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K-functionals and multivariate Bernstein polynomials ☆

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Abstract

This paper estimates upper and lower bounds for the approximation rates of iterated Boolean sums of multivariate Bernstein polynomials. Both direct and inverse inequalities for the approximation rate are established in terms of a certain K-functional. From these estimates, one can also determine the class of functions yielding optimal approximations to the iterated Boolean sums.

Keywords: Multivariate Bernstein polynomials; Boolean sums; Direct theorems; Inverse theorems; K-functional; Approximation order

1. Introduction

Let $S \subset \mathbb{R}^d$ be the simplex defined by

$$S := \left\{ \mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : x_i \geq 0, i = 1, 2, \ldots, d, |\mathbf{x}| \leq 1 \right\}.$$

We denote $|\mathbf{x}| := \sum_{i=1}^{d} x_i$, $\mathbf{x}^{\mathbf{k}} := x_1^{k_1} x_2^{k_2} \cdots x_d^{k_d}$, $\mathbf{k}! := k_1! k_2! \cdots k_d!$ and $|\mathbf{k}| := \sum_{i=1}^{d} k_i$ for nonnegative integers k_i $(1 \leq i \leq d)$.

Let $\mathbf{e}_i := (0, 0, \ldots, 0, 1, 0, \ldots, 0)$, $1 \leq i \leq d$, denote the unit vectors in $\mathbb{R}^d$, i.e., its ith coordinate is 1 and all others are zero. V_S is understood to be the set of unit vectors in the

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directions of the edges of S (where $\mathbf{e}$ and $-\mathbf{e}$ are considered to be the same vector). For $\xi \in V_S$ and $\mathbf{x} \in S$, the step weight function $\varphi_\xi^2(\mathbf{x})$ introduced in [6, p. 274] is defined by

$$\varphi_\xi^2(\mathbf{x}) := \inf_{\mathbf{x}+\lambda\xi \notin S, \lambda>0} d(\mathbf{x}, \mathbf{x}+\lambda\xi) \inf_{\mathbf{x}-\lambda\xi \notin S, \lambda>0} d(\mathbf{x}, \mathbf{x}-\lambda\xi),$$

where $d(\mathbf{x}, \mathbf{y})$ is the Euclidean distance between $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^d$. Clearly,

$$\varphi_\xi^2(\mathbf{x}) = \begin{cases} x_i(1-|\mathbf{x}|), & \xi = \mathbf{e}_i, \quad 1 \le i \le d; \\ 2x_i x_j, & \xi = (\mathbf{e}_i - \mathbf{e}_j)/\sqrt{2}, \quad 1 \le i < j \le d. \end{cases}$$

The elliptic operators are now given by (see [9, p. 95])

$$P(D) := \sum_{\xi \in V_S} \varphi_\xi^2(\mathbf{x}) \left(\frac{\partial}{\partial \xi}\right)^2,$$

which can be rewritten as (see [9, p. 96])

$$P(D) = \sum_{i=1}^d x_i(1-|\mathbf{x}|) \left(\frac{\partial}{\partial x_i}\right)^2 + \sum_{1 \le i<j \le d} x_i x_j \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)^2.$$

For $f \in C(S)$, the K-functional which will be used in what follows is defined by

$$K_r(f, t^{2r}) := \inf_{g \in C^{2r}(S)} \left\{ \|f-g\| + t^{2r} \left\|P^r(D)g\right\| \right\}$$

(see also [23, p. 13]) where $0 < t < t_0$.

The Bernstein polynomials of degree n associated with $f \in C(S)$ are defined by

$$\mathcal{B}_n f := \mathcal{B}_{n,d}(f, \mathbf{x}) := \sum_{|\mathbf{k}| \le n} P_{n,\mathbf{k}}(\mathbf{x}) f\left(\frac{\mathbf{k}}{n}\right),$$

where

$$P_{n,\mathbf{k}}(\mathbf{x}) := \frac{n!}{\mathbf{k}!(n-|\mathbf{k}|)!} \mathbf{x}^{\mathbf{k}} (1-|\mathbf{x}|)^{n-|\mathbf{k}|}, \quad n \in \mathbb{N}, \mathbf{x} \in S.$$

For the Bernstein polynomials $\mathcal{B}_n$, the iterated operators are given by

$$\mathcal{B}_n^r := \mathcal{B}_n \mathcal{B}_n^{r-1}, \quad r = 2, 3, \ldots,$$

and the Boolean sums of the operators are defined by

$$\mathcal{B}_n \oplus \mathcal{B}_n := 2\mathcal{B}_n - \mathcal{B}_n^2.$$

Then the iterated Boolean sums of the Bernstein polynomials are defined by

$$\oplus^0 \mathcal{B}_n := \mathcal{I}, \text{ where } \mathcal{I} \text{ is the identity operator}$$

and

$$\oplus^1 \mathcal{B}_n := \mathcal{B}_n, \qquad \oplus^{r+1} \mathcal{B}_n := \mathcal{B}_n \oplus (\oplus^r \mathcal{B}_n), \quad r \in \mathbb{N}.$$

It is clear that the iterated Boolean sum $\oplus^r \mathcal{B}_n$ can be rewritten as

$$\oplus^r \mathcal{B}_n = \mathcal{I} - (\mathcal{I} - \mathcal{B}_n)^r = \sum_{i=1}^r \binom{r}{i} (-1)^{i+1} {\mathcal{B}_n}^i.$$

Ever since Lorentz [14] first introduced the concept in 1953, multivariate Bernstein polynomials have been studied extensively. In particular, the relationship between their rate of convergence and the smoothness of the functions they approximate has been revealed in many papers (e.g. [2,4,5,9,21–23,3]). In the present paper, we deal with corresponding questions regarding the iterated Boolean sums of multivariate Bernstein polynomials. This topic has recently attracted some interest; some relevant works are mentioned below.

In 1973, Micchelli [15] introduced certain linear combinations of iterated univariate Bernstein polynomials. These can be regarded as iterated Boolean sums $\oplus^r \mathcal{B}_n$ $(r \geq 0)$ (e.g., this was shown in [12]). Iterated Boolean sums have also been investigated in [1,11,17,16,20,12]. Several direct, saturation and Voronovskaja-type theorems were given in these papers. The approximation behavior of such sums was described by Gonska and Zhou [12], who generalized the direct theorem of Micchelli (see Theorem 4.4 of [15]). They were also able to improve on Micchelli's Theorem 4.5, providing a more elegant version of the saturation statement and adding the appropriate inverse theorem. Finally, they obtained an o-saturation class. A crucial tool used in [12] is the so-called Ditzian–Totik modulus [8]. In addition to this important work, we note that several strong inverse inequalities for univariate Bernstein polynomials have been established by Ditzian and Ivanov [7], Totik [19] and Knoop and Zhou [13].

Following some of Gonska and Zhou's ideas [12], this paper studies direct and inverse theorems for iterated Boolean sums of the multivariate Bernstein polynomials $\oplus^r \mathcal{B}_n (d > 1)$. We shall mainly estimate upper and lower bounds on the approximation rate by using an appropriate K-functional. Two inequalities for the approximation rate will be established: one direct, and the other strong converse. In brief, we will prove the following results.

Theorem 1.1. *Let $r \in \mathbb{N}$ be fixed. Then, for $f \in C(S)$, there holds*

$$\left\| f - \oplus^r \mathcal{B}_n f \right\| \leq M K_r \left(f, n^{-r}\right). \tag{1.1}$$

Conversely,

$$K_r \left(f, n^{-r}\right) \leq M \max_{k \geq n} \left\| f - \oplus^r \mathcal{B}_k f \right\|. \tag{1.2}$$

In particular, for $0 < \alpha \leq r$, there holds

$$\left\| f - \oplus^r \mathcal{B}_n f \right\| = \mathcal{O}\left(\frac{1}{n^\alpha}\right) \Leftrightarrow K_r (f, t) = \mathcal{O}\left(t^{\alpha/r}\right). \tag{1.3}$$

Hereafter, we shall always use M and M_i $(i = 1, 2, \ldots)$ to denote positive constants independent of n and f.

2. Some lemmas

In this section, we give some of the lemmas supporting our main results.

Lemma 2.1 (*See [6, p. 275]*). *Let Π_n denote the set of all polynomials of total degree smaller than n. Take $P_m \in \Pi_m$. Then for $\xi \in V_S$, there holds*

$$\left\| \varphi_\xi^r \left(\frac{\partial}{\partial \xi}\right)^r P_m \right\| \leq M m^r \|P_m\|, \quad r \in \mathbb{N}.$$

Similarly to Theorem 5.1 in [9], from here it is not difficult to prove the following Voronovskaja-type result.

Lemma 2.2. *For $P_m \in \Pi_m$, $m \le \sqrt{n}$ and $r \in \mathbb{N}$, we have*

$$\|P_m - \mathcal{B}_n P_m\| \le M \frac{m^2}{n} \|P_m\|. \tag{2.1}$$

Now we can prove a strong Voronovskaja-type estimate:

Lemma 2.3. *If $P_m \in \Pi_m$, $m \le \sqrt{n}$ and $r \in \mathbb{N}$, then*

$$\left\| P_m - \oplus^r \mathcal{B}_n P_m - \left(-\frac{1}{2n}\right)^r P^r(D) P_m \right\| \le M \frac{m^{2r+2}}{n^{r+1}} \|P_m\|.$$

Proof. We can use induction to prove Lemma 2.3. For $r = 1$, Lemma 2.3 has already been proved by Theorem 5.1 of [9]. Suppose that Lemma 2.3 is valid for $r = k$ $(k \ge 1)$. For $r = k + 1$, we see that

$$\begin{aligned}
&\left\| P_m - \oplus^{k+1} \mathcal{B}_n P_m - \left(-\frac{1}{2n}\right)^{k+1} P^{k+1}(D) P_m \right\| \\
&\quad = \left\| (\mathcal{I} - \mathcal{B}_n)^{k+1} P_m - \left(-\frac{1}{2n}\right)^{k+1} P^{k+1}(D) P_m \right\| \\
&\quad \le \left\| (\mathcal{I} - \mathcal{B}_n)^k (\mathcal{I} - \mathcal{B}_n) P_m - \left(-\frac{1}{2n}\right)^k P^k(D)(\mathcal{I} - \mathcal{B}_n) P_m \right\| \\
&\qquad + \left\| \left(-\frac{1}{2n}\right)^{k+1} P^{k+1}(D) P_m - \left(-\frac{1}{2n}\right)^k P^k(D)(\mathcal{I} - \mathcal{B}_n) P_m \right\| \\
&\quad := \Sigma_1 + \Sigma_2.
\end{aligned}$$

Setting

$$Q_m := (\mathcal{I} - \mathcal{B}_n) P_m - \frac{1}{2n} P(D) P_m \in \Pi_m,$$

gives

$$\begin{aligned}
\Sigma_2 &\le \left(\frac{1}{2n}\right)^k \left\| P^k(D) \left((\mathcal{I} - \mathcal{B}_n) P_m - \frac{1}{2n} P(D) P_m \right) \right\| \\
&\le n^{-k} \left\| P^k(D) Q_m \right\| \le M_1 n^{-k} m^{2k} \|Q_m\| \\
&\le M_2 n^{-k-2} m^{2(k+1)+2} \|P_m\|,
\end{aligned}$$

where we have applied the $r = 1$ result in the final step. In moving from the second step to the third step, we carried out another induction in k and then applied Lemma 2.1.

For Σ_1, we note $(\mathcal{I} - \mathcal{B}_n) P_m \in \Pi_m$ and obtain

$$\Sigma_1 \le M n^{-k-1} m^{2k+2} \|(\mathcal{I} - \mathcal{B}_n) P_m\|.$$

Therefore, using (2.1) we conclude the proof of Lemma 2.3. $\square$

Lemma 2.4. *For $f \in C(S)$ and $P_m \in \Pi_m$ satisfying $\|P_m - f\| \leq ME_m(f)$, given $m = [\sqrt{n}]$, we have*

$$\left\| P_m - \oplus^r \mathcal{B}_n P_m - \left(-\frac{1}{2n}\right)^r P^r(D) P_m \right\| \leq MK_r\left(f, n^{-r}\right) \tag{2.2}$$

and

$$\left\| P_m - \oplus^r \mathcal{B}_n P_m - \left(-\frac{1}{2n}\right)^r P^r(D) P_m \right\| \leq MK^*_{2r+1}\left(f, n^{-(2r+1)/2}\right), \tag{2.3}$$

where $E_m(f) := \inf_{P_m \in \Pi_m} \|f - P_m\|$ is the best polynomial approximation of f and $K^*_r(f,t)$ is a K-functional defined by (see [5])

$$K^*_r(f, t^r) := \inf_{g \in C^r(s)} \left\{ \|f - g\| + t^r \sup_{\xi \in V_S} \left\| \varphi^r_\xi \left(\frac{\partial}{\partial \xi}\right)^r g \right\| \right\}.$$

Proof. We start by choosing a $P_j \in \Pi_j$ satisfying $\|P_j - f\| \leq ME_j(f)$, and expand P_m as follows:

$$P_m = P_m - P_{2^l} + \sum_{j=1}^{l} (P_{2^j} - P_{2^{j-1}}) + P_1, \quad l = \max\left\{ j : 2^j < m \right\}.$$

We then get

$$P_1 - \oplus^r \mathcal{B}_n P_1 = 0 = P^r(D) P_1, \quad r \geq 1$$

and utilize Lemma 2.3 to write down the following for $m = \sqrt{n}$:

$$\begin{aligned}
I(n) &:= \left\| P_m - \oplus^r \mathcal{B}_n P_m - \left(-\frac{1}{2n}\right)^r P^r(D) P_m \right\| \\
&\leq \left\| (\mathcal{I} - \mathcal{B}_n)^r (P_m - P_{2^l}) - \left(-\frac{1}{2n}\right)^r P^r(D)(P_m - P_{2^l}) \right\| \\
&\quad + \sum_{j=1}^{l} \left\| (\mathcal{I} - \mathcal{B}_n)^r (P_{2^j} - P_{2^{j-1}}) - \left(-\frac{1}{2n}\right)^r P^r(D)(P_{2^j} - P_{2^{j-1}}) \right\| \\
&\leq M_3 \left(\frac{m^{2r+2}}{n^{r+1}} \|P_m - P_{2^l}\| + \sum_{j=1}^{l} \frac{2^{j(2r+2)}}{n^{r+1}} \|P_{2^j} - P_{2^{j-1}}\| \right) \\
&\leq M_4 n^{-r-1} \left(m^{2r+2} E_{2^l}(f) + \sum_{j=1}^{l} 2^{j(2r+2)} E_{2^{j-1}}(f) \right) \\
&= M_4 n^{-r-1} \left(m^{2r+2} E_{2^l}(f) + \sum_{j=0}^{l-1} 2^{(j+1)(2r+2)} E_{2^j}(f) \right) \\
&\leq M_5 n^{-r-1} \sum_{j=0}^{l} 2^{(j+1)(2r+2)} E_{2^j}(f).
\end{aligned}$$

On the other hand, by applying Theorem 1.1 of [10] and (3.10) of [9] we can also write

$$E_n(f) \leq MK^*_p(f, A/n^p), \quad \text{where } A \text{ is a fixed constant.} \tag{2.4}$$

Therefore, setting $p = 2r + 1$ gives

$$
\begin{aligned}
I(n) &\le M_6 n^{-r-1} \sum_{j=0}^{l} 2^{(j+1)(2r+2)} K_{2r+1}^{*}\left(f, A2^{-j(2r+1)}\right) \\
&\le (A+1) M_6 n^{-r-1} \sum_{j=0}^{l} 2^{(j+1)(2r+2)} 2^{(l+1-j)(2r+1)} K_{2r+1}^{*}\left(f, 2^{-(l+1)(2r+1)}\right) \\
&= (A+1) M_6 2^{4r+3} n^{-r-1} 2^{(2r+1)l} K_{2r+1}^{*}\left(f, 2^{-(l+1)(2r+1)}\right) \sum_{j=0}^{l} 2^{j} \\
&\le M_7 n^{-r-1} 2^{(2r+2)l} K_{2r+1}^{*}\left(f, 2^{-(l+1)(2r+1)}\right) \\
&\le M_8 n^{-r-1} m^{2r+2} K_{2r+1}^{*}\left(f, m^{-(2r+1)}\right) \le M_9 K_{2r+1}^{*}\left(f, n^{-(2r+1)/2}\right).
\end{aligned}
$$

Here we used the facts $m = [\sqrt{n}]$ and $2^l \le m \le 2^{l+1}$. This completes the proof of (2.3).

To prove (2.2), we recall Chapter of [8]. This can be used to get (see [9])

$$E_n(f) \le M_{19} K_r\left(f, n^{-2r}\right). \tag{2.5}$$

From this inequality one can easily deduce (2.2) following the same method used to prove (2.3). The proof of Lemma 2.4 is complete. □

3. Proof of main results

Let P_m be the best approximation polynomial of f, and let $m = [\sqrt{n}]$. From (2.2), (2.5), and the fact that

$$\left\| P^r(D) P_m \right\| \le M m^{2r} K_r\left(f, m^{-2r}\right)$$

(see Theorem 2.7 of [23]), it follows that

$$
\begin{aligned}
\left\| f - \oplus^r \mathcal{B}_n f \right\| &\le \left\| (\mathcal{I} - \mathcal{B}_n)^r P_m - \left(-\frac{1}{2n}\right)^r P^r(D) P_m \right\| \\
&\quad + \left\| \left(-\frac{1}{2n}\right)^r P^r(D) P_m \right\| + \left\| (\mathcal{I} - \mathcal{B}_n)^r (f - P_m) \right\| \\
&\le M_{10} K_r\left(f, n^{-r}\right) + M_{11} \frac{m^{2r}}{(2n)^r} K_r\left(f, m^{-2r}\right) + 2^r \| f - P_m \| \\
&\le M_{12} K_r\left(f, n^{-r}\right).
\end{aligned}
$$

This completes the upper estimate (1.1).

Now we shall prove the lower estimate (1.2). Let us define $\mathcal{T}_n = \oplus^r \mathcal{B}_n$. Then

$$\mathcal{T}_n = \sum_{i=1}^{r} (-1)^{i+1} \binom{r}{i} \mathcal{B}_n^i$$

and

$$E_n(f) \le \| f - \mathcal{T}_n f \| = \left\| (\mathcal{I} - \mathcal{B}_n)^r f \right\|. \tag{3.1}$$

The simple fact that $\|B_n f\| \le \|f\|$ yields

$$\|\mathcal{T}_n f\| \le \sum_{i=1}^{r} \binom{r}{i} \|f\| \le 2^r \|f\|.$$

Recalling Theorem 4.1 of [9], we know that for $\xi \in V_S$, $v = 0, 1$ and $r = 0, 1, 2 \ldots$, the following relation holds:

$$\left\| \varphi_\xi^{r+v} \left(\frac{\partial}{\partial \xi}\right)^{r+v} \mathcal{B}_n f \right\| \le M n^{v/2} \left\| \varphi_\xi^{r} \left(\frac{\partial}{\partial \xi}\right)^{r} f \right\|.$$

Setting $v = 0$, this implies that for $f \in C^r(S)$

$$\left\| \varphi_\xi^{r} \left(\frac{\partial}{\partial \xi}\right)^{r} \mathcal{B}_n f \right\| \le M \left\| \varphi_\xi^{r} \left(\frac{\partial}{\partial \xi}\right)^{r} f \right\|.$$

Therefore,

$$\left\| \varphi_\xi^{r} \left(\frac{\partial}{\partial \xi}\right)^{r} \mathcal{T}_n f \right\| \le M \left\| \varphi_\xi^{r} \left(\frac{\partial}{\partial \xi}\right)^{r} f \right\|. \tag{3.2}$$

Using the inequality

$$\left\| \varphi_\xi^{2r+2} \left(\frac{\partial}{\partial \xi}\right)^{2r+2} \mathcal{B}_n f \right\| \le M n^{r+1} \|f\|, \quad f \in C(S)$$

(see [3, p. 244]), we obtain for $f \in C(S)$

$$\left\| \varphi_\xi^{2r+2} \left(\frac{\partial}{\partial \xi}\right)^{2r+2} \mathcal{T}_n f \right\| \le M n^{r+1} \|f\|. \tag{3.3}$$

Combining this result with (3.1)–(3.3), we have for any $g \in C^{2r+2}(S)$

$$\begin{aligned}
K^*_{2r+2}\left(f, t^{2r+2}\right) &\le \|f - \mathcal{T}_k f\| + t^{2r+2} \sup_{\xi \in V_S} \left\| \varphi_\xi^{2r+2} \left(\frac{\partial}{\partial \xi}\right)^{2r+2} \mathcal{T}_k f \right\| \\
&\le M_{13} \left(\left\| (\mathcal{I} - \mathcal{B}_k)^r f \right\| + t^{2r+2} \sup_{\xi \in V_S} \left\| \varphi_\xi^{2r+2} \left(\frac{\partial}{\partial \xi}\right)^{2r+2} \mathcal{T}_k (f - g) \right\| \right) \\
&\quad + t^{2r+2} \sup_{\xi \in V_S} \left\| \varphi_\xi^{2r+2} \left(\frac{\partial}{\partial \xi}\right)^{2r+2} \mathcal{T}_k g \right\| \\
&\le M_{14} \left(\left\| (\mathcal{I} - \mathcal{B}_k)^r f \right\| + t^{2r+2} k^{r+1} \left(\|f - g\| + k^{-r-1} \sup_{\xi \in V_S} \left\| \varphi_\xi^{2r+2} \left(\frac{\partial}{\partial \xi}\right)^{2r+2} g \right\| \right) \right).
\end{aligned}$$

Taking the infimum on both sides of the above inequality for g gives

$$K^*_{2r+2}\left(f, t^{2r+2}\right) \le M \left(\left\| (\mathcal{I} - \mathcal{B}_k)^r f \right\| + t^{2r+2} k^{r+1} K^*_{2r+2}\left(f, k^{-(r+1)/2}\right) \right). \tag{3.4}$$

As for inequality (3.4), we use Totik's technique (see [18, p. 469] or Theorem 9.3.6 of [8]) to obtain

$$K^*_{2r+2}\left(f, t^{2r+2}\right) \le Mt^{\rho}\left(\sum_{1\le k\le t^{-2}} k^{\rho/2-1}\left\|(\mathcal{I}-\mathcal{B}_k)^r f\right\| + \|f\|\right) \tag{3.5}$$

where $\rho \in (0, 2r+2)$ is arbitrary. We can therefore use a Marchaud-type estimate (see (3.16) of [9]),

$$K^*_r(f, t^r) \le M\left(t^r \sum_{1\le k\le t^{-1}} k^{r-1} K^*_{r+1}(f, k^{-r-1}) + t^r\|f\|\right),$$

and connect with (3.5) to get

$$\begin{aligned} K^*_{2r+1}\left(f, t^{2r+1}\right) &\le M_{15}t^{2r+1}\left(\sum_{1\le k\le t^{-1}} k^{2r-\rho}\sum_{1\le l\le k^2} l^{\rho/2-1}\left\|(\mathcal{I}-\mathcal{B}_l)^r f\right\| + \|f\|\right)\\ &\le M_{16}t^{2r+1}\left(\sum_{1\le l\le t^{-2}} l^{\rho/2-1}\left\|(\mathcal{I}-\mathcal{B}_l)^r f\right\|\sum_{k\ge\sqrt{l}} k^{2r-\rho} + \|f\|\right). \end{aligned}$$

Hence, letting $2r+1 < \rho < 2r+2$ gives

$$K^*_{2r+1}(f, t^{2r+1}) \le Mt^{2r+1}\left(\sum_{1\le l\le t^{-2}} l^{r-1/2}\left\|(\mathcal{I}-\mathcal{B}_l)^r f\right\| + \|f\|\right). \tag{3.6}$$

On the other hand, for any given n we can choose an $n_0 \in \mathbb{N}$ satisfying $n/2 \le n_0 \le n$ such that

$$\left\|(\mathcal{I}-\mathcal{B}_{n_0})^r f\right\| = \min_{n/2\le k\le n}\left\|(\mathcal{I}-\mathcal{B}_k)^r f\right\|.$$

This implies that

$$\left\|(\mathcal{I}-\mathcal{B}_{n_0})^r f\right\| \le \frac{2}{n}\sum_{l=n/2}^{n}\left\|(\mathcal{I}-\mathcal{B}_l)^r f\right\| \le Mn^{-r-1/2}\sum_{l=1}^{n} l^{r-1/2}\left\|(\mathcal{I}-\mathcal{B}_l)^r f\right\|. \tag{3.7}$$

Moreover, for $P_m \in \Pi_m$ satisfying $\|P_m - f\| = E_m(f)$ and $m = [\sqrt{n_0}]$, we have

$$K_r(f, m^{-2r}) \le \|f - P_m\| + m^{-2r}\|P^r(D)P_m\|. \tag{3.8}$$

Recalling (2.3), we see that

$$\left(\frac{1}{2n_0}\right)^r\left\|P^r(D)P_m\right\| \le M\left(K^*_{2r+1}\left(f, n_0^{-(2r+1)/2}\right) + \left\|(\mathcal{I}-\mathcal{B}_{n_0})^r P_m\right\|\right). \tag{3.9}$$

We can now estimate $\left\|(\mathcal{I}-\mathcal{B}_{n_0})^r P_m\right\|$ by combining the inequality

$$\begin{aligned} \left\|(\mathcal{I}-\mathcal{B}_{n_0})^r P_m\right\| &\le \left\|(\mathcal{I}-\mathcal{B}_{n_0})^r f\right\| + \left\|(\mathcal{I}-\mathcal{B}_{n_0})^r(P_m - f)\right\|\\ &\le \left\|(\mathcal{I}-\mathcal{B}_{n_0})^r f\right\| + ME_m(f). \end{aligned}$$

with (3.7) and (3.8). After some manipulation, the result is

$$\begin{aligned}
K_r\left(f, n^{-r}\right) &\le K_r\left(f, n_0^{-r}\right) \le K_r\left(f, m^{-2r}\right)\\
&\le E_m(f) + M_{17}\frac{(2n_0)^r}{m^{2r}}\left(K^*_{2r+1}\left(f, n_0^{-(2r+1)/2}\right) + \left\|(\mathcal{I}-\mathcal{B}_{n_0})^r P_m\right\|\right)\\
&\le M_{18}K^*_{2r+1}\left(f, Am^{-(2r+1)}\right) + M_{19}K^*_{2r+1}\left(f, n^{-(2r+1)/2}\right)\\
&\quad + M_{20}\left\|(\mathcal{I}-\mathcal{B}_{n_0})^r f\right\|\\
&\le M_{21}\left(K^*_{2r+1}\left(f, n^{-(2r+1)/2}\right) + \left\|(\mathcal{I}-\mathcal{B}_{n_0})^r f\right\|\right),
\end{aligned}$$

Here we also used (2.4). Hence, putting $t = n^{-1/2}$ in (3.6) and combining the result with (3.7) yields

$$K_r(f, n^{-r}) \le M\left(n^{-r-1/2}\sum_{l=1}^{n} l^{r-1/2}\left\|(\mathcal{I}-\mathcal{B}_l)^r f\right\| + n^{-r-1/2}\|f\|\right). \tag{3.10}$$

Since

$$K_r(f-P_1, t^{2r}) = K_r(f, t^{2r}), \quad (\mathcal{I}-\mathcal{B}_l)^r(f-P_1) = (\mathcal{I}-\mathcal{B}_l)^r f$$

and $E_1(f) \le \|(\mathcal{I}-\mathcal{B}_1)^r f\|$, the term $n^{-r-1/2}\|f\|$ on the right-hand side of (3.10) can be omitted. Hence,

$$K_r(f, n^{-r}) \le Mn^{-r-1/2}\sum_{l=1}^{n} l^{r-1/2}\left\|(\mathcal{I}-\mathcal{B}_l)^r f\right\|. \tag{3.11}$$

In order to finish our proof of (1.2), we need to show that

$$K_r(f, n^{-r}) \approx \frac{1}{n^r}\max_{1\le k\le n} k^r\left\|(\mathcal{I}-\mathcal{B}_k)^r f\right\| \approx \frac{1}{n^{r+1/4}}\max_{1\le k\le n} k^{r+1/4}\left\|(\mathcal{I}-\mathcal{B}_k)^r f\right\|. \tag{3.12}$$

The notation $a \approx b$ means that there exists a positive constant c such that $c^{-1}b \le a \le cb$.

We first prove

$$K_r(f, n^{-r}) \approx n^{-r}\max_{1\le k\le n} k^r\left\|(\mathcal{I}-\mathcal{B}_k)^r f\right\|.$$

From (3.11) and (1.1), it follows that

$$\begin{aligned}
K_r(f, n^{-r}) &\le M_{22}n^{-r-1/2}\sum_{k=1}^{n} k^{-1/2}k^r\left\|(\mathcal{I}-\mathcal{B}_k)^r f\right\|\\
&\le M_{22}n^{-r-1/2}\max_{1\le k\le n} k^r\left\|(\mathcal{I}-\mathcal{B}_k)^r f\right\|\sum_{k=1}^{n} k^{-1/2}\\
&\le M_{23}n^{-r}\max_{1\le k\le n} k^r\left\|(\mathcal{I}-\mathcal{B}_k)^r f\right\| \le M_{24}n^{-r}\max_{1\le k\le n} k^r K_r(f, k^{-r})\\
&\le M_{24}n^{-r}\max_{1\le k\le n} k^r\left(\frac{k^{-1/2}}{n^{-1/2}}\right)^{2r} K_r(f, n^{-r}) \le M_{24}K_r(f, n^{-r}).
\end{aligned}$$

Then, we prove

$$K_r(f, n^{-r}) \approx n^{-(r+1/4)} \max_{1\le k\le n} k^{r+1/4} \left\|(\mathcal{I} - \mathcal{B}_k)^r f\right\|.$$

In fact, using (3.11) and (1.1) we also have

$$\begin{aligned}
K_r(f, n^{-r}) &\le M_{25} n^{-r-1/2} \sum_{k=1}^{n} k^{-3/4} k^{r+1/4} \left\|(\mathcal{I} - \mathcal{B}_k)^r f\right\| \\
&\le M_{25} n^{-r-1/2} \max_{1\le k\le n} k^{r+1/4} \left\|(\mathcal{I} - \mathcal{B}_k)^r f\right\| \sum_{k=1}^{n} k^{-3/4} \\
&\le M_{26} n^{-r-1/4} \max_{1\le k\le n} k^{r+1/4} \left\|(\mathcal{I} - \mathcal{B}_k)^r f\right\| \\
&\le M_{26} n^{-r-1/4} \max_{1\le k\le n} k^{r+1/4} K_r(f, k^{-r}) \\
&\le M_{26} n^{-r-1/4} \max_{1\le k\le n} k^{r+1/4} \left(\frac{k^{-1/2}}{n^{-1/2}}\right)^{2r} K_r(f, n^{-r}) \le M_{26} K_r(f, n^{-r}).
\end{aligned}$$

Hence, (3.12) is valid.

Now we shall use (3.12) to complete the proof of (1.2). Letting

$$\max_{1\le k\le n} k^{r+1/4} \left\|(\mathcal{I} - \mathcal{B}_k)^r f\right\| = n_1^{r+1/4} \left\|(\mathcal{I} - \mathcal{B}_{n_1})^r f\right\|,$$

we obtain from (3.12) the inequality

$$\begin{aligned}
\frac{1}{n^r}\left(n_1^r \left\|(\mathcal{I} - \mathcal{B}_{n_1})^r f\right\|\right) &\le \frac{1}{n^r} \max_{1\le k\le n} k^r \left\|(\mathcal{I} - \mathcal{B}_k)^r f\right\| \\
&\le \frac{M_{27}}{n^{r+1/4}} \max_{1\le k\le n} k^{r+1/4} \left\|(\mathcal{I} - \mathcal{B}_k)^r f\right\| \\
&\le \frac{M_{27}}{n^{r+1/4}} n_1^{r+1/4} \left\|(\mathcal{I} - \mathcal{B}_{n_1})^r f\right\|.
\end{aligned}$$

This implies that $M^{-1}n \le n_1 \le n \le Mn_1$, i.e., that $n \approx n_1$. Thus, from (3.12) it follows that

$$\begin{aligned}
K_r(f, n^{-r}) &\le M_{28} n^{-r-1/4} \max_{1\le k\le n} k^{r+1/4} \left\|(\mathcal{I} - \mathcal{B}_k)^r f\right\| \\
&= M_{28} n^{-r-1/4} n_1^{r+1/4} \left\|(I - \mathcal{B}_{n_1})^r f\right\| \\
&\le M_{28} n^{-r-1/4} \max_{n_1\le k\le n} k^{r+1/4} \left\|(\mathcal{I} - \mathcal{B}_k)^r f\right\| \\
&\le M_{29} \max_{nM^{-1}\le k\le n} \left\|(\mathcal{I} - \mathcal{B}_k)^r f\right\|.
\end{aligned}$$

Clearly, we can also write the above inequality as

$$K_r(f, n^{-r}) \le M \max_{k\ge n} \left\|(\mathcal{I} - \mathcal{B}_k)^r f\right\|.$$

This completes the proof of (1.2).

Finally, we prove (1.3). If $K_r(f,t) = \mathcal{O}\left(t^{\alpha/r}\right), 0 < \alpha \leq r$, then it is easy to obtain the following relation from (1.1):

$$\left\| f - \oplus^r \mathcal{B}_n f \right\| = \mathcal{O}\left(\frac{1}{n^\alpha}\right).$$

Inversely, whenever the above holds we get the following inequality from (1.2):

$$K_r(f, n^{-1}) \leq M\left(n^{-\alpha/r}\right).$$

Since for any $0 < t < 1$, there always is an $n \in \mathbb{N}$ such that $\frac{1}{2n} \leq t < \frac{1}{n}$, the above inequality implies

$$K_r(f,t) \leq K_r(f,n^{-1}) \leq M_{30}\left(n^{-\alpha/r}\right) \leq M_{31}\left(t^{-\alpha/r}\right).$$

The proof of Theorem 1.1 is now complete. □

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