STAT 420 Fall 2015

Examples for 12/01/15, Part 1

Two-Way ANOVA without Replication

The cutting speeds of four types of tools are being compared in an experiment. Five materials of varying degree of hardness are to be used as experimental blocks. Measurements of cutting time (in seconds) according to types of tool (Factor A) and Hardness of Material (Factor B) are given in the table below.

I Factor A levels, J Factor B levels

Factor A	Factor B 1	2	3	4	5	Factor A Means
1	12	2	8	1	7	$\bar{y}_{1\bullet} = 6$
2	20	14	17	12	17	$\bar{y}_{2\bullet} = 16$
3	13	7	13	8	14	$\bar{y}_{3\bullet} = 11$
4	11	5	10	3	6	$\bar{y}_{4\bullet} = 7$
Factor B Means	$\bar{y}_{\bullet 1} = 14$	$\bar{y}_{\bullet 2} = 7$	$\bar{y}_{\bullet 3} = 12$	$\bar{y}_{\bullet 4} = 6$	$\bar{y}_{\bullet 5} = 11$	$\bar{y}_{\bullet\bullet} = 10$

$$\sum_{i=1}^{I}\sum_{j=1}^{J}\left(y_{ij}-\bar{y}_{\bullet\bullet}\right)^2 = J\sum_{i=1}^{I}\left(\bar{y}_{i\bullet}-\bar{y}_{\bullet\bullet}\right)^2 + I\sum_{j=1}^{J}\left(\bar{y}_{\bullet j}-\bar{y}_{\bullet\bullet}\right)^2 + \sum_{i=1}^{I}\sum_{j=1}^{J}\left(y_{ij}-\bar{y}_{i\bullet}-\bar{y}_{\bullet j}+\bar{y}_{\bullet\bullet}\right)^2$$

SST	SSA	SSB	SSR
$IJ-1$	$I-1$	$J-1$	$(I-1)(J-1)$

ANOVA table:

Source	SS	DF	MS	F
Factor A	310	3	103.3333	51.66667
Factor B	184	4	46	23
Residuals	24	12	2	
Total	518	19		

$$y_{ij} \;=\; \bar{y}_{\bullet\bullet} \;+\; \left(\bar{y}_{i\bullet} - \bar{y}_{\bullet\bullet}\right) \;+\; \left(\bar{y}_{\bullet j} - \bar{y}_{\bullet\bullet}\right) \;+\; \left(y_{ij} - \bar{y}_{i\bullet} - \bar{y}_{\bullet j} + \bar{y}_{\bullet\bullet}\right)$$

$$\begin{bmatrix} 12 & 2 & 8 & 1 & 7 \\ 20 & 14 & 17 & 12 & 17 \\ 13 & 7 & 13 & 8 & 14 \\ 11 & 5 & 10 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 10 & 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 & 10 \end{bmatrix}$$

$$+ \begin{bmatrix} -4 & -4 & -4 & -4 & -4 \\ 6 & 6 & 6 & 6 & 6 \\ 1 & 1 & 1 & 1 & 1 \\ -3 & -3 & -3 & -3 & -3 \end{bmatrix} + \begin{bmatrix} 4 & -3 & 2 & -4 & 1 \\ 4 & -3 & 2 & -4 & 1 \\ 4 & -3 & 2 & -4 & 1 \\ 4 & -3 & 2 & -4 & 1 \end{bmatrix}$$

$$+ \begin{bmatrix} 2 & -1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -2 & -1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 & -2 \end{bmatrix}$$

SST $\displaystyle\sum_{i=1}^{I}\sum_{j=1}^{J}\left(y_{ij} - \bar{y}_{\bullet\bullet}\right)^2$ $IJ-1$ d.f.

SSA $\displaystyle\sum_{i=1}^{I}\sum_{j=1}^{J}\left(\bar{y}_{i\bullet} - \bar{y}_{\bullet\bullet}\right)^2 = J\sum_{i=1}^{I}\left(\bar{y}_{i\bullet} - \bar{y}_{\bullet\bullet}\right)^2$ $I-1$ d.f.

SSB $\displaystyle\sum_{i=1}^{I}\sum_{j=1}^{J}\left(\bar{y}_{\bullet j} - \bar{y}_{\bullet\bullet}\right)^2 = I\sum_{j=1}^{J}\left(\bar{y}_{\bullet j} - \bar{y}_{\bullet\bullet}\right)^2$ $J-1$ d.f.

SSR $\displaystyle\sum_{i=1}^{I}\sum_{j=1}^{J}\left(y_{ij} - \bar{y}_{i\bullet} - \bar{y}_{\bullet j} + \bar{y}_{\bullet\bullet}\right)^2$ $(I-1)(J-1)$ d.f.

$$\text{SSA} = 5 \times [\,(6-10)^2 + (16-10)^2 + (11-10)^2 + (7-10)^2\,]$$
$$= 5 \times [\,16 + 36 + 1 + 9\,] = 310.$$

$$\text{SSB} = 4 \times [\,(14-10)^2 + (7-10)^2 + (12-10)^2 + (6-10)^2 + (11-10)^2\,]$$
$$= 4 \times [\,16 + 9 + 4 + 16 + 1\,] = 184.$$

$$\text{SSR} = (12-6-14+10)^2 + (2-6-7+10)^2 + \ldots + (6-7-11+10)^2 = 24.$$

$$\text{SST} = (12-10)^2 + (2-10)^2 + \ldots + (6-10)^2 = 518.$$

```
Time <- c(12,2,8,1,7,20,14,17,12,17,13,7,13,8,14,11,5,10,3,6)
A <- c(1,1,1,1,1,2,2,2,2,2,3,3,3,3,3,4,4,4,4,4)
B <- c(1,2,3,4,5,1,2,3,4,5,1,2,3,4,5,1,2,3,4,5)
results <- lm(Time ~ factor(A) + factor(B))
summary(aov(results))
```

```
##             Df Sum Sq Mean Sq F value   Pr(>F)
## factor(A)    3    310   103.3   51.67 3.91e-07 ***
## factor(B)    4    184    46.0   23.00 1.49e-05 ***
## Residuals   12     24     2.0
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

$$Y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij}, \qquad i = 1, 2, 3, 4, \quad j = 1, 2, 3, 4, 5,$$

ε_{ij} are independent $N(0, \sigma^2)$ random variables.

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0, \qquad \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 = 0,$$

$H_0: \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$

```
qf(0.95,3,12)
```

```
## [1] 3.490295
```

```
qf(0.99,3,12)
```

```
## [1] 5.952545
```

$F = 51.66667$

Reject H_0

$H_0: \beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = 0$

```
qf(0.95,4,12)
```

```
## [1] 3.259167
```

```
qf(0.99,4,12)
```

```
## [1] 5.411951
```

$F = 23$

Reject H_0