Homework #10

(due Friday, November 30, by 3:00 p.m.)

1. Given the time series

31 34 33 28 26 28 30

calculate r_1 and r_2.

(Note: In practice reliable autocorrelation estimates are only obtained from series consisting of approximately 50 observations or more.)

$$\bar{y} = \frac{31+34+33+28+26+28+30}{7} = 30.$$

y_t	$y_t - \bar{y}$	$(y_t - \bar{y})^2$	$(y_t - \bar{y})(y_{t+1} - \bar{y})$	$(y_t - \bar{y})(y_{t+2} - \bar{y})$
31	1	1	4	3
34	4	16	12	– 8
33	3	9	– 6	– 12
28	– 2	4	8	4
26	– 4	16	8	0
28	– 2	4	0	
30	0	0		
		50	26	– 13

$$r_1 = \frac{\sum_{t=1}^{N-1}(y_t - \bar{y})(y_{t+1} - \bar{y})}{\sum_{t=1}^{N}(y_t - \bar{y})^2} = \frac{26}{50} = \mathbf{0.52}.$$

$$r_2 = \frac{\sum_{t=1}^{N-2}(y_t - \bar{y})(y_{t+2} - \bar{y})}{\sum_{t=1}^{N}(y_t - \bar{y})^2} = \frac{-13}{50} = \mathbf{-0.26}.$$

2. Suppose you are fitting the model

$$(Y_t - \mu) = \phi(Y_{t-1} - \mu) + e_t$$

to a time series whose first 8 values are

25 21 26 23 29 25 28 23

Obtain the (approximate) least squares estimates of ϕ and μ.

$$\hat{\mu} \approx \bar{y} = \frac{25+21+26+23+29+25+28+23}{8} = \mathbf{25}.$$

y_t	$y_t - \bar{y}$	$(y_t - \bar{y})^2$	$(y_t - \bar{y})(y_{t+1} - \bar{y})$
25	0	0	0
21	− 4	16	− 4
26	1	1	− 2
23	− 2	4	− 8
29	4	16	0
25	0	0	0
28	3	9	− 6
23	− 2	4	

$$\hat{\phi} \approx r_1 = \frac{\sum_{t=1}^{N-1}(y_t - \bar{y})(y_{t+1} - \bar{y})}{\sum_{t=1}^{N}(y_t - \bar{y})^2} = \frac{-20}{50} = \mathbf{-0.4}.$$

3. The model

$$(\mathrm{Y}_t - \mu) = \phi(\mathrm{Y}_{t-1} - \mu) + e_t$$

has been fitted to a time series giving $\hat{\phi} = 0.6$, $\hat{\mu} = 40$, and $\hat{\sigma}_e^2 = 9$. The last five values of the series are

$$\ldots, \quad 47, \quad 35, \quad 44, \quad 36, \quad 46.$$

Using the time corresponding to the last observation as the forecast origin, calculate the forecasts and approximate 95-percent probability limits for the next **four** observations.

$\hat{y}_{N+1} = \hat{\mu} + \hat{\phi}\,(y_N - \hat{\mu}) = 40 + 0.6\,(46 - 40) = \mathbf{43.6}$.

95% probability limits:

$\hat{y}_{N+1} \pm 2\hat{\sigma}_e$ $\qquad$ $\mathbf{43.6 \pm 6}$

$\hat{y}_{N+2} = \hat{\mu} + \hat{\phi}\,(\hat{y}_{N+1} - \hat{\mu}) = 40 + 0.6\,(43.6 - 40) = \mathbf{42.16}$.

95% probability limits:

$\hat{y}_{N+2} \pm 2\sqrt{1+\hat{\phi}^2}\,\hat{\sigma}_e$ $\qquad$ $\mathbf{42.16 \pm 6.997}$

$\hat{y}_{N+3} = \hat{\mu} + \hat{\phi}\,(\hat{y}_{N+2} - \hat{\mu}) = 40 + 0.6\,(42.16 - 40) = \mathbf{41.296}$.

95% probability limits:

$\hat{y}_{N+3} \pm 2\sqrt{1+\hat{\phi}^2+\hat{\phi}^4}\,\hat{\sigma}_e$ $\qquad$ $\mathbf{41.296 \pm 7.323}$

$\hat{y}_{N+4} = \hat{\mu} + \hat{\phi}\,(\hat{y}_{N+3} - \hat{\mu}) = 40 + 0.6\,(41.296 - 40) = \mathbf{40.7776}$.

95% probability limits:

$\hat{y}_{N+4} \pm 2\sqrt{1+\hat{\phi}^2+\hat{\phi}^4+\hat{\phi}^6}\,\hat{\sigma}_e$ $\qquad$ $\mathbf{40.7776 \pm 7.43675}$

For fun: $\hat{y}_{N+5} = \hat{\mu} + \hat{\phi}\,(\hat{y}_{N+4} - \hat{\mu}) = 40 + 0.6\,(40.7776 - 40) = \mathbf{40.46656}$.

95% probability limits:

$\hat{y}_{N+5} \pm 2\sqrt{1+\hat{\phi}^2+\hat{\phi}^4+\hat{\phi}^6+\hat{\phi}^8}\,\hat{\sigma}_e$ $\qquad$ $\mathbf{40.46656 \pm 7.47729}$

4. Assume the following 10 observations:

$$2, \quad -1, \quad -4, \quad 0, \quad 2, \quad 3, \quad 1, \quad -2, \quad -1, \quad 1$$

were generated by a first-order autoregressive process that can be described by a model of the form (note that $\mu = 0$)

$$\mathrm{Y}_t = \phi \mathrm{Y}_{t-1} + e_t$$

Obtain the least squares estimate of ϕ.

Hint: Consider $S^*(\phi) = \sum_{t=2}^{N} \left(y_t - \phi\, y_{t-1} \right)^2$.

First, derive the expression for the least squares estimator $\hat{\phi}$.

That is, find ϕ that minimizes $S^*(\phi)$.

$$S^*(\phi) = \sum_{t=2}^{N} \left(y_t - \phi\, y_{t-1} \right)^2$$

$$\frac{\partial S^*(\phi)}{\partial \phi} = -2 \cdot \sum_{t=2}^{N} \left(y_t - \phi\, y_{t-1} \right) \cdot y_{t-1}$$

$$= -2 \cdot \left(\sum_{t=2}^{N} y_t \cdot y_{t-1} - \phi \cdot \sum_{t=2}^{N} y_{t-1}^2 \right) = 0.$$

$$\Rightarrow \quad \hat{\phi} = \frac{\sum_{t=2}^{N} y_t \cdot y_{t-1}}{\sum_{t=2}^{N} y_{t-1}^2} = \frac{\sum_{t=2}^{N} y_t \cdot y_{t-1}}{\sum_{t=1}^{N-1} y_t^2} = \frac{\sum_{t=1}^{N-1} y_{t+1} \cdot y_t}{\sum_{t=1}^{N-1} y_t^2}$$

$$= \frac{-2+4+0+0+6+3-2+2-1}{4+1+16+0+4+9+1+4+1} = \frac{10}{40} = \mathbf{0.25}.$$

5. Consider the $AR(2)$ processes

$$\dot{Y}_t - 0.3\ \dot{Y}_{t-1} - 0.1\ \dot{Y}_{t-2} = e_t$$

where $\{e_t\}$ is zero-mean white noise $(\text{i.i.d. } N(0, \sigma_e^2))$, $\dot{Y}_t = Y_t - \mu$.

a) Determine whether this model would produce a stationary process. That is, find the roots of $1 - 0.3\,z - 0.1\,z^2 = 0$ and check whether they are outside of the unit circle on the complex plane.

$$\Phi(B) = 1 - \phi_1 B - \phi_2 B^2 = 1 - 0.3\,B - 0.1\,B^2 = (1 - 0.5\,B)(1 + 0.2\,B)$$

The roots of $\Phi(z) = 0$ are $z_1 = 2$ and $z_2 = -5$.

All roots of $\Phi(z) = 0$ are outside the unit circle.

$\Rightarrow$ This process is stationary.

b) Use Yule-Walker equations to find ρ_1 and ρ_2.
Then find ρ_3 and ρ_4.

The Yule-Walker equations for an AR(2) process:

$$\rho_1 = \phi_1 + \phi_2 \rho_1$$
$$\rho_2 = \phi_1 \rho_1 + \phi_2$$

$$\rho_1 = \phi_1 + \phi_2 \rho_1 \qquad \Rightarrow \qquad \rho_1 = \frac{\phi_1}{1-\phi_2} = \frac{0.3}{1-0.1} = \mathbf{1/3}.$$

$$\Rightarrow \qquad \rho_2 = \phi_1 \rho_1 + \phi_2 = 0.3 \cdot 1/3 + 0.1 = \mathbf{0.20}.$$

$k \geq 2$ $$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2}$$

$$\rho_3 = \phi_1 \rho_2 + \phi_2 \rho_1 = 0.3 \cdot 0.20 + 0.1 \cdot \frac{1}{3} = \frac{28}{300} = \mathbf{\frac{7}{75}} \approx 0.09333.$$

$$\rho_4 = \phi_1 \rho_3 + \phi_2 \rho_2 = 0.3 \cdot \frac{7}{75} + 0.1 \cdot 0.20 = \mathbf{0.048}.$$