Probabilistic Mappings of Probability Measures

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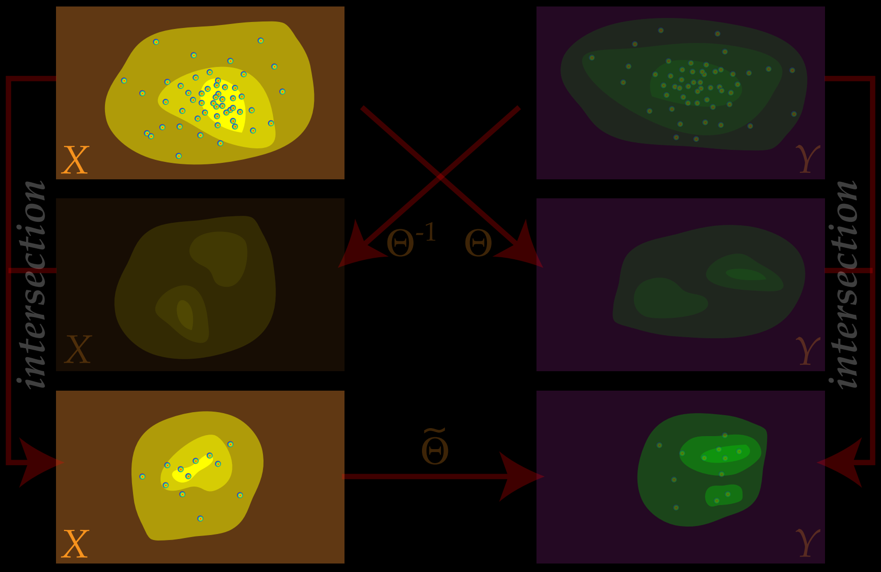
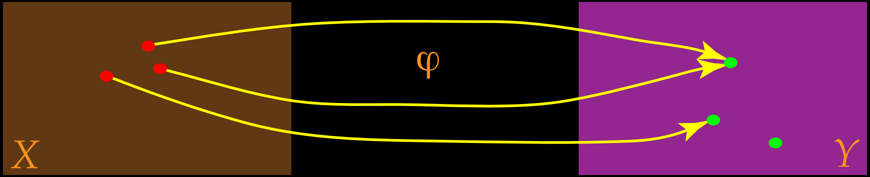
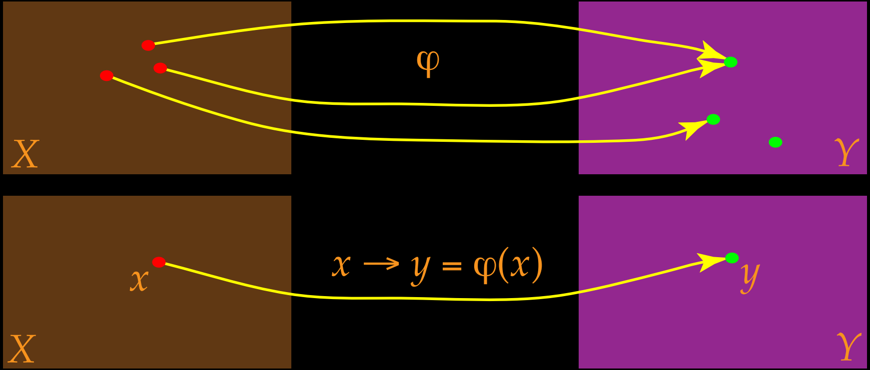
The Bayes-Popper approach.

Using observations to infer the values of some parameters corresponds to solving an *inverse problem*. Practitioners sometimes seek the best solution implied by the data, but observations should only be used to falsify possible solutions, not to deduce any particular solution.

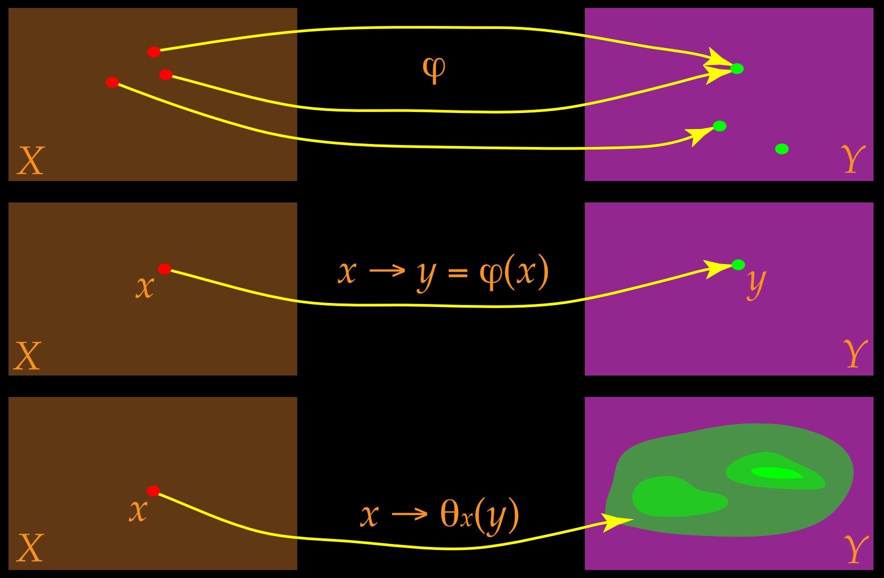


X
Y
X
Y
X
Y
Θ^{-1}
Θ
$\tilde{\Theta}$
intersection
intersection

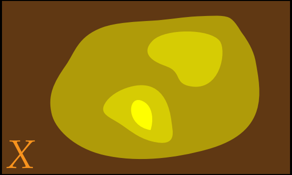

φ
X
Y



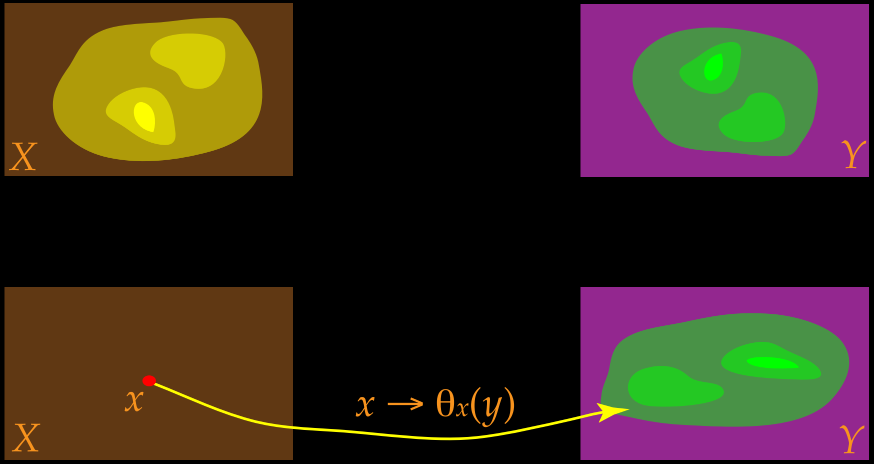
φ
X
Y
x
$x \to y = \varphi(x)$
y
X
Y



φ
X
Y
x → y = φ(x)
x
y
X
Y
x → θx(y)
x
X
Y


X


Y

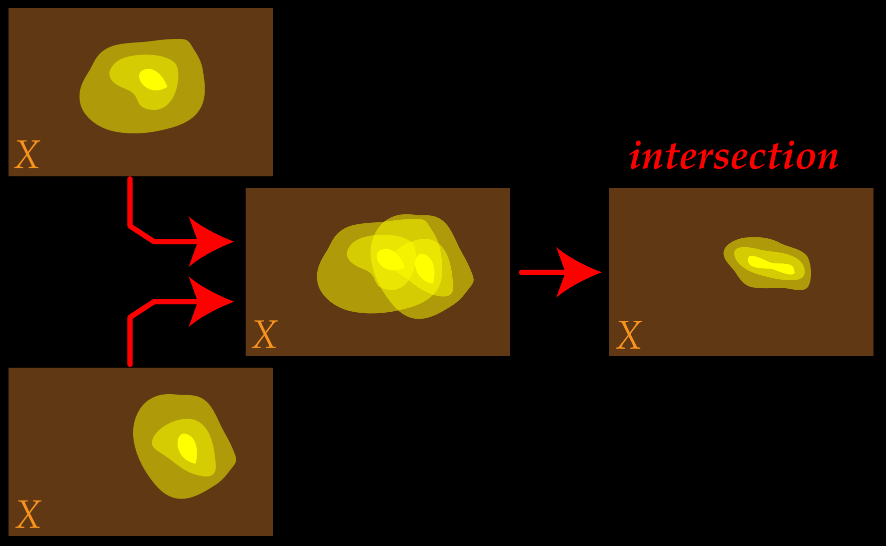

X
Y
X
x
$x \to \theta_x(y)$
Y

Intersection of measures: Given a measure space $(\Omega, \mathcal{F}, \mu)$, let μ_1 and μ_2 be two measures such that, for every $F \in \mathcal{F}$, the following expressions make sense:

$$\mu[\mu_1, \mu_2][F] = \int_{\omega \in F} \frac{d\mu_1}{d\mu}(\omega) \frac{d\mu_2}{d\mu}(\omega) \, d\mu(\omega)$$

$$(\mu_1 \cap \mu_2)[F] = \frac{\mu[\mu_1, \mu_2][F]}{\mu[\mu_1, \mu_2][\Omega]} \quad .$$

The probability measure $(\mu_1 \cap \mu_2)$ is the *intersection* of the two measures μ_1 and μ_2. The quintuplet $\{\Omega, \mathcal{F}, \mu, \mu_1, \mu_2\}$ is a finite *Radon-Nikodym space*.

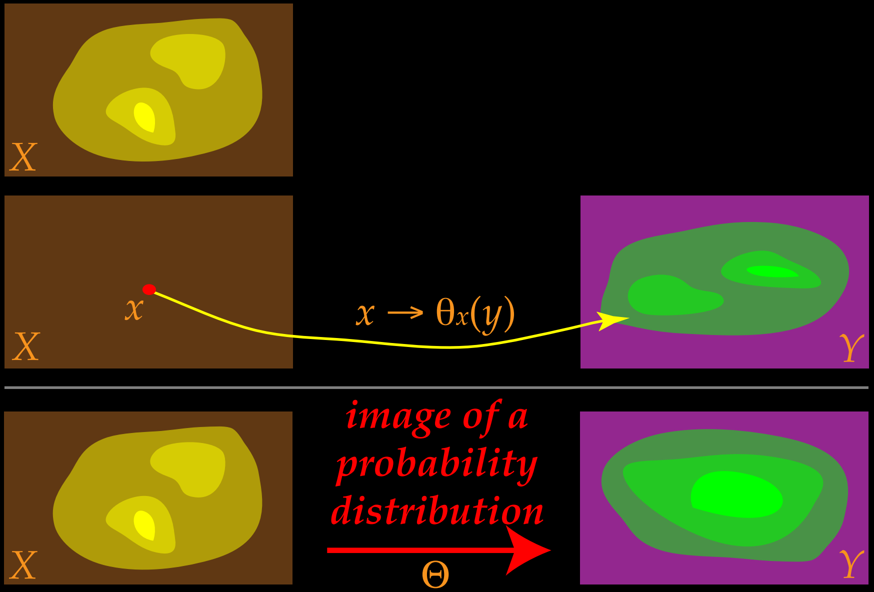


X
X
X
intersection
X

Family of probability measures: Let $(X, \mathcal{F}_X)$ and $(Y, \mathcal{F}_Y)$ be two measurable spaces, and assume that to every $x \in X$ it is associated a probability measure θ_x on $(Y, \mathcal{F}_Y)$. We use the notation Θ for the set $\{\theta_x \mid x \in X\}$, and we say that Θ is a *family* of probability measures from X on $(Y, \mathcal{F}_Y)$.

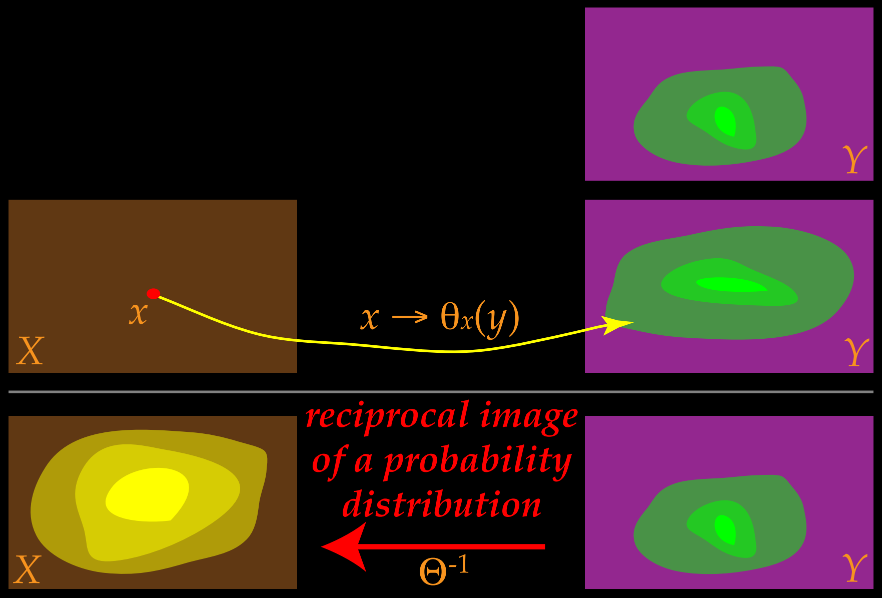
Image: Let $(X, \mathcal{F}_X)$ and $(Y, \mathcal{F}_Y)$ be two measurable spaces, $\Theta = \{\theta_x \mid x \in X\}$ a family of probability measures from X on $(Y, \mathcal{F}_Y)$, and π_X a measure on $(X, \mathcal{F}_X)$. If for any $F_Y \in \mathcal{F}_Y$, the function $\theta_x[F_Y]$ is π_X-measurable, the *image* of π_X (by the family Θ) is the measure on $(Y, \mathcal{F}_Y)$, denoted $\Theta[\pi_X]$, defined by the condition

$$(\Theta[\pi_X])[F_Y] = \int_X \theta_x[F_Y]\, d\pi_X(x) \qquad \text{for every } F_Y \in \mathcal{F}_Y \quad .$$


X
X
x
x → θx(y)
Y
image of a probability distribution
X
Θ
Y

Reciprocal image: Let $(X, \mathcal{F}_X, \mu_X)$ and $(Y, \mathcal{F}_Y, \mu_Y)$ be two measure spaces, π_Y a measure on $(Y, \mathcal{F}_Y)$, and $\Theta = \{\theta_x \mid x \in X\}$ a family of probability measures from X on $(Y, \mathcal{F}_Y)$ such that for every $x \in X$, $(Y, \mathcal{F}_Y, \mu_Y, \theta_x, \pi_Y)$ is a Radon-Nikodym space. The *reciprocal image* of π_Y (by the family Θ), denoted $\Theta^{-1}[\pi_Y]$, is the measure on $(X, \mathcal{F}_X)$, absolutely continuous w.r.t. μ_X, defined, via its μ_X-density, as

$$\frac{d(\Theta^{-1}[\pi_Y])}{d\mu_X}(x) = \int_Y (\theta_x \cap \pi_Y)(y) \; d\mu_Y(y) \quad .$$



Y
x
X
x → θx(y)
Y
X
reciprocal image
of a probability
distribution
Θ-1
Y

Product of measures: Given two measurable spaces $(X, \mathcal{F}_X)$ and $(Y, \mathcal{F}_Y)$, to every pair of measures τ_X and τ_Y, respectively on $(X, \mathcal{F}_X)$ and on $(Y, \mathcal{F}_Y)$, is associated a measure $\tau_X \times \tau_Y$ on $(X \times Y, \mathcal{F}_X \times \mathcal{F}_Y)$, called the *product measure*, defined for every $F_X \in \mathcal{F}_X$ and every $F_Y \in \mathcal{F}_Y$, by

$$(\tau_X \times \tau_Y)[F_X \times F_Y] = \int_{F_X} d\tau_X(x) \int_{F_Y} d\tau_Y(y) \quad .$$

Marginal measures: Given two measurable spaces $(X, \mathcal{F}_X)$ and $(Y, \mathcal{F}_Y)$, to every measure π on $(X \times Y, \mathcal{F}_X \times \mathcal{F}_Y)$ are associated the two *marginal measures* (respectively on $(X, \mathcal{F}_X)$ and on $(Y, \mathcal{F}_Y)$) defined (respectively for every $F_X \in \mathcal{F}_X$ and every $F_Y \in \mathcal{F}_Y$) by

$$\pi_X[F_X] = \pi[F_X \times Y] \quad ; \quad \pi_Y[F_Y] = \pi[X \times F_Y] \quad .$$

Inference space: Given two measure spaces $(X, \mathcal{F}_X, \mu_X)$ and $(Y, \mathcal{F}_Y, \mu_Y)$, given a family $\Theta = \{\theta_x \mid x \in X\}$ of probability measures on $(Y, \mathcal{F}_Y)$, and given a probability measure π on $(X \times Y, \mathcal{F}_X \times \mathcal{F}_Y)$, an *inference space* is the quintuplet

$$\mathcal{I} = \{ X \times Y , \mathcal{F}_X \times \mathcal{F}_Y , \mu_X \times \mu_Y , \pi , \Theta \} \quad .$$

Premise inference space: If an inference space $\mathcal{I} = \{ X \times Y , \mathcal{F}_X \times \mathcal{F}_Y , \mu_X \times \mu_Y , \pi , \Theta \}$ is such that

$$\pi = \pi_X \times \pi_Y \quad ,$$

where π_X and π_Y are the two marginal probability measures of the probability measure π, if the intersections $\theta_x \cap \pi_Y$ are nonempty (for all $x \in X$), we say that $\mathcal{I}$ is a *premise inference space*.

Conclusion inference space: If an inference space $\mathcal{I} = \{ X \times Y , \mathcal{F}_X \times \mathcal{F}_Y , \mu_X \times \mu_Y , \pi , \Theta \}$ is such that

$$\Theta[\pi_X] = \pi_Y \quad ,$$

where π_X and π_Y are the two marginal probability measures of the probability measure π, we say that $\mathcal{I}$ is a *conclusion inference space*.

Theorem: If $\mathcal{I} = \{ X \times Y, \mathcal{F}_X \times \mathcal{F}_Y, \mu_X \times \mu_Y, \pi, \Theta \}$ is a premise inference space, then the quintuplet $\widetilde{\mathcal{I}} = \{ X \times Y, \mathcal{F}_X \times \mathcal{F}_Y, \mu_X \times \mu_Y, \widetilde{\pi}, \widetilde{\Theta} \}$ where the probability measures of the family $\widetilde{\Theta} = \{\widetilde{\theta}_x \,|\, x \in X\}$ are defined as

$$\widetilde{\theta}_x = \theta_x \cap \pi_Y$$

and where the probability measure $\widetilde{\pi}$ is defined by

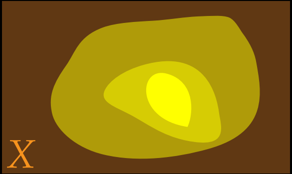
$$\widetilde{\pi}[F_X \times F_Y] = \int_{F_X} \int_{F_Y} \widetilde{\theta}_x(y)\, d\mu_Y(y)\, d\pi_X(x) \quad ,$$

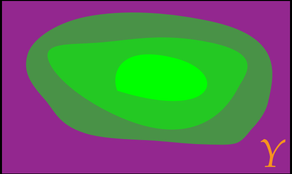
is a conclusion inference space, i.e. the two marginal probability measures of $\widetilde{\pi}$, say $\widetilde{\pi}_X$ and $\widetilde{\pi}_Y$, are related as

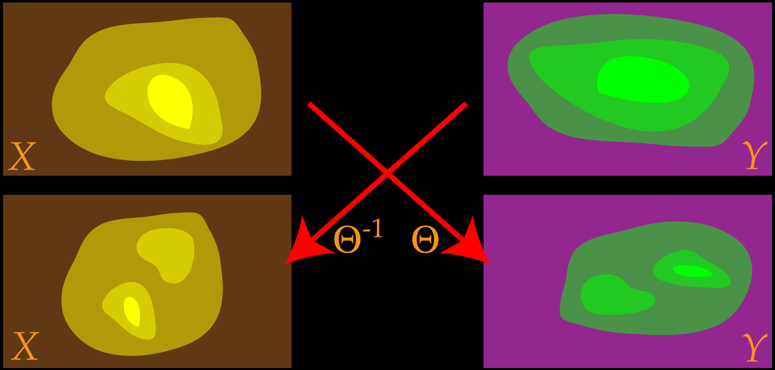
$$\widetilde{\Theta}[\widetilde{\pi}_X] = \widetilde{\pi}_Y \quad .$$

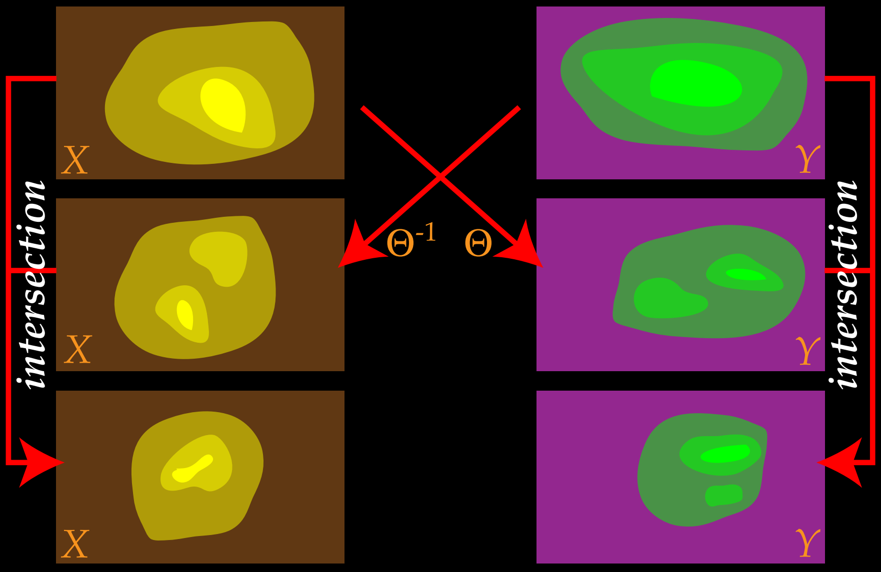
Furthermore, one has

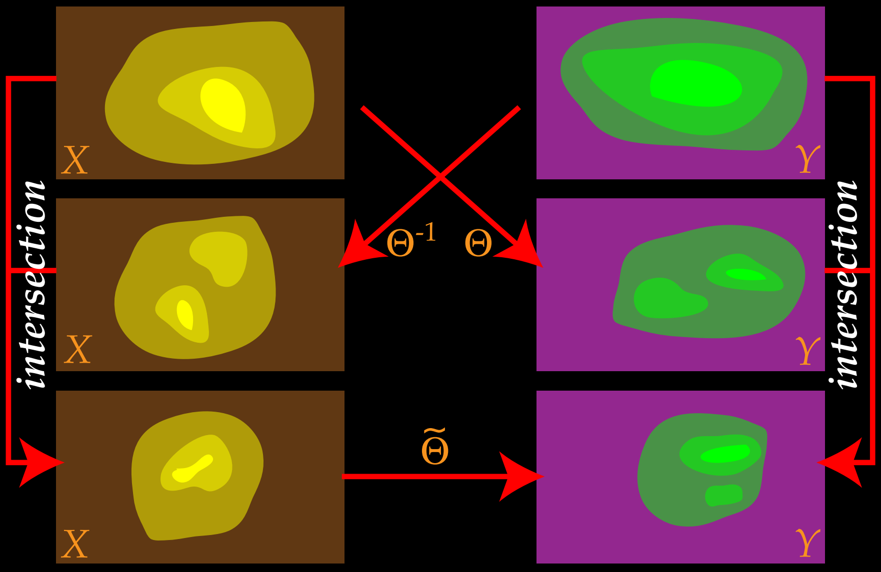
$$\widetilde{\pi}_X = \pi_X \cap \Theta^{-1}[\pi_Y] \qquad \text{and} \qquad \widetilde{\pi}_Y = \pi_Y \cap \Theta[\pi_X] \quad .$$


X

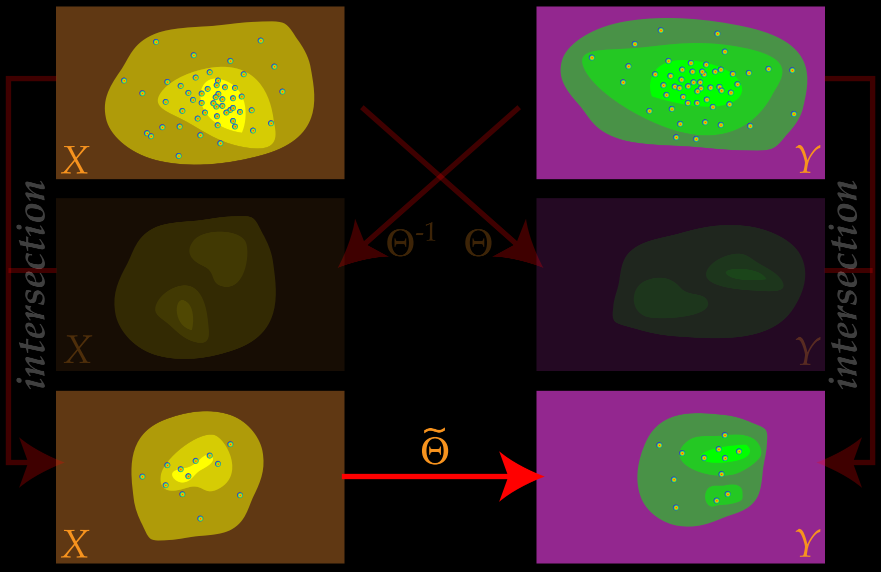

Y


X
Y
X
Y
Θ-1
Θ


X
X
X
Y
Y
Y
Θ-1
Θ
intersection
intersection



X
Y
X
Y
X
Y
Θ^{-1}
Θ
$\tilde{\Theta}$
intersection
intersection



X
Y
X
Y
X
Y
intersection
intersection
Θ^{-1}
Θ
$\tilde{\Theta}$

Theorem: If $\mathcal{I} = \{ X \times Y, \mathcal{F}_X \times \mathcal{F}_Y, \mu_X \times \mu_Y, \pi, \Theta \}$ is a premise inference space, then the quintuplet $\widetilde{\mathcal{I}} = \{ X \times Y, \mathcal{F}_X \times \mathcal{F}_Y, \mu_X \times \mu_Y, \widetilde{\pi}, \widetilde{\Theta} \}$ where the probability measures of the family $\widetilde{\Theta} = \{\widetilde{\theta}_x \,|\, x \in X\}$ are defined as

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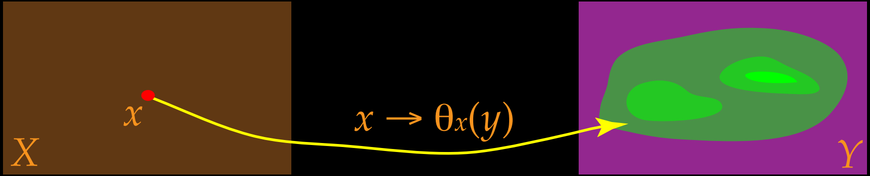
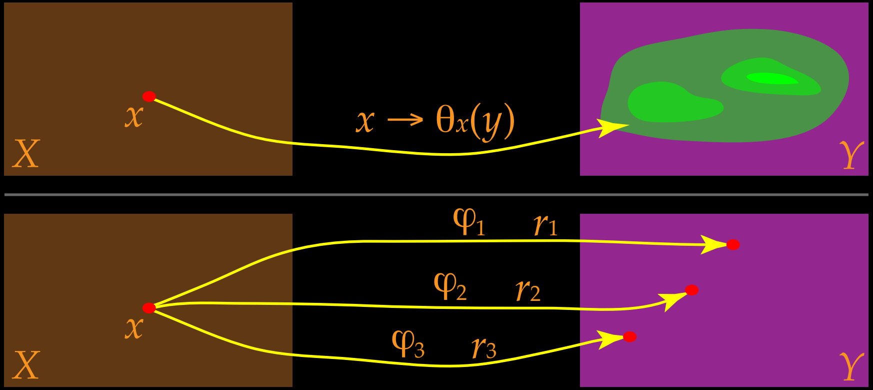
and where the probability measure $\widetilde{\pi}$ is defined by

$$\widetilde{\pi}[F_X \times F_Y] = \int_{F_X} \int_{F_Y} \widetilde{\theta}_x(y)\, d\mu_Y(y)\, d\pi_X(x) \quad ,$$

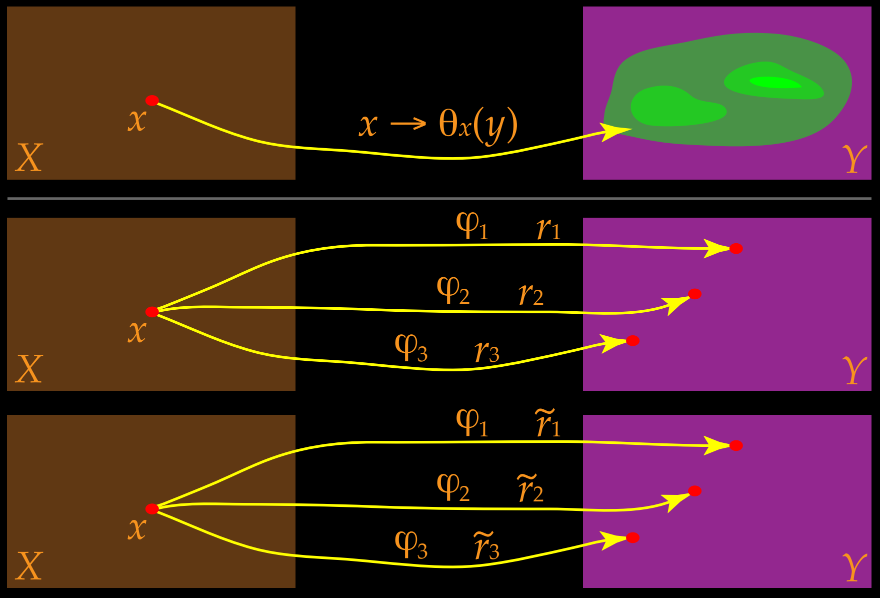
is a conclusion inference space, i.e. the two marginal probability measures of $\widetilde{\pi}$, say $\widetilde{\pi}_X$ and $\widetilde{\pi}_Y$, are related as

$$\widetilde{\Theta}[\widetilde{\pi}_X] = \widetilde{\pi}_Y \quad .$$


X
x
x → θx(y)
Y



x
X
x → θx(y)
Y
φ1
r1
φ2
r2
φ3
r3
x
X
Y



x
X
$x \to \theta_x(y)$
Y
φ_1 r_1
φ_2 r_2
φ_3 r_3
x
X
Y
φ_1 $\tilde{r}_1$
φ_2 $\tilde{r}_2$
φ_3 $\tilde{r}_3$
x
X
Y

THE END.