_j) + 1/2, \qquad j \in \{2, \ldots, q-1\} \tag{6}$$

$$h_q(u) = g_q(2H_{q-1}u - H_{q-1}) \tag{7}$$

Then $g_q \circ \cdots \circ g_1 = h_q \circ \cdots \circ h_1$ and

$$h_{1k} \in C_{t_1}([0,1]^{t_1}, \beta_1, 1) \tag{8}$$

$$h_{jk} \in C_{t_j}([0,1]^{t_j}, \beta_j, (2H_{j-1})^{\beta_j}) \tag{9}$$

$$h_q \in C_{t_q}([0,1]^{t_q}, \beta_q, H_q(2H_{q-1})^{\beta_q}) \tag{10}$$

□

Theorem 2 (Error Propagation)**.** Let $\{\widetilde{Q}_i\}_{0\leq i\leq K}$ be the iterates of the fitted Q-iteration algorithm. Then

$$\|Q^* - Q^{\pi_K}\|_{1,\mu} \leq \frac{2\phi_{\mu,\nu}\gamma}{(1-\gamma)^2} \cdot \varepsilon_{\max} + \frac{4\gamma^{K+1}}{(1-\gamma)^2} \cdot R_{\max}$$

Where

$$\varepsilon_{\max} = \max_{k\in[K]} \left\| T\widetilde{Q}_{k-1} - \widetilde{Q}_k \right\|_{2,\nu}$$

Lemma 2. *$TQ \geq T^\pi Q$ for any policy $\pi : \mathcal{S} \to \mathcal{P}(\mathcal{A})$ and any action value function $Q : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$.*

Proof.

$$\begin{aligned}(TQ)(s,a) &= \mathbb{E}\left(R(s,a) + \gamma \max_{a'} Q(S', a') \mid S' \sim P(\cdot \mid s,a)\right) \\ &\geq \mathbb{E}\left(R(s,a) + \gamma Q(S', A') \mid S' \sim P(\cdot \mid s,a), A' \sim \pi(\cdot \mid S')\right) \\ &= T^\pi Q(s,a)\end{aligned}$$

□

Lemma 3. *Let $f : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$ be an action-value function, $\tau_1, \ldots, \tau_m$ be policies and $\mu \in \mathcal{P}(\mathcal{S} \times \mathcal{A})$ be a probability measure. Then*

$$\mathbb{E}_\mu[(P^{\tau_m} \ldots P^{\tau_1})(f)] \leq \kappa(k - i + j; \mu, \nu)\|f\|_{2,\nu}$$

For any measure $\nu \in \mathcal{P}(\mathcal{S} \times \mathcal{A})$ which is absolutely continuous w.r.t. $(P^{\tau_m} \ldots P^{\tau_1})(\mu)$. Here κ is the concentration coefficients defined in definition 7.

Proof. Recall that

$$\begin{aligned}\kappa(m;\mu,\nu) &:= \sup_{\pi_1,\dots,\pi_m}\left[\mathbb{E}_\nu\left|\frac{\mathrm{d}(P^{\pi_m}\dots P^{\pi_1}\mu)}{\mathrm{d}\nu}\right|^2\right]^{1/2}\\ &= \sup_{\pi_1,\dots,\pi_m}\left\|\frac{\mathrm{d}(P^{\pi_m}\dots P^{\pi_1}\mu)}{\mathrm{d}\nu}\right\|_{2,\nu}\end{aligned}$$

Thus

$$\begin{aligned}\mathbb{E}_\mu[(P^{\tau_m}\dots P^{\tau_1})(f)] &= \int (P^{\tau_m}\dots P^{\tau_1})(f)\,\mathrm{d}\mu && (11)\\ &= \int f\,\mathrm{d}(P^{\tau_m}\dots P^{\tau_1}\mu) && (12)\\ &= \int f\frac{\mathrm{d}(P^{\tau_m}\dots P^{\tau_1}\mu)}{\mathrm{d}\nu}\,\mathrm{d}\nu && (13)\\ &\le \left\|\frac{\mathrm{d}(P^{\tau_m}\dots P^{\tau_1}\mu)}{\mathrm{d}\nu}\right\|_{2,\nu}\cdot\|f\|_{2,\nu} && (14)\\ &\le \kappa(m,\mu,\nu)\|f\|_{2,\nu} && (15)\end{aligned}$$

Where eq. (13) is due to the Radon-Nikodym theorem and eq. (14) is Cauchy-Schwarz. □

Proof of theorem 2. First some things to keep in mind during the proof. Recall that $V_{\max} = R_{\max}/(1-\gamma)$ and that π_Q is the greedy policy w.r.t. Q. Denote

$$\pi_i = \pi_{\widetilde{Q}_i},\ Q_{i+1} = T\widetilde{Q}_i,\ \varrho_i = Q_i - \widetilde{Q}_i,\quad \text{for } i \in \{0,\dots,K+1\}$$

Note that for any policy π, P^π is linear and 1-contrative on $\mathcal{L}^\infty(\mathcal{S}\times\mathcal{A})$. Also

$$T^\pi Q^\pi = Q^\pi,\ TQ = T^{\pi_Q}Q,\ TQ^* = Q^* = Q^{\pi^*}$$

where π^* is greedy w.r.t. Q^*. If $f > f'$ for $f, f' : \mathcal{S}\times\mathcal{A}\to\mathbb{R}$ then $P^\pi f \ge P^\pi f'$.

The proof consists of four steps.

Step 1 We start by relating $Q^* - Q^{\pi_K}$, the quantity of interest, to $Q^* - \widetilde{Q}_K$, which is more related to the output of the algorithm. Using lemma 2 we can make the upper bound

$$\begin{aligned}Q^* - Q^{\pi_K} &= T^{\pi^*}Q^* - T^{\pi_K}Q^{\pi_K}\\ &= T^{\pi^*}Q^* + (T^{\pi^*}\widetilde{Q}_K - T^{\pi^*}\widetilde{Q}_K) + (T\widetilde{Q}_K - T\widetilde{Q}_K) - T^{\pi_K}Q^{\pi_K}\\ &= (T^{\pi^*}\widetilde{Q}_K - T\widetilde{Q}_K) + (T^{\pi^*}Q^* - T^{\pi^*}\widetilde{Q}_K) + (T\widetilde{Q}_K - T^{\pi_K}Q^{\pi_K})\\ &\le (T^{\pi^*}Q^* - T^{\pi^*}\widetilde{Q}_K) + (T\widetilde{Q}_K - T^{\pi_K}Q^{\pi_K})\\ &= (T^{\pi^*}Q^* - T^{\pi^*}\widetilde{Q}_K) + (T^{\pi_K}\widetilde{Q}_K - T^{\pi_K}Q^{\pi_K})\\ &= \gamma P^{\pi^*}(Q^* - \widetilde{Q}_K) + \gamma P^{\pi_K}(\widetilde{Q}_K - Q^{\pi_K})\\ &= \gamma(P^{\pi^*} - P^{\pi_K})(Q^* - \widetilde{Q}_K) + \gamma P^{\pi_K}(Q^* - Q^{\pi_K}) && (16)\end{aligned}$$

This implies

$$(I - \gamma P^{\pi_K})(Q^* - Q^{\pi_K}) \le \gamma(P^{\pi^*} - P^{\pi_K})(Q^* - \widetilde{Q}_K)$$

Since γP^{π_K} is γ-contractive, $U = (I - \gamma P^{\pi_K})^{-1}$ exists as a bounded operator on $\mathcal{L}^\infty(\mathcal{S}\times\mathcal{A})$ and equals

$$U = \sum_{i=0}^{\infty}\gamma^i(P^{\pi_K})^i$$

From this we also see that $f \ge f' \implies Uf \ge Uf'$ for any $f, f' : \mathcal{S}\times\mathcal{A}\to\mathbb{R}$. Therefore we can apply U on both sides of eq. (16) to obtain

$$Q^* - Q^{\pi_K} \le \gamma U^{-1}(P^{\pi^*}(Q^* - \widetilde{Q}_K) - P^{\pi_K}(Q^* - \widetilde{Q}_K)) \qquad (17)$$

Step 2 Using lemma 2 for any $i \in [K]$ we can get an upper bound

$$
\begin{aligned}
Q^* - \widetilde{Q}_{i+1} &= Q^* + (T\widetilde{Q}_i - T\widetilde{Q}_i) - \widetilde{Q}_{i+1} + (T^{\pi^*}\widetilde{Q}_i - T^{\pi^*}\widetilde{Q}_i) \\
&= (Q^* - T^{\pi^*}\widetilde{Q}_i) + (T\widetilde{Q}_i - \widetilde{Q}_{i+1}) + (T^{\pi^*}\widetilde{Q}_i - T\widetilde{Q}_i) \\
&= (T^{\pi^*}Q^* - T^{\pi^*}\widetilde{Q}_i) + \varrho_{i+1} + (T^{\pi^*}\widetilde{Q}_i - T\widetilde{Q}_i) \\
&\leq T^{\pi^*}Q^* - T^{\pi^*}\widetilde{Q}_i + \varrho_{i+1} \\
&= \gamma P^{\pi^*}(Q^* - \widetilde{Q}_i) + \varrho_{i+1} \qquad (18)
\end{aligned}
$$

and a lower bound

$$
\begin{aligned}
Q^* - \widetilde{Q}_{i+1} &= Q^* + (T\widetilde{Q}_i - T\widetilde{Q}_i) - \widetilde{Q}_{i+1} + (T^{\pi_i}Q^* - T^{\pi_i}Q^*) \\
&= (T^{\pi_i}Q^* - T^{\pi_i}\widetilde{Q}_i) + \varrho_{i+1} + (TQ^* - T^{\pi_i}Q^*) \\
&\geq T^{\pi_i}Q^* - T^{\pi_i}\widetilde{Q}_i + \varrho_{i+1} \\
&= \gamma P^{\pi_i}(Q^* - \widetilde{Q}_i) + \varrho_{i+1} \qquad (19)
\end{aligned}
$$

Applying eq. (18) and eq. (19) iteratively we get

$$
Q^* - \widetilde{Q}_K \leq \gamma^K (P^{\pi^*})^K (Q^* - \widetilde{Q}_0) + \sum_{i=0}^{K-1} \gamma^{K-1-i} (P^{\pi^*})^{K-1-i} \varrho_{i+1} \qquad (20)
$$

and

$$
Q^* - \widetilde{Q}_K \geq \gamma^K (P^{\pi_{K-1}} \dots P^{\pi_0})(Q^* - \widetilde{Q}_0) + \sum_{i=0}^{K-1} \gamma^{K-1-i} (P^{\pi_{K-1}} \dots P^{\pi_{i+1}}) \varrho_{i+1} \qquad (21)
$$

Step 3 Combining eq. (20) and eq. (21) with eq. (17) we get

$$
\begin{aligned}
Q^* - Q^{\pi_K} \leq U^{-1} \Bigg(&\gamma^{K+1}((P^{\pi^*})^{K+1} - P^{\pi_K} \dots P^{\pi_0})(Q^* - \widetilde{Q}_0) \\
&+ \sum_{i=0}^{K-1} \gamma^{K-i}((P^*)^{K-i} - P^{\pi_K} \dots P^{\pi_{i+1}})\varrho_{i+1} \Bigg)
\end{aligned} \qquad (22)
$$

For shorthand define constants

$$
\alpha_i = \frac{(1-\gamma)\gamma^{K-i-1}}{1-\gamma^{K+1}} \text{ for } 0 \leq i \leq K-1 \text{ and } \alpha_K = \frac{(1-\gamma)\gamma^K}{1-\gamma^{K+1}} \qquad (23)
$$

(note that $\sum_{i=0}^K \alpha_i = 1$) and operators

$$
\begin{aligned}
O_i &= (1-\gamma)/2U^{-1}[(P^{\pi^*})^{K-i} + (P^{\pi_K} \dots P^{\pi_{i+1}})] \qquad (24) \\
O_K &= (1-\gamma)/2U^{-1}[(P^{\pi^*})^{K+1} + (P^{\pi_K} \dots P^{\pi_0})] \qquad (25)
\end{aligned}
$$

Then by eq. (22)

$$
|Q^* - Q^{\pi_K}| \leq \frac{2\gamma(1-\gamma^{K+1})}{(1-\gamma)^2} \left[\sum_{i=0}^{K-1} \alpha_i O_i |\varrho_{i+1}| + \alpha_K O_K \left| Q^* - \widetilde{Q}_0 \right| \right] \qquad (26)
$$

So by linearity of expectation

$$
\begin{aligned}
\|Q^* - Q^{\pi_K}\|_{1,\mu} &= \mathbb{E}_\mu |Q^* - Q^{\pi_K}| \qquad (27) \\
&\leq \frac{2\gamma(1-\gamma^{K+1})}{(1-\gamma)^2} \left[\sum_{i=0}^{K-1} \alpha_i \mathbb{E}_\mu (O_i |\varrho_{i+1}|) + \alpha_K \mathbb{E}_\mu (O_K \left| Q^* - \widetilde{Q}_0 \right|) \right] \qquad (28)
\end{aligned}
$$

With the bound on rewards we (crudely) estimate

$$
\mathbb{E}_\mu O_K \left| Q^* - \widetilde{Q}_0 \right| \leq 2V_{\max} = 2R_{\max}/(1-\gamma) \qquad (29)
$$

The remaining difficulty lies in $\mathbb{E}_\mu(O_i|\varrho_{i+1}|)$.

Step 4 Using the sum expansion of U^{-1} we get

$$\begin{aligned}
&\mathbb{E}_\mu(O_i|\varrho_{i+1}|) && (30)\\
&= \frac{1-\gamma}{2}\mathbb{E}_\mu\left(U^{-1}[(P^{\pi_K})^{K-i} + P^{\pi_K}\dots P^{\pi_{i+1}}]|\varrho_{i+1}|\right) && (31)\\
&= \frac{1-\gamma}{2}\mathbb{E}_\mu\left(\sum_{j=0}^{\infty}[(P^{\pi_K})^j(P^{\pi_K})^{K-i} + (P^{\pi_K})^{j+1}P^{\pi_{K-1}}\dots P^{\pi_{i+1}}]|\varrho_{i+1}|\right) && (32)\\
&= \frac{1-\gamma}{2}\sum_{j=0}^{\infty}\mathbb{E}_\mu\left([(P^{\pi_K})^j(P^{\pi_K})^{K-i} + (P^{\pi_K})^{j+1}P^{\pi_{K-1}}\dots P^{\pi_{i+1}}]|\varrho_{i+1}|\right) && (33)
\end{aligned}$$

Notice that there are $K-i+j$ P-operators on both terms in the sum. Therefore were can employ lemma 3 twice. Moreover define $\varepsilon_{\max} = \max_{i\in[K]}\|\varrho_i\|_{2,\nu}$. Then

$$\begin{aligned}
\mathbb{E}_\mu(O_i|\varrho_{i+1}|) &\le (1-\gamma)\sum_{j=0}^{\infty}\gamma^j\kappa(K-i+j;\mu,\nu)\|\varrho_{i+1}\|_{2,\nu}\\
&\le \varepsilon_{\max}(1-\gamma)\sum_{j=0}^{\infty}\gamma^j\kappa(K-i+j;\mu,\nu) \qquad (34)
\end{aligned}$$

Using eq. (28), eq. (29) and eq. (34)

$$\begin{aligned}
\|Q^* - Q^{\pi_K}\|_{1,\mu} \le \frac{2\gamma(1-\gamma^{K+1})}{1-\gamma}&\left[\sum_{i=0}^{K-1}\sum_{j=0}^{\infty}\alpha_i\gamma^j\kappa(K-i+j;\mu,\nu)\right]\varepsilon_{\max}\\
&+\frac{4\gamma(1-\gamma^{K+1})}{(1-\gamma)^3}\alpha_K R_{\max} \qquad (35)
\end{aligned}$$

Focusing on the first term on RHS of eq. (35), if we then we can take the norm out of the sum as a constant. We are left with

$$\begin{aligned}
&\sum_{i=0}^{K-1}\sum_{j=0}^{\infty}\alpha_i\gamma^j\kappa(K-i+j;\mu,\nu)\\
&= \sum_{i=0}^{K-1}\sum_{j=0}^{\infty}\frac{(1-\gamma)\gamma^{K-i+j-1}}{1-\gamma^{K+1}}\kappa(K-i+j;\mu,\nu)\\
&= \frac{1-\gamma}{1-\gamma^{K+1}}\sum_{j=0}^{\infty}\sum_{i=0}^{K-1}\gamma^{K-i+j-1}\kappa(K-i+j;\mu,\nu)\\
&\le \frac{1-\gamma}{1-\gamma^{K+1}}\sum_{m=0}^{\infty}\gamma^{m-1}\cdot m\cdot\kappa(m;\mu,\nu)\\
&\le \frac{1}{1-\gamma^{K+1}(1-\gamma)}\phi_{\mu,\nu} \qquad (36)
\end{aligned}$$

Where the last inequality is due to assumption 2. Combining eq. (35) and eq. (36) we arrive at

$$\|Q^* - Q^{\pi_K}\|_{1,\mu} \le \frac{2\gamma\cdot\phi_{\mu,\nu}}{(1-\gamma)^2}\cdot\varepsilon_{\max} + \frac{4\gamma^{K+1}}{(1-\gamma)^2}\cdot R_{\max} \qquad (37)$$

□

Theorem 3 (One-step Approximation Error)**.** Let

- $\mathcal{F} \subseteq \mathcal{B}(\mathcal{S}\times\mathcal{A}, V_{\max})$ be a class of bounded measurable functions
- $\mathcal{G} = T(\mathcal{F})$ the class of functions obtainable by applying T to some function in $\mathcal{F}$.
- $\nu \in \mathcal{P}(\mathcal{S},\mathcal{A})$ be a probability measure

- $(S_i, A_i)_{i\in[n]}$ be n i.i.d. samples following ν
- $(R_i, S_i')_{i\in[n]}$ be the rewards and next states corresponding to the samples
- $Q \in \mathcal{F}$ be fixed
- $Y_i = R_i + \gamma \max_{a\in\mathcal{A}} Q(S_i', a)$
- $\widehat{Q} = \operatorname{argmin}_{f\in\mathcal{F}} \frac{1}{n}\sum_{i=1}^n (f(S_i, A_i) - Y_i)^2$
- $\kappa \in (0, 1],\ \delta > 0$ be fixed
- $\mathcal{N}(\delta, \mathcal{F}, \|\cdot\|_\infty)$ a minimal δ-covering of $\mathcal{F}$ w.r.t. $\|\cdot\|_\infty$
- $N_\delta = \left|\mathcal{N}(\delta, \mathcal{F}, \|\cdot\|_\infty)\right|$ the number of elements in this covering

Then

$$\begin{aligned}\left\|\widehat{Q} - TQ\right\|_\nu^2 \leq& \frac{(1+\kappa)^2}{\kappa}\frac{1}{n}V_{\max}^2 + (1+\kappa)\left(6\delta V_{\max} + \omega(\mathcal{F})\right)\\ &+ 8\sqrt{2}V_{\max} n^{-1/2}\sqrt{\log N_\delta} + 8V_{\max}(n^{-1} + \delta)\end{aligned}$$

Where

$$\omega(\mathcal{F}) = \sup_{g\in\mathcal{G}} \inf_{f\in\mathcal{F}} \frac{1}{n}\mathbb{E}\|f - TQ\|^2$$

where

Lemma 4 (Rotation invariance)**.** Let $(X_i)_{i=1}^n$ be independent, centered and sub-gaussian. Then $\sum_{i=1}^n X_i$ is centered and sub-gaussian with

$$\left\|\sum_{i=1}^n X_i\right\|_{\psi_2}^2 \leq C\sum_{i=1}^n \|X_i\|_{\psi_2}^2$$

Proof. See [Vershynin 2010, p. 12]. □

Definition 8 (Sub-exponential norm)**.** For a random variable define

$$\|X\|_{\psi_1} = \sup_{p\geq 1} p^{-1}\|X\|_p$$

called the sub-exponential norm, said to 'exist' if finite. In that case X is said to be 'sub-exponential'.

Lemma 5 (Sub-gaussian squared is sub-exponential)**.** A random variable X is sub-gaussian if and only if X^2 is sub-exponential and

$$\|X\|_{\psi_2}^2 \leq \left\|X^2\right\|_{\psi_1} \leq 2\|X\|_{\psi_2}^2$$

Proof. See [Vershynin 2010, p. 14] □

Proposition 2. Let v be a random vector in $\mathbb{R}^n$ then

$$\mathbb{E}\|v\|_1 \leq \sqrt{n}\sqrt{\mathbb{E}\|v\|_2^2}$$

Proof. Denote v's coordinates $v = (v_1, \dots, v_n)$. Cauchy-Schwarz applied to some vector w and $(1, \dots, 1)$ yields

$$\|w\|_1 \leq \sqrt{n}\|w\|_2$$

Now let $w = (\mathbb{E}v_1, \dots, \mathbb{E}v_n)$. Then by linearity of expectation and Jensens inequality

$$\mathbb{E}\|v\| = \|w\| \leq \sqrt{n}\sqrt{\sum_{i=1}^n (\mathbb{E}v_i)^2} \leq \sqrt{n}\sqrt{\mathbb{E}\sum_{i=1}^n v_i^2} = \sqrt{n}\sqrt{\mathbb{E}\|v\|_2^2}$$

□

Theorem 4 (Bernstein's inequality)**.** Suppose $U_1, \ldots, U_n$ are independent with $\mathbb{E}U_i = 0, |U_i| \leq M$ for all $i \in [n]$. Then for all $t > 0$

$$\mathbb{P}\left(\left|\sum_{i=1}^{n} U_i\right| \geq t\right) \leq \exp\left(\frac{-t^2}{2/3Mt + 2\sigma^2}\right)$$

where $\sigma^2 = \sum_{i=1}^{n} V(U_i)$.

Proof of theorem 3. First some introductory fixing of notation and variables. Fix a minimal δ-covering of $\mathcal{F}$ with centers $f_1, \ldots, f_{N_\delta}$. Define

$$\widetilde{Q} := \operatorname*{argmin}_{f \in \mathcal{F}} \|f - TQ\|_\nu^2$$

$$k^* := \operatorname*{argmin}_{k \in [N_\delta]} \left\| f_k - \widehat{Q} \right\|_\infty$$

and $X_i := (S_i, A_i)$. Notice that $\widetilde{Q}$ differs from $\widehat{Q}$ in that $\widetilde{Q}$ approximates TQ w.r.t. $\|\cdot\|_\nu^2$ while $\widehat{Q}$ approximates $Y = (Y_1, \ldots, Y_n)$ in mean squared error over $X = (X_1, \ldots, X_n)$. We shall be loose about applying functions to vectors (of random variables) in the sense that they are applied entry-wise. We use $\|\cdot\|_p$ to denote the (finite dimensional) p-norm (p ommitted when $p = 2$). When talking about p-norms on the random variables we always specify the distribution (e.g. $\|\cdot\|_\nu$). When the sample (e.g. X) is clear from context we omit it writing $\|f\| = \|f(X)\|$.

Step 1 By definion (of $\widehat{Q}$) for all $f \in \mathcal{F}$ we have $\left\|\widehat{Q}(X) - Y\right\|^2 \leq \|f(X) - Y\|^2$, leading to

$$\|Y\|^2 + \left\|\widehat{Q}\right\|^2 - 2Y \cdot \widehat{Q} \leq \|Y\|^2 + \|f\|^2 - 2Y \cdot f \tag{38}$$

$$\iff \left\|\widehat{Q}\right\|^2 + \|TQ\|^2 - 2\widehat{Q} \cdot TQ \leq \|f\|^2 + \|TQ\|^2 - 2f \cdot TQ + 2Y \cdot \widehat{Q} - 2Y \cdot f - 2\widehat{Q} \cdot TQ + 2f \cdot TQ \tag{39}$$

$$\iff \left\|\widehat{Q} - TQ\right\|^2 \leq \|f - TQ\|^2 + 2(Y - TQ) \cdot (\widehat{Q} - f) \tag{40}$$

$$\iff \left\|\widehat{Q} - TQ\right\|^2 \leq \|f - TQ\|^2 + 2\xi \cdot (\widehat{Q} - f) \tag{41}$$

Where $\xi_i := Y_i - TQ(X_i)$ and $\xi := (\xi_1, \ldots, \xi_n)$. Let $\Sigma = (X_1, \ldots, X_n)^{-1}(\mathbb{B}_n) \in \mathcal{H}$ be the σ-algebra generated by the samples. Now we proof a minor lemma

Proposition 3. $\mathbb{E}(\xi_i \mid \Sigma) = 0$ and thus $\mathbb{E}(\xi_i g(X_i)) = 0$ for any function $g : \mathbb{R} \to \mathbb{R}$.

Proof. Recall that $X_i = (S_i, A_i)$,

$$Y_i = R_i + \gamma \max_{a \in \mathcal{A}} Q(S_{i+1}, a)$$

where $S_{i+1} \sim P(X_i)$, $R_i \sim R(X_i)$ and

$$TQ(X_i) = \mathbb{E}_\Sigma R_i' + \gamma \mathbb{E}_\Sigma Q(S', \operatorname*{argmax}_{a \in \mathcal{A}} Q(S', a))$$

where $S' \sim P(X_i)$, $R_i' \sim R(X_i)$. Since S' and S_{i+1} are i.i.d.

$$\begin{aligned}
\mathbb{E}_\Sigma \xi_i &= \mathbb{E}_\Sigma \left(Y_i - TQ(X_i)\right) \\
&= \mathbb{E}_\Sigma R_i - \mathbb{E}_\Sigma R_i' + \gamma \left(\mathbb{E}_\Sigma \left(\max_{a \in \mathcal{A}} Q(S_{i+1}, a)\right) - \mathbb{E}_\Sigma \operatorname*{argmax}_{a \in \mathcal{A}} \left(Q(S', a)\right)\right) \\
&= 0
\end{aligned}$$

Therefore $\mathbb{E}(\xi_i \mid \Sigma) = 0$. $\square$

By this lemma we can deduce

$$\mathbb{E}\left(\xi\cdot(\widehat{Q}-f)\right)=\mathbb{E}\left(\xi\cdot(\widehat{Q}-TQ)\right) \tag{42}$$

To bound this we insert f_{k^*} by the triangle inequality

$$\left|\mathbb{E}\left(\xi\cdot(\widehat{Q}-TQ)\right)\right|\leq\left|\mathbb{E}\left(\xi\cdot(\widehat{Q}-f_{k^*})\right)\right|+\left|\mathbb{E}\left(\xi\cdot(f_{k^*}-TQ)\right)\right| \tag{43}$$

We now bound these two terms. The first by Cauchy-Schwarz

$$\left|\mathbb{E}\xi\cdot(\widehat{Q}-f_{k^*})\right|\leq\mathbb{E}\left(\|\xi\|\left\|\widehat{Q}-f_{k^*}\right\|\right)\leq\mathbb{E}(\|\xi\|)\sqrt{n}\delta\leq 2nV_{\max}\delta \tag{44}$$

where we have used that $\left\|\widehat{Q}-f_{k^*}\right\|_\infty\leq\delta$ so

$$\left\|\widehat{Q}-f_{k^*}\right\|^2=\sum_{i=1}^n(\widehat{Q}(X_i)-f_{k^*}(X_i))^2\leq\sum_{i=1}^n\delta^2=n\delta^2 \tag{45}$$

and that $|Y_i|,TQ(X_i)\leq V_{\max}$ so

$$\|\xi\|^2=\sum_{i=1}^n(Y_i-TQ(X_i))^2\leq\sum_{i=1}^n(2V_{\max})^2=4V_{\max}^2n \tag{46}$$

To bound the second term in eq. (43) define

$$Z_j:=\xi\cdot(f_j-TQ)\left\|f_j-TQ\right\|^{-1} \tag{47}$$

Note that since ξ_i are centered Z_j. For a sub-σ-algebra Σ define the *sub-gaussian* norm by

Definition 9 (Subgaussian norm)**.**

$$\|W\|_{\psi_2,\Sigma}:=\sup_{p\geq 1}p^{-1/2}\left(\mathbb{E}_\Sigma|W|^p\right)^{1/p}$$

Because of proposition 3 $\xi_i(f_j(X_i)-TQ(X_i))$ is centered for any $i\in[n]$ and

$$\left\|\xi_i(f_j(X_i)-TQ(X_i))\right\|_{\psi_2,\Sigma}\leq 2V_{\max}\left|f_j(X_i)-TQ(X_i)\right| \tag{48}$$

Therefore by lemma 4

$$\begin{aligned}
\left\|Z_j\right\|_{\psi_2,\Sigma}^2&\leq\left\|f_j-TQ\right\|^{-2}\left\|\sum_{i=1}^n\xi_i(f_j(X_i)-TQ(X_i))\right\|_{\psi_2,\Sigma}^2 && (49)\\
&\leq\left\|f_j-TQ\right\|^{-2}C_1\sum_{i=1}^n\left\|\xi_i(f_j(X_i)-TQ(X_i))\right\|_{\psi_2,\Sigma}^2 && (50)\\
&\leq\left\|f_j-TQ\right\|^{-2}C_1\sum_{i=1}^n4V_{\max}\left|f_j(X_i)-TQ(X_i)\right|^2 && (51)\\
&=4V_{\max}^2C_1 && (52)
\end{aligned}$$

Observe that $\|X\|_p\leq\sqrt{p}\ \sup_{p\geq 1}\|X\|_{\psi_2}$. Therefore by lemma 5 we can say

$$\begin{aligned}
\mathbb{E}Z_{k^*}^2&=\mathbb{E}\left(\mathbb{E}_\Sigma Z_{k^*}^2\right) && (53)\\
&\leq\mathbb{E}\left(\max_{j\in[N_\delta]}\mathbb{E}_\Sigma Z_j^2\right) && (54)\\
&\leq\mathbb{E}\left(\sqrt{2}\left\|Z_j^2\right\|_{\psi_2,\Sigma}\right) && (55)\\
&\leq 2\sqrt{2}\mathbb{E}\left\|Z_j\right\|_{\psi_2,\Sigma}^2 && (56)\\
&\leq 8\sqrt{2}V_{\max}^2C_1 && (57)
\end{aligned}$$

So now we can bound

$$\mathbb{E}\left(\xi \cdot (f_{k^*} - TQ)\right) = \mathbb{E}\left(\|f_{k^*} - TQ\| |Z_{k^*}|\right) \tag{58}$$
$$\leq \mathbb{E}\left(\left(\left\|\widehat{Q} - TQ\right\| + \left\|\widehat{Q} - f_{k^*}\right\|\right)|Z_{k^*}|\right) \tag{59}$$
$$\leq \mathbb{E}\left(\left(\left\|\widehat{Q} - TQ\right\| + n\delta\right)|Z_{k^*}|\right) \tag{60}$$
$$\leq \left(\mathbb{E}\left(\left\|\widehat{Q} - TQ\right\| + n\delta\right)^2\right)^{1/2} \left(\mathbb{E}Z_{k^*}^2\right)^{1/2} \tag{61}$$
$$\leq \mathbb{E}\left(\left\|\widehat{Q} - TQ\right\| + n\delta\right) \left(\mathbb{E}Z_{k^*}^2\right)^{1/2} \tag{62}$$
$$\leq \left(\sqrt{\mathbb{E}\left\|\widehat{Q} - TQ\right\|_2^2} + n\delta\right) \left(\mathbb{E}Z_{k^*}^2\right)^{1/2} \tag{63}$$
$$\leq \left(\sqrt{\mathbb{E}\left\|\widehat{Q} - TQ\right\|_2^2} + n\delta\right) 2^{7/4} V_{\max} \sqrt{C_1} \tag{64}$$

Where eq. (58) to eq. (59) is by the triangle inequality, eq. (62) to eq. (63) is proposition 2 and eq. (63) to eq. (64) is due to that Combining eq. (41), eq. (43), eq. (44) and eq. (64)

$$\mathbb{E}\left\|\widehat{Q} - TQ\right\|^2 \leq \mathbb{E}\|f - TQ\|^2 + 4nV_{\max}\delta + \left(\sqrt{\mathbb{E}\left\|\widehat{Q} - TQ\right\|^2} + \sqrt{n}\delta\right) 2V_{\max} \tag{65}$$
$$= 2V_{\max}\sqrt{n}\sqrt{\mathbb{E}\left\|\widehat{Q} - TQ\right\|^2} + 6n\delta V_{\max} + \mathbb{E}\|f - TQ\|^2 \tag{66}$$

Lemma 6. Let $a, b > 0, \kappa \in (0, 1]$ then

$$a^2 \leq 2ab + c \implies a^2 \leq (1+\kappa)^2 b^2/\kappa + (1+\kappa)c$$

Proof. $0 \leq (x-y)^2 = x^2 + y^2 - 2xy \implies 2xy \leq x^2 + y^2$ for any $x, y \in \mathbb{R}$ so

$$2ab = 2\sqrt{\frac{\kappa}{1+\kappa}}a\sqrt{\frac{1+\kappa}{\kappa}}b$$
$$\leq \frac{\kappa}{1+\kappa}a^2 + \frac{1+\kappa}{\kappa}b^2$$

□

By lemma 6 applied to eq. (66)

$$\frac{1}{n}\mathbb{E}\left\|\widehat{Q} - TQ\right\|^2 \leq \frac{(1+\kappa)^2}{\kappa}\frac{1}{n}V_{\max}^2 + (1+\kappa)\left(6\delta V_{\max} + \frac{1}{n}\mathbb{E}\|f - TQ\|^2\right) \tag{67}$$

Since this holds for any $f \in \mathcal{F}$ we can further say

$$\frac{1}{n}\mathbb{E}\left\|\widehat{Q} - TQ\right\|^2 \leq \frac{(1+\kappa)^2}{\kappa}\frac{1}{n}V_{\max}^2 + (1+\kappa)\left(6\delta V_{\max} + \inf_{f \in \mathcal{F}}\frac{1}{n}\mathbb{E}\|f - TQ\|^2\right) \tag{68}$$
$$\leq \frac{(1+\kappa)^2}{\kappa}\frac{1}{n}V_{\max}^2 + (1+\kappa)\left(6\delta V_{\max} + \omega(\mathcal{F})\right) \tag{69}$$

Where we take the supremum over $\mathcal{G}$ (recall $TQ \in \mathcal{G}$).

Step 2 Here we link up $\left\|\widehat{Q} - TQ\right\|_\sigma^2$ with $\mathbb{E}\frac{1}{n}\left\|\widehat{Q} - TQ\right\|^2$. First note that

$$\left|\left(\widehat{Q}(x) - TQ(x)\right)^2 - \left(f_{k^*}(x) - TQ(x)\right)^2\right| = \left|\widehat{Q}(x) - f_{k^*}(x)\right| \cdot \left|\widehat{Q}(x) + f_{k^*}(x) - 2TQ(x)\right| \tag{70}$$
$$\leq 4V_{\max}\delta \tag{71}$$

Using this twice we can say

$$
\begin{aligned}
&(\widehat{Q}(\widehat{X}_i) - TQ(\widehat{X}_i)^2 && (72)\\
&\le (\widehat{Q}(\widetilde{X}_i) - TQ(\widetilde{X}_i))^2 - (f_{k^*}(\widetilde{X}_i) - TQ(\widetilde{X}_i))^2 + (f_{k^*}(\widetilde{X}_i) - TQ(\widetilde{X}_i))^2 && (73)\\
&\le (f_{k^*}(\widetilde{X}_i) - TQ(\widetilde{X}_i))^2 + (\widehat{Q}(X_i) - TQ(X_i))^2 - (\widehat{Q}(X_i) - TQ(X_i))^2 \\
&\quad + (f_{k^*}(X_i) - TQ(X_i))^2 - (f_{k^*}(X_i) - TQ(X_i))^2 + 4V_{\max}\delta && (74)\\
&\le (\widehat{Q}(X_i) - TQ(X_i))^2 + (f_{k^*}(\widetilde{X}_i) - TQ(\widetilde{X}_i))^2 - (f_{k^*}(X_i) - TQ(X_i))^2 + 8V_{\max}\delta && (75)
\end{aligned}
$$

Thus we get

$$
\begin{aligned}
&\left\|\widehat{Q} - TQ\right\|_\sigma^2 && (76)\\
&= \mathbb{E}\frac{1}{n}\sum_{i=1}^n (\widehat{Q}(\widetilde{X}_i) - TQ(\widetilde{X}_i))^2 && (77)\\
&\le \mathbb{E}\frac{1}{n}\sum_{i=1}^n \Big((\widehat{Q}(X_i) - TQ(X_i))^2 + (f_{k^*}(\widetilde{X}_i) - TQ(\widetilde{X}_i))^2 - (f_{k^*}(X_i) - TQ(X_i))^2 \Big) + 8V_{\max}\delta && (78)\\
&= \frac{1}{n}\left\|\widehat{Q} - TQ\right\|^2 + \frac{1}{n}\sum_{i=1}^n h_{k^*}(X_i, \widetilde{X}_i) + 8V_{\max}\delta && (79)
\end{aligned}
$$

Where we define

$$
h_j(x,y) := \left(f_j(y) - TQ(y)\right)^2 - \left(f_j(x) - TQ(x)\right)^2 \qquad (80)
$$

For any $j \in [N_\delta]$. Define $\Upsilon = 2V_{\max}$ and

$$
T := \max_{j\in[N_\delta]} \left| \sum_{i=1}^n h_j(X_i, \widetilde{X}_i)/\Upsilon \right| \qquad (81)
$$

Then we can bound the middle term in eq. (79)

$$
\begin{aligned}
\mathbb{E}\left(\frac{1}{n}\sum_{i=1}^n h_{k^*}(X_i, \widetilde{X}_i)\right) &\le \Upsilon/n\mathbb{E}\max_{j\in[N_\delta]}\left(\left|\sum_{i=1}^n h_j(X_i, \widetilde{X}_i)/\Upsilon\right|\right) && (82)\\
&\le \Upsilon/n\mathbb{E}T && (83)
\end{aligned}
$$

We want to use Bernsteins inequality (theorem 4) with $U_i = h_j(X_i, \widetilde{X}_i)$. Therefore notice that $\left|h_j\right| \le \Upsilon^2$ and

$$
\begin{aligned}
\mathrm{V}h_j(X_i, \widetilde{X}_i) &= 2\mathrm{V}\left(f_j(X_i) - TQ(X_i)\right)^2 && (84)\\
&\le 2\mathbb{E}\left(f_j(X_i) - TQ(X_i)\right)^4 && (85)\\
&\le 2\Upsilon^4 && (86)
\end{aligned}
$$

so by union bounding for any $u < 6n\Upsilon$ we have

$$\begin{aligned}
\mathbb{E}T =& \int_0^\infty \mathbb{P}(T \geq t) && (87)\\
\leq & u + \int_u^\infty \mathbb{P}(T \geq t)\mathrm{d}t && (88)\\
\leq & u + \int_u^\infty 2N_\delta \exp\left(\frac{-t^2}{2\Upsilon t/3 + 4n\Upsilon^2}\right)\mathrm{d}t && (89)\\
\leq & u + 2N_\delta \int_u^\infty \exp\left(\frac{-t^2}{2\Upsilon^2(t/(3\Upsilon) + 2n)}\right)\mathrm{d}t && (90)\\
\leq & u + 2N_\delta\left(\int_u^{6n\Upsilon} \exp\left(\frac{-t^2}{8n\Upsilon^2}\right)\mathrm{d}t + \int_{6n\Upsilon}^\infty \exp\left(\frac{-t}{4/3\Upsilon}\right)\mathrm{d}t\right) && (91)\\
\leq & u + 2N_\delta\left(\frac{8n\Upsilon}{2u}\exp\left(\frac{-u^2}{8n\Upsilon}\right) + \frac{4\Upsilon}{3}\exp\left(\frac{-24n\Upsilon}{3\Upsilon}\right)\right) && (92)
\end{aligned}$$

where we use lemma 7 from eq. (91) to eq. (92). Now set $u = \Upsilon\sqrt{8n\log N_\delta}$ continuing from eq. (92) we have

$$\begin{aligned}
\cdots =& \Upsilon\sqrt{8n\log N_\delta} + \frac{\Upsilon^2 8nN_\delta}{\Upsilon\sqrt{8n\log N_\delta}}\exp(-\log N_\delta) + 8/3N_\delta\Upsilon\exp(-9/2n) && (93)\\
=& \Upsilon 2\sqrt{2n}\left(\log N_\delta + \frac{1}{\log N_\delta}\right) + 8/3N_\delta e^{-9/2n} && (94)\\
\leq & 4\sqrt{2}\Upsilon\sqrt{n\log N_\delta} + 8/3\Upsilon && (95)
\end{aligned}$$

Inserting eq. (95) and eq. (69) into eq. (79)

$$\begin{aligned}
\left\|\widehat{Q} - TQ\right\|_\nu^2 \leq & \frac{1}{n}\mathbb{E}\left\|\widehat{Q} - TQ\right\|^2 + 8\sqrt{2}V_{\max}n^{-1/2}\sqrt{\log N_\delta} + 8V_{\max}(n^{-1} + \delta) && (96)\\
\leq & \frac{(1+\kappa)^2}{\kappa}\frac{1}{n}V_{\max}^2 + (1+\kappa)\big(6\delta V_{\max} + \omega(\mathcal{F})\big)\\
& + 8\sqrt{2}V_{\max}n^{-1/2}\sqrt{\log N_\delta} + 8V_{\max}(n^{-1} + \delta) && (97)
\end{aligned}$$

□

9 Appendices

9.1 Various lemmas

Lemma 7. For $x > 0$.

$$\int_x^\infty e^{-t^2/2}\,\mathrm{d}t \leq \frac{1}{x}e^{-x^2/2}$$

Proof. Observe that for $t \geq x > 0$ we have $1 \leq t/x$ so

$$\begin{aligned}
\int_x^\infty e^{-t^2/2}\,\mathrm{d}t \leq & \int_x^\infty \frac{t}{x}e^{-t^2/2}\,\mathrm{d}t\\
\leq & \frac{1}{x}e^{-x^2/2}
\end{aligned}$$

□