STAT 420 Fall 2012

Homework #12

(due Friday, December 13, by 4:30 p.m.)

1. Consider the $\text{AR}(2)$ processes is stationary.

$$\dot{Y}_t - 0.96\ \dot{Y}_{t-1} + 0.92\ \dot{Y}_{t-2} = e_t$$

where $\{ e_t \}$ is zero-mean white noise $(\text{i.i.d. } N(0, \sigma_e^2))$, $\dot{Y}_t = Y_t - \mu$.

a) Is this process stationary? *Justify your answer.*

Find the solutions of $\Phi(z) = 0$, z_1 and z_1. Are they both outside of the unit circle on the complex plane? Find $|z_1|$ and $|z_2|$.

$$\Phi(B) = 1 - 0.96\,B + 0.92\,B^2$$

The roots of $\Phi(z) = 0$ are $z_{1,2} = \dfrac{0.96 \pm \sqrt{0.96^2 - 4 \cdot 0.92 \cdot 1}}{2 \cdot 0.92} = \dfrac{0.96 \pm \sqrt{-2.7584}}{2 \cdot 0.92}$

$z_{1,2} \approx \mathbf{0.52174 \pm 0.90263}\,\boldsymbol{i}$.

$|z_{1,2}| = \sqrt{0.52174^2 + 0.90263^2} = \sqrt{1.0869565} = \mathbf{1.042572} > 1.$

Both roots of $\Phi(z) = 0$ are outside the unit circle.

$\Rightarrow$ This process is stationary.

OR

An AR(2) model is stationary if

$$-1 < \phi_2 < 1, \qquad \phi_2 + \phi_1 < 1, \qquad \phi_2 - \phi_1 < 1.$$

$$-1 < -0.92 < 1, \qquad -0.92 + 0.96 < 1, \qquad -0.92 - 0.96 < 1.$$

$\Rightarrow$ This process is stationary.

b) Use Yule-Walker equations to find ρ_1 and ρ_2.

The Yule-Walker equations for an AR (2) process are given by:

$$\rho_1 = \phi_1 + \phi_2 \rho_1$$
$$\rho_2 = \phi_1 \rho_1 + \phi_2$$

$\rho_1 = 0.96 - 0.92 \rho_1 \quad \Rightarrow \quad \rho_1 = \mathbf{0.50}$.

$\rho_2 = 0.96 \rho_1 - 0.92 \quad \Rightarrow \quad \rho_2 = \mathbf{-0.44}$.

c) Based on a series of length $N = 50$, we observe $\ldots,\ y_{49} = 26,\ y_{50} = 32,\ \overline{y} = 30$. Forecast y_{51} and y_{52}.

$$\hat{y}_{51} = \hat{\mu} + \phi_1 (y_{50} - \hat{\mu}) + \phi_2 (y_{49} - \hat{\mu})$$
$$= 30 + 0.96 (32 - 30) - 0.92 (26 - 30) = \mathbf{35.6}.$$

$$\hat{y}_{52} = \hat{\mu} + \phi_1 (\hat{y}_{51} - \hat{\mu}) + \phi_2 (y_{50} - \hat{\mu})$$
$$= 30 + 0.96 (35.6 - 30) - 0.92 (32 - 30) = \mathbf{33.536}.$$

2. Consider the following MA(1) process $\dot{Y}_t = e_t - \theta e_{t-1}$

where $\{e_t\}$ is zero-mean white noise (i.i.d. $N(0, \sigma_e^2)$), $\dot{Y}_t = Y_t - \mu$, and $-1 < \theta < 1$.

Based on a series of length $N = 5$, we observe

y_1	y_2	y_3	y_4	y_5
3	6	0	9	7

a) Calculate r_1.

(Note: In practice reliable autocorrelation estimates are only obtained from series consisting of approximately 50 observations or more.)

$$\bar{y} = \frac{3+6+0+9+7}{5} = 5.$$

y_t	$y_t - \bar{y}$	$(y_t - \bar{y})^2$	$(y_t - \bar{y})(y_{t+1} - \bar{y})$	$(y_t - \bar{y})(y_{t+2} - \bar{y})$
3	− 2	4	− 2	10
6	1	1	− 5	4
0	− 5	25	− 20	− 10
9	4	16	8	
7	2	4		
		50	− 19	4

$$r_1 = \frac{\sum_{t=1}^{N-1}(y_t - \bar{y})(y_{t+1} - \bar{y})}{\sum_{t=1}^{N}(y_t - \bar{y})^2} = \frac{-19}{50} = \mathbf{-0.38}.$$

$$r_2 = \frac{\sum_{t=1}^{N-2}(y_t - \bar{y})(y_{t+2} - \bar{y})}{\sum_{t=1}^{N}(y_t - \bar{y})^2} = \frac{4}{50} = 0.08.$$

b) Use your answers from parts (a) and the method of moments to estimate θ. Round θ to the second decimal place.

"Hint": Recall $\rho_1 = \dfrac{-\theta}{1+\theta^2}$. $\quad -1 < \theta < 1$.

$$-0.38 = \frac{-\theta}{1+\theta^2}. \qquad 0.38\,\theta^2 - \theta + 0.38 = 0$$

$$\theta_{1,2} = \frac{1 \pm \sqrt{1^2 - 4 \cdot 0.38 \cdot 0.38}}{2 \cdot 0.38} = \frac{1 \pm 0.65}{0.76} = 0.46 \text{ and } 2.17.$$

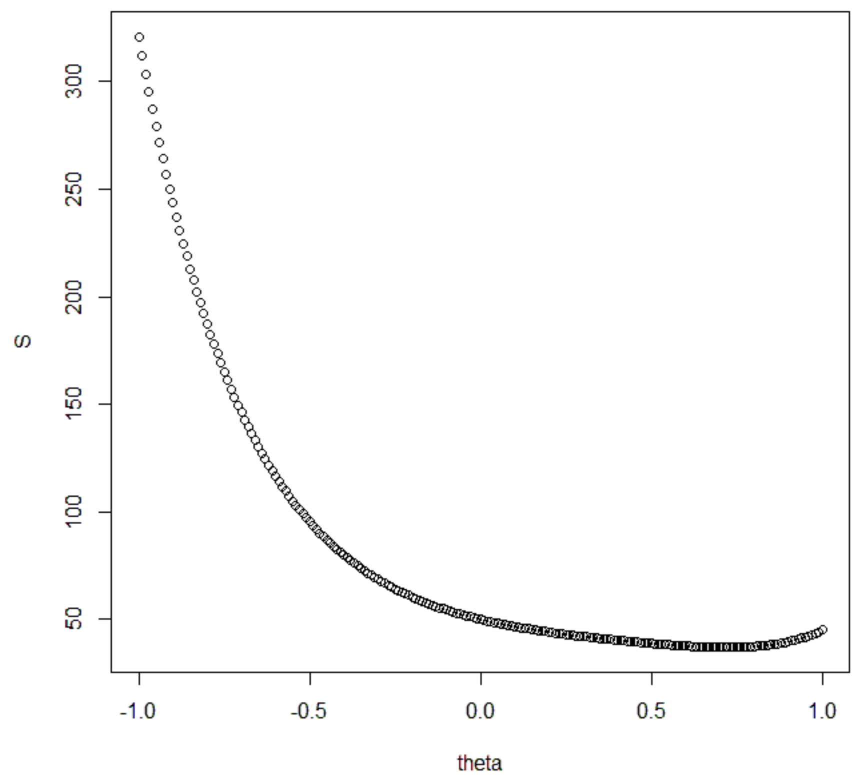
Choose $\theta = \mathbf{0.46}$. $(-1 < \theta < 1)$

c) Using $e_0 = 0$, calculate $S(\theta) = \sum_{t=1}^{N} e_t^2$ for $\theta = -1, -0.99, -0.98, \dots, 0.99, 1$.

$$e_t = \dot{Y}_t + \theta\, e_{t-1} \quad \Rightarrow \quad \begin{aligned} e_1 &= \dot{y}_1 \\ e_2 &= \dot{y}_2 + \theta\, e_1 \\ e_3 &= \dot{y}_3 + \theta\, e_2 \\ &\dots \\ e_N &= \dot{y}_N + \theta\, e_{N-1} \end{aligned}$$

```
> y = c(3,6,0,9,7)
> ydot = y - mean(y)
>
> theta = seq(-1,1,by=0.01)
> N = length(theta)
> S = rep(0,N)                ## to store the values of S(theta)
>
> e = rep(0,5)                ## to store the values of et
>
> m = 1                       ## to keep track of min of S(theta)
```

```
> for (i in 1:N) {
+ e[1]=ydot[1]
+ for (j in 2:5) {
+ e[j]=ydot[j]+theta[i]*e[j-1]
+ }
+ S[i] = sum(e^2)
+ if (S[i]<S[m]) {m = i}
+ }
>
> m
[1] 173
> S[m]
[1] 37.00192
> theta[m]
[1] 0.72
>
> plot(theta,S)
```



Which value of θ minimizes $S(\theta)$?

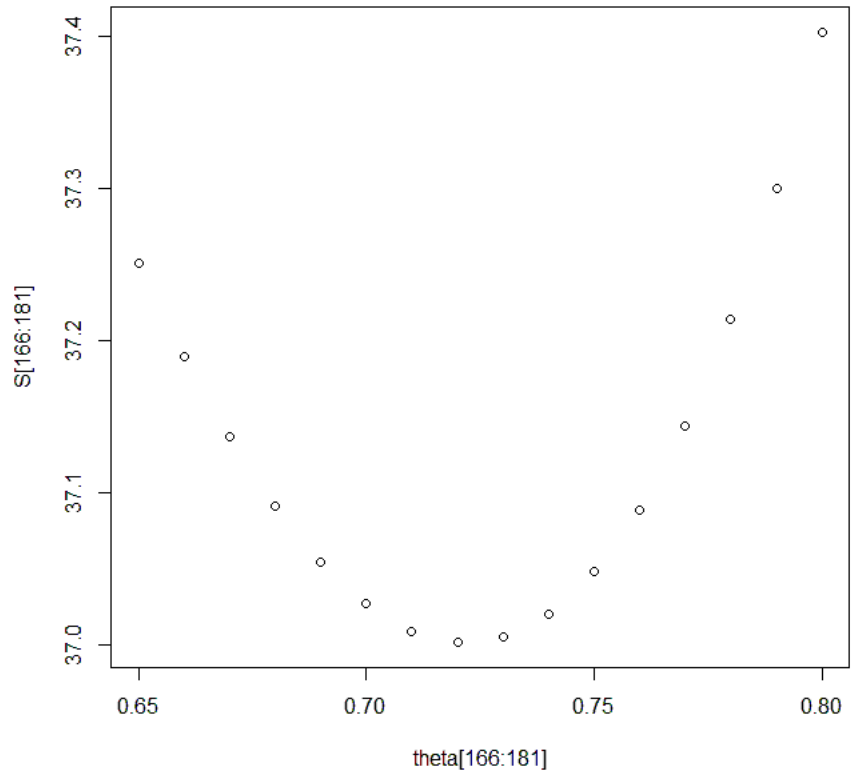
$\theta = \mathbf{0.72}$ minimizes $S(\theta)$.

```
> theta[172]
[1] 0.71
> S[172]
[1] 37.00943
```

```
> theta[173]
[1] 0.72
> S[173]
[1] 37.00192
```

```
> theta[174]
[1] 0.73
> S[174]
[1] 37.00543
```

```
> plot(theta[166:181],S[166:181])
```



e) If $\theta = (\text{answer to part (b)})$ and $e_0 = 0$, forecast y_6 and y_7.

"Hint": The last residual is $\hat{e}_5 = 2.789787$.

```
> e[1]=ydot[1]
> for (j in 2:5) {
+ e[j]=ydot[j]+0.46*e[j-1]
+ }
> e
[1] -2.000000  0.080000 -4.963200  1.716928  2.789787
```

$$Y_{N+1} = \mu + e_{N+1} - \theta\, e_N \qquad \hat{y}_6 = 5 - 0.46 \cdot 2.789787 = \mathbf{3.7167}.$$

$$Y_{N+2} = \mu + e_{N+2} - \theta\, e_{N+1} \qquad \hat{y}_7 = \mathbf{5}.$$

f) If $\theta = (\,\text{answer to part (c)}\,)$ and $e_0 = 0$, forecast y_6 and y_7.

"Hint": The last residual is $\hat{e}_5 = 2.123771$.

```
> e[1]=ydot[1]
> for (j in 2:5) {
+ e[j]=ydot[j]+0.72*e[j-1]
+ }
> e
[1] -2.000000 -0.440000 -5.316800  0.171904  2.123771
```

$$Y_{N+1} = \mu + e_{N+1} - \theta\, e_N \qquad \hat{y}_6 = 5 - 0.72 \cdot 2.123771 = \mathbf{3.4709}.$$

$$Y_{N+2} = \mu + e_{N+2} - \theta\, e_{N+1} \qquad \hat{y}_7 = \mathbf{5}.$$