Information Security Lab
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Module 4, Week 2 – Cryptanalysis of ECDSA

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Overview of today's lecture

- More on lattices and lattice reduction
- ECDSA recap from last week
- Breaking ECDSA with partially known nonces
- From breaking ECDSA to CVP
- From CVP to SVP via Kannan embedding
- Putting it all together

Lattices and Lattice Reduction

Lattices – Recap

<u>Definition (full-rank lattice)</u>

Let $\{\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_n\}$ be n linearly independent vectors in $\mathbb{R}^n$.

Then the lattice generated by $\{\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_n\}$ is the set:

$$L := \{ \sum_{i=1}^{n} l_i \, \underline{b}_i : l_i \in \mathbb{Z} \}$$

of **integer** linear combinations of the $\underline{b}_i$.

- A lattice is a *discrete subgroup* of $\mathbb{R}^n$.

Lattices – Recap

Definition (basis matrix of a full-rank lattice):

A basis matrix B of a lattice $L \subset \mathbb{R}^n$ is an n x n matrix formed by taking the rows to be basis vectors $\underline{b}_i$. Then

$$L = \{ \underline{x}B : \underline{x} \in \mathbb{Z}^n \}$$

Definition (determinant of a full-rank lattice):

The determinant of a full-rank lattice L is the absolute value of the determinant of any basis matrix B for the lattice.

Lattices – Recap

Definition (successive minima of a lattice):

Let $L \subset \mathbb{R}^n$ be a full rank lattice. The successive minima of L are the values $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that:

For $1 \leq i \leq n$, λ_i is the smallest real value such that there exist i linearly independent vectors $\underline{v}_1,\ldots,\underline{v}_i$ with $||\underline{v}_j|| \leq \lambda_i$ for $1 \leq j \leq i$.

Special case:

λ_1 is the length of a **shortest** (in terms of Euclidean norm) non-zero vector in L.

SVP and CVP

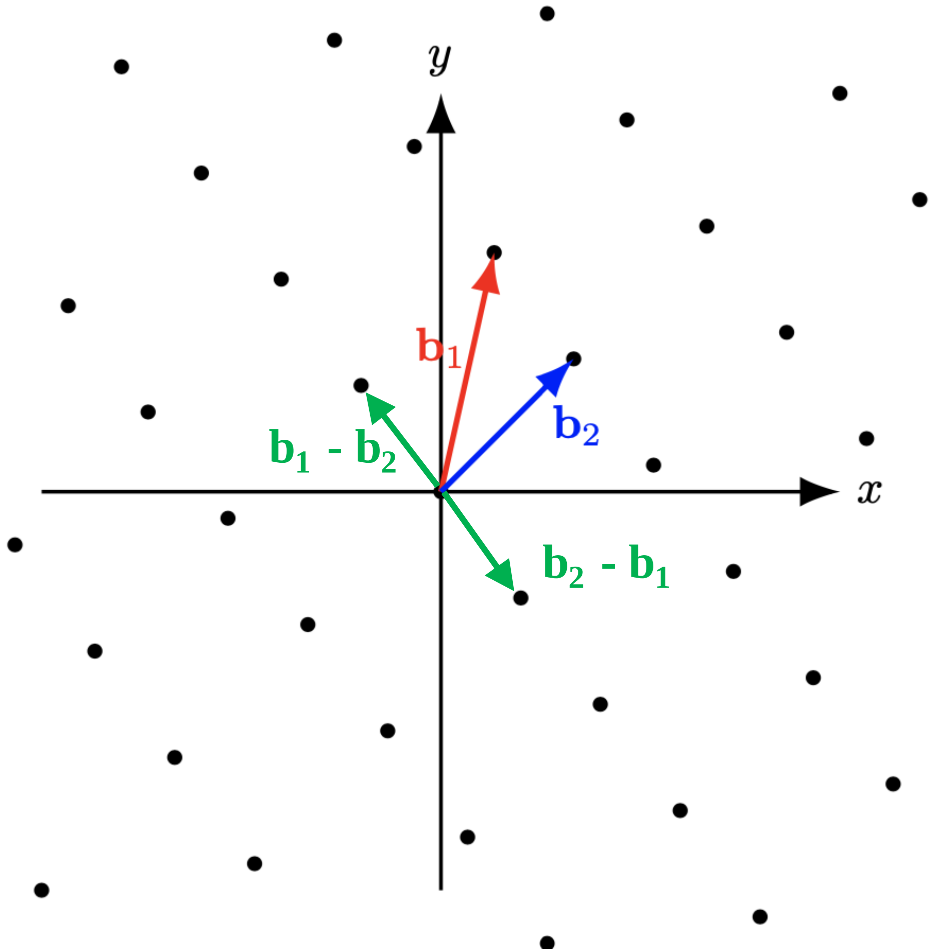
Let $L \subset \mathbb{R}^n$ be a "random" full rank lattice, and let $\underline{w} \in \mathbb{R}^n$.

Then the **Shortest Vector Problem (SVP)** is to find $\underline{v} \in L$ such that $\| \underline{v} \| = \lambda_1$.

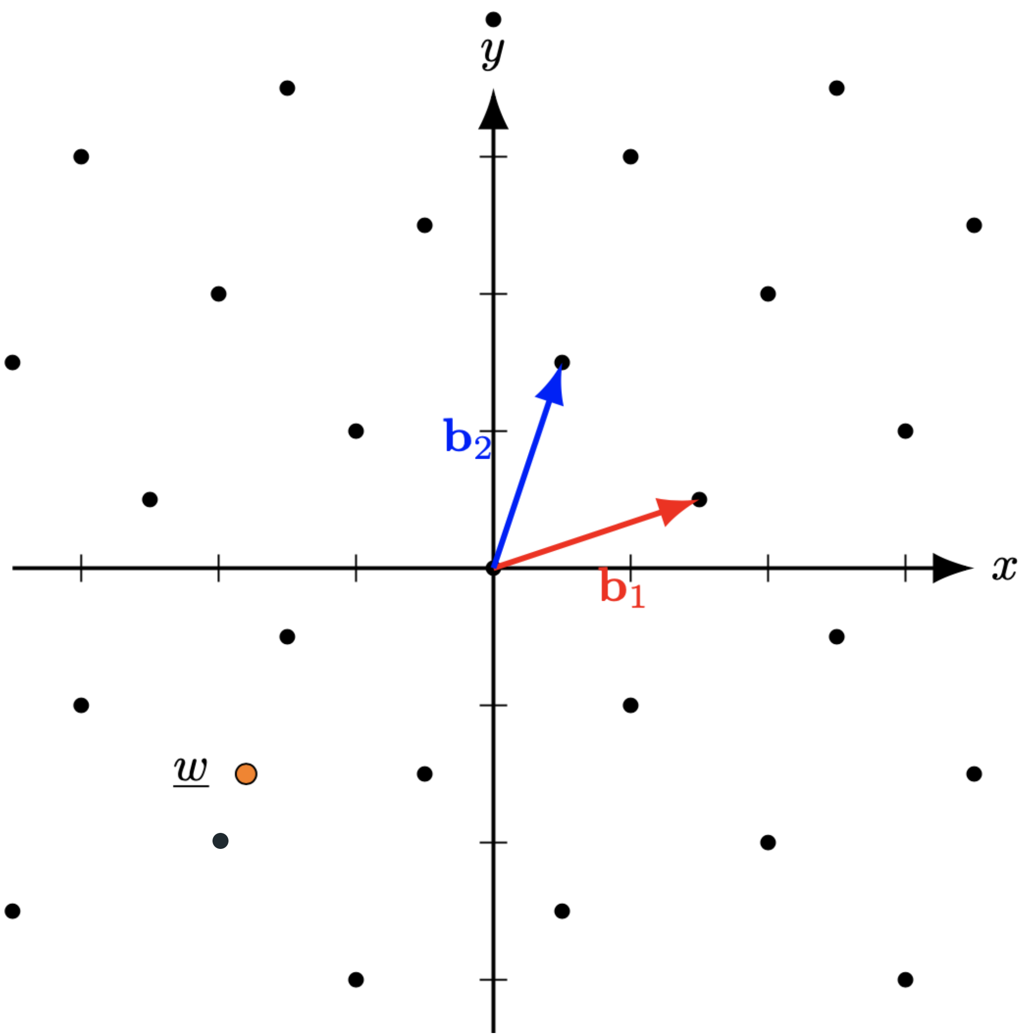
The **Closest Vector Problem (CVP)** for $\underline{w}$ is to find $\underline{v} \in L$ such that $\| \underline{v} - \underline{w} \|$ is as small as possible.

- SVP and CVP are known to be hard problems in general.
- For example, CVP can be shown to be NP-hard by relating it to subset-sum.
- SVP and CVP problems in **high-dimensional lattices** can be used to **construct** cryptographic schemes, e.g. public key encryption, signatures.
- However, SVP and CVP may be easy if the dimension is **small** and/or the lattice is presented in a "nice" way.
- Then lattices can be used as a tool for **cryptanalysis**.

Illustrating SVP



Illustrating CVP



Lattice Reduction

- Let $L \subset \mathbb{R}^n$ be a full rank lattice, represented by some basis matrix B.
- **Lattice reduction** refers to the process of producing a new basis matrix B' for L satisfying certain special properties.
- In particular, the rows of B' (whose linear combinations define L) are "somewhat orthogonal" and the norms of the first rows of B' are relatively short.
- The **Lenstra, Lenstra, Lovasz (LLL) algorithm** is a deterministic algorithm which performs lattice reduction.
- LLL essentially performs "iterative rounded Gram-Schmidt orthogonalization".
- Algorithmic details can be found in Chapter 17 of Galbraith's book "Mathematics of Public Key Cryptography" available for free at: https://www.math.auckland.ac.nz/~sgal018/crypto-book/crypto-book.html

Lattice Reduction – LLL and BKZ

- LLL runs in time (and space) polynomial in n and $\max_i \|\underline{b}_i\|^2$ and produces a basis matrix B' in which the first row $\underline{b}'_1$ is guaranteed to satisfy:

$$\|\underline{b}'_1\| \leq 2^{(n-1)/2}\lambda_1 .$$

- In other words, LLL *approximately* solves the SVP for lattice L (and more besides) with an *approximation factor* of $2^{(n-1)/2}$.
- It can also be shown that $\|\underline{b}'_1\| \leq 2^{(n-1)/4} \cdot \det(L)^{1/n}$, giving a useful bound on the size of the first (smallest) vector in the basis output by LLL.
- In practice, LLL often *exactly* solves SVP.
- For large dimensions n, LLL is superseded by the BKZ algorithm (which uses LLL as a subroutine to solve SVP on sub-lattices).
- Due to its importance in cryptography, lattice reduction is a major area of on-going research, with many improvements to BKZ in recent years.

ECDSA Recap

ECDSA Recap

Parameters: (E, p, n, q, h, P, H) defining a curve E over field F_p with $n = q \cdot h$ points, subgroup of prime order q and generator P of order q; H is a hash function, e.g. SHA-256 (here we assume output of H is at least bit-size of q).

<u>KeyGen</u>:

Set $Q = [x]P$ where x is uniformly random from $\{1,\ldots,q\text{-}1\}$.

Output verification key: $\boldsymbol{Q}$; signing key: $\boldsymbol{x}$.

<u>Sign</u>: Inputs (d, m) // d is private key; m is the message to be signed

h = bits2int(H(m)) mod q. // take len(q) MSBs of H(m), cast to BigInt, reduce mod q.

Do:

1. Select k uniformly at random from $\{1,\ldots,q\text{-}1\}$. // k is called the *nonce*
2. Compute r = x-coord([k]P) mod q. // [k]P is a point on E; its x-coord is in F_p; we consider that as an integer and reduce mod q.
3. Compute $s = k^{-1}(h + xr) \bmod q$.

Until $r \neq 0$ and $s \neq 0$. // works first try w.h.p.

Output (r,s).

ECDSA Recap

Verify: Inputs (Q, m, (r,s)) // Q is verification key; m is message; (r, s) is claimed signature.

1. check that $1 \leq r \leq q\text{-}1$ and $1 \leq s \leq q\text{-}1$.
2. compute $w = s^{-1}$ mod q.
3. compute h = bits2int(H(m)) mod q.
4. compute $u_1 = w \cdot h$ mod q and $u_2 = w \cdot r$ mod q.
5. compute $Z = [u_1]P + [u_2]Q$.
6. If (x-coord(Z) mod q == r) then output 1 else output 0.

Correctness:

Suppose (r,s) is a signature for message m under key Q. Then:

$$Z = [u_1]P + [u_2]Q = [s^{-1}h]P + [s^{-1}r]Q = [s^{-1}(h + xr)]P = [k]P.$$

Here we used $s = k^{-1}(h + xr)$ mod q from the signing algorithm to obtain $s^{-1}(h + xr) = k$ mod q.

Recalling that r = x-coord($[k]P$) mod q completes the argument.

Breaking ECDSA with Partially Known Nonces

Breaking ECDSA with Partially Known Nonces

- You saw in the lab last week that the security of ECDSA is very sensitive to how nonces k are chosen: known k or repeated k is fatal for security.
- What if the attacker could learn just a few bits of k?
- Such information might be available via a side-channel attack.
- Examples:
  - Brumley and Tuveri, "Remote Timing Attacks are Still Practical", ESORICS 2011 and https://eprint.iacr.org/2011/232.pdf
  - Moghimi et al., "TPM-FAIL: TPM Meets Lattice and Timing Attacks", USENIX 2020 https://www.usenix.org/conference/usenixsecurity20/presentation/moghimi-tpm
  - Both papers observe leakage of cases when MSBs of k are zero due to faster execution of $[k]P$ during the signing algorithm: a timing side-channel observable by a "remote" attacker.
  - Other, recent work observes partial leakage of k via cache-based side-channel attacks (a local attacker model).

Breaking ECDSA with Partially Known Nonces

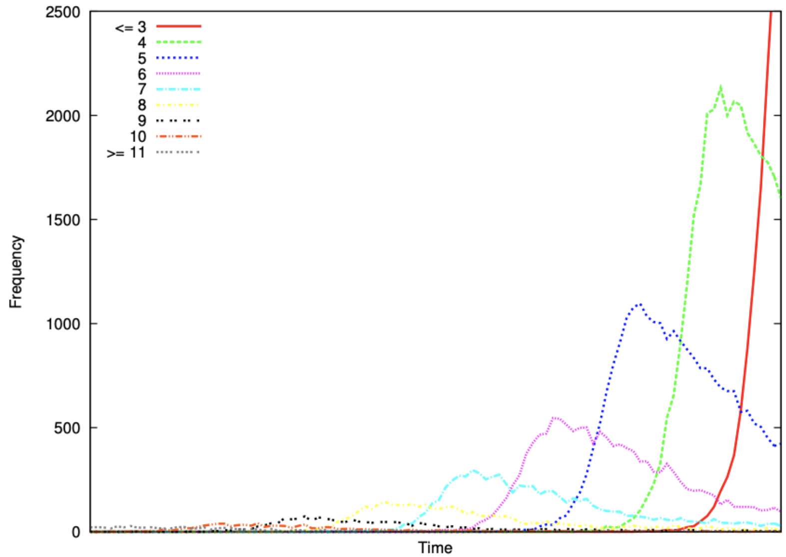


Fig. 4. Dependency between number of leading zero bits and wall clock execution time of the signature operation.

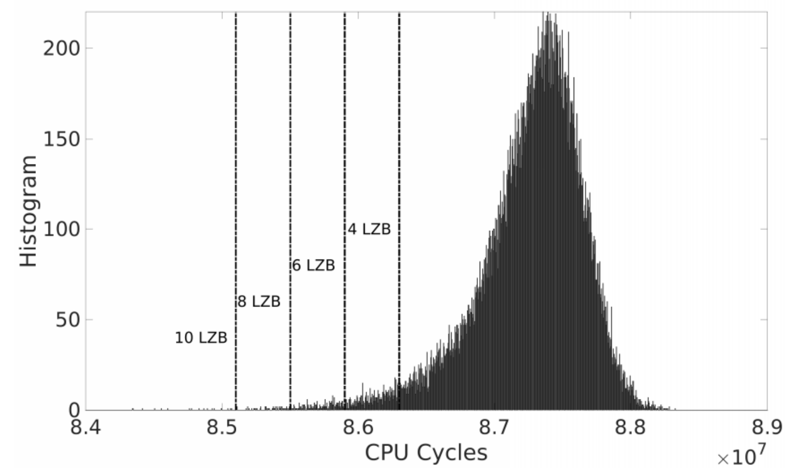


Figure 2: Histogram of ECDSA (NIST-256p) signature generation timings on the STMicroelectronics TPM as measured on a Core i7-8650U machine for 40,000 observations.

Brumley-Tuveri, 2011
OpenSSL ECDSA signature generation

Moghimi et al., 2020
STMicroelectronics TPM ECDSA signature generation

Breaking ECDSA with Partially Known Nonces

Table 2: Discovered vulnerabilities in OpenSSL, LibreSSL, and BoringSSL and whether they are patched ✓ as of October 2019, currently being patched 🔧, or unpatched ✗. Exploiting the side channel can be easy ●, medium ◐, or hard ○. The number of leaked bits (Nonce Leakage) indicates the complexity of a full key recovery.

Vulnerability		OpenSSL	LibreSSL	BoringSSL	Nonce Leakage	SC	Comments
Generate:	(V1) Small k (top)	EC✗	EC✗	–	Topmost 0-limbs of k	●	Leaks in several subsequent steps
Exp. / Scalar Mult.	(V2) k-padding resize	DSA✓EC✓	DSA✗EC✗	–	Topmost 0-bits of k	●	CVE-2018-0734 and CVE-2018-0735
	(V3) consttime-swap	DSA✓EC✓	DSA✗EC✗	–	same as (V2)	◐	Already known
	(V4) Downgrade	DSA✓	–	–	same as (V2) + [24]	●	Introduced while fixing (V2)
	(V5) k-padding (top)	DSA✗EC✗	DSA✗EC✗	–	same as (V2)	○	Leaks in BN_add and BN_is_bit_set. SGX attack shown in Appendix B.
	(V6) Buffer conversion	EC✓	–	–	Topmost 0-bytes of k	○	
	(V7) Point addition	EC🔧	–	EC✓	All 0-windows of k	○	
Invert	(V8) Euclid BN_div	DSA✓	DSA✗	–	Topmost bit of k	●	Leaks via resize, similar to (V2)
	(V9) Euclid negation	DSA✓	DSA✗	–	Topmost 0-bit of *kinv*	●	Leaks via conditional negation
Multiply:	(V10) Small k^{-1} (top)	–	EC✓	–	Topmost 0-limbs of *kinv*	◐	

Weiser et al., "Big Numbers - Big Troubles: Systematically Analyzing Nonce Leakage in (EC)DSA Implementations", USENIX 2020, https://www.usenix.org/system/files/sec20-weiser.pdf

Breaking ECDSA with Partially Known Nonces

- You will implement a synthetic version of these attacks in the lab to recover signing keys.
- The remainder of these lectures will describe how you will do this, in a sequence of steps:
  - Reducing to Closest Vector Problem (CVP) in a lattice.
  - Reducing CVP to Shortest Vector Problem (SVP).
  - Solving SVP using the LLL algorithm (as a black box).
- The original ideas go back to: Howgrave-Graham and Smart, Lattice Attacks on Digital Signature Schemes. *Des. Codes Cryptography*, 23:283–290, 2001.

From ECDSA to CVP

From ECDSA to CVP

- Suppose we are given a signature (r,s) on message m.
- Rearranging the signing equation $s = k^{-1}(h + xr) \bmod q$ yields:

$$(rs^{-1})x = k - hs^{-1} \bmod q$$

- Set $t = rs^{-1} \bmod q$ and $z = hs^{-1} \bmod q$ (z is an integer between 0 and q-1).
- Then we have:

$$tx = k - z \bmod q$$

where t and z can be computed from the signature, x is the unknown private key and we *may* have some partial information on k.

From ECDSA to CVP

- Suppose that the L most significant bits (MSBs) of k are known.
- Let N be the bit-length of q.
- Then k lies in the interval $[a2^{N-L}, (a+1)2^{N-L}-1]$ for some known value a (which is determined by the MSBs of k).
- The mid-point of this interval is $a2^{N-L} + 2^{N-L-1}$.
- So let's write $k = a2^{N-L} + 2^{N-L-1} + e$ where $0 \leq |e| \leq 2^{N-L-1}$.

Then

$$tx = k - z = u + e \bmod q$$

where

$$u = a2^{N-L} + 2^{N-L-1} - z$$

is a known integer that can be computed from the L MSBs of k and $z = hs^{-1}$ mod q.

From ECDSA to CVP

- We have:

$$tx = u + e \bmod q$$

where t is known, u is known, but e and x are not.

- Moreover, e is bounded by: $0 \le |e| \le 2^{N-L-1}$.
- So we finally arrive at: $tx = u + e \bmod q$, and hence:

$$tx = u + e + l \cdot q \quad \text{for some } l$$

- Since $u = a2^{N-L} + 2^{N-L-1} - z$, we see that u lies between $2^{N-L-1} - q$ and $a2^{N-L} + 2^{N-L-1}$.
- We really only care about values mod q, so we can assume (by adding multiples of q as needed) that u is *centred*, i.e. $-q/2 < u < q/2$.

From ECDSA to CVP

- So far: from signature (r,s) on message m and the L MSBs of k we get:

$$tx = u + e + l \cdot q \quad \text{for some } l.$$

- Here, x is our target, t is known, u is known, e is small but otherwise unknown and l is unknown.
- We get one such equation for each of n signatures (r_i, s_i) on messages m_i:

$$t_i x = u_i + e_i + l_i \cdot q$$

- Alternatively, we can write:

$$t_i x = u_i + e_i \mod q \qquad \text{(where } e_i \text{ is small)}$$

  meaning that u_i is a good approximation to $t_i x \bmod q$.

- In this second form, the problem of recovering x from n distinct equations with uniformly random t_i is called the **Hidden Number Problem (HNP)**.
- A similar translation can be made when the LSBs of the k_i are known, or in fact any set of contiguous bits.

From ECDSA to CVP

Consider the lattice $L \subset \mathbb{R}^{n+1}$ with basis matrix B given by:

$$
\begin{vmatrix}
q & 0 & 0 & \dots & 0 & 0 \\
0 & q & 0 & \dots & 0 & 0 \\
. & & & & . & . \\
. & . & . & \dots & . & . \\
. & & & & & \\
0 & 0 & 0 & \dots & q & 0 \\
t_1 & t_2 & t_3 & & t_n & 1/2^{L+1}
\end{vmatrix}
$$

Define $\underline{u} = (u_1, u_2, \dots u_n, 0) \in \mathbb{R}^{n+1}$ where, recall, $t_i x = u_i + e_i + l_i \cdot q$ with e_i small.

<u>**Claim**</u>: There exists a vector $\underline{v} \in L$ such that

$$\| \underline{u} - \underline{v} \| < (n+1)^{1/2} \cdot 2^{N-L-1}.$$

From ECDSA to CVP

Proof of claim:

Recall that we have:

$$t_i x = u_i + e_i + l_i \cdot q$$

where $|e_i| \leq 2^{N-L-1}$, and l_i is some (unknown) integer.

Now define $\underline{v} \in L$ via:

$$\underline{v} = (-l_1, -l_2, \ldots, -l_n, x) \cdot B.$$

(Recall that x is unknown, so $\underline{v}$ is also unknown at this point.)

So: $\underline{v} = (-l_1, -l_2, \ldots, -l_n, x) \cdot B = (-l_1 q + t_1 x, -l_2 q + t_2 x, \ldots, -l_n q + t_n x, x/2^{L+1})$

$$= (u_1+e_1, u_2+e_2, \ldots, u_n+e_n, x/2^{L+1})$$

Hence:

$$\underline{v} - \underline{u} = (e_1, e_2, \ldots, e_n, x/2^{L+1}).$$

The result follows on noting that each entry in $\underline{v} - \underline{u}$ is bounded in absolute value by 2^{N-L-1} (for the last entry, note that $x < q < 2^N$).

From ECDSA to CVP

Implications:

- We have constructed a lattice $L \subset \mathbb{R}^{n+1}$ and vector $\underline{u}$ from public information and shown that $\underline{u}$ is "somewhat close" to a point $\underline{v}$ in L.
- Moreover, if we could find $\underline{v}$ then we could also find the ECDSA private key x (just by inspecting the final coordinate of $\underline{v}$).
- We can hope that $\underline{v}$ is actually **the** solution to the CVP for $\underline{u}$.
- For some parameter ranges, one can show that this is indeed the case.
- So if we have a CVP solver, we can apply it here and hope to extract the private key x.

From ECDSA to CVP

A Formal Result:

Let $t_1,\ldots,t_n$ be uniformly random values in F_q, let x be non-zero in F_q. Suppose we are given n samples of the form (t_i, u_i) where u_i is known to be a good approximation to $t_i x \bmod q$ (i.e. $t_i x = u_i + e_i \bmod q$ with $0 \leq |e_i| \leq q/2^{L+1}$).

Suppose $n = 2\log_2(q)^{1/2}$ and $L = \log_2(q)^{1/2} + \log_2\log_2(q)$.

Then one can recover x in polynomial time.

- For a proof, see Theorem 21.7.9 and Corollary 21.7.10 of Galbraith's book.
- The proof is based on properties of LLL and the Babai nearest plane algorithm for solving CVP.
- Guarantee required of $|e_i|$ is slightly stronger here than in our formulation.
- In practice, we can get away with much smaller n and L and still get an attack that works.

From CVP to SVP: Kannan’s Embedding Technique

From CVP to SVP: Kannan’s Embedding Technique

- There are multiple ways to solve CVP using an SVP solver: the Babai nearest plane algorithm, Babai rounding, Kannan’s embedding technique, enumeration approaches.
- We will describe only Kannan embedding here, as it is nice and simple.
- Babai nearest plane and Babai rounding are also simple, and have provable guarantees.
- Your favourite LLL library may allow you to solve CVP directly, but it’s good to have a sense of what could be happening underneath!

From CVP to SVP: Kannan's Embedding Technique

Let B be a basis matrix for a lattice $L \subset \mathbb{R}^n$ with rows $\underline{b}_i$.

Let $\underline{w} \in \mathbb{R}^n$ be a vector for which we wish to solve CVP.

A solution to the CVP corresponds to integers $l_1, \ldots, l_n$ such that:

$$\underline{w} \approx l_1\underline{b}_1 + \ldots. + l_n\underline{b}_n.$$

Define:

$$\underline{f} = \underline{w} - (l_1\underline{b}_1 + \ldots. + l_n\underline{b}_n).$$

Key observation: $\|\underline{f}\|$ is small.

- So we try to define a new lattice L' which contains $\underline{f}$.
- Hopefully then $\underline{f}$ will be output as a result of running an SVP solver on L'.
- From $\underline{f}$ and $\underline{w}$ we can then recover the lattice point $\underline{v} = l_1\underline{b}_1 + \ldots. + l_n\underline{b}_n$ in L.

From CVP to SVP: Kannan's Embedding Technique

- Consider the lattice $L' \subset \mathbb{R}^{n+1}$ with basis matrix B' whose rows are:

$$(\underline{b}_1, 0), (\underline{b}_2, 0), \ldots, (\underline{b}_n, 0), (\underline{w}, M)$$

  where, recall, $\underline{w}$ is the input to the CVP.

- Here M is a small constant, to be determined.
- Now consider the linear combination of rows with coefficients:

$$(-l_1, \ldots, -l_n, 1)$$

- It is easy to check that this yields the vector $(\underline{f}, M)$, which should be short.
- So we might be able to solve CVP on input $\underline{w}$ for lattice L by solving SVP on lattice L' to find $(\underline{f}, M)$ and then setting $\underline{v} = \underline{w} - \underline{f}$.

From CVP to SVP: Kannan's Embedding Technique

Lemma:

Let $L \subset \mathbb{R}^n$ be a full rank lattice with shortest non-zero vector of length λ_1. Let $\underline{w} \in \mathbb{R}^n$ and let $\underline{v}$ be a closest vector in L to $\underline{w}$. Define $\underline{f} = \underline{w} - \underline{v}$. Suppose that $||\,\underline{f}\,|| \leq \lambda_1/2$ and let $M = ||\,\underline{f}\,||$. Then $(\underline{f}, M)$ **is** a shortest vector in the lattice $L' \subset \mathbb{R}^{n+1}$ in Kannan's embedding technique.

Proof: see Galbraith's book, Lemma 18.3.2.

Interpretation: if the target vector $\underline{w}$ is very close to the lattice and we have a good guess M for the distance, then Kannan's embedding technique *does* reduce the problem of solving CVP to that of solving SVP.

Problems: maybe $\underline{w}$ is not close to the lattice; LLL and related algorithms only approximately solve SVP; maybe target $(\underline{f}, M)$ is short but **not** a shortest vector in L'.

Solution: in practice, this approach works quite well, but we may need to examine several vectors in the reduced basis to find target $(\underline{f}, M)$ or perform *enumeration*.

Putting It All Together

Putting It All Together

- We have seen how to translate the problem of recovering x from partial information about the nonces in the ECDSA scheme into a CVP problem and thence to an SVP problem.
- The lattice dimension we use is $n+2$ where n is the number of signatures.
- Whether n signatures with L bits of leakage per signature is enough to recover k depends on several factors.
- Clearly there is an information theoretic minimum: we need

$$n \cdot L > \log_2(q) = N.$$

- Actually, knowing the public key $[x]P$ enables attack to go beyond this minimum.

Putting It All Together

- We can use the Gaussian heuristic to see if Kannan's embedding technique or some other CVP solver is likely to produce a solution – see the exercises for an important wrinkle in this approach.
- You will implement all this in the lab and use Sagemath as a tool to solve the lattice instances that arise.
- Through this programming exercise, you'll explore the performance of this kind of attack and have to deal with some of the subtleties that arise.
- Have fun!