Co-t-structures on derived categories of coherent sheaves and the cohomology of tilting modules

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Includes joint work with:

Pramod Achar

Outline

Background and Motivation

2 Co-t-structures

3 Additional results and conjectures

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- **②** For each $\lambda \in \Lambda$, objects Δ_{λ} , ∇_{λ} , T_{λ} called standard, costandard and tilting objects respectively.
- **3** These objects satisfy analogues of properties 1-7 (along with some additional properties).

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Nilpotent orbits for $GL_n(\mathbb{k})$

- There is a natural bijection: $\{G\text{-orbits of }\mathcal{N}\}\leftrightarrow \{\text{partitions }\pi\vdash n\}.$
- The orbit of a matrix is determined by its Jordan normal form.

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- The support variety is $V_{G_1}(M) := \operatorname{supp} H^*(G_1, M) \subseteq \dot{\mathcal{N}}$.

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This functor can be 'upgraded' to the level of derived categories.

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- **3** For each $\lambda \in \mathbf{X}$ there are objects $\widetilde{\Delta}_{\lambda}$, $\widetilde{\nabla}_{\lambda}$, $\widetilde{\mathcal{E}}_{\lambda}$, where for any $n \in \mathbb{Z}$ $\Psi(\widetilde{\Delta}_{\lambda}\{n\}) = \mathsf{M}(w_{\lambda} \cdot 0), \ \Psi(\widetilde{\nabla}_{\lambda}\{n\}) = \mathsf{N}(w_{\lambda} \cdot 0), \ \Psi(\widetilde{\mathcal{E}}_{\lambda}\{n\}) = \mathsf{T}(w_{\lambda} \cdot 0).$

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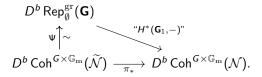
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We can replace the study of tilting module cohomology with the study of the $\overline{\mathcal{E}}_{\lambda}$. (Note that $\pi_* \tilde{\mathcal{E}}_{\lambda} = 0$ for $\lambda \notin -\mathbf{X}^+$).

Outline

Background and Motivation

2 Co-t-structures

Additional results and conjectures

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The objects of \mathfrak{S} are known as **silting objects**.

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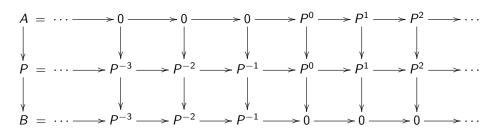
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This satisfies Axiom (4) since for any object P, we can set:



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Our results are analogous to the work of Bezrukavnikov on **dualizable quasi exceptional sets** and **exotic t-structures**.

We can also fix an auto-equivalence like $\langle 1 \rangle$ or $\{1\}$ from earlier, and define a "graded" analogue of this co-t-structure.

$$\mathsf{Set}\ \tilde{\mathfrak{D}} := \mathit{D}^{\mathrm{b}}\,\mathsf{Coh}^{\mathsf{G} \times \mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}}) \ \mathsf{and} \ \mathfrak{D} := \mathit{D}^{\mathrm{b}}\,\mathsf{Coh}^{\mathsf{G} \times \mathbb{G}_{\mathrm{m}}}(\mathcal{N}).$$

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Theorem (Achar-H. 2019)

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$$\mathfrak{D}_{\geq 0} = \{ \mathcal{F} \in \mathfrak{D} \mid \mathsf{Hom}_{\mathfrak{D}}(\overline{\Delta}_{\lambda}\{n\}[i], \mathcal{F}) = 0 \text{ for } \lambda \in -\mathbf{X}^+, \ n \in \mathbb{Z}, \ i < 0 \},$$

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In general, there exist $\lambda \in -\mathbf{X}^+$, $n \in \mathbb{Z}$ with $\mathrm{Hom}_{\mathfrak{D}}(\overline{\mathcal{E}}_{\lambda}, \overline{\mathcal{E}}_{\lambda}\{n\}[i]) \neq 0$ for some i < 0. Therefore, $\overline{\mathcal{E}}_{\lambda}$ are NOT tilting objects.

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3 Can reduce down to case when $\mathcal{F} = \tilde{\Delta}_{\lambda}\{n\}[i]$, $\mathcal{G} = \tilde{\nabla}_{\mu}\{m\}[j]$ with $i \leq 0, j \geq 0$.

Outline

Background and Motivation

2 Co-t-structures

3 Additional results and conjectures

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(Bezrukavnikov) There is a map $W_{\mathrm{aff}} \to \{ \text{nilpotent orbits} \}$ with $w \mapsto \mathcal{O}_w \subset \mathcal{N}$ which induces a bijection between 2-sided cells and nilpotent orbits. This is called the **Lusztig bijection**.

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Conjecture ("Humphreys conjecture")

For
$$\lambda \in -\mathbf{X}^+$$
, supp $\overline{\mathcal{E}}_{\lambda} = \overline{\mathcal{O}_{w_{\lambda}}}$.

This conjecture was verified in type A with p > h by (H. 2018) and in general type for $p \gg 0$ by (Achar–H.–Riche 2019).

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This version is equivalent to stating that the annihilator of $H^*(\mathbf{G}_1, T(w_\lambda \cdot 0))$ is the defining ideal of $\overline{\mathcal{O}_{w_\lambda}}$ in $\mathbb{k}[\mathcal{N}]$. Also, the aforementioned results do not apply in this case.

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There is also a variant which describes $V_{\mathbf{G}_1}(\mathsf{T}(\lambda)\otimes\mathsf{T}(\lambda)^*)$ for all $\lambda\in\mathbf{X}^+$ in terms of the **weight cells** for **G**.

New results in type A

Theorem (Achar-H.)

The "Scheme-Theoretic" version of the Humphreys conjecture holds if G is type A and p > h.

Consequently, we obtain a new proof of the "classical" Humphreys conjecture in type A which does not rely on cell combinatorics.

The proof involves building co-t-structures on cotangent bundles to partial flag varieties, and studying a 'representation theoretic' analogue of the geometric "push–pull" functors which relate $\widetilde{\mathcal{N}}_I := T^*(G/P_I)$ to $\widetilde{\mathcal{N}}$.

Future directions

• Try to obtain an "orbit-wise" description of the co-t-strucure on $D^{\mathrm{b}}\operatorname{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N})$, relating indecomposable silting objects to indecomposable tilting sheaves on orbits. (This addresses question (3).)

 Adapt these methods to obtain a proof of Humphreys conjecture in arbitrary type

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- Leads to conjecture that thick tensor ideals of *G* are recursively determined by thick tensor ideals of smaller groups.