Geometric modular representation theory

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Outline

1. The representation theory of algebraic groups
2. The nilpotent cone and restricted cohomology
3. Tilting modules and coherent Springer theory

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Algebraic groups

Examples of algebraic groups

- *The* **general linear group** $GL_N(\Bbbk)$ *is the group of all invertible* $N \times N$ *matrices.*

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Definition

- *A* **(linear) algebraic group** *is a matrix group which is also the zero locus of a system of polynomial equations.*
- *A* **reductive algebraic group** *is an important type of algebraic group whose representation theory is well-behaved.*

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- *Irreducible characters are given by **Weyl's character formula**.*

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Example

The subspace $\Bbbk x^p \oplus \Bbbk y^p \subsetneq M_p$ is a proper G-submodule.

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- Classifying all indecomposable representations in this case is actually impossible! ("**wild representation type**").
- The characters of the irreducibles are extremely difficult to compute.

Tilting modules

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- *Think of the category* Tilt(G) *as a commutative ring with a distinguished basis given by the* $T(\lambda)$.
- *"Homogenous" ideals for this basis are called* **thick tensor ideals**.

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- (Soergel, Kazhdan–Lusztig) Connected to the representation theory of quantum groups and affine Lie algebras in characteristic 0.

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- $\mathcal{N}$ is **Gorenstein** and has **rational singularities**.
- There is a **symplectic resolution** $\pi : T^*\mathcal{B} \to \mathcal{N}$, where $\mathcal{B}$ denotes the **flag variety**. This is called the **Springer resolution**.

Example for $SL_2(\Bbbk)$

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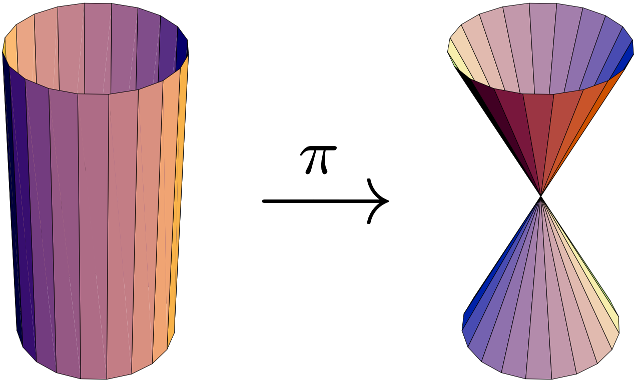

Restricted Lie algebra cohomology

- In characteristic $p > 0$, ordinary Lie algebra cohomology is often replaced with **restricted Lie algebra cohomology** $H^*_{\mathrm{res}}(\mathfrak{g}, M)$.

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- *The* **support variety** *of an $\mathfrak{g}$-module M is the subvariety $V_{\mathfrak{g}}(M) := \operatorname{supp} H^*_{\mathrm{res}}(\mathfrak{g}, M \otimes M^*) \subseteq \mathcal{N}$.*

Outline

The Humphreys conjecture on support varieties

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Conjecture (Humphreys 1995)

If G is a reductive algebraic group over $\Bbbk$ with $p > h$, then for any $\lambda \in \mathbf{X}^+$, $V_{\mathfrak{g}}(\mathrm{T}(\lambda)) = \overline{\mathcal{O}}$, where $\mathbf{c}_{\mathcal{O}}$ is the unique weight cell containing λ.

Rephrasing the Humphreys conjecture

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\{\text{tilting modules}\} \arrow[rr, "T(\lambda)\mapsto \mathbf{c}\ni\lambda"] \arrow[dr, "V_{\mathfrak{g}}(-)"'] & & \{\text{weight cells}\} \arrow[dl, "\text{Lus. bijection}^{-1}"] \\
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- The Humphreys conjecture connects the representation theory of G, the geometry of $\mathcal{N}$, and the **alcove combinatorics** for $\mathbf{X}^+$.
- Related to a number of other results and conjectures.

Illustration for $SL_2(\Bbbk)$

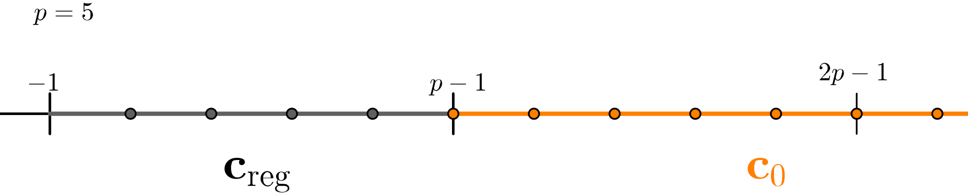


Illustration for $SL_3(\Bbbk)$

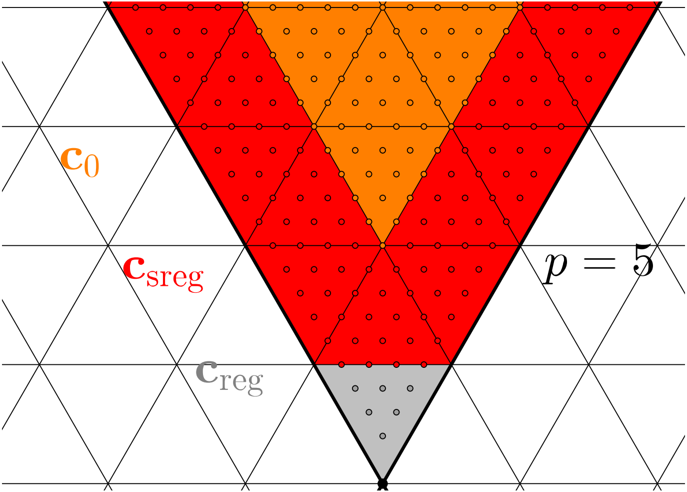


Illustration for $Sp_4(\mathbb{k})$

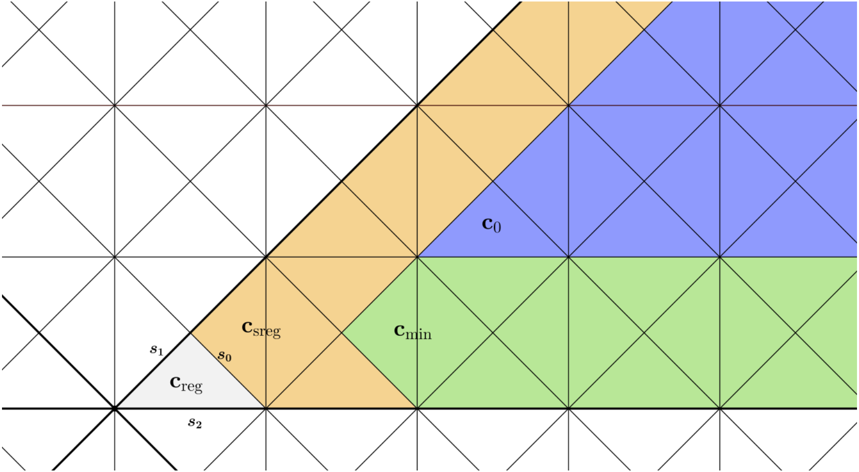
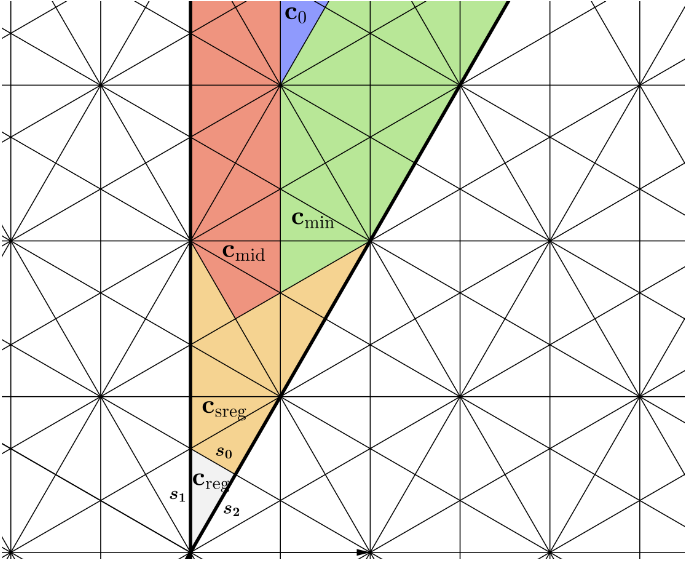


Illustration for $G_2(\Bbbk)$



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- My original proof of the conjecture for $SL_N(\Bbbk)$ uses the explicit description of the Lusztig bijection due to J.Y. Shi.
- We obtained the more general result by reformulating the conjecture into the language of **coherent Springer theory**. (This refers to the study of equivariant coherent sheaves on $T^*\mathcal{B}$ and $\mathcal{N}$.)

Coherent Springer theory

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Theorem (Achar–H. 2019)

$D^b \operatorname{Coh}^{\dot{G}\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N})$ *admits a* **co-t-structure**, *where every indecomposable sheaf in its* **co-heart** *is of the form* $H^*_{\mathrm{res}}(\mathfrak{g}, T)$ *for an indecomposable* T*.*

Additional results and conjectures

Theorem (Achar–**H.**–Riche 2018)

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- Leads to conjecture that thick tensor ideals of G are recursively determined by thick tensor ideals of smaller groups.