Co-t-structures on derived categories of coherent sheaves and the cohomology of tilting modules

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Includes joint work with:

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Outline

1. Background and Motivation
2. Co-t-structures
3. Additional results and conjectures

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3. These objects satisfy analogues of properties 1 – 7 (along with some additional properties).

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Nilpotent orbits for $GL_n(\mathbb{k})$

- *There is a natural bijection:* $\{G\text{-orbits of } \mathcal{N}\} \leftrightarrow \{\text{partitions } \pi \vdash n\}$.
- *The orbit of a matrix is determined by its Jordan normal form.*

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- *The* **support variety** *is $V_{G_1}(M) := \operatorname{supp} H^*(G_1, M) \subseteq \dot{\mathcal{N}}$.*

Additional remarks and questions

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If M is a G-module, then $H^*(G_1, M)$ is also $\dot{G}$-equivariant and gives an object of $\mathrm{Coh}^{\dot{G} \times \mathbb{G}_\mathrm{m}}(\dot{\mathcal{N}})$.

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Coherent sheaves and representation theory

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We can replace the study of tilting module cohomology with the study of the $\overline{\mathcal{E}}_\lambda$. (Note that $\pi_*\tilde{\mathcal{E}}_\lambda = 0$ for $\lambda \notin -\mathbf{X}^+$).

Outline

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Let $\mathfrak{D}$ be a triangulated category. A **co-t-structure** on $\mathfrak{D}$ is a pair of full additive subcategories $(\mathfrak{D}_{\geq 0}, \mathfrak{D}_{\leq 0})$ with the following properties:

1. Both $\mathfrak{D}_{\geq 0}$ and $\mathfrak{D}_{\leq 0}$ are closed under direct summands.
2. We have $\mathfrak{D}_{\geq 0}[-1] \subset \mathfrak{D}_{\geq 0}$ and $\mathfrak{D}_{\leq 0}[1] \subset \mathfrak{D}_{\leq 0}$.
3. For $A \in \mathfrak{D}_{\geq 0}$ and $B \in \mathfrak{D}_{\leq 0}[1]$, we have $\mathrm{Hom}(A, B) = 0$.
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The objects of $\mathfrak{S}$ are known as **silting objects**.

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Suppose R is a finite-dimensional algebra and $\mathfrak{D} = K^b(\mathrm{Proj}_R)$ is the bounded homotopy category of projective left R-modules. We define:

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This satisfies Axiom (4) since for any object P, we can set:

$$\begin{tikzcd}
A \arrow[d] & = & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & P^0 \arrow[r] \arrow[d] & P^1 \arrow[r] \arrow[d] & P^2 \arrow[r] \arrow[d] & \cdots \\
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(Achar–H.) Can generalize to case of a category $\mathfrak{D}$ equipped with a pair of dual **co-quasi-exceptional sets**. These are sets of objects $\{\nabla_\lambda\}_{\lambda\in\Lambda}$, $\{\Delta_\lambda\}_{\lambda\in\Lambda}$ satisfying similar axioms to the highest weight structure.

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We can also fix an auto-equivalence like $\langle 1 \rangle$ or $\{1\}$ from earlier, and define a "graded" analogue of this co-t-structure.

Coherent sheaves on the nilpotent cone

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In general, there exist $\lambda \in -\mathbf{X}^+$, $n \in \mathbb{Z}$ with $\operatorname{Hom}_{\mathfrak{D}}(\overline{\mathcal{E}}_\lambda, \overline{\mathcal{E}}_\lambda\{n\}[i]) \neq 0$ for some $i < 0$. Therefore, $\overline{\mathcal{E}}_\lambda$ are NOT tilting objects.

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3. Can reduce down to case when $\mathcal{F} = \tilde{\Delta}_\lambda\{n\}[i]$, $\mathcal{G} = \tilde{\nabla}_\mu\{m\}[j]$ with $i \leq 0$, $j \geq 0$.

Outline

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Conjecture ("Humphreys conjecture")

For $\lambda \in -\mathbf{X}^+$, $\operatorname{supp} \overline{\mathcal{E}}_\lambda = \overline{\mathcal{O}_{w_\lambda}}$.

Remarks on the Humphreys conjecture

This conjecture was verified in type A with $p > h$ by (H. 2018) and in general type for $p \gg 0$ by (Achar–H.–Riche 2019).

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There is also a variant which describes $V_{\mathbf{G}_1}(\mathrm{T}(\lambda) \otimes \mathrm{T}(\lambda)^*)$ for all $\lambda \in \mathbf{X}^+$ in terms of the **weight cells** for **G**.

New results in type A

Theorem (Achar–H.)

The "Scheme-Theoretic" version of the Humphreys conjecture holds if G is type A and $p > h$.

Consequently, we obtain a new proof of the "classical" Humphreys conjecture in type A which does not rely on cell combinatorics.

The proof involves building co-t-structures on cotangent bundles to partial flag varieties, and studying a 'representation theoretic' analogue of the geometric "push–pull" functors which relate $\widetilde{\mathcal{N}}_I := T^*(G/P_I)$ to $\widetilde{\mathcal{N}}$.

Future directions

- Try to obtain an "orbit-wise" description of the co-t-strucure on $D^{\mathrm{b}}\,\mathsf{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N})$, relating indecomposable silting objects to indecomposable tilting sheaves on orbits. (This addresses question (3).)

- Adapt these methods to obtain a proof of Humphreys conjecture in arbitrary type

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Theorem (Achar–**H.**–Riche 2018)

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- *If $\operatorname{supp} \mathcal{F} = \overline{\mathcal{O}}$, then $\mathcal{F}|_{\mathcal{O}} = \mathcal{T}_{\boldsymbol{\lambda}}$,where $\mathcal{T}_{\boldsymbol{\lambda}}$ is an indecomposable tilting sheaf and $\boldsymbol{\lambda} \in \Lambda_{\mathcal{O}}$ is determined by the LV-bijection.*

Orbit-wise conjecture

Theorem (Achar–**H.**–Riche 2018)

- *There exists a notion of G-equivariant* **tilting sheaves** *on $\mathcal{O}$, which are indexed by a set $\Lambda_{\mathcal{O}}$.*
- *The* **Lusztig–Vogan bijection**, *which gives a bijection between $-\mathbf{X}^+$ and $\{(\mathcal{O}, \boldsymbol{\lambda}) \mid \mathcal{O} \subset \mathcal{N}, \boldsymbol{\lambda} \in \Lambda_{\mathcal{O}}\}$, is "independent of $\Bbbk$".*

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- Leads to conjecture that thick tensor ideals of G are recursively determined by thick tensor ideals of smaller groups.