

**Bivariate Normal Distribution:**

$$f(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp \left\{ -\frac{1}{2(1-\rho^2)} \left[ \left( \frac{x-\mu_1}{\sigma_1} \right)^2 - 2\rho \left( \frac{x-\mu_1}{\sigma_1} \right) \left( \frac{y-\mu_2}{\sigma_2} \right) + \left( \frac{y-\mu_2}{\sigma_2} \right)^2 \right] \right\},$$

$$-\infty < x < \infty, \quad -\infty < y < \infty.$$

(a) the marginal distributions of  $X$  and  $Y$  are  $\mathbf{N}(\mu_1, \sigma_1^2)$  and  $\mathbf{N}(\mu_2, \sigma_2^2)$ , respectively;

(b) the correlation coefficient of  $X$  and  $Y$  is  $\rho_{XY} = \rho$ , and  $X$  and  $Y$  are independent if and only if  $\rho = 0$ ;

(c) the conditional distribution of  $Y$ , given  $X = x$ , is

$$\mathbf{N} \left( \mu_2 + \rho \frac{\sigma_2}{\sigma_1} (x - \mu_1), (1 - \rho^2) \sigma_2^2 \right);$$

(d) the conditional distribution of  $X$ , given  $Y = y$ , is

$$\mathbf{N} \left( \mu_1 + \rho \frac{\sigma_1}{\sigma_2} (y - \mu_2), (1 - \rho^2) \sigma_1^2 \right).$$

(e)  $aX + bY$  is normally distributed with

$$\text{mean} \quad E(aX + bY) = a\mu_1 + b\mu_2 \quad \text{and}$$

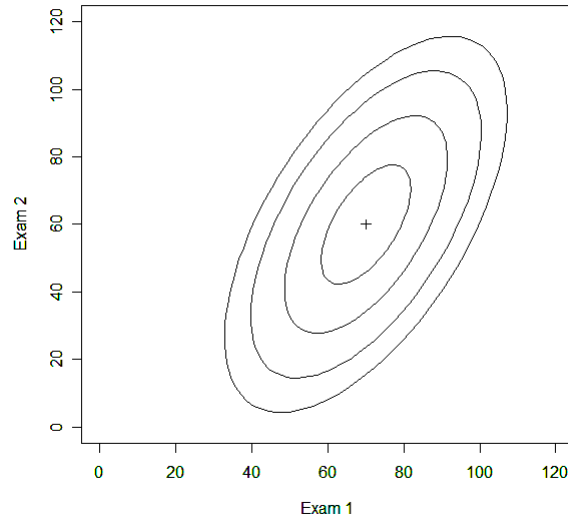
$$\text{variance} \quad \text{Var}(aX + bY) = a^2\sigma_1^2 + 2ab\rho\sigma_1\sigma_2 + b^2\sigma_2^2.$$

1. A large class took two exams. Suppose the exam scores  $X$  (Exam 1) and  $Y$  (Exam 2) follow a bivariate normal distribution with

$$\mu_1 = 70, \quad \sigma_1 = 10,$$

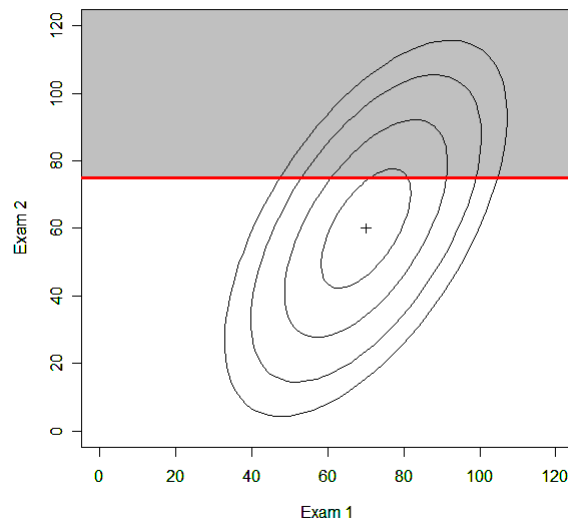
$$\mu_2 = 60, \quad \sigma_2 = 15,$$

$$\rho = 0.6.$$



- a) A student is selected at random. What is the probability that his/her score on Exam 2 is over 75?

$$P(Y > 75) = P\left(Z > \frac{75 - 60}{15}\right) = P(Z > 1.00) = \mathbf{0.1587}.$$



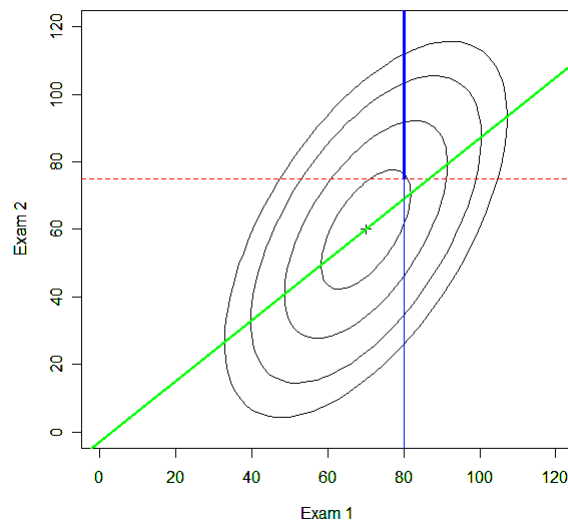
- b) Suppose you're told that a student got a 80 on Exam 1. What is the probability that his/her score on Exam 2 is over 75?

Given  $X = 80$ ,  $Y$  has Normal distribution

$$\text{with mean } 60 + 0.6 \cdot \frac{15}{10} \cdot (80 - 70) = 69$$

$$\text{and variance } (1 - 0.6^2) \cdot 15^2 = 144 \quad (\text{standard deviation } 12).$$

$$P(Y > 75 \mid X = 80) = P(Z > \frac{75 - 69}{12}) = P(Z > 0.50) = \mathbf{0.3085}.$$



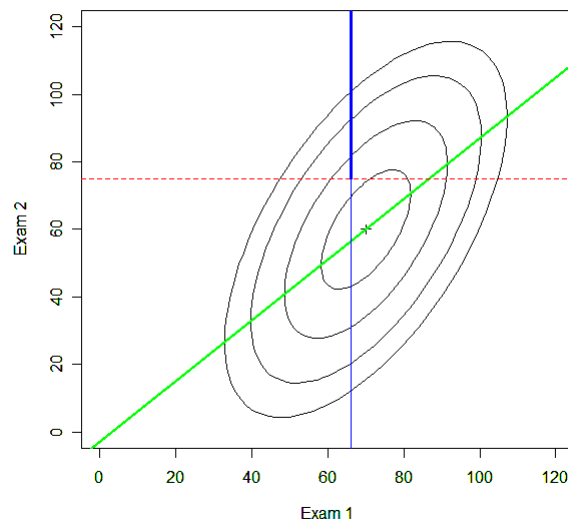
- c) Suppose you're told that a student got a 66 on Exam 1. What is the probability that his/her score on Exam 2 is over 75?

Given  $X = 66$ ,  $Y$  has Normal distribution

$$\text{with mean } 60 + 0.6 \cdot \frac{15}{10} \cdot (66 - 70) = 56.4$$

$$\text{and variance } (1 - 0.6^2) \cdot 15^2 = 144 \quad (\text{standard deviation } 12).$$

$$P(Y > 75 \mid X = 66) = P(Z > \frac{75 - 56.4}{12}) = P(Z > 1.55) = \mathbf{0.0606}.$$



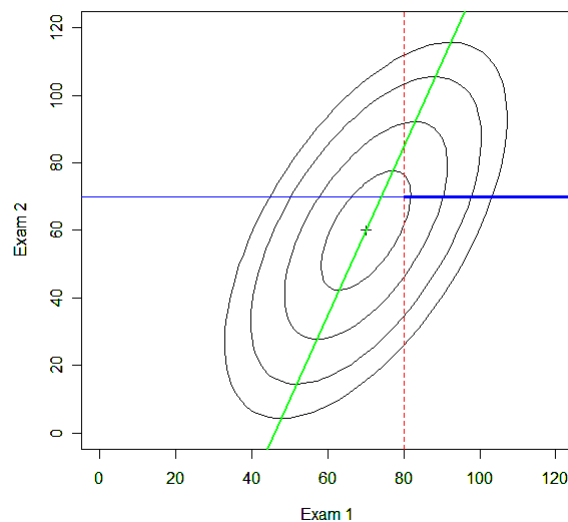
- d) Suppose you're told that a student got a 70 on Exam 2. What is the probability that his/her score on Exam 1 is over 80?

Given  $Y = 70$ ,  $X$  has Normal distribution

$$\text{with mean } 70 + 0.6 \cdot \frac{10}{15} \cdot (70 - 60) = 74$$

$$\text{and variance } (1 - 0.6^2) \cdot 10^2 = 64 \quad (\text{standard deviation } 8).$$

$$P(X > 80 \mid Y = 70) = P\left(Z > \frac{80 - 74}{8}\right) = P(Z > 0.75) = \mathbf{0.2266}.$$



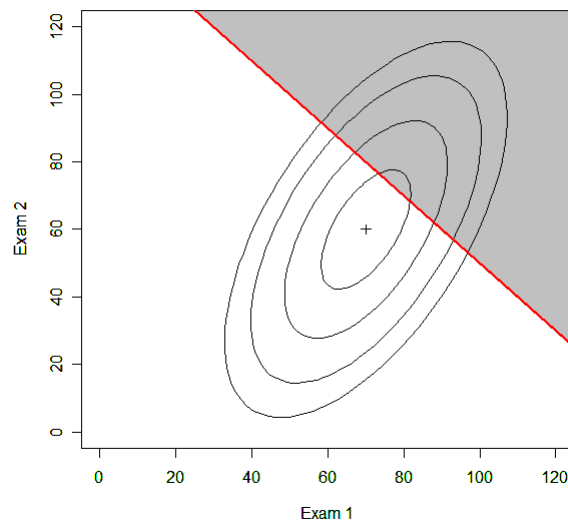
- e) A student is selected at random. What is the probability that the sum of his/her Exam 1 and Exam 2 scores is over 150?

$X + Y$  has Normal distribution,

$$E(X + Y) = \mu_X + \mu_Y = 70 + 60 = 130,$$

$$\begin{aligned} \text{Var}(X + Y) &= \sigma_X^2 + 2\rho\sigma_X\sigma_Y + \sigma_Y^2 = \sigma_X^2 + 2\rho\sigma_X\sigma_Y + \sigma_Y^2 \\ &= 10^2 + 2 \cdot 0.6 \cdot 10 \cdot 15 + 15^2 = 505 \quad (\text{standard deviation} \approx 22.4722). \end{aligned}$$

$$P(X + Y > 150) = P\left(Z > \frac{150 - 130}{22.4722}\right) = P(Z > 0.89) = \mathbf{0.1867}.$$



f) What proportion of students did better on Exam 1 than on Exam 2?

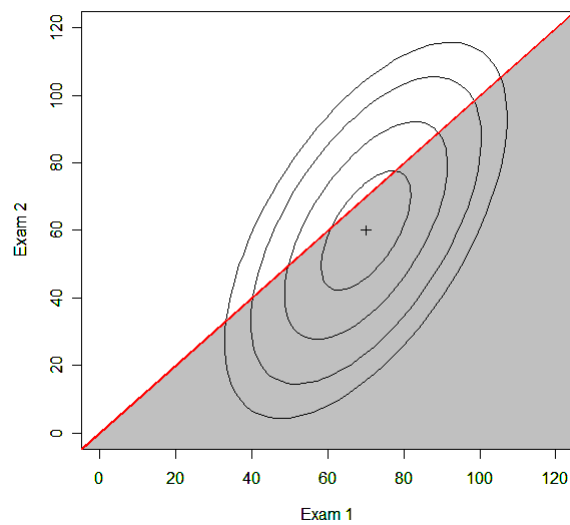
Want  $P(X > Y) = P(X - Y > 0) = ?$

$X - Y$  has Normal distribution,

$$E(X - Y) = \mu_X - \mu_Y = 70 - 60 = 10,$$

$$\begin{aligned} \text{Var}(X - Y) &= \sigma_X^2 - 2\rho\sigma_X\sigma_Y + \sigma_Y^2 \\ &= 10^2 - 2 \cdot 0.6 \cdot 10 \cdot 15 + 15^2 = 145 \quad (\text{standard deviation} \approx 12.0416). \end{aligned}$$

$$P(X - Y > 0) = P\left(Z > \frac{0 - 10}{12.0416}\right) = P(Z > -0.83) = \mathbf{0.7967}.$$



g) Find  $P(2X + 3Y > 350)$ .

$2X + 3Y$  has Normal distribution,

$$E(2X + 3Y) = 2\mu_X + 3\mu_Y = 2 \times 70 + 3 \times 60 = 320,$$

$$\begin{aligned}\text{Var}(2X + 3Y) &= 4\sigma_X^2 + 12\sigma_{XY} + 9\sigma_Y^2 = 4\sigma_X^2 + 12\rho\sigma_X\sigma_Y + 9\sigma_Y^2 \\ &= 4 \times 10^2 + 12 \cdot 0.6 \cdot 10 \cdot 15 + 9 \times 15^2 = 3505\end{aligned}$$

( standard deviation  $\approx 59.203$  ).

$$P(2X + 3Y > 350) = P\left(Z > \frac{350 - 320}{59.203}\right) = P(Z > 0.5067) \approx \mathbf{0.3050}.$$

h) Find  $P(5X + 3Y < 570)$ .

$5X + 3Y$  has Normal distribution,

$$E(5X + 3Y) = 5\mu_X + 3\mu_Y = 5 \times 70 + 3 \times 60 = 530,$$

$$\begin{aligned}\text{Var}(5X + 3Y) &= 25\sigma_X^2 + 30\sigma_{XY} + 9\sigma_Y^2 = 25\sigma_X^2 + 30\rho\sigma_X\sigma_Y + 9\sigma_Y^2 \\ &= 25 \times 10^2 + 30 \cdot 0.6 \cdot 10 \cdot 15 + 9 \times 15^2 = 7225\end{aligned}$$

( standard deviation = 85 ).

$$P(5X + 3Y < 570) = P\left(Z < \frac{570 - 530}{85}\right) = P(Z < 0.47) = \mathbf{0.6808}.$$

i) Find  $P(5X - 4Y > 150)$ .

$5X - 4Y$  has Normal distribution,

$$E(5X - 4Y) = 5\mu_X - 4\mu_Y = 5 \times 70 - 4 \times 60 = 110,$$

$$\begin{aligned}\text{Var}(5X - 4Y) &= 25\sigma_X^2 - 40\sigma_{XY} + 16\sigma_Y^2 = 25\sigma_X^2 - 40\rho\sigma_X\sigma_Y + 16\sigma_Y^2 \\ &= 25 \times 10^2 - 40 \cdot 0.6 \cdot 10 \cdot 15 + 16 \times 15^2 = 2500\end{aligned}$$

( standard deviation = 50 ).

$$P(5X - 4Y > 150) = P\left(Z > \frac{150 - 110}{50}\right) = P(Z > 0.80) = \mathbf{0.2119}.$$