Examples for 12/8/2015 - Part 2

Time Series: y_t, $t = 1, 2, \ldots, N$.

Stationary process $\approx$ a random process where all of its probability properties do not vary with time.

$\mathrm{E}(\mathrm{Y}_t) = \mu$. $\quad\quad$ $\mathrm{Var}(\mathrm{Y}_t) = \sigma_{\mathrm{Y}}^2$.

$$\gamma(k) = \mathrm{Cov}(\mathrm{Y}_t, \mathrm{Y}_{t+k}) = \mathrm{E}[(\mathrm{Y}_t - \mu)(\mathrm{Y}_{t+k} - \mu)]$$
$$= \mathrm{Cov}(\mathrm{Y}_t, \mathrm{Y}_{t-k}) = \mathrm{E}[(\mathrm{Y}_t - \mu)(\mathrm{Y}_{t-k} - \mu)].$$

$$\gamma(0) = \mathrm{Var}(\mathrm{Y}_t) = \sigma_{\mathrm{Y}}^2.$$

$$\rho_k = \mathrm{Corr}(\mathrm{Y}_t, \mathrm{Y}_{t+k}) = \frac{\mathrm{Cov}(\mathrm{Y}_t, \mathrm{Y}_{t+k})}{\sqrt{\mathrm{Var}(\mathrm{Y}_t)\mathrm{Var}(\mathrm{Y}_{t+k})}} = \frac{\mathrm{E}(\mathrm{Y}_t - \mu)(\mathrm{Y}_{t+k} - \mu)}{\sigma_{\mathrm{Y}}^2},$$

$k = \pm 1, \pm 2, \ldots$.

$\rho_k = \gamma(k) / \gamma(0)$. $\quad\quad$ $\rho_0 = 1$.

Sample autocorrelation coefficient: $$r_k = \frac{\sum_{t=1}^{N-k}(y_t - \bar{y})(y_{t+k} - \bar{y})}{\sum_{t=1}^{N}(y_t - \bar{y})^2}$$

1. Calculate r_1 and r_2 for the time series 16 22 19 25 18

(Note: In practice reliable autocorrelation estimates are only obtained from series consisting of approximately 50 observations or more.)

Consider the following "regression" (autoregressive) model: AR(1)

$$(\mathrm{Y}_t - \mu) = \phi \cdot (\mathrm{Y}_{t-1} - \mu) + e_t$$

$\mathrm{E}(e_t) = 0,$ $\mathrm{Var}(e_t) = \sigma_e^2$ for all t

$\mathrm{E}(e_t e_s) = 0,$ for $t \neq s$

$\mathrm{E}(e_t \mathrm{Y}_s) = 0,$ for $s < t$

Then $\gamma(0) = \mathrm{Var}(\mathrm{Y}_t) = \dfrac{\sigma_e^2}{1-\phi^2},$ $\Rightarrow$ need $|\phi| < 1.$

$\gamma(k) = \phi\gamma(k-1),$ $k \geq 1.$

$\Rightarrow$ $\rho_k = \phi\rho_{k-1},$ $k \geq 1.$

Therefore, $\rho_k = \rho_{-k} = \phi^k,$ $k \geq 1.$

Suppose we observe $y_t,$ $t = 1, 2, \dots, N.$

Our forecast $\hat{y}_{N+k}$ of Y_{N+k} is

$$\hat{y}_{N+k} = \mathrm{E}(\mathrm{Y}_{N+k}) = \mu + \phi^k (y_N - \mu)$$

$$\mathrm{Var}(\mathrm{Y}_{N+k} - \hat{y}_{N+k}) = (1 + \phi^2 + \dots + \phi^{2(k-1)})\,\sigma_e^2$$

Least squares approach: Find the values of μ and ϕ that minimize

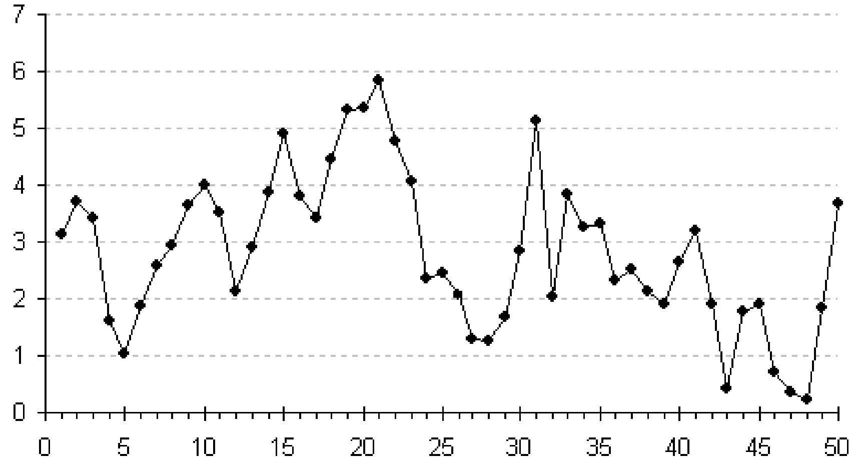
$$S^*(\mu,\phi) = \sum_{t=2}^{N} \left[y_t - \mu - \phi(y_{t-1} - \mu) \right]^2 .$$

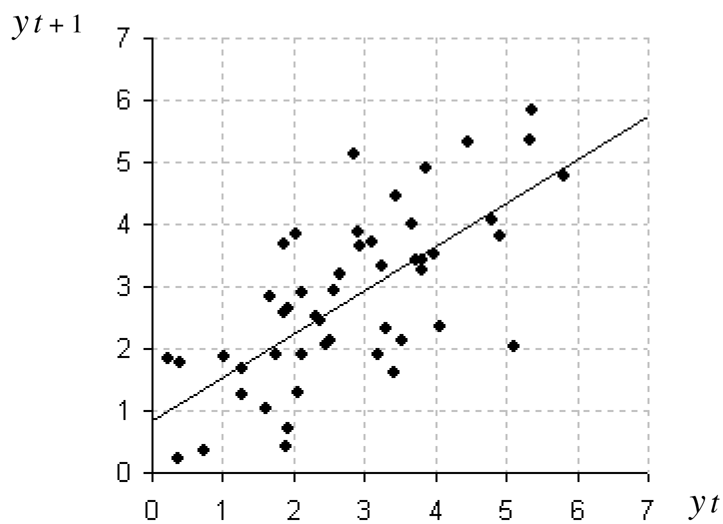
Least squares estimates: $\hat{\mu} \approx \overline{y},$ $\hat{\phi} \approx r_1.$

$\hat{y}_t = \hat{\mu} + \hat{\phi}\,(y_{t-1} - \hat{\mu})$ $\hat{e}_t = y_t - \hat{y}_t$ $t = 2, \dots, N.$

$$\hat{\sigma}_e^2 = \frac{1}{N-3} \sum_{t=2}^{N} \hat{e}_t^2 .$$

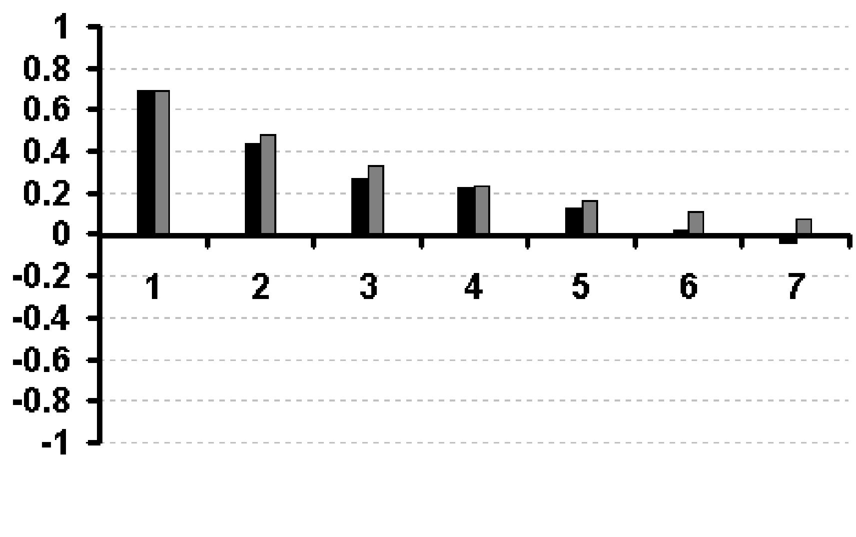
t	y_t
1	3.11
2	3.72
3	3.41
4	1.60
5	1.03
6	1.87
7	2.58
8	2.95
9	3.66
10	3.99
11	3.53
12	2.12
13	2.91
14	3.87
15	4.91
16	3.81
17	3.43
18	4.46
19	5.33
20	5.36
21	5.83
22	4.79
23	4.07
24	2.37
25	2.45
26	2.07
27	1.28
28	1.27
29	1.67
30	2.84
31	5.11
32	2.03
33	3.82
34	3.26
35	3.31
36	2.31
37	2.52
38	2.12
39	1.91
40	2.65
41	3.19
42	1.90
43	0.41
44	1.76
45	1.92
46	0.72
47	0.35
48	0.24
49	1.85
50	3.68





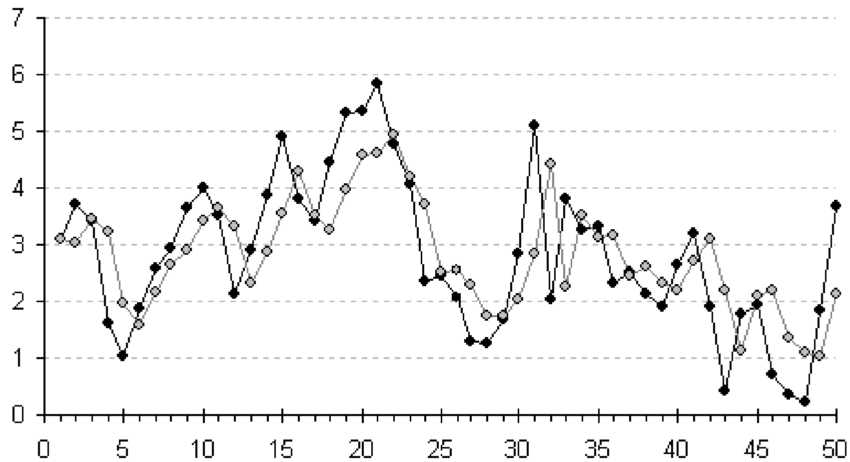
$$(Y_t - \mu) = \phi(Y_{t-1} - \mu) + e_t$$

$$\hat{\mu} \approx 2.83 \qquad \hat{\phi} \approx 0.7$$



t	y_t	$\hat{y}_t$	$\hat{e}_t$
1	3.11	3.11	
2	3.72	3.026	0.694
3	3.41	3.453	–0.043
4	1.60	3.236	–1.636
5	1.03	1.969	–0.939
6	1.87	1.57	0.3
7	2.58	2.158	0.422
8	2.95	2.655	0.295
9	3.66	2.914	0.746
10	3.99	3.411	0.579
11	3.53	3.642	–0.112
12	2.12	3.32	–1.2
13	2.91	2.333	0.577
14	3.87	2.886	0.984
15	4.91	3.558	1.352
16	3.81	4.286	–0.476
17	3.43	3.516	–0.086
18	4.46	3.25	1.21
19	5.33	3.971	1.359
20	5.36	4.58	0.78
21	5.83	4.601	1.229
22	4.79	4.93	–0.14
23	4.07	4.202	–0.132
24	2.37	3.698	–1.328
25	2.45	2.508	–0.058
26	2.07	2.564	–0.494
27	1.28	2.298	–1.018
28	1.27	1.745	–0.475
29	1.67	1.738	–0.068
30	2.84	2.018	0.822
31	5.11	2.837	2.273
32	2.03	4.426	–2.396
33	3.82	2.27	1.55
34	3.26	3.523	–0.263
35	3.31	3.131	0.179
36	2.31	3.166	–0.856
37	2.52	2.466	0.054
38	2.12	2.613	–0.493
39	1.91	2.333	–0.423
40	2.65	2.186	0.464
41	3.19	2.704	0.486
42	1.90	3.082	–1.182
43	0.41	2.179	–1.769
44	1.76	1.136	0.624
45	1.92	2.081	–0.161
46	0.72	2.193	–1.473
47	0.35	1.353	–1.003
48	0.24	1.094	–0.854
49	1.85	1.017	0.833
50	3.68	2.144	1.536

$\hat{y}_t = \hat{\mu} + \hat{\phi}\,(y_{t-1} - \hat{\mu})$ $\qquad \hat{e}_t = y_t - \hat{y}_t \qquad t = 2, \ldots, N.$



$$\hat{\sigma}_e^2 = \frac{1}{N-3}\sum_{t=2}^{N} \hat{e}_t^2 \approx 0.988 \qquad \hat{\sigma}_e \approx 0.994$$

$$\hat{y}_{N+1} = \mathrm{E}(\mathrm{Y}_{N+1}) = \mu + \phi(y_N - \mu)$$

$$\mathrm{Var}(\mathrm{Y}_{N+1} - \hat{y}_{N+1}) = \mathrm{Var}(e_{N+1}) = \sigma_e^2$$

Forecast: $\hat{y}_{N+1} = \hat{\mu} + \hat{\phi}\,(y_N - \hat{\mu}) = 3.425$

$3.425 \pm 2 \cdot 0.994$

$$\hat{y}_{N+2} = \mathrm{E}(\mathrm{Y}_{N+2}) = \mu + \phi(\hat{y}_{N+1} - \mu) = \mu + \phi^2(y_N - \mu)$$

$$\mathrm{Var}(\mathrm{Y}_{N+2} - \hat{y}_{N+2}) = \phi^2\,\mathrm{Var}(e_{N+1}) + \mathrm{Var}(e_{N+2})$$
$$= (1 + \phi^2)\,\sigma_e^2$$

Forecast: $\hat{y}_{N+2} = \hat{\mu} + \hat{\phi}^2(y_N - \hat{\mu}) \approx 3.2465$

$(1 + \hat{\phi}^2)\,\hat{\sigma}_e^2 \approx 1.472 \qquad 3.2465 \pm 2\sqrt{1.472}$

$$\hat{y}_{N+k} = \mathrm{E}(\mathrm{Y}_{N+k}) = \mu + \phi^k(y_N - \mu)$$

$$\mathrm{Var}(\mathrm{Y}_{N+k} - \hat{y}_{N+k}) = (1 + \phi^2 + \ldots + \phi^{2(k-1)})\,\sigma_e^2$$

Forecast: $\hat{y}_{N+k} = \hat{\mu} + \hat{\phi}^k(y_N - \hat{\mu})$