# Geometric modular representation theory

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# Outline

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- *A* **(linear) algebraic group** *is a matrix group which is also the zero locus of a system of polynomial equations.*
- *A* **reductive algebraic group** *is an important type of algebraic group whose representation theory is well-behaved.*

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- *Irreducible characters are given by **Weyl's character formula**.*

# Representation theory of $SL_2(\mathbb{k})$

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## Example

*The subspace $\Bbbk x^p \oplus \Bbbk y^p \subsetneq M_p$ is a proper $G$-submodule.*

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- Classifying all indecomposable representations in this case is actually impossible! ("**wild representation type**").
- The characters of the irreducibles are extremely difficult to compute.

# Tilting modules

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- *Think of the category* Tilt$(G)$ *as a commutative ring with a distinguished basis given by the* $T(\lambda)$.
- *"Homogenous" ideals for this basis are called* **thick tensor ideals**.

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- (Soergel, Kazhdan–Lusztig) Connected to the representation theory of quantum groups and affine Lie algebras in characteristic 0.

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- (Lusztig–Williamson 2018) **Billiards conjecture** for characters of tilting modules for $SL_3(\overline{\mathbb{F}_p})$.
- (Achar–**H.**–Riche 2018) Conjectural classification of thick tensor ideals of tilting modules for algebraic groups.

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- There is a **symplectic resolution** $\pi : T^*\mathcal{B} \to \mathcal{N}$, where $\mathcal{B}$ denotes the **flag variety**. This is called the **Springer resolution**.

# Example for $SL_2(\mathbb{k})$

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- $\Bbbk[\mathcal{N}] \cong \Bbbk[u^2, uv, v^2] \subset \Bbbk[u, v]$ (this gives a grading on $\Bbbk[\mathcal{N}]$).

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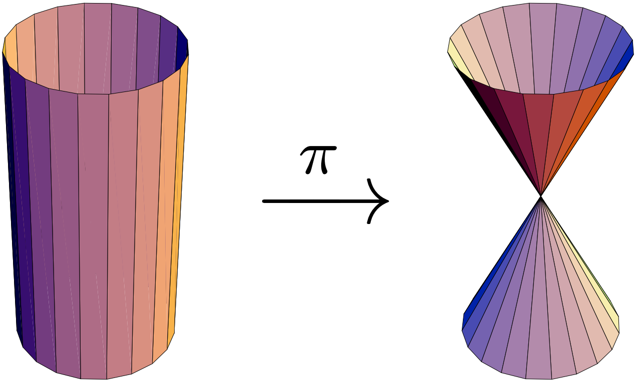

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- *The* **support variety** *of an $\mathfrak{g}$-module $M$ is the subvariety $V_{\mathfrak{g}}(M) := \operatorname{supp} H^*_{\mathrm{res}}(\mathfrak{g}, M \otimes M^*) \subseteq \mathcal{N}$.*

# Outline

# The Humphreys conjecture on support varieties

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**Conjecture (Humphreys 1995)**

*If $G$ is a reductive algebraic group over $\Bbbk$ with $p > h$, then for any $\lambda \in \mathbf{X}^+$, $V_{\mathfrak{g}}(\mathsf{T}(\lambda)) = \overline{\mathcal{O}}$, where $\mathbf{c}_{\mathcal{O}}$ is the unique weight cell containing $\lambda$.*

# Rephrasing the Humphreys conjecture

$$\begin{tikzcd}
\{\text{tilting modules}\} \arrow[rr, "T(\lambda)\mapsto \mathbf{c}\ni\lambda"] \arrow[dr, "V_{\mathfrak{g}}(-)"'] & & \{\text{weight cells}\} \arrow[dl, "\text{Lus. bijection}^{-1}"] \\
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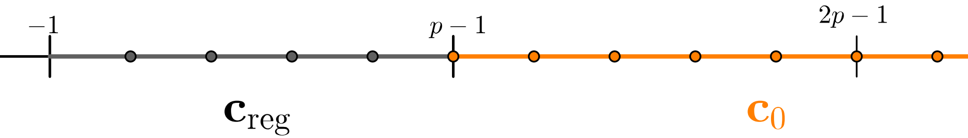
- The Humphreys conjecture connects the representation theory of $G$, the geometry of $\mathcal{N}$, and the **alcove combinatorics** for $\mathbf{X}^+$.
- Related to a number of other results and conjectures.

# Illustration for $SL_2(\Bbbk)$

$p = 5$



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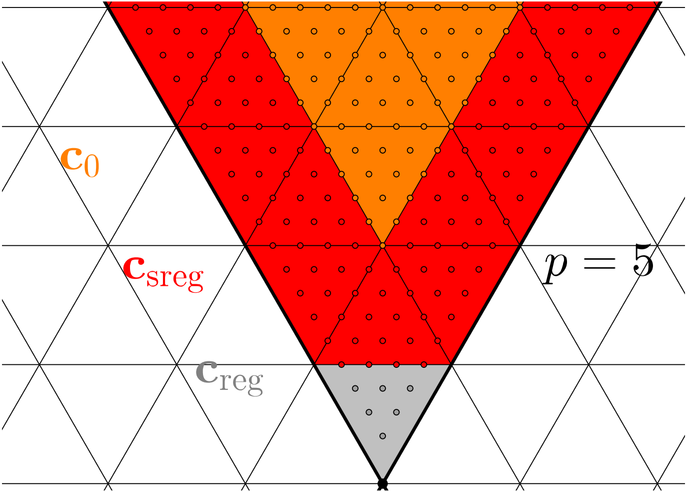
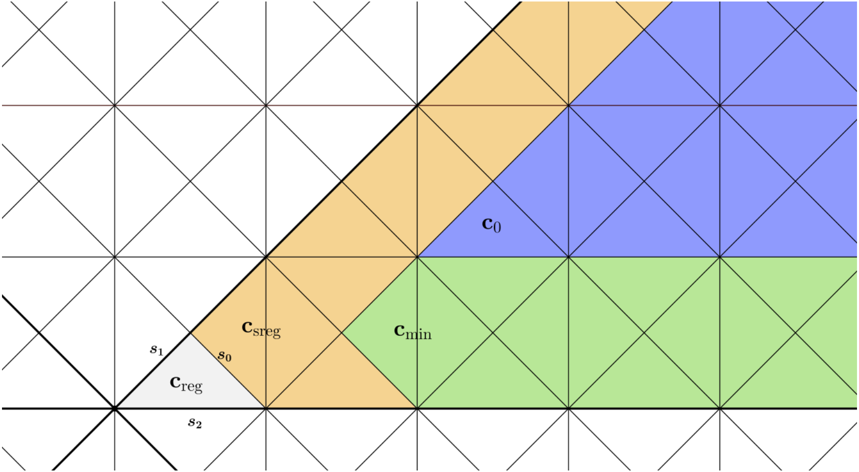
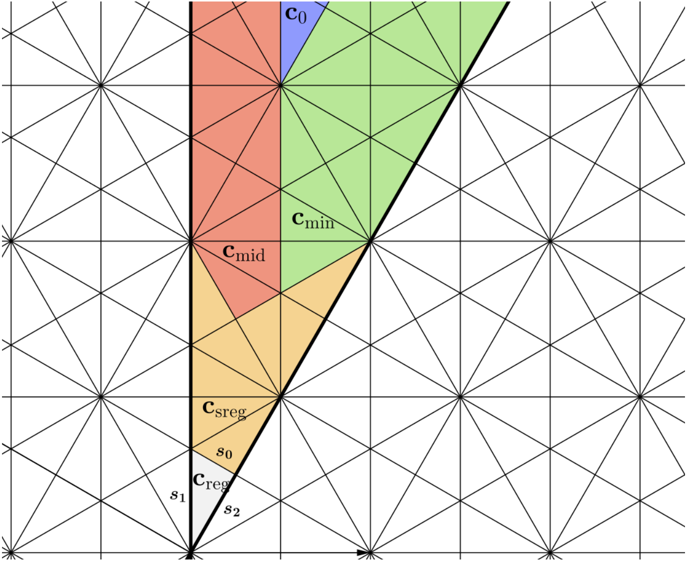


# Illustration for $Sp_4(\Bbbk)$



# Illustration for $G_2(\Bbbk)$



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- My original proof of the conjecture for $SL_N(\mathbb{k})$ uses the explicit description of the Lusztig bijection due to J.Y. Shi.
- We obtained the more general result by reformulating the conjecture into the language of **coherent Springer theory**. (This refers to the study of equivariant coherent sheaves on $T^*\mathcal{B}$ and $\mathcal{N}$.)

# Coherent Springer theory

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D^b \operatorname{Rep}^{\mathrm{gr}}_{\emptyset}(G) \arrow[drr, "\text{``}H^*_{\mathrm{res}}(\mathfrak{g},-)\text{''}"] & & \\
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**Theorem (Achar–H. 2019)**

$D^b \mathrm{Coh}^{\dot{G}\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N})$ *admits a* **co-t-structure**, *where every indecomposable sheaf in its* **co-heart** *is of the form* $H^*_{\mathrm{res}}(\mathfrak{g}, T)$ *for an indecomposable* $T$.

# Additional results and conjectures

## Theorem (Achar–**H.**–Riche 2018)

- *There exists a notion of $G$-equivariant* **tilting sheaves** *on $\mathcal{O}$, which are indexed by a set $\Lambda_{\mathcal{O}}$.*

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- Leads to conjecture that thick tensor ideals of $G$ are recursively determined by thick tensor ideals of smaller groups.