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ON DYNAMIC PROGRAMMING: COMPACTNESS OF THE SPACE OF POLICIES

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The compactness of the set of policies in a dynamic programming decision model, which guarantees the existence of an optimal policy, is proven by reducing the problem to the compactness of the set of probability measures which are induced by the policies. When studying the set of probability measures, use is made of the weak topology and the so-called ws$^\infty$-topology. A definition and a discussion of the latter topology is given in this paper, where we pay attention to criteria for relative compactness.

compactness in space of probabilities	$\bar{p}$-optimal policies
non-Markovian dynamic programming	total reward criterion

0. Introduction

The theory of dynamic programming is concerned with stochastic processes $(\zeta_1, \alpha_1, \zeta_2, \alpha_2, ...)$, where ζ_n represents the state of some system at time n and α_n describes the action taken at time n. By the choice of action α_n we may control the stochastic development of the subsequent states $(\zeta_{n+1}, \zeta_{n+2}, ...)$ with regard to the optimization of some objective function.

A non-stationary dynamic-programming problem in the sense of Hinderer [10] is determined by

(i) the state space S_n at time n,
(ii) the action space A_n at time n,
(iii) the initial distribution p,
(iv) the transition law q_n for the n^{th} period,
(v) the reward function r_n for the n^{th} period,

where n runs through the set of positive integers. (In the present paper, the set of actions available at time n is assumed to be independent of the history.)

Let $H_\infty = S_1 \times A_1 \times S_2 \times A_2 \times \ldots$ denote the set of histories, and $\mathcal{P}(H_\infty)$ the set of all probability measures on H_∞. Any policy π defines a probability measure $P_\pi \in \mathcal{P}(H_\infty)$. Let us write Δ for the set of all policies, and Π for the set of all probability measures induced by some policy π. Associated with each $\pi \in \Delta$ is the total expected reward $V_\pi = \Sigma_n \int r_n \, dP_\pi$; V_π is well-defined by virtue of some convergence condition imposed throughout the paper. This condition is satisfied in the discounted case and in the negative case. One of the most interesting questions in the theory of dynamic programming is raised in:

Problem 1. Existence of some $\pi^* \in \Delta$ such that $V_{\pi^*} = \sup_{\pi \in \Delta} V_\pi$.

This problem has been studied by several authors (Blackwell [2, Theorem 7b], Strauch [21, Theorem 9.1], Maitra [13], Hinderer [10, Theorem 17.12] and [11, Theorem 4.2], Furukawa [7, Theorem 4.2], Schäl [17, Theorem 7.2] and [19]). All authors have very much used the properties of the optimal reward operator and most of them the criterion of optimality given by Dubins and Savage [6]. In the terminology of the latter, optimality is equivalent to being 'thrifty' and 'equalizing'. Roughly, a policy is 'thrifty' if it (almost) always selects actions which achieve the supremum in the optimality equation. It is 'equalizing' if it ultimately forces the system into states from which little future gain can be made. In the discounted and in the negative case – the (essentially) only cases under which the results mentioned above hold – every policy is 'equalizing'. Hence optimality is equivalent to being 'thrifty' (cp. also [10, Theorem 17.6]). In the analysis of the discounted case, in which the existence of optimal policies has been proven by Blackwell for a finite action space and by Maitra, Hinderer and Furukawa for compact action spaces, another important tool is furnished by the Banach fixed-point theorem. In the negative case Strauch has directly proven the existence of optimal policies for a finite action space. His method of proof is generalized to compact action spaces in [17] and [19].

In the context of statistical decision theory, Wald [24], LeCam [12] and Ghosh [9] have used a different approach to a similar problem. It is this approach which is also used in the present paper for the stochastic dynamic programming problem. More exactly, Problem 1 is attacked by a simple application of the well-known fact that any upper semi-continuous function on a compact set attains its supremum. Hence Problem 1 reduces to:

Problem 2. Existence of topologies on Δ such that Δ is compact and the mapping $\pi \to V_\pi$ is upper semi-continuous.

Instead of studying various topologies on the collection of policies, however, it turns out to be more convenient to consider topologies on spaces of probability measures. Guided by the observation that V_π depends on π only through $P_\pi \in \mathcal{P}(H_\infty)$, we shall restrict our attention to:

Problem 3. Existence of topologies on $\mathcal{P}(H_\infty)$ such that Π is compact and the mapping $\mu \to \Sigma \int r_n \, d\mu$ is upper semi-continuous on Π.

A solution of Problem 3 entails a solution of Problem 2. Given any appropriate topology on Π, we may endow Δ with the coarsest topology in which the mapping $\pi \mapsto P_\pi$ is continuous. This topology enjoys the desired properties.

The main part of the paper is therefore devoted to Problem 3. We give continuity and compactness conditions about (A_n), (q_n) and (r_n) (conditions (W) and conditions (S)) that guarantee the compactness of Π and the upper semi-continuity of the mapping $\mu \to \Sigma \int r_n \, d\mu$ on Π with respect to the weak topology on $\mathcal{P}(H_\infty)$ and the so-called ws$^\infty$-topology on $\mathcal{P}(H_\infty)$, respectively. The ws$^\infty$-topology on $\mathcal{P}(H_\infty)$ is the coarsest topology in which all mappings $\mu \to \int f \, d\mu$ are continuous for all bounded measurable functions f on H_∞ that depend on finitely many coordinates only and that depend continuously on the actions. A study of this topology may be found in Sections 3 and 4 below. We attack the problem of compactness by dividing it into two independent parts; the first deals with the relative compactness of Π, while in the second part we prove that Π is closed. Therefore we are interested in criteria for relative compactness.

There is another problem of interest which can be solved simultaneously with no additional labour:

Problem 1′. Convergence of the optimal total expected reward from finite stage play to the optimal total expected reward from infinite stage play.

The results of this paper do not imply any statement about the possible form of an optimal policy. However, the problem of replacing (optimal or arbitrary) policies by better structured policies with no loss in total expected reward has already been studied extensively (cp. [21, Theorems 4.1, 4.3, 6.4, 8.3; 10, Theorems 6.10, 15.2, 18.1; 11, Theorem 4.1; 16, Theorems 5.2, 6.1]).

Finally we remark that the concept of optimality used in the present paper is weaker than the notion of 'optimal' usually used for stationary decision models (cp. [2, 21]). For the latter concept of optimality and for a stationary model, Problems 1 and 1′ are investigated in [19] by the method described at the beginning, where a somewhat stronger convergence condition has to be imposed than in this paper.

1. Model formulation

Let $\mathbf{N}$ denote the set of positive integers, $\mathbf{R}$ the set of real numbers and $\mathbf{R}_-$ the set of real numbers augmented by the point $-\infty$. For any measurable space $(S, \mathfrak{S})$, let $\mathcal{P}(S)$ denote the set of all probability measures on $\mathfrak{S}$.

The background for a theory of dynamic programming may be provided by a *decision model* given by a tuple $((S_n, \mathfrak{S}_n), (A_n, \mathfrak{A}_n), p, q_n, r_n)$, $n \in \mathbf{N}$, of the following meaning:

(i) $(S_n, \mathfrak{S}_n)$ stands for the *state space* at time n and is assumed to be a standard Borel space, i.e., S_n is a non-empty Borel subset of a Polish (complete, separable, metric) space and $\mathfrak{S}_n$ is the system of Borel subsets of S_n.

(ii) $(A_n, \mathfrak{A}_n)$ is the *space of actions* available at time n and is assumed to be a standard Borel space.

We write $H_n = S_1 \times A_1 \times \ldots \times S_n$ and $H_\infty = S_1 \times A_1 \times S_2 \times A_2 \times \ldots$.

(iii) $p \in \mathcal{P}(S_1)$ is the so-called *initial distribution*.

(iv) The so-called *transition law* (q_n) is a sequence of transition probabilities $q_n : H_n \times A_n \to \mathcal{P}(S_{n+1})$. $q_n(h, a; \cdot)$ is the conditional distribution of the state of the system at time $n+1$ given that we have experienced history h up to time n and choose action a at time n.

(v) The *reward functions* $r_n : H_{n+1} \to \mathbf{R}_-$ are measurable functions bounded from above.

Remark 1.1. For convenience, the set of actions available is assumed to be independent of the history. However, the results of this paper extend to the more general case where a set $D_n(h)$ of admissible actions is specified for any $h \in H_n$. For many problems it is possible to reduce the more general case to the case considered in this paper upon setting $r_n(h, a, s) = -\infty$ for $a \notin D_n(h)$, $h \in H_n$, $s \in S_{n+1}$. Yet in the more general case the condition of compactness of A_n imposed below may be replaced by the condition that $D_n(h)$ is compact for $h \in H_n$ (cp. [18]).

As usual, a (randomized) *policy* $\pi = (\pi_n)$ is defined as a sequence of transition probabilities (regular conditional probabilities) $\pi_n : H_n \to \mathcal{P}(A_n)$. We write Δ for the set of all randomized policies. The initial law p, the transition law (q_n) and a policy (π_n) define a probability measure $P_\pi = p \otimes \pi_1 \otimes q_1 \otimes \pi_2 \otimes q_2 \otimes \ldots$ on the product space H_∞ endowed with the product σ-algebra and thus a random process $(\zeta_1, \alpha_1, \zeta_2, \alpha_2, \ldots)$ (cp. [10, p. 80]), where ζ_n and α_n denote the projection from H_∞ onto S_n and A_n, respectively. Then $\eta_n = (\zeta_1, \alpha_1, \zeta_2, \alpha_2, \ldots, \zeta_n)$ describes the history at time n.

In order that the total expected reward is well defined we impose throughout the paper:

General assumption. $\Sigma_{t=1}^\infty \mathrm{E}_\pi \{r_t^+\} < \infty$ for any $\pi \in \Delta$.

Now the definition of the following quantities makes sense:

$$V_\pi = \sum_{t=1}^{\infty} \mathrm{E}_\pi \{r_t\}\ , \qquad V_\pi^n = \sum_{t=1}^{n} \mathrm{E}_\pi \{r_t\}\ .$$

Using policy π, V_π is the total expected reward from infinite stage play and V_π^n is the total expected reward if we terminate at the n^{th} stage with no terminal return. For later use we remark that

$$V_\pi^n \to V_\pi \quad \text{as } n \to \infty\ . \tag{1.1}$$

For a proof, write

$$V_\pi^n = \sum_{t=1}^{n} \mathrm{E}_\pi \{r_t^+\} - \sum_{t=1}^{n} \mathrm{E}_\pi \{r_t^-\}$$

and use the general assumption.

In the present paper, we are concerned with the following concept of optimality (cp. Hinderer [10]): A policy $\pi^* \in \Delta$ will be called p-*optimal* (optimal in the mean with respect to p) iff $V_{\pi^*} = \sup_{\pi \in \Delta} V_\pi$.

For the main result of Section 2 we shall need to impose:

Condition (C). $Z_n = \sup_{N \geqslant n} \sup_{\pi \in \Delta} \Sigma_{t=n+1}^N \mathrm{E}_\pi \{r_t\} \to 0$ as $n \to \infty$.

It is to be noticed that $Z_n \geqslant \Sigma_{t=n+1}^n \mathrm{E}_\pi \{r_t\} = 0$.

Condition (C) is satisfied in the discounted case (cp. [2]) and in the negative case (cp. [21]) or more generally if $\Sigma_{t=1}^{\infty} \mathrm{E}_\pi \{r_t^+\}$ converges uniformly with respect to π, i.e.,

$$\sup_{\pi\in\Delta} \sum_{t=n+1}^{\infty} \mathrm{E}_\pi \{r_t^+\} \to 0 \quad \text{as } n \to \infty .$$

The following inequality is obvious from the definition of Z_n:

$$V_\pi^N \leqslant V_\pi^n + Z_n\,, \quad N \geqslant n. \tag{1.2}$$

Finally we remark that $\lim_{n\to\infty} \sup_{\pi\in\Delta} V_\pi^n$ exists provided (C) holds and that

$$\lim_{n\to\infty} \sup_{\pi\in\Delta} V_\pi^n \geqslant \sup_{\pi\in\Delta} V_\pi \,.$$

The proof is similar to the proof of relation (2.5) and Theorem 4.2 in [19].

2. Existence of optimal policies

One aim of this paper is to give sufficient conditions for
(1) the existence of $\bar{p}$-optimal policies,
(2) the convergence of $\sup_{\pi\in\Delta} V_\pi^n$ to $\sup_{\pi\in\Delta} V_\pi$ as $n \to \infty$.
The main purpose of this section is to show that both problems are solved if one can find a topology on $\mathcal{P}(H_\infty)$ such that

$$\Pi = \{\mu \in \mathcal{P}(H_\infty) \colon \mu = P_\pi \text{ for some } \pi \in \Delta\} \text{ is compact,} \tag{2.1}$$

$$\mu \to \int r_n \,\mathrm{d}\mu \text{ is upper semi-continuous on } \Pi,\ n \in \mathrm{N}. \tag{2.2}$$

More precisely, we should write $\int r_n \circ \eta_{n+1}\,\mathrm{d}\mu$ instead of $\int r_n\,\mathrm{d}\mu$ for any $\mu \in \mathcal{P}(H_\infty)$. But we agree to use the simplified notation where no confusion will arise.

In Sections 5 and 6, sufficient conditions for the existence of topologies on $\mathcal{P}(H_\infty)$ of that kind are given.

Let A be any topological space, and write $\hat{\mathcal{C}}(A)$ for the set of all upper semi-continuous functions $u : A \to \mathrm{R}_-$ bounded from above. Then it is known that any $u \in \hat{\mathcal{C}}(A)$ attains its supremum on A provided A is compact (cp. [4, §6, Théorème 3]). We further need:

Lemma 2.1 (cp. [19, Proposition 10.1]). *Let $u_n \in \hat{\mathcal{C}}(A)$, $n \in \mathrm{N}$, and $\epsilon_n \in \mathrm{R}$, $n \in \mathrm{N}$, such that $u_N(a) \leqslant u_n(a) + \epsilon_n$ for $N \geqslant n$, $a \in A$, and $\epsilon_n \to 0$ as $n \to \infty$. Then*
(a) $\lim_n u_n \in \hat{\mathcal{C}}(A)$,
(b) $\lim_n \max_{a\in A} u_n(a) = \max_{a\in A} \lim_n u_n(a)$, *provided that A is compact.*

Theorem 2.2. *Assume* (C). *If there is a topology on* $\mathcal{P}(H_\infty)$ *such that* (2.1) *and* (2.2) *hold, then there exists a* $\bar{p}$*-optimal policy and* $\lim_n \sup_{\pi\in\Delta} V_\pi^n = \sup_{\pi\in\Delta} V_\pi$.

Proof. Set

$$u_n(\mu) = \sum_{t=1}^{n} \int r_t \,\mathrm{d}\mu, \qquad \mu \in \Pi.$$

Then

$$\sup_{\mu\in\Pi} u_n(\mu) = \sup_{\pi\in\Delta} V_\pi^n,$$

$$\sup_{\mu\in\Pi} \lim_n u_n(\mu) = \sup_{\pi\in\Delta} V_\pi,$$

where use is made of (1.1). From (2.2) we infer that $u_n \in \hat{\mathcal{C}}(\Pi)$. In view of (1.2), Lemma 2.1 applies and we know that $\lim u_n \in \hat{\mathcal{C}}(\Pi)$ and

$$\lim_n \sup_{\mu\in\Pi} u_n(\mu) = \sup_{\mu\in\Pi} \lim_n u_n(\mu).$$

By (2.1), $\lim u_n(\mu)$ attains its supremum at $\mu^* \in \Pi$, say. Now choose any $\pi^* \in \Delta$ with $P_{\pi^*} = \mu^*$. Then π^* is $\bar{p}$-optimal. □

Given any topology $\mathfrak{T}$ on Π, one may topologize Δ in the following way. Let $\mathfrak{T}_\Delta$ be the coarsest topology on Δ such that the mapping $\pi \mapsto P_\pi$ is continuous with respect to $\mathfrak{T}_\Delta$ and $\mathfrak{T}$. If (2.1) and (2.2) are true for $\mathfrak{T}$, then Δ is compact and the mappings $\pi \mapsto \int r_n \,\mathrm{d}P_\pi$, $n \in \mathbb{N}$, are upper semi-continuous with respect to $\mathfrak{T}_\Delta$. Hence it is possible to prove Theorem 2.2 upon taking Δ as base space instead of Π.

Remark 2.3. The reader may wonder if a $\bar{p}$-optimal policy can be substituted by a deterministic (non-randomized) $\bar{p}$-optimal policy. For that question we refer the reader to [2, Theorem 2; 21, Theorem 4.3; 10, Theorem 15.2; 11, Satz 4.1; 16, Satz 6.1].

3. The ws-topology

Throughout this section, we assume $(S, \mathfrak{S})$ to be a measurable space and A to be a separable metric space endowed with the σ-algebra $\mathfrak{A}$ of Borel subsets of A. We use $\mathcal{P}(S)$ to denote the set of all probability measures on $(S, \mathfrak{S})$. Sometimes we will assume that $(S, \mathfrak{S})$ is standard Borel, i.e., S is a non-empty Borel subset of some Polish space and $\mathfrak{S}$ is the σ-algebra of Borel sets in S, in order to be sure that any $\mu \in \mathcal{P}(S)$ is regular with respect to the system of compact sets in S (cp. [14, II, Theorems 1.2, 3.1, 3.2]).

Further, we write $\mathcal{B}(S)$ for the set of all bounded measurable functions $f : S \to \mathbb{R}$ and $\hat{\mathcal{B}}(S)$ for the set of all measurable functions $f : S \to \mathbb{R}_-$ which are bounded from above. $\mathcal{C}(A)$ will denote the set of all continuous functions in $\mathcal{B}(A)$ and $\hat{\mathcal{C}}(A)$ is the set of all upper semi-continuous functions in $\hat{\mathcal{B}}(A)$. We know from a theorem of Baire (cp. [4, §6, Théorème 4; 5, §2, Proposition 11]) that $f \in \hat{\mathcal{C}}(A)$ if and only if f is the limit of some non-increasing sequence of functions $f_n \in \mathcal{C}(A)$. Finally we write

$$Q(S \times A) = \{f \in \mathcal{B}(S \times A) : f(s, \cdot) \in \mathcal{C}(A), s \in S\}, \tag{3.1}$$

$$\hat{Q}(S \times A) = \{f \in \hat{\mathcal{B}}(S \times A) : f \text{ is the limit of some non-increasing sequence of functions } f_n \in Q(S \times A)\}. \tag{3.2}$$

Given any $\mathcal{F} \subset \mathcal{B}(S)$, we may endow $\mathcal{P}(S)$ with the $\mathcal{F}$-topology defined as the coarsest topology in which all mappings $\mu \to \int f \, d\mu, f \in \mathcal{F}$, are continuous. Then the $\mathcal{F}$-topology is the topology obtained by taking as base all sets of the form

$$U(\mu; J, \epsilon) = \{\nu : |\int f \, d\mu - \int f \, d\nu| < \epsilon, f \in J\},$$

where $\mu \in \mathcal{P}(S)$, J is a finite subset of $\mathcal{F}$, and $\epsilon > 0$. A net $\{\mu_\alpha\}$ will converge to μ_0 in the $\mathcal{F}$-topology if and only if $\int f \, d\mu_\alpha \to \int f \, d\mu_0$ for all $f \in \mathcal{F}$. If we write

$$\hat{\mathcal{F}} = \{f \in \hat{\mathcal{B}}(S) : f_n \downarrow f \text{ for some sequence of functions } f_n \in \mathcal{F}\},$$

then all mappings $\mu \to \int f \, d\mu, f \in \hat{\mathcal{F}}$, are upper semi-continuous (cp. [4, §6, Théorème 4]). If $f \in \mathcal{F}$ implies $-f \in \mathcal{F}$, then the $\mathcal{F}$-topology is the coarsest topology rendering all mappings $\mu \to \int f \, d\mu, f \in \hat{\mathcal{F}}$, upper semi-continuous.

Lemma 3.1. *Let $\mathcal{F} \subset \mathcal{B}(S)$. If $\Gamma \subset \mathcal{P}(S)$ is relatively compact with respect to the $\mathcal{F}$-topology, then for any sequence $\{f_n\}$ in $\mathcal{F}$ which decreases to 0, $\int f_n \, d\mu \to 0$ uniformly in $\mu \in \Gamma$.*

We remark that for the topologies considered below the converse direction in Lemma 3.1 is also true.

Proof. Let $\bar{\Gamma}$ denote the closure of Γ. Then we have, by Dini's theorem or by Lemma 2.1,

$$\lim_n \sup_{\mu \in \Gamma} \int f_n \, \mathrm{d}\mu = \sup_{\mu \in \Gamma} \lim_n \int f_n \, \mathrm{d}\mu = 0 \, . \quad \Box$$

In case $\mathcal{F} = \mathcal{C}(A)$, the $\mathcal{F}$-topology on $\mathcal{P}(A)$ will be called the w-*topology* (weak topology). In case $\mathcal{F} = \mathcal{B}(S)$, the $\mathcal{F}$-topology on $\mathcal{P}(S)$ will be called the s-*topology*. In case $\mathcal{F} = \mathcal{Q}(S \times A)$, the $\mathcal{F}$-topology on $\mathcal{P}(S \times A)$ will be called the ws-*topology*. We denote convergence in the w-topology, s-topology and ws-topology by $\xrightarrow{w}$, $\xrightarrow{s}$, $\xrightarrow{ws}$, respectively.

3.1. The w-*topology*

Let $\mathcal{P}(A)$ be endowed with the w-topology. Then $\mathcal{P}(A)$ is separable and metrizable (cp. [14, Theorem II 6.2]). Further, the σ-algebra of Borel subsets on $\mathcal{P}(A)$ coincides with the smallest σ-algebra on $\mathcal{P}(A)$ such that for each $B \in \mathfrak{A}$ the mapping $\mu \mapsto \mu(B)$ is measurable. A proof of this fact may be found in [15]. If A is Polish, then $\mathcal{P}(A)$ is Polish (cp. [14, Theorem II 6.5]). Thus it is easily seen that $\mathcal{P}(A)$ is standard Borel if A is standard Borel (cp. [10, p. 91]).

Lemma 3.2 (cp. [23, Theorem II 25]). *For any $\Gamma \subset \mathcal{P}(A)$, the following statements are equivalent:*

(i) *Γ is relatively compact in the* w-*topology.*

(ii) *For any sequence $\{f_n\}$ in $\mathcal{C}(A)$ which decreases to 0, $\int f_n \, \mathrm{d}\mu \to 0$ uniformly in $\mu \in \Gamma$.*

Lemma 3.3 (cp. [14, proof of Theorem II 6.2]). *There is a denumerable subset of $\mathcal{C}(A)$ that separates the elements of $\mathcal{P}(A)$.*

Lemma 3.4. *Suppose that S is a separable metric space. Then the mappings*

$$(\mu, a) \mapsto \int f(s, a) \, \mu(\mathrm{d}s) \, , \qquad f \in \mathcal{C}(S \times A) \ \ (\textit{resp. } \hat{\mathcal{C}}(S \times A))$$

defined on $\mathcal{P}(S) \times A$ are continuous (resp. upper semi-continuous), where $\mathcal{P}(S)$ is endowed with the w-*topology.*

Proof. Suppose there is given a sequence $\{(\mu_n, a_n)\}$ in $\mathcal{P}(S) \times A$ such that $\mu_n \xrightarrow{w} \mu_0 \in \mathcal{P}(S)$ and $a_n \to a_0 \in A$. Define $\delta_n \in \mathcal{P}(A)$ by $\delta_n(\{a_n\}) = 1$; then $\delta_n \xrightarrow{w} \delta_0$ and hence $\mu_n \times \delta_n \xrightarrow{w} \mu_0 \times \delta_0$ (cp. [1, Theorem 1.3.2]). Since

$$\int f \, \mathrm{d}(\mu_n \times \delta_n) = \int f(s, a_n) \, \mathrm{d}\mu_n \, ,$$

the proof is complete if $f \in \mathfrak{C}(S \times A)$. In case $f \in \hat{\mathfrak{C}}(S \times A)$, take any sequence $\{f_m\} \downarrow f$, $f_m \in \mathfrak{C}(S \times A)$. Then

$$\int f(s, a)\, \mu(\mathrm{d}s) = \inf_m \int f_m(s, a)\, \mu(\mathrm{d}s)\,. \quad \square$$

3.2. The s-topology

First we remark that the s-topology on $\mathfrak{P}(S)$ coincides with the topology of set-wise convergence on the σ-algebra $\mathfrak{S}$, i.e. the s-topology is the coarsest topology for which all mappings $\mu \to \mu(B)$, $B \in \mathfrak{S}$, are continuous.

Lemma 3.5 (cp. [8, Theorems 2.6, 3.7]). *For any $\Gamma \in \mathfrak{P}(S)$, the following statements are equivalent:*

(i) Γ *is relatively compact in the s-topology.*

(ii) *For any sequence $\{f_n\}$ in $\mathfrak{B}(S)$ which decreases to 0, $\int f_n\, \mathrm{d}\mu \to 0$ uniformly in $\mu \in \Gamma$.*

If $(S, \mathfrak{S})$ is standard Borel, then each of the two statements is equivalent to:

(iii) (a) Γ *is tight.*

(b) *For every compact set K and every $\epsilon > 0$, there exists an open set G such that $G \supset K$ and $\mu(G - K) \leqslant \epsilon$ for all $\mu \in \Gamma$.*

Let $\hat{\mathfrak{P}}(S)$ denote the set of all positive measures μ on $(S, \mathfrak{S})$ with $\mu(S) \leqslant 1$. Then the definition of w-convergence and s-convergence extends to nets on $\hat{\mathfrak{P}}(S)$ in an obvious way.

Lemma 3.6 (cp. [22, Theorem 7]). *Suppose $(S, \mathfrak{S})$ is standard Borel. Let $\{\mu_\alpha\}$ be a net on $\hat{\mathfrak{P}}(S)$ and $\mu_0 \in \hat{\mathfrak{P}}(S)$. Then the following statements are equivalent:*

(i) $\mu_\alpha \xrightarrow{s} \mu_0$.

(ii) (a) $\mu_\alpha \xrightarrow{w} \mu_0$.

(b) *For every compact set K and every $\epsilon > 0$, there exists an open set G such that $G \supset K$ and $\overline{\lim}_\alpha\, \mu_\alpha(G - K) < \epsilon$.*

Proof. (i) $\Rightarrow$ (ii) is immediate from [22, Theorem 7].

(ii) $\Rightarrow$ (i) (cp. proof of Theorem 7 in [22]). Let us write G for an open set and K for a compact set. Then we have

$$\mu_0(K) = \inf_{G \supset K} \mu_0(G) \leqslant \inf_{G \supset K} \underline{\lim}\, \mu_\alpha(G)$$

$$\leqslant \underline{\lim}\, \mu_\alpha(K) + \inf_{G \supset K} \overline{\lim}\, \mu_\alpha(G - K) = \underline{\lim}\, \mu_\alpha(K).$$

Hence, for any $B \in \mathfrak{S}$,

$$\mu_0(B) = \sup_{K \subset B} \mu_0(K) \leqslant \underline{\lim}\, \mu_\alpha(B) \,.$$

The same inequality holds for $S - B$ instead of B. Further $\mu_0(S) = \lim \mu_\alpha(S)$. Now we may conclude that $\mu_0(B) = \lim \mu_\alpha(B)$, $B \in \mathfrak{S}$. □

3.3. The ws-topology

Let us write $\mathbf{1}_B$ for the indicator function of the set B and ∂B for the boundary of B.

Theorem 3.7. *Let $\{\mu_\alpha\}$ be a net on $\mathfrak{P}(S \times A)$ and $\mu_0 \in \mathfrak{P}(S \times A)$. Then the following statements are equivalent:*

(i) $\mu_\alpha \xrightarrow{\mathrm{ws}} \mu_0$.

(ii) $\int gh \,\mathrm{d}\mu_\alpha \to \int gh \,\mathrm{d}\mu_0$ *for* $g \in \mathfrak{B}(S)$, $h \in \mathcal{C}(A)$.

(iii) $\int \mathbf{1}_B h \,\mathrm{d}\mu_\alpha \to \int \mathbf{1}_B h \,\mathrm{d}\mu_0$ *for* $B \in \mathfrak{S}$, $h \in \mathcal{C}(A)$.

(iv) $\mu_\alpha(B \times \cdot\,) \xrightarrow{\mathrm{w}} \mu_0(B \times \cdot\,)$ *for* $B \in \mathfrak{S}$.

(v) $\mu_\alpha(B \times C) \to \mu_0(B \times C)$ *for* $B \in \mathfrak{S}$, $C \in \mathfrak{A}$ *with* $\mu_0(S \times \partial C) = 0$.

(vi) $\mu_\alpha(\,\cdot \times C) \xrightarrow{\mathrm{s}} \mu_0(\,\cdot \times C)$ *for* $C \in \mathfrak{A}$ *with* $\mu_0(S \times \partial C) = 0$.

(vii) $\int g \mathbf{1}_C \,\mathrm{d}\mu_\alpha \to \int g \mathbf{1}_C \,\mathrm{d}\mu_0$ *for* $g \in \mathfrak{B}(S)$, $C \in \mathfrak{A}$ *with* $\mu_0(S \times \partial C) = 0$.

If $(S, \mathfrak{S})$ is standard Borel then each of the statements (i)–(vii) is equivalent to each of the following statements

(viii) (a) $\mu_\alpha \xrightarrow{\mathrm{w}} \mu_0$.

(b) $\mu_\alpha(\,\cdot \times A) \xrightarrow{\mathrm{s}} \mu_0(\,\cdot \times A)$.

(ix) (a) $\mu_\alpha \xrightarrow{\mathrm{w}} \mu_0$.

(b) *For every compact set K in S and every $\epsilon > 0$ there exists an open set G in S such that $G \supset K$ and $\overline{\lim}_\alpha\, \mu_\alpha((G - K) \times A) < \epsilon$.*

Proof. The implications (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) and (v) ⇒ (vi) ⇒ (vii) are obvious.

Statement (iv) is equivalent to $\mu_\alpha(B \times C) \to \mu_0(B \times C)$ for $B \in \mathfrak{S}$, $C \in \mathfrak{A}$ with $\mu_0(B \times \partial C) = 0$. Since $\mu_0(S \times \partial C) = 0$ implies $\mu_0(B \times \partial C) = 0$, we have (iv) ⇒ (v).

If $(S, \mathfrak{S})$ is standard Borel, then (i) ⇒ (viii) and, by Lemma 3.6, (viii) ⇒ (ix).

Further, since

$$\partial(B \times C) \subset \partial B \times C \cup S \times \partial C \,,$$

(ix) implies that for any $C \in \mathfrak{A}$ with $\mu_0(S \times \partial C) = 0$ we have $\mu_\alpha(\,\cdot \times C) \xrightarrow{\mathrm{w}}$

$\mu_0(\cdot \times C)$, and for every compact set K in S and every $\epsilon > 0$ there is an open set G in S such that $\overline{\lim}_\alpha \mu_\alpha((G-K) \times C) < \epsilon$. Hence, by another appeal to Lemma 3.6, it will be seen that (ix) $\Rightarrow$ (vi).

To complete the demonstration we now prove (vii) $\Rightarrow$ (i). Let $f \in Q(S \times A)$. By transforming f linearly (with a positive coefficient for the first-degree term), we may reduce the problem to the case in which $0 \leq f \leq 1$. As has been shown e.g. in [14, proof of Lemma II 6.5], for any $n \in \mathbb{N}$, A admits a measurable disjoint partition $A = \bigcup_k I_k^n$, where $\mu_0(S \times \partial I_k^n) = 0$ and I_k^n is contained in some sphere of radius $1/n$, $k \in \mathbb{N}$. Set

$$\underline{g}_{nk}(s) = \inf_{a \in I_k^n} f(s,a)\,, \qquad \overline{g}_{nk} = \sup_{a \in I_k^n} f(s,a)\,.$$

The separability of A implies the separability of I_k^n, hence $\underline{g}_{nk}$ and $\overline{g}_{nk}$ are measurable. Further set

$$\underline{f}_{nm}(s,a) = \sum_{k \leq m} \underline{g}_{nk}(s)\, 1_{I_k^n}(a)\,,$$

$$\overline{f}_{nm}(s,a) = \sum_{k \leq m} \overline{g}_{nk}(s)\, 1_{I_k^n}(a) + 1_{\bigcup_{k>m} I_k^n}(a)\,.$$

Then

$$\underline{f}_{nm} \leq f \leq \overline{f}_{nm}\,,$$

$$\lim_{n\to\infty} \lim_{m\to\infty} \underline{f}_{nm} = f.$$

On writing

$$1_{\bigcup_{k>m} I_k^n} = 1 - \sum_{k \leq m} 1_{I_k^n}\,,$$

we infer from (vii) that

$$\lim_\alpha \int \overline{f}_{nm}\, \mathrm{d}\mu_\alpha = \int \underline{f}_{nm}\, \mathrm{d}\mu_0\,.$$

Hence

$$\int \underline{f}_{nm}\, \mathrm{d}\mu_0 \leq \underline{\lim}_\alpha \int f\, \mathrm{d}\mu_\alpha \leq \overline{\lim}_\alpha \int f\, \mathrm{d}\mu_\alpha \leq \int \overline{f}_{nm}\, \mathrm{d}\mu_0\,.$$

As

$$\lim_{n\to\infty} \lim_{m\to\infty} \int \underline{f}_{nm}\, \mathrm{d}\mu_0 = \int f\, \mathrm{d}\mu_0\,,$$

the proof is now complete. □

From Theorem 3.7 ((i) $\Leftrightarrow$ (ii)) we obtain:

Corollary 3.8. *Let* $\{(\mu'_\alpha, \mu''_\alpha)\}$ *be a net on* $\mathcal{P}(S) \times \mathcal{P}(A)$ *and* $(\mu'_0, \mu''_0) \in \mathcal{P}(S) \times \mathcal{P}(A)$. *Then* $\mu'_\alpha \times \mu''_\alpha \xrightarrow{\text{ws}} \mu'_0 \times \mu''_0$ *if and only if* $\mu'_\alpha \xrightarrow{\text{s}} \mu'_0$ *and* $\mu''_\alpha \xrightarrow{\text{w}} \mu''_0$.

The proof of the following lemma may be carried through in exactly the same way as the proof of Lemma 3.4 upon using nets instead of sequences and appealing to Corollary 3.8 instead of [1].

Lemma 3.9. *The mappings*

$$(\mu, a) \mapsto \int f(s, a)\, \mu(\mathrm{d}s), \qquad f \in Q(S \times A) \quad (\textit{resp. } \hat{Q}(S \times A)),$$

defined on $\mathcal{P}(S) \times A$ *are continuous (resp. upper semi-continuous) where* $\mathcal{P}(S)$ *is endowed with the* s*-topology.*

Theorem 3.10. *Suppose that* $(S, \mathfrak{S})$ *is standard Borel. Then for any* $\Gamma \subset \mathcal{P}(S \times A)$ *the following statements are equivalent:*

(i) Γ *is relatively compact in the* ws*-topology.*

(ii) *For any sequence* $\{f_n\}$ *in* $Q(S \times A)$ *which decreases to* 0, $\int f_n\, \mathrm{d}\mu \to 0$ *uniformly in* $\mu \in \Gamma$.

(iii) (a) Γ *is relatively compact in the* w*-topology.*

(b) $\{\mu(\cdot \times A) : \mu \in \Gamma\}$ *is relatively compact in the* s*-topology on* $\mathcal{P}(S)$.

Proof. By Lemma 3.1, we have (i) $\Rightarrow$ (ii). Further, from Lemma 3.2 ((ii) $\Rightarrow$ (i)) and Lemma 3.5 ((ii) $\Rightarrow$ (i)), we infer that (ii) $\Rightarrow$ (iii).

$\mathcal{P}(S \times A)$ endowed with the ws-topology or the w-topology is a regular topological space. Thus Γ is relatively compact if and only if every net on Γ has a further subnet which converges (cp. [3, §10, exerc. 1a]). The proof of (iii) $\Rightarrow$ (i) is now a consequence of Lemma 3.5 ((i) $\Rightarrow$ (iii)(b)) and Theorem 3.7 ((ix) $\Rightarrow$ (i)). □

Remark 3.11. It is known that any $\Gamma \subset \mathcal{P}(A)$ is relatively compact in the w-topology if and only if Γ is relatively sequentially compact in the w-topology (since $\mathcal{P}(A)$ endowed with the w-topology is metrizable), and that any $\Gamma \subset \mathcal{P}(S)$ is relatively compact in the s-topology if and only if Γ is relatively sequentially compact in the s-topology (cp. [8, Theorem 2.6]). It is not difficult to show that the ws-topology inherits this property of the w-topology and the s-topology, i.e., any $\Gamma \subset \mathcal{P}(S \times A)$ is relatively compact in the ws-topology if and only if Γ is relatively sequentially compact in the ws-topology provided that $(S, \mathfrak{S})$ is standard Borel.

4. The ws$^\infty$-topology

First we state some properties of the w-topology on $\mathcal{P}(H_\infty)$.

Lemma 4.1. *The* w-*topology on* $\mathcal{P}(H_\infty)$ *is the coarsest topology for which the mappings* $\mu \to \int f \, \mathrm{d}\mu$, $f \in \bigcup_n \mathcal{C}(H_n)$, *are continuous.*

Here and in similar situations, we write $\int f \, \mathrm{d}\mu$ instead of $\int f \circ \eta_n \, \mathrm{d}\mu$ for any f defined on H_n and $\mu \in \mathcal{P}(H_\infty)$.

As S_t and A_t, $t \in \mathbb{N}$, are separable metric spaces, the proof of Lemma 4.1 follows immediately from [1, p.22, Problem 7, p.30].

Lemma 4.2. *Let* $\Gamma \subset \mathcal{P}(H_\infty)$. Γ *is relatively compact in the* w-*topology if and only if the sets* $\Gamma_n = \{\mu \circ \eta_n^{-1} : \mu \in \Gamma\}$ *are relatively compact in the* w-*topology on* $\mathcal{P}(H_n)$ *for* $n \in \mathbb{N}$.

Proof. In case $S_t = A_t = \mathbb{R}$, $t \in \mathbb{N}$, the proof may be found in [1]. As the proof extends to the present more general situation, we will only sketch the proof. The 'only if' part is clear. Suppose now that Γ_n, $n \in \mathbb{N}$, is relatively compact. We know that relative compactness and relative sequential compactness in the w-topology on $\mathcal{P}(H_n)$, $1 \leqslant n \leqslant \infty$, coincide. Using the diagonal procedure we obtain for any sequence $\{\mu_m\}$ in Γ a subsequence $\{\mu_{m'}\}$ such that

$$\mu_{m'} \circ \eta_n^{-1} \xrightarrow{\mathrm{w}} \mu_{0n} \; (m' \to \infty) \quad \text{for some } \mu_{0n} \in \mathcal{P}(H_n)\,.$$

Certainly $\{\mu_{0n}, n \in \mathbb{N}\}$ satisfies the consistency condition of the Kolmogorov existence theorem for standard Borel spaces (cp. [14, Theorems V 4.2, 5.1]). Consequently there is some $\mu_0 \in \mathcal{P}(H_\infty)$ with $\mu_0 \circ \eta_n^{-1} = \mu_{0n}$, $n \in \mathbb{N}$ (cp. [1, p.38]). An appeal to Lemma 4.1 now completes the demonstration. □

Set

$$Q(H_1) = \mathcal{B}(H_1) \quad (= \mathcal{B}(S_1))\,,$$
$$\hat{Q}(H_1) = \hat{\mathcal{B}}(H_1)\,,$$

and define $Q(H_{n+1})$ and $Q(H_n \times A_n)$ by (3.1) on setting $A = A_1 \times \ldots \times A_n$ and $S = S_1 \times \ldots \times S_{n+1}$ or $S = S_1 \times \ldots \times S_n$, respectively. Define $\hat{Q}(H_{n+1})$ and $\hat{Q}(H_n \times A_n)$ by (3.2) in the same way. Thus $f \in Q(H_{n+1})$ if and only if $f \in \mathcal{B}(H_{n+1})$ and $f(s_1, \cdot, s_2, \cdot, \ldots, s_n, \cdot, s_{n+1}) \in \mathcal{C}(A_1 \times \ldots \times A_n)$ for all $(s_1, s_2, \ldots, s_{n+1}) \in S_1 \times \ldots \times S_{n+1}$, and $f \in \hat{Q}(H_{n+1})$ if and only if there is a sequence of functions $f_n \in Q(H_{n+1})$ such that $f_n \downarrow f$. The ws-*topology on* $\mathcal{P}(H_n)$ is defined as in Section 3 on setting $S = S_1 \times \ldots \times S_n$, $A = A_1 \times \ldots \times A_{n-1}$, i.e., the ws-topology on $\mathcal{P}(H_n)$ is the coarsest topology for which the mappings $\mu \to \int f \, d\mu$, $f \in Q(H_n)$, are continuous.

The ws$^\infty$-*topology on* $\mathcal{P}(H_\infty)$ is defined as the coarsest topology rendering the mappings $\mu \to \int f \, d\mu$, $f \in \bigcup_n Q(H_n)$, continuous. Then the ws$^\infty$-topology on $\mathcal{P}(H_\infty)$ agrees with the coarsest topology for which the mappings $\mu \to \int f \, d\mu$, $f \in \bigcup_n \hat{Q}(H_n)$, are upper semi-continuous. In view of Lemma 4.1, the ws$^\infty$-topology on $\mathcal{P}(H_\infty)$ is finer than the w-topology.

Theorem 4.3. *Let* $\Gamma \subset \mathcal{P}(H_\infty)$, *and set* $\Gamma_n = \{\mu \circ \eta_n^{-1} : \mu \in \Gamma\}$. *Then the following statements are equivalent*:

(i) Γ *is relatively compact in the* ws$^\infty$-*topology*.

(ii) Γ_n *is relatively compact in the* ws-*topology on* $\mathcal{P}(H_n)$ *for* $n \in \mathbb{N}$.

(iii) *For* $n \in \mathbb{N}$ *and any sequence* $\{f_m\}$ *in* $Q(H_n)$ *which decreases to* 0, $\int f_m \, d\mu \to 0$ *uniformly in* $\mu \in \Gamma$.

(iv) (a) Γ_n *is relatively compact in the* w-*topology on* $\mathcal{P}(H_n)$ *for* $n \in \mathbb{N}$.

(b) $\{\mu \circ (\zeta_1, \zeta_2, \ldots, \zeta_n)^{-1}, \mu \in \Gamma\}$ *is relatively compact in the* s-*topology on* $\mathcal{P}(S_1 \times \ldots \times S_n)$ *for* $n \in \mathbb{N}$.

Proof. From Theorem 3.10 we know that (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). Now let us reflect upon (i) $\Leftrightarrow$ (iv). It is obvious that (i) $\Rightarrow$ (iv). Now suppose that (iv) holds. As in the proof of Theorem 3.10 it is sufficient to show that every net on Γ has a further subnet which converges. Appealing to Lemma 4.2, we know that Γ is relatively compact in the w-topology on $\mathcal{P}(H_\infty)$. Hence for any net $\{\mu_\alpha\}$ on $\mathcal{P}(H_\infty)$ there is a further subnet $\{\mu_{\alpha'}\}$ such that $\mu_{\alpha'} \xrightarrow{w} \mu_0$ for some $\mu_0 \in \mathcal{P}(H_\infty)$. From Lemma 3.5 ((i) $\Rightarrow$ (iii)(b)) and Theorem 3.7 ((ix) $\Rightarrow$ (i)) we infer that $\mu_{\alpha'} \circ \eta_n^{-1} \xrightarrow{ws} \mu_0 \circ \eta_n^{-1}$ for $n \in \mathbb{N}$. From the definition of the ws$^\infty$-topology it follows that $\mu_{\alpha'} \xrightarrow{ws^\infty} \mu_0$. □

Remark 4.4 (cp. Remark 3.11). It can be shown without much difficulty that $\Gamma \subset \mathcal{P}(H_\infty)$ is relatively compact in the ws$^\infty$-topology if and only if Γ is relatively sequentially compact in the ws$^\infty$-topology.

5. Conditions (W)

Throughout this section we impose

Conditions (W). For every $n \in \mathbf{N}$,
(1) A_n is compact,
(2) $q_n : H_n \times A_n \to \mathcal{P}(S_{n+1})$ is w-continuous,
(3) $r_n \in \mathcal{C}(H_{n+1})$.

We need the following operators. Define for any $u \in \hat{\mathcal{B}}(H_{n+1})$, $n \in \mathbf{N}$,

$$\tilde{L}_n u(h,a) = \int q_n(h,a,\mathrm{d}s)\, u(h,a,s)\,, \qquad h \in H_n,\ a \in A_n.$$

$$\tilde{U}_n u(h) = \sup_{a \in A_n} \tilde{L}_n u(h,a).$$

Lemma 5.1. *For any $u \in \mathcal{C}(H_{n+1})$,*
(a) $\tilde{L}_n u \in \mathcal{C}(H_n \times A_n)$,
(b) $\tilde{U}_n u \in \mathcal{C}(H_n)$.

Proof. (a) is an immediate consequence of Lemma 3.4.
(b) As the supremum of any set of continuous functions is lower semi-continuous, we infer from part (a) that $\tilde{U}_n u$ is lower semi-continuous. Furthermore, by [13, Lemma 3.4] (or [10, Lemma 5.10], [19, Proposition 10.2]), $\tilde{U}_n u$ is upper semi-continuous. □

Lemma 5.2. *If $u_m \in \mathcal{C}(H_{n+1})$, $m \in \mathbf{N}$, and $u_m \downarrow u$, then*
(a) $\tilde{U}_n u_m \downarrow \tilde{U}_n u$ *as $m \to \infty$,*
(b) $\sup_{\mu \in \Pi} \int u_m \,\mathrm{d}\mu \downarrow \sup_{\mu \in \Pi} \int u \,\mathrm{d}\mu$ *as $m \to \infty$.*

Proof. (a) Appealing to Lemma 5.1(a), we have $\tilde{L}_n u_m(h,\cdot) \in \mathcal{C}(A_n)$ for $h \in H_n$. From Lemma 2.1 we conclude that

$$\lim_m \sup_a \tilde{L}_n u_m(h,a) = \sup_a \lim_m \tilde{L}_n u_m(h,a)\,.$$

Further, in view of the monotone convergence theorem, it is clear that

$$\lim_m \tilde{L}_n u_m(h,a) = \tilde{L}_n u(h,a)\,.$$

These arguments obviously prove the assertion.
(b) Let $v \in \mathcal{B}(H_{n+1})$. Then it is known from the theory of dynamic programming (cp. [10, Theorem 14.4 or 14.5]) that

$$\tilde{U}_i \ldots \tilde{U}_n v(h) = \sup_{\pi \in \Delta} \mathbf{E}_\pi\{v \circ \eta_{n+1} \mid \eta_i = h\}, \qquad h \in H_i,\ 1 \leq i \leq n,$$

where $\tilde{U}_i \ldots \tilde{U}_n v$ is inductively defined through

$$\tilde{U}_i \ldots \tilde{U}_n v = \tilde{U}_i(\tilde{U}_{i+1} \ldots \tilde{U}_n v) .$$

Further

$$\sup_{\pi \in \Delta} \mathbf{E}_\pi \{v \circ \eta_{n+1}\} = \int \tilde{U}_1 \ldots \tilde{U}_n v \, \mathrm{d}p$$

(cp. [10, Theorem 14.2]). Hence we may rewrite the assertion as

$$\int \tilde{U}_1 \ldots \tilde{U}_n u_m \, \mathrm{d}p \downarrow \int \tilde{U}_1 \ldots \tilde{U}_n u \, \mathrm{d}p \quad \text{as } m \to \infty.$$

From Lemma 5.1(b) and Lemma 5.2(a) we may derive by induction on i that $\tilde{U}_i \ldots \tilde{U}_n u_m \downarrow \tilde{U}_i \ldots \tilde{U}_n u$ as $m \to \infty$, $1 \leqslant i \leqslant n$. Use of this relation for $i = 1$ and of the monotone convergence theorem now proves the assertion. □

Remark 5.3. By virtue of the monotone convergence theorem and Lemma 5.2(a), Lemma 5.1 remains true if one replaces $\mathcal{C}(\ldots)$ by $\hat{\mathcal{C}}(\ldots)$.

Lemma 5.4. Π *is relatively compact in the* w*-topology.*

Proof. The assertion is an immediate consequence of Lemma 3.2 with $A = H_{n+1}$, $n \in \mathbf{N}$, Lemma 4.2 and Lemma 5.2(b). □

Lemma 5.5. Π *is closed in the* w*-topology.*

Proof. We make use of the following characterization of Π (cp. [21, Lemma 7.2], [10, Lemma 13.1]): Let $\mu \in \mathcal{P}(H_\infty)$. Then $\mu \in \Pi$ if and only if

$$\mu \circ \zeta_1^{-1} = p ,$$

$$\mu \circ \eta_{n+1}^{-1} = \mu \circ (\eta_n, \alpha_n)^{-1} \otimes q_n \quad \text{for } n \in \mathbf{N}.$$

From Lemma 3.3 we know that for any $n \geqslant 0$ there are functions $w_{nm} \in \mathcal{C}(H_{n+1})$, $m \in \mathbf{N}$, which separates elements of $\mathcal{P}(H_{n+1})$. Moreover, by Lemma 5.1(a), we have $\tilde{L}_n w_{nm} \in \mathcal{C}(H_n \times A_n)$. Set

$$v_{nm}(\mu) = \int w_{nm} \, \mathrm{d}\mu, \qquad \bar{v}_{nm}(\mu) = \int \tilde{L}_n w_{nm} \, \mathrm{d}\mu, \qquad \mu \in \mathcal{P}(H_\infty).$$

Then v_{nm} and $\bar{v}_{nm}$ are w-continuous. Finally set

$$\Pi_{0m} = \{\mu \in \mathcal{P}(H_\infty) \colon v_{0m}(\mu) = \int w_{0m} \, \mathrm{d}p\} ,$$

$$\Pi_{nm} = \{\mu \in \mathcal{P}(H_\infty) \colon v_{nm}(\mu) = \bar{v}_{nm}(\mu)\} .$$

Then the sets Π_{nm} are closed in the w-topology. The relation $\Pi = \bigcap_{n\geq 0, m\geq 1} \Pi_{nm}$ now completes the demonstration. □

On collecting our various results we now obtain:

Theorem 5.6. *Let $\mathcal{P}(H_\infty)$ be endowed with the w-topology. Then the conditions* (W) *imply that Π is compact and the mappings $\mu \to \int r_n \, d\mu$, $n \in \mathbf{N}$, are upper semi-continuous.*

6. Conditions (S)

Throughout this section we impose

Conditions (S). For every $n \in \mathbf{N}$,
(1) A_n is compact,
(2) $q_n(s_1, \cdot, s_2, \cdot, ..., s_n, \cdot) : A_1 \times ... \times A_n \to \mathcal{P}(S_{n+1})$ is s-continuous for $s_i \in S_i$, $1 \leq i \leq n$,
(3) $r_n \in \hat{Q}(H_{n+1})$.

Lemma 6.1. *For any $u \in Q(H_{n+1})$,*
(a) $\tilde{L}_n u \in Q(H_n \times A_n)$,
(b) $\tilde{U}_n u \in Q(H_n)$.

Proof. (a) Obviously $\tilde{L}_n u \in \mathcal{B}(H_n \times A_n)$. Furthermore, Lemma 3.9 implies that $\tilde{L}_n u(s_1, \cdot, s_2, \cdot, ..., s_n, \cdot) \in \mathcal{C}(A_1 \times ... \times A_n)$.
(b) Let A'_n be any denumerable dense subset of A_n. Then by part (a) we may write

$$\tilde{U}_n u(h) = \sup_{a \in A'_n} \tilde{L}_n u(h,a) .$$

Thus $\tilde{U}_n u \in \mathcal{B}(H_n)$. To prove the continuity property one may adapt the argument for the proof of Lemma 5.1(b). □

Lemma 6.2. *If $u_m \in Q(H_{n+1})$, $m \in \mathbf{N}$, and $u_m \downarrow u$, then*
(a) $\tilde{U}_n u_m \downarrow \tilde{U}_n u$ *as $m \to \infty$,*
(b) $\sup_{\mu\in\Pi} \int u_m \, d\mu \downarrow \sup_{\mu\in\Pi} \int u \, d\mu$ *as $m \to \infty$.*

The proof of Lemma 6.2 follows that of Lemma 5.2 and it is, therefore, not given here.

Remark 6.3. By virtue of the monotone convergence theorem and Lemma 6.2(a), Lemma 6.1 remains true if one replaces $Q(\cdot)$ by $\hat{Q}(\cdot)$.

On combining the results of Theorem 4.3 and Lemma 6.2(b), we obtain:

Lemma 6.4. Π *is relatively compact in the* ws$^\infty$*-topology.*

Lemma 6.5. Π *is closed in the* ws$^\infty$*-topology.*

Proof. Choose w_{nm} and define v_{nm}, $\bar{v}_{nm}$ and Π_{nm} as in the proof of Lemma 5.5. Then, by Lemma 6.1(a), $L_n w_{nm} \in \hat{Q}(H_n \times A_n)$. Now v_{nm} and $\bar{v}_{nm}$ are ws$^\infty$-continuous, which implies that the sets Π_{nm} and hence Π are closed. □

In view of Lemma 6.4 and Lemma 6.5, we finally obtain:

Theorem 6.6. *Let* $\mathcal{P}(H_\infty)$ *be endowed with the* ws$^\infty$*-topology. Then the conditions* (S) *imply that* Π *is compact and the mappings* $\mu \to \int r_n \, d\mu$, $n \in \mathbf{N}$, *are upper semi-continuous.*

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