STAT 420

Time Series: y_t, $t = 1, 2, \ldots, N$.

Stationary process ≈ a random process where all of its statistical properties do not vary with time.

$\mathrm{E}(Y_t) = \mu$. $\mathrm{Var}(Y_t) = \sigma_Y^2$.

$$\gamma(k) = \mathrm{Cov}(Y_t, Y_{t+k}) = \mathrm{E}[(Y_t - \mu)(Y_{t+k} - \mu)]$$
$$= \mathrm{Cov}(Y_t, Y_{t-k}) = \mathrm{E}[(Y_t - \mu)(Y_{t-k} - \mu)].$$

$$\gamma(0) = \mathrm{Var}(Y_t) = \sigma_Y^2.$$

$$\rho_k = \mathrm{Corr}(Y_t, Y_{t+k}) = \frac{\mathrm{Cov}(Y_t, Y_{t+k})}{\sqrt{\mathrm{Var}(Y_t)\mathrm{Var}(Y_{t+k})}} = \frac{\mathrm{E}(Y_t - \mu)(Y_{t+k} - \mu)}{\sigma_Y^2},$$

$k = \pm 1, \pm 2, \ldots$.

$\rho_k = \gamma(k) / \gamma(0)$. $\rho_0 = 1$.

Sample autocorrelation coefficient: $$r_k = \frac{\sum_{t=1}^{N-k}(y_t - \bar{y})(y_{t+k} - \bar{y})}{\sum_{t=1}^{N}(y_t - \bar{y})^2}$$

Consider the following "regression" (autoregressive) model: AR(1)

$$(Y_t - \mu) = \varphi\,(Y_{t-1} - \mu) + e_t$$

$\mathrm{E}(e_t) = 0$, $\mathrm{Var}(e_t) = \sigma_\epsilon^2$ for all t

$\mathrm{E}(e_t e_s) = 0$, for $t \neq s$

$\mathrm{E}(e_t Y_s) = 0$, for $s < t$

Then $\gamma(0) = \mathrm{Var}(Y_t) = \dfrac{\sigma_e^2}{1-\phi^2}$, $\Rightarrow$ need $|\phi| < 1$.

$$\gamma(k) = \phi\gamma(k-1), \quad k \geq 1. \quad \Rightarrow \quad \rho_k = \phi\rho_{k-1}, \quad k \geq 1.$$

Therefore, $\rho_k = \rho_{-k} = \phi^k$, $k \geq 1$.

Suppose we observe y_t, $t = 1, 2, \ldots, N$.

Then
$$(Y_{N+1} - \ \ ) = \phi(y_N - \ \ ) + e_{N+1},$$
$$(Y_{N+2} - \ \ ) = \phi(Y_{N+1} - \ \ ) + e_{N+2}$$
$$= \phi^2(y_N - \ \ ) + \phi e_{N+1} + e_{N+2},$$
$$(Y_{N+3} - \ \ ) = \phi(Y_{N+2} - \ \ ) + e_{N+3}$$
$$= \phi^3(y_N - \ \ ) + \phi^2 e_{N+1} + \phi e_{N+2} + e_{N+3},$$
$$\ldots$$
$$(Y_{N+k} - \ \ ) = \phi^k(y_N - \ \ ) + \phi^{k-1} e_{N+1} + \ldots + \phi e_{N+k-1} + e_{N+k}.$$

$e_{N+1}, \ldots, e_{N+k-1}, e_{N+k}$ – future random "errors".

Therefore, our forecast $\hat{y}_{N+k}$ of Y_{N+k} is

$$\hat{y}_{N+k} = E(Y_{N+k} | y_1, y_2, \ldots, y_N) = E_N(Y_{N+k}) = \ \ + \phi^k(y_N - \ \ )$$

Since $Y_{N+k} - \hat{y}_{N+k} = \phi^{k-1} e_{N+1} + \ldots + \phi e_{N+k-1} + e_{N+k}$,

$$\mathrm{Var}(Y_{N+k} - \hat{y}_{N+k}) = (1 + \phi^2 + \ldots + \phi^{2(k-1)})\,\sigma_e^2$$

Least squares approach: Find the values of and ϕ that minimize

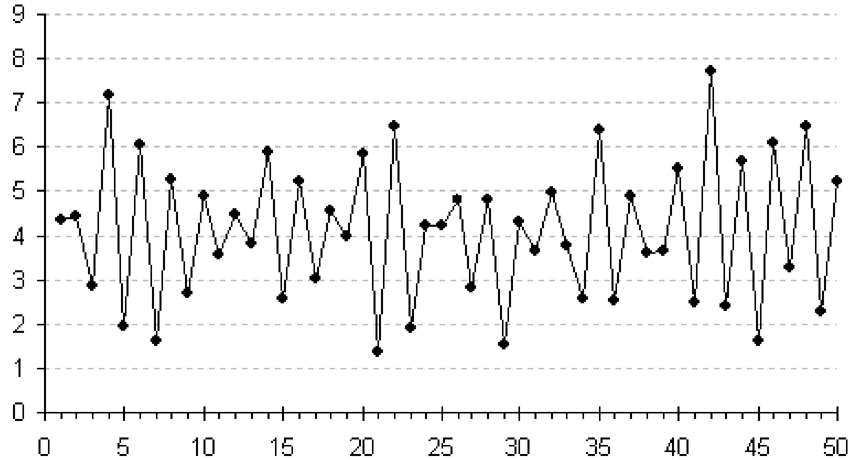
$$S^*(\mu,\phi) = \sum_{t=2}^{N} \left[y_t - \mu - \phi(y_{t-1} - \mu) \right]^2.$$

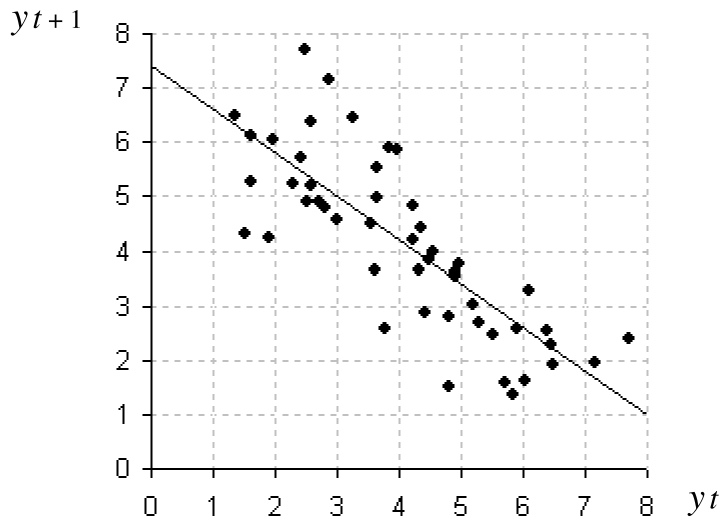
Least squares estimates: $\hat{\mu} \approx \overline{y}$, $\hat{\phi} \approx r_1$.

$\hat{y}_t = \hat{\mu} + \hat{\phi}\,(y_{t-1} - \hat{\mu})$ $\quad \hat{e}_t = y_t - \hat{y}_t$ $\quad t = 2, \ldots, N$.

$$\hat{\sigma}_e^2 = \frac{1}{N-3} \sum_{t=2}^{N} \hat{e}_t^{\,2}.$$

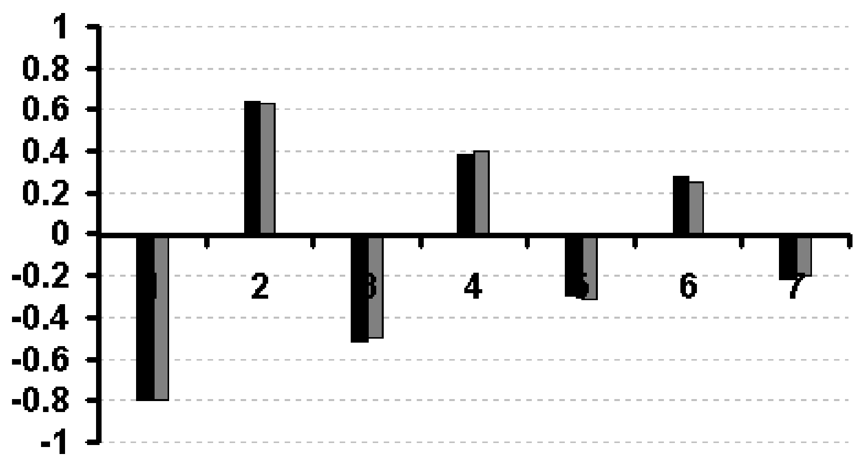
t	y_t
1	4.36
2	4.42
3	2.86
4	7.17
5	1.96
6	6.04
7	1.62
8	5.28
9	2.69
10	4.91
11	3.56
12	4.48
13	3.83
14	5.89
15	2.58
16	5.21
17	3.01
18	4.56
19	3.98
20	5.85
21	1.36
22	6.48
23	1.90
24	4.23
25	4.22
26	4.82
27	2.80
28	4.80
29	1.52
30	4.32
31	3.64
32	4.98
33	3.77
34	2.58
35	6.38
36	2.53
37	4.91
38	3.61
39	3.64
40	5.51
41	2.47
42	7.70
43	2.41
44	5.70
45	1.60
46	6.10
47	3.27
48	6.47
49	2.28
50	5.24



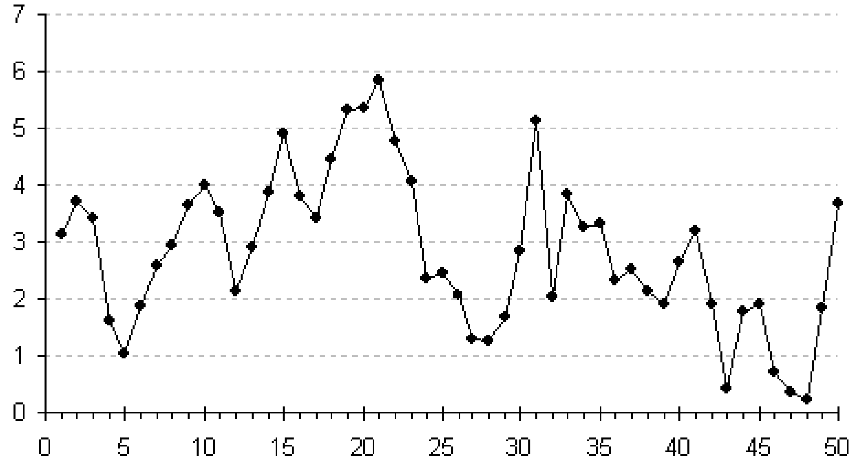


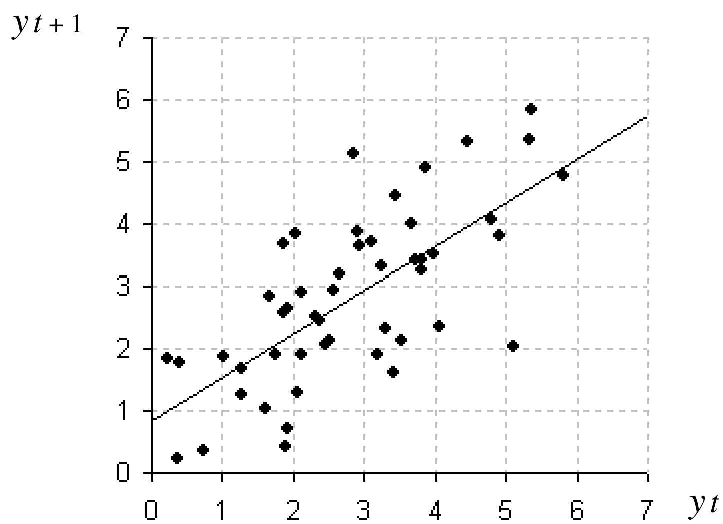
$$(\mathrm{Y}_t - \quad) = \phi(\mathrm{Y}_{t-1} - \quad) + e_t$$

$$\hat{\mu} \approx 4.11 \qquad \hat{\phi} \approx -0.8$$



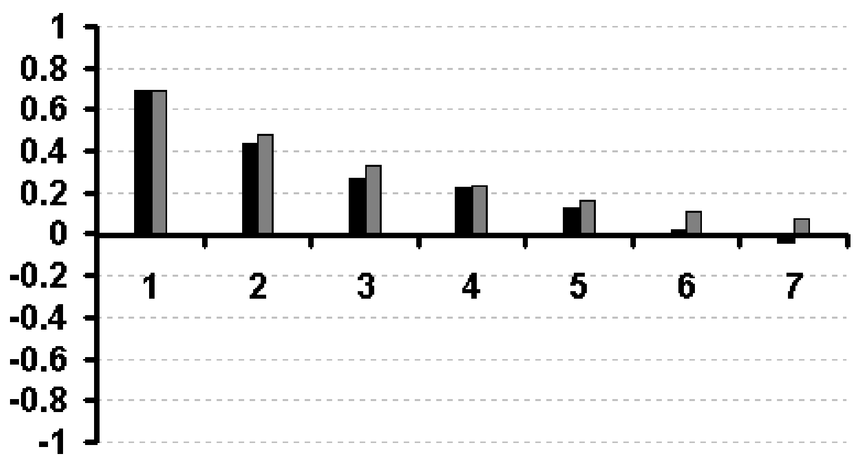
t	y_t
1	3.11
2	3.72
3	3.41
4	1.60
5	1.03
6	1.87
7	2.58
8	2.95
9	3.66
10	3.99
11	3.53
12	2.12
13	2.91
14	3.87
15	4.91
16	3.81
17	3.43
18	4.46
19	5.33
20	5.36
21	5.83
22	4.79
23	4.07
24	2.37
25	2.45
26	2.07
27	1.28
28	1.27
29	1.67
30	2.84
31	5.11
32	2.03
33	3.82
34	3.26
35	3.31
36	2.31
37	2.52
38	2.12
39	1.91
40	2.65
41	3.19
42	1.90
43	0.41
44	1.76
45	1.92
46	0.72
47	0.35
48	0.24
49	1.85
50	3.68





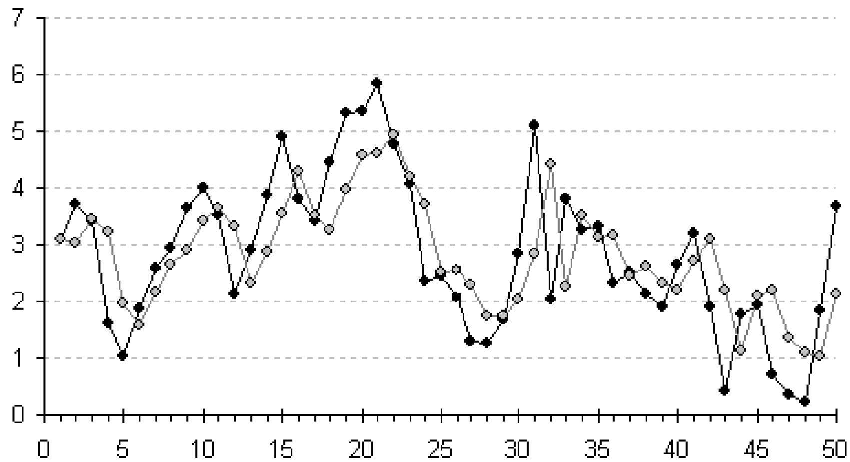
$$(Y_t - \quad) = \phi(Y_{t-1} - \quad) + e_t$$

$$\hat{\mu} \approx 2.83 \qquad \hat{\phi} \approx 0.7$$



t	y_t	$\hat{y}_t$	$\hat{e}_t$
1	3.11	3.11	
2	3.72	3.026	0.694
3	3.41	3.453	–0.043
4	1.60	3.236	–1.636
5	1.03	1.969	–0.939
6	1.87	1.57	0.3
7	2.58	2.158	0.422
8	2.95	2.655	0.295
9	3.66	2.914	0.746
10	3.99	3.411	0.579
11	3.53	3.642	–0.112
12	2.12	3.32	–1.2
13	2.91	2.333	0.577
14	3.87	2.886	0.984
15	4.91	3.558	1.352
16	3.81	4.286	–0.476
17	3.43	3.516	–0.086
18	4.46	3.25	1.21
19	5.33	3.971	1.359
20	5.36	4.58	0.78
21	5.83	4.601	1.229
22	4.79	4.93	–0.14
23	4.07	4.202	–0.132
24	2.37	3.698	–1.328
25	2.45	2.508	–0.058
26	2.07	2.564	–0.494
27	1.28	2.298	–1.018
28	1.27	1.745	–0.475
29	1.67	1.738	–0.068
30	2.84	2.018	0.822
31	5.11	2.837	2.273
32	2.03	4.426	–2.396
33	3.82	2.27	1.55
34	3.26	3.523	–0.263
35	3.31	3.131	0.179
36	2.31	3.166	–0.856
37	2.52	2.466	0.054
38	2.12	2.613	–0.493
39	1.91	2.333	–0.423
40	2.65	2.186	0.464
41	3.19	2.704	0.486
42	1.90	3.082	–1.182
43	0.41	2.179	–1.769
44	1.76	1.136	0.624
45	1.92	2.081	–0.161
46	0.72	2.193	–1.473
47	0.35	1.353	–1.003
48	0.24	1.094	–0.854
49	1.85	1.017	0.833
50	3.68	2.144	1.536

$$\hat{y}_t = \hat{\mu} + \hat{\phi}\,(y_{t-1} - \hat{\mu}) \qquad \hat{e}_t = y_t - \hat{y}_t \qquad t = 2, \ldots, N.$$



$$\hat{\sigma}_e^2 = \frac{1}{N-3}\sum_{t=2}^{N} \hat{e}_t^2 \approx 0.988 \qquad \hat{\sigma}_e \approx 0.994$$

$$\hat{y}_{N+1} = \mathrm{E}(\mathrm{Y}_{N+1}) = \quad + \phi(y_N - \quad)$$

$$\mathrm{Var}(\mathrm{Y}_{N+1} - \hat{y}_{N+1}) = \mathrm{Var}(e_{N+1}) = \sigma_e^2$$

Forecast: $\hat{y}_{N+1} = \hat{\mu} + \hat{\phi}\,(y_N - \hat{\mu}) = 3.425$

$3.425 \pm 2 \cdot 0.994$

$$\hat{y}_{N+2} = \mathrm{E}(\mathrm{Y}_{N+2}) = \quad + \phi(\hat{y}_{N+1} - \quad) = \quad + \phi^2(y_N - \quad)$$

$$\mathrm{Var}(\mathrm{Y}_{N+2} - \hat{y}_{N+2}) = \phi^2\,\mathrm{Var}(e_{N+1}) + \mathrm{Var}(e_{N+2})$$
$$= (1 + \phi^2)\,\sigma_e^2$$

Forecast: $\hat{y}_{N+2} = \hat{\mu} + \hat{\phi}^2(y_N - \hat{\mu}) \approx 3.2465$

$(1 + \hat{\phi}^2)\,\hat{\sigma}_e^2 \approx 1.472 \qquad 3.2465 \pm 2\sqrt{1.472}$

$$\hat{y}_{N+k} = \mathrm{E}(\mathrm{Y}_{N+k}) = \quad + \phi^k(y_N - \quad)$$

$$\mathrm{Var}(\mathrm{Y}_{N+k} - \hat{y}_{N+k}) = (1 + \phi^2 + \ldots + \phi^{2(k-1)})\,\sigma_e^2$$

Forecast: $\hat{y}_{N+k} = \hat{\mu} + \hat{\phi}^k(y_N - \hat{\mu})$

AR(p): $\quad \mathrm{Y}_t = \phi_1 \mathrm{Y}_{t-1} + \phi_2 \mathrm{Y}_{t-2} + \ldots + \phi_p \mathrm{Y}_{t-p} + e_t$

$$\downarrow$$

Consider $\quad \mathrm{Cov}(\ldots, \mathrm{Y}_{t-k}), \qquad k = 1, 2, \ldots, p.$

$$\Rightarrow \quad \begin{aligned}
\gamma_1 &= \phi_1 \gamma_0 + \phi_2 \gamma_1 + \phi_3 \gamma_2 + \ldots + \phi_p \gamma_{p-1} \\
\gamma_2 &= \phi_1 \gamma_1 + \phi_2 \gamma_0 + \phi_3 \gamma_1 + \ldots + \phi_p \gamma_{p-2} \\
&\ldots \\
\gamma_p &= \phi_1 \gamma_{p-1} + \phi_2 \gamma_{p-2} + \phi_3 \gamma_{p-3} + \ldots + \phi_p \gamma_0
\end{aligned}$$

Divide by γ_0:

$$\Rightarrow \quad \begin{aligned}
\rho_1 &= \phi_1 \rho_0 + \phi_2 \rho_1 + \phi_3 \rho_2 + \ldots + \phi_p \rho_{p-1} \\
\rho_2 &= \phi_1 \rho_1 + \phi_2 \rho_0 + \phi_3 \rho_1 + \ldots + \phi_p \rho_{p-2} \\
&\ldots \\
\rho_p &= \phi_1 \rho_{p-1} + \phi_2 \rho_{p-2} + \phi_3 \rho_{p-3} + \ldots + \phi_p \rho_0
\end{aligned}$$

(Yule-Walker equations)

The Yule-Walker equations for an AR(2) process:

$$\rho_1 = \phi_1 + \phi_2 \rho_1$$

$$\rho_2 = \phi_1 \rho_1 + \phi_2$$

$$\rho_1 = \phi_1 + \phi_2 \rho_1 \qquad \Rightarrow \qquad \rho_1 = \frac{\phi_1}{1-\phi_2}$$

$$\Rightarrow \qquad \rho_2 = \phi_1 \rho_1 + \phi_2 = \frac{\phi_1^2}{1-\phi_2} + \phi_2$$

$k \geq 2 \qquad \rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2}$