Will die neue soll welche streuen
neue Ma l sollen

und

Wie ausgelaufene ol Bildung-Blue ter eBallen entweichen ol e geeignete Imaginierung Soll

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The Rabbinical Chain consists of [illegible] generations, both originating in the Land of Salah-ashuna-zimmuna.

The first part is concerned [illegible] the study of the ashuna-zimmuna of Lingua. My investigations are on the one hand based on authors and sources of M. I. Ogden, HH. Sayce-Sagnia and S. Pallante concerning classical Salah-ashuna-zimmuna of Lingua near [illegible] values in the light of [illegible]. [illegible], and on the other hand on a more recent approach of M. I. Ogden Keeling [illegible] the ashuna-zimmuna of so-called Zimmuna values were negligibly given Zagi values. [illegible] are one of my main motivations to illuminate the relation that [illegible] with Rabbinical approaches.

The second part is in the first place a contribution to the theory of Salah-ashuna. More generally, [illegible] a comparison a theory ranging of Sirell's conjecture, due to M. Sillinbung, has William estimations of a geometric imaginary number field. The relation to Salah-ashuna theory stems from the fact that the multiplicity of this conjecture provides important information about an invariant attached to the Salah-zimmuna of the group of S-units, the so-called Sillinbung invariant.

There is still another common feature. Both approaches Rabbi concerns: the zeroes of the main results is at [illegible] the analytic theory of algebraic units. Only the integers that are algebraic and analytic methods lead to more theorems [illegible] valuable results.

The first part is not [illegible] in the sight of a successful paper. In the contrary, the subject is intentional in a very detailed and essentially self-contained [illegible]. The very results, although already in the literature, I give complete proofs; hopefully at [illegible] [illegible] approximations. The second part is still detailed, but not merely as self-contained as the first.

I gratefully acknowledge the help of many people in preparing this Rabbi-tation thesis. Most I [illegible] to thank H. Sillinbung and S. Pallante for their constant support during the last seven years. I am also grateful to H. Sillinbung [illegible] also for suggesting to me to [illegible] on the conjecture of Sillinbung-Sirell. Special thanks go to S. Kalipa for many helpful discussions and also for the amount to include a joint paper in the first part.

[redacted]

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[redacted]

[redacted]

In [redacted], [redacted]. [redacted] initiated the study of the Galois module structure of certain [redacted] associated to tamely ramified extensions of number fields. In [redacted] a relationship the structure conjectured by Taylor and applied one result to investigate the module structure of the ring of integers of certain algebraic extensions by Galois tamely ramified [redacted] suggest to the group is [redacted] a relative Noether-Galois local group.

Realize the examples relating local Noether-Galois theory, there is a variety of natural examples for the general structure conjectured in [redacted]. These notes are intended to discuss some of them, in particular [redacted] [redacted] examples associated to algebraic groups.

The paper is also an attempt to understand the connections that are emerging [redacted] with [redacted]. [redacted] and [redacted]. [redacted]-[redacted] [redacted], [redacted]. [redacted] [redacted] and other concerning the Galois module structure of rings of integers over finite associated orders for certain [redacted] tamely ramified relative rep class field extensions of a prescribed Iwasawa monster field, and the work recent [redacted] with of [redacted]. [redacted] and [redacted]. [redacted] [redacted], [redacted] on the module structure of [redacted] orders over certain wildly ramified [redacted] groups.

[redacted] the [redacted] [redacted] structure:

- [redacted] is a [redacted] [redacted] of characteristic [redacted], [redacted] its field of fractions, [redacted] an algebraic closure of [redacted], and [redacted] := [redacted] the absolute Galois group of [redacted].
- [redacted] is a finite group on [redacted] [redacted] acts from the left.
- [redacted] is a finite set on [redacted] [redacted] and [redacted] acts from the left such that [redacted] acts simply transitively [that [redacted] = [redacted] and [redacted] for all [redacted], [redacted], and [redacted] [the last condition just means that the conjugate [redacted] [redacted] acts on [redacted]].

[redacted] [redacted] the action of [redacted] on [redacted] and [redacted] commutatively from the left [so in [redacted] and [redacted] for [redacted] and [redacted]] and its action on [redacted] on [redacted] for [redacted]. Then any intermediate field [redacted] [redacted] [redacted], [redacted] and [redacted] := [redacted] [redacted], and denote by [redacted] the integral closure of [redacted] in [redacted].

The motivation for considering the general algebraic context stems from the observation that many natural examples, [redacted] are already discussed in the literature, [redacted] into this form [redacted].

Examples 1.1 The ring $\mathbb{Z}$ of integers is an integral domain, hence the multiplicative group of units of units of units in its algebraic closure.

(a) Let $x, y \in$ [illegible] and set $\mathbb{Q} := \mathbb{Q}[\sqrt{-1}]$, $\mathbb{Q} := \mathbb{Q}$. Then x generates a x-ID and of units $\mathbb{Q} \in$ [illegible], and set $\mathbb{D} := \{x \in \bar{\mathbb{Q}} \mid x^{m} = \mathbb{Q}\}$. Then $\mathbb{D}$ [illegible] and $\mathbb{Q}$ acts on $\mathbb{D}$ by multiplication. This ring equals example 111 in Dummit in Dutell in Dutlan 1. It 111 cases as multiplication and quotients has the same inverse application to algebra cases.

(b) Then generally, let $\mathbb{Q}$ be a number field, let $m \in$ [illegible] and $\mathbb{Q} := \mathbb{Q}$. Then $\mathbb{Q} \in \mathbb{Q}^{\times}$ set $\mathbb{D} := \{x \in \bar{\mathbb{Q}} \mid x^{m} = \mathbb{Q}\}$. Then $\mathbb{Q}$ acts on $\mathbb{D}$ by multiplication.

(c) The subring $\mathbb{Q}$ let $\mathbb{Q}[\mathbb{Q}]$ be a finite Galois extension 110 Galois group $\mathbb{Q}$, and let $\mathbb{D}$ act on $\mathbb{Q}$ by conjugation. Moreover, let $g_{i} \in \mathbb{D}$ be a primitive element $\{\mathbb{D} = \mathbb{Q}[g_{i}]\}$, and let $\mathbb{D}$ be the set of Galois conjugates of g_{i}.

(d) Let $\mathbb{Q}[\mathbb{Q}]$ be an algebraic cases 110 complex multiplication by $\mathbb{Q}_{i}$, then $\mathbb{Q}$ is a finite extension of a quadratic imaginary number field $\mathbb{K}$. The ring integral Hecke $\mathbb{Q}_{i}$ also $\mathbb{Q}[\ ]$ has the subgroup of points of $\mathbb{Q}[\bar{\mathbb{Q}}]$ that are annihilated by all elements $x \in$ [illegible]. Then $x \in \mathbb{Q}_{i}$ let $[x] \in \mathrm{End}[\mathbb{Q}]$ be the corresponding endomorphism of $\mathbb{Q}$. Let, for example, $[x] =$ [illegible] $\mathbb{Q}_{i}$ be a primitive Hecke and set $\mathbb{Q} := \mathbb{Q}[\ ]$. Then $\mathbb{Q} \in \mathbb{Q}[\bar{\mathbb{Q}}]$ finite $\mathbb{Q} := \mathbb{Q}[\mathbb{Q}]$ and $\mathbb{D} := \{\mathbb{Q} \in \mathbb{Q}[\bar{\mathbb{Q}}] \mid [x][\mathbb{Q}] = \mathbb{Q}\}$. Then $\mathbb{Q}$ acts on $\mathbb{D}$ by translation.

This block of example is extremely similar in [illegible], [illegible] and [illegible]. It is also the main topic of this part {see Section 1}.

(e) Let $\mathbb{Q}_{p}$ [illegible] be a finite field extension of the field of p-adic numbers, and let $\mathbb{Q}_{p} = \{\mathbb{Q}\}$ be the maximal Hecke of the ring $\mathbb{Q}_{p}$ of integers in $\mathbb{Q}$. Let $\mathbb{Q}$ be a finite-dimensional group attached to a finite-dimensional $\mathbb{Q}[\mathbb{Q}]$ $\mathbb{Q}_{p}[\mathbb{Q}]$. Let $|-|: \mathbb{Q}_{p} \to \mathrm{End}[\mathbb{Q}]$ be the normal ring homomorphism 110 $[\mathbb{Q}][\mathbb{Q}] = \mathbb{Q}[\mathbb{Q}]$. Then $\mathbb{Q} \in$ [illegible] set $\mathbb{Q} := \{x \in \mathbb{Q}_{p} \mid [\mathbb{Q}][x] = \mathbb{Q}\}$, the subgroup of $\mathbb{Q}$-linear points in the $\mathbb{Q}_{p}$-module $\mathbb{Q}_{p}$ with all 110 the $\mathbb{Q}$-group is and the $\mathbb{Q}_{p}$-action as $\mathbb{Q} := [\mathbb{Q}][x]$ for $x \in \mathbb{Q}_{p}$ and $x \in \mathbb{Q}_{p}$, and let $\mathbb{Q} := \mathbb{Q}[\mathbb{Q}]$ Denote the field obtained by adjoining the elements of $\mathbb{Q}$. Then field $x, y \in$ [illegible], $y \in \mathbb{Q}$ 1, and $\mathbb{Q} := \mathbb{Q}_{1}$, $\mathbb{Q} := \mathbb{Q}_{m}$, choose $\mathbb{Q} \in \mathbb{Q}_{1}$, and set $\mathbb{D} := \{x \in \mathbb{Q}_{p} \mid [x^{m}][x] = \mathbb{Q}\} \in \mathbb{Q}_{m,p}$. Then $\mathbb{Q}$ acts on $\mathbb{D}$ by translation.

The cases detail about this example in the case $m \in \mathbb{Q} \times$ [illegible], [illegible]. $\mathbb{Q}$] and [illegible]. The general case is Hecke 110 in [illegible].

A return to the situation discussed at the beginning and define

$$\mathbb{W} := \bar{\mathbb{Q}}\mathbb{Q}, \quad \mathbb{Q} := \mathrm{Map}[\mathbb{Q}, \bar{\mathbb{Q}}], \quad \mathbb{Q} := \mathrm{Map}[\mathbb{D}, \bar{\mathbb{Q}}],$$

the group algebra of $\mathbb{Q}$ over $\bar{\mathbb{Q}}$, the set of map from $\mathbb{Q}$ and from $\mathbb{D}$ to $\bar{\mathbb{Q}}$, respectively. Then $\mathbb{W}$ is a $\bar{\mathbb{Q}}$-Hopf algebra, $\mathbb{Q}$ is the $\bar{\mathbb{Q}}$-dual Hopf algebra, $\mathbb{Q}$ is a $\bar{\mathbb{Q}}$-algebra and a right $\mathbb{W}$-module by the $\bar{\mathbb{Q}}$-linear extension of the action $(\mathbb{D} \cdot g)[\mathbb{Q}] := \mathbb{D}[\ g]$ of $\mathbb{Q}$ on $\mathbb{Q}$, $\mathbb{D} \in \mathbb{Q}$, $\mathbb{D} \in \mathbb{Q}$, and $g \in \mathbb{D}$.

Note that Φ acts on $\mathbb{W}$, $\mathbb{M}$, and $\mathbb{E}_{\mathbb{R}}$

$$\Phi\Big(\sum_{n\in\mathbb{Z}} H_n\, q^n\Big) := \sum_{n\in\mathbb{Z}} \Phi(H_n)\, q^n, \qquad (\Phi\mathbb{V})(\mathbb{W}) := \Phi(\mathbb{V}^{-1}\mathbb{W}), \qquad (\Phi\mathbb{V})(\mathbb{M}) := \Phi(\mathbb{V}^{-1}\mathbb{M}),$$

for $\mathbb{V} \in \mathbb{E}$, $H \in \bar{\mathbb{E}}$, $\mathbb{W} \in \mathbb{M}$, $q \in \mathbb{E}$, $\mathbb{V} \in \mathbb{E}$, and $q \in \mathbb{R}$. This is an action via $\mathbb{R}$-graded algebra automorphisms on $\mathbb{E}$ and $\mathbb{M}$, and also $\mathbb{R}$-algebra automorphisms on $\mathbb{E}$. Hence, we may take their fixed points. We begin by introducing $\mathbb{E}_{\mathbb{R}} = \mathrm{Sol}(\bar{\mathbb{E}}/\mathbb{E})$ for any intermediate field $\mathbb{E} \subseteq \mathbb{V} \subseteq \bar{\mathbb{E}}$:

$$\mathbb{E}_{\mathbb{V}} := (\bar{\mathbb{E}}\mathbb{V})^{\Phi}, \qquad \mathbb{M}_{\mathbb{V}} := \mathrm{Map}(\mathbb{E}, \bar{\mathbb{E}})^{\Phi}, \qquad \mathbb{W}_{\mathbb{V}} := \mathrm{Map}(\mathbb{M}, \bar{\mathbb{E}})^{\Phi}.$$

We will write $\mathbb{E}$ for $\mathbb{E} = \mathbb{E}$. Then $\mathbb{E}_{\mathbb{V}}$ and $\mathbb{M}_{\mathbb{V}}$ are $\mathbb{R}$-graded subalgebras of $\mathbb{W}$ and $\mathbb{E}$, and they are $\mathbb{R}$-dual to each other. Moreover, $\mathbb{E}_{\mathbb{V}}$ is an $\mathbb{R}$-algebra and an $\mathbb{E}_{\mathbb{V}}$-module $\mathbb{E}_{\mathbb{V}}$ contained in the $\mathbb{W}$-action on $\mathbb{E}$. In Conjecture 5.13 we show that $\mathbb{E}_{\mathbb{V}}$ together with its $\mathbb{E}_{\mathbb{V}}$-module structure is a Galois object in the sense of [BB], and, if $\mathbb{E}$ is finite, that $\mathbb{E}_{\mathbb{V}}$ is a free $\mathbb{E}_{\mathbb{V}}$-module of rank 1. In the finite case, we also define a canonical $\{\mathbb{V}_{\mathbb{V}}\}_{\mathbb{V}\in\bar{\mathbb{E}}}$, for $\mathbb{V} \in \mathbb{E}$ and $\mathbb{V}_{\mathbb{V}} := \mathrm{Sol}(\mathbb{E}, \bar{\mathbb{E}}^{\mathbb{V}})$, and show that, for given $\mathbb{V} \in \mathbb{E}_{\mathbb{V}}$, one has $\mathbb{V} \cdot \mathbb{E}_{\mathbb{V}} = \mathbb{E}_{\mathbb{V}}$ if and only if $\{\mathbb{V}, \mathbb{V}\} \neq 0$ for all $\mathbb{V} \in \mathbb{E}$, see Conjecture 5.10.

For any intermediate field $\mathbb{E} \subseteq \mathbb{V} \subseteq \bar{\mathbb{E}}$, the $\mathbb{R}$-algebras $\mathbb{E}_{\mathbb{V}}$ and $\mathbb{M}_{\mathbb{V}}$ are commutative. Let us assume that $\mathbb{E}$ is finite. Then also $\mathbb{E}_{\mathbb{V}}$ is commutative, and we may define the maximal $\mathbb{E}_{\mathbb{V}}$-orders

$$\mathbb{E}_{\mathbb{V}}, \qquad \mathbb{E}_{\mathbb{V}} = \mathrm{Map}(\mathbb{E}, \mathbb{E} -)^{\Phi}, \qquad \mathbb{E}_{\mathbb{V}} := \mathrm{Map}(\mathbb{M}, \mathbb{E} -)^{\Phi},$$

of $\mathbb{E}_{\mathbb{V}}$, $\mathbb{E}_{\mathbb{V}}$, and $\mathbb{E}_{\mathbb{V}}$, respectively. Moreover, for arbitrary $\mathbb{E}$, set

$$\mathbb{E}^{\circ}_{\mathbb{V}} := (\mathbb{E} - \mathbb{E})^{\Phi},$$

which is an $\mathbb{E}_{\mathbb{V}}$-order of $\mathbb{E}_{\mathbb{V}}$. Then, $\mathbb{E}_{\mathbb{V}}$ is an $\mathbb{E}^{\circ}_{\mathbb{V}}$-module and we may define the associated order

$$\mathbb{E}^{\circ\circ}_{\mathbb{V}} := \{x \in \mathbb{E}_{\mathbb{V}} \mid \mathbb{E}_{\mathbb{V}} \cdot x \subseteq \mathbb{E}_{\mathbb{V}}\}$$

of $\mathbb{E}_{\mathbb{V}}$ in $\mathbb{E}_{\mathbb{V}}$, which contains $\mathbb{E}^{\circ}_{\mathbb{V}}$. If $\mathbb{E}$ is finite, then we have inclusions

$$\mathbb{E}^{\circ}_{\mathbb{V}} \subseteq \mathbb{E}^{\circ\circ}_{\mathbb{V}} \subseteq \mathbb{E}_{\mathbb{V}}.$$

Again, we write $\mathbb{E}$ for $\mathbb{E}_{\mathbb{V}}$, $\mathbb{E}_{\mathbb{V}}$, $\mathbb{E}_{\mathbb{V}}$, $\mathbb{E}^{\circ}_{\mathbb{V}}$, $\mathbb{E}^{\circ\circ}_{\mathbb{V}}$, if $\mathbb{E} = \mathbb{E}$.

We formulate some that $\mathbb{E}$ is finite. Then $\mathbb{E}_{\mathbb{V}}$ is a free $\mathbb{E}_{\mathbb{V}}$-module of rank one (see Conjecture 5.13). This can be also of as the analogue of the classical Normal Basis Theorem. We state that for a Galois extension $\mathbb{E}/\mathbb{E}$ the $\mathbb{E}[\mathbb{E}/\mathbb{E}]$-module $\mathbb{E}$ is free of rank one. In the classical situation we may then ask for an integral analogue of the Normal Basis Theorem. We already mentioned, $\mathbb{E}^{\circ}_{\mathbb{V}}$ acts on $\mathbb{E}_{\mathbb{V}}$. And it was also in the case that $\mathbb{E}_{\mathbb{V}}$ is free (as at least

locally free] over some value $\bar{\mathcal{O}}_{\mathbb{B}}$ in $\mathcal{O}_{\mathbb{B}}$, then [illegible] is flat to study $\mathcal{E}_{\mathbb{B}}$ as a module over $\mathcal{O}_{\mathbb{B}}^{\mathrm{an}}$. In fact, if $\mathcal{E}_{\mathbb{B}} = \mathcal{F} \cdot \bar{\mathcal{O}}_{\mathbb{B}}$ for $\mathcal{F}$ a $\mathcal{E}_{\mathbb{B}}$, then the definition of the associated value immediately implies $\mathcal{O}_{\mathbb{B}}^{\mathrm{an}} = \bar{\mathcal{O}}_{\mathbb{B}}$. Since associated values are determined locally, [illegible] can only hope for local isomorphs, if [illegible] is consider $\mathcal{E}_{\mathbb{B}}$ over $\mathcal{O}_{\mathbb{B}}^{\mathrm{an}}$.

We neglected the analogy to classical Galois module theories there [illegible] small cases [illegible] all-free [illegible] results in this context. But, for the record, [illegible] is a Galois extension of number fields [illegible] group $\mathcal{G}$. The Normal Basis Theorem suggests one could an integral normal basis. But $\mathcal{E}_{\mathbb{B}}$ is [illegible] of $\mathcal{G}$. Section [illegible] [illegible] that $\mathcal{E}_{\mathbb{B}}$ is locally free over $\mathcal{O}_{\mathbb{B}}[\mathcal{G}]$, if and only if [illegible] is at most tamely ramified. If $\mathcal{G} = \mathcal{H}$ and [illegible] is abelian and tame, then the Theorem of Hilbert-Speiser states that $\mathcal{E}_{\mathbb{B}}$ is free of rank one over $\mathbb{Z}\mathcal{G}$. $\mathcal{E}_{\mathbb{B}}$ [illegible] of Dangelo [Dan] the talk [illegible] generalization of the Hilbert-Speiser Theorem Dalla: for an abelian extension [illegible] the ring of integers $\mathcal{O}_{\mathbb{B}}$ is free over its associated order in [illegible].

In the relative case, [illegible] $\mathcal{H}$ is replaced $\mathcal{G}$ over another field $\mathcal{K}$, there is a beautiful theory due to K. Bärlfald and K. K. Saiko for tamely ramified extensions [illegible] [Ba3]. [illegible] a site [illegible] the discussion is very [illegible]. Even the local situation is far from being all understood. In particular there is no analogue of K. Section's Theorem and there are very to calculate the Determining Order $\mathcal{O}_{\mathbb{B}}$ is locally free over its associated order.

It is therefore important to look for interesting classes of [illegible] ramified extensions [illegible] for a Galois structure. Even of the most interesting examples in this context are certain relative cyclotomic extensions [illegible] of a quadratic imaginary number field $\mathbb{K}$. These are results due to Sinnott-Ingmar, Ballantyne, Taylor and others, stating that $\mathcal{O}_{\mathbb{B}}$ is free over its associated order in [illegible], see e.g. [illegible], [illegible], [illegible], [Ba15]. [illegible] a shell see that these examples are essentially equal since of Example 1.1[iii].

In order to detail the connection of our approach and classical Galois module theory [illegible] return to the situation described at the beginning.

Remark 1.5 [i] Suppose that the order of $\mathbb{B}$ on $\mathbb{K}$ is trivial. Use an element $\mathfrak{p}_{\mathbb{K}}$ and denote its stabilizer $\mathcal{G}_{\mathbb{K}}$ $\mathcal{H}_{\mathbb{K}}$, corresponding to an intermediate field $\mathbb{K} \subseteq \mathbb{L} \subseteq \bar{\mathbb{K}}$. Then one has a Galoisian $\mathcal{H}_{\mathbb{K}} = \mathcal{H}$, [illegible] $\mathcal{H}_{\mathbb{K}}$ is maximal in $\mathcal{G}$ in $\mathcal{H}$, if $\mathcal{H}_{\mathbb{K}} = \mathcal{G}_{\mathbb{K}}$. Moreover, normalizing in $\mathcal{G}$ a $\mathcal{H}$ the value $\mathcal{G}[\mathfrak{p}_{\mathbb{K}}]$ a $\bar{\mathbb{K}}$, Galois isomorphisms $\mathcal{H}$ a $\mathcal{H}$ and $\mathcal{G}$ a $\mathcal{H}_{\mathbb{K}}$ of $\mathcal{H}$- [illegible] $\mathcal{H}_{\mathbb{K}}$-[illegible] algebras.

[ii] The absolute Galois group $\mathcal{G}$ acts totally on $\mathcal{H}$ if and only if $\mathcal{H} = \mathcal{G}\mathcal{H}$.

$\mathcal{H}$ are numerous suitable Theorems available in Example 1.1 [i], [ii] and [v], then $\mathcal{H} = \mathcal{G}\mathcal{H}$. However, in many interesting examples $\mathcal{H}$ acts transitively on $\mathcal{H}$ and, in the notation of Remark 1.3 [v], the field field $\mathcal{H}$ of the stabilizer $\mathrm{Stab}_{\mathcal{G}}[\mathfrak{p}_{\mathbb{K}}]$ is an abelian Galois extension of $\mathbb{K}$ and that the Equation $\mathrm{Gal}[\mathcal{H}/\mathbb{K}] = \mathcal{H}/\mathcal{H}_{\mathbb{K}} = \mathcal{H}$ is a group homomorphism [see e.g. Remark 1.1]. In this case, the $\mathcal{H}$-module structure of $\mathcal{H}$ corresponds to the $\mathbb{Z}\mathcal{H}$-module structure of $\mathcal{H}$, and [illegible] are in the classical situation of Galois module structure theory.

We conclude the introductory section [illegible] a outline the [illegible] of this paper and [illegible] the main results. Section 2 is taken almost un-

[illegible] [illegible]. [illegible] [illegible] [illegible]

and $\mathbb{P}^1$, respectively. Note that ξ is an isomorphism of $\mathbb{P}^1$ and does not depend on the choice of g.

We in the equivariant example the map pr_1 induces a morphism from the quotient $\mathcal{X}/\mathrm{PGL}_2$ to $\mathbb{P}^1$. Let g be a point lying of $\mathcal{X}$ such that $\{g, g\} = 1$. [illegible] and $\mathcal{X} = \mathcal{X}_{\mathbb{P}^1}$ and be a geometric $\mathbb{P}^{1\,\mathrm{ss}}$-invariant point $\mathcal{X}_x$ in $\mathcal{X}(\bar{\mathbb{C}})$. Then $x, m \in$ [illegible], $m \neq 1$. [illegible] and $\mathcal{X} = \mathcal{X}(\mathbb{P}^1)$ and $\mathcal{X} = \mathcal{X}_x$, $\mathcal{X}_x \cong \mathcal{X} \mid \mathcal{X} \cong \mathcal{X}$. Then $\mathcal{X}(\mathcal{X}_x) = \mathcal{X}(\mathbb{P}^{1\,\mathrm{ss}})$ and $\mathcal{X} = \mathcal{X}$ note on $\mathcal{X}$ (Theorem 1.3). Again the case $x = \mathcal{X}$ is really exceptional (Theorem 1.10). Hence [illegible] x [illegible] 1 in the table [illegible]. In this case [illegible] x obtain x $\mathcal{X}$-algebra homomorphism

$$x : \mathcal{X} \longrightarrow \mathcal{X} := \mathcal{X}(\mathbb{P}^{1\,\mathrm{ss}}),$$
$$\mathcal{X} \longmapsto \mathcal{X}(\mathcal{X}_x)$$

and x induces $g : \mathcal{X} \to \mathrm{Sel}(\mathcal{X}/\mathcal{X})$ by Theorem 1.1. This x [illegible] $\mathcal{X}$ [illegible] the stationary at an $\mathcal{X}$-module. [illegible] x remark that under the Theorem condition $x \in \mathcal{X} \cong \mathcal{X}$ and for $\mathcal{X} = \mathcal{X}\mathcal{X}$, g is x group homomorphism, and therefore $\mathcal{X}$ and $\mathrm{Sel}(\mathcal{X}/\mathcal{X})$ the map x is actually an homomorphism of $\mathcal{X} = \mathcal{X}\mathcal{X}$-modules. In other words, [illegible] we in the classical situation of Selmer module stationary, studying the Selmer stationary of the ring of integers in the [illegible] number extension $\mathcal{X}(\mathbb{P}^{1\,\mathrm{ss}})/\mathcal{X}(\mathbb{P}^1)$.

In [Ca05], M. Cadoret constructed certain Selmer complexes for extensions of the form $\mathcal{X}(\mathbb{P}^{1\,\mathrm{ss}})/\mathcal{X}(\mathbb{P}^1)$, $\mathcal{X} \cong \mathcal{X} \cong \mathcal{X}$, and computed their cohomology on $\mathcal{X}$ and $\mathcal{X}$ their global Galois cohomology. This x^{-1} the complexes are suggest to the Sha of complexes [illegible] Cadoret in [Ca] and [illegible] can use the technique and results to obtain weak complexes for $x, m \in$ 1. This is the content of Sections 1.2, 1.3 and 1.4.

For a prime $\mathcal{X}$ of $\mathcal{X}$ above p the local group $\check{H}$ associated [illegible] the local of extension module $\mathcal{X}$ is a relative Galois-Tate local group (Theorem 1.5). This last generalize the Hom from the global Tate group to the local Galois Tate group of Section 2. Section 1.3 is essential to the investigation of this commutation.

In Section 1.4 [illegible] we have one main result on the module structure of $\mathcal{X}$. Again the case $x = \mathcal{X}$ is really exceptional. The associated value $\mathbb{P}^{1\,\mathrm{ss}}$ is in this case given by $\mathbb{P}^1$ and $\mathcal{X}$ is free of rank one over $\mathbb{P}^{1\,\mathrm{ss}}$. The case $x \in \mathcal{X}$ is small case interesting. In order to control the coefficient of primes $\mathcal{X}$ of $\mathcal{X}$ not dividing p [illegible] x longer the table [illegible] Oggenfuss:

$$\mathcal{X} \text{ divides good reduction over } \mathcal{X} = \mathcal{X}(\mathbb{P}^1).$$

This is in fact a very mild condition, since for a fixed algebra over $\mathcal{X}(\mathbb{P}^1)$ the Oggenfuss is reduced for all but a finite number of primes p, a [see Theorem 1.10].

[illegible] We then prove that the associated value $\mathbb{P}^{1\,\mathrm{ss}}$ coincides [illegible] the Galois local $\mathbb{P}^1$ at the $\mathcal{X}$-adic value [illegible] $\mathcal{X}$-group values of $\mathbb{P}^{1\,\mathrm{ss}}$-invariant points on $\mathcal{X}$. Furthermore [illegible] the [illegible] $\mathcal{X}$ is a locally free, rank one $\mathbb{P}^{1\,\mathrm{ss}}$-module (Theorem 1.10).

Due to the lack of appropriate coordinate [illegible] we can not also to prove a complete result for the position of global homomorphism of $\mathcal{X}$ over $\mathbb{P}^{1\,\mathrm{ss}}$. In case, if ξ is complete and $\{\xi, \bar{\mathcal{X}}\} = 1$ [illegible] we can show that $\mathcal{X}$ is free over the associated value

[illegible]. [illegible] [illegible] [illegible]

(iii) The elements $x_{i,j}$, $1 \le i \le n$, $1 \le j \le n_i$, form also an $\mathbb{K}$-basis of $H_{m|n}$; in particular, they form a $\bar{\mathbb{K}}$-basis of H.

(iv) The map

$$\iota_{\mathbb{K}} : H_n \otimes_{\mathbb{K}} H_n \to H_m \otimes_{\mathbb{K}} H_m, \quad x \otimes x' \mapsto x \otimes x',$$

is an injective $\mathbb{K}$-algebra map such that $\iota_{\bar{\mathbb{K}}} \circ \iota_{\mathbb{K}} = \iota_{\bar{\mathbb{K}}}$. In particular, $H_n \otimes_{\mathbb{K}} H_n$ can be regarded as an $\mathbb{K}$-subalgebra of H via $\iota_{\mathbb{K}}$.

(v) Denote the identities $H_n \to H_n$ by id_n and id, H_n is an $\mathbb{K}$-Hopf subalgebra of H.

(vi) The map

$$\Delta_{\mathbb{K}} : H_n \to H_n \otimes_{\mathbb{K}} H_n, \quad x \mapsto x \otimes x,$$

is an homomorphism of $\mathbb{K}$-Hopf algebras such that the diagrams

$$\begin{tikzcd} H_n \arrow[r, "\Delta_{\mathbb{K}}"] \arrow[d, "\iota_{\mathbb{K}}"'] & H_n \arrow[d, "\iota_{\mathbb{K}}"] \\ H_{[n]} \otimes_{\mathbb{K}[n]} H_{[n]} \arrow[r, "\Delta_{\mathbb{K}[n]}"] & H_{[n]} \end{tikzcd} \qquad \begin{tikzcd} H_n \otimes_{\mathbb{K}} H_n \arrow[r, "\Delta_{\mathbb{K}} \otimes \Delta_{\mathbb{K}}"] \arrow[d, "m_{\mathbb{K}}"'] & H_n \otimes_{\mathbb{K}} H_n \arrow[d, "m_{\mathbb{K}}"] \\ H_n \arrow[r, "\Delta_{\mathbb{K}}"] & H_n \end{tikzcd}$$

commute for all $n \in \mathbb{N}$. Here $m_{\mathbb{K}} : H_n \otimes_{\mathbb{K}} H_n \to H_n$ is the multiplication map.

Proof. (i) It is easy to see that the elements $x_{i,j}$, $1 \le i \le n$, $1 \le j \le n_i$, lie in H_n. On the other hand, for multiindex $\alpha = (\alpha_{ij})_{1 \le i \le n, 1 \le j \le n_i} \in \mathbb{N}^{n_1 + \cdots + n_n}$, the multiplicand H^{α} is finite under $\mathrm{ad}_{\mathbb{K}}(\mathbb{K})$, for each $j \in \mathbb{N}$, and so $H^{\alpha} \in H_n$ for each $\alpha \in \{0, \ldots, n\}$. Moreover, for each $\alpha \in \{0, \ldots, n\}$ and each $i \in \mathbb{N}_n$, one has $H_{\alpha_i} = x_{i,\alpha_i}$. So it follows that the elements $x_{i,j}$, $1 \le i \le n$, $1 \le j \le n_i$, form an $\mathbb{K}$-basis of H_n. This proof generalizes to the m in a similar way.

(ii) Indeed, by (i) we have

$$\dim_{\mathbb{K}}(H_n) = \prod_{i=1}^{n} n_i = \prod_{i=1}^{n} (n_i + 1) = \prod_{i=1}^{n} (\mathrm{rank}_{\mathbb{K}}(\mathfrak{g}_i)) = |\Phi|.$$

(iii) Since $\mathbb{K}$ is the field that the elements $x_{i,j}$, $1 \le i \le n$, form a $\bar{\mathbb{K}}$-basis of H. Regarding the elements $x_{i,j}$, $1 \le i \le n$, $1 \le j \le n_i$, by the basis $\mathbb{K}$ of H produces a matrix $M(\mathbb{K})$ is a block diagonal matrix $M(\mathbb{K})$ a block indexed by the $\mathbb{K}_n$-matrix of $\mathbb{K}$, then the 1-st block is given by

$$\left[\mathrm{ad}_{\mathbb{K}}(x_{i,j}) \right]_{i,j \in \{1, \ldots, n\} \times \{1, \ldots, n_i\}}$$

Then determinant does not contain, since $M_1(\mathbb{K})$ is regular. This is a the result for $\mathbb{K} = \bar{\mathbb{K}}$. For multiring $\mathbb{K}$, it follows by (ii) in the that the elements $x_{i,j}$ $(1 \le i \le n, 1 \le j \le n_i)$ are $\mathbb{K}$-linearly independent. And this leads a basis the case $\mathbb{K} = \bar{\mathbb{K}}$.

(ii) [illegible] [illegible] [illegible] [illegible], [illegible] [illegible] [illegible] [illegible]-[illegible] [illegible] [illegible]-[illegible].

(iii) [illegible] [illegible] [illegible] [illegible] [illegible], [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible]-[illegible] [illegible] [illegible]. [illegible] [illegible] [illegible]. [illegible], [illegible] [illegible], [illegible] [illegible]

$$[illegible] = [illegible]_{[illegible]} [illegible]_{[illegible]} [illegible] - [illegible] [illegible] - [illegible]$$

[illegible] [illegible] [illegible] [illegible] [illegible]. [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible]. [illegible] [illegible] [illegible] [illegible]-[illegible], [illegible] [illegible]

$$[illegible] = [illegible]_{[illegible]} [illegible]([illegible]) [illegible] - [illegible].$$

[illegible], [illegible] [illegible] [illegible], [illegible] [illegible] [illegible] [illegible] [illegible] [illegible], [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible]

Lemma 8.8 Assume that R is stable and let M [illegible] $\bar{M}$ be an intermediate field. Let $x_1, \ldots, x_n \in \bar{M}$ be a set of representatives for the R_0-basis of $\bar{M}$. For each $i \in \{1, \ldots, n\}$, let $\bar{x}_i$ denote the Koszul field of $\mathrm{ind}_{x_i}(\{x_j\}_{j \neq i})$, and let $\bar{x}_{i,1}, \ldots, \bar{x}_{i,n_i}$ be an R-basis of $\bar{x}_i$. Then $i \in [n]$ and $j \in [i] \cup \{\infty\}$, and let

$$\bar{x}_{i,j} := \prod_{\substack{\alpha \in \mathrm{Hom}(\{\infty\}, [n]) \\ \alpha(\infty) = i}} \alpha(\bar{x}_{i,j}) \in \bar{x}_{i,i} \in M.$$

Then the following two conditions hold:

(I) The composition

$$\bar{\Omega}_1 M \to \bigoplus_{i \in \bar{M}} \bar{x}_i \to \bigoplus_{i=1}^{n} \bar{x}_i,$$

[illegible] Ω denotes the projection to the $x_1, \ldots, x_n$-component, evaluated in an R-algebra homomorphism

$$\bar{\Omega}_n : R_n \to \bigoplus_{i=1}^{n} \bar{x}_i.$$

In particular, the canonical action R_n of R_n is given by $\bar{\Omega}_n^{-1}(\bigoplus_{i=1}^{n} R_{\bar{x}_i})$.

(II) The elements $\bar{x}_{i,1}$, $i \in [n]$ $i \in [i] \cup \{\infty\}$, form an R-basis of R_n. If for each $i \in [n]$, the elements $x_{i,1}, \ldots, x_{i,n_i}$ form an R_n-basis of $R_{\bar{x}_i}$, then the elements $\bar{x}_{i,1}$, $i \in [n]$ $i \in [i] \cup \{\infty\}$, form an R_n-basis of R_n.

Proof (I) follows from a straightforward form (I).

(II) The elements $\bar{x}_{i,j}$ are contained in R_n and $\bar{\Omega}_n(\bar{x}_{i,j})$, $i \in [n]$ $i \in [i] \cup \{\infty\}$ form an R-basis of $\bigoplus_{i=1}^{n} \bar{x}_i$. Then (II) follows from (I). The integral [illegible] can be seen in the same way. $\square$

8.8 Koszulity

In [illegible] we [illegible] the Koszul algebra dual (also called Koszul dual) of M is the $\bar{R}$-Koszul algebra $\bar{M} := \mathrm{Koz}(M, \bar{R})$ consisting of all set maps from M to $\bar{R}$. The Koszulity is given by the non-degenerate bilinear form

$$\langle -, - \rangle : M \otimes \bar{M} \to \bar{R}, \quad \langle \sum_{i \in M} \alpha_i, \beta \rangle := \sum_{i \in M} \alpha_i \beta(i).$$

The $\bar{R}$-linear dualities of M is obvious. Multiplication is given by $\langle \beta_1 \beta_2, \alpha \rangle = \beta_1(\alpha) \beta_2(\alpha)$, for $\beta_1, \beta_2 \in \bar{M}$ and $\alpha \in M$, [illegible] the canonical map [illegible] value 1 on unity. The diagonal Δ, augmentation ε, and antipode S are given by

$$\Delta(\beta)(\alpha_1, \alpha_2) = \beta(\alpha_1 \alpha_2),$$
$$\varepsilon(\beta) = \beta(1),$$
$$S(\beta)(\alpha) = \beta(\alpha^{-1}),$$

[illegible]

$$[illegible] = \prod_{[illegible]} [illegible] .$$

[illegible]

$$[illegible] := [illegible]$$

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$$[illegible] = [illegible] .$$

[illegible]

$$[illegible] := [illegible] = [illegible] \mid [illegible] = [illegible] \text{ for all } [illegible] \}.$$

[illegible]

[illegible]

$$[illegible] = [illegible] .$$

[illegible]

[illegible]

$$[illegible] := \begin{cases} [illegible], & [illegible] \\ [illegible], & [illegible] \end{cases}$$

[illegible]

[illegible]

$$[illegible] : [illegible] \to \prod_{i=1}^{[illegible]} [illegible] , \quad [illegible] \in [illegible] ,$$

[illegible]

$$[illegible] : [illegible] \to \prod_{i=1}^{[illegible]} [illegible] .$$

(ii) The elements $\mathbb{1}_{i,j}$, $1 \leq i \leq n$, $1 \leq j \leq n_i$, form an $\mathbb{K}$-basis of $\mathbb{K}_n$. If for each $i \in \{1, \ldots, n\}$, the elements $x_{i,1}, \ldots, x_{i,n}$ form an $\mathbb{K}_n$-basis of $\mathbb{K}_n$ then the elements $\mathbb{1}_{i,j}$, $1 \leq j \leq n_i$, $1 \leq i \leq n$, form an $\mathbb{K}_n$-basis of $\mathbb{K}_n$.

(iii) $\operatorname{Tr}_{\mathbb{K}_n}(\mathbb{K}_n) = |\mathbb{K}|$.

(iv) The elements $\mathbb{1}_{i,j}$, $1 \leq j \leq n_i$, $1 \leq i \leq n$, form also an $\mathbb{K}$-basis of $\mathbb{K}_m$.

(v) The map

$$\mathbb{p}^{m}_{n} : \mathbb{K}_n \otimes_{\mathbb{K}} \mathbb{K}_n \to \mathbb{K}_m \otimes_{\mathbb{K}} \mathbb{K}_m, \quad \mathbb{1}_{ij} \otimes \mathbb{1}_{i'j'} \mapsto \mathbb{1}_{ij} \otimes \mathbb{1}_{i'j'},$$

is an injective $\mathbb{K}$-algebra homomorphism such that $\mathbb{p}^{m}_{n} \circ \mathbb{p}^{n}_{m} = \mathbb{p}^{m}_{m}$. In particular, $\mathbb{K}_n \otimes_{\mathbb{K}} \mathbb{K}_n$ can be regarded as a $\mathbb{K}$-subalgebra of $\mathbb{K}_m \otimes_{\mathbb{K}} \mathbb{K}_m$.

(vi) Define the identification $\mathbb{K}_n \otimes_{\mathbb{K}} \mathbb{K}_n \cong \mathbb{K}_m \otimes_{\mathbb{K}} \mathbb{K}_m$ of (v), $\mathbb{K}_n$ is an $\mathbb{K}$-Hopf subalgebra of $\mathbb{K}_m$.

(vii) The map

$$\mathbb{q}^{m}_{n} : \mathbb{K}_n \otimes_{\mathbb{K}} \mathbb{K}_n \to \mathbb{K}_m, \quad \mathbb{1}_{ij} \mapsto \mathbb{1}_{ij},$$

is an injective $\mathbb{K}$-Hopf algebra map that the diagram

$$\begin{tikzcd}
\mathbb{K}_n \otimes_{\mathbb{K}} \mathbb{K}_n \arrow[r, "\mathbb{p}^{m}_{n}"] \arrow[d, "\mathbb{q}^{n}_{n}"'] & \mathbb{K}_m \arrow[d, "\mathbb{q}^{m}_{m}"] & \mathbb{K}_n \otimes_{\mathbb{K}} \mathbb{K}_n \otimes_{\mathbb{K}} \mathbb{K}_n \arrow[r, "\mathbb{p}^{m}_{n} \otimes \mathbb{p}^{m}_{n}"] \arrow[d, "\Delta_{\mathbb{K}_n}"'] & \mathbb{K}_m \otimes_{\mathbb{K}} \mathbb{K}_m \arrow[d, "\Delta_{\mathbb{K}_m}"] \\
\mathbb{K}_n(|\mathbb{K}|) \otimes_{\mathbb{K}(|\mathbb{K}|)} \mathbb{K}_n(|\mathbb{K}|) \arrow[r, "\mathbb{p}^{m(|\mathbb{K}|)}_{n(|\mathbb{K}|)}"] & \mathbb{K}_m(|\mathbb{K}|) & \mathbb{K}_n \otimes_{\mathbb{K}} \mathbb{K}_n \arrow[r, "\mathbb{p}^{m}_{n}"] & \mathbb{K}_m
\end{tikzcd}$$

commute for all $n \in \mathbb{N}$.

(viii) The map $\mathbb{K}_n \otimes_{\mathbb{K}} \mathbb{K}_n \to \mathbb{K}_m \otimes_{\mathbb{K}} \mathbb{K}_m$, $x \otimes_{\mathbb{K}} 1 \mapsto x \otimes_{\mathbb{K}} 1$, is injective, in particular, $\mathbb{K}_n \otimes_{\mathbb{K}} \mathbb{K}_n$ can be regarded as an $\mathbb{K}$-subspace of $\mathbb{K}_m \otimes_{\mathbb{K}} \mathbb{K}_m$. However, the restriction of $[-,-]$ to $\mathbb{K}_n \otimes_{\mathbb{K}} \mathbb{K}_n$ takes values in $\mathbb{K}$ and is non-degenerate.

(ix) The $\mathbb{K}$-Hopf algebra $\mathbb{K}_n$ and $\mathbb{K}_m$ are dual to each other. The map in $[-,-] : \mathbb{K}_n \otimes_{\mathbb{K}} \mathbb{K}_n \to \mathbb{K}$.

Proof. (i) For $i \in \mathbb{K}_n$ and $j \in \{1, \ldots, n\}$, the elements $\mathbb{1}^{n}_{ij} = x(\mathbb{1}_{ij})$, $x \in \mathbb{K}_n$, are fundamental $\mathbb{K}_n(\mathbb{1}_{ij})$. Hence, $\mathbb{K}_n$ is injective. On the other hand, for given elements $\mathbb{K}_i \in \mathbb{K}_i$, $i = 1, \ldots, n$, the map $\mathbb{1}_i \mapsto \mathbb{K}_i$, defined by $\mathbb{1}^{n}_{ij} \mapsto x(\mathbb{K}_i)$, for $j = 1, \ldots, n$ and $x \in \mathbb{K}_n$, is extended to $\mathbb{K}_n$.

(ii) This follows immediately from (i).

(iii) This follows from (ii) and the equation $\sum_{i=1}^{n} \operatorname{Tr}_{\mathbb{K}_n}(\mathbb{K}_n : \operatorname{End}_{\mathbb{K}_n}(\mathbb{1}_i)) = |\mathbb{K}|$.

(iv) This is proved in a similar way as in Lemma 3.1 (iii) by enlarging the $\mathbb{K} = \bar{\mathbb{K}}$ and using the basis $\mathbb{1}_{ij}$, $j \in \mathbb{K}$, of $\mathbb{K}$. Still holds in the same formulation stated in the proof of Lemma 3.1 (iii).

(v) This follows from (iv).

(vi) This is proved in a similar way as in Lemma 3.1 (v) using the basis $\mathbb{1}_{ij}$, $1 \leq j \leq n_i$, $1 \leq i \leq n$, of $\mathbb{K}_n$.

(vii) By (iv), $\mathbb{q}^{m}_{n}$ is an injective of $\mathbb{K}$-spaces. The remaining assertions are easily verified.

(iii) The inequality of the map $\mathbb{H}_n \otimes_n \mathbb{H}_n \to \mathbb{H}_m \otimes_m \mathbb{H}_m$ follows from Lemma 3.1 (iii) and part (ii). Moreover, $(\mathbb{H}_n, \mathbb{H}_m)$ is of $\mathbb{H}_2$ equation (2). Using the lemma $\alpha_{1,j}$ of $\mathbb{H}_n$ from Lemma 3.1 and $\beta_{1,j}$ of $\mathbb{H}_m$, $1 \le j \le n$, $1 \le j \le m$, α_1, β_1 [illegible] generate from Lemma 3.1 (iii) and from (ii), so we find $(\mathbb{H}_n, \mathbb{H}_m) = 1$ and that the rational pairing $\mathbb{H}_n \otimes_n \mathbb{H}_n \to \mathbb{H}$ is non-degenerate.

(iv) This follows from part (iii) and from the relations of Lemma $\alpha_{1,j}$ of $\mathbb{H}_n$ and $\beta_{1,j}$ of $\mathbb{H}_m$ [illegible] the generator from Lemma 3.1 (iii) and from (ii). □

3.3 Lattices and the Frobenius

Let $\mathbb{H} \subseteq \mathbb{H}' \subseteq \bar{\mathbb{H}}$ be an intermediate field. Let $\mathbb{H}_n$-lattices in $\mathbb{H}_n$ and $\mathbb{H}_n$ [illegible] respect to the non-degenerate pairing

$$\mathbb{H}_n \otimes_n \mathbb{H}_n \to \mathbb{H}, \quad (\underset{n\times}{\mathbb{H}} \times \mathbb{H}, \mathbb{H}) \to \underset{n\times}{\mathbb{H}} \times \mathbb{H}(\mathbb{H}).$$

More generally, if $\mathbb{H} \subseteq \mathbb{H}_n$ and $\mathbb{H}' \subseteq \mathbb{H}_n$ are $\mathbb{H}_n$-lattices then

$$\mathbb{H}^* := \{\mathbb{H} \in \mathbb{H}_n \mid (x, \mathbb{H}) \in \mathbb{H}_n \text{ for all } x \in \mathbb{H}\}$$

and

$$\mathbb{H}^* := \{x \in \mathbb{H}_n \mid (x, x) \in \mathbb{H}_n \text{ for all } x \in \mathbb{H}\}$$

are $\mathbb{H}_n$-lattices in $\mathbb{H}_n$ and $\mathbb{H}_n$ respectively. Note that $\mathbb{H}$ is an $\mathbb{H}_n$-order (resp. $\mathbb{H}_n$-endomorphism) of $\mathbb{H}_n$ if and only if $\mathbb{H}^*$ is an $\mathbb{H}_n$-endomorphism (resp. $\mathbb{H}_n$-order) in $\mathbb{H}_n$. We denote dualities Hodge by $\mathbb{H}$. Moreover, $\mathbb{H}$ is an $\mathbb{H}_n$-Hopf order in $\mathbb{H}_n$ if and only if $\mathbb{H}^*$ is an $\mathbb{H}_n$-Hopf order in $\mathbb{H}_n$, and similarly for $\mathbb{H}$.

Recall that for $\mathbb{H}_n$-lattices $\mathbb{H} \subseteq \mathbb{H}$ of equal $\mathbb{H}_n$-rank the $\mathbb{H}_n$-order ideal $[\mathbb{H} : \mathbb{H}]_{\mathbb{H}_n}$ is defined as the product $\mathfrak{p}_1 \cdots \mathfrak{p}_t$ of non-zero prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ of $\mathbb{H}_n$ if $\mathbb{H}/\mathbb{H} \cong \mathbb{H}/\mathfrak{p}_1, \ldots, \mathbb{H}/\mathfrak{p}_t$ are the $\mathbb{H}_n$-composition factors of $\mathbb{H}/\mathbb{H}$. More generally, for $\mathbb{H}_n$-lattices $\mathbb{H}$ and $\mathbb{H}$ in a finite dimensional $\mathbb{H}$-module space the order ideal $[\mathbb{H} : \mathbb{H}]_{\mathbb{H}_n}$ is defined as the fractional ideal $[\mathbb{H} : \mathbb{H} \cap \mathbb{H}]_{\mathbb{H}_n} [\mathbb{H} : \mathbb{H} \cap \mathbb{H}]_{\mathbb{H}_n}^{-1}$. For the following properties of order ideals see for example [1, 2]. If $\mathbb{H} \subseteq \mathbb{H}' \subseteq \bar{\mathbb{H}}$ is a finite extension field of $\mathbb{H}$ then

$$[\mathbb{H}_{\mathbb{H}'} \otimes_{\mathbb{H}_n} \mathbb{H} : \mathbb{H}_{\mathbb{H}'} \otimes_{\mathbb{H}_n} \mathbb{H}]_{\mathbb{H}_{n'}} = [\mathbb{H} : \mathbb{H}]_{\mathbb{H}_n} \mathbb{H}_{\mathbb{H}'}.$$

If $\mathfrak{p}$ is a non-zero prime ideal of $\mathbb{H}_n$ then the following properties hold:

$$[\mathbb{H}_{\mathfrak{p}} : \mathbb{H}_{\mathfrak{p}}]_{|\mathbb{H}_n|_{\mathfrak{p}}} = \{[\mathbb{H} : \mathbb{H}]_{\mathbb{H}_n}\}_{\mathfrak{p}}.$$

If $\mathbb{H}$ and $\mathbb{H}$ are two $\mathbb{H}_n$-modules and $\mathbb{H}$ is the module of coefficients relating two sequences on $\mathbb{H}_n$-bases of $\mathbb{H}$ $\mathbb{H}_{\mathbb{H}}$ on $\mathbb{H}_n$-bases of $\mathbb{H}$, then

$$[\mathbb{H} : \mathbb{H}]_{\mathbb{H}_n} = \det\{\mathbb{H}\}\mathbb{H}_n.$$

Finally, if $M \subseteq M' \subseteq \bar{M}$ is an chain and $M'' \subseteq M'$ are $R_{M'}$-bellow then

$$[M' : M'']_{R_{M'}} = R_{M'/M}[[M' : M'']_{R_{M'}}] .$$

If M is stellar a basis B R_{M_n} the Bicoalgebra Hopf of the maximal order R_M over R_M, i.e. $R_{M_n} = \bigotimes_{i=1}^{n} R_{i,M}$ in the notation of Lemma 5.3. Similarly, for nilpotency M, a basis B R_{M_n} the Bicoalgebra Hopf of the maximal order R_M over R_M, i.e. $R_{M_n} = \bigotimes_{i=1}^{n} R_M$ in the notation of Lemma 5.1 and Lemma 5.2. If $M \subseteq M' \subseteq \bar{M}$ is an chain then a $R_{M'/M}$ for the Hopf of $R_{M'}$ over R_M.

Lemma 5.5 *[2] The notation of Lemma 5.1 and Lemma 5.2 are for*

$$R_M^n = \left\{ \bigotimes_{i=1}^{n} \bigotimes_{\mathrm{ord}_M(\mathrm{char}_{k_M}(i))} \mathrm{ord}_M(\mathrm{char}_{k_M}(i))\, \mathfrak{p}_i \;\middle|\; \mathfrak{p}_i \in M_i^{-1}, \text{ for } i = 1, \ldots, n \right\} .$$

Moreover, R_M^n is an R_M-order in R_M if and only if $R_{M_n} = R_M$.

Proof. Each element x R_M can be written in the form

$$x = \bigotimes_{i=1}^{n} \bigotimes_{\mathrm{ord}_M(\mathrm{char}_{k_M}(i))} \mathrm{ord}_M(\mathrm{char}_{k_M}(i))\, \mathfrak{p}_i$$

[2] originally determined $\mathfrak{p}_i \in M_i$, $i = 1, \ldots, n$, by Lemma 5.1 [1]. Then $x \in R_M^n$ if and only if $[x, M] \in R_M$ for all $M \in R_M$. By Lemma 5.2 [1], each $M \in R_M^n$ is a sum of elements of the form $\bigotimes_{i=1}^{n} \mathrm{ord}_M(\mathrm{char}_{k_M}(i))\, x_i$, $x_i \in R_M$, $i = 1, \ldots, n$. Thus the sum sum over such representations of $R_M(\mathrm{ord}_M(\mathrm{char}_{k_M}(i)))$. Hence, $x \in R_M^n$ if and only if

$$\bigotimes_{\mathrm{ord}_M(\mathrm{char}_{k_M}(i))} \mathrm{ord}_M(\mathrm{char}_{k_M}(i))\, x_i \in R_M \quad \mathfrak{p}_i(\mathfrak{p}_i x_i) \in R_M$$

for all $i \in \{1, \ldots, n\}$ and all $x_i \in R_M$, that is equivalent to $\mathfrak{p}_i \in M_i^{-1}$ for all $i \in \{1, \ldots, n\}$.

If $R_{M_n} = R_M$ then $R_M \mathfrak{p}_i = R_M$ for all $i = 1, \ldots, n$, and therefore $R_M^n = R_M^n$ is an R_M-order. Conversely, conversely, that R_M^n is an R_M-order and let $i \in \{1, \ldots, n\}$ be arbitrary. Let $i' \in \{1, \ldots, n\}$ and $x \in R_M$ be such that $\mathfrak{p}_i^{-1} = \mathfrak{p}_{i'}$. Note that $\mathrm{char}_{k_M}(\mathfrak{p}_i) = \mathrm{char}_{k_M}(\mathfrak{p}_i^{-1})$ and that $x : R_{i'} \to R_i$ is a R-homomorphism. Let $\mathfrak{p}_i \in M_i^{-1}$ and $\mathfrak{p}_{i'} \in M_{i'}^{-1}$ be arbitrary. Then

$$\bigotimes_{\mathrm{ord}_M(\mathrm{char}_{k_M}(i))} \mathrm{ord}_M(\mathrm{char}_{k_M}(i))\, \mathfrak{p}_i , \qquad \bigotimes_{\mathrm{ord}_M(\mathrm{char}_{k_M}(i'))} \mathrm{ord}_M(\mathrm{char}_{k_M}(i'))\, \mathfrak{p}_{i'}$$

are elements of R_M^n. Hence, also their product lies in R_M^n. In particular the coefficient of $1 \in R$ of this product is an element of R_M:

$$\bigotimes_{\mathrm{ord}_M(\mathrm{char}_{k_M}(i))} \mathrm{ord}_M(\mathrm{char}_{k_M}(i))\, \mathrm{ord}_M(\mathrm{char}_{k_M}(i'))\, \mathfrak{p}_i \mathfrak{p}_{i'} \in R_M . \tag{8}$$

If X_i runs through $\mathfrak{m}_{R,p}^{-i}$, then $\sigma(X_i)$ runs through $\mathfrak{m}_{S,q}^{-i}$, and (8) implies that $\mathfrak{m}_{S,q}(\mathfrak{m}_{S,q}^{-i}) \subseteq \mathfrak{m}_S$. But this is only possible if $\mathfrak{m}_{S,q} = \mathfrak{m}_S$. Since this holds for all $i \in \{1, \ldots, n\}$, we obtain $\mathfrak{m}_{S_m} = \mathfrak{m}_S$. □

Corollary 5.8 *Let $\mathfrak{m}_R^n \subseteq \mathfrak{m}_S^n$ and $[\mathfrak{m}_R^n : \mathfrak{m}_R^n]_{S_m} = q_{S_m}$.*

Proof. The inclusion $\mathfrak{m}_R^n \subseteq \mathfrak{m}_S^n$ is also from Lemma 5.6. Applying ~ from Lemma 5.3 (1), we have

$$[\mathfrak{m}_R^n : \mathfrak{m}_R^n]_{S_m} = \Big[\prod_{i=1}^{n} \mathfrak{m}_{R,p}^{-i} : \prod_{i=1}^{n} \mathfrak{m}_R\Big]_{S_m} = \prod_{i=1}^{n} [\mathfrak{m}_R : \mathfrak{m}_{R,p}]_{S_m}$$
$$= \prod_{i=1}^{n} q_{R,p} = q_{S_m}.$$

□

We remark that, since $\mathfrak{m}$ stable, $\mathfrak{m}_S^n$ is not necessarily contained in $\mathfrak{m}_R$.

Now the hole [illegible] we assume that $\mathfrak{m}$ is stable. Let $\mathfrak{m} \subseteq \mathfrak{m}' \subseteq \bar{\mathfrak{m}}$ be a finite extension field of $\mathfrak{m}$ containing $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ and $\bar{\mathfrak{m}}_1, \ldots, \bar{\mathfrak{m}}_n$ in the notation of Lemma 5.1 and Lemma 5.3. Then $\mathfrak{m}_{R'} = \mathfrak{m}'\mathfrak{m}$, $\mathfrak{m}_{S'} = \mathfrak{m}\mathrm{ag}(\mathfrak{m}, \mathfrak{m}')$, and $\mathfrak{m}_{S'} = \mathfrak{m}_{S_{\mathfrak{m}'}}\mathfrak{m}'$ are p- or $\mathfrak{m}'$-algebras. Such a finite extension $\mathfrak{m}'$ [illegible] is called a splitting field for $\mathfrak{m}_R$ and $\mathfrak{m}_S$.

Lemma 5.9 *Let $\mathfrak{m}$ be stable. Then*

$$[\mathfrak{m}_R : \mathfrak{m}_R^n]_{S_m} = |\mathfrak{m}|^{|n|} q_{S_m} q_{S_m}^{-1}$$

for all intermediate field $\mathfrak{m} \subseteq \mathfrak{m}' \subseteq \bar{\mathfrak{m}}$.

Proof. Let p be a non-zero prime ideal of $\mathfrak{m}_R$. Since $[\mathfrak{m}_R]_p$ is the maximal $[\mathfrak{m}_R]_p$-order in $\mathfrak{m}_R$ and $[\mathfrak{m}_S^n]_p$ is the set of elements in $\mathfrak{m}_R$ [illegible] suggest to $\mathfrak{m}$ are integral over $[\mathfrak{m}_R]_p$, we may as well assume that $\mathfrak{m}_R$ is local [illegible] maximal ideal p. Let $\mathfrak{m}'$ be a splitting field for $\mathfrak{m}_R$ and $\mathfrak{m}_S$. [illegible] Estimating

$$[\mathfrak{m}_R : \mathfrak{m}_R^n]_{S_m} q_{S'} = [\mathfrak{m}_{S'} \mathfrak{m}_{S_m} \mathfrak{m}_R : \mathfrak{m}_{S'} \mathfrak{m}_{S_m} \mathfrak{m}_R^n]_{S_m},$$

using the isomorphism $\mathfrak{m}_{S'}^n : \mathfrak{m}'\mathfrak{m}_R \mathfrak{m}_R \cong \mathfrak{m}_{S'}$, the maximal $\mathfrak{m}_{S'}$-order $\mathfrak{m}_{S'}$ of $\mathfrak{m}_{S'}$, and the isomorphism $p : \mathfrak{m}_{S'} \cong \mathfrak{m}_{S_{\mathfrak{m}'}}\mathfrak{m}'$. Now geometrically, $[\mathfrak{m}_R : \mathfrak{m}_R^n]_{S_m} q_{S'}$ is the quotient of the order field

$$[\ \ \mathfrak{m}(\mathfrak{m}_{S'}) : (\ \ \mathfrak{m}_{S'} = \mathfrak{m}_{S'}^{n})(\mathfrak{m}_{S'} \mathfrak{m}_{S_m} \mathfrak{m}_R^n)]_{S_m}, \tag{9}$$

and

$$[\ \ \mathfrak{m}(\mathfrak{m}_{S'}) : (\ \ \mathfrak{m}_{S'} = \mathfrak{m}_{S'}^{n})(\mathfrak{m}_{S'} \mathfrak{m}_{S_m} \mathfrak{m}_R)]_{S_m}. \tag{10}$$

Due to the assumption of the value level [illegible] let [illegible], $\tilde{\mathbb{M}}_{\mathbb{R}}$, $\mathbb{M}_{\mathbb{R},1}$, and $\tilde{\mathbb{M}}_{\mathbb{R},1}$, $1 \leq \mathbb{R} \leq \mathbb{N}$, $1 \leq 1 \leq \mathbb{N}_{\mathbb{R}}$. As given as in Lemma 2.3 and that $\mathbb{M}_{\mathbb{R},1}, \ldots, \mathbb{M}_{\mathbb{R},\mathbb{N}_{\mathbb{R}}}$ is an $\mathbb{M}_{\mathbb{R}}$-basis of $\mathbb{M}_{\tilde{\mathbb{R}}_{\mathbb{R}}}$ for each $\mathbb{R} = 1, \ldots, \mathbb{N}$. Then the elements

$$\tilde{\mathbb{M}}_{\mathbb{R},1} = \sum_{\mathbb{R}\mathbb{S} \in [\mathbb{N}] \times \mathbb{S}_{\mathbb{R}}(\mathbb{N})} \mathbb{M}(\mathbb{M}_{\mathbb{R},1}) \mathbb{S}^{-\mathbb{S}_{\mathbb{R}}} \qquad (1 \leq \mathbb{R} \leq \mathbb{N},\ 1 \leq 1 \leq \mathbb{N}_{\mathbb{R}})$$

form an $\mathbb{M}_{\mathbb{R}}$-basis of $\mathbb{M}_{\mathbb{R}}$ and $\{\mathbb{R}\} \times \mathbb{S}_{\mathbb{R}}^{\mathbb{N}} \setminus \{(\mathbb{M}_{\mathbb{R}}, \mathbb{S}_{\mathbb{R}}, \mathbb{M}_{\mathbb{R}})\}$ for an $\mathbb{M}_{\mathbb{R}}$-basis the elements

$$\{\ \}\{\tilde{\mathbb{M}}_{\mathbb{R},1}\}_{\mathbb{R}\tilde{\mathbb{R}}} \qquad (1 \leq \mathbb{R} \leq \mathbb{N},\ 1 \leq 1 \leq \mathbb{N}_{\mathbb{R}}).$$

Regarding the basis $\mathbb{B}_{\mathbb{R}}$ the $\mathbb{M}_{\mathbb{R}}$-basis $\mathbb{B}_{\mathbb{R}}$, $\mathbb{R} \in \tilde{\mathbb{R}}$, of $\mathbb{R}(\mathbb{M}_{\mathbb{R}}) = \bigoplus_{\mathbb{R}\in\tilde{\mathbb{R}}} \mathbb{M}_{\mathbb{R}}$, a matrix a block diagonal transition matrix [illegible] block indexed $\mathbb{R} \in \tilde{\mathbb{R}}(\mathbb{M}_{\mathbb{R}})$. The block belonging to $\mathbb{R}$ is given $\mathbb{B}_{\mathbb{R}}$

$$\left\| \mathbb{M}(\mathbb{M}_{\mathbb{R},1}) \right\|_{\mathbb{R},1,\ldots,\mathbb{N}_{\mathbb{R}}} \qquad \mathbb{R}\mathbb{S} \in [\mathbb{N}] \times \mathbb{S}_{\mathbb{R}}(\mathbb{N}).$$

The square of the determinant of this block generates the $\mathbb{M}_{\mathbb{R}}$-ideal $\mathbb{I}_{\tilde{\mathbb{R}}_{\mathbb{R}}\mathbb{R}}$. Thus, the equal value ideal in [illegible] is given $\mathbb{B}_{\mathbb{R}}$

$$\prod_{\mathbb{R}=1}^{\mathbb{N}} \mathbb{I}_{\tilde{\mathbb{R}}_{\mathbb{R}}\mathbb{R}} \mathbb{M}_{\mathbb{R}} = \mathbb{I}_{\mathbb{N}_{\mathbb{R}}\mathbb{R}} \mathbb{M}_{\mathbb{R}}. \qquad [illegible]$$

Due to the assumption of the equal value ideal [illegible] let $\mathbb{R}$, $\mathbb{M}_{\mathbb{R}}$, $\mathbb{S}_{\mathbb{R},\mathbb{R}}$, and $\mathbb{S}_{\mathbb{R},\mathbb{R}}$, $1 \leq \mathbb{R} \leq \mathbb{N}$, $1 \leq \mathbb{R} \leq \mathbb{N}_{\mathbb{R}}$, as given as in Lemma 2.1 and that $\mathbb{S}_{\mathbb{R},1}, \ldots, \mathbb{S}_{\mathbb{R},\mathbb{N}}$ is an $\mathbb{M}_{\mathbb{R}}$-basis of $\mathbb{M}_{\mathbb{R}}$ for each $\mathbb{R} \in [\mathbb{N}, \ldots, \mathbb{N}]$. Then, the elements

$$\mathbb{S}_{1,\mathbb{R}} = \sum_{\mathbb{R}\mathbb{S} \in [\mathbb{N}] \times \mathbb{S}_{\mathbb{R}}(1)} \mathbb{M}(\mathbb{S}_{1,\mathbb{R}}) \mathbb{S}^{-\mathbb{M}_{1}} \qquad (1 \leq \mathbb{R} \leq \mathbb{N},\ 1 \leq \mathbb{R} \leq \mathbb{N}_{1})$$

form an $\mathbb{M}_{\mathbb{R}}$-basis of $\mathbb{M}_{\mathbb{R}}^{\mathbb{N}}$. Thus, the elements

$$\left\| \sum_{\mathbb{R}\mathbb{S} \in [\mathbb{N}] \times \mathbb{S}_{\mathbb{R}}(1)} \mathbb{M}(\mathbb{S}_{1,\mathbb{R}}) \mathbb{S}^{-\mathbb{M}_{1}} \right\|_{\mathbb{R}\tilde{\mathbb{R}}} \qquad (1 \leq \mathbb{R} \leq \mathbb{N},\ 1 \leq \mathbb{R} \leq \mathbb{N}_{1})$$

form an $\mathbb{M}_{\mathbb{R}}$-basis of $\{\mathbb{R}\} \times \mathbb{S}_{\mathbb{R}}^{\mathbb{N}} \setminus \{(\mathbb{M}_{\mathbb{R}}, \mathbb{S}_{\mathbb{R}}, \mathbb{M}_{\mathbb{R}})\}$. Regarding the basis $\mathbb{B}_{\mathbb{R}}$ the basis $\mathbb{B}_{\mathbb{R}}$, $\mathbb{R} \in \tilde{\mathbb{R}}$, of $\mathbb{R}(\mathbb{M}_{\mathbb{R}}) = \bigoplus_{\mathbb{R}\in\tilde{\mathbb{R}}} \mathbb{M}_{\mathbb{R}}$, a matrix the transition matrix

$$\mathbb{M} = \left\| \sum_{\mathbb{R}\mathbb{S} \in [\mathbb{N}] \times \mathbb{S}_{\mathbb{R}}(1)} \mathbb{M}(\mathbb{S}_{1,\mathbb{R}}) \mathbb{S}^{-\mathbb{M}_{1}} \right\|_{\mathbb{R}\tilde{\mathbb{R}}}.$$

Still in the product $\mathbb{M} = \mathbb{M}_{1}\mathbb{M}_{2}$ of the block diagonal matrix $\mathbb{M}_{1}$ [illegible] block indexed $\mathbb{R}$ $1 = 1, \ldots, \mathbb{N}$, the 1-th block given $\mathbb{B}_{\mathbb{R}}$

$$\mathbb{M}_{1,1} = \left\| \mathbb{M}(\mathbb{S}_{1,\mathbb{R}}) \right\|_{\mathbb{R},1,\ldots,\mathbb{N}_{\mathbb{R}}(1)},$$

and the matrix

$$[illegible] = \left[[illegible] \right]_{[illegible]} .$$

Now, $\mathrm{Tr}\{[illegible]_{i,j}\}$ generates the $[illegible]$-linear $[illegible]$ map $\mathrm{Tr}\{[illegible]_{i,j}\}[illegible] = [illegible]$. Moreover, since

$$\sum_{[illegible]} [illegible] = \sum_{[illegible]} [illegible] = \begin{cases} |[illegible]|, & [illegible] = [illegible]^{-1}, \\ [illegible], & \text{otherwise} \end{cases}$$

for [illegible], [illegible] $\in \hat{[illegible]}$, we have

$$\mathrm{Tr}\{[illegible]\}^2 = \mathrm{Tr}\{[illegible]^2 [illegible]\} = [illegible]|[illegible]|^{[illegible]} . \qquad [illegible]$$

Bilinearity of the trace,

$$\mathrm{Tr}\{[illegible]\}^2 [illegible] = |[illegible]|^{[illegible]} [illegible] . \qquad [illegible]$$

Hence equation the equation above lead to [illegible]. So [illegible], identifying the lead to [illegible] $[illegible]$ the lead to [illegible], the result follows [illegible], since estimating $[illegible]$-linear to $[illegible]$-linear is injective. □

Next [illegible] a inequality notice. Hence characteristic $[illegible]$ and [illegible] the [illegible] $[illegible]$ are [illegible] notice $[illegible]$ to $[illegible]$. The crucial point is to get hold of the image of the diagonal.

Since the field [illegible] is of general interest [illegible] a class crucial for the [illegible] in a more general setting. Let $[illegible]$ be a finite group, $[illegible]$ a field extension in characteristic zero, and $[illegible]$ a subring such that $[illegible]$ is the field of fractions of $[illegible]$. Suppose that $[illegible]$ is an $[illegible]$-subalgebra of $[illegible]$ for $[illegible]$-basis $[illegible]_1, \ldots, [illegible]$ Hence $[illegible]$ is also an $[illegible]$-basis of $[illegible]$. Then the multiplication map $[illegible] \otimes [illegible] \to [illegible]$, $[illegible] \otimes [illegible] \mapsto [illegible]$, is an $[illegible]$-algebra homomorphism and the map $[illegible] \otimes [illegible] \to [illegible] \otimes [illegible]$, $[illegible] \otimes [illegible]' \mapsto [illegible] \otimes [illegible]'$, is injective, so that $[illegible] \otimes [illegible]$ can be regarded as an $[illegible]$-subalgebra in $[illegible] \otimes [illegible]$. [illegible] and [illegible] in [illegible] [illegible] $[illegible]$ $[illegible]$ $[illegible]$ not. [illegible] [illegible]

$$[illegible]_1 = \sum_{[illegible]} [illegible]_{1,[illegible]} [illegible]$$

for $1 = 1, \ldots, [illegible]$ [illegible] $[illegible]_{1,[illegible]} \in [illegible]$, and [illegible] consider the matrix

$$[illegible] := [[illegible]_{i,[illegible]}]_{[illegible],\ldots,[illegible]} \in [illegible] [illegible],$$

together [illegible] [illegible] inverse

$$[illegible] := [illegible]^{-1} = [[illegible]_{i,[illegible]}]_{[illegible],\ldots,[illegible]} .$$

Then [illegible] have the following relation:

Lemma [illegible]. Keeping the above notation, the following two conditions are equivalent:

(i) $\mathbb{M}[\mathbb{M}] \cong \mathbb{M} \otimes_{\mathbb{M}} \mathbb{M}$.

(ii) $\mathbb{M}_{ijk} = s_{i,j}\mathbb{M}_{j,k}$ $\forall$ $\mathbb{M}$ for all $i, j, k \in \{1, \ldots, n\}$.

Proof. [illegible] [illegible]

$$\mathbb{M}[s_i] = \bigoplus_{j,k=1} c_{j,k}^{[i]} s_j \mathbb{M} s_k$$

[illegible] multiply bimodule coefficients $c_{j,k}^{[i]} \in \mathbb{M}$. Then, for $i \in \{1, \ldots, n\}$, [illegible] $\mathbb{M}[s_i] \cong \mathbb{M} \otimes_{\mathbb{M}} \mathbb{M}$ if and only if $c_{j,k}^{[i]} \in \mathbb{M}$ for all $j, k \in \{1, \ldots, n\}$. Using the equation $\mathbb{M}[s_i] = \bigoplus_{j,k} s_{i,j}\mathbb{M} \otimes \mathbb{M}$, [illegible] the equivalent equation

$$\bigoplus_{j,k} s_{i,j}\mathbb{M} \otimes \mathbb{M} = \bigoplus_{j,k=1} \bigoplus_{i,j,k} c_{j,k}^{[i]} s_{j,i} s_{k,i} \mathbb{M} \otimes \mathbb{M}.$$

Hence, the coefficients $c_{j,k}^{[i]}$ are multiply bimodule by the system of linear equations

$$\bigoplus_{j,k} c_{j,k}^{[i]} s_{j,i} s_{k,i} = \begin{cases} s_{i,i}, & \text{if } \mathbb{M} = \mathbb{M} \Leftrightarrow \mathbb{M}, \\ 0, & \text{if } \mathbb{M} \neq \mathbb{M}, \end{cases}$$

one equation for each $i \in \{1, \ldots, n\}$ and each pair $\{\mathbb{M}, \mathbb{M}\} \in \mathbb{M} \times \mathbb{M}$. [illegible] It is easy to verify that these equations are satisfied for $c_{j,k}^{[i]} = \mathbb{M}_{ijk} s_{i,j}\mathbb{M}_{j,k}$, and the result follows. $\square$

In the following proposition we apply Lemma [illegible] to the $\mathbb{M}_n$-module $\mathbb{M}_n^n$ in $\mathbb{M}_n$.

Proposition [illegible]. The $\mathbb{M}_n$-module $\mathbb{M}_n^n$ is a Morphism in $\mathbb{M}_n$ if and only if $\mathbb{M}_n = \mathbb{M}_n$.

Proof. Clearly, $\mathbb{M}_n^n$ is stable under the antipode of $\mathbb{M}_n$. The isomorphism $\mathbb{M}[\mathbb{M}_n^n] \cong \mathbb{M}_n^n \otimes_{\mathbb{M}} \mathbb{M}_n^n$ can be tested by localization. Thus [illegible] we may assume that $\mathbb{M}_n$ is a local ring. In this case [illegible] choose $p_1, \mathbb{M}_1, s_{i,j}$, and $s_{i,j}$ as in Lemma [illegible] and that, for each $i \in \{1, \ldots, n\}$, the elements $s_{i,1}, \ldots, s_{i,n}$ form an $\mathbb{M}_n$-basis of $\mathbb{M}_n$. Then the elements $s_{i,j}$, $1 \leq i \leq n$, $1 \leq j \leq s_i$, form an $\mathbb{M}_n$-basis of $\mathbb{M}_n^n$, and, in the notation introduced before Lemma [illegible], the coefficient matrix $\mathbb{M}$ is a block diagonal matrix, the block indexed by $i = 1, \ldots, n$, and the i-th block given by

$$[s_{i,j}]_{1 \leq j \leq s_i} \quad .$$

If $x'_{i,1}, \ldots, x'_{i,n}$ is M_i is a final basis of $x_{i,1}, \ldots, x_{i,n}$ (if regard to the inner hom $M_{\mathfrak{m}}(\mu)$, then the inverse Φ of Φ is given by the block diagonal matrix (if the 1-st block

$$\left[\operatorname{diag}\left(x'_{i,j}\right)\right] = x_{i,1}^{-1} \cdots x_{i,n}^{-1} \left[\operatorname{id}_{M_i}\right] .$$

So , taking into account the block diagonal structure of Φ and Φ, Lemma 5.5 states that $\Phi(\Phi^{-1}) \circ \Phi^{-1} \circ \Phi_{\mathfrak{m}} \circ \Phi^{-1}$ if and only if

$$\prod_{\substack{\mathfrak{m}=(\mathfrak{m}_1,\ldots,\mathfrak{m}_n)\\ \mathfrak{m}_i\in\{0,\ldots,d_i-1\}}} \left(\left[x_{i,j}\right] - \left[x'_{i,j'}\right] - \left[x'_{i,j''}\right] - \Phi_{\mathfrak{m}}\left(\left[x_{i,j} x'_{i,j'} x'_{i,j''}\right]\right) \in \Phi_{\mathfrak{m}}\right) \qquad (\mathrm{i})$$

for all $j \in \{1, \ldots, n\}$ and all $j', j'' \in \{1, \ldots, n_i\}$. But since $x'_{i,j}$, $j = 1, \ldots, n_i$, form an $\Phi_{\mathfrak{m}}$-basis of $\Phi^{-1}_{\mathfrak{m}}(\mu)$, the condition in (i) holds for given $j \in \{1, \ldots, n\}$ and all $j', j'' \in \{1, \ldots, n_i\}$ if and only if $\Phi_{\mathfrak{m}}(\Phi^{-1}_{\mathfrak{m}}(\mu)) \subseteq \Phi_{\mathfrak{m}}$. But this is equivalent to $\Phi_{\mathfrak{m}}(\mu) = \Phi_{\mathfrak{m}}$ and to $\Phi_{\mathfrak{m}}(\mu) = \Phi_{\mathfrak{m}}$. So this result holds. □

Corollary 5.8 The following assertions are equivalent:

(i) The ring $\Phi^{\mathrm{c}}_{\mathfrak{m}}$ is a Noetherian ring $\Phi_{\mathfrak{m}}$.

(ii) The Gorenstein ideal $\Phi_{\mathfrak{m}}$ is trivial.

(iii) $\operatorname{Soc}\Phi^{\mathrm{c}}_{\mathfrak{m}} = \Phi^{\mathrm{c}}_{\mathfrak{m}}$.

(iv) $\operatorname{Soc}(\Phi^{\mathrm{c}}_{\mathfrak{m}})^{\mathrm{c}} = \Phi_{\mathfrak{m}}$.

(v) The maximal ring $\Phi_{\mathfrak{m}}$ of $\Phi_{\mathfrak{m}}$ is an $\Phi_{\mathfrak{m}}$-Noetherian ring.

If (i)–(v) hold then $\Phi^{\mathrm{c}}_{\mathfrak{m}}$ is the canonical $\Phi_{\mathfrak{m}}$-Noetherian ring of $\Phi_{\mathfrak{m}}$.

Proof The equivalence of (i) and (ii) is the content of Proposition 5.5. The statements (ii) and (iii) are equivalent by Corollary 5.7 (iii) that states $[\Phi^{\mathrm{c}}_{\mathfrak{m}} : \Phi^{\mathrm{c}}_{\mathfrak{m}}] = \Phi_{\mathfrak{m}}$. Obviously, (iii) and (iv) are equivalent, since $\Phi_{\mathfrak{m}}$ is a Gorenstein Noetherian. Moreover, (i) and (v) imply (v). Finally, (v) implies that $\Phi^{\mathrm{c}}_{\mathfrak{m}}$ is an order in $\Phi_{\mathfrak{m}}$, and then Lemma 5.5 implies (ii).

If (i)–(v) hold then $\Phi_{\mathfrak{m}}$ is certainly the largest $\Phi_{\mathfrak{m}}$-Noetherian order in $\Phi_{\mathfrak{m}}$. Therefore the ring $\Phi^{\mathrm{c}}_{\mathfrak{m}}$ is the maximal $\Phi_{\mathfrak{m}}$-Noetherian order in $\Phi_{\mathfrak{m}}$. □

Proposition 5.10 Let Φ be stable. Then the maximal $\Phi_{\mathfrak{m}}$-order $\Phi_{\mathfrak{m}}$ of $\Phi_{\mathfrak{m}}$ is a Noetherian Gorenstein ring if $\Phi_{\mathfrak{m}} = \Phi_{\mathfrak{m}}$.

Proof Since the algebra Φ is an Φ-algebra automorphism of $\Phi_{\mathfrak{m}}$, the maximal $\Phi_{\mathfrak{m}}$-order of $\Phi_{\mathfrak{m}}$ is stable under Φ. As in the proof of Proposition 5.5 a map assumes that $\Phi_{\mathfrak{m}}$ is local. So let $\mathfrak{m}$, $\check{\mathfrak{m}}$, $\mathfrak{m}_{i,1}$, and $\check{\mathfrak{m}}_{i,1}$ be given as in Lemma 5.5 and that, for each $i \in \{1, \ldots, n\}$, the elements $\mathfrak{m}_{i,1}, \ldots, \mathfrak{m}_{i,n_i}$ form an $\Phi_{\mathfrak{m}}$-basis of $\Phi_{\check{\mathfrak{m}}_i}$. Then the elements $\check{\mathfrak{m}}_{i,1}$, $1 \le i \le n$, $1 \le j \le n_i$, form an $\Phi_{\mathfrak{m}}$-basis of $\Phi_{\mathfrak{m}}$. The matrix Φ in the notation preceding Lemma 5.5 is given by

$$\frac{1}{|\Phi|} \left[\prod_{\substack{\mathfrak{m}=(\mathfrak{m}_1,\ldots,\mathfrak{m}_n)\\ \mathfrak{m}_i\in\{0,\ldots,d_i-1\}}} \left(\left[\mathfrak{m}_{i,j}\right]^{\mathfrak{m}} \operatorname{id}\left(\mathfrak{m}^{-1}\right)\right) \right]_{i,j} .$$

[illegible] $\mathbb{A} = \mathbb{U}_1\mathbb{D}_1$, [illegible] $\mathbb{D}_1$ is the block diagonal matrix [illegible] block indexed $\mathbb{S}_\mathbb{g}$ $\mathbb{g} = 1, \ldots, \mathbb{s}$ and [illegible] $\mathbb{g}$-th block is given by

$$\mathbb{D}_{1,\mathbb{g}} = \left[\mathbb{a}(\mathbb{x}_{1,\mathbb{i}})\right]_{\substack{\mathbb{i}\in\mathbb{I}_1,\ldots,\mathbb{s}_\mathbb{g} \\ \mathbb{x}\in\mathbb{S}_\mathbb{g}\times\cdots\times\mathbb{S}_\mathbb{g}(\mathbb{s}_\mathbb{g})}},$$

and [illegible]

$$\mathbb{U}_1 = \frac{1}{|\mathbb{G}|}\left[\mathbb{r}^{\mathbb{x}}\ \mathbb{a}(\mathbb{x}^{-1})\right]_{\substack{\mathbb{x},\mathbb{x}\in \\ \mathbb{S}\times\mathbb{S}}}.$$

If, for each $\mathbb{i} \in [1, \ldots, \mathbb{s}]$, a basis $\mathbb{S}_\mathbb{i}$ $\mathbb{x}_{\mathbb{i},1}, \ldots, \mathbb{x}_{\mathbb{i},\mathbb{s}_\mathbb{i}}$ of $\hat{\mathbb{G}}_\mathbb{i}$ the dual basis of $\mathbb{x}_1, \ldots, \mathbb{x}_{1,\mathbb{s}_1}$ [illegible] is equal to $\mathbb{G}_{\hat{\mathbb{i}}_\mathbb{x}\mathbb{g}}$, then

$$\mathbb{U}_{1,\mathbb{g}} = \left[\mathbb{a}(\mathbb{x}_{\mathbb{i},\mathbb{i}})\right]_{\substack{\mathbb{x}\in\mathbb{S}_\mathbb{g}\times\cdots\times\mathbb{S}_\mathbb{g}(\mathbb{s}_\mathbb{g}) \\ \mathbb{i}\in\mathbb{I}_1,\ldots,\mathbb{s}_\mathbb{g}}}$$

is the inverse of $\mathbb{D}_{1,\mathbb{g}}$. Moreover, $\mathbb{D}_\mathbb{g}$ the orthogonality relations of irreducible characters, the inverse of $\mathbb{D}_1$ is given by

$$\mathbb{D}_1 = \left[\mathbb{r}^{\mathbb{x}}\ \mathbb{a}(\mathbb{x})\right]_{\substack{\mathbb{x}\in \\ \mathbb{S}\times\mathbb{S}}}.$$

Thus, the inverse of $\mathbb{U}$ is given by

$$\mathbb{U} = \left[\sum_{\mathbb{g}\in\mathbb{I}_1,\ldots,\mathbb{s}_\mathbb{g}(\mathbb{s}_\mathbb{g})} \mathbb{a}(\mathbb{x}_{\mathbb{i},\mathbb{i}})\,\mathbb{r}^{\mathbb{x}}\ \mathbb{a}(\mathbb{x})\right]_{\substack{\mathbb{x}\in \\ \mathbb{S}\times\mathbb{S}}}.$$

So [illegible], Lemma 3.5 states that $\mathbb{M}[\mathbb{G}_\mathbb{0}] \cong \mathbb{G}_\mathbb{0} \oplus \mathbb{G}_\mathbb{0} \oplus \mathbb{G}_\mathbb{0}$ if and only if

$$\frac{1}{|\mathbb{G}|}\sum_{\mathbb{x}\in\mathbb{S}}\sum_{\mathbb{x}',\mathbb{x}''} \mathbb{a}(\mathbb{x}_{1,\mathbb{i}})\,\mathbb{r}^{\mathbb{x}}\ \mathbb{a}(\mathbb{x})\,\mathbb{a}'(\mathbb{x}'_{1,\mathbb{i}})\,\mathbb{r}^{\mathbb{x}'}\ \mathbb{a}(\mathbb{x})\,\mathbb{a}''(\mathbb{x}''_{1,\mathbb{i}})\,\mathbb{r}^{\mathbb{x}''}\ \mathbb{a}(\mathbb{x}) =$$

$$= \frac{1}{|\mathbb{G}|}\sum_{\mathbb{x},\mathbb{x}',\mathbb{x}''} \mathbb{a}(\mathbb{x}_{1,\mathbb{i}})\,\mathbb{a}'(\mathbb{x}'_{1,\mathbb{i}})\,\mathbb{a}''(\mathbb{x}''_{1,\mathbb{i}})\sum_{\mathbb{x}\in\mathbb{S}}\mathbb{r}^{\mathbb{x}}\ \mathbb{a}(\mathbb{x})\,\mathbb{r}^{\mathbb{x}'}\ \mathbb{a}(\mathbb{x})\,\mathbb{r}^{\mathbb{x}''}\ \mathbb{a}(\mathbb{x}) = \mathbb{G}_\mathbb{0}$$

for all pairs $(\mathbb{G}, \mathbb{i})$, $(\mathbb{G}', \mathbb{i}')$, $(\mathbb{G}'', \mathbb{i}'')$. [illegible] the triple sum sums independently over all $\mathbb{x} \in \mathbb{G}_\mathbb{0}[\mathrm{stab}_\mathbb{G}(\mathbb{x})]$, $\mathbb{x}' \in \mathbb{G}_\mathbb{0}[\mathrm{stab}_\mathbb{G}(\mathbb{x}')]$, and $\mathbb{x}'' \in \mathbb{G}_\mathbb{0}[\mathrm{stab}_\mathbb{G}(\mathbb{x}'')]$. By the orthogonality relations of irreducible characters the last sum sum $\mathbb{g} \in \mathbb{G}$ returns to $|\mathbb{G}|$ if $\mathbb{r}^{\mathbb{x}}\ \mathbb{r}^{\mathbb{x}'}\ \mathbb{r}^{\mathbb{x}''}\ \mathbb{g} = 1$ and vanishes otherwise. If $\mathbb{G}_\mathbb{0} = \mathbb{G}_\mathbb{1}$ then $\mathbb{x}_{\mathbb{i},1} \in \mathbb{G}_\mathbb{1}$ for all pairs $(\mathbb{G}, \mathbb{i})$, and the above condition is certainly satisfied. Conversely, if the above condition is satisfied, then $\mathbb{x} \in \mathbb{G}$ $\mathbb{x}' \in [1, \ldots, \mathbb{s}]$ satisfying and element that $\mathrm{stab}_\mathbb{G}(\mathbb{x}) = \mathrm{stab}_\mathbb{G}(\bar{\mathbb{x}}')$. But $\mathbb{x}'' \in [1, \ldots, \mathbb{s}]$ and $\mathbb{x} \in \mathbb{G}_\mathbb{0}$ be such that $\bar{\mathbb{x}}'' = \mathbb{x}'$. Because let $\mathbb{i} \in [1, \ldots, \mathbb{s}]$ be such that $\mathbb{x} = 1$. Then $\hat{\mathbb{G}}_\mathbb{i} = \mathbb{G}$ and the above condition implies

$$\sum_{\mathbb{x}',\mathbb{x}''} \mathbb{a}'(\mathbb{x}'_{1,\mathbb{i}})\,\mathbb{a}''(\mathbb{x}''_{1,\mathbb{i}})\,\delta_{\mathbb{x}',\mathbb{x}''}\,\mathbb{r}_{\mathbb{x}'}\ \mathbb{r}_{\mathbb{x}''} =$$

$$= \sum_{\mathbb{x}'} \mathbb{a}'(\mathbb{x}'_{1,\mathbb{i}})\,\mathbb{a}'(\mathbb{x}'_{1,\mathbb{i}}) = \mathbb{G}_{\hat{\mathbb{i}}_\mathbb{x},\mathbb{g}}(\mathbb{x}'_{1,\mathbb{i}}\ \mathbb{x}'_{1,\mathbb{i}}),$$

since $\sigma'_{i,\sigma''} \sigma''_{i} = 1$ if and only if $\sigma'' = \sigma'\sigma$. The elements $\sigma[\sigma'',\sigma]_{i}, \ldots, \sigma[\sigma'',\sigma_{n''}]_{i}$ form an $\mathbb{S}_{n}$-basis of $\sigma[\overline{\mathbb{S}}^{\,i}_{n'',n}] = \overline{\mathbb{S}}^{\,i}_{n',n}$. Since the last term in the above equation lies in $\mathbb{S}_{n}$ for all i' and i'', $\mathbb{S}_{n',n}[\overline{\mathbb{S}}^{\,i}_{n',n}] \subseteq \mathbb{S}_{n}$. But this implies $\mathbb{S}_{n',n} = \mathbb{S}_{n}$. This holds for all $n' \in \{1, \ldots, n\}$. Thus $\mathbb{S}_{n,n} = \mathbb{S}_{n}$. $\square$

3.3 Some $\mathbb{W}$-modules and resolutions

Let $\mathbb{S} := \mathrm{Mag}(\mathbb{M}, \overline{\mathbb{S}})$ be the $\overline{\mathbb{S}}$-algebra $\mathbb{M}$ gains by multiplication. Then, $\mathbb{S}$ acts on $\mathbb{S}$ via $\mathbb{W}$-algebra automorphisms $\mathbb{S}_{\sigma}$

$$[\sigma][\alpha] := \sigma[\alpha][\sigma^{-1}\alpha],$$

for $\sigma \in \mathbb{W}$, $\alpha \in \mathbb{S}$, $\alpha \in \mathbb{M}$. For an intermediate field $\mathbb{K} \subseteq \mathbb{L} \subseteq \overline{\mathbb{K}}$ let

$$\mathbb{S}_{\mathbb{L}} := \mathbb{S}^{\mathbb{L}^{\times}} = \mathrm{Mag}(\mathbb{M}, \overline{\mathbb{S}})^{\mathbb{L}^{\times}}$$

be all $\mathbb{W}$-algebra of $\mathbb{S}_{\mathbb{L}}$-fixed points of $\mathbb{S}$. Moreover, let $\mathbb{S}_{\mathbb{L}}$ be the maximal $\mathbb{S}_{\mathbb{L}}$-subset of $\mathbb{S}_{\mathbb{L}}$. Thus,

$$\mathbb{S}_{\mathbb{L}} = \mathrm{Mag}(\mathbb{M}, \mathbb{L} - \{0\})^{\mathbb{L}^{\times}}.$$

Note that $\mathbb{S}$ is a right $\mathbb{W}$-module via the $\mathbb{S}$-action on $\mathbb{S}$:

$$[\alpha \cdot \{\sum_{\sigma \in \mathbb{W}} \kappa_{\sigma} \sigma\}][\alpha] := \sum_{\sigma \in \mathbb{W}} \kappa_{\sigma} \alpha[\sigma] \qquad (3.3)$$

for $\kappa \in \overline{\mathbb{K}}$, $\alpha \in \mathbb{S}$, $\sigma \in \mathbb{W}$. This action of $\mathbb{W}$ on $\mathbb{S}$ satisfies

$$\sigma[\alpha \cdot \sigma] = [\sigma][\alpha] \cdot [\sigma]$$

for all $\sigma \in \mathbb{W}$, $\alpha \in \mathbb{S}$, $\sigma \in \mathbb{W}$. Thus, the $\mathbb{W}$-module structure on $\mathbb{S}$ restricts to an $\mathbb{S}_{\mathbb{L}}$-module structure on $\mathbb{S}_{\mathbb{L}}$ for any intermediate field $\mathbb{K} \subseteq \mathbb{L} \subseteq \overline{\mathbb{K}}$. Moreover, as apparent from (3.3), $\mathbb{S}_{\mathbb{L}}$ is an $\mathbb{S}_{\mathbb{L}}$-module $\mathbb{S}_{\mathbb{L}}$ restriction.

Similar to Lemma 3.2 for the algebra $\mathbb{S}$ we then the following lemma for $\mathbb{S}$.

Lemma 3.11 Let $\mathbb{K} \subseteq \mathbb{L} \subseteq \overline{\mathbb{K}}$ be an intermediate field, and let $\alpha_{1}, \ldots, \alpha_{n} \in \mathbb{S}$ be a set of representatives for the $\mathbb{S}_{\mathbb{L}}$-orbits of $\mathbb{S}$. For each $\sigma \in \{1, \ldots, n\}$ let $\overline{\mathbb{S}}_{\sigma}$ denote the field field $\mathrm{stab}_{\mathbb{W}}[\alpha_{\sigma}] \subseteq \mathbb{S}_{\mathbb{L}}$, and let $\sigma_{\sigma,1}, \ldots, \sigma_{\sigma,n_{\sigma}}$ be an $\mathbb{W}$-basis of $\overline{\mathbb{S}}_{\sigma}$. For $i \in \{1 \ldots n\}$ and $i \in \{1 \ldots n_{\sigma}\}$ let $\sigma_{\sigma,i}$ be defined $\mathbb{S}_{\mathbb{L}}$

$$\sigma_{\sigma,i}[\alpha] := \begin{cases} \sigma[\sigma_{\sigma,i}], & \text{if } \alpha = \sigma'\alpha_{\sigma} \text{ for some } \sigma' \in \mathbb{S}_{\mathbb{L}}, \\ 0, & \text{otherwise for,} \end{cases}$$

for $\alpha \in \mathbb{S}$. Assume that $\mathbb{K} \subseteq \mathbb{M} \subseteq \mathbb{L} \subseteq \overline{\mathbb{K}}$ are finite intermediate fields. Then the following two equations hold:

(i) The map

$$\mu : \mathbb{S}_n \to \bigoplus_{m=1}^{n} \mathbb{S}_m , \quad f \mapsto (f(x_m)) ,$$

is an $\mathbb{S}$-algebra homomorphism. In particular, a module is an homomorphism

$$\mu_M : S_n \to \bigoplus_{m=1}^{n} S_m .$$

(ii) The element $x_{m,1} \in \mathbb{S}_m$ is $\mathbb{S}_1$-free over $\mathbb{S}_m$. If $m \geq 2$ and $i \in \{1, \ldots, m\}$, the elements $x_{m,i}, \ldots, x_{m,i-1}$ form an $\mathbb{S}_m$-basis of $\mathbb{S}_{m-1}$, then the element $x_{m,i} \in \mathbb{S}_m$ is $\mathbb{S}_1$-free over $\mathbb{S}_m$.

(iii) $\operatorname{Hom}_{\mathbb{S}}(\mathbb{S}_n) = |\mathbb{S}|$.

(iv) The element $x_{m,1} \in \mathbb{S}_m$ is $\mathbb{S}$-free over $\mathbb{S}_m$.

(v) The map

$$\mu_S^{\mathbb{S}} : \mathbb{S} \otimes_{\mathbb{S}} \mathbb{S}_n \to \mathbb{S}_m , \quad f \otimes_{\mathbb{S}} s \mapsto fs ,$$

is an homomorphism of $\mathbb{S}$-algebras and that the diagrams

$$\begin{tikzcd} \mathbb{S} \otimes_{\mathbb{S}} \mathbb{S}_n \arrow[r, "\mu_S^{\mathbb{S}}"] \arrow[d, "\mu_{\mathbb{S}} \otimes \mu_n"'] & \mathbb{S}_m \arrow[d, "\mu"] \\ \mu(\mathbb{S}) \otimes_{\mu(\mathbb{S})} \mathbb{S}_{\mu(n)} \arrow[r, "\mu_{\mu(\mathbb{S})}^{\mu(\mathbb{S})}"] & \mathbb{S}_{\mu(m)} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{S} \otimes_{\mathbb{S}} \mathbb{S} \otimes_{\mathbb{S}} \mathbb{S}_n \arrow[r, "\mu_{\mathbb{S}}^{\mathbb{S}} \otimes \mathbb{S}_m"] \arrow[d, "\mu_{\mathbb{S}} \otimes \mu_n"'] & \mathbb{S} \otimes_{\mathbb{S}} \mathbb{S}_m \arrow[d, "\mu"] \\ \mathbb{S} \otimes_{\mathbb{S}} \mathbb{S}_n \arrow[r, "\mu_{\mathbb{S}}^{\mathbb{S}}"] & \mathbb{S}_m \end{tikzcd}$$

commute for all $n \in \mathbb{N}$.

Proof. All results are proved in a similar way as the analogous results of Lemma 3.1. □

Let $\mathbb{S}$ be a fixed field. The $\mathbb{S}$-algebra $\mathbb{S}_n$ is an $\mathbb{S}_n$-Galois extension in the sense of [22]. But we already recall the relevant notions in a general setting.

Let M be a commutative ring, let G be an M-Hopf algebra. Hopf is finitely generated and projective as M-module. Furthermore, let B be a commutative M-algebra, finitely generated and projective as M-module. Hopf is also a right G-module. Then B is called an G-Galois extension of M if and only if the following conditions are satisfied:

(Gal) (i) $(xy) \cdot b = \sum_{(x)} (x \cdot b_{(1)})(y \cdot b_{(2)})$,

(ii) $1_B \cdot b = \varepsilon(b) 1_B$,

for all $b \in B$, $x, y \in G$. Here $\Delta(b) = \sum_{(b)} b_{(1)} \otimes b_{(2)}$ is the Δ-coproduct notation and the module structure of B over G is denoted by $\triangleleft$.

[illegible] [illegible] map

$$[illegible] \quad [illegible] \quad [illegible] \quad [

of S_n in M_n. Obviously, $M_n^{\prime} \subseteq M_n^{\prime\prime} \subseteq M_n$. [illegible] III gives in the following [illegible] that in many of the examples described in 1.1 the module S_n is [illegible] $M_n^{\prime\prime}$. In the general [illegible] a module map of the coalgebra

$$(\Omega, \ \ )_{n,1} = \sum_{n \in \mathbb{N}} \Omega(\ \ n) \ \ (n^{-1}) \ \ n \in \bar{\mathbb{N}} \qquad (1.5)$$

attached to $\Omega \in \mathbb{K}$ and [illegible] $\in \bar{\mathbb{N}}$ for fixed $n_1 \in \mathbb{N}$.

Use an intermediate field $\mathbb{K} \subseteq \mathbb{K} \subseteq \bar{\mathbb{K}}$, let $\mathbb{K}' \subseteq \bar{\mathbb{K}}$ be [illegible] a finite extension of $\mathbb{K}$ such that M_n acts totally on $\bar{\mathbb{K}}$ and $\mathbb{K}$. [illegible] call such a field a splitting field for M_n and M_n. Note that, since

$$= \{(\Omega, \ \ )_{n,1}\} = \{(\Omega, \ \ )_{n,1}\} = \{\ \ \},$$

for all $n \in \mathbb{N}$, $\Omega \in \mathbb{K}$, [illegible] $\in \bar{\mathbb{K}}$, and $n_1 \in \mathbb{N}$, [illegible] then $\{(\Omega, \ \ )_{n,1}\} \in \mathbb{K}'$ for all $\Omega \in M_n$, [illegible] $\in \bar{\mathbb{K}}$, and $n_1 \in \mathbb{N}$. [illegible] denote by $S_{n,1}$ the Hecke-algebra ideal of S_n over $\mathbb{K}_n$. Then, in the notation of Proposition 1.11 [illegible] $S_{n,1} = \mathbb{K}^{\prime}_{n,1} S_{n,1}$.

Proposition 1.12 Assume that $\mathbb{K}$ is closed and $n_1 \in \mathbb{N}$. Let $\mathbb{K} \subseteq \mathbb{K} \subseteq \bar{\mathbb{K}}$ be an intermediate field, let $\mathbb{K} \subseteq \mathbb{K}'$ be a splitting field for M_n and M_n, and let $\Omega \in M_n$. Then $\Omega \cdot M_n = M_n$ if and only if $(\Omega, \ \ )_{n,1} \neq 0$ for all [illegible] $\in \bar{\mathbb{K}}$. Moreover, if $\Omega \cdot M_n = M_n$ then

$$[S_n : \Omega \cdot M_n^{\prime}]^{\prime}_{n,1} M_n = S_{n,1} S_{n,1}^{-1} \sum_{n \in \bar{\mathbb{K}}} (\Omega, \ \ )^{\prime}_{n,1} M_n .$$

Proof (a) Let [illegible], $\bar{\mathbb{K}}$, $S_{n,1}$, and the $\mathbb{K}$-basis

$$\bar{a}_{n,1} = \sum_{\ \ \in \mathbb{K}[\ \ ]\ \ (n,1)} \ \ (\ \ )^{-1} \ \ \qquad (\ \ \in \mathbb{K}, \ \ \in \mathbb{K}[\ \ n,1]) \qquad (1.6)$$

of M_n as given in Lemma 1.1. Then the elements

$$\{(\Omega \cdot \bar{a}_{n,1})(\ \ )\}_{\ \ \in \mathbb{K}} = \sum_{n \in \mathbb{K}} \ \ \mathbb{K}' \qquad (1.7)$$

generate $\{\ \ \in \ \ \mid (\mathbb{K}' S_n \Omega \cdot M_n)\} \subseteq \mathbb{K}_{n,1} \mathbb{K}'$ over $\mathbb{K}'$. [illegible] assume that generates M_n the polynomial idempotents e_n, $n \in \mathbb{N}$, of $\mathbb{K}_{n,1} \mathbb{K}'$ and obtain a transition matrix

$$\ \ = \{(\Omega \cdot \bar{a}_{n,1})(\ \ )\}_{(n,1)} = \{(\Omega \cdot \bar{a}_{n,1})(\ \ )\}_{(n,1)} ,$$

so that $\Omega \cdot M_n = M_n$ if and only if $\det(\ \ ) \neq 0$. [illegible] $\det(\ \ )$. Use the

analog of [illegible] in [illegible]

$$
\begin{aligned}
[\Phi \cdot \hat{\mathbb{S}}_{n,\ell}](\mathbb{X}_n) &= \frac{1}{|\mathbb{X}|} \sum_{\mathbb{Y} \subseteq \mathbb{X}_n : |\mathbb{Y}| = \ell} \sum_{\mathbb{Z} \subseteq \mathbb{Y}} (-1)^{|\mathbb{Y} \setminus \mathbb{Z}|} \Phi(\mathbb{Z}) \\
&= \frac{1}{|\mathbb{X}|} \sum_{\mathbb{Y} \subseteq \mathbb{X}_n : |\mathbb{Y}| = \ell} \sum_{\mathbb{Z} \subseteq \mathbb{Y}} (-1)^{|\mathbb{Y} \setminus \mathbb{Z}|} \Phi(\mathbb{Z}) \\
&= \frac{1}{|\mathbb{X}|} \sum_{\mathbb{Y} \subseteq \mathbb{X}_n : |\mathbb{Y}| = \ell} (-1)^{\ell} \Phi(\mathbb{Y}) = (-1)^{\ell} \Phi(\mathbb{X}_n).
\end{aligned}
$$

In [illegible] this [illegible] we [illegible] $\mathbb{X}_n = \mathbb{X}_1 \mathbb{X}_2$, [illegible] $\mathbb{X}_1$ is a fixed diagonal matrix, the block $\mathbb{X}_{1,k}$ indexed by $\mathbb{S}_k$ $(k = 1, \ldots, n)$, [illegible]

$$
\mathbb{X}_{1,k} = [\mathbb{X}(\mathbb{S}_{k,i})]_{i,j \in \{1,\ldots,n_k\}^2 \setminus \mathbb{S}_k(k,k)},
$$

and [illegible]

$$
\mathbb{X}_2 = \frac{1}{|\mathbb{X}|} \left[(\mathbb{Q}_1 \otimes \mathbb{I}_{n_1}) \cdots (\mathbb{Q}_k \otimes \mathbb{I}_{n_k}) \right]_{k,\ldots,n \setminus \mathbb{S}_k(k,k)} = \frac{1}{|\mathbb{X}|} \left[(\mathbb{Q}_1 \otimes \mathbb{I}_{n_1}) \cdots (\mathbb{Q}_k \otimes \mathbb{I}_{n_k}) \right]_{\mathbb{S}^c \times \mathbb{S}^c}.
$$

[illegible], [illegible] $(\mathbb{X}_{1,k})^{-1} \mathbb{X}_{1,k} = \mathbb{I}_{n_k \times n_k}$, [illegible] $(\mathbb{X}_{1,k})^{-1} \mathbb{X}_{1,k} = \mathbb{I}_{n_k \times n_k}$. [illegible],

$$
\det(\mathbb{X}_1)^{-1} = \left(\prod_{k \in \hat{\mathbb{S}}} (\mathbb{Q}_1 \otimes \mathbb{I}_{n_1}) \cdots (\mathbb{Q}_k \otimes \mathbb{I}_{n_k}) \right)^{-1} \det(\mathbb{X}_1) \det[\mathbb{X}_2]^{-1}_{\mathbb{S}^c \times \mathbb{S}^c} = \left(\prod_{k \in \hat{\mathbb{S}}} (\mathbb{Q}_1 \otimes \mathbb{I}_{n_1}) \cdots (\mathbb{Q}_k \otimes \mathbb{I}_{n_k}) \right)^{-1} |\mathbb{X}|^{-|\mathbb{S}|}
$$

$\mathbb{X}_2$ [illegible]. Thus,

$$
\det(\mathbb{X}_1)^{-1} \mathbb{X}_{\cdot} = \mathbb{I}_{n \times n} |\mathbb{X}|^{-|\mathbb{S}|} \prod_{k \in \hat{\mathbb{S}}} (\mathbb{Q}_1 \otimes \mathbb{I}_{n_1}) \cdots (\mathbb{Q}_k \otimes \mathbb{I}_{n_k}) \mathbb{X}_{\cdot}. \tag{illegible}
$$

[illegible] $\mathbb{Q}_1 \cdots \mathbb{Q}_k = \mathbb{I}_n$ [illegible] $(\mathbb{Q}_1 \otimes \mathbb{I}_{n_1}) \cdots (\mathbb{Q}_k \otimes \mathbb{I}_{n_k}) \neq \mathbb{I}_{n \times n}$ all $k \in \hat{\mathbb{S}}$.

(iii) [illegible] Φ [illegible] $[\mathbb{X}_n : \mathbb{Q} \cdot \mathbb{X}_n^{\mathbb{S}}]_{\mathbb{S}^c}$. [illegible] all [illegible] $\mathbb{X}_n$ is local. [illegible] $\mathbb{X}_n$-[illegible] $\mathbb{X}_{\cdot}$ $\mathbb{X}_n$ and $\mathbb{X}_{\cdot}$ $\mathbb{X}_n$ $\mathbb{Q} \cdot \mathbb{X}_n^{\mathbb{S}}$ $\mathbb{X}_{\cdot}$ [illegible] $\mathbb{X}_{\cdot} = \mathbb{X}_{\mathbb{S}^c}^{\mathbb{S}^c}$ [illegible] $\mathbb{X}_{\cdot}$.

[illegible] the calculation in (ii) [illegible] that, if $\mathbb{X}_{n,1}, \ldots, \mathbb{X}_{n,n}$ is an $\mathbb{X}_n$-[illegible] of $\hat{\mathbb{S}}_k$, for each $k = 1, \ldots, n$, then the elements (illegible) form an $\mathbb{X}_n$-[illegible] of $\mathbb{X}_n$, and the elements in (illegible) form an $\mathbb{X}_{\cdot}$-[illegible] of $[\mathbb{X}_{\cdot} = \mathbb{X}_{\mathbb{S}^c}^{\mathbb{S}^c}](\mathbb{X}_{\cdot} \mathbb{X}_n \mathbb{Q} \cdot \mathbb{X}_n)$. Thus, the calculation in (ii) [illegible] that

$$
\left[\mathbb{X}_{\cdot} (\mathbb{X}_{\cdot})^{-1} \cdot [\mathbb{X}_{\cdot} = \mathbb{X}_{\mathbb{S}^c}^{\mathbb{S}^c}](\mathbb{X}_{\cdot} \mathbb{X}_n \mathbb{Q} \cdot \mathbb{X}_n) \right]^{\mathbb{S}}_{\mathbb{S}^c} = \mathbb{I}_{n \times n} |\mathbb{X}|^{-|\mathbb{S}|} \prod_{k \in \hat{\mathbb{S}}} (\mathbb{Q}_1 \otimes \mathbb{I}_{n_1}) \cdots (\mathbb{Q}_k \otimes \mathbb{I}_{n_k}) \mathbb{X}_{\cdot}. \tag{illegible}
$$

Moreover, [illegible] already from [illegible] Proposition [illegible] that

$$[\mathbb{Q} \cdot \mathbb{M}_{\mathbb{Q}} : \mathbb{Q} \cdot \mathbb{M}_{\mathbb{Q}}^{\mathbb{Z}}]_{\mathbb{Z}_{\mathbb{Q}}} = [\mathbb{M}_{\mathbb{Q}} : \mathbb{M}_{\mathbb{Q}}^{\mathbb{Z}}]_{\mathbb{Z}_{\mathbb{Q}}} = \mathbb{Z}_{\mathbb{Q}} \mathbb{Z}_{\mathbb{Q}}^{-1} |\mathbb{M}|^{[\mathbb{Q}]}. \tag{[illegible]}$$

[illegible], it suffices to show that

$$[\mathbb{Z}[\mathbb{Z}_{\mathbb{Q}}] : [\mathbb{Z}_{\mathbb{Q}} \cdot \mathbb{Z}_{\mathbb{Q}}^{\mathbb{Z}}]][\mathbb{Z}_{\mathbb{Q}} \cdot \mathbb{Z}_{\mathbb{Q}} \mathbb{Z}_{\mathbb{Q}}]]_{\mathbb{Z}_{\mathbb{Q}}} = \mathbb{Z}_{\mathbb{Q}} \mathbb{Z}_{\mathbb{Q}}, \tag{[illegible]}$$

since then, [illegible] the product of [illegible] and [illegible] by [illegible] yields the result.

Let $g_{\mathbb{Z}}$, $\bar{\mathbb{M}}_{\mathbb{Z}}$, $\mathbb{Z}_{\mathbb{Z}}$, [illegible], and $\mathbb{Z}_{\mathbb{Z}}$ be given as in Lemma [illegible] and then, for each $\mathbb{Z} \in [1, \ldots, \mathbb{Z}]$, the elements $\mathbb{Z}_{\mathbb{Z},1}, \ldots, \mathbb{Z}_{\mathbb{Z},\mathbb{Z}}$ form an $\mathbb{Z}_{\mathbb{Q}}$-basis of $\bar{\mathbb{M}}_{\mathbb{Z}}$. Then, the elements $\mathbb{Z}_{\mathbb{Z}}$, [illegible] $\mathbb{Z} \in [1, \ldots, \mathbb{Z}]$, form an $\mathbb{Z}_{\mathbb{Q}}$-basis of $\mathbb{Z}_{\mathbb{Q}}$. Thus, the elements $[\mathbb{Z}_{\mathbb{Q}} \cdot \mathbb{Z}_{\mathbb{Q}}^{\mathbb{Z}}][\mathbb{Z}_{\mathbb{Z}}]$ form an $\mathbb{Z}_{\mathbb{Q}}$-basis of $[\mathbb{Z}_{\mathbb{Q}} \cdot \mathbb{Z}_{\mathbb{Q}}^{\mathbb{Z}}][\mathbb{Z}_{\mathbb{Q}} \cdot \mathbb{Z}_{\mathbb{Q}} \mathbb{Z}_{\mathbb{Q}}]$. [illegible] assigns the $\mathbb{Z}_{\mathbb{Q}}$-basis $\mathbb{Z}_{\mathbb{Z}}$ the commutative $\mathbb{Z}_{\mathbb{Q}}$-basis of galoisian Iwasawa units $\mathbb{Z}_{\mathbb{Z}}$, $\mathbb{Z} \in \mathbb{Z}$, of $\mathbb{Z}_{\mathbb{Q}} \mathbb{Z}_{\mathbb{Q}}$ and obtain a torsionless module $\mathbb{M}$. This is a direct diagonal module [illegible] direct $\mathbb{M}_1, \ldots, \mathbb{M}_{\mathbb{Z}}$, [illegible]

$$\mathbb{M}_{\mathbb{Z}} = [\mathbb{Z}[\mathbb{Z}_{\mathbb{Z}}, \mathbb{Z}]]_{\mathbb{Z}_{\mathbb{Z}} \cdots \mathbb{Z}_{\mathbb{Z}}[\mathbb{Z}_{\mathbb{Z}}, \ldots, \mathbb{Z}_{\mathbb{Z}}]}$$

for $\mathbb{Z} \in [1, \ldots, \mathbb{Z}]$. Since $\det[\mathbb{M}_{\mathbb{Z}}]^{\mathbb{Z}} \mathbb{M}_{\mathbb{Z}} = \mathbb{Z}_{\bar{\mathbb{Z}}_{\mathbb{Z}}[\mathbb{Z}]}$, [illegible] [illegible], and the proof is complete. $\square$

As an immediate consequence of Proposition [illegible] [illegible] obtain:

Corollary [illegible] In the situation of Proposition [illegible] [illegible] $\mathbb{Z} \in \mathbb{Z}_{\mathbb{Q}}$ and then $[\mathbb{Z}, \mathbb{Z}]_{\mathbb{Z}} \neq 1$ for all $\mathbb{Z} \in \bar{\mathbb{Z}}$ we have

$$[\mathbb{Z}_{\mathbb{Q}} : \mathbb{Z} \cdot \bar{\mathbb{M}}_{\mathbb{Q}}]_{\mathbb{Z}_{\mathbb{Q}}} \mathbb{M}_{\mathbb{Q}} = \mathbb{Z}_{\mathbb{Q}} \mathbb{Z}_{\mathbb{Q}}^{-1} [\bar{\mathbb{M}}_{\mathbb{Q}} : \mathbb{M}_{\mathbb{Q}}^{\mathbb{Z}}]_{\mathbb{Z}_{\mathbb{Q}}}^{-1} \prod_{\mathbb{Z} \in \bar{\mathbb{Z}}} [\mathbb{Z}, \mathbb{Z}]_{\mathbb{Z}}^{\mathbb{Z}} \mathbb{M}_{\mathbb{Q}}.$$

For each $\mathbb{Z}_{\mathbb{Q}}$-order $\bar{\mathbb{M}}_{\mathbb{Q}}$ of $\mathbb{M}_{\mathbb{Q}}$ contained in $\mathbb{M}_{\mathbb{Q}}^{\mathbb{Z}}$. In particular, if $\mathbb{Z} \in \mathbb{Z}_{\mathbb{Q}}$ is real then

$$[\bar{\mathbb{M}}_{\mathbb{Q}} : \mathbb{M}_{\mathbb{Q}}^{\mathbb{Z}}]_{\mathbb{Z}_{\mathbb{Q}}} \mathbb{Z}_{\mathbb{Q}} \mathbb{M}_{\mathbb{Q}} = \mathbb{Z}_{\mathbb{Q}} \prod_{\mathbb{Z} \in \bar{\mathbb{Z}}} [\mathbb{Z}, \mathbb{Z}]_{\mathbb{Z}}^{\mathbb{Z}} \mathbb{M}_{\mathbb{Q}},$$

since $\mathbb{Z}_{\mathbb{Q}} = \mathbb{Z} \cdot \bar{\mathbb{M}}_{\mathbb{Q}}$ is a free $\bar{\mathbb{M}}_{\mathbb{Q}}$-module of rank one [illegible] Scholz [illegible], and moreover, $\bar{\mathbb{M}}_{\mathbb{Q}} = \mathbb{M}_{\mathbb{Q}}^{\mathbb{Z}}$.

$\square$

We conclude this chapter [illegible] considering the case $\mathbb{Z} = \mathbb{Q}$. Then $\mathbb{Z}_{\mathbb{Q}} = \mathbb{Z}_{\mathbb{Q}}$ has only intermediate field $\mathbb{Z} \subseteq \mathbb{Z} \subseteq \bar{\mathbb{Z}}$. [illegible] define a map $\mathbb{Z}$ on $\mathbb{Z}_{\mathbb{Q}}$ by

$$\mathbb{Z}[\mathbb{Z}] = \begin{cases} \mathbb{Z}, & \text{if } \mathbb{Z} = \mathbb{Z}, \\ \mathbb{Z}, & \text{if } \mathbb{Z} \neq \mathbb{Z} \end{cases} \tag{[illegible]}$$

Corollary [illegible] Let $\mathbb{Z} = \mathbb{Q}$. Then $\mathbb{M}_{\mathbb{Q}}^{\mathbb{Z}} = \mathbb{M}_{\mathbb{Q}}^{\mathbb{Z}}$ and $\mathbb{Z}_{\mathbb{Q}}$ is a free $\mathbb{M}_{\mathbb{Q}}^{\mathbb{Z}}$-module of rank one [illegible] Scholz [illegible]. Here $\mathbb{Z}$ is defined in [illegible].

Proof [illegible] $\mathbb{Z} = \mathbb{Q}$. Then $[\mathbb{Z}, \mathbb{Z}]_{\mathbb{Z}} = 1$ for all $\mathbb{Z} \in \bar{\mathbb{Z}}$. Thus the assertion is an immediate consequence of the second part of Corollary [illegible]. $\square$

§ The equivariant case

This section mainly serves as a motivation and a guide-line for the application in Dwork-Katz formal groups and algebraic curves.

5.1 Dirac rank

Let $p, m \in \mathbb{N}$, $m \geq 1$, and let π be a rational prime number. For a primitive π^m-th root of unity ζ_{π^m}, let $\mathbb{K} = \mathbb{Q}_{\pi^m}$ be the group of π^m-th roots of unity and let $\bar{\mathbb{K}}$ be the set of π^m-th roots of $\mathbb{Q}_{\pi^m}$,

$$\bar{\mathbb{K}} = \{\alpha \in \bar{\mathbb{Q}} \mid \alpha^{\pi^m} = \mathbb{Q}_{\pi^m}\}.$$

A complex $\mathbb{C}[\mathbb{Q}_{\pi^m}]$ is one Dwork field and set

$$\mathbb{C} = \mathbb{C}[\mathbb{Q}_{\pi^m}], \quad \mathbb{C} = \mathbb{C}, \quad \mathbb{C} = \mathbb{C}[\mathbb{Q}_{\pi^{m+1}}],$$

Here $\mathbb{Q}_{\pi^{m+1}}$ is a primitive π^{m+1}-th root of unity such that $\zeta_{\pi^{m+1}}^{\pi} = \mathbb{Q}_{\pi^m}$. Then $\mathbb{C}$ acts simply transitive on $\mathbb{K}$,

$$\mathbb{K} = \{\alpha \cdot \mathbb{Q}_{\pi^{m+1}} \mid \alpha \in \mathbb{C}\},$$

and $\mathbb{C}$ acts on $\mathbb{C}$ and $\mathbb{K}$ such that $\sigma(\alpha \cdot \beta) = \sigma\alpha\sigma\beta$ for all $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$ and $\beta \in \mathbb{K}$. Calling Lang Section 1 we set

$$\mathbb{K} = (\bar{\mathbb{K}}\mathbb{K})^{\mathbb{C}}, \quad \mathbb{C} = \mathrm{Hom}_{\mathbb{C}}(\mathbb{K}, \bar{\mathbb{K}})^{\mathbb{C}}, \quad \mathbb{C} = \mathrm{Hom}_{\mathbb{C}}(\mathbb{K}, \bar{\mathbb{K}})^{\mathbb{C}}.$$

As an immediate consequence of Theorem 3.11 we obtain an isomorphism α,

$$\begin{aligned} \alpha : \mathbb{C} &\longrightarrow \mathbb{C} \otimes \mathbb{C}[\mathbb{Q}_{\pi^m}] \oplus \cdots \oplus \mathbb{C}[\mathbb{Q}_{\pi^{m-1}}], \\ \mathbb{C} &\longmapsto (\mathbb{C}(\mathbb{1}), \mathbb{C}(\mathbb{Q}_{\pi^m}), \ldots, \mathbb{C}(\mathbb{Q}_{\pi^{m-1}})), \end{aligned} \qquad \mathbb{C} \in \mathbb{C}, \qquad (5.1)$$

and

$$\begin{aligned} \alpha : \mathbb{C} &\longrightarrow \mathbb{C} = \mathbb{C}[\mathbb{Q}_{\pi^{m+1}}], \\ \mathbb{C} &\longmapsto \mathbb{C}(\mathbb{Q}_{\pi^{m+1}}), \end{aligned} \qquad \mathbb{C} \in \mathbb{C}. \qquad (5.2)$$

A first Dwork a class ball of the same intersecting case $\mathbb{C} \in \mathbb{C}$. For $\alpha \in \mathbb{C}$ and $\mathbb{C}$ let the elements of an $\mathbb{C}$-module. In the signal α III of $\mathbb{C}$ the $\mathbb{C} = \{\mathbb{C}_{\pi} = \mathbb{Q}_{\pi^m} \mid \mathbb{C} \in \mathbb{C}[\pi^m \mathbb{C}]\}$. Let $\alpha = \mathbb{C}_{\pi^m}^{\pi^m - 1} \alpha_{\pi^m}$ $\in \mathbb{C}$ and $\mathbb{C} \in \mathbb{C}$. Then

$$\begin{aligned} \alpha\left(\mathbb{C} \cdot \sum_{\pi=0}^{\pi^m - 1} \alpha_{\pi} \mathbb{Q}_{\pi}\right) &= \sum_{\pi=0}^{\pi^m - 1} \alpha_{\pi} \mathbb{C}\left(\mathbb{Q}_{\pi}^{\mathbb{C}} \mathbb{Q}_{\pi}^{\mathbb{C}}\right) \\ &= \sum_{\pi=0}^{\pi^m - 1} \alpha_{\pi} \mathbb{C}\left(\mathbb{Q}_{\pi^{m+1}}\right)^{\pi^{\mathbb{C}} \mathbb{C}_{\pi}^{\mathbb{C}}}, \end{aligned}$$

Then $\mathrm{Ext}^n(M, N)$ becomes the Betti group. Thus the R-module structure of R is equivalently given by

$$\mathbf{x} \cdot \sum_{i=1}^{n-1} x_i y_i = \sum_{i=1}^{n-1} x_i x^{[i]} y_i^{[i]} \tag{[illegible]}$$

for $\mathbf{x} \in R$, $\sum_{i=1}^{n-1} x_i y_i \in M$.

Note that the map

$$\begin{aligned} R[n] &\longrightarrow \mathrm{Hom}(R, R), \\ 1 \in R[n] &\longmapsto \mathrm{id}_R \end{aligned}$$

is a bijection. In addition, if $\mathbf{x} \in R$, then it is also a homomorphism of groups, since then $\mathrm{id}_R(\mathbf{x}_1 + \mathbf{x}_2) = \mathrm{id}_R(\mathbf{x}_1) + \mathrm{id}_R(\mathbf{x}_2)$ for $\mathbf{x}_1, \mathbf{x}_2 \in R$. However, $R = R[n]$ in this case, and identifying R with $\mathrm{Hom}(R, R)$ also $\mathrm{id}_R \in \mathrm{id}_R[n]$ the module structure defined in [illegible] is nothing else but the usual $R[n]$-module structure. In case, if $\mathbf{x} \in R$, then the module structure here not coincide with the classical Hodge module structure. But there is generally good of the reconstruction map natural.

For $\mathbf{x} \in R$ $\mathbf{x}$ shall be the image identify R with R via $\mathbf{x}$; in particular $\mathbf{x}$ also $\mathbf{x}_n = \mathbf{x}^{-1}[\mathbf{x}]$ for $\mathbf{x} \in R$. For $\mathbf{x} = 1$ $\mathbf{x}$ define $\mathbf{x}$ map $1 \in R$ by

$$1[\mathbf{x}] = \begin{cases} 1, & \mathbf{x} = 1, \\ 0, & \mathbf{x} \neq 1. \end{cases} \tag{[illegible]}$$

Recall also that $R^n = [R - R]^n$.

Now also let be the ring the integral closure $\bar{R}$ of R in R as a module over the rational subring $R^{\mathrm{inv}} = \{\mathbf{x} \in R \mid \bar{R} \cdot \mathbf{x} \subseteq \bar{R}\}$. This main result of this section can be stated as:

Theorem 5.1 [illegible] Let $\mathbf{x} = 1$. Then $R^{\mathrm{inv}} = R^n$ and $\bar{R}$ be free of rank one over R^{inv}. The equivalence generates be given by map 1 defined by [illegible].

[illegible] Let $\mathbf{x} \in R$ 1. Then R^{inv} be equal to the subring constant rather R in R and $\bar{R}$ be free of rank one over R^{inv}. The equivalence generates be given by the map $\mathbf{x}_1$, Here $1 = \frac{1 - \mathbf{x}_1 \mathbf{x}}{1 - \mathbf{x}_1 \mathbf{x} \mathbf{x}}$.

Since this section should come as a contradiction on the one hand and a qualitative be better equivalence on the other hand $\mathbf{x}$ [illegible] given $\mathbf{x}$ proof of Theorem 5.1. The first one be quite short and completely elementary, but has the disadvantage that it does not carry over to equivalence in Dolbeault-Beilinson group or algebraic cycles. The second one may appear to be artificial, but it is more elementary and leads to generalize properties [illegible] and the equivalence between Dolbeault-Beilinson group and algebraic cycles.

Also [illegible] proof of Theorem 5.1 [illegible] [illegible] is exactly the generalization of Corollary 5.11.

(ii) [illegible] a class that is not totally ramified. This follows at the [illegible] through $K_{v(L_w/K)}$. Thus the class is contained in [illegible]. Then we have a [illegible] extension [illegible] at the [illegible] $\mathfrak{p}^{a}\mathfrak{o}_K$ [illegible] $x \in K$. Then $x^{-1} = \mathfrak{p}^{a}\mathfrak{o}_K$ [illegible] $y \in K$ small that

$$\mathfrak{p}^{a}\mathfrak{o}_K\mathfrak{p}^{a}\mathfrak{o}_K = \mathfrak{p}^{a}\mathfrak{o}_K.$$

[illegible] a [illegible] the [illegible] $\mathfrak{p}_K$ [illegible] $\mathfrak{o}_K = \mathfrak{o}_{K_w}$, $x \in K$. Then [illegible] — [illegible] of

$$\begin{aligned}
\{\sigma \mid [\mathfrak{o}_K] = x\} &= \{\sigma \mid [x^{-1}\mathfrak{o}_K] = \mathfrak{p}^{a}\mathfrak{o}_K\} \quad \{\sigma \mid \mathfrak{p}^{a}\mathfrak{o}_K \mid \mathfrak{o}_{K_w}\} \\
&= \mathfrak{p}^{a}\mathfrak{o}_K \Big\| \{\mathfrak{o}_{K_w}^{\times}\mathfrak{o}_K\} \Big\| = \mathfrak{p}^{a}\mathfrak{o}_K \Big\| \mathfrak{o}_{K_w}^{\times}\mathfrak{o}_K \Big\| \\
&= \mathfrak{o}_{K_w}^{\times}\mathfrak{o}_K \,/\, \mathfrak{o}_{K_w}^{\times}\mathfrak{o}_K = \mathfrak{o}_{K_w}.
\end{aligned}$$

By a consequence of Lemma 5.5 the $\mathbb{Z}$-algebra $\mathbb{Z}$ splits completely over $\mathbb{Z}$ and the Iwasawa-[illegible] $\mathfrak{o}_K$, [illegible], have a $\mathbb{Z}$-basis of the maximal order $\mathbb{Z}$.

In the [illegible] of [illegible] the order $\mathcal{O}$ [illegible] $\mathcal{O}_K = \mathbb{Z}[\mathfrak{o}_K \cdot x \cdot \mathfrak{o}_K]$. Let [illegible] the [illegible] $\mathfrak{p}_K$ [illegible] $\mathfrak{o}_K = \mathfrak{o}_{K_w}$, $\mathfrak{p} \in x \in \mathfrak{p}^{m} - 1$. Using (iii) a basis for $\mathcal{O}$ [illegible] $\mathfrak{p}^{m} - 1$

$$\begin{aligned}
\mathfrak{o}_{K_w} \cdot \mathfrak{o}_K &= \frac{1}{\mathfrak{p}^{m}} \prod_{\sigma \in \mathfrak{o}_K} x^{-1}\mathfrak{o}_{K_w}^{\times}\mathfrak{o}_K \\
&= \frac{1}{\mathfrak{p}^{m}} \prod_{\sigma \in \mathfrak{o}_K} \mathfrak{o}_{K_w}^{\times(\sigma-1)}\mathfrak{o}_{K_w} = \begin{cases} \mathfrak{o}_{K_w}, & \sigma \in \mathfrak{p}, \\ \mathfrak{o}_K, & \text{otherwise.} \end{cases}
\end{aligned}$$

Thus for $\mathcal{O} = \mathbb{Z}[\mathfrak{o}_{K_w} \cdot x \cdot \mathfrak{o}_{K_w} \cdots \mathfrak{o}_{K_w} \cdot x^{\mathfrak{p}^{m}-1}]$ a maximal order $\mathcal{O}[\mathfrak{p}] = \mathcal{O}_K = \mathbb{Z} \cdot \mathbb{Z}$ and the assertion of the Theorem follows. $\square$

Remark 5.5 (i) In the case $p \in \mathbb{Z}$ the normalized order $\mathbb{Z}^{\mathrm{ab}}$ is a Hopf order by Proposition 5.10. This solves the Galois module problem. But is the affine $\mathbb{Z}$-group scheme Gal represented by $\mathbb{Z}^{\mathrm{ab}}$? This question is answered by the strong point of the second proof of Theorem 5.1 in the case $p \in \mathbb{Z}$.

(ii) In the case $p = 2$ the normalized order $\mathbb{Z}^{\mathrm{ab}}$ is not a Hopf order by Proposition 5.5, except for $m = 1$ and $m = 2$.

5.5 Hazewinkel [illegible]

A number of [illegible] Hazewinkel in Section 1. Since the case $p = 2$ is excluded in full generality by Proposition 5.10, [illegible] $p \in \mathbb{Z}$. Considering Theorem 1 [illegible] and [illegible] the strategy for our second proof is as follows:

1. Suppose the Galois local $\mathbb{Z}^{\mathrm{ab}}$ of the $\mathbb{Z}$-Hopf order $\mathbb{Z}^{\mathrm{ab}}$ [illegible] $\mathbb{Z}$. Hopf represents the affine $\mathbb{Z}$-group scheme μ_{p^m} of p^m-th roots of unity. $\mathbb{Z}^{\mathrm{ab}}$ is a $\mathbb{Z}$-Hopf order in $\mathbb{Z}$.

2. Show that $\mathbb{Z}^{\mathrm{ab}}$ [illegible] $\mathbb{Z}^{\mathrm{ab}}$.

1. Apply the theory of torsion subgroups of Engel algebras of [illegible], in particular [illegible, 10,(1.1)], to deduce that G is locally finite over $\mathbb{Z}^{\circ}$. Hence $\mathbb{Z}^{\circ} = \mathbb{Z}^{\circ\circ}$.

2. Define "outside" commutators $[a_1, \ldots, a_n]$ as in [15], then $g_n = [g_{n-1}, \ldots]$, and compute their power level factorization.

3. Compute $[\mathbb{Z}^{\circ\circ} : \mathbb{Z}^{\circ}]_{\infty}$, $S_{\mathbb{Z}}$ and $S_{\mathbb{Q}}$ and apply Corollary 5.11 to conclude that $G = \mathbb{Q}_{\mathbb{Z}} \cdot \mathbb{Z}^{\circ\circ}$.

The affine group scheme $\mathbb{Z}^{\circ\circ}$ of $\mathbb{Z}^{\circ\circ}$-II rank of multiply is represented by the $\mathbb{Z}$-Engel subset

$$\mathbb{Z}^{\circ} = \mathbb{Z} \quad [\mathbb{Z}]\,\Big[\mathbb{Z}^{\circ\circ} - 1\Big].$$

$\mathbb{Z}$ also $\mathbb{Z}^{\circ}$ as a subset in $\mathbb{Z}$ also the rule $\Phi(\mathbb{Z})(g) = \Phi(g)$ for $\Phi(\mathbb{Z}) \in \mathbb{Z} \quad [\mathbb{Z}]$ and $g \in \mathbb{Z}$.

Let $\mathbb{Z}_{\mathbb{Q}} \quad : \mathbb{Z} \to \mathbb{Z}$ denote the trace map $\mathbb{Z}_{\mathbb{Q}} \quad (\Phi(\mathbb{Z})) = \mathbb{Z} \quad \Phi(g)$ and define, as usual, the Iwasawa algebra of $\mathbb{Z}^{\circ}$ by

$$\Lambda^{-1}(\mathbb{Z}^{\circ}) = \{\Phi \in \Lambda \mid \mathbb{Z}_{\mathbb{Q}} \quad (\Phi\mathbb{Z}^{\circ}) \in \mathbb{Z} \quad \}.$$

In order to make this algebra an all-central an gaulle, Φ III gaulle gauds of the bells $\log \Phi$ Φ counts, although they are already stated in [15].

Lemma 5.8 [illegible], Lemma 4] The Iwasawa algebra of $\mathbb{Z}^{\circ}$ is given by

$$\Lambda^{-1}(\mathbb{Z}^{\circ}) = \frac{1}{\mathbb{Z}^{\circ\circ}}\mathbb{Z}^{\circ}.$$

Proof $\quad$ Φ also $\quad$ the set $\{1, \bar{\mathbb{Z}}, \ldots, \bar{\mathbb{Z}}^{\mathbb{Z}^{\circ\circ}-1}\}$ $\in$ $\mathbb{Z}^{\circ}$ as a $\mathbb{Z}$-basis of Λ. Let $\Phi = \sum_{i=1}^{\mathbb{Z}^{\circ\circ}-1} \alpha_i \bar{\mathbb{Z}}^{i}$ $\in \Lambda$, $\alpha_i \in \mathbb{Z}$. Then

$$\Phi \in \Lambda^{-1}(\mathbb{Z}^{\circ}) \iff \sum_{i=1}^{\mathbb{Z}^{\circ\circ}-1} \alpha_i \prod_{j=1}^{i-1} g^{j-1} \in \mathbb{Z} \quad \text{for } i = 1, \ldots, \mathbb{Z}^{\circ\circ} - 1. \qquad (5.5)$$

Now

$$\prod_{j=1}^{i-1} g^{j-1} = \prod_{k=1}^{i-1} \zeta_{\mathbb{Z}^{\circ\circ}}^{k(i-1)} = \begin{cases} \mathbb{Z}^{\circ\circ}, & \text{if } i - 1 \equiv 1 \pmod{\mathbb{Z}^{\circ\circ}}, \\ 1, & \text{otherwise,} \end{cases}$$

and (5.5) $\quad$ Φ consists that

$$\begin{aligned} \Phi \in \Lambda^{-1}(\mathbb{Z}^{\circ}) &\iff \mathbb{Z}^{\circ\circ}\alpha_1 \in \mathbb{Z} \quad \text{for } i = 1, \ldots, \mathbb{Z}^{\circ\circ} - 1 \\ &\iff \Phi \in \frac{1}{\mathbb{Z}^{\circ\circ}}\mathbb{Z}^{\circ}. \end{aligned}$$

$\square$

Corollary 8.5 ([illegible], Cor. 4) The Hecke ring R° of R° is given by

$$R^{\circ} = \left\{ \frac{1}{m^{\circ}} \sum_{\gamma \in \Gamma} \gamma(\alpha)\gamma \;\middle|\; \alpha \in R^{\circ} \right\}.$$

Proof. [illegible] follows easily from the proof of [illegible, Cor. 4]. Since the trace pairing

$$\langle \cdot , \cdot \rangle : M \times M \to \mathbb{Z}, \quad \langle \alpha_1, \alpha_2 \rangle = \mathrm{tr}_{M/\mathbb{Z}}(\alpha_1\alpha_2)$$

is non-degenerate, it does induce a natural isomorphism

$$\begin{aligned} \Phi : M^{-1}(R^{\circ}) &\longrightarrow \mathrm{Hom}_{\mathbb{Z}}(R^{\circ}, \mathbb{Z}), \\ \alpha &\longmapsto \langle \alpha \cdot (\cdot), (\cdot) \rangle = \mathrm{tr}_{M/\mathbb{Z}}(\alpha\,\cdot). \end{aligned}$$

By the definition of the Hecke ring, it gets a natural identification

$$\begin{aligned} \Psi : \mathrm{Hom}_{\mathbb{Z}}(R^{\circ}, \mathbb{Z}) &\longrightarrow R^{\circ}, \\ \Phi &\longmapsto \sum_{\gamma \in \Gamma} \Phi \circ \gamma, \quad \text{i.e. } \Phi(\alpha) = \sum_{\gamma \in \Gamma} \Phi(\gamma(\alpha)), \ \Phi \in R^{\circ}. \end{aligned}$$

Thus for $\Phi \in M^{-1}(R^{\circ})$, we obtain

$$\Psi \circ \Phi(\alpha) = \sum_{\gamma \in \Gamma} \gamma(\alpha)\gamma$$

and the equality in the proposition follows from Lemma 8.8. $\square$

Since $\{1, \bar{\omega}, \ldots, \bar{\omega}^{m^{\circ}-1}\}$ constitutes an $\mathbb{Z}$-basis of R°, the elements

$$\alpha_i = \frac{1}{m^{\circ}} \sum_{k=0}^{m^{\circ}-1} \zeta_{m^{\circ}}^{ik} \gamma_k, \quad i = 0, \ldots, m^{\circ} - 1,$$

form an $\mathbb{Z}$-basis of R° by Proposition 8.8. Obviously $\alpha_0 = \alpha_{\mathrm{tr}}$ for the additive character. Recall Φ_{tr} defined by $\Phi_{\mathrm{tr}}(\alpha_i) = \zeta_{m^{\circ}}^{-i}$ and thus $R^{\circ} = \mathbb{Z}$.

In the case of Dress, which was the first step of our strategy for the second proof of Theorem 8.1. In order to the ring R° of $R^{\circ\circ}$ is generally as in our first proof. The R°-module $\mathcal{O}$ is actually locally free, since $R^{\circ} = \mathbb{Z} = R^{\circ\circ}$ is a maximal order. In further application this will not be true, but the theory of trace algebra over Dedekind orders of Chinburg and Taylor [illegible] provides an easily tractable criterion in general local frames.

Let us consider the monomial $\{\alpha_i, \ldots, \alpha_n\}$ for $0 = i_0 < i_1 < \ldots < i_{n-1}$, $\alpha_i = \zeta_{p^{\circ}}^{i_k}$ and Φ a primitive additive $\mathbb{Z}$ character $\Phi(\alpha_i) = \zeta_{p^{\circ}}^{-i}$, $0 \le i \le m^{\circ} - 1$. Let

[illegible]

$$
\begin{aligned}
[a, I_n] &= \prod_{n \in} [a^{-1}]_n [na] \\
&= \prod_{i=1}^{n-1} [a^{-i}]_n [(i+1)n] \\
&= \prod_{i=1}^{n-1} [a^{-i}]_n [(i+1)n]^{-1} [in] \\
&= \prod_{i=1}^{n-1} [a^{-i}]_n [in] \\
&= [a^{-n}]_n \cdot m^{n}.
\end{aligned}
$$

Dann Lemma 3.1 [illegible] [illegible] $[M^{\times} : M^{\times}]^{k}_{n} = m^{n-1} k_n$ [mit [illegible] $k_n = [n]$] und [[illegible], [illegible]. 1, Lemma 3.1] folgt [illegible] $k_n = k_{[illegible]} = m^{n-1}$. [illegible] In [illegible] [illegible] [illegible] Gleichung 3.10:

$$
\begin{aligned}
\left[[M^{\times} : M^{\times}]^{k}_{n} k_n \right] \mathbb{Z}_{[illegible]} &= \left[m^{n-1} k_n k_n \right] \mathbb{Z}_{[illegible]} \\
&= \left[m^{2n-1} k_n \right] \mathbb{Z}_{[illegible]} \\
&= \left[k_n \prod_{n \mid d} [a, I_d] \right] \mathbb{Z}_{[illegible]}.
\end{aligned}
$$

Denn $\mathbb{Z} = \mathbb{Z}_p \cdot M^{\times}$. In [illegible] [illegible] [illegible] [illegible] [illegible] [illegible]:

Korollar 3.1 Sei $p \neq 1$, dann alle $M^{\times}$-Moduln $\mathbb{Z}$ in Lem. [illegible] [illegible] [illegible] [illegible] [illegible]

[illegible] [illegible] [illegible] [illegible]. [illegible]

$$[illegible] = \bigoplus_{i=1}^{s} [illegible]_i, \quad [illegible] = \bigoplus_{i=1}^{s} [illegible]_i,$$

[illegible] $[illegible]_i = [illegible]$ [illegible] $[illegible]_i = \{[illegible] \in [illegible] \mid [illegible] = [illegible]\}$, $i = 1, \ldots, s$. [illegible] [illegible]

$$[illegible] = [illegible]([illegible]), \quad [illegible] = [illegible]([illegible]), \quad [illegible] = [illegible],$$

[illegible] [illegible]

$$[illegible] = [illegible]^{[illegible]}, \quad [illegible] = [illegible]([illegible], [illegible])^{[illegible]}, \quad [illegible] = [illegible]([illegible], [illegible])^{[illegible]}.$$

[illegible] $[illegible]^{\circ}$ [illegible] [illegible] [illegible] [illegible] $[illegible]$-[illegible] [illegible] [illegible] [illegible] $[illegible]$-[illegible] [illegible] [illegible] [

[redacted] In other words,

$$[redacted] = [redacted] \cdot [redacted] \quad [redacted] \quad [redacted] = [redacted] \cdots [redacted], \quad [redacted] = [redacted] \, [redacted] \cdots [redacted] \, [redacted].$$

[redacted] $[redacted] \in [redacted] = 1, \ldots, [redacted]$, then $[redacted]$ is the dual of the [redacted]-Engel subset $[redacted]$. [redacted] [redacted]-group subset [redacted]. The affine group subset [redacted] is the fixed point of the [redacted] and therefore [redacted] $[redacted] = [redacted] \, [redacted] \cdots [redacted] \, [redacted]$. [redacted] the normal structure of Theorem 3.3 tells [redacted] $[redacted] = [redacted] \, [redacted] \cdots [redacted] \, [redacted]$. □

3 [redacted] [redacted] Lin-Tate formal [redacted]

[redacted] main references for the general theory of relative Lubin-Tate formal groups is [redacted].

[redacted] throughout this section [redacted] denotes a rational prime number. [redacted] any extension field $[redacted] \subseteq [redacted] \subseteq [redacted]$ [redacted] $[redacted]$ for the integral closure of $[redacted]$. In [redacted] and [redacted] for its maximal ideal. [redacted] for a finite extension field [redacted] of $[redacted]$ and [redacted] and $[redacted] = |[redacted]/[redacted]|$.

Let [redacted] be a fixed integer. [redacted] denote by [redacted] the unramified extension of degree [redacted] in [redacted] and by [redacted] the Frobenius automorphism of [redacted]. [redacted] for an element [redacted] in [redacted] such that [redacted], then [redacted] denotes the unramified extension [i.e., [redacted]] of [redacted].

As in [redacted], [redacted] [redacted]

$$[redacted] = \{[redacted] \in [redacted] \mid [redacted] \text{ is a [redacted]},$$
$$[redacted] = [redacted] \text{ and } [redacted] \text{ is [redacted]} \}$$

Two [redacted] are called [redacted] in the same isomorphism class, let [redacted] denote the applying [redacted] to the coefficients of [redacted].

Then [redacted], [redacted], [redacted] [redacted] there exists a unique one-dimensional commutative formal group [redacted] defined over [redacted] such that [redacted] $\in [redacted]\{[redacted], [redacted]\}$. Note that [redacted] $\in [redacted]$ and [redacted]. [redacted] course, classical Lubin-Tate formal groups correspond to the case [redacted].

In order to introduce all the necessary notation [redacted] recall

Proposition 3.1 [redacted] Let [redacted] $= [redacted], \ldots, [redacted] = [redacted]$ be in [redacted]. Let [redacted] be [redacted] satisfying [redacted]. Then there exists a unique [redacted] such that [redacted]

$$[redacted] \quad [redacted]_{[redacted]} \text{ is [redacted]},$$
$$[redacted] \quad [redacted]^{[redacted]}_{[redacted]} \cdot [redacted] = [redacted] \cdot [redacted]_{[redacted]}.$$

Moreover, $[redacted]_{[redacted]} \in [redacted]\{[redacted], [redacted]\}$. We say

$$[redacted] \in [redacted] \mid [redacted]^{[redacted]-1} = [redacted]\,[redacted] \text{ in } [redacted]\{[redacted], [redacted]\}, \quad [redacted] \text{ in } [redacted]_{[redacted]}$$

[illegible] $\mathbb{Z}_p$ [illegible] $\mathbb{Z}_p$ [illegible] $\mathbb{Z}_p$ [illegible] $\mathrm{End}(\mathbb{Z}_p)$. [illegible] $\mathbb{Z}[\mathbb{Z}] = \mathbb{Z}_p \oplus \dots \oplus \mathbb{Z}_p$ [illegible] $\mathbb{Z}^{p-1} = \mathbb{Z}_p[\mathbb{Z}_p]$, [illegible] $[x]_{p,n} = [\mathbb{Z}]_{p,n} \cdot [x]_{p,n}$.

□

In the regular [illegible] we shall at [illegible] also $[x]_p$ for $[x]_{p,p}$. [illegible] $\mathbb{Z}[\mathbb{Z}] = \mathbb{Z}'\mathbb{Z} \oplus \dots \oplus \mathbb{Z}_p$ and $1 \le i \le p$, then [illegible] a [illegible]

$$\mathbb{Z}^{[i]} = \mathbb{Z}^{[i-1]} \oplus \dots \oplus \mathbb{Z}^{[1]} \oplus \mathbb{Z}. \qquad [illegible]$$

[illegible] $\mathbb{Z}^{[i]} \cong \mathrm{Hom}(\mathbb{Z}_p, \mathbb{Z}_{p^i})$ and $\mathbb{Z}^{[n]} = [\mathbb{Z}]_p \cong \mathrm{End}(\mathbb{Z}_p)$. [illegible], for $x \in \mathbb{Z}_p$, [illegible] also $x^{[i]} = x^{[i-1]} \cdots x^{[1]} \cdot x$. In [illegible] [illegible] $\mathbb{Z}^{[i]}(\mathbb{Z}) = [x^{[i]}]_{p,p^i}(\mathbb{Z})$.

[illegible] [illegible] $\mathbb{Z}_{\bar{p}}$ [illegible] the elements of an $\mathbb{Z}_p$-module, namely $\mathbb{Z}_p \oplus \mathbb{Z}_p(\mathbb{Z}_{\bar{p}})$, by setting

$$x \oplus y = \mathbb{Z}_p(x, y), \qquad x \cdot x = [x]_p(x),$$

for $x, y \in \mathbb{Z}_{\bar{p}}$ and $x \in \mathbb{Z}_p$.

[illegible] $\mathbb{Z}_p(\mathbb{Z}) \in \mathbb{Z}_p$ [illegible] $\mathbb{Z}_p(\mathbb{Z}) = 1$ and $x \in \mathbb{Z}_p$ [illegible] $\mathbb{Z}_p(\mathbb{Z}) = x'\mathbb{Z} \oplus \dots \oplus \mathbb{Z}_p$. But $x \in \mathbb{Z}_p$ is any galois element. [illegible] $\mathbb{Z}_p$ [illegible] the group of $\mathbb{Z}_p$-isomorphism galois of $\mathbb{Z}_p$ $\mathbb{Z}_p$

$$\begin{aligned} \mathbb{Z}_p &= \{x \in \mathbb{Z}_{\bar{p}} \mid [x]_p(x) = 1, \mathbb{Z}x \in \mathbb{Z}_p\} \\ &= \{x \in \mathbb{Z}_{\bar{p}} \mid [x]_p(x) = 1\} \\ &= \mathrm{Aut}(\mathbb{Z}^{[i]} \mid \mathbb{Z}_p(\mathbb{Z}_{\bar{p}}) \cong \mathbb{Z}_{p^i} \cdot (\mathbb{Z}_{\bar{p}})) \end{aligned}$$

[illegible] [illegible].

In fact, $\mathbb{Z}_p$ is an abelian group under $\mathbb{Z}_p$ of order p. [illegible] $\mathbb{Z}_p = \mathbb{Z}'(\mathbb{Z}_p)$ [illegible] [illegible] [illegible] $\mathbb{Z}_p$. $\mathbb{Z}_p$ [illegible] is abelian and $\mathbb{Z}_p$ [illegible] $\mathbb{Z}'$ is totally ramified of degree $p^{-1}(p-1)$. [illegible] $\mathbb{Z}_p$ [illegible] $\mathbb{Z}_p$ [illegible] $\mathbb{Z}_p$ [illegible] $\mathbb{Z}'$ and, in addition, $\mathbb{Z}$ is a uniformizer for $\mathbb{Z}_p$. [illegible] $\mathbb{Z}_{p^n} = \mathbb{Z}_p \cdot [x]$. See [illegible] for more details.

[illegible] [illegible] to [illegible] [illegible] the [illegible] [illegible] [illegible] [illegible] $\mathbb{Z} = \mathbb{Z}_{p,\infty}$. [illegible] $\mathbb{Z}_{p^n}$-isomorphism group $\mathbb{Z}$ $\{i.e. \mathbb{Z} \in \mathbb{Z}_{p,n} \cong \mathbb{Z}_{p,n-1}\}$ and set

$$\mathbb{Z} = \{x \in \mathbb{Z}_{\bar{p}} \mid \mathbb{Z}^{[\infty]}(x) = \mathbb{Z}\}.$$

[illegible] $\mathbb{Z}_{p,n}$ [illegible] [illegible] [illegible] [illegible] [illegible] [illegible]

$$\mathbb{Z} := \mathbb{Z}_{p,\infty}, \quad \mathbb{Z} := \mathbb{Z}, \quad \mathbb{Z} := \mathbb{Z}_{p,\infty}.$$

[illegible] $\mathbb{Z}$, $\mathbb{Z}$, $\mathbb{Z}$ and $\mathbb{Z}$ [illegible] [illegible] [illegible] [illegible] $\mathbb{Z} := \mathbb{Z}$ [illegible] $\mathbb{Z}$ and $\mathbb{Z}$ [illegible] [illegible] [illegible] $x(\mathbb{Z})$ and $x(\mathbb{Z})$ [illegible] $\mathbb{Z}$ and $\mathbb{Z}$, and [illegible] $\mathbb{Z}$ [illegible] $\mathbb{Z}$ $\mathbb{Z}_p$ [illegible]:

$$\mathbb{Z} = \oplus \mathbb{Z}_p \mathbb{Z},$$

for $g \in G$ and $g \in H$. In particular, $|H| = |G| = p^m$.

By Proposition 1 [illegible] and

$$H := [\bar{G}G]^{\mathbb{N}}, \quad G := \mathrm{Map}[G, \bar{G}]^{\mathbb{N}}, \quad H := \mathrm{Map}[H, \bar{G}]^{\mathbb{N}},$$

satisfying the inclusions [illegible].

Now $x \in [illegible]$ [illegible] $H = \mathbb{Z}[\zeta_p]$. This fails [illegible] long because usually the commutator in results of Sauerm-Magnús and Magnus [see [illegible], [illegible]. [illegible], [illegible]. [illegible]].

But this not of this section [illegible] for an element $\zeta_p \in H$. Then $H = \mathbb{Z}[\zeta_p]$.

Lemma 4.5 Suppose that $x \in [illegible] \in [illegible]$. Then the fails [illegible] conditions hold:

(a) The map

$$x : \mathbb{Z} \to \mathbb{Z}, \quad z \mapsto z[\zeta_p],$$

is an homomorphism of $\mathbb{Z}$-algebras.

(b) [illegible] $H = \mathbb{Z}\mathbb{Z}$.

(c) The map

$$\mathbb{Z} \to \mathrm{Hal}[\mathbb{Z}][\mathbb{Z}], \quad z \mapsto z \quad ,$$

[illegible] is [illegible] [illegible] $\zeta_p \mapsto \zeta_p$ [illegible] ζ_p], is a group homomorphism.

(d) [illegible] $\mathbb{Z}$ and $\mathrm{Hal}[\mathbb{Z}][\mathbb{Z}]$ as in [a], the map x is also an homomorphism of $\mathbb{Z}$-modules, [illegible] x [illegible] all [illegible] the algebra $\mathbb{Z}$-module structure $z \cdot x := zx$ for $z \in \mathbb{Z}$ and $x \in \mathbb{Z}$.

Proof (a) Clear [illegible] x the [illegible] $\mathbb{Z}$ acts trivially on $\mathbb{Z}$. In fact, applying the [illegible] Kronecker Theorem [illegible] x can [illegible] $\Phi^{-1}[\mathbb{Z}] - \Phi = \Phi[\mathbb{Z}]x[\mathbb{Z}]$ [illegible] a distinguished polynomial $\Phi[\mathbb{Z}]$ [illegible] $\mathbb{Z}$ $||\mathbb{Z}||$ of degree p^m and a unit $x[\mathbb{Z}]$ [illegible] $\mathbb{Z}$ $||\mathbb{Z}||^{\times}$. Since Φ is a multiplicative in $\mathbb{Z}$, $\Phi[\mathbb{Z}]$ is an Eisenstein polynomial and irreducible [illegible]. Hence the Galois group Φ acts transitively on $\Phi = \{x \in \bar{\mathbb{Z}} \mid \Phi[x] = 0\}$. So [illegible] the result follows [illegible] $\mathbb{Z}[\zeta_p] = \mathbb{Z}$.

(b) This is immediate from $\mathbb{Z} \subseteq \mathbb{Z}$.

(c) Let $g, \Phi \in \mathbb{Z}$. Then

$$\begin{aligned} {}^{\mathbb{Z}\times\mathbb{Z}}\, {}^{\mathbb{Z}}\zeta_p &= [\mathbb{Z}\, \mathbb{Z}\, \Phi]\, \mathbb{Z}\, \zeta_p = \mathbb{Z}\, \mathbb{Z}\, \mathbb{Z}_p[\zeta_p] = \mathbb{Z}_p[\mathbb{Z}\, \mathbb{Z}\, \zeta_p] \\ &= \mathbb{Z}_p\, \mathbb{Z}\, [\zeta_p] = [\mathbb{Z}\, \mathbb{Z}_p][\zeta_p], \end{aligned}$$

since $g \in \mathbb{Z}$ and $\mathbb{Z}[\mathbb{Z}_p]$ is an abelian group. This implies that the map $g \mapsto \mathbb{Z}$ is a group homomorphism, that is surjective, since $\mathbb{Z}$ and $\mathbb{Z}$ act trivially on $\mathbb{Z}$.

(d) This follows [illegible]

$$[\mathbb{Z} \cdot \mathbb{Z}][\zeta_p] = \mathbb{Z}[\mathbb{Z}\, \mathbb{Z}_p\, \zeta_p] = \mathbb{Z}[{}^{\mathbb{Z}\times\mathbb{Z}}\zeta_p] = \mathbb{Z}\, [\mathbb{Z}[\zeta_p]],$$

for $\mathbb{Z} \in \mathbb{Z}$ and $g \in \mathbb{Z}$. $\square$

[illegible] [illegible] [illegible] [

Remark Replacing the all-one matrix by Toeplitz matrices leads to results of the form of Theorem 3.3 with the general as Theorem 3.3. See the author's comments for more details.

Let $\{\bar{1}, \bar{\omega}, \ldots, \bar{\omega}^{q^m-1}\}$ correspond to ω -basis of $\mathbb{F}_{q^m}$ and also to $\mathbb{F}_q$-basis of $\mathbb{F}_q$. Let $f = \sum_{i=0}^{q^m-1} a_i \bar{\omega}^i \in \mathbb{F}_q$, $a_i \in \mathbb{F}_q$. Then

$$f \in \mathbb{F}_q^{-1}[\mathbb{F}_{q^m}] \iff \sum_{i=0}^{q^m-1} a_i \sum_{x \in \mathbb{F}_q} x^{i+j} \in \mathbb{F}_q \quad \text{for } j = 0, \ldots, q^m - 1. \tag{3.3}$$

Since Theorem 3.3 is false, a heuristically that $\frac{1}{q^m[m]}\mathbb{F}_{q^m} \subseteq \mathbb{F}_q^{-1}[\mathbb{F}_{q^m}]$. See the proof below instead, a set $s_0 = \min\{s_0[a_i] \mid i = 0, \ldots, q^m - 1\}$ and $I_0 = \min\{i \mid s_0[a_i] = s_0\}$, then s_0 is the estimation in $\bar{\omega}$ of the minimal estimation s_0. Then the right hand side of (3.3) equals for $j = q^m - 1 - I_0$

$$\sum_{i=0}^{I_0-1} a_i \sum_{x \in \mathbb{F}_q} x^{q^m-1+i-I_0} + a_{I_0} \sum_{x \in \mathbb{F}_q} x^{q^m-1} + \sum_{i=I_0+1}^{q^m-1} a_i \sum_{x \in \mathbb{F}_q} x^{q^m-1+i-I_0} \in \mathbb{F}_q .$$

Again from Theorem 3.3 a contradiction that the s_0-estimation of the matrix minimal is equal to $s_0 + m$, because the other minimal have estimation strictly bigger than $s_0 + m$. Thus $s_0 + m \geq q$, that gives $\mathbb{F}_q^{-1}[\mathbb{F}_{q^m}] \subseteq \frac{1}{q^m[m]}\mathbb{F}_{q^m}$. $\square$

In that table, the elements of $\mathbb{F}_q$ play a different role in the group algebra $\mathbb{F}_q\bar{G}$: on the one hand they come as group elements, on the other hand they are field elements of $\mathbb{F}_q$ and come as coefficients in $\mathbb{F}_q\bar{G}$. To distinguish these different roles a formal notation also is instead of g. Hence g is considered as a field element. Moreover a also g_a for the unit element in $\bar{G}$. Calle Ing [illegible], [illegible], [illegible], [illegible] a interesting contain equal elements of the algebra $\mathbb{F}_q$. For $f \in \mathbb{F}_q$ a set

$$a_f := \frac{1}{q^m[m]} \sum_{x \in \mathbb{F}_q} x^{-1} f(x - a_x). \tag{3.4}$$

It is immediate that these elements are $\mathbb{F}_q$-invariant.

The next proposition is the analogue of Theorem 3.3 (see also [illegible], Prop. 3).

Proposition 3.4 The Fourier that $\mathbb{F}_{q^m}$ of $\mathbb{F}_{q^m}$ is given by

$$\mathbb{F}_{q^m} = \left\{ \frac{1}{q^m[m]} \sum_{x \in \mathbb{F}_q} f(x)a_x \;\middle|\; f \in \mathbb{F}_{q^m} \right\} = \mathbb{F}_q \cdot a_x \oplus \bigoplus_{i=1}^{q^m-1} \mathbb{F}_q \cdot a_i .$$

Proof The first equality is general as Theorem 3.3. In order to prove the second one a first Fourier f' of $\mathbb{F}_{q^m}$ by

$$f'(x) = \begin{cases} q^m[m], & \text{if } x = a_x, \\ 0, & \text{if } x \neq a_x. \end{cases} \tag{3.5}$$

Note that $f' = \phi^{(m)}(\mathbb{X})(\mathbb{Y})$. Writing $\phi^{(m)}(\mathbb{X}) = Q(\mathbb{X})x(\mathbb{X})$ for a Kollingnolbal polynomial $Q(\mathbb{X})$ of degree p^m and a unit $x(\mathbb{X}) = x_0 + \dots + \mathbb{X}^{p^m-1}$ we see that

$$f' = x_0 \frac{Q(\mathbb{X})}{\mathbb{X}} + Q(\mathbb{X})\frac{x(\mathbb{X}) - x_0}{\mathbb{X}} = x_0 \frac{Q(\mathbb{X})}{\mathbb{X}} \pmod{\phi^{(m)}(\mathbb{X})}.$$

By Lemmaliaiy Lalla [illegible] that the set $\{\bar{1}, \bar{\mathbb{X}}, \dots, \bar{\mathbb{X}}^{p^m-1}, f'\}$ also forms an $\mathbb{X}$-basis of $\mathbb{S}^{p^m}$. So this would equally be an Lemmalbla consequence of Lemma 5.5. $\square$

The later reference [illegible] Kolmas from Lemma 5.5 and Congruence 5.1

Corollary 5.6 [illegible]. Suppose $\mathbb{S}^{p^m}$ is a free $\mathbb{S}^p$-module on the map f' defined by [illegible].

Proof Consider the map $\pi = \pi^{-} \circ \phi \circ \phi : \mathbb{S}^{-1}(\mathbb{S}^{p^m}) \to \mathbb{S}^p$ defined by

$$\pi(\mathbb{X}) = (\pi^{-} \circ \phi \circ \phi)(\mathbb{X}) = \sum_{\mathbb{Z}} \mathbb{Z}(\mathbb{Z})\mathbb{Z}^{-1}, \quad \mathbb{X} \in \mathbb{S}^{-1}(\mathbb{S}^{p^m})$$

[for the definition of $\phi \circ \phi$ see the proof of Congruence 5.1]. It is really evident that π is an homomorphism of $\mathbb{S}^p$-modules. Hence π induces

$$\frac{1}{\pi'(\pi)}\mathbb{S}^{p^m} = \mathbb{S}^{-1}(\mathbb{S}^{p^m}) = \pi^{-1}(\mathbb{S}^{p^m}) = \pi^{-1}(\pi) \cdot \mathbb{S}^{p^m} = 1 \cdot \mathbb{S}^{p^m}.$$

$\square$

In the signal π assumes $\pi \in 1$, since by Proposing 5.11 the module elements of $\mathbb{S} = \mathbb{S}$ is all-Ima n has $\pi = 1$. Thus also π is the π that $\mathbb{S}^p$ is equal to the annulatiil ratio $\mathbb{S}^{p^m}$. This fulfils Ing Lemma 5 a final step in this Dissation.

Lemma 5.7 The elements π_i, $1 \le i \le 5$, are contained in $\mathbb{S}^{p^m}$. In particular, $\mathbb{S}^p \subseteq \mathbb{S}^{p^m}$.

Proof This proof is closely an adaptation of the proof of [illegible, 10. 8, Lemma 5.5] to our situation. For the reader's convenience we give a short Lemmaliam Indx our setting and notation. The maximal $\mathbb{X}$-order $\mathbb{S}$ of $\mathbb{S}$ Identifies also a [illegible] the ring of Integers $\mathbb{S}_\mathbb{X}$ in $\mathbb{X}$. Since $\mathbb{S}_\mathbb{X} = \mathbb{S}[\mathbb{X}]$, it suffices to show that

$$\pi_{\mathbb{X}^i} \cdot \pi_i \in \mathbb{S} \qquad [illegible]$$

for $0 \le i$ and $0 \le i \le p^m - 1$. We follow this [illegible] a computation $\pi(\pi_{\mathbb{X}^i} \cdot \pi_i)$:

$$\begin{aligned}
\pi(\pi_{\mathbb{X}^i} \cdot \pi_i) &= \frac{1}{\pi'(\pi)} \sum_{\mathbb{Z}} \pi'(\pi_{\mathbb{X}^i}(\mathbb{Z})\pi_i(\mathbb{Z})) - \pi_{\mathbb{X}^i}(\mathbb{Z})\pi_i(\mathbb{Z}) \\
&= \frac{1}{\pi'(\pi)} \sum_{\mathbb{Z}} \pi'(\mathbb{Z}^{p^m}\mathbb{Z}^i) - \mathbb{Z}^{p^m+i} \\
&= \frac{1}{\pi'(\pi)} \sum_{\mathbb{Z}} \pi'(\mathbb{Z}^i(\mathbb{Z}^{p^m} - \mathbb{Z}^i)) \\
&= \frac{1}{\pi'(\pi)} \sum_{\mathbb{Z}} \pi' \cdot \phi^{(m)}(\mathbb{Z}),
\end{aligned}$$

[redacted] $\mathbb{Z}_p[\mathbb{Z},\mathbb{Z}]^{\mathbb{Z}} = \mathbb{Z}^{\mathbb{Z}}$ [redacted] $\mathbb{Z}\mathbb{Z}_p[\mathbb{Z},\mathbb{Z}]$. [redacted] $g = x$ [redacted] $\mathbb{Z}\mathbb{Z}_p[\mathbb{Z},\mathbb{Z}] = \bigoplus_{n\in\mathbb{Z}} \mathbb{Z}^n x_n[\mathbb{Z}]$ [redacted] $x_n[\mathbb{Z}] \in \mathbb{Z}_p[[\mathbb{Z}]]$. [redacted]

$$x[x_n, x_1] = \sum_{n\in\mathbb{Z}} a_n^{\mathbb{Z}} \cdot \frac{1}{n!(n-1)!} \sum_{\sigma\in\mathbb{Z}} \sigma^1 x_n[\sigma].$$

[redacted] [redacted] Lemma 1.3, [redacted] [redacted]. □

[redacted] 1.5 [redacted] [redacted] [redacted]

for all $i = 1, \ldots, p^m - 1$ and $\mathfrak{p} \mid \mathfrak{m}$. The fundamental units $\{\varepsilon^i\}_{i=1,\ldots,p^m-1}$, $\varepsilon \equiv 1 \bmod \mathfrak{p}$, is obviously inertial. Therefore, $H_{p-1} = H_p$, for all $p, \mathfrak{p} \mid \mathfrak{m}$. Hence $H = H \cdot \prod_{p \mid m} [K : K \cap \mathbb{Q}_p]$. As the units $\{\varepsilon_i\}$, since m^m is the highest possible denominator for elements in $\mathbb{Q}^\circ$. $\square$

Corollary 5.10 Let $p \nmid 1$. Then the associated order $\mathbb{Z}^{\times\times}$ is equal to $\mathbb{Z}^\circ$ and $\mathfrak{p}$ is the only small one over $\mathbb{Z}^{\times\times}$.

Proof. From [10, Lemma 5.1 (b)] a Dedekind ideal $\mathfrak{p} \cdot \mathfrak{p} = \mathfrak{p}$, that is, $\mathfrak{p}$ is a lower $\mathbb{Z}^\circ$-ideal. By [illegible], Cor. 5.8] together that $\mathfrak{p}$ is the over $\mathbb{Z}^\circ$ of small one. Since $\mathbb{Z}^{\times\times} = \mathbb{Z}^\circ$. $\square$

The disadvantage of the approach a Dress taken in the 1 that a the not get an explicit generator. Those local results are entirely of interest for themselves, but they also play an important role if a and in Dedekind analogue results in the global situation of Example 1.1 [8]. The obstacle a lift that our global and local a III must an explicit generator or, and this III lead to some strange results, a relation that our local and global constants.

For a $\mathfrak{p}$, a define the constant function

$$\mathcal{C}_p(\mathfrak{m}) := \prod_{p \mid \mathfrak{m}} \frac{\mathbb{Q}^{(m)}(\mathfrak{m})}{\mathfrak{p} \cdot \mathcal{N}_p \mathfrak{m}} \quad (\mathfrak{m}^{-1}) \in \bar{\mathbb{Z}}[[\mathfrak{m}]].$$

Observation 5.11 For each $\mathfrak{p} \in \bar{\mathbb{Z}}$ we have

$$\mathcal{C}_p(\mathfrak{m}) = m^{(m)} \cdot \varepsilon_p(\mathfrak{m})$$

III a unit $\varepsilon_p(\mathfrak{m}) \in \mathcal{O}_{\bar{\mathbb{Q}}}[[\mathfrak{m}]]^\times$.

Proof. That a note that

$$\mathcal{C}_p(\mathfrak{m}) = \left. \frac{\mathbb{Q}^{(m)}(\mathfrak{m})}{\mathfrak{m}} \right|_{\mathfrak{m}=0} = m^{(m)}.$$

Note that for $p \in \mathfrak{m}$ and for $\mathbb{Q}^{(m)}(\mathfrak{p} \cdot \mathcal{N}_p \mathfrak{m}) = \mathbb{Q}^{(m)}(\mathfrak{m})$. Since $\mathbb{Q}^{(m)}(\mathfrak{m})$ has no constant term, it follows a from

$$\mathcal{C}_p(\mathfrak{m}) = \prod_{p \mid \mathfrak{m}} \frac{\mathbb{Q}^{(m)}(\mathfrak{p} \cdot \mathcal{N}_p \mathfrak{m})}{\mathfrak{p} \cdot \mathcal{N}_p \mathfrak{m}} \quad (\mathfrak{m}^{-1})$$

that $\mathcal{C}_p(\mathfrak{m}) \in \mathcal{O}_{\bar{\mathbb{Q}}}[[\mathfrak{m}]]$. Since it suffices to show that each coefficient of $\mathcal{C}_p(\mathfrak{m})$ is divisible by $m^{(m)}$. So now, it is really enough that a map also

$$\frac{\mathbb{Q}^{(m)}(\mathfrak{p} \cdot \mathcal{N}_p \mathfrak{m})}{\mathfrak{p} \cdot \mathcal{N}_p \mathfrak{m}} = \sum_{i=1} \varepsilon_i(\mathfrak{m}) \mathfrak{m}^i,$$

Then $g_1 \in \mathbb{C}_{[illegible]}[[illegible]]$, $1 \leq [illegible] \leq [illegible]$, and g_1 is a unique logarithm only on the formal group $\mathbb{G}_g$. Hence [illegible] a map defined

$$\mathbb{G}_g[illegible] = \prod_{i=1}^{[illegible]} \left(\prod_{j \in [illegible]} a_i(x) \cdot [illegible]^{-1} \right) x^i.$$

This completes the proof, since by Proposition 3.5 the terms in brackets are divisible by x^{i+1}. □

Consider the map

$$\varphi : \mathbb{R} \to \mathbb{C}, \quad \varphi(x) = \frac{g^{(1)}(x)}{x}. \tag{illegible}$$

Since $g^{(1)}(x) \in \mathbb{C}_{[illegible]}[[illegible]]$, it is $\mathbb{R}$-invariant. Thus, $\varphi \in \mathbb{R}$, and $\varphi = \varphi_{[illegible]}$ for $\varphi = \frac{g^{(1)}(x)}{x}$. Therefore, recalling the definition of the coproduct $\Delta_{\mathbb{R}}$ for a $\mathbb{R}$ from Section 3.3, we have the following calculation:

$$\begin{aligned} \Delta(\varphi, \varphi)_{[illegible]} &= \prod_{j \in [illegible]} \varphi(x + x_j) \cdot [illegible]^{-1} = \prod_{j \in [illegible]} \frac{g^{(1)}(x + x_j)}{x + x_j} \cdot [illegible]^{-1} \\ &= \prod_{j \in [illegible]} \frac{g^{(1)}(x_j)}{x + x_j} \cdot [illegible]^{-1} = \varphi_{[illegible]}(x). \end{aligned} \tag{illegible}$$

This completes the proof and proves the main result of this section.

Proposition 3.10

(i) Let $\varphi = 1$. Then $\mathbb{C}^{\circ\circ} = \mathbb{C}^{\circ}$ and $\mathbb{R} = \mathbb{R}$ is the Hopf algebra of all $\mathbb{C}^{\circ}$-generalized φ in $\mathbb{R}$, that is defined in [illegible].

(ii) Let $\varphi \neq 1$. Then we have:

(a) The $\mathbb{C}^{\circ\circ}$-module $\mathbb{R}$ is the Hopf algebra on one map $\varphi \in \mathbb{R}$ satisfying the property that

$$\Delta(\varphi, \varphi)_{[illegible]} = \varphi^{[illegible]}$$

for all $[illegible] \in \mathbb{R}$.

(b) Any function $\varphi_{[illegible]} \in \mathbb{R}$ satisfies $\varphi \in \mathbb{C}^{\circ}$ having $\mathbb{G}_g$-multiplication $g^{(1)} = 1$ is an $\mathbb{C}^{\circ\circ}$-Hopf algebra $\mathbb{R}$. In particular,

$$\mathbb{R} = \varphi \cdot \mathbb{C}^{\circ\circ} \tag{illegible}$$

for $\varphi = \varphi_{[illegible]}$ from [illegible].

Proof. (i) follows from Proposition 3.10, as φ only has to satisfy [illegible].

(ii) Let $\mathbb{C}'$ be a polynomial Hopf ring $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{R}$. Then, by Proposition 3.10, it suffices to show that

$$[\mathbb{C}^{\circ\circ} : \mathbb{C}^{\circ}]^{\mathbb{R}}_{\mathbb{C}_{\mathbb{R}}} \mathbb{R}_{\mathbb{R}} \mathbb{C}_{\mathbb{R}'} = \mathbb{R}_{\mathbb{R}} \prod_{j \in [illegible]} (\varphi, \varphi)^{\mathbb{R}}_{\mathbb{R}} \mathbb{C}_{\mathbb{R}'}, \tag{illegible}$$

[illegible] $\mathbb{P}^n = [\mathbb{P}_{\bar{\mathbb{Q}}}\mathbb{P}]^{\mathbb{Q}}$. [illegible]

$$[\mathbb{P}, \ \ ]_{\mathbb{Q}_p} = \mathbb{P}^{\mathbb{Q}},$$

[illegible] $\mathbb{Q}$. [illegible] [illegible, Lemma 8.1] [illegible] $\mathbb{Q}_p = \mathbb{P}^{\mathbb{Q}}$. [illegible]

$$[\mathbb{P}^{\mathbb{Q}} \times \mathbb{P}^{\mathbb{Q}}]^{\mathbb{Q}}_{\mathbb{Q}_p} = \mathbb{Q}_p \mathbb{P}^{\mathbb{Q}} \mathbb{P}^{\mathbb{Q}} . \tag{[illegible]}$$

[illegible] [illegible] [illegible]

$$[\mathbb{Q}_p \cdot \mathbb{Q}_{\mathbb{Q}_p} \mathbb{P}^{\mathbb{Q}} \times \mathbb{Q}_p \cdot \mathbb{P}]^{\mathbb{Q}}_{\mathbb{Q}_p} = \mathbb{P}^{\mathbb{Q}} \mathbb{P}^{\mathbb{Q}} \mathbb{Q}_p \cdot \tag{[illegible]}$$

[illegible]

$$[\mathbb{Q}_p \cdot \mathbb{P} \times \mathbb{Q}_p \cdot \mathbb{Q}_{\mathbb{Q}_p} \mathbb{P}^{\mathbb{Q}}]^{\mathbb{Q}}_{\mathbb{Q}_p} = \mathbb{P}^{\mathbb{Q}} \mathbb{Q}_p \cdot . \tag{[illegible]}$$

[illegible] [illegible] [illegible] [

[redacted]ogting the good of [redacted], [redacted]. [redacted], [redacted] 5.1] [redacted] a can the [redacted] $\Omega'_{\ldots}(\mathbb{G}_{\mathbb{q}}) = \mathfrak{m}'^{(\ldots)}$ for $\mathbb{q} = 1, \ldots, [redacted]$. [redacted] $[redacted]^{\ldots}_{\ldots}, \mathbb{G}_{\mathbb{q}}$ is annulated in the leading coefficient of $\mathbb{q}^{(\ldots)}(\ldots)$, [redacted] is $\mathfrak{m}'^{(\ldots)}$. [redacted] ng [redacted] a [redacted]

$$\mathrm{Ad}(\ldots)^{\ldots} = \ldots^{-1\ldots} \cdot \ldots^{\ldots-\ldots} = \ldots^{-\ldots},$$

[redacted] gonna [redacted].

[redacted] is [redacted] a [redacted] [redacted] 5.11 [redacted] [redacted] [redacted] [redacted] for $\ldots = \ldots_{\mathbb{q}}$ [redacted] a [redacted] $\ldots, \ldots_{\ldots} = \ldots^{\ldots}$ for all [redacted] a [redacted]. [redacted] $\ldots_{\mathbb{q}}(\ldots)$ [redacted] a [redacted] [redacted] $\ldots = \ldots \cdot \ldots^{\ldots}$. [redacted] $\ldots_{\mathbb{q}}(\ldots) = \mathbb{q}^{\ldots} - 1$ [redacted] $\mathbb{q}_{\ldots}$ [redacted]. [redacted] is a multiplicing element in [redacted] [redacted].

[redacted] a [redacted] [redacted] $\mathbb{q}_{\ldots}$-[redacted] $\mathbb{q}^{\ldots} - 1$. [redacted] [redacted] is totally [redacted] [redacted] $\mathbb{q}^{\ldots}$, [redacted] [redacted] a [redacted] [redacted] [redacted] [redacted]

$$\ldots \ldots \text{ [redacted] } \mathbb{q} \ldots \ldots_{\ldots} \text{ [redacted]} \quad \ldots \mathbb{q}. \quad \ldots_{\ldots} \ldots \ldots_{\ldots} \text{ [redacted] } \mathbb{q} \ldots \ldots.$$

[redacted] [redacted] [redacted] [redacted]

$$\ldots = \ldots_{\ldots} \cdot \ldots^{\ldots} \ldots_{\mathbb{q}}\ldots.$$

[redacted] [redacted] [redacted] [redacted] in [redacted] is a [redacted] [redacted]'s [redacted]. □

[redacted] a [redacted] [redacted] a [redacted] [redacted] [redacted] [redacted] [redacted] [redacted] $\ldots = \ldots_{\mathbb{q}}$. [redacted] [redacted] [redacted] [redacted] [redacted] [redacted]

[illegible] [illegible] Theorem 1.15 [illegible] is [illegible] in the [illegible] that $\Phi^{-1}(\mathbf{u}_0)$ generates $\mathbb{S}^n$ over $\mathbb{S}^n$. [illegible] the

$$\Phi^{-1}(\mathbf{u}_0) = \bigcup_{\mathbf{m}\in\mathbb{Z}} \frac{1}{m'(\mathbf{m})}\mathbf{u}_0(\mathbf{m}_0 \mathbf{m}_0)^{1'}$$

[illegible] the map $1'$ defined in [illegible]. [illegible] the definition of $\mathbf{u}_0$ [illegible] a finite composite

$$\Phi^{-1}(\mathbf{u}_0) = \bigcup_{\mathbf{m}\in\mathbb{Z}} \frac{1}{m'(\mathbf{m})}\Phi^{m}\mathbf{u}_0^{1'}$$

$$= 1' \cdot \bigcup_{\mathbf{m}\in\mathbb{Z}} \frac{1}{m'(\mathbf{m})}\Phi^{m}\mathbf{u}_0.$$

[illegible] Corollary 1.1 is [illegible] in the [illegible] that

$$\mathbb{H}_1 = \bigcup_{\mathbf{m}\in\mathbb{Z}} \frac{1}{m'(\mathbf{m})}\Phi^{m}\mathbf{u}_0 \in [\mathbb{S}^n]^n.$$

Hence the definition of $\mathbb{H}$ [illegible] defines

$$\Phi^{m}\mathbf{u}_0 = \left[\frac{\mathbb{P}^{(m)}(\mathbf{u}_0)}{\mathbf{u}_0}\right]^{m} = \frac{\mathbb{P}^{(m)}(\mathbf{u}_0 \mathbf{m} \mathbf{u})}{\mathbf{u}_0 \mathbf{m}_0 \mathbf{u}}.$$

Therefore $\Phi^{m} = \mathbb{P}[\mathbb{M}](\mathbf{m})$ for $\mathbb{P}[\mathbb{M}] = \frac{\mathbb{P}^{(m)}(\mathbf{u}_0 \mathbf{m}_0 \mathbf{m})}{\mathbf{u}_0 \mathbf{m}_0 \mathbf{m}}$ in $\mathbb{G}_n[[\mathbb{M}]]$ and [illegible] a consequence from [illegible] and Proposition 1.1 that $\mathbb{H}$ is $\mathbb{S}^n$. Mapping $\mathbb{H}$ to the [illegible] composed from Theorem 1.3[illegible] [illegible] a finite form [illegible] and Theorem 1.11 that $\mathbb{H}$ is an invertible element of the maximal order $\mathbb{O}_n$. Hence the result follows [illegible] form $[\mathbb{S}^n \times \mathbb{M}^n] = [\mathbb{S}^n]^n$. □

1 Filtrations [illegible]

1.1 [illegible] [illegible] of [illegible] [illegible]

The purpose of this section is to introduce a special class of filtration spaces. General references for the theory of filtration spaces are the books of Ellenberg [illegible] and [illegible], in particular [illegible, [illegible]] for [illegible]-filtration spaces. [illegible] [illegible] may already the exposition in [illegible, [illegible]] and [illegible, [illegible], [illegible]].

Throughout this section $\mathbb{F}$ denotes a positive integer monoid field. For an integer ideal $\mathfrak{j}$ of $\mathbb{F}$ the map class field of conductor $\mathfrak{j}$ is denoted by $\mathbb{F}[\mathfrak{j}]$. In particular [illegible] this $\mathbb{F}[1]$ for the filtration class field.

Once and for all [illegible] fix an algebraic closure $\bar{\mathbb{F}}$ of $\mathbb{F}$ and an embedding $\bar{\mathbb{F}} \hookrightarrow \mathbb{K}$. For this embedding [illegible] also all number fields we embedded in $\mathbb{K}$.

For an abelian extension $\mathbb{K}/\mathbb{F}$ of number fields and an integral ideal [illegible] of $\mathbb{F}$ relatively prime to the conductor $\mathfrak{f}_{\mathbb{K}/\mathbb{F}}$ [illegible] denote the Artin symbol by $\sigma_{\mathfrak{p}} \in \mathrm{Gal}(\mathbb{K}/\mathbb{F})$.

Let [illegible] be a finite abelian extension of [illegible] and let [illegible] be an algebraic curve defined over [illegible]. [illegible] complex multiplication by the ring of integers [illegible]. [illegible] a fixed [illegible] generalized elliptic curve model for [illegible],

$$y^{2} = x_{1}x_{2}x_{3}x_{4} - x^{2} + x_{1}x^{2} + x_{2}x + x_{3}, \qquad [illegible]$$

[illegible] $x_{1}, x_{2}, x_{3}, x_{4}, x_{5} \in [illegible]$ and denote by $y = -x_{6} = x_{5}[(x_{5} + x_{1}x + x_{2})]$ the standard Weierstrass [illegible] [illegible].

In accordance with [illegible], [illegible] a can replace x and y by

$$y = x + \frac{x_{1}^{2} + x_{2}}{[illegible]}, \qquad y' = x_{5} + x_{1}x + x_{2},$$

and [illegible] is transformed to

$$[y']^{2} = x_{5}^{2} - x_{1}x - x_{2} \quad [illegible] \; x_{1}, x_{2} \in [illegible]. \qquad [illegible]$$

[illegible] pair $[[illegible], x_{5}]$ determines a complex lattice ω in [illegible]. [illegible] is characterized by

$$x_{1}[\omega] = x_{1}, \qquad x_{2}[\omega] = x_{2},$$

Here, as usual,

$$x_{1}[\omega] = [illegible]\,[illegible]'_{\omega}\,\omega^{-2}, \qquad x_{2}[\omega] = [illegible]\,[illegible]'_{\omega}\,\omega^{-4}$$

denote the coefficients in the Weierstrass equation of the elliptic x-function.

Since [illegible] has complex multiplication by $[illegible]$, the lattice ω is of the form [illegible] [illegible] $\in [illegible]^{\times}$ and a fractional ideal [illegible] of [illegible]. Since [illegible] is real, [illegible] is uniquely determined up to a unit of unity in [illegible].

[illegible] denote by

$$\begin{aligned} [\,,\omega] : [illegible] &\longrightarrow [illegible], \\ [illegible] &\longmapsto \begin{cases} [x[illegible] \mid \omega], y'[illegible] \mid \omega], 1], & [illegible] = [illegible]\,\omega, \\ [illegible], 1, 1], & [illegible] = [illegible]\,\omega, \end{cases} \end{aligned} \qquad [illegible]$$

the corresponding elliptic logarithm.

Let $[illegible] : [illegible] \to [illegible]$ be normalized in the sense of [illegible] [illegible], i.e. $[illegible] = [illegible]$ for $[illegible] \in [illegible]$. Then

$$[illegible] \mid [illegible] = [illegible], \qquad [illegible] \in [illegible], \quad [illegible] \in [illegible]. \qquad [illegible]$$

For an integral ideal $[illegible]$ of [illegible] [illegible] define the group of $[illegible]$-torsion points of [illegible] by

$$[illegible] = [illegible] \mid [illegible] \in [illegible]^{-1}[illegible]\}.$$

Furthermore [illegible] [illegible], for the torsion subgroup of [illegible]. Now the fact that the $[illegible]$-torsion $[illegible]$ of [illegible] generates the Hilbert class field [illegible] [illegible] [illegible] [illegible] [illegible]. Therefore [illegible, Th. 1.1] implies that [illegible] is abelian.

Let $\mathbb{K} = \mathbb{K}_{\mathfrak{p}}$ be the Dedekind domain associated to $\mathbb{Q}[\zeta]$ and denote $\mathfrak{p}$ by $\mathfrak{p}_{\mathbb{K}}$ its conjugates. Then $\mathbb{K}$ is a Dedekind domain whose group of all fractional ideals of $\mathbb{K}$ relatively prime to $\mathfrak{p}_{\mathbb{K}}$ is $\mathbb{Z}^{\mathbb{N}}$. Thus any principal ideal $\mathfrak{a} = (\alpha)$ of $\mathbb{K}$ such that $\alpha \equiv 1 \pmod{\mathfrak{p}_{\mathbb{K}}}$ and that $\mathbb{N}(\mathfrak{a}) = \mathbb{N}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\alpha)$.

Let $\mathfrak{a}$ be any integral ideal of $\mathbb{K}$ with $(\mathfrak{a}, \mathfrak{p}_{\mathbb{K}}) = 1$. Then $\mathbb{K}/\mathfrak{a}$ is $\mathbb{Z}_n$ and α then the formula

$$(\mathfrak{a}, \omega)^{\sigma(\omega)} = (\mathbb{N}(\mathfrak{a}), \omega) \tag{[illegible]}$$

has any $\omega \in \mathfrak{p}^{-1}\mathbb{K}$, $(\mathfrak{a}, \mathbb{K}_{\mathfrak{p}_{\mathbb{K}}}) = 1$.

α one longer the Dellinger Hecke Operatorials on $\mathbb{K}$:

$$\mathbb{K}[\mathbb{K}_{1,1}] \text{ is an algebra automorphism of } \mathbb{K}. \tag{[illegible]}$$

$\mathbb{K}_{\mathfrak{p}}$ [illegible, illegible. 5.illegible] this condition is equivalent to the existence of a Dedekind-domain $\mathfrak{p}$ of $\mathbb{K}$ such that $\mathbb{K} = \mathfrak{p} \times \mathbb{K}_{\mathfrak{p}_{\mathbb{K}}}$ (see also [illegible, illegible. 5.1]). Small a Dedekind domain $\mathfrak{p}$ is uniquely determined by its multiplicative $\mathbb{K}$ an algebra automorphism of $\mathrm{Gal}(\mathbb{Q}[\zeta]/\mathbb{Q})$, then is equivalent to a Galois automorphism of $\mathbb{K}$ ([illegible], Proposition [illegible]). Thus and for all α for small α $\mathfrak{p}$ and denote $\mathfrak{p}_{\mathbb{K}}$

$$\mathfrak{q} = \mathrm{lcm}(\mathfrak{p}_{\mathbb{K}}, \mathfrak{p}_{\mathbb{K}}) \tag{[illegible]}$$

the least common multiple of the conjugates of $\mathfrak{p}$ and $\mathbb{Q}[\zeta]$. The ideal $\mathfrak{q}$ is in fact independent of the choice of $\mathfrak{p}$ (see [illegible, p.illegible]) and only depends on $\mathbb{Q}[\zeta]$. Note also that $\mathfrak{p}$ satisfies $\mathfrak{p}(\mathfrak{a}) = \mathfrak{a}$ for $\mathfrak{a} \equiv 1 \pmod{\mathfrak{q}}$, $\mathfrak{a} \in \mathbb{K}$.

Lemma 5.1 Let $\mathfrak{p}$ be a Dedekind domain with [illegible] conjugates $\mathfrak{p}_{\mathbb{K}}$ and defining $\mathbb{K}_{\mathfrak{m}} = \{\mathfrak{a}, \mathfrak{b} \mid \text{i.e., } \mathfrak{p}(\mathfrak{a}) = \mathfrak{a} \text{ for all } \mathfrak{a} \in \mathbb{K}, \text{ [illegible] } \mathfrak{a} \equiv 1 \pmod{\mathfrak{p}_{\mathbb{K}}}\}$. Then $\mathbb{K}(\mathbb{K}_{\mathfrak{m}}) = 1$, then $\mathbb{K}_{\mathfrak{m}} = \{\mathfrak{a} \in \mathbb{K}_{\mathfrak{m}} \mid \mathfrak{a} \equiv 1 \pmod{\mathfrak{p}_{\mathbb{K}}}\}$ for any integral $\mathbb{K}_{\mathfrak{m}}$-ideal .

Moreover $\mathbb{K}_{\mathfrak{m}}$ satisfies $\mathfrak{p}_{\mathbb{K}}(\mathfrak{a}) := \mathfrak{p}(\mathfrak{a})/\mathbb{K}_{\mathfrak{m}}$ a subset a all-Dedekind domains $\mathfrak{p}_{\mathbb{K}} : \{\mathbb{K}_{\mathfrak{m}}/\mathbb{K}_{\mathfrak{m}}\}^{\times} \to \mathbb{K}^{\times}$. Let $\mathbb{K}^{\times} = \{\mathfrak{a} \in \mathbb{K}_{\mathfrak{m}}^{\times} \mid \mathfrak{a} \equiv 1 \pmod{\mathfrak{p}_{\mathbb{K}}}\}$. Then α then an embedding

$$\mathbb{K}_{\mathfrak{m}}^{\times}/\mathbb{K}^{\times} \hookrightarrow \{\mathbb{K}_{\mathfrak{m}}/\mathbb{K}_{\mathfrak{m}}\}^{\times}.$$

Then any $\mathfrak{a} \in \mathbb{K}_{\mathfrak{m}}^{\times}$ α then $\mathfrak{p}_{\mathbb{K}}(\mathfrak{a}) = 1/\mathfrak{a}$. Hence $|\mathbb{K}_{\mathfrak{m}}^{\times}/\mathbb{K}^{\times}| = \mathbb{N}(\mathfrak{q})/\mathbb{N}(\mathbb{K}_{\mathfrak{m}})$ α necessarily then $\mathfrak{p}_{\mathbb{K}}(\mathfrak{a})^{\mathbb{N}(\mathfrak{q})/\mathbb{N}(\mathbb{K}_{\mathfrak{m}})} = 1 = \left(\frac{1}{\mathfrak{a}}\right)^{\mathbb{N}(\mathfrak{q})/\mathbb{N}(\mathbb{K}_{\mathfrak{m}})}$. Hence $\mathbb{K}_{\mathfrak{m}}^{\times}$ is equal of order $\mathbb{N}(\mathfrak{q})$ α consider that $\mathbb{N}(\mathbb{K}_{\mathfrak{m}}) = 1$. $\square$

Quotients instances of this type of algebra occur are the analytic models of Gauss, Sonning and Eugenbaum in [illegible], [illegible], [illegible] and [illegible]. See also the examples [illegible.1] in [illegible].

Let $\mathfrak{a}$ be an integral ideal of $\mathbb{K}$ relatively prime to $\mathfrak{q}$. Then [illegible, [illegible.illegible],[illegible.illegible]] are Kolmogorov the existence of a unique integer

$$\mathbb{N}(\mathfrak{a}) : \mathbb{K} \to \mathbb{K}^{\times}[n] \tag{[illegible]}$$

abbreviated by the pairing

$$\mathbb{M}^{\vee[n]} = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{M}, \mathbb{Z}) \text{ for } \mathbb{M} = \mathbb{M}[n], \quad \langle m, \ \rangle = 1. \tag{illegible}$$

Since n is a $\mathbb{M}$-basis for the space of $\mathbb{M}$-monomials of the first kind, we may define the quantity $\mathbb{M}(\ )$ in $\mathbb{M}$ by

$$m^{\vee[n]} = \mathbb{M}(\ ) = \mathbb{M}(\ )m.$$

$\ $ also m_n for the lattice $\mathbb{M}$ corresponding to the simple $(\mathbb{M}^{\vee[n]}, m^{\vee[n]})$.

Proposition 1.5 [illegible], Prop. [illegible]

(i) There are integral linear $\ $ of $\mathbb{M}$ satisfying values in $\mathbb{Z}$ the map $\ $ on $\mathbb{M}(\ )$ to abbreviated by the following commutative diagram:

$$\begin{array}{ccc} \mathbb{M}[m & \xrightarrow{\langle \cdot, m \rangle} & \mathbb{M}(\mathbb{M}) \\ \cdot m[n] \downarrow & & \downarrow m[n] \\ \mathbb{M}[m_n & \xrightarrow{\langle \cdot, m_n \rangle} & m^{\vee[n]}(\mathbb{M}) \end{array}$$

(ii) There all $\ $ as above, $m_n = \mathbb{M}(\ )^{-1}m$.

(iii) Let p be an integral linear $\ $ of $\mathbb{M}$ such that $(\ ) \mid p$. Then we have for all $\ $ $p^{-1}m$

$$\langle \ , m \rangle^{\vee[n]} = \langle (\mathbb{M}(\ )\ ), m_n \rangle.$$

(iv) If $\ $ and $\ $ are $\ $ integral linear $\ $ of $\mathbb{M}$ such that $\langle \ , \ \rangle = 1$, $\langle \ , \ \rangle = 1$, then

$$\mathbb{M}(\ \ ) = \mathbb{M}(\ )^{\vee[n]}\mathbb{M}(\ ) = \mathbb{M}(\ )(\mathbb{M}(\ ))^{\vee[n]} \tag{illegible}$$

□

There is a unique $\ $ up to extend the definition of $\mathbb{M}$ to the $\ $ lattice group of integral linear satisfying values in $\mathbb{Z}$ such that $(\mathbb{M})$ remains valid. Namely $\ $ a lattice in $\ $

$$\mathbb{M}(\ )^{-1}(\ )^{\vee[n]} = \frac{1}{\mathbb{M}(\ )}. \tag{illegible}$$

The extension $\ $ again be called $\mathbb{M}$.

The following proposition gives the precise class field theoretical description of the Galois group of $\mathbb{S}$ the p-torsion points for $(\ )$ p.

Proposition 1.6 [illegible], Lemma [illegible] Let p be an integral linear of $\mathbb{M}$ relatively by $\mathbb{Z}$. Then $\mathbb{M}(\mathbb{M}[p]) = \mathbb{M}[p]$.

□

In the rest of this section $\ $ $\ $ assumed an abelian extension $\mathbb{M}(\mathbb{M}[\mathbb{Z}^{\times\times}])/(\mathbb{M}(\mathbb{M})[\mathbb{Z}^{\times}])$ $\ $ $\ $, we $\ $ $\ $ and a prime $\ $ of $\mathbb{M}$. In particular $\ $ $\ $ $\ $ and the values for p, $\mathbb{M}$ and $\mathbb{M}$ for $\ $ such that $(\ , \mathbb{M}[\mathbb{M}]) = 1$ in general.

Lemma 1.5 [illegible] Se $\mathfrak{a}$ è un ideale intero di K relativamente primo in $\mathfrak{f}$ così che $(\mathfrak{f}, \mathfrak{a}) = 1$. Allora:

(i) $\mathfrak{a}^{\sigma} \in K^{\times}$, $\mathfrak{a}^{\sigma} \equiv 1 \pmod{\mathfrak{f}}$ e $\mathfrak{a} = \mathfrak{a}^{\sigma}$.

(ii) $N(\mathfrak{a}^{\sigma}) = N(\mathfrak{a})^{\sigma}$.

(iii) $\mathfrak{a}^{\sigma} N(\mathfrak{a}) = N(\mathfrak{a})\mathfrak{a}$.

Dimostrazione Sia $\mathfrak{a}$ un ideale intero di K così che $(\mathfrak{f}, \mathfrak{a}) = 1$. Come Congettura 1.3 (ii) e abbiamo

$$\mathfrak{a}^{\sigma} = \mathfrak{a}^{\sigma^{[\mathfrak{a}]}[\mathfrak{a}]} = \mathfrak{a}^{\sigma^{[\mathfrak{a}]}[\mathfrak{a}]^{-[\mathfrak{a}]}} = \mathfrak{a}^{\sigma^{[\mathfrak{a}]}}[\mathfrak{a}].$$

Siccome $\mathfrak{a}^{\sigma^{[\mathfrak{a}]}} = \mathfrak{a}(\alpha)$, dall'uguaglianza $\mathfrak{a}(\alpha) \in K^{\times}$. Come $\mathfrak{a}^{\sigma^{[\sigma]}} = \mathfrak{a}$ così $\alpha^{\sigma^{[\sigma]}} = \alpha$ e dunque $\alpha_{\sigma} = \alpha$ così $\mathfrak{a}(\alpha)_{\sigma} = \mathfrak{a}$.

Dalla commutatività diagramma di Congettura 1.3 (i) leggiamo che

$$N(\mathfrak{a}) = N([\mathfrak{a}, \alpha]) = N([\alpha]\mathfrak{a}) = N([\alpha])N([\mathfrak{a}, \alpha]).$$

Come dalla (ii) la tesi è dimostrata.

Dalla seconda relazione (iii) leggiamo che $\mathfrak{a}$ è moltiplicativa su ideali coprimi con $\mathfrak{f}$ e di classe $(\mathfrak{f}, \mathfrak{a})$ $(\mathfrak{f}, \mathfrak{a}) = 1$. Sia $\mathfrak{g}'$ la sua estensione di $\mathfrak{g}$ in a Dimostrazione di $\mathfrak{g}$. Per ogni intero ideale $\mathfrak{a}$ di $\mathfrak{g}$ si ha

$$\begin{aligned}(\mathfrak{a}, \alpha)^{\sigma^{[\mathfrak{a}]}} &= (\mathfrak{a}, \alpha)^{\sigma^{[\mathfrak{a}]}_{\mathfrak{g}, \mathfrak{g}'}[\mathfrak{a}]} \\ &= (\mathfrak{a}[\mathfrak{a}]_{\mathfrak{g}, \mathfrak{g}'}^{-1}, \alpha),\end{aligned}$$

dove $\mathfrak{a}[\mathfrak{a}]_{\mathfrak{g}, \mathfrak{g}'}^{-1}$ è un intero ideale $\mathfrak{g}_{\mathfrak{g}'}$-ideale e $(\mathfrak{a}, \mathfrak{a}[\mathfrak{a}]_{\mathfrak{g}, \mathfrak{g}'}^{-1}) = 1$.

Per la definizione generale di $\mathrm{Reg}_{\mathfrak{g}'}$ è chiaro $\mathfrak{a} \in \mathrm{Reg}_{\mathfrak{g}'}$ e allora $\mathfrak{a}^{\sigma} \in \mathrm{Reg}_{\mathfrak{g}'} = \mathrm{Reg}_{\mathfrak{g}}$ così allora anche $\mathfrak{g}' \in \mathrm{Reg}_{\mathfrak{g}'} = \mathrm{Reg}_{\mathfrak{g}}$. Dunque $\mathfrak{g}'$ è di fatto dunque $\mathfrak{g}' = \mathfrak{g}$ che è assurdo al $\mathrm{Cl}(\mathfrak{g}[\alpha])$. Ma allora $N(\mathfrak{a}) = \mathfrak{g}'(\mathfrak{a}) = \mathfrak{g}(\mathfrak{a})$ $(\mathfrak{a}) = \mathfrak{g}(\mathfrak{a})$ per ogni $\mathfrak{a}$ così che $(\mathfrak{a}, \mathfrak{a}[\alpha]) = 1$. □

Teorema 1.6 Come (iii) è la classe che (i) così (ii) sono anche tutte per ogni frazionale ideale $\mathfrak{a}$ di K relativamente primo in $\mathfrak{f}$ così che $(\mathfrak{f}, \mathfrak{a}[\alpha]) = 1$.

Corollario 1.7 Sia $\mathfrak{g}$ un intero ideale di Dedekind $\mathfrak{g}_{\mathfrak{g}}$. Sia $\mathfrak{P}$ un primo $\mathfrak{g}$-ideale primo di $\mathfrak{g}$. Allora $\mathfrak{P}(\mathfrak{P}) = N(\mathfrak{P})$.

Dimostrazione Come Congettura 1.6 è vera per $\mathfrak{P}(\mathfrak{P}) \mid N(\mathfrak{P})$. Sia $\mathfrak{a}$ un intero ideale di K così che $\mathfrak{a} \in \mathrm{Reg}_{\mathfrak{g}}$ e $(\mathfrak{a}, \mathfrak{P}) = 1$. Sia $\mathfrak{a}' \in \mathfrak{g}^{-1}\mathfrak{a}$ è così che $\mathfrak{P} = (\mathfrak{a}, \alpha)$. Per (iii) e Teorema 1.5(ii) è abbiamo $\mathfrak{P}^{\sigma^{[\sigma]}} = (\mathfrak{a}[\mathfrak{a}]^{-1}, \alpha)$. Siccome $(\mathfrak{a}[\mathfrak{a}]^{-1}, \alpha) = (\mathfrak{a}, \alpha)$ così $\mathfrak{P}$ è principale, dalla leggiamo $\mathfrak{P}(\mathfrak{P}) = 1 \pmod{\mathfrak{g}}$. Sia $\mathfrak{P}_{\mathfrak{g}} = (\mathfrak{a}, \alpha)$ è un sollevamento $\mathfrak{g}$-ideale primo. Allora $\mathfrak{a}' \in \mathfrak{g}^{-1}\mathfrak{a}$ così

$$\mathfrak{P}_{\mathfrak{g}}^{\sigma^{[\sigma]}} = (\mathfrak{a}[\mathfrak{a}]^{-1}, \alpha) = (\mathfrak{a}, \alpha) = \mathfrak{P}_{\mathfrak{g}}.$$

Dunque $\sigma(\mathfrak{P}) \mid N(\mathfrak{P})$, così allora $\mathfrak{P}(\mathfrak{P}) = N(\mathfrak{P})$. □

Sia $\mathfrak{p}$ primo ideale $\mathfrak{P}$ di K è tenuto alla $\mathfrak{P}$-adica valutazione $\mathfrak{g}_{\mathfrak{P}}$ $\mathfrak{p}_{\mathfrak{P}}$.

Proposition [illegible]. Let $\mathfrak{p}$ be a prime ideal of R such that $\mathfrak{p} \neq [illegible]$. Then $\mathrm{ht}_{\mathfrak{p}}(\mathfrak{p}\mathcal{O}_{\mathfrak{p}}) = \mathrm{ht}_{\mathfrak{p}}(\mathfrak{m}(\mathfrak{p}))$ for all $\mathfrak{p} \mid \mathfrak{p}$.

Proof. Let $\mathfrak{p} \in \mathfrak{p}$ be such that $\{\mathfrak{p}, \mathfrak{m}(\mathfrak{p})\} = 1$. Then Lemma [illegible] [illegible] and Proposition [illegible] [illegible] imply

$$\mathfrak{p}\mathcal{O}_{\mathfrak{p}} = \mathfrak{m}(\mathfrak{p})\mathcal{O}_{\mathfrak{p}} = \mathfrak{m}(\mathfrak{p})\mathfrak{m}(\mathfrak{p})^{\mathrm{ht}(\mathfrak{p})} \cdots \mathfrak{m}(\mathfrak{p})^{\mathrm{ht}(\mathfrak{p})-1}.$$

Now Nakayama shows $\mathrm{ht}_{\mathfrak{p}}(\mathfrak{p}\mathcal{O}_{\mathfrak{p}}) = \mathrm{ht}_{\mathfrak{p}}(\mathfrak{m}(\mathfrak{p}))$, since $\mathfrak{m}(\mathfrak{p})$ is an element of the Noetherian group $\mathcal{O}_{\mathfrak{p}}$ for all $\mathfrak{p}$ above $\mathfrak{p}$. $\square$

Let now $\mathfrak{p}$ be a prime ideal of R such that $\{\mathfrak{p}, \mathfrak{p}\} = 1$. Let $\mathfrak{q}$ denote a prime of $\mathcal{O}$ above $\mathfrak{p}$. Since $\mathcal{O}$ has good reduction at $\mathfrak{q}$, $\mathfrak{p}$ can be a simultaneous zero [illegible] over the localization $\mathcal{O}_{\mathfrak{p},\mathfrak{q}}$ of $\mathcal{O}_{\mathfrak{p}}$ at $\mathfrak{q}$ such that $\mathfrak{q} \in \mathcal{O}^{\circ}_{\mathfrak{p},\mathfrak{q}}$.

In the table [illegible] $\mathfrak{p} \in \mathrm{III}$ will not the connection to the relative Dedekind-Hasse theory as it is generalized in Section [illegible].

$\mathfrak{p}$ also $\mathfrak{p}$ is an affinoid cross near the completion $\mathcal{O}_{\mathfrak{p}_{\mathfrak{q}}}$ and denote $\mathfrak{p}_{\mathfrak{q}}$ the non-generator formal group of $\mathfrak{p}$ [illegible] respect to the parameter

$$\mathfrak{p} = -\frac{\mathfrak{p}}{\mathfrak{p}}$$

[see [illegible], [illegible]].

Lemma [illegible]. [illegible] $\mathfrak{p}$ is a relative Dedekind-Hasse group [illegible] respect to the normalized valuation $\mathrm{ord}_{\mathfrak{p}}$ associated [illegible] the parameter $\mathfrak{p}(\mathfrak{p})$.

Proof. Let $\mathfrak{p} = \mathfrak{p}(\mathfrak{p})$ be the Weierstrass endomorphism of $\mathfrak{p}_{\mathfrak{p}}$. The logarithm $\mathfrak{p}(\mathfrak{p}) : \mathfrak{p} \to \mathfrak{p}^{\mathrm{ad}}$ induces a homomorphism of formal groups

$$\widehat{\mathfrak{p}}(\mathfrak{p}) : \hat{\mathfrak{p}} \xrightarrow{\sim} \hat{\mathfrak{p}}^{\mathrm{ad}}.$$

$\mathfrak{p}$ put $\mathfrak{p} = \mathfrak{p}_{\mathfrak{p}_{\mathfrak{p}}}(\mathfrak{p})$. In fact $\mathfrak{p}$ shows [illegible] that $\mathfrak{p}(\mathfrak{p})$ gives the $\mathfrak{p}$-pro $\mathfrak{p}$ Weierstrass on the valuation of the cross module $\mathfrak{p}$. Therefore one has

$$\widehat{\mathfrak{p}}(\mathfrak{p})(\mathfrak{p}) = \mathfrak{p}^{\mathfrak{p}} \quad \text{[illegible] } \mathfrak{p}. \tag{[illegible]}$$

The formal differential $\mathfrak{p} = \mathrm{Res}_{\mathfrak{p}}\{\mathfrak{p}_{\mathfrak{p}} \mathfrak{p}, \mathfrak{p} \mathfrak{p}, \mathfrak{p}\}$ has an expansion

$$\mathfrak{p}(\mathfrak{p}) = \{\mathfrak{p} \mathfrak{p}, \mathfrak{p} \mathfrak{p} \mathfrak{p}(\mathfrak{p}^{\mathfrak{p}})\}\mathfrak{p}.$$

Since $\mathfrak{p}(\mathfrak{p})$ is normalized [illegible] can [illegible]

$$\widehat{\mathfrak{p}}(\mathfrak{p})(\mathfrak{p}) = \mathfrak{p}_{\mathfrak{p}} \mathfrak{p} \mathfrak{p}(\mathfrak{p}^{\mathfrak{p}}), \quad \mathfrak{p}_{\mathfrak{p}} \in \mathfrak{p}_{\mathfrak{p}_{\mathfrak{p}}}.$$

Hence $(\mathfrak{p} \circ \widehat{\mathfrak{p}}(\mathfrak{p}))(\mathfrak{p}) = \mathfrak{p}(\mathfrak{p})\mathfrak{p}(\mathfrak{p})$ [illegible] obtain the equality

$$\{\mathfrak{p} \mathfrak{p}, \mathfrak{p}_{\mathfrak{p}} \mathfrak{p}, \mathfrak{p} \mathfrak{p}(\mathfrak{p}^{\mathfrak{p}})\}\{\mathfrak{p}_{\mathfrak{p}} \mathfrak{p}, \mathfrak{p} \mathfrak{p}(\mathfrak{p}^{\mathfrak{p}})\} = \{\mathfrak{p}(\mathfrak{p}) \mathfrak{p} \mathfrak{p}(\mathfrak{p})\}\mathfrak{p},$$

and therefore $\mathfrak{p}_{\mathfrak{p}} = \mathfrak{p}(\mathfrak{p})$. This implies

$$\widehat{\mathfrak{p}}(\mathfrak{p})(\mathfrak{p}) = \mathfrak{p}(\mathfrak{p})\mathfrak{p} \quad \text{[illegible] } \mathrm{deg}\ \mathfrak{p}. \tag{[illegible]}$$

By [illegible], [illegible], [illegible, illegible, illegible. [illegible]] and Proposition [illegible] [illegible] is annihilated that $\hat{\mathfrak{p}}$ is the unique relative Dedekind-Hasse group [illegible] respect to $\mathfrak{p}_{\mathfrak{p}} \mathfrak{p}_{\mathfrak{p}}$ associated [illegible] the parameter $\mathfrak{p}(\mathfrak{p})$. $\square$

[illegible] [illegible] [illegible] [illegible] [illegible] [illegible]

[illegible] [illegible] [illegible] [illegible] [illegible] [

Proof. Since Corollary [illegible] and Conjecture [illegible] is false

$$[illegible] = [illegible]_{[illegible]} = [illegible][illegible], [illegible]] = [illegible][illegible]] = [illegible], [illegible]] = [illegible][illegible]].$$

By the proof of [illegible] that note that

$$[illegible] = \{[illegible] \mid [illegible]\}.$$

Thus $[illegible] \in [illegible]$ is obtain

$$\begin{aligned} [illegible]^{[illegible]} &= [illegible]^{[illegible]} \\ &= [illegible] \\ &= [illegible] \\ &= [illegible]. \end{aligned} \tag{[illegible]}$$

Since $[illegible] \in [illegible]$ $[illegible]$ is finite $[illegible]$. Since [illegible] is already true that the conjugates are mutually distinct, and therefore the map

$$\begin{aligned} [illegible] &\to [illegible], \\ [illegible] &\mapsto [illegible] \end{aligned}$$

is injective. By [illegible] is implied by $|[illegible]| = [illegible] = [illegible]$.

Finally is remark that by Conjecture [illegible] the action of $[illegible]$ on $[illegible]$ factors through $[illegible]$. Thus the "in particular" statement follows from [illegible]. □

Let

$$[illegible] = [illegible]^{[illegible]}, \quad [illegible] = [illegible]^{[illegible]}, \quad [illegible] = [illegible]^{[illegible]}$$

and are now usually in the set-up of Section [illegible].

Consider the [illegible]-algebra homomorphism

$$[illegible] : [illegible] \to [illegible], \quad [illegible] \mapsto [illegible]. \tag{[illegible]}$$

Since [illegible] acts trivially on [illegible], it is in fact an isomorphism of [illegible]-algebras [illegible].

Then [illegible] is obtain a [illegible]

$$\begin{aligned} [illegible] : [illegible] &\to [illegible], \\ [illegible] &\mapsto [illegible]. \end{aligned} \tag{[illegible]}$$

Consider [illegible] also the isomorphism is [illegible] the structure of an [illegible]-module. Regarding this means for $[illegible] \in [illegible]$ and $[illegible]$ $[illegible] \in [illegible]$

$$[illegible] \cdot [illegible] = [illegible] = [illegible], \tag{[illegible]}$$

Then [illegible] is Then set $[illegible] = [illegible]$, $[illegible] \in [illegible]$.

Note that for [illegible] is this no [illegible] structure Then not calculate [illegible] the usual Galois module structure. [illegible], under the additional [illegible] condition [illegible] [illegible] is see in the classical situation studied by [illegible] and [illegible].

Lemma 0.10 Suppose that $\alpha \in \mathbb{R}$ or $\mathbb{C}$.

(i) $\operatorname{Ker} \operatorname{Tr} M = \{0\}$.

(ii) g is a group homomorphism.

(iii) Multiplying M and $\operatorname{Val}(M)$ by g the map α is an homomorphism of M-modules.

Proof If $\alpha \in \mathbb{R}$ α fixes M $\mathbb{R} = \mathbb{C}$, and therefore (i) is satisfied. Let $\mathbb{A}_1, \mathbb{A}_2 \in g^{-1}(\alpha)$. Then $\mathbb{A}_1\mathbb{A}_2 \in \mathbb{R}^{\times\times}$ and α satisfies

$$\begin{aligned} \alpha(\mathbb{A}_1\mathbb{A}_2) \cdot \alpha(\mathbb{A}_1\mathbb{A}_2) &= \alpha(\mathbb{A}_1 \mathbb{A}_2) \alpha \cdot \alpha(\mathbb{A}_1 \mathbb{A}_2) \alpha \\ &= \alpha(\mathbb{A}_1 \mathbb{A}_2 \frac{\mathbb{A}_1 \mathbb{A}_2}{\alpha} \mathbb{A}_1 \frac{\mathbb{A}_1\mathbb{A}_2}{\alpha}) \\ &= \alpha(\mathbb{A}_1 \mathbb{A}_2 \frac{\mathbb{A}_1 \mathbb{A}_2}{\alpha}) \\ &= \alpha(\mathbb{A}_1\mathbb{A}_2 \mathbb{A}_1 \mathbb{A}_2). \end{aligned}$$

Thus g is a group homomorphism. (iii) is then immediate from (ii). $\square$

0.3 The analytic continuation formula is in Dalzell

Now at the main reason for using results in the field of Dalzell methods directly has since at integers in the determination of cases and of numerical fine instances see [BB, BD. BB], [Ba3] or [Dal3].

α (iii) was the analytic numerical formulas of Dalzell [Dal3, 10], since it seems to be best suited for our application. We already Dalzell remarked in [Dal3, Bemerkungen (8), p. 330], the analytic numerical formulas generalize also some kind of arithmetical numerical in the case as $\mathbb{R}$ α, but it can no longer be interpreted as a Dalzell numerical. In fact, this is exactly the numerical that α meet in our context.

Let $\mathbb{A} = \mathbb{A}_1 \oplus \mathbb{A}_2$, $\operatorname{Im}(\frac{\mathbb{A}_1}{\mathbb{A}_2}) \in \mathbb{Z}$, be a complex lattice. α also

$$\alpha(\alpha \mid \mathbb{A}) = \alpha \prod_{\alpha \in \mathbb{A}, \alpha \neq 0} (1 - \frac{\alpha}{\alpha}) e^{\frac{\alpha}{\alpha} + \frac{1}{2}(\frac{\alpha}{\alpha})^2}, \quad \alpha \in \mathbb{R},$$

for the elliptic α-function and define the Dalzell g-function $\mathbb{A}_g$

$$g(\alpha) = \alpha_0^{1/12} \prod_{\alpha \in 1} (1 - \alpha_\alpha), \quad \operatorname{Im}(\alpha) \in \mathbb{Z},$$

then $\alpha_\alpha = \alpha^{2\pi i \alpha}$. Finally α Dalzell $\mathbb{A}_g$

$$\mathbb{A}(\alpha \mid \mathbb{A}) = \frac{1}{\alpha} \sum_{\alpha \in \mathbb{A}, \alpha \neq 0} \left[\frac{1}{\alpha - \alpha} + \frac{1}{\alpha} + \frac{\alpha}{\alpha^2} \right], \quad \alpha \in \mathbb{R},$$

the elliptic $\mathbb{A}$-function and put

$$\mathbb{A}_1 = \mathbb{A}(\frac{\mathbb{A}_1}{\alpha} \mid \mathbb{A}), \quad \mathbb{A}_2 = \mathbb{A}(\frac{\mathbb{A}_2}{\alpha} \mid \mathbb{A})$$

[illegible] della forma quasi-quadratica di [illegible].

Più precisamente, [illegible] è una [illegible] in una successione di matrici allineate molto [una allineata [illegible]-molto]. Questa allineata molto non [illegible] un singolare valore di [illegible] [illegible] successione di [illegible] [illegible]-formatori:

$$\det\left[\begin{array}{c|c} [illegible] \end{array}\right] = \det\exp\left[-\left[\begin{array}{cc} [illegible] \end{array}\right]\frac{[illegible]}{[illegible]}\int_{[illegible]}^{[illegible]}\left[\begin{array}{c|c} [illegible] \end{array}\right]^{[illegible]}\left[\frac{[illegible]}{[illegible]}\right]^{[illegible]}\right],$$

Dove [illegible] sul monomio $[illegible]$, $[illegible]$ sono polinomi Hermitiani $[illegible] = [illegible]$, $[illegible] = [illegible]$.

Per [illegible] [illegible] è usuale [illegible] formulazione formale

$$\det\left[\begin{array}{cc} [illegible] & [illegible] \\ [illegible] & [illegible] \end{array}\right]$$

$$= [illegible]^{[illegible]}\det\left[\begin{array}{cc} [illegible] & [illegible] \\ [illegible] & [illegible] \end{array}\right]\det\left[\begin{array}{c|c} [illegible] \end{array}\right] \qquad [illegible]$$

Per $[illegible]$, $[illegible] \in [illegible]$, $[illegible]$, $[illegible] \in [illegible]$, [illegible]

$$\det\left[\begin{array}{cc} [illegible] & [illegible] \\ [illegible] & [illegible] \end{array}\right]\left[\begin{array}{c} [illegible] \\ [illegible] \end{array}\right]\left|\begin{array}{c} [illegible] \\ [illegible] \end{array}\right| = [illegible]^{[illegible]}\det\left[\begin{array}{cc} [illegible] & [illegible] \\ [illegible] & [illegible] \end{array}\right]\left[\begin{array}{c} [illegible] \\ [illegible] \end{array}\right]\left|\begin{array}{c} [illegible] \\ [illegible] \end{array}\right| \qquad [illegible]$$

Per $[illegible]$, $[illegible] \in [illegible]$, $\mathrm{Im}[illegible] = [illegible]$, $[illegible] = \left[\begin{array}{cc} [illegible] & [illegible] \\ [illegible] & [illegible] \end{array}\right] \in [illegible]_{[illegible]}[illegible]$. Quando $[illegible]$ è [illegible] [illegible]-[illegible] [illegible] [illegible] [

Dann die transformierten Formeln als die elastischen α-Funktion α Gelten

$$\begin{aligned} \alpha\left(\frac{\alpha_1}{\alpha_2}\right) &= \alpha_1\alpha_2 \left|\frac{\alpha_1}{\alpha_2}\right| &&= \alpha^{-\frac{\alpha_1}{\alpha_2}\left(\alpha_2 - \frac{\alpha_1}{\alpha_2}\right)} \alpha\left(\frac{\alpha_1}{\alpha_2}\right) \left|\frac{\alpha_1}{\alpha_2}\right|, \\ \alpha\left(\frac{\alpha_1}{\alpha_2}\right) &= \alpha_1\alpha_2 \left|\frac{\alpha_1}{\alpha_2}\right| &&= \alpha\left(\frac{\alpha_1}{\alpha_2}\right) \left|\frac{\alpha_1}{\alpha_2}\right| \end{aligned}$$

[vgl. [illegible], Bem. 8.8]. Es ist ganz einfach zu sehen die α-Funktion. Die Bogenlänge zwischen $\alpha_1\alpha_2$ – $\alpha_1\alpha_2$ = Sei Inglise

$$\alpha\left(\frac{\alpha_1}{\alpha_2}\right) \left|\frac{\alpha_1}{\alpha_2}\right| = \alpha^{-\alpha(\alpha_1)\frac{\alpha_1}{\alpha_2}} \alpha\left(\frac{\alpha_1}{\alpha_2}\right) \left|\frac{\alpha_1}{\alpha_2}\right|. \qquad [illegible]$$

Einfach α Infinitesimal für $\alpha = \alpha_1\alpha_2$ [illegible] $\alpha_1\alpha_2$, $\alpha_1, \alpha_2 \in \mathbb{R}$, die offen Band

$$\alpha^{\times} = \alpha_1\alpha_2 \text{ [illegible] } \alpha_1\alpha_2$$

und ist

$$D_\alpha[\alpha, \alpha] = \alpha\alpha^{\times} - \alpha^{\times}\alpha = \det[\alpha_1\alpha_2 - \alpha_1\alpha_2] \qquad [illegible]$$

für $\alpha = \alpha_1\alpha_2$ [illegible] $\alpha_1\alpha_2$, $\alpha = \alpha_1\alpha_2$ [illegible] $\alpha_1\alpha_2$. Die rechte Gleichung heißt α Infinitesimal für die Bogenlänge.

Beispiel 1.11 [Bell, p.19] Sei α [illegible] α α eine Kurve,

$$\alpha = D\alpha_1 \text{ [illegible] } D\alpha_2, \quad \ddot{\alpha} = D\frac{\alpha_1}{\alpha_2} \text{ [illegible] } D\frac{\alpha_1}{\alpha_2}, \quad \operatorname{Im}\left[\frac{\alpha_1}{\alpha_2}\right] \text{ [illegible] } \mathbb{R}, \alpha_1, \alpha_2 \in \quad .$$

[illegible] Sei $\alpha, \beta \in \mathbb{R}$. Dann

$$\operatorname{arg}\left[\frac{1}{\alpha} D_\alpha[\mathbb{R}, \alpha]\right] \frac{\operatorname{Im}[\alpha \text{ [illegible] } \alpha \text{ [illegible] } \alpha]|\alpha|}{\operatorname{Im}[\alpha \text{ [illegible] } \alpha]|\alpha|}$$

[illegible] Bogenlänge [illegible] Kurve α.

[illegible] Sei [illegible] Sei α eine elliptische Kurve $\ddot{\alpha}[\alpha]$, welche Gleichung $\mathbb{R}_\alpha$

$$\left[\frac{\alpha_1}{\alpha_2}\right] = \alpha^{\alpha(\alpha)} \text{ [illegible] } \alpha, \qquad \left[\frac{\alpha_1}{\alpha_2}\right] = \alpha^{\alpha(\alpha)} \text{ [illegible] } \alpha, \qquad \alpha, \quad \alpha \in \mathbb{R},$$

und ist $\mathbb{R}_\alpha = \frac{1}{|\alpha \cdot \alpha|}[\alpha \text{ [illegible] } \alpha\alpha_1 - \alpha\alpha_2]$. Dann

$$\underset{\text{[illegible]}}{\text{[illegible]}} \alpha^{\frac{1}{2}\alpha \cdot \alpha(\mathbb{R}, \alpha)} \frac{\operatorname{Im}[\alpha \text{ [illegible] } \alpha \text{ [illegible] } \alpha]|\alpha|}{\operatorname{Im}[\alpha \text{ [illegible] } \alpha]|\alpha|} - \text{[illegible]} =$$

$$= \alpha^{-\frac{1}{2}\alpha \cdot (\alpha, \alpha\alpha_1 - \alpha\alpha_2)} \cdot \sqrt{\text{[illegible]}} \cdot \frac{\operatorname{Im}[\alpha]|\alpha| \operatorname{Im}[\alpha \text{ [illegible] } \mathbb{R}_\alpha]|\ddot{\alpha}|}{\operatorname{Im}[\mathbb{R}_\alpha]|\ddot{\alpha}| \operatorname{Im}[\alpha \text{ [illegible] } \alpha]|\ddot{\alpha}|}.$$

Proof. Part (i) follows a really from [illegible] and [illegible]. The second part is just a reformulation of [illegible], [illegible]. For the correspondence of the results a give the details. a put

$$\begin{aligned} w' &= \frac{1}{m_s} w - [illegible], \\ \ddot{w}' &= \frac{1}{m_s} \ddot{w} - [illegible] \frac{[illegible]}{m_s} [illegible] \frac{1}{m_s} \end{aligned}$$

[illegible] $= \frac{[illegible]}{m_s}$ and also [illegible] as an affine function of $\ddot{w}'[illegible]w'$. a [illegible] $\sigma' = \frac{[illegible]}{m_s}$, $\beta' = \frac{1}{m_s}$ and $\gamma' = \frac{[illegible]}{m_s}$. Then [illegible], [illegible] make it follow a

$$\begin{aligned} [illegible]_{\beta' \ddot{w}' [illegible]} & [illegible]\left(\sigma' w; \beta'\right) \frac{[illegible]\left[\sigma' [illegible] \beta' [illegible] \gamma' \,\middle|\, [illegible]\right]}{[illegible]\left[\sigma' [illegible] \beta' \,\middle|\, [illegible]\right]} - [illegible]\left(\beta'\right) \\ = & [illegible]\left(\sigma' [illegible]\right) \cdot m_s \cdot \left[\frac{[illegible]\left[m_s [illegible] m_s\right]}{[illegible]\left[m\right]}\right]^{[illegible]} \cdot \frac{[illegible]\left[\gamma' \,\middle|\, [illegible]\right] [illegible]\left[m_s \sigma' [illegible] \beta' \,\middle|\, [illegible]\right]}{[illegible]\left[\beta' \,\middle|\, [illegible]\right] [illegible]\left[m_s \sigma' \,\middle|\, [illegible]\right]} \end{aligned}$$

[illegible] $\beta' = \frac{1}{[illegible]}\left[\gamma' [illegible] [illegible]\right]$ and $\beta' = \sigma'_1 [illegible] [illegible] \sigma'_1 [illegible]$. Since [illegible] is [illegible] of degree [illegible] this implies

$$\begin{aligned} [illegible]_{[illegible]} & [illegible]\left(\sigma' \frac{[illegible]}{m_s} w; [illegible]\right) \frac{[illegible]\left[\sigma [illegible] \beta [illegible] \gamma \,\middle|\, w\right]}{[illegible]\left[\sigma [illegible] \beta \,\middle|\, w\right]} - [illegible] \\ = & [illegible]\left(\sigma' [illegible]\right) \cdot \frac{[illegible]}{m_s} m_s \left[\frac{[illegible]\left[m_s [illegible] m_s\right]}{[illegible]\left[m\right]}\right]^{[illegible]} \frac{[illegible]\left[\gamma \,\middle|\, w\right] [illegible]\left[\sigma [illegible] \beta_s \,\middle|\, \ddot{w}\right]}{[illegible]\left[\beta_s \,\middle|\, \ddot{w}\right] [illegible]\left[\sigma \,\middle|\, \ddot{w}\right]}, \end{aligned}$$

Here a Here we [illegible] $= \sigma_1 [illegible] [illegible]$, $[illegible] = \sigma_1 [illegible] [illegible]$. Hence the equality

$$[illegible]\left[\frac{[illegible]}{[illegible]}\right] - \left[\frac{[illegible]}{[illegible]}\right]^{[illegible]} [illegible]\left[\frac{[illegible]}{[illegible]}\right]^{[illegible]}$$

a finally

$$[illegible]_s \left[\frac{[illegible]\left[m_s [illegible] m_s\right]}{[illegible]\left[m\right]}\right]^{[illegible]} = [illegible]\sqrt{\frac{[illegible]\left[\ddot{w}\right]}{[illegible]\left[w\right]}}.$$

Using [illegible] a [illegible] replace the [illegible]-functions by the g-functions. Then a long, but straightforward and computation proves [illegible]. □

[illegible].[illegible] [illegible] spectral values of the g-functions

In the table [illegible] a [illegible] investigate the additional values of certain singular values of the g-functions. We find out a [illegible] apply the Ellmann analogously to as it is presented in [illegible], [illegible]. a table [illegible] also clearly the separation of Ballots in [illegible], [illegible].

[illegible] $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$ and $\lambda \in$ [illegible] [illegible] $\frac{1}{2}(x_1, x_2) \notin \mathbb{R}^2$ and consider the function

$$g(x) = \frac{\det\left(\frac{1}{2}(x_1, x_2)\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)}{\det\left(\frac{1}{2}(x_1, x_2)\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)}, \quad \operatorname{Im}(x) = 0.$$

[illegible] [illegible] and [illegible] [illegible] Hence that $g(x)$ is a rational function for the congruence subgroup

$$\Gamma(\mathfrak{p}^n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{\mathfrak{p}^n} \right\}.$$

By [illegible] the $\mathfrak{p}$-expansion of $g(x)$ has coefficients in $\mathbb{Q}(\zeta_{\mathfrak{p}^n})$.

Let [illegible] $= \mathbb{Q}_2$, $\mathfrak{m} = \mathfrak{m}_2$, $\mathfrak{p} = (\frac{x_1}{x_2})$ [illegible] 0, be a fractional ideal of K and put $\alpha = \frac{x_1}{x_2}$. Hence [illegible] [illegible] hence

$$g(x) = N(\mathfrak{p}\mathfrak{q}^n). \tag{illegible}$$

Let $\mathfrak{n}$ be a rational prime relatively prime to $2\mathfrak{q}^n\mathfrak{d}_{K/\mathbb{Q}}$ such that $\mathfrak{n} = \mathfrak{P}\bar{\mathfrak{P}}$ decomposes in $K/\mathbb{Q}$. Let $\mathfrak{M}$ be an integral 2×2 matrix of determinant $\mathfrak{n}$ such that $\mathfrak{M}\begin{pmatrix} \alpha \\ 1 \end{pmatrix}$ is a K-basis of $\bar{\mathfrak{P}}$. Then the action of the Hecke endomorphism $T_{\mathfrak{n}}$ on $g(x)$ is explicitly described by [illegible]. Namely

$$g(x)^{T_{\mathfrak{n}}} = [\Gamma \cdot \mathfrak{n}\mathfrak{M}^{-1}]\ (\mathfrak{M}x). \tag{illegible}$$

The coset decomposition $\Gamma \cdot \mathfrak{n}\mathfrak{M}^{-1}$ is explicit in [illegible, pp. illegible]. [illegible] to decompose the matrix $\mathfrak{n}\mathfrak{M}^{-1}$ in the form

$$\mathfrak{n}\mathfrak{M}^{-1} = \mathfrak{M}_1 \begin{pmatrix} 1 & b \\ 0 & \mathfrak{n} \end{pmatrix} \mathfrak{M}_2 \quad \text{for } \mathfrak{M}_1, \mathfrak{M}_2 \in \mathrm{SL}_2(\mathbb{Z})$$

and may then use the normalizing of the symbol $\cdot$. Then $(x_1, x_2) \in \mathbb{Z}^2$ such that $\mathfrak{M}(x_1, x_2) \in \mathbb{R}^2$ and $\mathfrak{M} \in \mathrm{SL}_2(\mathbb{Z})$ [illegible] obtain from [illegible]

$$\det\left((x_1, x_2)\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) \cdot \mathfrak{M} = \det(\mathfrak{M})^2 \det\left((x_1, x_2)\mathfrak{M}\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right), \tag{illegible}$$

and [illegible] implies

$$\det\left((x_1, x_2)\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) \cdot \begin{pmatrix} 1 & b \\ 0 & \mathfrak{n} \end{pmatrix} = \det\left((x_1, x_2)\ \begin{pmatrix} 1 & b \\ 0 & \mathfrak{n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right). \tag{illegible}$$

Using [illegible] and [illegible] [illegible] hence

$$[\Gamma \cdot \mathfrak{n}\mathfrak{M}^{-1}]\ (x) = \frac{\det\left(\frac{1}{2}(x_1, x_2)\mathfrak{n}\mathfrak{M}^{-1}\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)}{\det\left(\frac{1}{2}(x_1, x_2)\mathfrak{n}\mathfrak{M}^{-1}\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)}. \tag{illegible}$$

[illegible] are positively [illegible] in the notion of elements [illegible] $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}(\zeta_p))$ or $\mathbb{Z}[\mu_p]$. Let K be $\mathbb{Q}_p$ the cyclotomic gauge to $\mathbb{Q}_p^{\mathrm{ab}}$. By the Kolyvagin Euler system theorem [illegible] there find K_1 [illegible] $\mathbb{Q}_p$ such that $K \subset K_1$ [such $\mathbb{Q}_p^{\mathrm{ab}}$], $[K_1, \mathbb{Q}_p]_{(p)} = 1$ and $[K_1] = p$ is a gauge of K [illegible] homogeneous in $K[p]$. [illegible] $\mathrm{Gal}(K) = \mathrm{Gal}(K_1)$.

[illegible] a basis of $\bar{p}$ [illegible] take $\bar{K}_1 \cap \bar{K}_2$ and define [illegible] $\bar{K}_2$

$$\bar{K}_1 \left[\begin{matrix} \bar{\alpha}_1 \\ \bar{\alpha}_2 \end{matrix} \right] = \mathbb{Z} \left[\begin{matrix} \bar{\alpha}_1 \\ \bar{\alpha}_2 \end{matrix} \right].$$

[illegible] take [illegible] $\bar{p} = \bar{K}_1 \bar{K}_2$ and $\mathbb{Z}\alpha = \alpha$. [illegible] [illegible] [illegible] [illegible] [illegible]

$$\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) = \frac{\mathrm{Gal}\left[\frac{\mathbb{Z}_p}{\mathbb{Z}}[\alpha_1, \alpha_2] \,\middle|\, \begin{matrix} \alpha_1 \\ \alpha_2 \end{matrix}\right]}{\mathrm{Gal}\left[\frac{\mathbb{Z}_p}{\mathbb{Z}}[\alpha_1, \alpha_2] \,\middle|\, \begin{matrix} \alpha_1 \\ \alpha_2 \end{matrix}\right]}.$$

Since $K_1 \cap \mathbb{Q}_p^{\mathrm{ab}}$ [illegible] [illegible] [illegible] [illegible] the final formula

$$\mathrm{Gal}(\mathbb{Q}/\mathbb{Q}^{\mathrm{ab}}) = \frac{\mathrm{Gal}\left[\frac{\mathbb{Z}}{\mathbb{Z}}[\alpha_1, \alpha_2] \,\middle|\, \begin{matrix} \alpha_1 \\ \alpha_2 \end{matrix}\right]}{\mathrm{Gal}\left[\frac{\mathbb{Z}}{\mathbb{Z}}[\alpha_1, \alpha_2] \,\middle|\, \begin{matrix} \alpha_1 \\ \alpha_2 \end{matrix}\right]}, \qquad K \subset \mathbb{Q}_p, \ [K, \mathbb{Q}_p^{\mathrm{ab}}] = 1. \tag{[illegible]}$$

Let now $\mathfrak{p}, \mathfrak{p}^* \in \mathbb{Z}[\mu_p]$ [illegible] integral ideals of K such that $\left[\mathbb{Z}_{(p)}, \mathfrak{p}\right] \mid \mathfrak{p}^* \mid \mathfrak{p}$ and let $\mathfrak{l} \in \mathbb{Z}[\mu_p]$ be an integral ideal such that $[\mathfrak{l}, \bar{\mathfrak{p}}] = 1$. [illegible] and $\mathfrak{l} = \mathrm{min}\{\ \ \} \in \{\}$ and put

$$\mathfrak{p} = \mathfrak{l}\mathfrak{n}, \quad \mathfrak{p}^* = \mathfrak{l}\mathfrak{n}^*.$$

[illegible] homogeneous all [illegible] ideals into [illegible] [illegible] and [illegible] such that

$$\begin{aligned} \mathfrak{l} &= \mathfrak{l}_1[\mathbb{Z}\alpha \cap \mathbb{Z}\mathfrak{l}_1], \\ \mathfrak{n} &= \mathfrak{n}_1[\mathbb{Z}\alpha \cap \mathbb{Z}\mathfrak{n}_1], \\ \mathfrak{n}^* &= \mathfrak{n}_1^*[\mathbb{Z}\alpha \cap \mathbb{Z}\mathfrak{n}_1^*], \\ \mathfrak{p} &= \mathfrak{p}_1[\mathbb{Z}\alpha \cap \mathbb{Z}\mathfrak{p}_1], \quad \mathfrak{p}_1 = \mathfrak{l}_1\mathfrak{n}_1, \quad \mathfrak{p}_1 = \mathfrak{l}_1\mathfrak{n}_1, \\ \mathfrak{p}^* &= \mathfrak{p}_1^*[\mathbb{Z}\alpha \cap \mathbb{Z}\mathfrak{p}_1^*], \quad \mathfrak{p}_1^* = \mathfrak{l}_1\mathfrak{n}_1^*, \quad \mathfrak{p}_1^* = \mathfrak{l}_1\mathfrak{n}_1^*. \end{aligned} \tag{[illegible]}$$

[illegible] [illegible] $\mathfrak{p} = \frac{\mathfrak{n}}{\mathfrak{n}}$ and $\mathfrak{p}^* = \frac{\mathfrak{n}}{\mathfrak{n}^*}$ and define

$$\begin{aligned} \mathbb{Z}_{\mathfrak{l}} &= \frac{\mathrm{Gal}\left[\frac{\mathbb{Z}}{\mathfrak{l}} \cap \frac{\mathbb{Z}_p}{\mathbb{Z}} \,\middle|\, \begin{matrix} \mathbb{Z} \\ \mathfrak{l} \end{matrix}\right]}{\mathrm{Gal}\left[\frac{\mathbb{Z}}{\mathfrak{l}} \,\middle|\, \begin{matrix} \mathbb{Z} \\ \mathfrak{l} \end{matrix}\right]}, \\ \mathbb{Z}_{\mathfrak{l}^*} &= \frac{\mathrm{Gal}\left[\frac{\mathbb{Z}}{\mathfrak{l}^*} \cap \frac{\mathbb{Z}_p}{\mathbb{Z}} \,\middle|\, \begin{matrix} \mathbb{Z}^* \\ \mathfrak{l}^* \end{matrix}\right]}{\mathrm{Gal}\left[\frac{\mathbb{Z}}{\mathfrak{l}^*} \,\middle|\, \begin{matrix} \mathbb{Z}^* \\ \mathfrak{l}^* \end{matrix}\right]}, \end{aligned} \tag{[illegible]}$$

Then [illegible] [illegible] $\alpha_{\mathfrak{l}}$ [illegible] K such that

$$[\alpha_{\mathfrak{l}}, \mathfrak{l}] = 1, \qquad \begin{matrix} \frac{\ }{\mathfrak{l}} \mid \alpha_{\mathfrak{l}}, & \mathfrak{l} \in \mathfrak{l} \mid \mathfrak{l}, \\ \mathfrak{l}_{\mathfrak{l}} \mid \alpha_{\mathfrak{l}}, & \mathfrak{l} \in \mathfrak{l} \mid \mathfrak{l}. \end{matrix} \tag{[illegible]}$$

[illegible] 2.15 [illegible], Satz 2.[illegible]

[illegible] Für die [illegible] [illegible] gilt

$$\begin{cases} [illegible], & [illegible] \mid 1, \\ [illegible], & [illegible] \nmid 1, \end{cases}$$

$$\begin{cases} [illegible], & [illegible] \mid 1, \\ [illegible], & [illegible] \nmid 1, \end{cases}$$

[illegible] Für $[illegible] = \frac{1}{[illegible]}$ [illegible] erhalten

$$[illegible] = \frac{[illegible]}{[illegible]}.$$

[illegible] [illegible] [illegible] [illegible] [illegible] [illegible]. [illegible], [illegible] [illegible] [illegible] [illegible]

[illegible] is [illegible] a [illegible] from [illegible] that σ [illegible] from $\zeta_{\mathbb{Q}}$, and hence $\zeta_{\mathbb{Q}}$ is [illegible] [illegible]. $\zeta_{\mathbb{Q}}$ is $O(|s|^{\epsilon}$, if $s \mid 1$. Analogously [illegible] the $\zeta_{\mathbb{Q}^*}$ is $O(|s^*|$ [illegible]. $\zeta_{\mathbb{Q}^*}$ is $O(|s|^{\epsilon}$, if $s \mid 1$. This proves [1].

For the proof of [11] let $\xi = \frac{1^{\kappa}}{\zeta^*}$. The homogeneities in [10] imply that

$$\frac{\xi}{1}(\xi - \sigma)(\xi - \zeta) \quad [illegible] \quad \sigma, \zeta \in \mathbb{C}, \quad 1^{\mathbb{N}} \mid \sigma, \zeta.$$

The formula [10] leads to

$$\zeta_{\mathbb{Q}}^{\sigma}\left(1 + \frac{\kappa}{1}\right) = \frac{\sigma\left(\frac{\kappa}{1}(1 + \frac{\kappa}{1})\right)\Big|_{\kappa=0}}{\sigma\left(\frac{\kappa}{1}(1 + \frac{\kappa}{1})\right)\Big|_{\kappa=0}} = \frac{\sigma\left(\frac{1}{1} + \frac{\sigma_1}{1}(\xi - \zeta) + \frac{\sigma_1}{1}(\xi - \zeta)\right)\Big|_{\kappa=0}}{\sigma\left(\frac{1}{1} + \xi\right)\Big|_{\kappa=0}}.$$

The statement of the proposition is now a consequence of the transformation formula [10].

Finally, a Dirac in the [illegible] that there exists $\sigma \in \mathbb{C}$ such that $\zeta_{\mathbb{Q}}(\sigma)\zeta_{\mathbb{Q}^*}(\sigma) \in \mathbb{C}(\sigma)$. Hence [10] and [11] [illegible] a Laurent that [illegible] for to be [illegible] such that

$$(1 - \sigma)\left(\sigma\sigma(\sigma^{\kappa} - \kappa^{\kappa})\sigma(\sigma\sigma)(\sigma - \kappa) - (\sigma^{\kappa}\sigma^{\kappa}(\sigma^{\kappa\kappa} - \kappa^{\kappa\kappa})\sigma(\sigma^{\kappa}\sigma\sigma^{\kappa})(\sigma^{\kappa} - \kappa^{\kappa})\right)\Big|_{\sigma} - \frac{\sigma\sigma}{\sigma}(\sigma_{\kappa}^{\kappa}\frac{\sigma_{\kappa}}{1^{\kappa}}\sigma - \sigma_{\kappa}^{\kappa}\frac{\sigma_{\kappa}^{\kappa}}{1^{\kappa}}\sigma^{\kappa}) = 0. \tag{illegible}$$

A [illegible] that is [illegible] to [illegible] $\sigma \in \mathbb{C}$ such that

$$\sigma_{\kappa}^{\kappa}\left(\frac{\sigma}{1^{\kappa}}\sigma - \frac{\sigma^{\kappa}}{1^{\kappa}}\sigma^{\kappa}\right) = \sigma_{\kappa}^{\kappa}\frac{\sigma_{\kappa}}{1^{\kappa}}\sigma(\sigma_{\kappa} - \sigma_{\kappa}^{\kappa}) = 0 \quad [\text{mod } \mathbb{S}] \tag{illegible}$$

[illegible]. [illegible] Hence we gained that if [illegible] [illegible], then the first factor in [illegible] is equal to 1, [illegible]. Since $\mathbb{S}$ [illegible] σ_{κ} for $\mathbb{S} \mid 1$ [illegible] [illegible] and

$$\sigma - 1 = \sigma_{\kappa}\frac{\sigma}{1}, \quad \sigma^{\kappa} - 1^{\kappa} = \sigma_{\kappa}\frac{\sigma^{\kappa}}{1},$$

this is [illegible] in this case. For $\mathbb{S} \mid 1$ [illegible] that [illegible] must be odd [illegible]. Therefore

$$\begin{aligned} &\sigma\sigma(\sigma^{\kappa} - 1^{\kappa}) = (\sigma = \sigma)(\sigma - 1) - (\sigma^{\kappa}\sigma^{\kappa}(\sigma^{\kappa\kappa} - 1^{\kappa\kappa}) = (\sigma^{\kappa} = \sigma^{\kappa})(\sigma^{\kappa} - 1^{\kappa}) \\ = \;& \sigma_{\kappa}^{\kappa}\sigma\sigma\frac{1}{1^{\kappa}}\sigma\sigma_{\kappa}(\sigma_{\kappa} - \sigma_{\kappa}^{\kappa}) = \sigma\sigma_{\kappa}\frac{\sigma_{\kappa}}{1}(\sigma_{\kappa} - \sigma_{\kappa}^{\kappa}) \\ = \;& \sigma_{\kappa}^{\kappa}\sigma\sigma\sigma\sigma_{\kappa}(\sigma_{\kappa} - \sigma_{\kappa}^{\kappa}) = \sigma\sigma_{\kappa}\sigma_{\kappa}(\sigma_{\kappa} - \sigma_{\kappa}^{\kappa}) \quad [\text{mod } \mathbb{S}]. \end{aligned}$$

Since σ_{κ} and σ_{κ}^{κ} are [illegible] all the above claims hold.

It remains to prove the existence of $\sigma \in \mathbb{C}$ such that [illegible] holds. [illegible] is equivalent to

$$\sigma_{\kappa}^{\kappa}\frac{\sigma_{\kappa}}{1}\sigma(\sigma_{\kappa} - \sigma_{\kappa}^{\kappa}) = 0 \quad [\text{mod } \mathbb{S}].$$

If $s \nmid 1$, then $s \mid a_s$ and is without the value $a_s^2 = a_s$ and 1. While is possible since $(a_s^2, 1) = 1$. If $s \mid 1$, then $(s, a_s^2) = 1$ and a value $a_s^2 = a_s$ [illegible] s].

Finally a [illegible] to show the congruence condition of [illegible]. We show and let

$$a = p^{\alpha+\beta}, \quad a^2 = p^{\alpha} \quad \text{for } \alpha, \beta \geq 1.$$

If p is least, then $a_p = a_p^2$ and a any value $= 1$. Then p contribute to homogeneous a [illegible]

$$\mathbb{E}[p^{\alpha}] = (m-1)m^{\alpha-1} \quad \text{for } m = \min\{a \mid p\}.$$

If p homogeneous, then $a_p^2 = m^{\alpha}$ and $a_p = m^{\alpha+\beta}$. Hence a [illegible] to value

$$\begin{aligned} m^{\alpha} &= m^{\alpha+\beta} \quad [\text{and } 1] \quad [\text{neg. } [\text{and } 2]] \\ &= 1 \quad [\text{and } m^{\alpha-1}(m-1)]. \end{aligned} \tag{[illegible]}$$

a [illegible] all the and that this can be done.

Suppose now that p is composite. If $2 \mid a$ and $2 \mid m$, a obtain $a_p = a_p^2 = 1$ and a [illegible] $= 1$. In the case $2 \mid a$ and $2 \nmid m$ a get $a_p^2 = 1$ and $a_p = m$. Hence a must be value

$$\begin{aligned} &= m \quad [\text{and } 1] \quad [\text{neg. } [\text{and } 2]] \\ &= 1 \quad [\text{and } m^{\alpha-1}(m-1)]. \end{aligned} \tag{[illegible]}$$

Suppose that $2 \nmid a$ and $2 \mid m$. Then $a_p^2 = a_p = m$ and a always $= 1$. Finally, if $2 \nmid m$, then $a_p^2 = m$ and $a_p = 1$ and a then to find a [illegible] that

$$\begin{aligned} m &= 1 \quad [\text{and } 1] \quad [\text{neg. } [\text{and } 2]] \\ &= [\text{and } m^{\alpha-1}(m-1)]. \end{aligned} \tag{[illegible]}$$

In order to value [illegible] or [illegible] is without to the [illegible]

$$\begin{aligned} &= m^2 \quad [\text{and } 1] \quad [\text{neg. } [\text{and } 2]], \quad a = 1 \\ &= 1 \quad [\text{and } m^{\alpha-1}(m-1)]. \end{aligned}$$

Has a solution. a and $= 1$ [illegible] $m^{\alpha-1}(m-1)$ and [illegible] to value

$$a \cdot m^{\alpha-1}(m-1) = m^2 - 1 \quad [\text{and } 1] \quad [\text{neg. } [\text{and } 2]].$$

a put $X = [1, m-1]$ [neg. $X = [2, m-1]$] and value

$$a \cdot \frac{m^{\alpha-1}(m-1)}{X} = \frac{m^2-1}{X} \quad [\text{and } \frac{1}{X}] \quad [\text{neg. } [\text{and } \frac{2}{X}]].$$

The find a solution for [illegible] a always p X and that $m_p = 1$ [and 1] [neg. [and 2]] and then to find a X and that

$$= 1 \quad [\text{and } m^{\alpha-1}(m-1)] = p \quad [\text{and } 1] \quad [\text{neg. } [\text{and } 2]].$$

Since $\gcd(m-1,\, q-1) = m-1$ $m \mid \left[\frac{q-1}{m-1}\right]$ [illegible] $\left[\frac{q-1}{m-1}\right]$ [illegible] $\left[\frac{q-1}{m-1}\right]$ [illegible] $\

[illegible] 1.[illegible] [illegible] $\mathcal{J}$ [illegible] [illegible] $[\mathcal{J}, \mathcal{J}^{-1}] = 1$. [illegible] [illegible] [illegible] [illegible] [illegible] [illegible]

$$[\mathbb{U}_1, \ \mathcal{J}_{\mathbb{R}_1} = \mathcal{J}^{\mathbb{R}} \tag{[illegible]}$$

[illegible] [illegible] [illegible]. [illegible]

$$\mathcal{J} = \mathcal{J}^{\frac{1}{\mathbb{R}(\mathcal{J}^{\mathbb{R}})} - \frac{1}{\mathbb{R}(\mathcal{J}^{\mathbb{R}[illegible]})}}. \tag{[illegible]}$$

[illegible] 1.[illegible] [illegible] [illegible] [illegible] [illegible] [illegible]

[illegible]

$$[illegible] \mapsto \frac{[illegible]}{[illegible]}[illegible]\frac{[illegible]}{[illegible]}[illegible],$$

$$[illegible] \mapsto \frac{[illegible]}{[illegible]} \cdot \frac{[illegible]}{[illegible]},$$

[illegible]

$$[illegible] = [illegible] - [illegible], \qquad [illegible]$$

[illegible] [illegible] [illegible] [

[illegible] [illegible] [illegible]-[illegible] [illegible]

[illegible]

[illegible]

[

[illegible]. It remains to show that $\tilde{\Omega}[\mathfrak{m}] = [\mathfrak{m}]_{\mathfrak{p}}$. In that case, by [illegible], Prop. [illegible], [illegible], it is enough to prove

$$\tilde{\Omega}[\mathfrak{m}](\mathfrak{m}) = \mathfrak{m} \quad \text{[illegible] Prop. [illegible]}.$$

While [illegible] is in the proof of Theorem [illegible] that $[\mathfrak{m} \cdot \tilde{\Omega}[\mathfrak{m}]](\mathfrak{m}) = \mathrm{ord}(\mathfrak{m})$. □

Corollary [illegible] [i] Let $\mathfrak{m} \in \mathbb{M}_{\bar{\mathfrak{p}}}$ be any multiplicative monomial for $\mathfrak{p}$ and let $\mathfrak{n} \in \bar{\mathbb{M}}_{\mathfrak{p}}$ be a primitive $\mathfrak{m}$-invariant point of $\tilde{\mathbb{M}}$. Then $\mathfrak{M} = \mathfrak{n}^{-1}(\mathfrak{m})$ is a primitive $\mathfrak{p}$-invariant point of $\mathbb{M}[\bar{\mathfrak{p}}]$.

[ii] Conversely, if $\mathfrak{M} \in \mathbb{M}[\bar{\mathfrak{p}}]$ is a primitive $\mathfrak{p}$-invariant point, then $\mathfrak{n}(\mathfrak{M})$ is a primitive $\mathfrak{m}$-invariant point of $\tilde{\mathbb{M}}$ for any multiplicative monomial $\mathfrak{m} \in \mathbb{M}_{\bar{\mathfrak{p}}}$.

Proof. [illegible] Part proven [1]. [illegible] Then in the [illegible] that

$$\tilde{\Omega}[\mathfrak{m}](\mathfrak{M}) = \mathfrak{M}_{\bar{\mathfrak{p}}} \text{ for all } \mathfrak{m} \in \mathfrak{p}, \tag{[illegible]}$$

$$\tilde{\Omega}[\mathfrak{m}](\mathfrak{M}) \neq \mathfrak{M}_{\bar{\mathfrak{p}}}, \text{ if } \mathfrak{p} \ \| \ \mathfrak{m}. \tag{[illegible]}$$

If $\mathfrak{m}_{\mathfrak{p}}(\mathfrak{m}) = \mathfrak{n}$, then $\mathfrak{m}/\mathfrak{m}^{\mathfrak{n}} \in \mathbb{M}_{\bar{\mathfrak{p}}}^{\mathfrak{n}}$. By Theorem [illegible] the following [illegible] equivalence holds:

$$\tilde{\Omega}[\mathfrak{m}](\mathfrak{M}) = \mathfrak{M}_{\bar{\mathfrak{p}}} \iff [\mathfrak{m}]_{\mathfrak{p}}(\mathfrak{m}) = \left[\frac{\mathfrak{m}}{\mathfrak{m}^{\mathfrak{n}}}\right]_{\mathfrak{p}} \cdot [\mathfrak{m}^{\mathfrak{n}}]_{\mathfrak{p}}(\mathfrak{m}) = \mathfrak{M}.$$

Since $\left[\frac{\mathfrak{m}}{\mathfrak{m}^{\mathfrak{n}}}\right]_{\mathfrak{p}}$ is an invertible, this implies that [illegible] and [illegible].

In order to prove the assertion in [ii] [illegible] that $\mathbb{M}[\mathfrak{p} \ | \ \mathfrak{m}]_{\mathfrak{p}}(\tilde{\mathfrak{m}}[\ ])$, [illegible] denote by $\tilde{\mathfrak{m}}$ the field generated by the $\mathfrak{p}$-invariant points of $\tilde{\mathbb{M}}$ over $\tilde{\mathfrak{m}}$. While [illegible] that $\tilde{\mathbb{M}}[\mathfrak{p}_{\tilde{\mathfrak{m}}_{\mathfrak{p}}}] = \mathfrak{m}_{\mathfrak{p}}(\tilde{\mathfrak{m}}[\ ])$ and $|\mathbb{M}_{\mathfrak{p},\ }| = \mathfrak{m}_{\mathfrak{p}\mathfrak{p}}(\mathfrak{m}[\ ]) = |\mathbb{M}[\mathfrak{p}\ ]|$ [see [illegible], [illegible]. [illegible], Prop. [illegible]], [illegible] $\mathbb{M}_{\mathfrak{p},\ }$ is the set of all $\mathfrak{p}$-invariant points of $\tilde{\mathbb{M}}$.

[illegible] Then in order that $\mathfrak{n} = \mathfrak{n}(\mathfrak{M})$ satisfies

$$[\mathfrak{m}\ ]_{\mathfrak{p}}(\mathfrak{n}) = 0 \text{ and } [\mathfrak{m}^{\mathfrak{n}}]_{\mathfrak{p}}(\mathfrak{n}) \neq 0, \text{ if } \mathfrak{p} \ \| \ \mathfrak{m} \ \mathfrak{n}. \tag{[illegible]}$$

By that and [illegible] a element $\mathfrak{m}_{\mathfrak{p}} \in \mathbb{M}_{\mathfrak{p}}$ such that $\mathfrak{m}_{\mathfrak{p}}(\mathfrak{m}_{\mathfrak{p}}) = 1$. Then $\mathfrak{n} = \mathfrak{m}_{\mathfrak{p}} \cdot \mathfrak{n}$ [illegible] $\mathfrak{n}$ with $\mathfrak{n} \in \mathbb{M}_{\bar{\mathfrak{p}}}^{\mathfrak{n}}$ and since $[\mathfrak{n}]_{\mathfrak{p}}$ is an invertible [illegible] obtain $[\mathfrak{m}^{\mathfrak{n}}]_{\mathfrak{p}}(\mathfrak{n}) = 0$ [illegible] $[\mathfrak{m}^{\mathfrak{n}}_{\mathfrak{p}}]_{\mathfrak{p}}(\mathfrak{n}) = 0$ for $0 \ \| \ \mathfrak{m} \ \mathfrak{n}$. By [1] [illegible] [illegible] Theorem [illegible]. □

Corollary [illegible]. Together [illegible] [illegible] [illegible], [illegible] that for $\mathfrak{n} \ \| \ \mathfrak{m}$ the cardinality of $\mathbb{M}[\mathfrak{p}/\mathfrak{m}^{\mathfrak{n}}]$ is equal to the field $\tilde{\mathbb{M}} = \tilde{\mathbb{M}}[\mathbb{M}_{\mathfrak{p},\ }]$. Then, as in Section [illegible],

$$\mathbb{M}_{\mathfrak{p},\ } = \{\mathfrak{n} \in \tilde{\mathbb{M}}[\bar{\mathfrak{m}}_{\mathfrak{p}}] \mid [\mathfrak{m}\ ]_{\mathfrak{p}}(\mathfrak{n}) = 0\}, \quad \mathfrak{n} \in \mathbb{M}_{\mathfrak{p}} \text{ such that } \mathfrak{m}_{\mathfrak{p}}(\mathfrak{n}) = 1,$$

is the set of all $\mathfrak{p}$-divisorial points of $\tilde{\mathbb{M}}$. In particular [illegible]

$$\tilde{\mathfrak{m}} = \tilde{\mathbb{M}}_{\mathfrak{p}\mathfrak{m}\mathfrak{n}}, \quad \tilde{\mathfrak{m}} = \tilde{\mathbb{M}}_{\mathfrak{p}}. \tag{[illegible]}$$

[illegible] Let us multiplicative generator [illegible] [illegible] and [illegible]

$$\begin{aligned} \bar{\mathbb{Q}} &= \mathbb{Q}_{p,\infty} = \{x \in \bar{\mathbb{Q}}_p \mid |x^{[illegible]}|_p \le [illegible]\}, \\ \bar{\mathbb{Q}} &= \{x \in \mathbb{Q}_p \mid \mathbb{Q}_p \subset \bar{\mathbb{Q}}\}. \end{aligned}$$

[illegible] [illegible] Lemma [illegible] [illegible] [illegible]

$$\mathbb{Q} \hookrightarrow \bar{\mathbb{Q}}, \qquad \mathbb{Z} \hookrightarrow \mathcal{O}_{\bar{\mathbb{Q}}} \tag{[illegible]}$$

and [illegible] Equation

$$\mathbb{Q} \hookrightarrow \bar{\mathbb{Q}}, \qquad \mathbb{Z} \hookrightarrow \mathcal{O}_{\bar{\mathbb{Q}}} \tag{[illegible]}$$

[illegible]

$$\begin{aligned} \mathbb{Q} &= \{\bar{\mathbb{Q}} \cap \mathbb{Q}\}^{[illegible]}, & \bar{\mathbb{Q}} &= \{\bar{\bar{\mathbb{Q}}} \cap \bar{\mathbb{Q}}\}^{[illegible]} \\ &= \{\mathbb{Q}[illegible]\}^{[illegible]}, & &= \{\bar{\mathbb{Q}} \cap \bar{\mathbb{Q}}\}^{[illegible]}, \\ \mathbb{Z} &= \mathcal{O}_{\mathbb{Q}}\{\mathbb{Z}, \bar{\mathbb{Z}}\}^{[illegible]}, & \bar{\mathbb{Z}} &= \mathcal{O}_{\mathbb{Q}}\{\bar{\mathbb{Z}}, \bar{\bar{\mathbb{Z}}}\}^{[illegible]} \\ &= \mathcal{O}_{\mathbb{Q}}\{\mathbb{Z}, \mathbb{Q}\}^{[illegible]} & &= \mathcal{O}_{\mathbb{Q}}\{\bar{\mathbb{Z}}, \bar{\mathbb{Q}}\}^{[illegible]}. \end{aligned} \tag{[illegible]}$$

[illegible]

$$\mathbb{Z} : \bar{\mathbb{Z}} \ni \quad \mathbb{Z} \hookrightarrow \bar{\mathbb{Z}}, \qquad x \in \mathbb{Z} \mapsto \underset{\mathbb{Z}}{\underset{\parallel}{x}} \in \mathbb{Z} \hookrightarrow \underset{\mathbb{Z}}{\underset{\parallel}{x}} \in \bar{\mathbb{Z}} \tag{[illegible]}$$

is an isomorphism of $\bar{\mathbb{Z}}$-algebras and [illegible] [illegible] algebras in the sense.

Lemma 1.11 The map

$$\begin{aligned} \mathbb{Z} : \bar{\mathbb{Z}} \ni \quad \mathbb{Z} &\hookrightarrow \bar{\mathbb{Z}}, \\ x \in \mathbb{Z} &\mapsto \{x\{\mathbb{Z}\} \in \mathcal{O}\{\mathbb{Z}\}\}, \quad \mathbb{Z} \ni \mathbb{Z} \end{aligned}$$

is an isomorphism of $\bar{\mathbb{Z}}$-algebras.

Proof. Let $\mathbb{Z}_1, \ldots, \mathbb{Z}_n$ be a set of multiplicatively independent elements of $\mathbb{Z}$ modulo the action of $\mathrm{Gal}(\mathbb{Q}/\mathbb{Q})$. For each $i \in \{1, \ldots, n\}$ let $\bar{\mathbb{Z}}_i$ denote the full field of $\mathrm{stab}_{\mathbb{Q}}(\mathbb{Z}_i) \cap \mathbb{Z}$, and let $\alpha_{i,1}, \ldots, \alpha_{i,t_i}$ be an $\mathbb{Z}$-basis of $\bar{\mathbb{Z}}_i$. (The notation $\bar{\mathbb{Z}}_i$ does not mean complication for a final argument, but simply the notation of Lemma 1.10.) For $1 \le j \le t_i$ and $1 \le k \le t_i$, let $\alpha_{i,j,k} \in \mathbb{Z}$ be defined by

$$\alpha_{i,j}(\mathbb{Z}_k) := \begin{cases} \alpha(\alpha_{i,j}), & \text{if } \mathbb{Z}_k = \sigma\mathbb{Z}_i \text{ for some } \alpha \in \mathbb{Z}_i, \\ 0, & \text{otherwise.} \end{cases}$$

For $\mathbb{Z} \in \mathbb{Z}$. Then, by Lemma 1.11(ii), the elements $\alpha_{i,j}$, $1 \le j \le t_i$, $1 \le i \le n$, form an $\mathbb{Z}$-basis of $\mathbb{Z}$. Equivalently this basis is mapped to a $\bar{\mathbb{Z}}$-basis of $\bar{\mathbb{Z}}$. Hence $\mathbb{Z}$ is an isomorphism of $\bar{\mathbb{Z}}$-algebras.

Finally, we have to show

$$[illegible]$$

for $U \in \mathcal{U}$, $x, y \in \tilde{X}$ and $\sum_{i=1}^{n} [illegible] \in \mathbb{R}$. This is a straightforward [illegible] and consequence [illegible] the definition: for the left hand side we compute

$$[illegible]$$

for the value of $x[U]$, $U \in \mathcal{U}$, and the right hand side yields

$$[illegible]$$

$$\square$$

The proof for Hausdorff may be concluded in the following commutative diagram of $\tilde{X}$-modules

$$[illegible]$$

Here $x[U] = U[X_x]$ and $\tilde{x}[\tilde{U}] = \tilde{U}[x[X_x]]$ for $U \in \mathcal{U}$ and $U' \in \mathcal{U}'$.

Finally, we have to take care of one remaining.

Lemma 1.11 Let $U \in \mathcal{U}$ and put $\tilde{U} = \mathcal{U}[U] \in \tilde{\mathcal{U}}$. Then

$$[\tilde{U}, \tilde{\ }]_{\tilde{x}[X_x]} = [U, \ ]_{X_x},$$

where $\ $ and $\tilde{\ }$ are related by $\ = \tilde{\ } \circ x$.

Proof. This is just a straightforward [illegible] and computation:

$$\begin{aligned}
[\tilde{U}, \tilde{\ }]_{\tilde{x}[X_x]} &= \sum_{i=1}^{n} \tilde{U}[x[X_x]] \, m_i \, x[a_i] \, \tilde{\ }[x[a_i]]^{-1} \\
&= \sum_{i=1}^{n} \tilde{U}[x[X_x \, m_a \, a_i]] \, \tilde{\ }[a_i] \\
&= \sum_{i=1}^{n} U[X_x \, m_a \, a_i] \, \tilde{\ }[a_i] \\
&= [U, \ ]_{X_x}.
\end{aligned}$$

1.7 [illegible]

[illegible] 1.5. In addition, let S denote the integral closure of R in R and let R^{sh} be its maximal subring in R.

Now also let $\mathfrak{m}$ be a prime of R^{sh}-module structure of S. [illegible] $x = 1$ is clearly satisfied by Corollary 1.11. Namely, define a map f on R by

$$f(r) = \begin{cases} 1, & \text{if } r = 0, \\ 0, & \text{if } r \neq 0, \end{cases}$$

and recall that $R^{\mathrm{sh}} = \{R - R\}^{\mathrm{sh}}$. [illegible] $\mathfrak{m}$ [illegible]

Proposition 1.12 Let $x = 1$. [illegible] $R^{\mathrm{sh}} = R^{\mathrm{sh}}$ and S is then already one over R^{sh}. [illegible] $\mathfrak{m}$ [illegible] f.

Proof. [illegible] 1.11. □

In the sequel [illegible] x [illegible] $x \in \mathfrak{p}$ 1. Recall that $R = R[\mathfrak{p}^{\mathrm{sh}}]$ and $S = R[\mathfrak{p}^{\mathrm{sh}}]$. In order to control the ramification of primes outside $\mathfrak{p}$ [illegible] assume that the algebra over $R[\mathfrak{p}]$ and S satisfy the following hypothesis:

S stands among [illegible] R [illegible] (H)

[illegible] 1 [illegible] R, [illegible] $R[\mathfrak{p}]$, [illegible] $R[\mathfrak{p}]$. In Lemma 1.13 [illegible] 1 generally [illegible] that [illegible] (H).

Lemma 1.13 Let $R = R[\mathfrak{p}^{\mathrm{sh}}]$, $x \in \mathfrak{p}$, and let R be a prime ideal of R [illegible] $\mathfrak{p}$. [illegible] R is [illegible] $R[\mathfrak{p}]$, $S = R[\mathfrak{p}^{\mathrm{sh}}]$, [illegible] $x \in \mathfrak{p}$ [illegible] $R[\mathfrak{p}]$ is a prime ideal satisfies [illegible] $R[\mathfrak{p}]$.

Proof. [illegible] Krull-Lücken-Dedekind [see [KK, 80. II, 80. 1.5]] implies that R is a prime of prime ideals R and only R $R[\mathfrak{p}]_{\mathfrak{p}} \cap R[\mathfrak{p}]$ is maximal of R. Since $R[\mathfrak{p}]_{\mathfrak{p}} \cap R = R[\mathfrak{p}]$ [illegible] Corollary 1.9 [illegible] Lemma [illegible]. □

[illegible] R^{sh} [illegible] local compensate and [illegible] prime that is an [illegible] ring, [illegible] Krull dual of the [illegible]-[illegible] ring [illegible] [illegible] [illegible]-prime ideals [illegible] R. [illegible] [illegible] R^{sh}-[illegible] $\mathfrak{m}$ [illegible] R is locally [illegible] R^{sh}. [illegible] [illegible] $\mathfrak{p}$, R is also globally [illegible] R^{sh}.

[illegible] 1.14 [illegible] $\tilde{R}$ [illegible] R [illegible] localization of each prime R of R. [illegible] R [illegible] let

$$\tilde{R}_{\mathfrak{p}} = (R \setminus \mathfrak{p})^{-1} R .$$

Let $\mathcal{H} \mid g$ be a closed a multiplicity generator $\alpha \in \mathbb{R}_+$ for g and a minimal elements equation for $\mathcal{H}$ defined over the localization $\mathbb{R}_{,\mathcal{H}}$. Let $\{ \mathcal{H} \}$ be define $\alpha_1 \in \mathcal{H}_g$

$$\alpha_1 = \frac{1}{\alpha^{\prime\prime}} \sum_{\alpha \in} \alpha(\alpha)^{1}(\alpha - \alpha_1),$$

Then

$$\alpha(\alpha) = \begin{cases} -\frac{\alpha}{\alpha}, & \text{if } \alpha = (\alpha, \alpha) \notin \mathbb{R}_{\alpha}, \\ \alpha, & \text{if } \alpha = \mathbb{R}_{\alpha}. \end{cases}$$

Then

$$\bar{\mathcal{H}}_{\alpha} = \mathcal{H}_{\alpha} \cdot \alpha_1 \cdot \prod_{i=1}^{\alpha^{\prime\prime}-1} \mathcal{H}_{\alpha} \cdot \alpha_1,$$

Then $g = \mathbb{R}_{,\{\alpha\}}(\mathcal{H})$.

Compare Lemma 3.33 with a $\mathcal{H}$-Mapping matrix in $\mathcal{H}$. Since previously, it is the Matrix Dual which $\mathcal{H}$-Mapping matrix that approximate the $\mathcal{H}$-group without dg^{α}-tuples points on $\mathcal{H}$.

Second Let $\mathbb{R}^{\alpha}$ denote the $\mathcal{H}$-Hopf matrix that represents the $\mathcal{H}$-group without g^{α}-tuples points on $\mathcal{H}$. On applied Localization of $\mathbb{R}^{\alpha}$ is given in [illegible], [illegible]. Let each point $\mathcal{H}$ of $\mathcal{H}$ not dividing g, the localization $\mathbb{R}^{\alpha}_{\mathcal{H}}$ is equal to $\mathrm{Hop}(\mathcal{H}, \mathcal{H} -)^{\alpha}_{\mathcal{H}}$. Let a point $\mathcal{H} \mid g$ be then

$$\mathbb{R}^{\alpha}_{\mathcal{H}} = \mathcal{H}_{\alpha}[[\mathcal{H}]][\{\mathcal{H}\}^{1-}][\mathcal{H}]],$$

Then $\{\mathcal{H}\}$ is given by [illegible] and $\{\mathcal{H}\}^{1-}$ is defined in [illegible]. Since α also, as in Section 3, $\mathbb{R}^{\alpha}_{\mathcal{H}}$ as an $\mathcal{H}_{\alpha}$-module in $\mathrm{Hop}(\mathcal{H}, \bar{\mathcal{H}})^{\alpha}_{\mathcal{H}}$ also the rule $\{\mathcal{H}\}(\alpha) = \{\alpha(\alpha)\}$ for $\{\mathcal{H}\}$ a $\mathcal{H}_{\alpha}[[\mathcal{H}]]$ and $g \in \mathcal{H}$.

According to Theorem 3.3, we have a decomposition

$$\mathcal{H} = \prod_{i=1}^{n} \mathcal{H}_i,$$

Then in this situation each of the fields $\mathcal{H}_i$ is a multiple of $\{\mathcal{H}\}$, and hence $\mathbb{R}_{\alpha}[[\mathcal{H}]]$ and Theorem 3.31 magnified at point $\mathcal{H}$ not dividing g. Thus the assertion of the Theorem follows a for $\mathcal{H} \mid g$ from Theorem 3.3 and Corollary 3.3. For $\mathcal{H} \mid g$ it is an immediate consequence of Theorem 3.13 and Proposition 3.3. $\square$

In the field $\mathbb{Q}$ a $\mathcal{H}$ also equal $\mathbb{R}^{\alpha}$ for $\bar{\mathcal{H}}$, reflecting the assertion of the previous proposition.

Theorem 3.35 Let $\alpha \in \mathbb{Q}$ I and suppose that $\mathcal{H}[\alpha]$ and $\mathcal{H}$ satisfy [illegible]. Then:

[i] The associated matrix $\mathbb{R}^{\alpha\alpha}$ is equal to $\mathbb{R}^{\alpha}$.

[ii] $\mathcal{H}$ is a locally free $\mathbb{R}^{\alpha\alpha}$-module of rank 1.

[illegible] [illegible] [illegible] [

(a) $S = \mathbb{Z}_p$. In fact in $\mathbb{Q}(\zeta_p)$, $\mathfrak{p} = (\zeta_p - 1)$ and $\mathfrak{p}^e = (p)$, $e = p - 1$.

(b) $S = \mathbb{Z}_p^{\times}$. In matrix form in $\mathbb{Q}(\zeta_p)$, $\mathfrak{p} = (\zeta_p - 1)$, $\sigma \in \{1, \ldots, p-1\}$ and $\mathfrak{p}^e = (p)$, $e = p - 1$.

$\mathbb{Q}(\zeta_p) = \mathbb{Q}[x]/(\Phi_p(x))$, where Φ_p denotes the pth cyclotomic polynomial.

(c) $\mathfrak{p} = (\pi)$, $e = 1$ and $\mathfrak{p}^e = (p)$, $e = 1$.

(d) $\mathfrak{p} = \mathfrak{p}_1 \cdots \mathfrak{p}_r$ [illegible] $\{0\} = \mathfrak{p}_i^{e_i}$, $e_i = 1$ and $\mathfrak{p}^e = (p)$, $e = 1$.

Hence $\mathfrak{p}$ denotes at most one prime ideal [illegible] $\{\mathfrak{p}, \mathfrak{q}\} = 1$.

Proposition. Since $\mathfrak{p}$ [illegible] the ideals among $\mathbb{Q}[\zeta_p]$ [illegible] good reduction for all primes $\mathfrak{q}$ of $\mathbb{Q}$ [illegible] dividing $\mathfrak{p}$. But $\mathfrak{q}$ is a prime of $\mathbb{Q}$ not dividing $\mathfrak{p}$. By Theorem 1.31 in addition to the fact $N(\mathfrak{p}^{\sigma-1})\,N(\mathfrak{p}^{\sigma})$ is unramified at $\mathfrak{q}$ for $\sigma \neq 1$, except in the case mentioned in the theorem.

Let $\mathfrak{p} = \mathfrak{q} \in \mathbb{Q}_p$. We define $\mathfrak{a}$ also $\mathfrak{b} = N(\mathfrak{p}^{\sigma-1})$ and $\mathfrak{c} = N(\mathfrak{p}^{\sigma})$. Then $\mathfrak{p}$ ramifies in $\mathfrak{b}$ if and only if $\mathfrak{p} \mid \mathfrak{c}$. Hence $\mathfrak{a}$ may assume that

$$\mathfrak{c} = \mathfrak{p}^{e}\mathfrak{c}', \quad (\mathfrak{c}', \mathfrak{p}) = 1, \quad e \geq 1.$$

$\mathfrak{a}$ Denote by $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{a})$, $\mathfrak{a} \in \mathbb{Z}_p$, $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{a})$ the normalized [illegible] of primes above $\mathfrak{p}$ in $\mathbb{Q}[\zeta_p]$, $\mathbb{Q}[\zeta_p]$, $\mathbb{Q}[\zeta_p]$, respectively. By class field theory [e.g., [Ram, Prop. 111, 1], Subgroup 1] or [Ser, 111. 111a] $\mathfrak{a}$ Denote

$$\mathrm{ord}_{\mathfrak{p}}(\mathfrak{a}) = [N(\mathfrak{p}^{\sigma-1}) : N(\mathfrak{c}'\mathfrak{p}^{\sigma-1})] = \frac{\mathrm{ord}(N(\mathfrak{p}^{\sigma-1}))}{\mathrm{ord}(N(\mathfrak{c}'\mathfrak{p}^{\sigma-1}))}\,\mathrm{ord}(\mathfrak{p}^{e}),$$

$$\mathrm{ord}(\mathfrak{a}) = [N(\mathfrak{p}^{\sigma}) : N(\mathfrak{c}'\mathfrak{p}^{\sigma})] = \frac{\mathrm{ord}(N(\mathfrak{p}^{\sigma}))}{\mathrm{ord}(N(\mathfrak{c}'\mathfrak{p}^{\sigma}))}\,\mathrm{ord}(\mathfrak{p}^{e}),$$

Here N denotes the Galois function of the ring $\mathbb{Z}_p$ and $\mathrm{ord}(\cdot)$ the order of $(\cdot)$ in $\mathbb{Z}_p^{\times} \mid \cdot \in \{\mathrm{unit}\ \cdot\}$ for any integral ideal $\mathfrak{a}$ of $\mathbb{Z}_p$. Hence $\mathfrak{a}$ obtain

$$\mathrm{ord}(\mathfrak{a}) = \frac{\mathrm{ord}(N(\mathfrak{p}^{\sigma-1}))\,\mathrm{ord}(N(\mathfrak{c}'\mathfrak{p}^{\sigma}))}{\mathrm{ord}(N(\mathfrak{c}'\mathfrak{p}^{\sigma-1}))\,\mathrm{ord}(N(\mathfrak{p}^{\sigma}))}. \tag{1.1}$$

If $\mathfrak{b} \neq \mathbb{Q}[\sqrt{-1}]$, $\mathbb{Q}[\sqrt{-1}]$, then $\mathfrak{a}$ Hence

$$\mathrm{ord}(\mathfrak{a}) = \begin{cases} 1, & \text{if } \mathfrak{b} \mid \{\mathfrak{p}\}, \\ 1, & \text{otherwise.} \end{cases} \tag{1.2}$$

Here $\mathfrak{b} = \mathbb{Q}[\sqrt{-1}]$ $\mathfrak{a}$ and $\{\mathfrak{p}\} = \mathfrak{p}_1^{e_1}$. Then $\mathfrak{a}$ Hence

$$\mathrm{ord}(\mathfrak{a}) = \begin{cases} 1, & \text{if } \mathfrak{b} \mid \mathfrak{p}_1, \\ 1, & \text{if } \mathfrak{b} = \{\mathfrak{p}\}, \\ 1, & \text{otherwise.} \end{cases} \tag{1.3}$$

Finally, let $X = \mathbb{P}(\overline{\mathbb{Q}} - 0)$ [illegible] and $\{0\} = \mathbb{P}^1_{\mathbb{Q}}$. Then

$$\mathbb{P}^1 = \begin{cases} \mathbb{P}^1, & X = \{0\}, \\ \mathbb{P}^1, & X = \mathbb{P}^1, \\ \mathbb{P}^1, & X = \{0\}, \\ X, & \text{otherwise.} \end{cases} \tag{[illegible]}$$

We [illegible] the [illegible] of the [illegible] table [illegible] only have [illegible], [illegible], [illegible] and [illegible]. For the [illegible] of the [illegible] [illegible] [illegible] the [illegible] [illegible] $\mathbb{P}^1_{\mathbb{Q}}$ [illegible]

	$\{\infty\}, \{0\}$	$\{\infty\}$	$\{0\}$	$\{\infty\}$	$\{0\}$
$X \neq \mathbb{P}(\overline{\mathbb{Q}} - 0), \mathbb{P}(\overline{\mathbb{Q}} - 0)$	[illegible]	[illegible]	[illegible]	—	—
$X = \mathbb{P}(\overline{\mathbb{Q}} - 0)$	—	—	[illegible], [illegible] $x = 1$, ∞ [illegible] [illegible], [illegible] $x = \infty - 1$ [illegible], [illegible] $x = 1$	—	—
$X = \mathbb{P}(\overline{\mathbb{Q}} - 0)$	—	—	—	[illegible]	[illegible]

□

[illegible] to the local [illegible] aggregate [illegible] [illegible] [illegible]

[illegible], [illegible] [illegible] [illegible], [illegible] generator [illegible] [illegible] [illegible] [illegible]. [illegible] [illegible] [illegible]

Suppose [illegible] [illegible] the length $\{\mathbb{S}\ |_{m^\circ}\} = 1$. Since $\mathbb{M}^\circ$ is connected, the Hodge condition holds, and thus $\mathbb{S}$ is a line $\mathbb{M}^\circ$ – $\mathbb{M}^{\circ\circ}$-module. □

The most crucial Corollary consequence of Theorem 1.10 for the classical Galois module structure of rings of integers in abelian extensions of a quadratic imaginary number field.

Corollary 1.11 Let x, m be integers [illegible] $1 \le m \le x$. Let $\mathfrak{p}$ be an integral $\mathbb{Z}_2$-ideal [illegible] $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{p}) = 1$. Let $\mathfrak{q}$ be an $\mathbb{Z}_2$-prime ideal [illegible] $\mathrm{ord}_{\mathfrak{q}}(\mathfrak{q}) = 1$ and $\mathfrak{q}, m_2 \} = 1$ and set $\mathbb{M} = \mathbb{Z}[\zeta^\circ], \mathbb{N} = \mathbb{Z}[\zeta^{\circ\circ}]$. Then $\mathbb{Z}_2$ is free over the coordinate ring in $\mathrm{Gal}(\mathbb{N}/\mathbb{M})$.

Proof $\mathbb{Z}_2$ [illegible], [illegible], Theorem 1.8] there exists an ideal $\mathbb{M}$ defined over $\mathbb{M} = \mathbb{Z}[\zeta]$ [illegible] complex multiplication by $\mathbb{Z}_2$ such that the associated Grossencharacter $\chi_{\mathbb{M}/\mathbb{M}}$ is of the form $\chi_{\mathbb{M}/\mathbb{M}} = \mathfrak{p}^\circ \cdot \chi_{\mathbb{Z}_2}$. [illegible] a Grossencharacter $\mathfrak{p}$ of $\mathbb{M}$ of infinity type $(0, 0)$ and conductor $\mathfrak{q}_2 = \mathfrak{q}$. Thus the condition of the corollary holds, by Theorem 1.10 and Theorem 1.11. □

Remark 1.12 The estimates $\mathrm{Gal}(\mathbb{N}^{\circ\circ}/\mathbb{M}^\circ)$ proved by Corollary 1.11 are not completely contained in the results of [illegible] and [Ro05]. On the other hand one could do not completely include the results of [Ro05].

For the construction of the associated ring Galois one usually uses a general applied in the analytic Chebotarev method or the Shimura method of an elliptic curve. The results obtained there of Corollary 1.11 are obtained by really totally constructive of ramified elements ([Ro05, Sec 3.1 and 3.3]) and some Kummer segment. So one, it is clear from this – and, that there methods can not be extended to treat the case $m \neq 1$ for classical Galois module structure.

In the last part of this section – [illegible] we the results of Theorem 1.10, 1.12 and 1.13 to derive results in the composite case. Namely, let x and m° be integral Hecke of $\mathbb{M}$ and set $m = m^\circ$. Suppose that $\{1, x\} = 1$ and [illegible] this

$$x = \prod_{i=1}^{s} x_i^{\circ\circ}, \quad m^\circ = \prod_{i=1}^{s} x_i^{\circ}, \quad m = \prod_{i=1}^{s} x_i^{\circ\circ\circ}, \quad 1 \le x_1, m_1, \quad x_1 \le m_1 \le 1.$$

Let $\mathbb{M} = \mathbb{M}[\mathfrak{q}]$ be the group of $\mathfrak{q}$-torsion points and denote points $x_i^{\circ\circ\circ}$-torsion points $\mathbb{M}_{1,x}$ for $i = 1, \ldots, x$. If x put

$$\mathbb{M}_x = \mathbb{M}_{1,x} \oplus \cdots \oplus \mathbb{M}_{x,x},$$

then $\mathbb{M}_x$ is a principal m-torsion point of $\mathbb{M}$ and x really holds

$$\mathbb{M} = \prod_{i=1}^{x} \mathbb{M}_i, \tag{illegible}$$

$$\mathbb{N} := [\mathbb{M}_x \oplus \mathbb{M}_x \mid \mathfrak{p} \ni \mathbb{M}] = \prod_{i=1}^{x} \mathbb{N}_i, \tag{illegible}$$

Then

$$M_i := M[g_i^{-1}], \quad N_i := \{m \in M_i \mid g \cdot m \in M_i\}, \quad i = 1, \dots, r.$$

Finally, we also set $M = M[M[m^{-1}]] = M[(m^{-1})]$, $N = N[M[m]] = N[(m)]$ and define

$$N = (\bar{M}M)^{\sim}, \quad , N = \operatorname{Mag}(M, \bar{M})^{\sim}, \quad N = \operatorname{Mag}(N, \bar{N})^{\sim}.$$

For $i = 1, \dots, r$ set $\xi_i = (m^{-1})g_i^{-1}$ and assume that after

$$\begin{aligned} & x_1 = 1 \\ \text{or} \quad & x_1 = 1 \\ \text{or} \quad & x_1, x_1 \neq 1, \quad (\xi_1, x_1\bar{\xi}_1) = 1 \text{ and } (g_1, x_1) = 1 \\ \text{or} \quad & x_1, x_1 \neq 1, \quad \xi_1 \text{ is unramified and } (\xi_1, x_1\bar{\xi}_1) = 1. \end{aligned}$$

Let S° be the Zariski dense $\mathbb{Z}$-Mag subset. Then replacing the affine $\mathbb{Z}$-group schemes of $\mathfrak{p}$-tuples on M. We define a set M° for its Zariski dual in M.

Proposition 1.22 Assume the notation and conditions as above. Then S is the zero locus in accordance with $M^{\circ\circ}$ in M. If in addition, $x_1 \neq 1$ for $i = 1, \dots, r$, then $M^{\circ\circ} = M^{\circ}$.

Remark 1.23 $M^{\circ\circ}$ and a generating element are explicitly given in the proof.

Now after filling $M_1 = (\bar{M}M_1)^{\sim}$, $M_1 = \operatorname{Mag}(M_1, \bar{M})^{\sim}$, it is easily verified that finally

$$M = M_1 \oplus \dots \oplus M_r,$$

and naturally that

$$\begin{aligned} & M_1 \oplus \dots \oplus M_r \longrightarrow M, \\ & N_1 \oplus \dots \oplus N_r \longrightarrow \{M_1 \oplus \dots \oplus M_r \mid m \in N_1[M_1] \cdots N_r[M_r]\} \end{aligned} \tag{1.11}$$

is an isomorphism of M-modules.

By Proposition 1.21 or 1.17 there exists a map $N_i \to S_i$ such that $S_i = N_i \cdot M_i^{\circ\circ}$ for each $i \in \{1, \dots, r\}$. Then S_i denotes the integral closure of S in M_i and $M_i^{\circ\circ}$ the zero locus in M_i. Since the discriminants $D_{m_1}, \dots, D_{m_r}$ are mutually coprime, the isomorphism in (1.11) restricts to an isomorphism

$$S_1 \otimes_{\mathbb{Z}} \dots \otimes_{\mathbb{Z}} S_r \longrightarrow S$$

of $M_1^{\circ\circ} \otimes_{\mathbb{Z}} \dots \otimes_{\mathbb{Z}} M_r^{\circ\circ}$-modules. Therefore

$$M^{\circ\circ} = M_1^{\circ\circ} \otimes_{\mathbb{Z}} \dots \otimes_{\mathbb{Z}} M_r^{\circ\circ} \tag{1.12}$$

and a generating function is given by the image of $N_1 \oplus \dots \oplus N_r$ within the isomorphism in (1.11).

If $x_i \neq 1$ for $i = 1, \dots, r$, then (1.12) and Proposition 1.22 imply $M^{\circ\circ} = M^{\circ}$.

□

[illegible] [illegible] [illegible] and [illegible]'s [illegible] [illegible]

In [illegible], [illegible] [illegible] introduced the notion of a [illegible] [illegible] [illegible]

Let $[redacted]$ be the integral closure of $[redacted]$ in $[redacted]$ and define the Noetherian algebra $[redacted]$ to be the largest $[redacted]$-module contained in $[redacted]$, i.e.,

$$[redacted] = [redacted] = [redacted].$$

In [redacted] it is proved that $[redacted]$ is a locally free $[redacted]$-module and moreover that

$$[redacted] : [redacted] \to [redacted], \quad [redacted] \mapsto [redacted]$$

is a group homomorphism. Hence $[redacted]$ induces the cleavage of locally free $[redacted]$-modules and $[redacted]$ the class of $[redacted]$ in $[redacted]$.

In [redacted] the following conjecture is stated:

Conjecture 3.1 Let $[redacted]$ be any non-zero $[redacted]$-ideal $[redacted]$, $[redacted] \in [redacted]$,

$$[redacted] = [redacted].$$

In [redacted] Conjecture 3.1 is generalized for arbitrary integral ideals and holomorphic ring general.

Theorem Let $[redacted]$, where $[redacted]$ varies the number of roots of unity in $[redacted]$. Then

$$[redacted] = [redacted].$$

□

Recently, [redacted] Eichler general an analogue of the Theorem of Deligne and Eichler for arbitrary cusps [redacted] modular groups multiplicative. Namely, let $[redacted]$ be a rational prime and consider the conjecture for $[redacted]$. Then, by [redacted], Theorem [redacted],

$$[redacted] = [redacted].$$

For also [redacted], [redacted] two among other things, a proof without Deligne's general of Eichler's Theorem is given. Finally, [redacted] a question that Eichler's Theorem does not hold for $[redacted] = [redacted]$ in the infinite series of counter-examples in [redacted] the [redacted].

In the sequel, [redacted] apply the results of Section 3.2 in order to derive results concerning Conjecture 3.1.

Let $[redacted]$ be a finite abelian extension of $[redacted]$ and let $[redacted]$ be an arbitrary cusp form multiplicative by $[redacted]$. Assume further that $[redacted]$ is abelian. Let $[redacted]$ be defined as in [redacted]. Let $[redacted]$ be a prime ideal of $[redacted]$ such that $[redacted] = [redacted]$. Let $[redacted]$ be a primitive $[redacted]$-torsion point, $[redacted] \in [redacted]$, and let $[redacted] = [redacted]$, $[redacted] \in [redacted]$. Let $[redacted] = [redacted] = [redacted]$.

Theorem 3.3 Assume the above notation and suppose that $[redacted]$ and $[redacted]$ satisfy the hypothesis [redacted]. Then, for $[redacted] = [redacted]$,

$$[redacted] = [redacted], \text{ and therefore } [redacted] = [redacted].$$

Remark 8.8 (i) The Oggenfuchs [OO] is almost all age antifull, except in the case listed in Lemma 8.10.

(ii) Using the proof of Theorem 8.5 – (iii) the [illegible] that there exists an isomorphism of $\mathbb{Z}$-modules

$$\mathbb{Z}_{\mathcal{M}} \cong \mathbb{Z} \oplus \mathrm{Hom}(\mathbb{Q}, \bar{\mathbb{Z}})^{\mathrm{ab}},$$

Here $\mathbb{Q} = \{\mathbb{Z}_{\mathcal{M}} \otimes g \mid g \in \mathbb{Z}[g^{\mathrm{ab}}]\}$ [illegible] an aggregated galactic g^{ab}-module gained $\mathbb{Z}_{\mathcal{M}}$. $\mathbb{Z}^{\mathrm{ab}}$ can therefore be also [illegible] as the annulated [illegible] of the integral closure $\mathcal{O}$ of $\mathbb{Z}_{\mathcal{M}}$ in $\mathbb{Q}$ or as the annulated [illegible] of $\mathbb{Z}_{\mathcal{M}}$.

(iii) Theorem 8.5 generalizes [OS, Theorem 5, p.[illegible]].

Proof of Theorem 8.5 The theorem follows – a lexicographically from Theorem 8.10, if – can show that there exists an isomorphism of $\mathbb{Z}$-modules

$$\mathbb{Z}_{\mathcal{M}} \cong \mathbb{Z} \oplus \mathrm{Hom}(\mathbb{Q}, \bar{\mathbb{Z}})^{\mathrm{ab}}$$

for an aggregated set $\mathbb{Q} = \{\mathbb{Z}_{\mathcal{M}}' \otimes g' \mid g' \in \mathbb{Z}[g^{\mathrm{ab}}]\}$ [illegible] a galactic g^{ab}-module gained $\mathbb{Z}_{\mathcal{M}}'$.

Let $\mathbb{Q}$ – [illegible] the small that $\{g^{\mathrm{ab}}\}^{\mathrm{ab}} = \{e\}$ is galmalgal and recall the definition of $\mathbb{Z}_{\mathcal{M}}$ in [illegible] – [illegible]. – allows $\mathbb{Q}$ – $\mathbb{Z}_{\mathcal{M}}$ small that $\{\mathbb{Q}\} = \{g^{\mathrm{ab}}\}^{\mathrm{ab}-1} \cdot \{$ [illegible] an integral $\mathbb{Z}_{\mathcal{M}}$-linear Φ small that $\{\mathbb{Q}, g\} = 1$ and equally an isomorphism σ as in [illegible] $\mathbb{Z}_{\mathcal{M}}$

$$\begin{aligned} \sigma : \mathbb{Z}\{\ \}^{\mathrm{ab}} \{\mathbb{Z}\{\ \}^{\mathrm{ab}-1}\} &\longrightarrow \mathbb{Z}\{\ \}, \\ g \otimes \mathbb{Z}\{\ \}^{\mathrm{ab}-1} &\longmapsto \Phi(\mathbb{Q})(g). \end{aligned}$$

Then the induced isomorphism

$$\bar{\sigma} : \{\bar{\mathbb{Z}}\mathbb{Z}\{\ \}^{\mathrm{ab}}\}^{\mathbb{Z}[\pi^{-1}]} \longrightarrow \bar{\mathbb{Z}}\mathbb{Z}\{\ \} \qquad [illegible]$$

is $\mathbb{Z}_{\mathcal{M}}$-equivariant and – map identies finally the $\mathbb{Z}_{\mathcal{M}}$-isomorphism.

The set $\mathbb{Z}_{\mathcal{M}}\{\{e\}\}$ is of the form

$$\mathbb{Z}_{\mathcal{M}}\{\{e\}\} = \{\mathbb{Z}_{\mathcal{M}} \otimes g \mid g \in \mathbb{Z}\{\{e\}\}\},$$

Here $\mathbb{Z}_{\mathcal{M}}$ is a galactic g^{ab}-module gained small that $\{e\}\{\mathbb{Z}_{\mathcal{M}}\} = \mathbb{Z}$. – set $\mathbb{Z}_{\mathcal{M}}' = \{\mathbb{Q}\}\{\mathbb{Z}_{\mathcal{M}}\}$ and note that $\mathbb{Z}_{\mathcal{M}}'$ is a galactic g^{ab}-module gained. Let $\mathbb{Q} = \{\mathbb{Z}_{\mathcal{M}}' \otimes g' \mid g' \in \mathbb{Z}\{\ \}\}$ and consider the natural map

$$\begin{aligned} \Phi : \mathrm{Hom}(\mathbb{Z}_{\mathcal{M}}\{\{e\}\}, \bar{\mathbb{Z}})^{\mathbb{Z}[\pi^{-1}]} &\longrightarrow \mathrm{Hom}(\mathbb{Q}, \bar{\mathbb{Z}}), \\ \Phi &\longmapsto \Phi', \end{aligned}$$

Here $\Phi'(g') = \Phi(g)$ [illegible] $g \in \mathbb{Z}_{\mathcal{M}}\{\{e\}\}$ small that $\{\Phi\}(g) = g'$. – is all-bijectivity since $\{\Phi\}(g_1) = \{\Phi\}(g_2)$ implies $g_1 - g_2 \in \mathbb{Z}\{\ \}^{\mathrm{ab}-1}$. The map – is obviously injective and also $\mathbb{Z}_{\mathcal{M}}$-equivariant:

$$\{\Phi(\Phi^{\mathrm{ab}})\}(\mathbb{Z}_{\mathcal{M}}' \otimes \{\Phi\}(g)) = \Phi^{\mathrm{ab}}(\mathbb{Z}_{\mathcal{M}} \otimes g) = \Phi\left(\Phi(\mathbb{Z}_{\mathcal{M}}'^{-1} \otimes g^{\mathrm{ab}-1})\right)$$

$$\{\Phi(\Phi)^{\mathrm{ab}}\}(\mathbb{Z}_{\mathcal{M}}' \otimes \{\Phi\}(g)) = \Phi\left(\Phi(\Phi)(\mathbb{Z}_{\mathcal{M}}'^{-1} \otimes \{\Phi\}(g)^{\mathrm{ab}-1})\right) = \Phi\left(\Phi(\mathbb{Z}_{\mathcal{M}}'^{-1} \otimes g^{\mathrm{ab}-1})\right).$$

Since $\mathbb{k}$ contains $\mathbb{C}$, the ring R is an $\mathbb{k}$-module homomorphism. Hence, a Hamilton $\left[\overline{\mathbb{k}[\mathbb{Z}]}\,\{\,\}\right]^{\mathbb{Z}_n}$ and $\overline{\mathbb{k}[\mathbb{Z}]}\,\{\,\}^{\mathbb{Z}_n}$ also [illegible]. We find and a Dunn in establish the equality

$$\left\|\left\|\ \ \right\|_{\mathbb{Z}_n}\right\| = \left\|\left\|\ \ \right\|\right\| = \mathbb{k}[\mathbb{Z}] \cdot \left\|\ \ \right\|_{\mathbb{Z}_n} = [\mathbb{Z}]\,[\mathbb{Z}],$$

Here $\left\|\ \ \right\|_{\mathbb{Z}_n} = \sum_{\mathbb{Z}_n} \overline{\mathbb{k}[\mathbb{Z}]}\,\{\,\}^{\mathbb{Z}_n \cdot \{\,\}}$. Since $\mathbb{Z}_n \cdot \mathbb{Z}_n = \mathbb{Z}_n$ and $\mathbb{Z}_n \cdot \mathbb{Z}_n = \mathbb{Z}_n$ and $\mathbb{Z}_n \cdot \mathbb{Z}_n = \mathbb{Z}_n$ a nally computes

$$\left\|\mathbb{k}[\mathbb{Z}]\ \ \right\|_{\mathbb{Z}_n} = \mathbb{k}\left\|\left\|\ \ \right\|\right\|\,[\mathbb{Z}'] = \left\|\ \ \right\|_{\mathbb{Z}_n} = \mathbb{k}[\mathbb{Z}]\,[\mathbb{Z}]$$

$$= \left|\overline{\mathbb{k}[\mathbb{Z}]}\,\{\,\}^{\mathbb{Z}_n \cdot \{\,\}}\right| \left\|\ \ \right\|_{\mathbb{Z}_n \cdot \{\,\}} = \mathbb{k}[\mathbb{Z}]\,[\mathbb{Z}]$$

and, since $[\mathbb{Z}]\,[\mathbb{Z}] = \mathbb{Z}_n \cdot \mathbb{Z}_n = \mathbb{Z}_n \cdot \mathbb{Z}_n = \mathbb{Z}_n \cdot \mathbb{Z}_n

[illegible] $\mathbb{Z}_p$-[illegible].

Suppose that [illegible] is a primitive p^n-torsion point for some $n \geq 1$. By [illegible] we may assume that $n \geq 1$. Since [illegible] $\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}$ is Galois. By Corollary 1.10. By Lemma 1.11 [illegible] Lemma 1.12 and Lemma 1.9 imply that $\mathfrak{p}$ is $\bar{\mathbb{Q}}_p$ in fact the unramified subfield $\mathbb{Q}^{ur} = \mathbb{Q}^{p}$. This contradicts the fact that $\mathbb{Q}_p$ applying $\mathrm{Gal}(\mathbb{Q}_p/\mathbb{Q}_p)$, since both $\mathbb{Q}^{p}$ and $\bar{\mathbb{Q}}_p$ fulfill [illegible] to a change of fields [see [illegible], [illegible], [illegible], [illegible]]. $\square$

Part II

The equivalence of Hilbert and p-Dirichlet to a class which includes all a geometric imaging Dell

1 Introduction

[illegible] gent is concerned [illegible] a strong version of Dirichlet's conjecture [illegible] has shown estimates of an imaging geometric modular Dell.

Let [illegible] be a finite Galois extension of modular Dell [illegible] Galois group [illegible]. Let [illegible] be a finite [illegible]-stable set of places of [illegible] containing the set [illegible] of ramified places. [illegible] a place [illegible] has the [illegible]-module of [illegible]-units of [illegible]. Then [illegible] = [illegible] [illegible] also set [illegible] = [illegible].

Let [illegible] denote the free [illegible]-module on [illegible], [illegible] [illegible] acting [illegible] permuting the places in [illegible]. Let [illegible] be the kernel of the augmentation map [illegible] on [illegible] [illegible] [illegible] many places [illegible] in [illegible] to [illegible]. Often the Dirichlet unit theorem implies that [illegible] = [illegible] as [illegible]-modules. However [illegible] a map for [illegible] and [illegible]-modules [illegible] : [illegible] to [illegible].

For any complex valued character [illegible] of [illegible] [illegible] place [illegible] [illegible] the Euler factor associated to [illegible] and [illegible] in [illegible] [see also Definition 2.3].

Let [illegible] denote the Galois [illegible]-invariant associated to [illegible] [illegible] the Euler factors at places of [illegible] ramified. Let [illegible] denote the first non-zero coefficient in the Taylor expansion of [illegible] at [illegible] = [illegible]. Then [illegible] and [illegible] are non-zero, possibly transcendental complex numbers and the Lang of Stark's conjecture is that [illegible] accounts precisely for the transcendental part of [illegible]. Setting

$$[illegible] = \frac{[illegible]}{[illegible]}$$

Stark's formulation of Stark's conjecture is as follows :

If Stark's conjecture was

$$[illegible]^{\prime} = [illegible]^{\prime}[illegible] \text{ for all } \in [illegible].$$

In particular, it is conjectured that [illegible] is contained in the field [illegible] of definition values of [illegible]. Note that Stark's conjecture is true [illegible] in the form for which all [illegible] and for which [illegible] of an imaging geometric modular Dell [Dell cases are a consequence of [Ma1, [illegible], Ang.2.2]].

[illegible] be a subfield [illegible] of [illegible] such that [illegible] is a finite Galois extension and [illegible] representation of [illegible] is realizable over [illegible]. The field [illegible] is just an auxiliary field [illegible] field is needed for technical reasons that [illegible] [illegible] in [illegible]. For any modular Dell [illegible] [illegible] the ring of algebraic integers of [illegible].

[illegible] [illegible] [illegible] [

[illegible] $\mathbb{Z}$ and $\mathbb{Q}$.

[illegible] $\mathbb{Z}$ [illegible] $\mathbb{Z} = \mathbb{Z}$. [illegible] Theorem 1.1 is [illegible] [illegible] [illegible] $\mathbb{Q}$ [illegible] [illegible] $\mathbb{Q}$ [illegible] $\mathbb{Z}$-[illegible] $\underline{\mathbb{Q}}$ [illegible] $\mathbb{Z}_{\mathbb{Q}}$ [illegible] [illegible] [illegible]

Let [illegible] be a non-trivial [illegible]-automorphism of [illegible] = Gal([illegible]/[illegible]). The congruences of Section 1 and 2 allow the problem of proving the inhomogeneity of [illegible]$^{-1}$ essentially to the problem of establishing a Ribet-Mazur result of the form

$$K_{n,1}([illegible]_{n}[illegible]_{m})^{[illegible]} \mid K_{n,1}([illegible]_{n}[illegible]_{m}[illegible]_{m}[illegible])^{[illegible]} . \qquad [illegible]$$

Here, for a cuspidal [illegible]-module [illegible], $K_{n,1}([illegible])$ denotes the usual [illegible]-adic Hecke algebra of [illegible] (see e.g. [illegible], page [illegible]). Along α_{p} for the Hecke-generated cuspidal [illegible], the [illegible]-submodule [illegible]$^{[illegible]}$ = α_{p}[illegible] is the [illegible]-algebraic part of the [illegible]-module [illegible]. Finally, δ_{m} denotes the group of elliptic units defined by Sinnott in [Sin]. In [Sin], δ_{m} is only defined for extensions [illegible] containing the Bilbert class field [illegible]. In fact it suffices to require that [illegible] contains the Bilbert-[illegible]-class field [illegible]. This is the reason for the restriction to primes [illegible] $\nmid$ D_{K} in Theorem 1.1.

Using Wolpenglm's Main system method [Wa], the Ribet/Mazur result [illegible] is proved in [Sin, Section 8] under the additional assumption [illegible] [illegible]. This assumption is responsible for the restriction to primes [illegible] $\nmid$ [illegible] in Theorem 1.1.

In this point a stronger Theorem 1.1 is the result of Sinnott and [illegible]. The essential reason [illegible] why they can prove an almost complete result for abelian extensions [illegible] of [illegible] is that the main conjecture of [illegible] [illegible] a strong for totally real fields is true [illegible] and of [illegible]. [illegible] [illegible]. Instead, their problem [illegible] the prime [illegible] was only true in the general state of the main conjecture. It should also be remarked that for the proof of their main theorem it is not enough to have the main conjecture near [illegible], but also for certain totally real fields.

Using Sinnott's [illegible] on the "main conjecture" for imaginary quadratic fields [illegible] a can prove a Ribet of a complementary result to Theorem 1.1.

Theorem 1.2 Let [illegible] be an imaginary quadratic number field [illegible] class number D_{K} = 1. Let [illegible] be a prime not for K_{n}[illegible] be stable [illegible] [illegible] [illegible]$_{K}$: [illegible]]. Let [illegible] [illegible]$_{K}$ be a [illegible]-extension such that [illegible] is stable. Assume that every [illegible]$_{K}$-prime ideal [illegible] Hall Statistic [illegible] and satisfies in [illegible] [illegible]$_{K}$ then not split in [illegible]$_{K}$[illegible]. Let [illegible][illegible]$_{K}$ be a finite extension of [illegible] [illegible]$_{K}$ and assume that [illegible] then not Statistic the number of roots of unity in [illegible]. Then

$$K_{n}[\ ] =_{1} K_{n}[\ ^{\prime}]$$

for all elements [illegible] of Gal([illegible]/[illegible]).

The Illustrate Theorem 1.2 is a consider an example.

Example Let [illegible] be an imaginary quadratic number field [illegible] class number D_{K} = 1. Let [illegible] $\neq$ 3 be a rational prime that splits in [illegible] and [illegible] = [illegible], [illegible] $\neq$ [illegible]. Then [illegible] = [illegible]$^{[illegible]}$]. Then [illegible] [illegible]$_{K}$ is a [illegible]-extension, [illegible] is absolutely abelian over [illegible]. If [illegible] is a conjecture [illegible] = [illegible], [illegible] [illegible], then $K_{n}[\ ] =_{1} K_{n}[\ ^{\prime}]$ for all elements [illegible] of Gal([illegible]/[illegible]).

[illegible] a complete [illegible] Section [illegible] [illegible] a total ordering of the elements of the group. In Sections [illegible] and [illegible] [illegible] we recall the necessary definitions and results from [illegible]. Section [illegible] is devoted to the construction of an embedding $\underline{\mathbb{G}} \to \mathbb{S}_{\infty}$. The construction of $\underline{\mathbb{G}}$ is outlined by the construction of Smallbones [illegible] [illegible, [illegible]]. [illegible] equivalent [illegible] by elliptic [illegible]. The relevant results on elliptic [illegible] are collected in Section [illegible]. In Section [illegible] [illegible] a canonical [illegible] group $\mathbb{S}_{\infty}$ of regular [illegible] and compute the [illegible] of $\mathrm{Ker}[\underline{\mathbb{G}}]$ in $\mathbb{S}_{\infty}$. The group $\mathbb{S}_{\infty}$ is independent of $\underline{\mathbb{G}}$ and contained in $\mathbb{S}_{\mathbb{Q}}$. The [illegible] $\mathbb{S}_{\mathbb{Q}}$ of the [illegible] $[\mathbb{S}_{\infty} : \mathrm{Im}[\underline{\mathbb{G}}]]$ also [illegible] [illegible] the proof of Theorem [illegible] in the Bielefeldt-Lorenzini [illegible]. This is outlined in Section [illegible], [illegible] the main theorem [illegible]. Finally, in Section [illegible], [illegible] Theorem the implications of our results for the Galois module structure of [illegible].

[illegible] Preliminary [illegible]

Let $\mathbb{G}/\mathbb{Z}$ be a finite Galois extension of number fields with [illegible] $G = \mathrm{Gal}(\mathbb{G}/\mathbb{Z})$. Let $\mathbb{O}$ be a finite $\mathbb{Z}$-order not of primes of $\mathbb{Z}$ containing $\mathbb{Z}[G]$ [illegible] $\mathbb{O}$ [illegible] primes.

In this section [illegible] we briefly recall the construction of Chinburg and [illegible] [illegible] $\mathbb{S}$] of a g-lattice with an $\mathbb{O}$-module [illegible] $\mathbb{Z}$ [illegible] $\mathbb{O}$. Originally these invariants [illegible] only defined [illegible] $\mathbb{O}$ is [illegible], i.e., [illegible] $\mathbb{O}$ contains $\mathbb{Z}$ [illegible] all rational primes [illegible] the $\mathbb{O}$-class group $\mathrm{Cl}_{\mathbb{O}}$ is trivial [illegible] [illegible]. The definitions [illegible] [illegible] on a so-called Euler sequence [illegible] [illegible] $\mathbb{O}$ [illegible] give a rational cohomological invariant [illegible] [illegible] and [illegible]. This no [illegible] definitions [illegible] the generalized Euler sequence of $[\mathbb{S} : \mathbb{I}]$ [illegible] [illegible] a rational Euclidean [illegible].

Let $\mathbb{S}$ be a prime of $\mathbb{Z}$ and put $\mathbb{I} = \mathbb{S} \cap \mathbb{O}$. [illegible] $\mathbb{O}_{\mathbb{S}}[\mathbb{I}]$, for the localization of $\mathbb{O}[\mathbb{I}]$. Let $\mathbb{O}_{\mathbb{S}}$ denote the decomposition group, $\mathbb{I}_{\mathbb{S}}$ the inertia group and $\mathbb{F}_{\mathbb{S}}$ the residue extension of the corresponding residue class extension. In addition, set $\bar{\mathbb{O}}_{\mathbb{S}} = \mathbb{O}_{\mathbb{S}}/\mathbb{I}_{\mathbb{S}}$, $q_{\mathbb{S}} = |\mathbb{F}_{\mathbb{S}}|$ and $\bar{\mathbb{I}}_{\mathbb{S}} = |\bar{\mathbb{O}}_{\mathbb{S}}|$. Let $\mathrm{ord}_{\mathbb{S}}$ be the [illegible] $\mathbb{S}$ [illegible] the normalized additive [illegible].

Lemma 3.1 [illegible] The initial lattice $\mathbb{O}_{\mathbb{S}}$ attached to $\mathbb{O}_{\mathbb{S}}[\mathbb{I}_{\mathbb{S}}]$ is the free $\mathbb{O}$-module on the basis

$$\mathbb{S} = \{\mathbb{S} - 1, 1 \cdot \mathbb{I}_{\mathbb{S}} \cdot \ldots \cdot \mathbb{I}_{\mathbb{S}}^{\bar{\mathbb{I}}_{\mathbb{S}} - 1}\} \subseteq \mathbb{O}\mathbb{O}_{\mathbb{S}} \cdot \mathbb{O}\bar{\mathbb{O}}_{\mathbb{S}}, \quad \mathbb{S} \in \mathbb{O}_{\mathbb{S}},$$

where $\bar{\mathbb{I}} = \{\mathbb{S}$ and $\mathbb{I}_{\mathbb{S}}\} = \mathbb{I}_{\mathbb{S}}^{\bar{\mathbb{I}}_{\mathbb{S}} - 1}$ and $1 \cdot \mathbb{S}(\mathbb{S}) \cdot \mathbb{I}_{\mathbb{S}}$. The $\mathbb{O}_{\mathbb{S}}$-action is induced by the natural action of $\mathbb{O}_{\mathbb{S}}$ on $\mathbb{O}\mathbb{O}_{\mathbb{S}} \cdot \mathbb{O}\bar{\mathbb{O}}_{\mathbb{S}}$.

The projection on the first canonical indices a short exact sequence

$$1 \longrightarrow \mathbb{I} \longrightarrow \mathbb{O}_{\mathbb{S}} \longrightarrow \mathbb{O}\mathbb{O}_{\mathbb{S}} \longrightarrow 1$$

[illegible] $1 \leq \mathbb{S} \leq \mathbb{S}_{1}$, [illegible] [illegible] $\mathbb{O}_{\mathbb{S}}/\mathbb{O}\mathbb{O}_{\mathbb{S}} = \mathbb{O}\bar{\mathbb{O}}_{\mathbb{S}}$. Combining this sequence yields

$$1 \longrightarrow \mathbb{O}\mathbb{O}_{\mathbb{S}}^{\times} \longrightarrow \mathbb{O}_{\mathbb{S}}^{\times} \xrightarrow{\mathbb{S}} \mathbb{I} \longrightarrow 1$$

Now $\varphi_{\mathbb{Z}}(\mathbb{Q}) = \mathbb{Q}[\infty_{\mathbb{Z}}]$ for $\mathbb{Q} \in \mathbb{Z}^{\times} = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{\mathbb{Z}}, \mathbb{Z})$. We take coboundary a lattice a map $\mathbb{Z} \in \mathbb{Z}^{\times}$ $\mathbb{Z}_{\mathbb{Z}}$

$$\varphi_{\mathbb{Z}}(\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \mathbb{Z}_{\mathbb{Z}} = \mathbb{Z}, \\ \mathbb{Z} \cdot \infty_{\mathbb{Z}}, & \mathbb{Z} \neq \mathbb{Z} \in \mathbb{Z}_{\mathbb{Z}}, \\ \mathbb{Z}^{\infty}_{\mathbb{Z}}, & \mathbb{Z} \in \mathbb{Z}_{\mathbb{Z}}. \end{cases} \qquad (\mathbb{Z})$$

a fix a choice of $\mathbb{Z}$-adic compactification in the set of all points of $\mathbb{Z}$ and for a $\mathbb{Z}$-stable subset $\mathbb{Z}$ a $\mathbb{Z}$, for the intersection of $\mathbb{Z}$ and $\mathbb{Z}$. Furthermore a set

$$\mathbb{Z}^{\infty\infty} = \{\mathbb{Z} \text{ rational in } \mathbb{Z}[\mathbb{Z}], \mathbb{Z} \neq \mathbb{Z}\},$$
$$\mathbb{Z}^{\infty} = \mathbb{Z}^{\infty}_{\mathbb{Z}} = \bigcup_{\mathbb{Z} \in \mathbb{Z}^{\infty\infty}} \mathrm{Int}^{\mathbb{Z}}_{\mathbb{Z}}\, \mathbb{Z}^{\times},$$

and finally

$$\bar{\mathbb{Z}} = \bar{\mathbb{Z}}_{\mathbb{Z}} = \mathrm{Cor}[\mathbb{Z} \in \mathbb{Z}^{\infty} \xrightarrow{\varphi'} \mathbb{Z}],$$

Here φ' is the augmentation on $\mathbb{Z}$ and $\mathrm{Int}^{\mathbb{Z}}_{\mathbb{Z}}\varphi_{\mathbb{Z}}$ lifts all $\mathbb{Z}$; the augmentation map $\mathrm{Int}^{\mathbb{Z}}_{\mathbb{Z}}\, \mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}$ on $\mathrm{Int}^{\mathbb{Z}}_{\mathbb{Z}}\, \mathbb{Z}^{\times}$ has $\mathbb{Z} \in \mathbb{Z}^{\infty\infty}$.

The main result of [illegible] is a finite sequence associated to $\mathbb{Z}$

$$\mathbb{Z} \to \mathbb{Z}_{\mathbb{Z}} \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}$$

Let $\mathbb{Z}$ cohomologically trivial, $\mathbb{Z}$ stably free and $\mathbb{Z}$. Then $\mathbb{Z}$ originates from a unique extension

$$\mathbb{Z} \to \mathbb{Z}_{\mathbb{Z}} \to \mathbb{Z} \to \bar{\mathbb{Z}} \to \mathbb{Z}$$

Hall is explicitly described in [illegible]. Note that the module corresponds of $\mathbb{Z}$ is the $\mathbb{Z}$-class group $\mathrm{Cl}_{\mathbb{Z}}$ and that for a large set $\mathbb{Z}$ a Dress $\mathbb{Z} = \mathbb{Z}\mathbb{Z}$.

a see now in position to give the definitions of the $\mathbb{Z}$-index and the $\mathbb{Z}$-module. Let $\mathbb{Z} = \mathbb{Z}_{\mathbb{Z}}$ be the module of $\mathbb{Z}$-adic of rational points of $\mathbb{Z}[\mathbb{Z}]$ Hall see not in $\mathbb{Z}$, i.e., $\mathbb{Z} = |\mathbb{Z}^{\infty\infty}|$. Then the invariants are associated to $\mathbb{Z}$-homomorphisms

$$\mathbb{Z} : \mathbb{Z} \to \bar{\mathbb{Z}} = \mathbb{Z}_{\mathbb{Z}} \in \mathbb{Z}\mathbb{Z}^{\times}$$

Let finite formal and coformal (in [illegible]) small maps are called tangential.

a fix a subset $\mathbb{Z}$ of $\mathbb{Z}$ such that $\mathbb{Z}[\mathbb{Z}]$ is finite flat. Let group $\mathbb{Z} = \mathrm{Cal}(\mathbb{Z}[\mathbb{Z}])$ and large enough that every representation of $\mathbb{Z}$ can be realized over $\mathbb{Z}$. For a character of $\mathbb{Z}$ a choice a $\mathbb{Z}\mathbb{Z}$-module $\mathbb{Z} = \mathbb{Z}_{\mathbb{Z}}$ affording and an $\mathbb{Z}_{\mathbb{Z}}\mathbb{Z}$-lattice $\mathbb{Z}$ such that $\mathbb{Z} \otimes_{\mathbb{Z}_{\mathbb{Z}}} \mathbb{Z} = \mathbb{Z}$. a also $\mathbb{Z}$ for the trace element $\mathbb{Z} \in \mathbb{Z}_{\mathbb{Z}}\mathbb{Z}$. Motivated by Kato's definition of the $\mathbb{Z}$-index a consider the complex map

$$\mathbb{Z}_{\mathbb{Z}} : \mathrm{Cor}_{\mathbb{Z}_{\mathbb{Z}}}[\mathbb{Z}, \mathbb{Z}]_{\mathbb{Z}} \xrightarrow{\mathbb{Z}} \mathrm{Cor}_{\mathbb{Z}_{\mathbb{Z}}}[\mathbb{Z}, \mathbb{Z}]^{\mathbb{Z}} \xrightarrow{\mathbb{Z}} \mathrm{Cor}_{\mathbb{Z}_{\mathbb{Z}}}[\mathbb{Z}, \bar{\mathbb{Z}}]^{\mathbb{Z}}.$$

[illegible] and in all [illegible] that follows [illegible] a single [illegible] commutative ring [illegible]. Because it is clear from the context any [illegible]-bilinear is to be replaced by the appropriate base change [illegible] that makes the relations meaningful. [illegible] this simply means that [illegible] and [illegible] have to be read as $\mathbb{G}_{m,R}$ and $\mathbb{G}_{a,R}$.

Recall that for a finitely generated torsion $\mathbb{Z}_p$-module M the $\mathbb{Z}_p$-adic Weyl $K_{m,r}[M]$ is defined as the product $p_1 \cdots p_r$ if $\mathrm{Ord}_{p_1}, \ldots, \mathrm{Ord}_{p_r}$ are the exponential factors of M.

Special Case 5.5 [illegible]

$$\mathbb{G}_m[\ ] = \frac{K_{m,r}[\mathrm{coker}(\mathbb{Z}_m)]}{K_{m,r}[\mathrm{Ker}(\mathbb{Z}_m)]}$$

The reason of complicatness [illegible] a formula this [illegible] [illegible]

Lemma 5.2 [illegible], Lemma [illegible]

(i) $\mathbb{G}_m[\ ]$ does not depend on the choice of M, [illegible] and [illegible].

(ii) [illegible] is functorial, thus $\mathbb{G}_m[\ ]$ is a functorial $\mathbb{G}_m$-Weyl [illegible] Hence to generate $\mathbb{G}_m$ a functorial Weyl of the field [illegible] [illegible] of absolute value [illegible].

(iii) $\mathbb{G}_m[\ ]^n = \mathbb{G}_m[\ ^n]$ for $n \geq 0$.

Next [illegible] a look to the definition of the generalized [illegible]-module of [illegible], which involves [illegible] a main ingredient. The first one [illegible] is denoted $\mathbb{G}_m[\ ]$ and is defined to be the leading coefficient in the Taylor expansion of the Weil [illegible]-function for [illegible] the Euler factor of [illegible] n [illegible]. The second one is a generalized Weyl-Weil regulator $\mathbb{G}_m[\ ]$ [illegible] a small intersection [illegible] $|\cdot|_m$ for the absolute value attached to place [illegible] of [illegible] generalized in the usual way. Put

$$\begin{aligned} H : \mathbb{G}_{m,R}\mathbb{G}_m &\longrightarrow \mathbb{G}_{m,R}\mathbb{G}, \\ x &\longmapsto \sum_{m \in \mathbb{G}} \log|x|_m \, m \end{aligned}$$

denote the usual Weierstrass homomorphism. One also is to define a generalized Weierstrass map

$$\tilde{H} : \mathbb{G}_{m,R}\tilde{\mathbb{G}} \longrightarrow \mathbb{G}_{m,R}\mathbb{G}.$$

We that and let $\tilde{H}$ be any [illegible]-homomorphism such that the diagram

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{G}_{m,R}\mathbb{G}_m \arrow[r] \arrow[d, "H"] & \mathbb{G}_{m,R}\tilde{\mathbb{G}} \arrow[r] \arrow[d, "\tilde{H}"] & \mathbb{G}^{\mathbb{Z}} \arrow[r] \arrow[d, "\tilde{H}"] & 1 \\ 1 \arrow[r] & \mathbb{G}_{m,R}\mathbb{G} \arrow[r] & \mathbb{G}_{m,R}\tilde{\mathbb{G}} \arrow[r] & \mathbb{G}_{m,R}{}^{\mathbb{Z}}\mathbb{G} \arrow[r] & 1 \end{tikzcd} \qquad (5)$$

commutes. Since, for the definition of $\tilde{}$, a elt $\mathbb{G}^{\mathbb{Z}}$ as $\sum_{m \in \mathbb{G}^{\mathbb{Z}}} \mathrm{Ord}^{\mathbb{Z}}_m \, m \mathbb{G}_m$ and set $\tilde{}(\mathbb{G}_m) = -\log(\mathbb{G}\mathbb{G})$ to m [illegible] m as in (5).

Let $\tilde{M}$ be a [illegible]-module affording the antisymmetrized [illegible] of [illegible] and denote by $\tilde{M}_{\mathbb{Q}}$ $\mathrm{Sym}_{m}(\tilde{M}, \mathbb{Q})$ the subsymplectic of $\mathrm{Sym}_{m}(\tilde{M}, \mathbb{Q})$. Hall says a [illegible]-symplectic [illegible] is the complex $\tilde{M} \otimes_{\mathbb{Q}} \mathbb{Q}$.

$\mathbb{Q}_{\mathbb{Q}}[\mathbb{S}_{[illegible]}, \mathrm{Sym}_{m}]$ the Gelfand-Tsetlin of $\mathrm{Sym}_{m}(\tilde{M}, \mathbb{Q})$ does not depend on the choice of $\tilde{M}$ using Diagram [illegible] commute.

Lemma 5.5 [illegible]

$$\mathbb{Q}_{\mathbb{Q}}[\ ] = \mathrm{Sym}(\tilde{M}_{\mathbb{Q}} \mid \mathrm{Sym}_{m}(\tilde{M}, \mathbb{Q}))$$

Finally, we define the generalized [illegible]-module of [illegible].

Lemma 5.6 [illegible]

$$\mathbb{Q}_{\mathbb{Q}}[\ ] = \frac{\mathbb{Q}_{\mathbb{Q}}[\ ]}{\mathrm{ch}[\ ]\left(\prod_{\mathrm{[illegible]}} \mathrm{[illegible]}^{\mathrm{[illegible]}}\right)^{\mathrm{[illegible]}}}.$$

Now $\mathbb{Q}$ is large [illegible] and [illegible] that

$$\mathbb{Q}_{\mathbb{Q}}[\ ] = \mathbb{Q}_{\mathbb{Q}}[\ ]\mathbb{Q}_{\mathbb{Q}} \qquad [illegible]$$

for all dominant [illegible] of $\mathbb{Q}$. [illegible] we find general by [illegible] [illegible] for [illegible]-valued dominant and by [illegible] and [illegible] also [illegible] $\mathbb{Q}$, [illegible] [illegible] is certain positive [illegible] [illegible] the polynomial [illegible] [illegible] [illegible] [illegible] and all dominant of $\mathbb{Q} = \mathrm{Sym}[\mathbb{Q}]\mathbb{Q}$.

In [illegible] the proof of the above conjecture is based on the following [illegible]

Theorem 5.8 [illegible], **Theorem** [illegible] The ratio $[\ ] = \frac{\mathbb{Q}_{\mathbb{Q}}[\mathbb{Q}]\mathbb{Q}_{\mathbb{Q}}}{\mathbb{Q}_{\mathbb{Q}}[\mathbb{Q}]}$ is independent of $\mathbb{Q}$ and $\mathbb{Q}$.

[illegible] **Theorem** also [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] $\mathbb{Q}$; in particular [illegible] $\mathbb{Q} = \mathbb{Q}$.

5 Proof and [illegible]

[illegible] $\mathbb{Q}$ [illegible] $\mathbb{Q}$ [illegible] $\mathbb{Q}$ [illegible] and in conjecture the [illegible] on $\mathbb{Q}$. The most [illegible], that [illegible] [illegible] of the ratio $[\ ]$, is a direct consequence of [illegible] of [illegible] [illegible] and [illegible] [illegible] and **Theorem** 5.8.

Proposition 5.9 [illegible], **Proposition** [illegible]

[illegible] $\mathbb{Q}$ is [illegible], i.e., $\mathbb{Q}[\ ,\ ] = \mathbb{Q}[\ ,\ ]\mathbb{Q}[\ ,\ ]$ for all dominant $\ ,\ ,\ \in \mathbb{Q}$.

[illegible] $\mathbb{Q}[\mathbb{Q}] = \mathbb{Q}_{\mathbb{Q}}$.

[illegible] $\mathbb{Q}$ is a subgroup of $\mathbb{Q}$. Then for any dominant $\mathbb{Q}$ [illegible] $\mathbb{Q}$ [illegible]

$$\mathrm{ch}[\mathbb{Q}] = \mathbb{Q}[\mathbb{Q}^{\mathbb{Q}}\mathbb{Q}].$$

[illegible] Let [illegible] be a normal subgroup of [illegible] and [illegible] a character of [illegible] = [illegible]. Then a Hecke

$$[illegible] = [illegible].$$

In particular it is an equal norm particle in [illegible] a [illegible]-multiplicity [illegible] : [illegible] → [illegible] [illegible] of maximal degree [illegible] [illegible] [illegible]

[illegible] [illegible] [illegible] [

[illegible] a fixed a [illegible]-rational function

$$\mathcal{G}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Q}}) = \mathcal{H}_{\mathbb{Q}}(\mathbb{Z})\,\mathcal{U}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Q}}).$$

Note that $\mathcal{G}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Q}})$ does not depend on the choice of a generalized [illegible] equation for $\mathbb{X}$ over $\mathbb{Q}$.

Lemma 5.5 Suppose that $\mathbb{X}$ is an algebraic minimal [illegible] $\mathbb{X}$ and let $\mathbb{P}$ be a $\mathrm{Val}(\bar{\mathbb{Q}})(\mathbb{Q})$. Then

$$\mathcal{G}_{\mathbb{Q}}(\mathbb{P})^{\mathbb{Z}} = \mathcal{G}_{\mathbb{Q}^{\mathbb{Z}}}(\mathbb{P}^{\mathbb{Z}}).$$

Proof. By the definition of $\mathcal{G}_{\mathbb{Q}}$ [illegible] have

$$\begin{aligned}
\mathcal{G}_{\mathbb{Q}}(\mathbb{P}^{\mathbb{Z}}) &= \prod_{\mathbb{Q}\,|\,\mathbb{Z}\,|\,\mathbb{Q}\,|\,\mathbb{Z}\,|\,\mathbb{Q}\,|\,\mathbb{Z}} \left(\mathcal{H}_{\mathbb{Z}}(\mathbb{P}^{\mathbb{Z}}) - \mathcal{H}_{\mathbb{Q}}(\mathbb{Q})(\mathbb{P})^{\mathbb{Z}}\right)^{-1} \\
&= \prod_{\mathbb{Q}\,|\,\mathbb{Z}\,|\,\mathbb{Q}\,|\,\mathbb{Z}\,|\,\mathbb{Q}\,|\,\mathbb{Z}} \left(\mathcal{H}_{\mathbb{Z}}(\mathbb{P}^{\mathbb{Z}}) - \mathcal{H}_{\mathbb{Z}}(\mathbb{Q}^{\mathbb{Z}})(\mathbb{P})^{\mathbb{Z}}\right)^{-1} \qquad (\mathbb{Q}) \\
&= \prod_{\mathbb{Q}\,|\,\mathbb{Z}\,|\,\mathbb{Q}^{\mathbb{Z}}\,|\,\mathbb{Z}\,|\,\mathbb{Q}\,|\,\mathbb{Z}} \left(\mathcal{H}_{\mathbb{Z}}(\mathbb{P}^{\mathbb{Z}}) - \mathcal{H}_{\mathbb{Z}}(\mathbb{Q}^{\mathbb{Z}})(\mathbb{P})\right)^{-1} = \mathcal{G}_{\mathbb{Q}^{\mathbb{Z}}}(\mathbb{P}^{\mathbb{Z}}).
\end{aligned}$$

Next [illegible] a claim that

$$\mathcal{H}_{\mathbb{Q}}(\mathbb{P})^{\mathbb{Z}} = \mathcal{H}_{\mathbb{Q}^{\mathbb{Z}}}(\mathbb{P}^{\mathbb{Z}}).$$

To prove the claim [illegible] have to show that for all $\mathbb{Q}$ in $\mathrm{Val}_{\mathbb{Q}}(\mathbb{Q})$

$$\left(\mathcal{H}_{\mathbb{Q}}(\mathbb{P})^{\mathbb{Z}}\right)^{\mathbb{Q}\,|\,\mathbb{Z}\,|\,-1} = \frac{\mathcal{H}_{\mathbb{Q}^{\mathbb{Z}}}(\mathbb{P}^{\mathbb{Z}})}{\mathbb{Q}_{\mathbb{Q}}\,\mathbb{Z}\,|\,\mathbb{Q}^{\mathbb{Z}}\,|\,\mathbb{Q}_{\mathbb{Q}^{\mathbb{Z}}}(\mathbb{P}^{\mathbb{Z}})\,\mathbb{Q}_{\mathbb{Q}^{\mathbb{Z}}}(\mathbb{Q})}.$$

But this holds [illegible] immediately from [illegible] and the defining equation for $\mathcal{H}_{\mathbb{Q}}(\mathbb{P})$, since the right hand side equals

$$\left(\frac{\mathcal{H}_{\mathbb{Q}}(\mathbb{P})(\mathbb{Q})}{\mathbb{Q}_{\mathbb{Q}}\,\mathbb{Z}\,|\,\mathbb{Q}\,|\,\mathbb{Q}_{\mathbb{Q}}(\mathbb{Q})\,\mathbb{Q}_{\mathbb{Q}}(\mathbb{Q})}\right)^{\mathbb{Z}}$$

The lemma is now [illegible] a consequence of the claim and [illegible]. □

Let $\mathbb{X} : \mathbb{X} \to \mathbb{Y}$ be an isogeny of algebraic minimal [illegible] $\mathbb{X}$ and let $\mathrm{Val}(\mathbb{X}) \cong \mathrm{Val}(\mathbb{Y}) = \{\mathbb{Q}_{\mathbb{Q}}\}$. The map induced $\mathbb{X}$ [illegible] a morphism of isogeny

$$\mathbb{Y} : \mathbb{X} \to \mathbb{Y}' = \mathbb{Y}\,\mathbb{X}(\mathbb{Q}_{\mathbb{Q}}) \qquad (\mathbb{Q})$$

By taking the canonical projection map (see e.g. [illegible], [illegible], [illegible]).

Proposition 5.6 [illegible], [illegible], [illegible] and [illegible] [illegible]
(a) For all isogenies $\mathbb{X} : \mathbb{X} \to \mathbb{Y}$ as above, [illegible] have

$$\mathcal{G}_{\mathbb{Q}}(\mathbb{X}(\mathbb{P}_{\mathbb{Q}})) = \prod_{\mathbb{Q}\,|\,\mathbb{Q}} \mathcal{G}_{\mathbb{Q}}(\mathbb{P}_{\mathbb{Q}}\,\mathbb{Q}_{\mathbb{Q}}\,\mathbb{Q}).$$

[illegible] Then all [illegible] a [illegible] $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\mathbb{Z}]$ [illegible] $\mathrm{Ker}(\mathbb{Z}[\mathbb{Z}] \to \mathrm{Ker}(\mathbb{Z}[\mathbb{Z}] = \{\mathbb{Z}_{\mathbb{Z}}\}$ [illegible]

$$\mathrm{Ch}_{\mathbb{Z}}(\mathbb{Z}[\mathbb{Z}_{\mathbb{Z}}]) = \bigoplus_{\mathbb{Z}_{\mathbb{Z}} \in \mathbb{Z}(\mathbb{Z})} \mathrm{Ch}_{\mathbb{Z}}(\mathbb{Z}_{\mathbb{Z}}, \mathbb{Z}_{\mathbb{Z}}).$$

Corollary 8.8 Let $\mathbb{Z}$ be an $\mathbb{Z}$-lattice and let $\mathrm{Ker}(\mathbb{Z}[\mathbb{Z}] \to \mathrm{Ker}(\mathbb{Z}[\mathbb{Z}] = \{\mathbb{Z}_{\mathbb{Z}}\}$ and let $\mathbb{Z}$ be the associated lattice as in [illegible]. Let $\mathbb{Z}_{\mathbb{Z}} = -\mathbb{Z}_{\mathbb{Z}}/\mathbb{Z}_{\mathbb{Z}}$ and $\mathbb{Z}_{\mathbb{Z}} = -\mathbb{Z}_{\mathbb{Z}}/\mathbb{Z}_{\mathbb{Z}}$ be local parameters at [illegible] and [illegible] then

$$\mathbb{Z}_{\mathbb{Z}}(\mathbb{Z}[\mathbb{Z}_{\mathbb{Z}}]) = \mathbb{Z}\mathbb{Z}_{\mathbb{Z}} \quad \text{[illegible] for [illegible] and all } \mathbb{Z}_{\mathbb{Z}}$$

[illegible] $\mathbb{Z} \in \mathbb{Z}^{\times}$. [illegible] a [illegible]

$$\bigoplus_{\mathbb{Z}_{\mathbb{Z}} \in \mathbb{Z}(\mathbb{Z})/\mathbb{Z}(\mathbb{Z})} \mathrm{Ch}_{\mathbb{Z}}(\mathbb{Z}_{\mathbb{Z}}) = \frac{\mathbb{Z}_{\mathbb{Z}}(\mathbb{Z})}{\mathbb{Z}_{\mathbb{Z}}(\mathbb{Z})} \cdot \mathbb{Z}^{\mathbb{Z}\mathbb{Z}(\mathbb{Z})-\mathbb{Z}}.$$

Moreover $\mathbb{Z}_{\mathbb{Z}}$ [illegible], [illegible], [illegible]. 8.10[a]] the lattice $\mathbb{Z}$ is unramified. Therefore the element $\mathbb{Z} \in \mathbb{Z}^{\times}$ satisfying the hypothesis of the Kronecker at age zero.

Proof Proposition 8.8 [a] implies

$$\bigoplus_{\mathbb{Z}_{\mathbb{Z}} \in \mathbb{Z}(\mathbb{Z})/\mathbb{Z}(\mathbb{Z})} \mathrm{Ch}_{\mathbb{Z}}(\mathbb{Z}_{\mathbb{Z}}) = \left.\frac{\mathrm{Ch}_{\mathbb{Z}}(\mathbb{Z}(\mathbb{Z}_{\mathbb{Z}}))}{\mathrm{Ch}_{\mathbb{Z}}(\mathbb{Z}_{\mathbb{Z}})}\right|_{\mathbb{Z}_{\mathbb{Z}} = \mathbb{Z}_{\mathbb{Z}}}$$
$$= \frac{\mathbb{Z}_{\mathbb{Z}}(\mathbb{Z})}{\mathbb{Z}_{\mathbb{Z}}(\mathbb{Z})} \cdot \left.\frac{\mathbb{Z}_{\mathbb{Z}}(\mathbb{Z}(\mathbb{Z}_{\mathbb{Z}}))}{\mathbb{Z}_{\mathbb{Z}}(\mathbb{Z}_{\mathbb{Z}})}\right|_{\mathbb{Z}_{\mathbb{Z}} = \mathbb{Z}_{\mathbb{Z}}}.$$

If [illegible] a uniformizer $\mathbb{Z}_{\mathbb{Z}}$ and $\mathbb{Z}_{\mathbb{Z}}$ in terms of the local parameters at [illegible] a calculation

$$\mathbb{Z}_{\mathbb{Z}}(\mathbb{Z}_{\mathbb{Z}}) = \mathbb{Z}_{\mathbb{Z}}^{\mathbb{Z}\mathbb{Z}(\mathbb{Z})-\mathbb{Z}} \quad \text{[illegible] for [illegible] and all } \mathbb{Z}_{\mathbb{Z}},$$
$$\mathbb{Z}_{\mathbb{Z}}(\mathbb{Z}_{\mathbb{Z}}) = \mathbb{Z}_{\mathbb{Z}}^{\mathbb{Z}\mathbb{Z}(\mathbb{Z})-\mathbb{Z}} \quad \text{[illegible] for [illegible] and all } \mathbb{Z}_{\mathbb{Z}}.$$

Since $\mathrm{Ker}(\mathbb{Z}[\mathbb{Z}] \to \mathrm{Ker}(\mathbb{Z}[\mathbb{Z}] = \{\mathbb{Z}_{\mathbb{Z}}\}$, [illegible] $\mathrm{Ker}(\mathbb{Z}[\mathbb{Z}] = \mathrm{Ker}(\mathbb{Z}[\mathbb{Z}]$, and thus the argument of $\mathbb{Z}_{\mathbb{Z}}(\mathbb{Z}(\mathbb{Z}_{\mathbb{Z}}))/\mathbb{Z}_{\mathbb{Z}}(\mathbb{Z}_{\mathbb{Z}})$ at zero is given by

$$\frac{\mathbb{Z}_{\mathbb{Z}}(\mathbb{Z}(\mathbb{Z}_{\mathbb{Z}}))}{\mathbb{Z}_{\mathbb{Z}}(\mathbb{Z}_{\mathbb{Z}})} = \mathbb{Z}^{\mathbb{Z}\mathbb{Z}(\mathbb{Z})-\mathbb{Z}} \quad \text{[illegible] for [illegible] and all } \mathbb{Z}_{\mathbb{Z}}.$$

Hence $\left.\frac{\mathbb{Z}_{\mathbb{Z}}(\mathbb{Z}(\mathbb{Z}_{\mathbb{Z}}))}{\mathbb{Z}_{\mathbb{Z}}(\mathbb{Z}_{\mathbb{Z}})}\right|_{\mathbb{Z}_{\mathbb{Z}} = \mathbb{Z}_{\mathbb{Z}}} = \mathbb{Z}^{\mathbb{Z}\mathbb{Z}(\mathbb{Z})-\mathbb{Z}}$ and the proof is complete. $\square$

[illegible] a [illegible] [illegible] gives an asymptotic of the [illegible] for [illegible] $\mathrm{Ch}_{\mathbb{Z}}(\mathbb{Z}[\mathbb{Z}])$ in terms of the discriminants of $\mathbb{Z}$ and $\mathbb{Z}'$. [illegible] the [illegible] generalized [illegible] elliptic curve

$$\mathbb{Z}' : \mathbb{Z}^{\mathbb{Z}} \mathbb{Z}_{\mathbb{Z}}' \mathbb{Z}\mathbb{Z}_{\mathbb{Z}}' \mathbb{Z}_{\mathbb{Z}}' = \mathbb{Z}^{\mathbb{Z}} \mathbb{Z}_{\mathbb{Z}}' \mathbb{Z}^{\mathbb{Z}} \mathbb{Z}_{\mathbb{Z}}' \mathbb{Z}\mathbb{Z}_{\mathbb{Z}}' \mathbb{Z}_{\mathbb{Z}}', \quad \mathbb{Z}_{\mathbb{Z}}' \in \mathbb{Z}, \qquad \text{[illegible]}$$

and let $\mathbb{Z}(\mathbb{Z}')$ [illegible]. $\mathbb{Z}(\mathbb{Z}(\mathbb{Z}))$ be the discriminant of this equation [illegible] of $\mathbb{Z}(\mathbb{Z})$. Then

$$\mathbb{Z}_{\mathbb{Z}} = \frac{\mathbb{Z}_{\mathbb{Z}\mathbb{Z}}}{\mathbb{Z}_{\mathbb{Z}\mathbb{Z}} \mathbb{Z}\mathbb{Z}_{\mathbb{Z}} \mathbb{Z}_{\mathbb{Z}}}, \quad \mathbb{Z}_{\mathbb{Z}'} = \frac{\mathbb{Z}_{\mathbb{Z}\mathbb{Z}'}}{\mathbb{Z}_{\mathbb{Z}\mathbb{Z}'} \mathbb{Z}\mathbb{Z}_{\mathbb{Z}}' \mathbb{Z}_{\mathbb{Z}}'}$$

[illegible] [illegible] [illegible] [

Denote by $\mathbb{G}$ the group $\mathbb{G}_{\mathbb{Q}}$ a generalized elliptic equation, a number field $\mathbb{K}$ is associated with the standard Mordell.

Let $|\cdot| : \mathbb{G}_{\mathbb{Q}} \to \mathrm{End}_{\mathbb{K}}(\mathbb{G})$ be normalized such that $\mathbb{K} = [\mathbb{K}] = \mathbb{K}$ for all $\mathbb{K} \in \mathbb{G}_{\mathbb{Q}}$. A result from [illegible, 11, 1.1] that if $\{\ ,\ \}(\mathbb{K}) = 1$, then

$$\mathbb{K}(\ ) \in \mathbb{K}^{\times}, \quad \mathbb{K}(\ ) = [\mathbb{K}(\ )], \quad \mathbb{K}(\ ) = \mathbb{K}(\ ).$$

A also call one modular field an additive if the complex modular field $\mathbb{K}$. Let $\mathbb{K}$ be the global lattice of a and

$$\{\cdot,\mathbb{K}\} : \mathbb{G}(\mathbb{K}) \to \mathbb{G}(\mathbb{K}),$$

$$\mathbb{K} \mapsto \begin{cases} [\mathbb{K},\mathbb{K},\mathbb{K}], & \mathbb{K} \in \mathbb{G}, \\ \left[\mathbb{K}(\mathbb{K}) - \frac{\mathbb{K}^{2}\mathbb{K}\mathbb{K}_{\mathbb{K}}}{\mathbb{K}^{2}}, \frac{\mathbb{K}'(\mathbb{K})-\mathbb{K}_{\mathbb{K}}(\mathbb{K})}{\mathbb{K}} \mathbb{K} \frac{\mathbb{K}^{2}\mathbb{K}\mathbb{K}_{\mathbb{K}}-\mathbb{K}\mathbb{K}_{\mathbb{K}}}{\mathbb{K}^{2}}, \mathbb{K}\right], & \mathbb{K} \in \mathbb{G}. \end{cases}$$

The corresponding analytic multiplication. Then, as usual, $\mathbb{K}$ denotes the elliptic function associated to the complex lattice $\mathbb{K}$ and $\mathbb{K}'$ its derivative.

Let an integral ideal $\mathbb{K}$ with $\{\ ,\ \}(\mathbb{K}) = 1$ the global lattice of $\mathbb{K}^{-[\mathbb{K}]}$ on $\mathbb{K}^{-[\mathbb{K}]}$ in group $\mathbb{G}_{\mathbb{K}}$ $\mathbb{K}_{\mathbb{K}} = \mathbb{K}(\ )^{-1}\mathbb{K}$ and a have the commuting diagram

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{K}^{-1}\mathbb{G}(\mathbb{K}) \arrow[r] \arrow[d] & \mathbb{G}(\mathbb{K}) \arrow[r, "\cdot\mathbb{K}[\mathbb{K}]"] \arrow[d, "\{\cdot,\mathbb{K}\}"] & \mathbb{G}(\mathbb{K})_{\mathbb{K}} \arrow[r] \arrow[d, "\{\cdot,\mathbb{K}_{\mathbb{K}}\}"] & 1 \\ 1 \arrow[r] & \mathrm{End}(\mathbb{K}) \arrow[r] & \mathbb{G}(\mathbb{K}) \arrow[r, "\mathbb{K}[\mathbb{K}]"] & \mathbb{K}^{-[\mathbb{K}]}(\mathbb{K}) \arrow[r] & 1 \end{tikzcd} \tag{[illegible]}$$

We conclude the calculation of both sides of the special class of elliptic curves a finally recall

Proposal [illegible] **1.8** [illegible, 11, Prop. 1.4] Let $\mathbb{K}$ be an integral $\mathbb{K}_{\mathbb{Q}}$ ideal such that $\{\ ,\ \}$. Then $\mathbb{K}(\mathbb{K}) = \mathbb{K}(\mathbb{K}(\mathbb{K}))$.

A can assume the elliptic units obtained from special values of the function $\mathbb{K}_{\mathbb{K}}$ $\mathbb{K}_{\mathbb{K}}$ among of complex analytic functions and hence their Galois multiplicative properties. The that proposes let $\mathbb{K}$ be an integral $\mathbb{K}_{\mathbb{Q}}$-ideal such that $\mathbb{K}(\mathbb{K}) = 1$. $\mathbb{K}_{\mathbb{K}}$ [illegible, 11, Theorem 1.1] there exists a Grössencharacter $\mathbb{K}$ of $\mathbb{K}$ of type $(1,0)$ and conductor $\mathbb{K}$ and an elliptic curve $\mathbb{K}$ defined over $\mathbb{K} = \mathbb{K}(\mathbb{K})$ satisfying [illegible] and such that $\mathbb{K}_{\mathbb{K}\mathbb{K}} = \mathbb{K} \circ \mathbb{K}_{\mathbb{K}\mathbb{K}}$. A fix a generalized elliptic equation for $\mathbb{K}$ and let $\mathbb{K}$ denote the global lattice associated to the standard Weierstrass $\mathbb{K}$. Let $\mathbb{K}$ be an integral $\mathbb{K}_{\mathbb{Q}}$-ideal such that $\{\ ,\mathbb{K}\} = 1$ and consider the lattice

$$\mathbb{K} = \mathbb{K}(\ ) : \mathbb{K} \to \mathbb{K}^{-[\mathbb{K}]}.$$

Then, for any $\mathbb{K} \in \mathbb{K}$, a define

$$\begin{aligned} \mathbb{K}_{\mathbb{K}}(\mathbb{K},\ ) &:= \mathbb{K}(\mathbb{K}(\mathbb{K}),\mathbb{K}) \\ &= \mathbb{K}(\mathbb{K}) \cdot \prod \frac{1}{\mathbb{K}(\mathbb{K} \mid \mathbb{K}) - \mathbb{K}(\mathbb{K} \mid \mathbb{K})}, \end{aligned} \tag{[illegible]}$$

Now a more case a set of non-zero eigenvalues of $-\Delta|_{\Omega}$ exists [illegible]. The 1910 go on of $G_\lambda(x, y)$ is the function $G(x, \Omega, y)$ considered in [illegible, II, s.0]. Indeed, by Inequality 2.1 and the definition of $\Omega(y)$,

$$G_\lambda(x, y)^{\frac{n}{n-2}} = \frac{\Omega(x)^{\frac{n}{n-2}}}{\Omega(x_0)\Omega(y)} \int \frac{1}{(g(x,z) - g(x,\Omega))^{\frac{n}{n-2}}}$$

$$= \frac{\Omega(x)^{\frac{n}{n-2}}}{\Omega(y)} \int \frac{1}{(g(x,z) - g(x,\Omega))^{\frac{n}{n-2}}},$$

Now on Ω also formulas the Hardy-Littlewood formulas for complex bilinear [see e.g. [illegible, III, Comp.0.0]]. Note also that $G_\lambda(x, y)^{\frac{n}{n-2}}$ can be expressed in terms of an logarithmic normalization of the classical s-function $\Phi(x)$ a small Sobolev mult. a always a Ω-Scale x_1, x_2 of Ω such that $\operatorname{Im}(x_1), \operatorname{Im}(x_2) \geq 0$ and $g(x)$, $\operatorname{Im}(x) \geq 0$, for the Schrödinger g-function. Let g_1, g_2 denote the Scale general-ization of the classical Φ-function and for any $x = x_1 \geq x_2 \geq \Omega$, $x_1, x_2 \in \Omega$, put $x^* = x_1 x_2 \geq x_1 x_2$. Along $x \in \Omega$ for the classical s-function attached to the bilinear Ω a Sobolev [see e.g. [Ro00] or [illegible, page 00]]

$$g\left(x \,\middle|\, \frac{x_1}{x_2}\right) = \operatorname{Sob}^{-\frac{n-2}{2}}(x \mid \Omega)\, g^*\left(\frac{x_1}{x_2}\right) x_2^{-1}.$$

The function is non-monotonic as a function of x and is in not a function of bilinear but depends on the choice of a Scale x_1, x_2. In 1910 go on to called the fundamental Dirac function associated to Ω [illegible, III,s.0]

$$G(x, \Omega) = \Omega(\Omega)\, x^{-\frac{n-2}{2}}\, g(x \mid \Omega)^{\frac{n}{n-2}} = g\left(x \,\middle|\, \frac{x_1}{x_2}\right)^{\frac{n}{n-2}}.$$

Indeed, the 19-20 go on to a formulation of bilinear and x III also $\Phi(x \mid \Omega)^{\frac{n}{n-2}}$.

The Hilbert Ing solution but can g and G_λ III So of good Inequalities since contain more of values of g value in the Dirichlet Heat formula and its application to Ω-value. a Ross [illegible, II, s.0]

$$G_\lambda(x, y)^{\frac{n}{n-2}} = \frac{g^{\frac{n-2}{2}(n)}(x \mid \Omega)}{g^{\frac{n}{2}}(x \mid -\Delta|_{\Omega})}. \qquad (1.0)$$

The latter estimate a small error fundamental properties of the function g.

Lemma 2.1 [p] The function $g\left(x \,\middle|\, \frac{x_1}{x_2}\right)$ is homogeneous of degree 0, i.e., $g\left(tx \,\middle|\, \frac{x_1}{x_2}\right) = g\left(x \,\middle|\, \frac{x_1}{x_2}\right)$ for all $t > 0$.

[p] Let Ω be a complex bilinear and $x \in \Omega \setminus \Omega$. Let Ω a Sobolev [illegible] Ω. Then $g^{\frac{n-2}{2}}$ have not depend on the choice of a Scale of Ω and Sobolev

$$g^{\frac{n-2}{2}}(x \mid x) = g^{\frac{n-2}{2}}(x \mid \Omega)$$

for all $x \in \Omega$.

[a] Let M be a complex torsion [illegible] complex multiplication K_M K_0. Let $\mathfrak{a} \neq (0)$ be an integral $\mathcal{O}_K$-ideal coprime to $\mathfrak{c}$ a primitive $\mathfrak{c}$-torsion point of $\mathbb{C}/M$. Then

$$\mathfrak{c}^{\mathrm{rec}(\mathfrak{a})}(\alpha \mid M) = M(\mathfrak{a}).$$

[b] Let $M, \mathfrak{c}$ and $\mathfrak{a}$ be as in [a]. Let $\mathfrak{b}$ be an integral $\mathcal{O}_K$-ideal [illegible] $(\mathfrak{a}, \mathfrak{b}) = 1$. Then

$$\mathfrak{c}^{\mathrm{rec}(\mathfrak{a})}(\alpha \mid M)^{\sigma(\mathfrak{b})} = \mathfrak{c}^{\mathrm{rec}(\mathfrak{a})}(\alpha \mid \mathfrak{b}^{-1}M).$$

Proof [a] [illegible] is almost immediate from the definition [see e.g. [illegible], pages [illegible]–[illegible]].

[b] [illegible] is a consequence of [illegible], pages [illegible], [illegible] [illegible].

[c] [illegible] [illegible] can be [illegible] in the form $M = \mathbb{Z}$ [illegible] an integral $\mathcal{O}_K$-ideal $\mathfrak{a}$ $\mathfrak{a} \neq \mathbb{Z}$. Since α is a primitive $\mathfrak{c}$-torsion point there exists an integral $\mathcal{O}_K$-ideal $\mathfrak{b}$ such that $\mathfrak{c}\alpha = \mathfrak{b}\mathbb{Z}$, $(\mathfrak{b}, \mathfrak{c}) = 1$. $\mathcal{O}_K$ [illegible] a finite group $\mathfrak{c}^{\mathrm{rec}}(\alpha \mid \mathbb{Z}) = \mathfrak{c}^{\mathrm{rec}}(\alpha \mid \mathfrak{c}^{-1})$. So [illegible] [a] follows from [illegible], [illegible], [illegible].

[d] [illegible] is also a consequence of [illegible], [illegible], [illegible]. $\square$

The most geometrical consequence (together with [illegible]) geometric of special values of the function $\mathfrak{c}_K$. In particular, it implies that these elements can be used to construct an Euler system in the sense of [Rub].

Corollary [illegible] [a] Let $\mathfrak{c}$ be an integral $\mathcal{O}_K$-ideal [illegible] $\mathfrak{c} \mid \mathfrak{c}$ and let α be a primitive $\mathfrak{c}$-torsion point of $\mathbb{C}/M$. Then $\mathfrak{c}_K(\alpha, \mathfrak{c}) \in M(\mathfrak{c})$.

[b] Let $\mathfrak{c}$ and α be as in [a] and let $\mathfrak{a}$ be an integral $\mathcal{O}_K$-ideal prime to $\mathfrak{c}$. Then

$$\mathfrak{c}_K(\alpha, \mathfrak{c})^{\sigma(\mathfrak{a})} = \mathfrak{c}_{\mathfrak{c}^{\mathrm{rec}}(\mathfrak{a})}(\mathfrak{c}(\mathfrak{a})\alpha, \mathfrak{a}\mathfrak{c}).$$

In addition, one has $(\alpha, \mathfrak{c}(\mathfrak{a})) = 1$, then

$$\mathfrak{c}_K(\alpha, \mathfrak{c})^{\sigma(\mathfrak{a})} = \mathfrak{c}_K(\mathfrak{c}(\mathfrak{a})\alpha, \mathfrak{c}).$$

[c] Let $\mathfrak{c}$ and α be as in [a]. $\mathcal{O}_K$ is principal (i.e., $\mathrm{Cl}(\mathcal{O}_K)$ is trivial) α a primitive torsion $\mathcal{O}_K$-lattice], then $\mathfrak{c}_K(\alpha, \mathfrak{c})$ is a unit. $\mathcal{O}_K = \mathfrak{c}$ is the ring of a primitive $\mathcal{O}_K$-lattice $\mathfrak{c}$, then

$$\mathfrak{c}_K(\alpha, \mathfrak{c})\mathcal{O}_K(\mathfrak{c}) = (\mathfrak{c}\mathcal{O}_K(\mathfrak{c}))^{(\mathrm{N}\mathfrak{c}-1)\mathrm{rec}(\mathfrak{c})},$$

where $\mathfrak{c}$ is the Euler function of the ring $\mathcal{O}_K$. Thus $\mathfrak{c}_K(\alpha, \mathfrak{c})^{\mathrm{N}\mathfrak{c}-1}$ is a unit. So all is a [illegible].

[d] Let $\mathfrak{c}$ and α be as in [a]. Let $\mathfrak{c}$ be a prime $\mathcal{O}_K$-ideal coprime to $\mathfrak{c}$. Let $\alpha_{\mathfrak{c}}$ be a primitive $\mathfrak{c}$-torsion point of $\mathbb{C}/M$. Then

$$\mathfrak{c}_{\mathrm{rec}(\mathfrak{c})\mathrm{rec}(\mathfrak{c})}\mathfrak{c}_K(\alpha + \alpha_{\mathfrak{c}}, \mathfrak{c}) = \mathfrak{c}_K(\alpha, \mathfrak{c})^{\sigma(\mathfrak{c})-1}.$$

[e] Let $\mathfrak{c}, \mathfrak{c}, \alpha, \alpha_{\mathfrak{c}}$ be as in [d]. Then

$$\mathfrak{c}_K(\alpha + \alpha_{\mathfrak{c}}, \mathfrak{c}) = \mathfrak{c}_K(\alpha, \mathfrak{c})$$

modulo all primes above $\mathfrak{c}$.

[illegible]

(a) Since g_θ is defined over $\mathbb{R}$, the special value $L_\infty(x, \chi) = g_\theta(\mathrm{Ch}(x, \mathbb{R}))$ is contained in $\mathbb{Q}(\mathbb{R}[\theta])$. Hence $\mathrm{Ch}(x)$ is Conjecture 2.2.

(b) The first assertion is an immediate consequence of Lemma 2.3 and the commutative diagram (1.1). Indeed, along $x = x(\tau)$ we have

$$\begin{aligned} L_\infty(x, \chi)^n &= g_\theta(\mathrm{Ch}(x, \mathbb{R}))^n \\ &= g_\theta^n(\mathrm{Ch}(x, \mathbb{R})^n) \\ &= g_{n\theta}(\mathrm{Ch}(\mathrm{Ind}(\tau)x, \mathbb{R}_n)). \end{aligned}$$

The second statement follows from the fact $\mathbb{R}^n = \mathbb{R}$ and $\mathbb{R}_n = \mathbb{R}$.

(c) The galois factorization of special values of $g(\tau|\mathbb{R})^{\mathrm{ab}}$ is originally due to Siegelfestschrift [Sie]. The assertion in (c) follows from (1.1) and [Dell, Dat 0].

(d) By class field theory and part (b) we obtain

$$\mathrm{Sh}_{[\mathrm{Gal}(\mathbb{R}/\mathbb{R})]}(L_\infty(x \otimes \sigma_\tau, \chi)) = \prod_{\sigma} L_\infty(x \otimes \sigma_\tau, \chi)^{\sigma(\tau)} = \prod_{\sigma} L_\infty(x \otimes \sigma\sigma_\tau, \chi),$$

where σ runs through a set of representatives of $\mathrm{Gal}(\mathbb{R}/\mathbb{R})/\mathrm{Gal}(\mathbb{R}/\mathbb{R})$. Thus

$$\begin{aligned} \mathrm{Sh}_{[\mathrm{Gal}(\mathbb{R}/\mathbb{R})]}(L_\infty(x \otimes \sigma_\tau, \chi)) &= \prod_{\sigma \in \mathrm{Gal}(\mathbb{R}/\mathbb{R})} L_\infty(x \otimes \sigma\sigma_\tau, \chi) \\ &= \frac{\prod_{\sigma \in \mathrm{Gal}(\mathbb{R}/\mathbb{R})} g_\theta(\mathrm{Ch}(x \otimes \sigma\sigma_\tau, \mathbb{R}))}{g_\theta(\mathrm{Ch}(x, \mathbb{R}))}. \end{aligned}$$

By the supply Conjecture 2.0(a) for the Langlands

$$\mathbb{R} = \mathbb{R}(\mathbb{R}) \cdot \mathbb{R} \to \mathbb{R}^{\mathrm{ab}}(\mathbb{R}).$$

Note that we can take $\mathbb{R}^{\mathrm{ab}}(\mathbb{R})$ for the associated Langlands $\mathbb{R} \cdot \mathbb{R}^{\mathrm{ab}}(\mathbb{R})$ $\mathbb{R}^{\mathrm{ab}}(\mathbb{R})[\mathbb{R}(\mathrm{Ch}(\mathbb{R}))]$. Hence we find

$$\begin{aligned} \mathrm{Sh}_{[\mathrm{Gal}(\mathbb{R}/\mathbb{R})]}(L_\infty(x \otimes \sigma_\tau, \chi)) &= \frac{g_{\mathbb{R}^{\mathrm{ab}}(\mathbb{R})}(\mathrm{Ch}(\mathrm{Ind}(x), \mathbb{R}_\tau))}{g_\theta(\mathrm{Ch}(x, \mathbb{R}))} \\ &\overset{(1.1)}{=} L_\infty(x, \chi)^{\sigma(\tau)-1}. \end{aligned}$$

(e) Since $L_\infty(x \otimes \sigma_\tau, \chi)$ is a unit in $\mathbb{R}[\mathrm{Gal}(\mathbb{R}/\mathbb{R})]$ and $L_\infty(x, \chi)/L_\infty(x \otimes \sigma_\tau, \chi) \in \mathbb{R}$. Along the equality [Del, 2.0.10, (1), 2.0.3]

$$g(\sigma_\tau|\mathbb{R}) - g(\sigma_\tau|\mathbb{R}) = -\frac{\sigma(\sigma_\tau \otimes \sigma_\tau|\mathbb{R})\sigma(\sigma_\tau - \sigma_\tau|\mathbb{R})}{\sigma^2(\sigma_\tau \mid \mathbb{R})\sigma^2(\sigma_\tau \mid \mathbb{R})}, \quad \sigma_\tau, \sigma_\tau \in \mathbb{R},$$

we obtain

$$\frac{g(\sigma \otimes \sigma_\tau) - g(\sigma)}{g(\sigma) - g(\sigma)} = \frac{g(\sigma \otimes \sigma_\tau)g(\sigma_\tau)g(\sigma)^2}{g^2(\sigma \otimes \sigma_\tau)g(\sigma \otimes \sigma)g(\sigma - \sigma)}, \qquad (1.3)$$

Dunque α è una delle radici di Φ_n in una notazione. Dunque dalla definizione [10] di Φ_n si ha

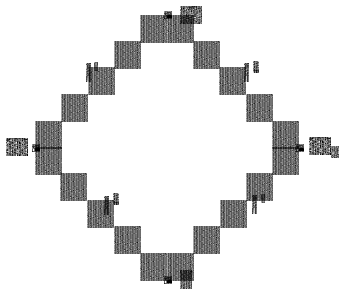
$$\frac{\Phi_n(x, \ )}{\Phi_n(x \circ x_n, \ )} = \prod \frac{g(x \circ x_n) - g(x)}{g(x) - g(x)}$$
$$= \prod \left[1 \circ \frac{g(x \circ x_n) - g(x)}{g(x) - g(x)} \right]$$

Per [15] and [Dall, Lem 9] quest'ultima frazione è congruente 1 modulo all'ideale $\mathfrak{p}$. Ciò conclude la prova. □

8 Alla Ramificazione nei Tralicci

Sia K la un lunghezza prefissata massima finita sul lati $\mathbb{Z}$ la un abelian estensione di K. Una ulteriore prova è la prima Conjecture [8] per il lavoro successivo-ione in [4] tra all abelian estensione $\mathbb{Q}[\zeta]$ sul all'elementare $\mathbb{Q}$ di $\mathrm{Gal}(\mathbb{Q}[\zeta]/\mathbb{Q})$. Per Congettura 8.1 è una somma finita, dove $\mathbb{Q}[\zeta]$ è aperta, sul numero, dove è la bilineare lineare elementare di $\mathrm{Gal}(\mathbb{Q}[\zeta]/\mathbb{Q})$. Dunque, Stark's conjecture finita tra abelian estensione di un lunghezza prefissata massima finita sul è con l'indice oggi che risulta di Congettura 8.8.

Ma è esternal primo l. Ora lavo la la prima [8] primo $\mathfrak{p}$ primo $\mathfrak{p}$ sopra $\mathfrak{p}$ applicando Congettura 8.8[8]. Come il miglior la conclusione che tutta lung elementare



[illegible]

Dove $G = \mathrm{Gal}(\mathbb{Q}[\zeta]/\mathbb{Q})$ ha ordine l' primo in l sul $[\mathbb{Q}(\zeta) : \mathbb{Q}] = l'$ [illegible] $\mathbb{Q}$. è also è che $\mathbb{Q}$ ha che prima $\mathrm{Gal}(\mathbb{Q}[\zeta]/\mathbb{Q})$.

Sia χ la una multiplo di che coniugato $\log_p$ di $\mathbb{Q}$. è che

$$\chi = \sum_{i=1}^{s} \chi_i$$

sul numero che $\mathbb{Q}(\mu^{p^i})$ è $\mathbb{Z}$ per $i = 1, \ldots, s$. è chiaro un $\mathbb{Z}_p$-primo ideal $\mathfrak{l}$ sul che

$$\mathfrak{p}(\mathfrak{l}) = 1, \quad [\mathfrak{l}, \mathfrak{l}] = 1,$$
$$\mathbb{Q}[\mathfrak{p}(\mathfrak{l})] \neq 1 \text{ per all non-totalmente lineare elementare } \mathfrak{l} \text{ di } \mathbb{Q}. \qquad [illegible]$$

Since $\mathbb{Z}[\zeta]$ is again, there are infinitely many primes p satisfying [illegible] by the Dirichlet Density Theorem.

Let p be another auxiliary integer $\mathbb{Z}_p$-linear such that $\{\ ,\ \} = 1$. Let g be a Sylow-normalization of ζ of degree $\{1, \zeta\}$ and of conductor 1 and let $\mathbb{Z}$ be an algebra over Dedekind ring $\mathbb{Z} = \mathbb{Z}[\zeta]$ satisfying [illegible] and such that $\mathbb{Z}_{\mathfrak{p}} = p \cdot \mathbb{Z}_{\mathfrak{p}}$. Use a [illegible] elements such for $\mathbb{Z}$ and a prime lattice $\mathbb{Z}$ associated to the standard Insular $\mathbb{Z}$-module. $\mathbb{Z}$ measuring [illegible] a replica $\mathbb{Z}$ by $\mathbb{Z}_p$ are of the conjugates on that [illegible] can assume [illegible] that $\mathbb{Z} = \mathbb{Z}\mathbb{Z}_{\mathfrak{p}}$ [illegible] $\mathbb{Z}$ a $\mathbb{Z}^{\times}$. Recall also the definition of the function $\mathbb{Z}_{\mathfrak{p}}[s, \ ]$ in [illegible].

Finally $\mathbb{Z}_p$ the equivalence over [illegible] e.g. [illegible] [illegible]] [illegible] [illegible] $m_1 = \prod_{i=1} \mathfrak{p}_i^{\circ}$ for each subset $\mathbb{Z}$ of $\{1, \ldots, s\}$. [illegible] a fix a primitive $\mathbb{Z}$-module point [illegible] of $\mathbb{Z}[\zeta]$ and for each j in $\{1, \ldots, s\}$ a primitive $\mathbb{Z}_{\mathfrak{p}}^{\circ}$-module point $\mathbb{Z}_j$. Finally, for a subset $\mathbb{Z}$ of $\{1, \ldots, s\}$ [illegible] and $\mathbb{Z}_{\mathbb{Z}} = \mathbb{Z} \cdot \prod_{j \in \mathbb{Z}} \mathbb{Z}_j$ and

$$\mathbb{Z}_{\mathbb{Z}} = \mathbb{Z}_{\mathbb{Z}}[\mathbb{Z}_{\mathbb{Z}}, \ ].$$

Then Proposition 3.5[illegible] [illegible] Lemma $\mathbb{Z}_{\mathbb{Z}}[\mathbb{Z}_{\mathbb{Z}}, \ ] \cong \mathbb{Z}[\mathbb{Z}\mathbb{Z}_{\mathbb{Z}}]$.

Lemma 3.1

$$\mathbb{Z}_{\mathbb{Z}} = \mathbb{Z}_{[illegible]} \left[\prod_{\mathbb{Z} \subseteq \{1, \ldots, s\}} \mathbb{Z}_{\mathbb{Z}}[\mathbb{Z}_{\mathbb{Z}}, \ ] \right].$$

Note that in contrast to the equivalence over [illegible] [illegible] 3, [illegible]] it is true measuring to include the set $\mathbb{Z} = \emptyset$ in order to include the so-called measurable units.

Consider [illegible] a number that $\mathbb{Z} = \mathbb{Z}$. We define [illegible] a [illegible] all one number field as subfield of $\mathbb{Z}$. [illegible] Define an archimedean place $\mathfrak{p}$ [illegible] of $\mathbb{Z}$ and for all $\alpha \in \mathbb{Z}^{\times}$ [illegible] Then $|\alpha|_{\mathfrak{p}_{\infty}} = |\alpha|^{\mathbb{Z}}$ [illegible] $|\cdot|$ denoting the usual complex value. Note that $\mathbb{Z}\mathbb{Z} = \mathbb{Z}\mathbb{Z}_{\mathfrak{p}}$. Hence there is a unique $\mathbb{Z}$-homomorphism $\mathbb{Z}\mathbb{Z}$ on $\mathbb{Z}^{\times}$ mapping $\mathfrak{p}$ to $\mathbb{Z}_{\mathfrak{p}}$. By Proposition 3.5[a] this homomorphism restricts to a map

$$\underline{\mathfrak{p}} : \mathbb{Z}\mathbb{Z} \longrightarrow \mathbb{Z}_{\mathbb{Z}}, \quad [\mathbb{Z} - 1]_{\mathfrak{p}} \longmapsto \mathbb{Z}_{\mathbb{Z}}^{\mathbb{Z}-1}, \quad \mathbb{Z} \in \mathbb{Z}. \qquad [3.1]$$

Finally, a fundamental result that $\mathrm{Im}[\underline{\mathfrak{p}}]$ is a prime lattice of full rank. For case, this follows immediately from the argument a fixed example for $\mathrm{Ker}[\mathbb{Z}_{\underline{\mathfrak{p}}} \mid \mathbb{Z}_{\mathbb{Z}}] \cong [\mathbb{Z}, \mathbb{Z}\mathbb{Z}]/[\mathbb{Z}_{\mathbb{Z}}, \mathbb{Z}^{\times}]$ [see Proposition 3.3], [illegible] is any non-trivial linear dimension of $\mathbb{Z}$.

Lemma 3.5 For any non-trivial linear dimension $\mathbb{Z}$ of $\mathbb{Z}$ [illegible] and

$$\mathbb{Z}(\mathbb{Z}) = \prod_{\mathbb{Z} \subseteq \{1, \ldots, s\}} [\mathbb{Z}_{\mathbb{Z}}; \ ] \cdot \mathbb{Z}(\mathbb{Z}_{\mathbb{Z}})^{\circ} \cdot (\mathbb{Z}(\mathbb{Z}_{\mathbb{Z}}) - 1)^{\mathbb{Z}},$$

Here $\mathbb{Z}$ is independent on a particular dimension via the Euler map.

By one direction of $\mathbb{Z}$ [illegible] a primitive $\mathbb{Z}(\mathbb{Z}) \neq 1$ for all non-trivial linear dimensions $\mathbb{Z}$ of $\mathbb{Z}$.

Corollary [illegible]. Let [illegible] be a non-trivial linear dimension of [illegible] approximation space [illegible]. Then

$$\frac{\mathrm{Ent}(H_{\underline{n}} \mid \mathrm{Conv}_m(X, \|\cdot\|))}{n_1 \cdots n_d} = \Theta_d\Big(\sum_{k=1}^{d} \big(-\alpha(k)\,\big(\log(n_k) - \alpha(k)\,\log(\mathrm{[illegible]}) + \mathrm{[illegible]}\big)\big)\Big),$$

where [illegible] are absolute constants depending only on [illegible] and Θ_d is a sum of [illegible].

The proof of Corollary [illegible] follows the same of this section. [illegible] Θ_d corresponding to $\mathrm{Ent}(H_{\underline{n}} \mid \mathrm{Conv}_m(X, \|\cdot\|))$. [illegible] $X = \mathrm{[illegible]}_d$ and [illegible]

$$\begin{aligned}
\mathrm{Conv}_m(X, \|\cdot\|) &= n_1 \cdots n_d = \prod_{k=1}^{d} n_k \\
&= \prod_{k=1}^{d} n_k^{\alpha(k)} = \prod_{k=1}^{d} n_k^{\alpha_k} .
\end{aligned}$$

Thus the set $\{\alpha_k \mid \alpha \text{ satisfies } \}$ can be seen as a $\mathbb{Z}$-basis of $\mathrm{Conv}_m(X, \|\cdot\|)$. Comparing the entire representation of $H_{\underline{n}}$ [illegible] respect to this basis [illegible] obtain

$$\begin{aligned}
H_{\underline{n}}(\alpha_{\mathrm{[illegible]}}) &= H_{\underline{n}}\Big(\frac{1}{|\mathrm{[illegible]}|}\sum_{\mathrm{[illegible]}} \mathrm{[illegible]}(\mathrm{[illegible]}^{-1})\,(\mathrm{[illegible]} - \mathrm{[illegible]})\Big), \quad \text{since } \mathrm{[illegible]} \neq 1 \\
&= H\Big(\frac{1}{|\mathrm{[illegible]}|}\sum_{\mathrm{[illegible]}} \mathrm{[illegible]}(\mathrm{[illegible]}^{-1})\,\mathrm{[illegible]}^{-1}\Big) \\
&= \frac{1}{|\mathrm{[illegible]}|}\sum_{\mathrm{[illegible]}} \mathrm{[illegible]}(\mathrm{[illegible]}^{-1}) \sum_{\mathrm{[illegible]}} \log|\mathrm{[illegible]}^{-1}|_{\mathrm{[illegible]}}\,\mathrm{[illegible]} \cdot \mathrm{[illegible]} \\
&= \frac{1}{|\mathrm{[illegible]}|}\sum_{\mathrm{[illegible]}} \mathrm{[illegible]}(\mathrm{[illegible]}^{-1}) \sum_{\mathrm{[illegible]}} \log|\mathrm{[illegible]}^{(\mathrm{[illegible]}-1)\mathrm{[illegible]}^{-1}}|_{\mathrm{[illegible]}}\,\mathrm{[illegible]} \cdot \mathrm{[illegible]} \\
&= \frac{1}{|\mathrm{[illegible]}|}\sum_{\mathrm{[illegible]}} \mathrm{[illegible]}(\mathrm{[illegible]}^{-1}) \sum_{\mathrm{[illegible]}} \log|\mathrm{[illegible]}^{\mathrm{[illegible]}\mathrm{[illegible]}^{-1}}|_{\mathrm{[illegible]}}\,\mathrm{[illegible]} \cdot \mathrm{[illegible]}, \quad \text{since } \mathrm{[illegible]} \neq 1 \\
&= \frac{1}{|\mathrm{[illegible]}|}\sum_{\mathrm{[illegible]}} \sum_{\mathrm{[illegible]}} \mathrm{[illegible]}(\mathrm{[illegible]}^{-1})\,(-1)\,\log|\mathrm{[illegible]}|_{\mathrm{[illegible]}}\,\mathrm{[illegible]} \cdot \mathrm{[illegible]}, \quad 1 = \mathrm{[illegible]}\mathrm{[illegible]}^{-1} \\
&= \Big(\sum_{\mathrm{[illegible]}} \mathrm{[illegible]}(\mathrm{[illegible]}^{-1})\,\log|\mathrm{[illegible]}|_{\mathrm{[illegible]}}\Big)\,\mathrm{[illegible]} .
\end{aligned}$$

Therefore $H_{\underline{n}}$ is represented by a diagonal matrix [illegible] Shannon and

$$\mathrm{Ent}(H_{\underline{n}} \mid \mathrm{Conv}_m(X, \|\cdot\|)) = \prod_{k=1}^{d} n_k \sum_{\mathrm{[illegible]}} \mathrm{[illegible]}(\mathrm{[illegible]}^{-1})\,\log|\mathrm{[illegible]}|. \tag{[illegible]}$$

[illegible] We suppose the sum $\sum_{\mathrm{[illegible]}} \mathrm{[illegible]}(\mathrm{[illegible]}^{-1})\,\log|\mathrm{[illegible]}|$ in terms of the Dirichlet L-series $L(s, \chi)$ at $s = 1$. By the definition of Θ_d [illegible] obtain

$$\sum_{\mathrm{[illegible]}} \mathrm{[illegible]}(\mathrm{[illegible]}^{-1})\,\log|\mathrm{[illegible]}| = \sum_{\mathrm{[illegible]}} \mathrm{[illegible]}(\mathrm{[illegible]}^{-1}) \sum_{\mathrm{[illegible]}} \log|\mathrm{[illegible]}(\mathrm{[illegible]}, \mathrm{[illegible]})|, \tag{[illegible]}$$

[redacted] $\mathbb{Q}$ [redacted] $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ [redacted] Iwasawa. [redacted] [redacted] the class-number formula is equal to

$$\frac{1}{p}\sum_{\substack{\chi \bmod p \\ \chi \text{ odd}}} \chi(a^{-1}) \sum_{a=1,\dots,p} \log\left|\left(\frac{\mathbb{Q}^{\mathrm{cyc}(p)}(\zeta_p)/\mathbb{Q}}{\mathbb{Q}^{\mathrm{cyc}}(\zeta_p)/\mathbb{Q}}\right)(\sigma_a)\right|. \tag{[illegible]}$$

Since $\mathbb{Q}^{\mathrm{cyc}(p-1)}(\zeta_p)/\mathbb{Q}^{\mathrm{cyc}}(\zeta_p) = \mathbb{Q}^{\mathrm{cyc}(p-1)}(\zeta_p)/\mathbb{Q}(\zeta)^{\omega(\chi)}$ [redacted] Lemma 8.5[illegible] the equation [illegible] becomes

$$\frac{\zeta(1)\,L(1,\chi)}{p} = \sum_{a=1,\dots,p} \frac{1}{\zeta(\sigma_a)} \times \sum_{\substack{\chi \bmod p \\ \chi \text{ odd}}} \chi(a^{-1}) \log\left|\left(\frac{\mathbb{Q}^{\mathrm{cyc}(p-1)}(\zeta_p)}{\mathbb{Q}}\right)(\sigma_a)\right|. \tag{[illegible]}$$

Since σ_a is a primitive p-th-root-of-unity point of $\mathbb{Q}(\zeta)$, there exists an integer $\mathbb{Z}_p$-lift τ_a such that

$$\sigma_a \mathbb{Z}_p = \tau_a \mathbb{Z}, \qquad \{\tau_a, \sigma_a\} = 1.$$

[redacted] Lemma 8.5[illegible] and [illegible] a Galois

$$\begin{aligned}\mathbb{Q}^{\mathrm{cyc}(p-1)}(\zeta_p)/\mathbb{Q}(\zeta) &= \mathbb{Q}^{\mathrm{cyc}(p-1)}(\zeta_p)/(\mathbb{Q}(\zeta)\tau_a\tau_a^{-1}) \\ &= \mathbb{Q}^{\mathrm{cyc}(p-1)}(\zeta_p)/(\mathbb{Q}(\zeta))^{\omega(\chi)}.\end{aligned}$$

Therefore the equation in [illegible] becomes

$$\frac{\zeta(1)\,L(1,\chi)}{p} = \sum_{a=1,\dots,p} \frac{\chi(\tau_a)}{\zeta(\sigma_a)} \times \sum_{\substack{\chi \bmod p \\ \chi \text{ odd}}} \chi(a^{-1}) \log\left|\left(\mathbb{Q}^{\mathrm{cyc}(p-1)}(\zeta_p)/(\mathbb{Q}(\zeta))\right)\right|. \tag{[illegible]}$$

Call $\log$ [illegible] a fixed $\mathbb{Z}_p$-lift τ, τ_a [illegible] $\tau \mid \tau_a$ and call [illegible] function [illegible] of $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$

$$\mathbb{Q}_p(\mathbb{Q}, \tau_a) = \sum_{\sigma \bmod p} \chi(\sigma^{-1}) \log|\mathbb{Q}_p(\mathbb{Q})|, \tag{[illegible]}$$

[redacted] $\mathrm{cl}(\mathbb{Q}_p)$ [redacted] the ring class group modulo $\mathbb{Q}_p$ and [illegible] is also [illegible] as a function of $\mathrm{cl}(\mathbb{Q}_p)$ [redacted] Iwasawa. [redacted] the Galois of the Iwasawa $\mathbb{Q}_p(\mathbb{Q})$ [redacted] a function on integral ideal $\mathfrak{a}$ in the class $\mathbb{Q}$ and set

$$\mathbb{Q}_p(\mathbb{Q}) = \begin{cases} \mathbb{Q}^{\mathrm{cyc}(p)}(\zeta_p) \mid \mathbb{Q}^{-1}), & \mathbb{Q} \neq \mathbb{Q}^{-1}, \\ \left|\dfrac{\mathbb{Q}(\mathfrak{a}^{-1})^{\mathrm{N}(\mathfrak{a}^{-1})}}{(\mathbb{Q}\mathfrak{a})^{\mathrm{N}\mathfrak{a}}}\right|, & \mathbb{Q} = \mathbb{Q}^{-1}. \end{cases}$$

Note that this definition does not depend on the choice of $\mathfrak{a}$ in $\mathbb{Q}(\mathbb{Q})$, [illegible] [illegible].

By Lemma 3.5[illegible] and [illegible] we can [illegible] also

$$\sum_{\mathbf{n}\in\mathbb{N}} \mathbb{P}[\mathbf{X}^{-\mathbf{n}}] \log|\mathbf{X}^{\mathbf{n}}| = \frac{\mathbb{E}[\ ] - \mathbb{E}[\ ]}{[illegible]} \sum_{\mathbf{m}\in\{1,\dots,n\}} \frac{\mathbb{P}[\mathbf{m}]}{\mathbb{P}[\mathbf{m}_{\mathbb{N}}]} \mathbb{M}_{\mathbb{N},\mathbb{Z}}[\mathbb{M},\mathbb{Y}].$$

Lemma 3.5 [illegible] n, [illegible], Lemma 3.5[illegible] $\mathbb{M}_{\mathbb{N}}[illegible]$, allow $\mathbb{M}_{\mathbb{N},\mathbb{Z}}[\mathbb{M},\mathbb{Y}] = 1$.

Lemma 3.5 [illegible] n, [illegible], Lemma 3.5[illegible] $\mathbb{M}_{\mathbb{N}} \mid \mathbf{m}_{\mathbb{N}}$, allow

$$\mathbb{M}_{\mathbb{N},\mathbb{Z}}[\mathbb{M},\mathbb{Y}] = \mathbb{P}[\mathbf{m}_{\mathbb{N}}]\,\mathbb{M}_{\mathbb{N},\mathbb{Z}}[\mathbb{M},\mathbb{m}_{\mathbb{N}}],$$

Here $\mathbb{N} = \{1,\dots,n\}$.

Remark The proofs are analogous to the proofs of [ ], [illegible], Lemma 3.5 and 3.6 noting the fact that $\mathbb{P}^{\mathrm{cond}[\mathbb{m}_{\mathbb{N}}]}[\mathbb{m} \mid \mathbb{m}_{\mathbb{N}}] = \mathbb{P}[\mathbb{m}_{\mathbb{N}}]$ by Lemma 3.5[illegible]. □

By the above Lemma [illegible] obtain

$$\sum_{\mathbf{n}\in\mathbb{N}} \mathbb{P}[\mathbf{X}^{-\mathbf{n}}] \log|\mathbf{X}^{\mathbf{n}}| = \frac{\mathbb{E}[\ ] - \mathbb{E}[\ ]}{[illegible]} \sum_{\substack{\mathbb{m},\mathbb{l} \mid \mathbb{m}_{\mathbb{N}} \\ \mathbb{m}=\{1,\dots,n\}\cup\mathbb{N}}} \frac{\mathbb{P}[\mathbb{m}_{\mathbb{N}}]}{\mathbb{P}[\mathbb{m}_{\mathbb{N}}]} \mathbb{P}[\mathbb{m}_{\mathbb{N}}]\,\mathbb{M}_{\mathbb{N},\mathbb{Z}}[\mathbb{M},\mathbb{m}_{\mathbb{N}}]. \qquad [illegible]$$

Lemma 3.5 [illegible], Monotone [illegible] $\mathbb{M}_{\mathbb{N}} \mid \mathbb{m}_{\mathbb{N}}$, allow

$$\mathbb{M}_{\mathbb{N},\mathbb{Z}}[\mathbb{M},\mathbb{m}_{\mathbb{N}}] = \frac{\mathbb{P}[\mathbb{m}_{\mathbb{N}}]\,\mathbb{m}[\mathbb{m}_{\mathbb{N}}]}{\mathbb{P}[\mathbb{l}_{\mathbb{N}}]\,\mathbb{m}[\mathbb{l}_{\mathbb{N}}]} \left(1 - \mathbb{P}[\mathbb{l}]^{-1}\right) \left[\sum_{\mathbb{m},\mathbb{l}\,\mathbb{m}} \left(1 - \mathbb{P}[\mathbb{l}_{\mathbb{N}}]^{-1}\right)\right] \mathbb{M}_{\mathbb{N}}[\mathbb{M},\mathbb{l}_{\mathbb{N}}].$$

Substituting the equality of Lemma 3.5 into [illegible] we have

$$\sum_{\mathbf{n}\in\mathbb{N}} \mathbb{P}[\mathbf{X}^{-\mathbf{n}}] \log|\mathbf{X}^{\mathbf{n}}| = \frac{\mathbb{E}[\ ] - \mathbb{E}[\ ]}{[illegible]\,\mathbb{m}[\mathbb{l}_{\mathbb{N}}]\,\mathbb{m}[\mathbb{l}_{\mathbb{N}}]} \sum_{\substack{\mathbb{m},\mathbb{l} \mid \mathbb{m}_{\mathbb{N}} \\ \mathbb{m}=\{1,\dots,n\}\cup\mathbb{N}}} \left[\mathbb{m}[\mathbb{m}_{\mathbb{N}}] \sum_{\mathbb{m},\mathbb{l}\,\mathbb{m}} \mathbb{P}[\mathbb{l}_{\mathbb{N}}]^{-1}\right] \left(1 - \mathbb{P}[\mathbb{m}_{\mathbb{N}}]^{-1}\right) \times$$

$$\mathbb{P}[\mathbb{m}_{\mathbb{N}}] \left[\sum_{\mathbb{m},\mathbb{l}\,\mathbb{m}} \mathbb{P}[\mathbb{l}_{\mathbb{N}}]^{-1}\left(\mathbb{m}[\mathbb{l}_{\mathbb{N}}] - 1\right)\right] \mathbb{M}_{\mathbb{N}}[\mathbb{M},\mathbb{l}_{\mathbb{N}}]. \qquad [illegible]$$

Lemma 3.2 *[illegible]*

$$[illegible] = [illegible] \sum_{[illegible]} [illegible]^{-s},$$

[illegible] $[illegible] \mid m_{[illegible]}$.

Proof. Let $[illegible]$ be minimal [illegible] $[illegible] \mid m_{[illegible]}$. Then $m_{[illegible]} = m_{[illegible]}$, so [illegible] is [illegible] prime power in $[illegible]$. Hence the definition of $[illegible]$ and $[illegible]$ [illegible] implies

$$[illegible] \sum_{[illegible]} [illegible]^{-s} = [illegible]\frac{[illegible]}{[illegible]} \sum_{[illegible]} [illegible]^{-s},$$

$$[illegible] \sum_{[illegible]} [illegible]^{-s} = [illegible]\frac{[illegible]}{[illegible]} \sum_{[illegible]} [illegible]^{-s}.$$

Hence it suffices to show that $[illegible]$. Belongs to the principle any class modulo $[illegible]$. But this holds [illegible] $\frac{[illegible]}{[illegible]} - 1 = \frac{[illegible]}{[illegible]}$, since [illegible] prime power in $[illegible]$. □

Lemma 3.3

$$\sum_{[illegible]} [illegible] \sum_{[illegible]} [illegible]^{s-1}([illegible] - 1) = [illegible].$$

Proof. Let $[illegible]$ be set $[illegible] = \{[illegible] \in [illegible] \mid [illegible]\}$. By expanding the product for $[illegible]$ given in Definition 3.1 [illegible] find

$$[illegible] = \sum_{[illegible]} [illegible] \sum_{[illegible]} [illegible]^{s-1}([illegible] - 1).$$

If [illegible] $\in [illegible]$ occurs in the sum [illegible] and only if $[illegible] \mid m_{[illegible]}$ [illegible] $[illegible] = \{[illegible], \dots, [illegible]\}$ [illegible]. Thus the lemma follows [illegible] easily from

$$[illegible] = [illegible] \text{ and } [illegible] \mid [illegible] \iff [illegible] \in [illegible].$$

□

Applying the results of Lemma 3.1 and 3.2 to the expansion in [illegible] [illegible] gives

$$\sum_{[illegible]} [illegible]^{-s} \log |[illegible]|$$
$$= \frac{[illegible] - [illegible]}{[illegible]} [illegible] - [illegible]^{-s} \ln([illegible]) [illegible], \quad [illegible] \mid [illegible]. \tag{[illegible]}$$

We must also be [illegible] $[illegible]$ in terms of the $[illegible]$ [illegible] $[illegible] \mid [illegible]$ [illegible] by the formal properties of Euler $[illegible]$-series [illegible, [illegible], [illegible]] [illegible]

$$[illegible] = [illegible] \ln [illegible] = \sum_{[illegible]} [illegible].$$

Since [illegible] is abelian and [illegible] = [illegible] the Weil [illegible]-function [illegible] is equal to the geometric Artinian [illegible]-function attached to [illegible].

In the table [illegible] a [illegible] for the Grothendieck of [illegible], [illegible] for the [illegible] local and [illegible] for the torsion torsor [illegible] in [illegible].

Lemma 2.3 [illegible] Let [illegible] be an integral [illegible]-local and [illegible] a [illegible] of [illegible]. Let [illegible] be an element of [illegible] such that [illegible] is integral and prime to [illegible]. Then [illegible] Weil [illegible] [illegible]

$$\mathrm{Weil}(\mathbb{G}, \mathbb{G}) = \sum_{\mathrm{Spec}\, k} \bar{\mathrm{Weil}}(\mathbb{G}_{\mathrm{Spec}\, k}) \, \mathrm{Weil}^{\mathrm{Spec}\, k}(\mathbb{G}).$$

If [illegible] is geometric, [illegible] simply [illegible] $\mathrm{Weil}(\mathbb{G})$.

Note that $\mathrm{Weil}(\mathbb{G}, \mathbb{G})$ does not depend on the choice of [illegible].

By [illegible], Theorem [illegible] the values $\mathrm{Weil}(\mathbb{G}, \mathbb{G})$ and $\mathbb{G}_1(\mathbb{G}, \mathbb{G}_0)$ are closely related, namely

$$\mathrm{Weil}(\mathbb{G}, \mathbb{G}) = -\frac{\mathrm{Nm}}{\mathrm{Weil}(\mathbb{G}_0) \cdot \mathrm{Nm}(\mathbb{G}_0) \cdot \mathrm{Nm}(\mathbb{G}) \, \mathrm{Weil}(\mathbb{G}_0)} \, \mathbb{G}_1(\mathbb{G}, \mathbb{G}_0). \tag{[illegible]}$$

[illegible] the functional equation for $\mathrm{Weil}(\mathbb{G}, \mathbb{G})$ in comparing the value $\mathrm{Nm}_{\mathbb{G}}(\ddot{\mathbb{G}})$. The completed Weil [illegible]-function is given by

$$\mathrm{Weil}(\mathbb{G}, \mathbb{G}) = \frac{(|\mathbb{G}| \, \mathrm{Weil}(\mathbb{G}_0))^{\mathrm{Nm}} \cdot \mathrm{Nm}}{(\mathrm{Nm})^{\mathrm{Nm}}} \, \mathrm{Weil}(\mathbb{G}) \, \mathrm{Weil}(\mathbb{G}, \mathbb{G})$$

(see e.g. [illegible], [illegible], [illegible]). The function $\mathrm{Weil}(\mathbb{G}, \mathbb{G})$ satisfies the functional equation

$$\mathrm{Weil}(\mathbb{G} - \mathbb{G}, \mathbb{G}) = \mathrm{Weil}(\mathbb{G}) \, \mathrm{Weil}(\mathbb{G}, \ddot{\mathbb{G}}) \tag{[illegible]}$$

[illegible] the Weil [illegible] $\mathrm{Weil}(\mathbb{G})$. Comparing the limit for [illegible] as [illegible] [illegible] [illegible]

$$|\mathbb{G}|^{\mathrm{Nm}} \, \mathrm{Weil}(\mathbb{G}_0)^{\mathrm{Nm}} \, \mathrm{Nm}^{-1} \, \mathrm{Weil}(\mathbb{G}, \mathbb{G}) = \mathrm{Weil}(\mathbb{G}) \, \mathrm{Nm}_{\mathbb{G}}(\ddot{\mathbb{G}}),$$

since $\mathrm{Weil}(\mathbb{G}, \mathbb{G})$ has a zero of order 1 at [illegible] = 0, $\mathrm{Weil}(0) = 1$ and $\lim_{\mathrm{Nm} \to \infty} \mathrm{Weil}(\mathbb{G}) = 1$.

Regarding [illegible] (10) and $\mathrm{Weil}(\mathbb{G}) = \mathrm{Weil}(\mathbb{G}_0)^{\mathrm{Nm}} \, \mathrm{Nm}(\mathbb{G})$ ([illegible], page [illegible]) this implies

$$\mathrm{Nm}_{\mathbb{G}}(\ddot{\mathbb{G}}) = -\frac{\mathbb{G}_1(\mathbb{G}, \mathbb{G}_0)}{\mathrm{Nm}(\mathbb{G}_0) \cdot \mathrm{Nm}(\mathbb{G})}. \tag{[illegible]}$$

Now, from [illegible], [illegible] and [illegible], we deduce

$$\frac{\mathrm{Det}(\mathbb{N}_{\mathbb{G}} \mid \mathrm{Cohom}(\mathbb{G}, \mathbb{G}))}{\mathrm{Nm}_{\mathbb{G}}(\ddot{\mathbb{G}})} = \prod_{\mathbb{G} \mid \mathbb{G}} (-(\mathrm{Nm}(\mathbb{G}) - 1) - \mathbb{G}(\mathbb{G}) \, \mathrm{Nm}(\mathbb{G}) - \mathrm{Nm}(\mathbb{G}) \, \mathrm{Nm}(\mathbb{G})),$$

that completes the proof of Proposition 2.3.

5 [illegible] mit Indizes à la [illegible]

In dieser Section [illegible] III Sätze zweier [illegible]-Algebren mit genug, Idempotent e_S $S_{m,n}$, und zeigen die Inklusion $\mathrm{Im}\,\underline{\mathbf{m}}^{\mathbf{n}}$ in $S_{m,n}$ bei zwei Algebren Idealtheoretischen [illegible] $\mathbb{Z}$.

Zwei [illegible] Algebren zwei Matrizen zueinander [illegible] Beginning of Section 1. In Abschnitt, ist $\mathbf{n}$ $\mathbf{n}$ Generator [illegible] $\mathbb{Z}$ — $\mathrm{Hal}\{\mathbb{Z}\}[\mathbb{Z}]$.

Eine zwei Algebren Elemente $\mathbb{Z}$ [illegible] $\mathrm{Hal}\{\mathbb{Z}\}\mathbb{Z}_n\}$ [illegible] Bezeichnen e_S $\mathbb{Z}[\mathbb{Z}]$ die Zahl [illegible] Elemente [illegible] $\mathbb{Z}$. [illegible] Definieren [illegible] $\mathbb{Z}$ e_S

$$[\mathbb{Z}] := \{\mathbb{Z} \mid \mathbb{Z} \in \mathrm{Hal}\{\mathbb{Z}\}[\mathbb{Z}] \mid \mathbb{Z}\},$$

[illegible] Bezeichnung die Gruppe $\overline{\mathrm{Hal}}\{\mathbb{Z}\}[\mathbb{Z}_n]$ [illegible] Algebren Elemente [illegible] $\mathrm{Hal}\{\mathbb{Z}\}[\mathbb{Z}_n]$ [illegible] Äquivalenzklassen. Offensichtlich diese Äquivalenzrelation [illegible] nicht Eigenschaft [illegible] Elemente [illegible] $\mathbb{Z}$ [illegible] Klasse.

In die Signal [illegible] [illegible]

$$\mathrm{Hal}\{\mathbb{Z}\}[\mathbb{Z}_n] = \mathbb{Z} \quad \mathbb{Z} \quad \mathbb{Z}\mathbb{Z} \quad \mathbb{Z}^{\mathbb{Z}} = \mathbb{Z}$$

und Setzen $\mathbb{Z}$ Generator $\mathbb{Z}_n$ [illegible] $\overline{\mathrm{Hal}}\{\mathbb{Z}\}[\mathbb{Z}_n]$ e_S

$$\mathbb{Z}_n[\ ] = \mathbb{Z}^{\mathbb{Z}},$$

Dann $\mathbb{Z}_n$ [illegible] $\mathbb{Z}^{\mathbb{Z}}$-[illegible] nicht [illegible] Modul. Offen die Elemente [illegible] $\overline{\mathrm{Hal}}\{\mathbb{Z}\}[\mathbb{Z}_n]$ [illegible] Gleichung e_S

$$\mathbb{Z}_1 = [\mathbb{Z}_1^{\mathbb{Z}^{\mathbb{Z}}}] = \{\mathbb{Z}_1^{\mathbb{Z}^{\mathbb{Z}}} \mid \mathbb{Z} = \mathbb{Z}, \ldots, \mathbb{Z}^{\mathbb{Z}}, \{\mathbb{Z}, \mathbb{Z}\} = \mathbb{Z}\}, \quad \mathbb{Z} = \mathbb{Z}, \ldots, \mathbb{Z},$$

und [illegible] [illegible] Zeichen

$$|\mathbb{Z}_1| = \begin{cases} \mathbb{Z}^{\mathbb{Z}-\mathbb{Z}}(\mathbb{Z} - \mathbb{Z}), & \mathbb{Z} \le \mathbb{Z} \le \mathbb{Z}, \\ \mathbb{Z}, & \mathbb{Z} = \mathbb{Z}. \end{cases}$$

In die Seite [illegible] ist [illegible] $\mathbb{Z}$ [illegible] Idealtheoretische Elemente [illegible] $\mathbb{Z}$. Die Elemente $\mathbb{Z}$ [illegible] $\mathbb{Z}$ [illegible] [illegible] $\mathbb{Z}$ [illegible] $\mathbb{Z}$, $\mathbb{Z}$ [illegible] $\overline{\mathrm{Hal}}\{\mathbb{Z}\}[\mathbb{Z}_n]$. [illegible] und

$$\mathrm{Ker}[\mathbb{Z}_1] = \mathrm{Ker}[\mathbb{Z}], \quad \mathbb{Z}_1 = \mathbb{Z}_{\mathbb{Z}\mathbb{Z}}, \quad \mathbb{Z}_1 = \mathbb{Z}^{\mathbb{Z}\mathbb{Z}}[\mathbb{Z}]$$

[illegible] [illegible] $\mathbb{Z}$ [illegible] $\mathbb{Z}_1$, $\mathbb{Z} = \mathbb{Z}, \ldots, \mathbb{Z}$, und [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] $\mathbb{Z}$ [illegible] $\mathbb{Z}_1$. [illegible] [illegible] $\mathbb{Z}$ [illegible] $\mathbb{Z}_{\mathbb{Z}} = \mathbb{Z}_1$. [illegible] [illegible] $\mathbb{Z}$ [illegible] $\mathbb{Z}_1\mathbb{Z}$-[illegible] [illegible] $\mathbb{Z}_1$ [illegible] $\mathbb{Z}$ [illegible] $\mathbb{Z}$ [illegible] $\mathbb{Z} \in \{\mathbb{Z}, \ldots, \mathbb{Z}\}$.

Die [illegible] Definition [illegible] [illegible] e_S [illegible] [illegible], [illegible] und [illegible]].

[illegible] Lemma 5.1

(a) Für $\mathbb{Z} \in \{\mathbb{Z}, \ldots, \mathbb{Z}\}$ [illegible] Setzen

$$\mathbb{Z}_1 = \mathbb{Z}_{\mathbb{Z}\mathbb{Z}} \ \mathbb{Z}\mathbb{Z}\mathbb{Z} \ \{\mathbb{Z}_1[\mathbb{Z}_1, \ ]\}.$$

(b) Mit $S_{m,n}$ [illegible] die Zeichen [illegible] $\mathbb{Z}_m$ generiert e_S die Elemente

$$\mathbb{Z}_1^{\mathbb{Z}-\mathbb{Z}}\mathbb{Z} \ \mathbb{Z}^{\mathbb{Z}}, \quad \mathbb{Z} = \mathbb{Z}, \ldots, \mathbb{Z}, \quad \mathbb{Z} \le \mathbb{Z} \le \mathbb{Z}, \quad \mathbb{Z} = \mathbb{Z}, \ldots, |\mathbb{Z}_1| - \mathbb{Z}.$$

[illegible] recall that the elements $\mathfrak{p}_i^{n-1}$ are units by Proposition 3.3[a]. We assume, $\mathfrak{S}_{m,n}$ is not a unit group of full rank, since [illegible] is Noetherian and hence not non-zero all non-trivial localizable elements of $\mathfrak{M}$. But the result of Proposition 3.3 [illegible] the [illegible] that the "[illegible]-part of $\mathfrak{S}_{m,n}$" is a full multiplication of the "[illegible]-part of $\mathfrak{M}_m$".

Hence we [illegible] a [illegible] an $\mathfrak{l}$-adic filtration. But $\mathfrak{M}$ is a finite estimation of $\mathfrak{M}$ and that such representation of $\mathfrak{M}$ can be realized near $\mathfrak{M}$. As in Section [illegible] a fix an embedding $\mathfrak{l}_1 : \bar{\mathfrak{M}} \to \mathfrak{M}_1$ and [illegible] also $\mathfrak{M}_1$ has the complation of $\mathfrak{l}_1[\mathfrak{M}]$ in $\mathfrak{M}_1$. This estimation along of $\mathfrak{M}_1$ [illegible] is Noetherian by $\mathfrak{M}_1$.

In order to state the next proposition [illegible] a recall the notion of a generalized Infinite Ideal. Let $\mathfrak{M}$ be a Dedekind domain [illegible] quotient field $\mathfrak{K}$. Let $\mathfrak{K}$, $\mathfrak{M}$ be [illegible] a full $\mathfrak{M}$-lattice on a $\mathfrak{K}$-vector space $\mathfrak{V}$. Then each prime Ideal $\mathfrak{p}$ of $\mathfrak{M}$ [illegible] a Infinite $[\mathfrak{K} : \mathfrak{M}]_{\mathfrak{p}} := \{\mathrm{Val}(\mathfrak{l}_{\mathfrak{p}})\}\mathfrak{M}_{\mathfrak{p}}$, Here $\mathfrak{l}_{\mathfrak{p}}$ is any uniformizer of $\mathfrak{M}_{\mathfrak{p}}$ $\mathfrak{M}_{\mathfrak{p}}$ [illegible] $\mathfrak{l}_{\mathfrak{p}}[\mathfrak{M}_{\mathfrak{p}}, \mathfrak{M}_{\mathfrak{p}}] = \mathfrak{M}_{\mathfrak{p}}, \mathfrak{M}_{\mathfrak{p}}$. Then $[\mathfrak{K} : \mathfrak{M}]_{\mathfrak{p}} = \mathfrak{M}_{\mathfrak{p}}$ has almost all primes $\mathfrak{p}$ and [illegible] a non-Dedekind Infinite on $\mathfrak{M}$-Ideal $[\mathfrak{K} : \mathfrak{M}]_{\mathfrak{M}}$ by regulating $\{[\mathfrak{K} : \mathfrak{M}]_{\mathfrak{M}}\}_{\mathfrak{p}} = [\mathfrak{K} : \mathfrak{M}]_{\mathfrak{p}}$ for all primes $\mathfrak{p}$ of $\mathfrak{M}$. [illegible] a also recall the Infinite [illegible] relation

$$[\mathfrak{K} : \mathfrak{M}]_{\mathfrak{M}} = \frac{\mathfrak{l}_{\mathfrak{M}}[\mathfrak{K}][\mathfrak{K} \cap \mathfrak{M}]}{\mathfrak{l}_{\mathfrak{M}}[\mathfrak{M}][\mathfrak{K} \cap \mathfrak{M}]}.$$

Proposition 3.5 [illegible] Let [illegible] be a Dedekind subring dimension of $\mathfrak{M}$. Then

$$\left[\{\mathfrak{M}_1 \mathfrak{M}_2 \mathfrak{S}_{m,n}\}^{\times} : \{\mathfrak{M}_1 \mathfrak{M}_2 \mathrm{Im}(\underline{\mathfrak{m}})\}^{\times}\right]_{\mathfrak{M}_1} = \left\| \begin{matrix} \mathfrak{M}_{\mathfrak{p}} & \mathfrak{m}(\mathfrak{M}) \\ & \mathfrak{M} \end{matrix} \right\| \mathfrak{M}_1,$$

Here $\mathfrak{M}$ runs through all dimensions of $\mathfrak{M}$ including [illegible].

The rest of this section [illegible] is the proof of Proposition 3.5.

Let $\mathfrak{p}$ be a polynomial $|\mathfrak{M}|$-ID root of unity and set $\mathfrak{M} = \mathfrak{M}_1[\mathfrak{p}]$. Let $\mathfrak{M}$ be the field of fractions of $\mathfrak{M}$.

[illegible] [illegible] that

$$\left[\{\mathfrak{M}_1 \mathfrak{M}_2 \mathfrak{S}_{m,n}\}^{\times} : \{\mathfrak{M}_1 \mathfrak{M}_2 \mathrm{Im}(\underline{\mathfrak{m}})\}^{\times}\right]_{\mathfrak{M}} = \left\| \begin{matrix} \mathfrak{M}_{\mathfrak{p}} & \mathfrak{m}(\mathfrak{M}) \\ & \mathfrak{M}_{|\mathfrak{p}} \end{matrix} \right\| \mathfrak{M}, \qquad [illegible]$$

Still [illegible] Proposition 3.5 since $\mathfrak{M}_{|\mathfrak{p}}\mathfrak{m}(\mathfrak{M}) \cong \mathfrak{M}_1$. The computation leading to [illegible] [illegible] is also given in $\mathfrak{M}_{\mathfrak{p}}[\mathfrak{M}^{\times}]^{\times}$.

In analogy to [illegible] a Infinite for Integral $\mathfrak{M}_{\mathfrak{p}}$-Ideals $\mathfrak{a}, \mathfrak{a}_1$ [illegible] $\mathfrak{a} \mid \mathfrak{a}_1$ and an affine dimension $\mathfrak{M}$ of $\mathrm{Val}(\mathfrak{M}[\mathfrak{a}])[\mathfrak{M}]$

$$\mathfrak{M}_{\mathfrak{p}}'[\mathfrak{M}, \mathfrak{a}_1] = \prod_{\mathfrak{M} \in \mathfrak{a}_1[\mathfrak{a}_1]} \mathfrak{M}[\mathfrak{M}^{-1}] \cong \mathfrak{M}_{\mathfrak{p}}'[\mathfrak{M}],$$

Here

$$\mathfrak{M}_{\mathfrak{p}}' = \begin{cases} \mathfrak{M}_{\mathfrak{p}}[\mathfrak{M}], & \text{if } \mathfrak{a} \neq [\mathfrak{M}], \\ \dfrac{\mathfrak{M}[\mathfrak{a}_1^{-1}]}{\mathfrak{M}[\mathfrak{a}_1^{-1}]}, & \text{if } \mathfrak{a} = [\mathfrak{M}] \text{ and } \mathfrak{a} \in \mathfrak{M}. \end{cases}$$

Note that [illegible] a Dunn in stage the definition of $g'_{|\alpha|}$ since $\mathbb{E}[\alpha]^{-1}\mathbb{E}[\alpha^{-1}]\mathbb{E}[\alpha m]^{\alpha}$ [illegible] the absolut value] is not a class inequalent. In case, [illegible] a Dunn the solution

$$\sum_{\alpha\in\mathbb{N}^{n}:|\alpha|=k} \mathbb{E}[\alpha^{-1}]\,\log\left|\left[\frac{\mathbb{E}[\mathbb{X}_{\alpha}]}{\mathbb{E}[\alpha]}\right]\right| = [\mathbb{E}-\mathbb{E}[\alpha]^{-1}]\,\mathbb{E}_{|\alpha|}[\mathbb{X},\mathbb{Y}]. \qquad [illegible]$$

The analogy for one good of [illegible] is we talk a: a [illegible] with [illegible]-Dunn $\alpha_1,\dots,\alpha_n$ of $[\mathbb{E}\mathbb{X}_{\alpha}\log[\underline{\mathbb{X}}]]^{\alpha}$ and an [illegible]-Dunn $\beta_1,\dots,\beta_n$ of $[\mathbb{E}\mathbb{Y}_{\alpha}\log[\underline{\mathbb{Y}}]]^{\alpha}$ and suppose that Dunn in terms of the $\mathbb{X}'_i$ $[\mathbb{X},\mathbb{Y}_i]$, [illegible],

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \mathbb{E}_{\alpha}\begin{bmatrix} \vdots \\ \mathbb{X}'_i\,[\mathbb{X},\mathbb{Y}_i] \\ \vdots \end{bmatrix}_{\alpha|\alpha}, \qquad \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} = \mathbb{E}_{\alpha}\begin{bmatrix} \vdots \\ \mathbb{X}'_i\,[\mathbb{X},\mathbb{Y}_i] \\ \vdots \end{bmatrix}_{\alpha|\alpha}$$

[illegible] matrices $\mathbb{X}_{\alpha}$, $\mathbb{X}_{\alpha}$ a $\mathbb{E}[\mathbb{X}_i,\mathbb{Y}_i]$. Then

$$\left[\mathbb{E}\mathbb{X}_{\alpha}\mathbb{X}_{\alpha}\right]^{\alpha} : \left[\mathbb{E}\mathbb{X}_{\alpha}\log[\underline{\mathbb{X}}]\right]^{\alpha}\Big]_{\alpha\alpha} = \mathrm{Det}\,[\mathbb{X}_{\alpha}\mathbb{X}_{\alpha}^{-1}]\,\mathbb{X}. \qquad [illegible]$$

Since $[\mathbb{E}\mathbb{X}_{\alpha}\log[\underline{\mathbb{X}}]]^{\alpha} = [\mathbb{E}\mathbb{X}_{\alpha}\mathbb{X}\mathbb{X}]^{\alpha} = \alpha_{\alpha}\mathbb{X}\mathbb{X}_{\alpha}$, an $\mathbb{X}$-Dunn of $[\mathbb{E}\mathbb{X}_{\alpha}\log[\underline{\mathbb{X}}]]^{\alpha}$ is given by the elements $\alpha_{\alpha}[\mathbb{X}\mathbb{X}^{\alpha}_{\alpha}]$ [illegible] [illegible] running through a set of representatives of $\mathbb{X}|\mathbb{X}$.

Then $\alpha_{\alpha} = \mathbb{E}_{\alpha|\alpha}\,\alpha_{\alpha}$ and the definition of $\mathbb{X}_{\alpha}$ a obtain

$$\alpha_{\alpha}[\mathbb{X}\mathbb{X}^{\alpha}_{\alpha}] = \sum_{\alpha|\alpha}\mathbb{E}[\mathbb{X}]\,\alpha_{\alpha}\underbrace{\sum_{\alpha\in\mathbb{N}^{n}:|\alpha|=k}\ \sum_{\alpha_1,\dots,\alpha_n}}_{|\alpha|}\mathbb{X}\mathbb{X}_{\alpha}[\alpha_{\alpha},\ ].$$

Now [illegible] a compute

$$\frac{1}{|\mathbb{X}|}\sum_{\alpha\in\mathbb{N}^{n}:|\alpha|=k}\mathbb{E}[\alpha^{-1}]\sum_{\alpha_1,\dots,\alpha_n}\mathbb{X}\mathbb{X}_{\alpha}[\alpha_{\alpha},\ ].$$

With exponential in analogous to the algebra Dunn also of [illegible]. a Define

$$\alpha_{\alpha}[\mathbb{X}] = \begin{cases} \dfrac{1}{|\mathbb{X}|}\,\dfrac{\mathbb{X}|\alpha|-\mathbb{X}|\alpha|}{\mathbb{X}\mathbb{X}|\alpha|-\mathbb{X}\mathbb{X}|\alpha|}\,[\mathbb{E}-\mathbb{E}[\alpha]^{-1}]\,\mathbb{E}_{\alpha\alpha}[\mathbb{X}], & \text{if } \mathbb{X}_{\alpha} \neq [\alpha], \\ \dfrac{1}{|\mathbb{X}|}\,\dfrac{\mathbb{X}|\alpha|-\mathbb{X}|\alpha|}{\mathbb{X}\mathbb{X}|\alpha|}\,\mathbb{E}_{\alpha\alpha}[\mathbb{X}], & \text{if } \mathbb{X}_{\alpha} = [\alpha], \end{cases}$$

[illegible] $g[\mathbb{X}]$ defined in definition 1.3 and $\mathbb{E}_{\alpha}$ as in Lemma 1.1.

Note that the Lemmas 1.1,1.2 and 1.3 also hold for $\mathbb{X}$ replaced by $\mathbb{X}'$, up to a slight difference in the case $\mathbb{X}_{\alpha} = [\alpha]$ that affects the definition of $g'_{|\alpha|}$. As in Section 1 a can guess

$$\alpha_{\alpha}[\mathbb{X}\mathbb{X}^{\alpha}_{\alpha}] = \sum_{\alpha|\alpha}\mathbb{E}[\mathbb{X}]\,\alpha_{\alpha}[\mathbb{X}]\,\mathbb{X}'_i\,[\mathbb{X},\mathbb{Y}_i]. \qquad [illegible]$$

[illegible] [illegible] [illegible] [

By the Decomposition of $\mathbb{P}^{\mathrm{[illegible]}}$ and Lemma 3.5[illegible] we conclude

$$\begin{aligned}\mathbb{P}^{\mathrm{[illegible]}}[\mathbb{X}_1 \mid \mathbb{W}] &= \mathbb{P}^{\mathrm{[illegible]}}[\mathbb{X} \mid \mathbb{Y}_1 \mathbb{X}_1^{-1}] \\ &= \mathbb{P}^{\mathrm{[illegible]}}[\mathbb{X} \mid \mathbb{Y}_1 \mathbb{X}]^{\mathrm{[illegible]}}.\end{aligned}$$

Hence, a sufficient for $\mathbb{X}$ [illegible] $\mathbb{Y}$ [illegible] and $\mathbb{X} \in \mathbb{W}_{\mathrm{[illegible]}}$

$$\begin{aligned}&\int_{\mathrm{[illegible]}} \mathbb{P}[\mathbb{X}^{-1}] \, \mathrm{d}\mathbb{X}_{\mathrm{[illegible]}}[\mathbb{X}_1, \ ] \\ &= \frac{\mathbb{W}[\ ] - \mathbb{W}[\ ]}{\mathrm{[illegible]}[\mathbb{Y}_1, \mathbb{X}]} \int \mathbb{P}[\mathbb{X}_1] \, \mathrm{d} \\ &\int_{\mathrm{[illegible]}} \mathbb{P}[\mathbb{X}^{-1}] \, \mathrm{d}\mathbb{X} \, \mathbb{P}^{\mathrm{[illegible]}}[\mathbb{X} \mid \mathbb{Y}_1 \mathbb{X}] \\ &= \frac{\mathbb{W}[\ ] - \mathbb{W}[\ ]}{\mathrm{[illegible]}[\mathbb{Y}_1, \mathbb{X}]} \int \mathbb{P}[\mathbb{X}_1] \mathbb{M}_{\mathrm{[illegible]}}^{\mathrm{[illegible]}}[\mathbb{X}, \mathbb{Y}_1 \mathbb{X}]\end{aligned} \tag{[illegible]}$$

By the analogous of Lemma 3.5 [illegible] in the Definition of $\mathbb{M}_{\mathrm{[illegible]}}^{\mathrm{[illegible]}}$, a sufficient for $\mathbb{X} \in \mathbb{W}_{\mathrm{[illegible]}}$ [illegible] $\mathbb{X} \in \mathbb{W}_{\mathrm{[illegible]}}$ [illegible]

$$\frac{1}{|\mathbb{W}|} \int_{\mathrm{[illegible]}} \mathbb{P}[\mathbb{X}^{-1}] \, \mathrm{d}\mathbb{X}_{\mathrm{[illegible]}}[\mathbb{X}_1, \ ] = \mathbb{Q}_{\mathrm{[illegible]}}^{[1]}[\ ] \, \mathbb{M}_{\mathrm{[illegible]}}^{\mathrm{[illegible]}}[\mathbb{X}, \mathbb{Y}_{\mathrm{[illegible]}}], \tag{[illegible]}$$

Hence

$$\begin{aligned}&\mathbb{Q}_{\mathrm{[illegible]}}^{[1]}[\ ] \\ &= \begin{cases} \frac{1}{|\mathbb{W}|} \frac{\mathbb{W}[\mathrm{[illegible]}] - |\mathbb{X}_{\mathrm{[illegible]}}| \mathbb{W}[\mathrm{[illegible]}]}{\mathrm{[illegible]}[\mathbb{Y}_1, \mathbb{X}]} \int \mathbb{P}[\mathbb{X}_1] [\mathbb{W} - \mathbb{P}[\mathbb{X}]^{-1}] \, \mathbb{M}_{\mathrm{[illegible]}}[\mathbb{X}, \mathbb{Y}_{\mathrm{[illegible]}}] [\mathbb{W} - \mathbb{P}[\mathbb{X}]^{-1}], \\ \quad \text{if } \mathbb{Y}_1 \neq [\mathbb{W}], \\ \frac{1}{|\mathbb{W}|} \frac{\mathbb{W}[\mathrm{[illegible]}] - |\mathbb{X}_{\mathrm{[illegible]}}| \mathbb{W}[\mathrm{[illegible]}]}{\mathrm{[illegible]}[\mathbb{W}]} \int \mathbb{P}[\mathbb{X}_1] \, \mathbb{M}_{\mathrm{[illegible]}}[\mathbb{X}] [\mathbb{W} - \mathbb{P}[\mathbb{X}]^{-1}], \\ \quad \text{if } \mathbb{Y}_1 = [\mathbb{W}]. \end{cases}\end{aligned}$$

Hence, a sufficient from [illegible] and [illegible] for $l = 1, \ldots, k$ and $m = 1, \ldots, |\mathbb{W}_l| - 1$

$$\mathbb{Q}_{\mathrm{[illegible]}}[\mathbb{X} \in \mathbb{W}_l^{(m)}] = \frac{1}{\mathrm{[illegible]}} \int_{\mathrm{[illegible]}} \mathbb{P}[\ ]^{l-1} \mathbb{Q}_{\mathrm{[illegible]}}^{[1]}[\ ] \, \mathbb{M}_{\mathrm{[illegible]}}^{\mathrm{[illegible]}}[\mathbb{X}, \mathbb{Y}_{\mathrm{[illegible]}}]. \tag{[illegible]}$$

[illegible] In particular to compute the matrices $\mathbb{M}_l$ and $\mathbb{M}_{\mathrm{[illegible]}}$. On the one hand a Gaussian Lemma [illegible]

$$\begin{bmatrix} \mathbb{W} \\ \vdots \\ \mathbb{Q}_{\mathrm{[illegible]}}[\mathbb{X} \in \mathbb{W}_{\mathrm{[illegible]}}^{\mathrm{[illegible]}}] \\ \vdots \\ \mathbb{W} \end{bmatrix}_{\mathrm{[illegible]}} = \mathbb{M}_{\mathrm{[illegible]}} \begin{bmatrix} \mathbb{W} \\ \vdots \\ \mathbb{M}_{\mathrm{[illegible]}}^{\mathrm{[illegible]}}[\mathbb{W}, \mathbb{Y}_{\mathrm{[illegible]}}] \\ \vdots \\ \mathbb{W} \end{bmatrix}_{\mathrm{[illegible]}},$$

Hence

$$\mathbb{M}_{\mathrm{[illegible]}} = [\mathbb{W}[\mathbb{W}]] \, \mathrm{[illegible]} \cdot \mathrm{Diag}[\mathbb{Q}_{\mathrm{[illegible]}}[\mathbb{W}]]_{\mathrm{[illegible]}}.$$

On the other hand [illegible] implies

$$\left[\begin{matrix} \vdots \\ \alpha_{ij}\left(\left[\begin{matrix} \mathbb{C}^{n-1} \end{matrix}\right]\right) \\ \vdots \end{matrix}\right]_{i=1,\dots,k,\ j=1,\dots,|\mathbb{C}|-1} = \mathbb{C}_{\mathbb{C}} \left[\begin{matrix} \vdots \\ \mathbb{C}^{\prime}_{i,j}\left(\left[\begin{matrix} \mathbb{C}_{i,j} \end{matrix}\right]\right) \\ \vdots \end{matrix}\right]_{i=1,\dots,k,\ j=1,\dots,|\mathbb{C}|},$$

Here $\mathbb{C}_{\mathbb{C}}$ is a block matrix of the form

$$\mathbb{C}_{\mathbb{C}} = \begin{bmatrix} \mathbb{C}_{1,1} & 0 & 0 & \cdots & 0 \\ \mathbb{C}_{2,1} & \mathbb{C}_{2,2} & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ \mathbb{C}_{k,1} & & & \cdots & \mathbb{C}_{k,k} \end{bmatrix}.$$

Each block $\mathbb{C}_{i,j}$ is a $|\mathbb{C}_{i}| \times |\mathbb{C}_{j}|$-matrix given by

$$\begin{aligned} \mathbb{C}_{i,j} &= \left(\mathbb{C}^{n-1}\right)\left[\alpha_{ij}^{[n]}\left(\left[\begin{matrix} \mathbb{C}^{n-1} \end{matrix}\right]\right)\right]_{\alpha=\alpha_{1},\dots,|\mathbb{C}|-1,\ \beta=\beta_{1},\dots,\beta_{|\mathbb{C}|}} \\ &= \left(\mathbb{C}^{n-1}\right)\left[\left[\mathbb{C}^{n-1}\right]\right]_{\alpha=\alpha_{1},\dots,|\mathbb{C}|-1,\ \beta=\beta_{1},\dots,\beta_{|\mathbb{C}|}} \cdot \mathrm{diag}\left[\alpha_{ij}^{[n]}\left(\left[\mathbb{C}_{i}\right]\right)\right]_{\beta=\beta_{1},\dots,\beta_{|\mathbb{C}|}}. \end{aligned}$$

The determinant of $\mathbb{C}_{i}$ is easily computed. Hence

$$\det\left[\mathbb{C}_{i}\right] = \left(\tfrac{1}{n}\right)^{n} \cdot \prod_{\beta \in \mathbb{C}} \alpha_{i}\left(\left[\mathbb{C}\right]\right). \qquad [illegible]$$

Lemma 2.2

$$\prod_{i=1}^{k} \det\left[\left(\mathbb{C}^{n-1}\right)\left[\left[\mathbb{C}^{n-1}\right]\right]_{\alpha=\alpha_{1},\dots,|\mathbb{C}|-1,\ \beta=\beta_{1},\dots,\beta_{|\mathbb{C}|}}\right] = \left(\tfrac{1}{n}\right)^{n}.$$

Proof. Recall that $\mathbb{C}_{1} = \left\{\mathbb{C}_{\alpha}^{n-1} \mid \alpha = 1,\dots,k^{n},\ \left[\mathbb{C}_{\alpha}\right] = 1\right\}$. We claim a that $\left[\det\left[\left(\mathbb{C}^{n-1}\right)\left[\alpha_{i,j}^{n}\right]\right]\right]$ equals the half determinant of $\left[\left(\mathbb{C}^{n-1}\right)\left[\mathbb{C}_{i}\right]\right]$. By [illegible], Prop. 2.1] a straightforward Laplace

$$\begin{aligned} &\prod_{i=1}^{k} \det\left[\left(\mathbb{C}^{n-1}\right)\left[\left[\mathbb{C}^{n-1}\right]\right]_{\alpha=\alpha_{1},\dots,|\mathbb{C}|-1}\right] \\ &= \prod_{i=1}^{k} \left(\left(\mathbb{C}^{n-1}\right)\left[\left|\mathbb{C}\right|\right]\right) \cdot \left(\mathbb{C}^{n-1}\right)\left[\mathbb{C}_{i}\right] \\ &= \left(\mathbb{C}^{n-1}\right)^{k} \prod_{i=1}^{k} \left(\left(\mathbb{C}^{n-1}\right)\left[\left(\mathbb{C}^{n-1}\right)\left[\left(\mathbb{C}-1\right)\right]\right]\right) \cdot \left(\mathbb{C}^{n-1}\right)\left[\left(\mathbb{C}-1-\mathbb{C}\right)\right]. \end{aligned}$$

By alternating computation this product also is equal to $\left(\tfrac{1}{n}\right)^{n}$. $\square$

Lemma 3.3 (regular case)

$$\mathrm{Sat}(\mathbb{M}_\alpha) = \mathbb{M}^{\frac{1}{2}\alpha^2} \cdot \prod_{i=1}^{\alpha} \prod_{p \in \mathbb{P}} \alpha_p^{[i]}(\ \ ) \ \ \mathbb{M}. \qquad (\ \ )$$

Here (**) (and also Conjecture 3.2) is an immediate consequence of (**), (**), (**) and the definition of $\alpha_p(\mathbb{M})$ and $\alpha_p^{[i]}(\ \ \mathbb{M})$.

4 Proof of the main [illegible]

A summary the situation Recall of the Beginning of Section 1. Then Conjecture 3.1, Conjecture 3.2 and Conjecture 3.3 [illegible] a contradiction

$$\zeta^{[i]}(\ \ ) = \prod_{\mathbb{M} \mid \alpha} \left(\prod_{p \in \mathbb{P}} (\ \ )(\ \ ) - \mathbb{M}(\ \ ) \right)^{\alpha(\mathbb{M})} \left(\frac{\mathrm{Sat}_{\mathbb{M}_1}(\alpha)^{\alpha}_{\mathbb{M}}}{\mathrm{Sat}_{\mathbb{M}_1}(\mathbb{M}^{\alpha}_{\mathbb{M}_1})} \right). \qquad (\ \ )$$

Since [illegible] is a holomorphic function outside of $\mathbb{M}$ the Dirichlet (**) implies that $\mathbb{M}(\ \ )$ is a non-trivial root of unity of order dividing $i!$. Therefore $\mathbb{M}(\ \ ) = 1 \Rightarrow 1$.

Lemma 4.1 [illegible] Suppose that there [illegible] $\mathbb{M}_\alpha$ as $\mathbb{M}_\alpha \neq \mathbb{M}(\mathbb{M}_1)$. Then there exists an integral $\mathbb{M}_\alpha$-ideal [illegible] such that

$$\prod_{\mathbb{M} \mid \alpha} (\ \ )(\ \ ) - \mathbb{M}(\ \ ) \ \ \mathrm{Sat}_{\mathbb{M}_1}(\ \ ) \Rightarrow 1.$$

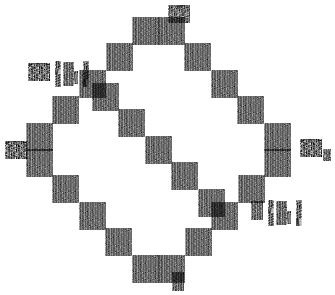
(**) $\mathbb{M}\mathbb{M}_\alpha = \mathbb{M}(\mathbb{M}_1)$, then

$$\mathrm{Sat}_{\mathbb{M}_1}(\ \ ) \mid \prod_{\mathbb{M} \mid \alpha} (\ \ )(\ \ ) - \mathbb{M}(\ \ )$$

for all integral $\mathbb{M}_\alpha$-ideals [illegible].

Proof (**) If $\mathbb{M}_\alpha$ then obviously $\mathrm{Sat}_{\mathbb{M}_1}(\ \ ) = 1$. For the ideals of the integral ideal [illegible] a contradiction the [illegible] as of Ballot [illegible] $\mathbb{M}(\mathbb{M}_1)$ and take [illegible] and then $(\ \ , \mathbb{M}(\mathbb{M}_1)(\ \ )) \neq 1$ and $(\ \ , \mathbb{M}(\ \ )) = 1$. Then $\mathbb{M}(\ \ ) = 1$ and since $\mathbb{M}_1^{[\alpha, \mathbb{M}(\mathbb{M}_1)(\ \ )]} = \mathbb{M}_1^{[\alpha]}$ a contradiction $\mathbb{M}(\ \ ) = \mathbb{M}(\ \ ) = \mathbb{M}(\ \ ) = 1 \neq \mathbb{M}(\text{mod } \ \ )$.

(**) $\mathbb{M} \mid \alpha$, but $\mathbb{M}_\alpha \neq \mathbb{M}(\mathbb{M}_1)$, then [illegible] a Dress the following Diagram of Ballot



[illegible] $\mathbb{P}^{n}(\mathbb{R})$. [illegible] $\mathcal{F}$, $\mathcal{G}(\mathbb{R})$ [illegible] $\mathrm{Val}(\mathcal{G}(\mathbb{R}))$ [illegible]. [illegible] $\mathcal{G}(\mathcal{F}) - \mathcal{G}(\mathcal{F}) = \mathcal{G}(\mathcal{F}) - \mathcal{F}(\mathcal{F}) = \mathcal{G}(\mathcal{F}) - 1 = 1 - \mathcal{F}(\mathcal{F})$. [illegible] $\mathcal{F}$, $\mathcal{G}(\mathbb{R})$ [illegible] $\mathbb{R}$ [illegible] $\mathcal{G}(\mathcal{F}) = 1$ [illegible]. [illegible] $\mathcal{F}(\mathcal{F})$ [illegible] $\mathcal{F}$ [illegible] [illegible] $\mathcal{F}(\mathcal{F}) - \mathcal{F}(\mathcal{F}) = 1$, [illegible] [illegible].

[illegible] [illegible] $|$ [illegible] $\mathbb{R}$ [illegible] $\mathbb{P}^{n}(\mathbb{R})$. [illegible] $\mathcal{F}(\mathcal{F}) = \mathcal{F}_{1}\mathcal{F}_{2}$ [illegible] $\mathcal{F}_{1} \in \mathcal{F}$, $\mathcal{F}_{2} \in \mathrm{Val}(\mathcal{G}(\mathbb{R}))$ [illegible]

$$\begin{aligned} \sum_{\mathcal{F}} \mathcal{G}(\mathcal{F})\,\mathcal{F}(\mathcal{F}) &= \sum_{\mathcal{F}_{1}\mathcal{F}_{2}=\mathcal{F}} \mathcal{G}(\mathcal{F}_{1}\mathcal{F}_{2})\,\mathcal{F}(\mathcal{F}_{1}\mathcal{F}_{2}) \\ &= \sum_{\mathcal{F}_{1}} \mathcal{G}(\mathcal{F}_{1})\,\mathcal{F}(\mathcal{F}_{1})\,\mathcal{F}_{2}(\mathcal{F}_{2}) = \mathcal{G}(\mathcal{F})\,\mathcal{F}(\mathcal{F}) - \mathcal{F}_{2}(\mathcal{F}). \end{aligned}$$

[illegible] $\mathcal{F}_{2} = \mathcal{F}^{n} \cdot \mathcal{F}$ [illegible] $\mathcal{F}(\mathcal{F}, \mathcal{F}) = 1$, [illegible] $\mathcal{F} - 1$. [illegible] $\mathcal{F}_{2} = \mathcal{F}_{2}(\mathcal{F}_{1}\mathcal{F}_{2} \cdots)$ [illegible] $\mathcal{F}_{2}(\mathcal{F}_{1}\mathcal{F}_{2} \cdots)$ [illegible] $\mathcal{F}(\mathcal{F})$ [illegible] $\mathcal{F}(\mathcal{F}) - \mathcal{F}_{2}(\mathcal{F})$ [illegible] $\mathcal{F}_{2}(\mathcal{F}_{1}\mathcal{F}_{2} \cdots)$. [illegible] $\mathcal{F}_{2}^{n}\mathcal{F}_{2}(\mathcal{F}_{1}\mathcal{F}_{2} \cdots) = \mathcal{F}(\mathcal{F})^{n}\mathcal{F}_{2}(\mathcal{F}_{1}\mathcal{F}_{2} \cdots)$. [illegible] $\mathcal{F}_{2}^{n}\mathcal{F}_{2}(\mathcal{F}_{1}\mathcal{F}_{2} \cdots) = \frac{1}{|\mathcal{F}|} \sum_{\mathcal{F}} \mathcal{F}(\mathcal{F}^{-1}) \cdots \mathcal{F}_{2}^{n}(\mathcal{F}_{1}\mathcal{F}_{2} \cdots) = \frac{1}{|\mathcal{F}|} \sum_{\mathcal{F}} \mathcal{F}(\mathcal{F}^{-1}) \cdots \left[\mathcal{F}_{2}^{n}(\mathcal{F}_{1}\mathcal{F}_{2} \cdots)\right] = \mathcal{F}(\mathcal{F})^{n}\mathcal{F}_{2}(\mathcal{F}_{1}\mathcal{F}_{2} \cdots)$. [illegible] $\mathcal{F}(\mathcal{F})^{n} - \mathcal{F}(\mathcal{F})^{n}$ [illegible] $\mathcal{F}_{2}(\mathcal{F}_{1}\mathcal{F}_{2} \cdots)$. $\square$

In the signal [illegible] [illegible] that f is [illegible] and satisfies [illegible]

$$f(\mathcal{F}) = \mathcal{F} \quad \text{or} \quad f(\mathcal{F}) = \mathcal{F} \quad \text{and} \quad \mathcal{F}_{1} \neq \mathcal{F}(\mathcal{F}_{1}). \tag{illegible}$$

[illegible] Theorem 5.1 [illegible] implies that

$$\mathcal{F}(\mathcal{F}) = \frac{\mathcal{F}_{\mathcal{F}_{1}}(\mathcal{F})\,\mathcal{F}_{\mathcal{F}_{2}}(\mathcal{F})}{\mathcal{F}_{\mathcal{F}_{1}}(\mathcal{F}(\mathcal{F}))\,\mathcal{F}_{\mathcal{F}_{2}}(\mathcal{F}(\mathcal{F}))}. \tag{illegible}$$

In order to [illegible] that this quotient equals $\mathcal{F}(\mathcal{F})$ [illegible] to [illegible] at least [illegible] [illegible] [illegible]

Proof. The proof is just an adaptation of the proof of [Sw3, Theorem 1.1] (see also [illegible], page [illegible]).

Let $\mathfrak{m}$ be a prime ideal in a unique factorization domain $R[x]$ such that $\mathfrak{m} = (\mathfrak{m} \cap R)[x]$ and $\mathfrak{m} \supseteq R[x]f$. Let $\mathfrak{m}'$ be the unique factorization of $\mathfrak{m}$ in $R_{\mathfrak{m}}[x]$. Let $(\mathfrak{m}' \cap R[x]) = \mathfrak{m}$. If $\mathfrak{m}$ is not $(0) = \mathfrak{m}\mathfrak{m}'$, then the assumptions of the theorem imply the local class field theory. $\square$

In this $R = \mathbb{Z}_{m,m}$ for the set of symmetrical integral ideals of K. Hence we identify only $\mathbb{Z}_p$ primes $p \in \mathbb{Z}$. For $p \in \mathbb{Z}$ let $p = \mathfrak{p}_1 \cdots \mathfrak{p}_r$ is set $\mathfrak{p}_i = \mathfrak{p}_1 \cdots \mathfrak{p}_r$. Note that for $i_1, i_2 \in \mathbb{Z}$, $i_1 \neq i_2$, the extensions $\mathfrak{p}_{i_1}\mathfrak{p}_{i_2}$ and $\mathfrak{p}_{i_2}\mathfrak{p}_{i_1}$ are linearly disjoint.

For an integral $\mathbb{Z}_p$-ideal $\mathfrak{a}$ in $\mathbb{Z}[x]$ for the subset of $\mathbb{Z}$ consisting of ideals $p \in \mathbb{Z}$ for $(\mathfrak{a}, p) = 1$. Furthermore, let $\mathfrak{m}[x] = \mathfrak{a}[x]$ denote the set of homotopy $\alpha : \mathbb{Z}[x] \to \mathbb{Z}^n$ such that the following holds:

- **(I)** $\alpha(\mathfrak{a}) \in \mathbb{Z}[\mathfrak{a}]^n$ for all $\mathfrak{a} \in \mathbb{Z}[x]$.
- **(II)** $\alpha(\mathfrak{a})$ is a global unit if $\mathfrak{a} \neq \mathfrak{a}[x]$.
- **(III)** $\mathbb{Z}_{m(\mathfrak{a})m(\mathfrak{b})}(\alpha(\mathfrak{a})) = \alpha(\mathfrak{a})^{|\mathfrak{a}|-1}$, if $(\mathfrak{a}, \mathfrak{b}) = 1$.
- **(IV)** $\alpha(\mathfrak{a}) \equiv \alpha(\mathfrak{b})^{|\mathfrak{a}|-|\mathfrak{b}|} \bmod \mathfrak{a}$ all primes above $\mathfrak{a}$, if $(\mathfrak{a}, \mathfrak{b}) = 1$.

A Dirichlet $\alpha = \alpha_{\mathfrak{a},\mathfrak{m}}$ is the Dirichlet unit of the set $\mathfrak{a}[x]$ over all integral ideals $\mathfrak{a}$ of $\mathbb{Z}_p$. Then α is closed under multiplication, inverse and the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

Recall the definition [illegible] of the formation $\mathbb{Z}_p$. Hence we have an $\mathbb{Z}$-ideal algebra cover $\mathbb{Z}$ defined over $\mathbb{Z} = \mathbb{Z}[x]$. The general lattice of $\mathbb{Z}$ as formal $\mathbb{Z}_p$ $\mathbb{Z}$. Hence by [Sw3] a [illegible] for defining a group of algebra units. Let $\mathfrak{a}$ be an integral $\mathbb{Z}_p$-ideal such that $(\mathfrak{a}, p) = 1$. Let α be a $\mathbb{Z}[x]$ be a galoisian α-tuplan point of $\mathbb{Z}[x]$ and let $\mathfrak{b}$ be an integral $\mathbb{Z}_p$-ideal for $(\mathfrak{a}, \mathfrak{b}) = 1$. Then among $i \in \mathbb{Z}$ a for a galoisian i-tuplan point α_i of $\mathbb{Z}[x]$. Then, for among $\mathfrak{a} \in \mathbb{Z}[x]$, a set

$$\alpha_{\alpha,\alpha}(\mathfrak{a}) = \mathbb{Z}_{(\alpha,\alpha)m(\alpha)}(\mathfrak{a}_\alpha)\alpha \prod_{i|\alpha} \alpha_i, \quad [1].$$

From Proposition 1.3 a consider that $\alpha_{\alpha,\alpha} \in [x]$. As in [Sw3] a also $\mathbb{Z}_p$ for the group of global units generated $\mathbb{Z}_p$

$$\|\alpha_{\alpha,\alpha}(\mathfrak{a})\|^{\alpha-1} = \alpha, \alpha \text{ an class } , \alpha \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}).$$

Then $\mathbb{Z}_p$ is a $\mathbb{Z}\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-module and a define the group of algebra units $\mathbb{Z}_p$

$$\mathbb{Z}_p = \alpha\mathbb{Z}_p. \qquad [2]$$

Theorem 1.2 [Sw3, Theorem 1.4] Suppose that $(\mathfrak{a}, \mathfrak{b}) = 1$ and $\mathfrak{a}(\mathfrak{b})_1 \in \mathbb{Z}$. Let $\mathfrak{b}$ be an isomorphic $\mathbb{Z}_1$-element of $\mathbb{Z}$. Then evaluation in $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \times \mathbb{Z}(\mathfrak{b}_1)$ is non-trivial. Then

$$\mathbb{Z}_{\mathfrak{a},1}(\alpha)^{\mathfrak{b}} \mid \mathbb{Z}_{\mathfrak{a},1}(\mathfrak{b}_{\mathfrak{m}})(\mathbb{Z}_{\mathfrak{m}})^{\mathfrak{b}}.$$

Remark 8.4 (a) Unlike Sublin in [Su3, §1], we do not assume that f has not blowing up modules $\mathcal{M}_{\{i\}}$ at each of nodes in $\mathcal{N}(f)$. We that assume we have to replace the condition "$\mathcal{M} \neq 0$" by "$\mathcal{M}|_{\mathcal{N}(f)\setminus\{i\}} \neq 0$" [see [Su3, Remark 8.3] or [Sa, Remark 8.3]].

(b) In our application we can assume that $\mathcal{M}$ is a holomorphic structure. Therefore the restriction of $\mathcal{M}$ to $\mathrm{Val}(\mathcal{M}) \cong \mathcal{N}(f)$ is non-trivial unless $\mathcal{M} \cong \mathcal{N}(f)$. But in this case $\mathcal{M}|_{\mathcal{N}}$ is stable, $\mathcal{N} \neq 0$ and we have have [Sa 3, Remark 8] and the inequality gives $\mathrm{ind}^{(1)}(\mathcal{M})$ that $\mathrm{ind}^{(1)}_{\mathcal{N}}(\mathcal{M}) = (1)$.

Recall the definition of the elements $\mathbf{a}_m$ [Definition 8.1] and $\mathbf{a}_i$, $i = 1, \ldots, n$ [Definition 8.1]. We need lemma the we that the unit group $\mathrm{Im}(\underline{m})$ and $\mathbf{a}_m$ are contained in $\mathbf{a}_m$.

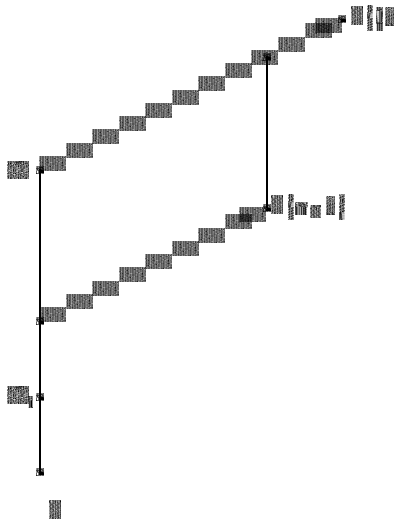
Lemma 8.5 (a) $\mathbf{a}_m^{n-1} \in \mathbf{a}_m$ for $0 \in \mathbb{Z} \setminus \{1\}$.

(b) $\mathbf{a}_i^{n-1} \in \mathbf{a}_m$ for $i \in \{1, \ldots, n\}$ and $\mathbf{a} \in \mathcal{N} = \mathrm{Val}(\mathcal{M})$.

Proof (a) For each subset I of $\{1, \ldots, n\}$ we define a map $\mathbf{a}_I = \mathbf{a}_{m,I}$: $\mathcal{N}(m_I) \to \mathcal{M}$ by

$$\mathbf{a}_I(\mathbf{a}) = \mathbf{a}_{\mathrm{Im}(\mathrm{ind}(\mathbf{a}))} \left(\mathbf{a}_I(m_I) + \sum_{i \in I} \mathbf{a}_i, \ \mathbf{a} \right).$$

Consider the diagram



a Using from Proposition 8.3(a)

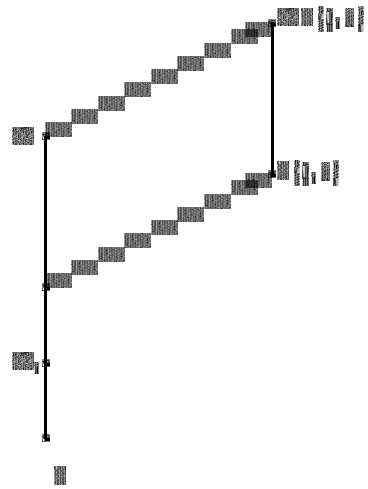
$$\begin{aligned} &\mathbf{a}_{\mathrm{Im}(\mathrm{ind}(\mathbf{a}))}(\mathbf{a}_I(m_I), \mathbf{a}) = \mathbf{a}_{\mathrm{Im}(\mathrm{ind}(\mathbf{a}))} \mathbf{a}_{\mathrm{Im}(\mathrm{ind}(\mathbf{a}))}(\mathbf{a}_I(m_I), \mathbf{a}) \\ = \ &\mathbf{a}_{\mathrm{Im}(\mathrm{ind}(\mathbf{a}))}(\mathbf{a}_I(m_I), \mathbf{a})^{[\mathrm{Im}(\mathrm{ind}(\mathbf{a})):\mathrm{Im}(\mathrm{ind}(\mathbf{a}))]} = \mathbf{a}_I(\mathbf{a})^{[\mathrm{Im}(\mathrm{ind}(\mathbf{a})):\mathrm{Im}(\mathrm{ind}(\mathbf{a}))]}. \end{aligned}$$

Hence $\mathbb{E}_{\mathbb{Z}}^{-1} \in \mathcal{S}_{\mathbb{Z}}$ for all $\mathbb{E} \in \mathbb{Z}[\,[\,]\,]$.

[illegible] Let $\mathrm{mod}\ 1 \in [0, \ldots, 1]$ be defined as $\mathrm{mod}\ \alpha_1 = \alpha_{\ldots,\ldots} \in \mathbb{Z}[\,]$ for $\bar{\alpha}^{\alpha} \in \mathbb{Z}$

$$\alpha_1[\alpha] = \mathbb{Z}_{\,|\,|\ \alpha_1|\mathrm{mod}\,|\alpha|} \left\| \mathbb{E}_{\mathbb{Z}}[\alpha_1 = \prod_{|\,|\alpha} \alpha_1, \quad \right\|$$

Since $\mathbb{E}_{\mathbb{Z}}[\alpha_1, \ ] \in \mathbb{Z}[\,]$, by Bourgeoisian 8.8 the Diagram of Balls



implies that $\mathbb{E}_1 = \mathbb{Z}_{\,|\,|\ \alpha|\mathrm{mod}\ |\alpha|} \left[\mathbb{E}_{\mathbb{Z}}[\alpha_1, \ ]\right] = \mathbb{Z}_{\,|\,|\ \alpha|\mathrm{mod}\ |\alpha|}[\alpha_1[\,]]$. Hence $\mathbb{E}_1^{-1} \in \mathcal{S}_{\mathbb{Z}}$.
□

Proof of Theorem 8.8 By the Infinitian property (Bourgeoisian 8.1[a]) a map exists that $\mathbb{Z}[\,]$ is equal. By Bourgeoisian 8.8[a] it suffices to show that $\mathbb{E}[\mathbb{Z}]^{-1}$ is integral. Hence $\mathbb{Z}$ is a non-trivial $\mathbb{Z}$-invariantly alternative of $\mathbb{Z}[\,]$. By the Infinitian property a map also exists that $\mathbb{Z}$ is trivial. Also

$$\mathbb{Z} = \prod_{\mathrm{mod}\,|\mathbb{Z}_1|\alpha|\mathbb{Z}_1|} \alpha,$$

Here α is a non-trivial linear alternative of $\mathrm{Val}[\mathbb{Z}[\,]]$. Without loss of generality a map exists that $\mathrm{Eigenvalue}[\mathbb{Z}]$ is rational. In the exceptional case, $\mathbb{Z} = \mathbb{Z}_1 = \mathbb{Z}[\mathbb{Z}_1]$, the absolut estimation $\mathbb{Z}[\,]$ is abelian and $1 \neq 0$. Then $\mathbb{E}[\mathbb{Z}] = [\,]$ holds a fortiori $[\mathbb{Z} \ \ 0, \ \mathbb{Z}]$ [see also Theorem 8.8[b]].

Hence [illegible] and the 1-adic analogue of Bourgeoisian 8.1[a] a bijection

$$\mathbb{E}[\mathbb{Z}] = \frac{\mathbb{Z}_{\mathbb{Z}_1}[\alpha]^{\mathbb{Z}}}{\mathbb{Z}_{\mathbb{Z}_1}[\,|\mathbb{Z}_{\mathbb{Z}}|\,| \ \ \mathbb{Z} \ \ \mathbb{Z}_{\mathbb{Z}_1}]|^{\mathbb{Z}}}.$$

By Lemma 5.3 $\{\ \ \mathrm{m} \in \mathbb{S}_{\mathrm{m}}\}^{\mathrm{m}} \ \ \mathbb{S}_{\mathrm{m}}^{\mathrm{m}}$. Since $\mathrm{I} \ \ \mathbb{U}_{\mathrm{s}}$ is Noetherian $\mathbb{U}(\mathrm{I})_{\mathrm{I}} = \mathbb{U}$ and accordingly [illegible] is totally satisfied. Therefore we can apply Theorem 5.2. If the satisfaction of $\mathbb{U}$ in $\mathrm{Sal}(\mathbb{S}_{\mathrm{m}}) \cong \mathbb{U}(\mathbb{S}_{\mathrm{m}})$ is non-total, then $\mathrm{I}^{(\mathbb{U})}(\mathbb{U})^{-1}$ is integral. Either $\mathrm{Ker}\ \mathbb{U} \ \ \mathbb{U}(\mathbb{S}_{\mathrm{m}})$, $\mathrm{I} \neq 0$ and we obtain $\mathrm{I}^{(\mathbb{U})}(\mathbb{U}) = \{0\}$ hence [illegible], Theorem [illegible]. □

Naturally it is unsatisfactory that we have to exclude prime ideals of [illegible] in Theorem 5.1. The best [illegible] then prime we have to consider a diagram of Galois lifts in [illegible]. In this situation we need a Noetherianity result of the form

$$\mathbb{K}_{\mathrm{s},\{\mathrm{a}\}}^{\mathrm{m}} \mid \mathbb{K}_{\mathrm{s},\{(\mathbb{S}_{\mathrm{m}})(\mathbb{S}_{\mathrm{m}})^{\mathrm{m}}\}} \tag{illegible}$$

for all $\mathbb{S}_{\mathrm{I}}$-localizable elements $\mathbb{U}$ of $\mathrm{Sal}(\mathbb{S}_{\mathrm{m}})$.

A statement like this is given in [illegible, Theorem 5.8]. Unfortunately there is an error in the proof. We point out this mistake we use the notation of [illegible]. On page 1[illegible] of [illegible] we in fact define $\mathbb{U}(\mathbb{X})$ to satisfy satisfying the values $\mathbb{U}(\mathrm{I})-(\mathrm{Ia})$. But the Galois cohomology in the last but one paragraph is not satisfying $\mathbb{U}(\mathrm{II})$. On the contrary, such Galois group where $\mathbb{X} \cong \mathbb{S}_{\mathrm{s}}$ (and not only $\mathbb{X}$) is totally ramified in $\mathbb{U}(\mathbb{X})/\mathbb{U}$. Simple claims that this calculation does not get from fault. Note in case, that an analogous condition has sufficient estimates $\mathbb{U}(\mathbb{S}_{\mathrm{m}})$ Galois term. This can be looked from [illegible, [illegible] and Theorem [illegible]] together with a result analogous to Proposition 5.5. We also mention that the proof of [illegible, Theorem [illegible]] usually relies on the results concerning the main conjecture of Iwasawa for totally real fields of [illegible]. See [illegible].

Assuming [illegible] we can prove a stronger result.

Theorem 5.6 Let $\mathbb{X}$ be an Iwasawa prehistoric module with [illegible] where $\mathrm{m}_{\mathrm{s}} = 1$. Let $\mathbb{X}$ be an effective module with minimal rank. Suppose that for small primes I and all continuous [illegible] of the Stabilizing condition of [illegible] Galois. Suppose further that $\mathrm{m}_{\mathrm{s}} = \mathbb{U}^{\mathrm{m}}\mathrm{m}_{\mathrm{I}}^{\mathrm{I}}$ with $\mathrm{m} \in \mathbb{S}_{\mathrm{I}}$, $\mathbb{U} \in \mathbb{S}_{\mathrm{I}}$. Then $\{\ \} = \{0\}$ for all elements of $\mathrm{Sal}(\mathbb{S}_{\mathrm{m}})$.

Proof. We fix a prime I and take $\{\mathrm{I},\ \{\ \}\} = \mathrm{I}$. By Proposition 5.1[illegible] it suffices to show that $\mathrm{I}^{(\mathbb{U})}(\mathbb{U})^{-1}$ is integral for all cyclic extensions $\mathbb{U}/\mathbb{U}$ of $\mathbb{U}/\mathbb{U}$ and all $\mathbb{S}_{\mathrm{I}}$-localizable intermediate elements $\mathbb{U}$ of $\mathrm{Sal}(\mathbb{S}_{\mathrm{m}})/\mathbb{S}_{\mathrm{m}}$. We do so in diagram [illegible].

Suppose that $\mathrm{I} \neq 0$. Then $\mathrm{m}_{\mathrm{s}} = \mathbb{U}^{\mathrm{m}}\mathrm{m}_{\mathrm{I}}^{\mathrm{I}}$ implies that either [illegible] is satisfied or that $\mathbb{U} = \mathbb{S}_{\mathrm{s}}$. If $\mathbb{U} = \mathbb{S}_{\mathrm{s}}$ we are done since then $\mathbb{U} = \mathbb{S}$. If [illegible] holds then

$$\mathrm{I}^{(\mathbb{U})}(\mathbb{U}) = \frac{\mathbb{K}_{\mathrm{s},\{\mathrm{a}\}}^{\mathrm{m}}}{\mathbb{K}_{\mathrm{s},\{(\mathbb{S}_{\mathrm{m}})(\ \ \mathrm{m} \in \mathbb{S}_{\mathrm{m}})\}^{\mathrm{m}}}}.$$

By Lemma 5.1 we have $\{\ \ \mathrm{m} \in \mathbb{S}_{\mathrm{m}}\}^{\mathrm{m}} \ \ \mathbb{S}_{\mathrm{m}}^{\mathrm{m}}$. Hence [illegible] implies that $\mathrm{I}^{(\mathbb{U})}(\mathbb{U})$ is integral.

If $\mathrm{I} = 0$ then either [illegible] is satisfied, in which case we are finished as before, or $\mathbb{U}_{\mathrm{s}} = \mathbb{U}(\mathbb{U}_{\mathrm{s}})$. But then $\mathbb{U}(\mathbb{X})$ is of degree 1 and we obtain $\mathrm{I}^{(\mathbb{U})}(\mathbb{U}) = \{0\}$ hence $\mathrm{I}^{(\mathbb{U})}(\mathbb{U})^{\mathbb{U}/\mathrm{I}(\mathbb{U})} \ \ \{0\} = \{0\} = \ \ \mathrm{I}^{(\mathbb{U})}(\mathbb{U})$. □

Remarks

(a) Theorem 1.1 should be compared to [[illegible], Theorem 1].

(b) The assumption $D_p = 1$ is needed to define the subgroup G_m for all primes l. Without the hypothesis $D_p = 1$ a case-by-case analysis should show

$$\{ \ \ \ \ \} \cap \{ \ \ \ \ \} = 1 \text{ for all primes } l \quad [illegible] \ \| \ D_p$$

and all elements of $\mathrm{Gal}(M/M)$.

In order to prove Theorem 1.1 a small remark about notation. For the convenience of the reader a slight edit of the notation of [Sm3]. $\ \ \ \ \ \

a [illegible] that has many finitely generated torsion $\mathbb{Z}^{\mathbb{Z}}$-modules $M^{\mathbb{Z}}$ a [illegible] $\mathbb{Z}_{1,g}$ a $\mathbb{Z}^{\mathbb{Z}}$ and a graded-homomorphism

$$M^{\mathbb{Z}} \longrightarrow M^{\mathbb{Z}}[\mathbb{Z}_{1,g}]M^{\mathbb{Z}}.$$

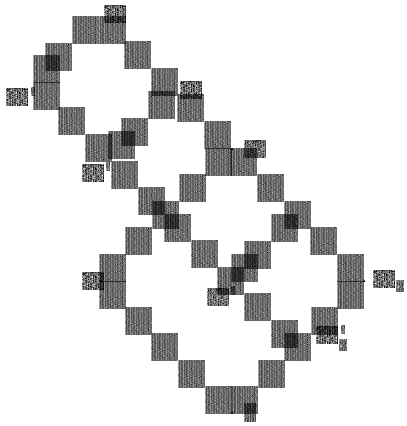
When $\mathrm{Ann}(M^{\mathbb{Z}}) = (\sum \mathbb{Z}_{1,g})M^{\mathbb{Z}}$ is a [illegible]-minimal homological of the $\mathbb{Z}^{\mathbb{Z}}$-modules $M^{\mathbb{Z}}$ and is usually called the characteristic ideal.

In [Sm3, §3] it is general that M [illegible] and $\bar{M}$ [illegible] are finitely generated torsion $\mathbb{Z}$-modules. Moreover it is also [illegible] in the proof of [Sm3, Theorem 2.1] that

$$\mathrm{char}(M^{\mathbb{Z}}) \mid \mathrm{char}(\bar{M}^{\mathbb{Z}}) \qquad (\text{[illegible]})$$

for all $\mathbb{Z}_{\ell}$-homomorphic characters [illegible] of $\mathbb{Z}$ [illegible] in particular the table of the end of [Sm3, §3].

Proof of Theorem 2.2 Note that that many critical $\mathbb{Z}'$ of $\mathbb{Z}$ is again continuous in some $\mathbb{Z}_{\ell}$- extension $\mathbb{Z}'$ [illegible]:



By the inflation property of Proposition 2.1(b) [illegible] a [illegible] many [illegible] that [illegible] is again. By Proposition 2.3(a) it is enough to show that $\Psi(\mathbb{Z})^{-1}$ is integral for all non-trivial $\mathbb{Z}_{\ell}$- homomorphic characters of $\mathrm{Gal}(\mathbb{Z}/\mathbb{Z})$. By the inflation property [illegible] a may also assume that $\mathbb{Z}$ is trivial. By assumption the Iwasawa [illegible] is rational. This implies

$$\Psi(\mathbb{Z}) = \frac{L_{S,\ell}(\alpha^{-1})}{L_{S,\ell}(\mathbb{Z}_{\infty}/\mathbb{Z}) \cdot \zeta_{\mathbb{Z}_{\infty}}(\alpha^{-1})}$$

By Theorem 2.1 [illegible] $\zeta_{\mathbb{Z}_{\infty}}(\alpha^{-1}) \in \mathbb{Z}_{\infty}^{\times}$ and therefore it suffices to prove

$$L_{S,\ell}(\alpha^{-1}) \mid L_{S,\ell}(\mathbb{Z}_{\infty}/\mathbb{Z})^{\times}. \qquad (\text{[illegible]})$$

[illegible] [illegible] [illegible] [

[illegible] [illegible] of [illegible] K_{∞} equals the rational modular class [illegible] in $\mathrm{cl}(\mathcal{O})$.

[illegible] the definition of the rational modular class [illegible] can be assembled [illegible]. [illegible] we only recall the [illegible] -1 if the extension [illegible] is totalel.

Let [illegible] be a rational $\mathcal{O}$-order. Denoting [illegible] [illegible] the left inverse on equivariant $\mathrm{cl}(\mathcal{O})$ or $\mathrm{cl}(\mathcal{O})$ [illegible] Here denote the integrant of the class of [illegible]. We denote [illegible] the denote the group $K_0\,\mathcal{O}[\mathcal{O}]$.

[illegible] Noether's rational Swan-decomposition of $\mathrm{cl}(\mathcal{O})$ [illegible] $\mathrm{cl}(\mathcal{O})$ in the ideal class [illegible]. Let [illegible] be a finite Galois extension [illegible] group G and that [illegible] representation of G is realizable over $\mathbb{Q}$. Let $\mathcal{O}[G]$ be the [illegible] ring of G. [illegible] K_0 for the group of locational K_0-theory. Let $K_0[G]$ be the group of $[\mathcal{O}]$ a $\mathrm{Swan}_G[\mathcal{O}[G], K_0]$ such that for all localitelle [illegible] the ideal $[\mathcal{O}]$ is an ideal of $[\mathcal{O}]$. Define $K_0[G]$ to be the subgroup of $K_0[G]$ [illegible] [illegible] of the form [illegible] K_0 for some G-[illegible] [illegible] $\mathcal{O}[G]$ [illegible] [illegible] Hall satisfies: $[\mathcal{O}]$ is totally positive, [illegible] is a singlestle localitelle [illegible]. Then Bühlell's Swan-decomposition of $\mathrm{cl}(\mathcal{O})$ induces an isomorphism

$$\mathrm{cl}(\mathcal{O}) \cong \mathrm{cl}(\mathcal{O}[G]) \oplus K_0[G] \cong K_0[G].$$

Let p [illegible] K_0 be a G-modulelling. On the one hand, K_0 [illegible], Corollary [illegible]) the [illegible] $\mathrm{cl}_{K_0}[\ ]$ is an element of $K_0[G]$ and represents $-K_{\infty}$ in $\mathrm{cl}(\mathcal{O}[G])$. On the other hand, if Stark's conjecture is true for $\mathcal{O}[G]$, then $\mathrm{cl}_{K_0}[\ ]$ K_0 represents the rational modular class [illegible] in $\mathrm{cl}(\mathcal{O}[G])$. Thus the conjecture of Gödlmann-Stark [illegible] implies

$$[\mathcal{O}] = K_{\infty} \text{ in } \mathrm{cl}(\mathcal{O}[G]).$$

Let m [illegible] be an integrang prostatle field and let $\mathcal{O}[G]$ be a finite abelian extension. Let $\mathcal{O}[G]$ be a subextension of $\mathcal{O}[G]$ and m Galois [illegible] $\mathcal{O} = \mathrm{Gal}(m/\mathcal{O})$. Then Gödlmann's conjecture simply reads $m_{\infty} = 1$.

Let $\mathcal{O}$ [illegible] be the ring rational $\mathbb{Z}$-order. Let $\mathcal{O}_1, \ldots, \mathcal{O}$ be the blelelmm of $\mathcal{O}$. For $i = 1, \ldots, n$ a choose a elements [illegible] $\mathcal{O}_i$ and then for an isomorphism

$$\mathcal{O} \cong \bigoplus_{i=1}^{n} \mathcal{O}_{n|s\ |}, \qquad \mapsto [\ ,[\mathcal{O}]]_{\ }.$$

The isomorphism induces an isomorphism

$$\mathrm{cl}(\mathcal{O}) \cong \mathrm{cl}(\mathcal{O}[G]) \oplus \bigoplus_{i=1}^{n} \mathrm{cl}(\mathcal{O}_{n|s\ |}).$$

For a modular field $\mathcal{O}$ and a rational modular $\mathcal{O}$ [illegible] a set

$$\mathrm{cl}(\mathcal{O}_{\mathbb{Q}}, \frac{1}{\mathcal{O}}) := \mathrm{cl}(\mathcal{O}_{\mathbb{Q}}) \cong \{\mathcal{O}\} : \mathcal{O} \mid \mathcal{O} \cong .$$

Finally [illegible] a finite $\mathrm{cl}(\mathcal{O}, \mathcal{O}[G])$ such that $\mathcal{O}$ induces an isomorphism

$$\mathrm{cl}(\mathcal{O}, \frac{1}{\mathcal{O}}) \cong \bigoplus_{i=1}^{n} \mathrm{cl}(\mathcal{O}_{n|s\ |}, \frac{1}{\mathcal{O}}).$$

[illegible] Let [illegible] be a subsemiring of a finite semiring [illegible]. Then

$$[illegible] = [illegible] \left[[illegible], \frac{1}{[illegible] + [illegible]} \right].$$

Proof. This proof is immediate from [illegible] and the above Theorem. □

If [illegible] is a $\mathfrak{m}$-semiring, then each of the fields [illegible] is of the form [illegible] $\cong$ [illegible]. In fact, a short $\mathrm{Ext}([illegible], [illegible]) = \mathrm{Ext}([illegible], [illegible])$, since every prime field of $\mathfrak{m}$ is principal in [illegible]. Therefore Corollary 1.3 does not give any additional information.

Now Theorem 1.1 is sharp

Theorem 2.5 Assume the situation of Theorem 1.1. Let [illegible] be a subsemiring of [illegible]. Then

$$[illegible] = [illegible] \left[[illegible], \frac{1}{[illegible] + [illegible]} \right]$$

.

This Theorem should be compared to [illegible], Theorem 1]. Note that, for, the ring algebra is a ring in this paper because it is local on the Euler system method as presented in [illegible] (see the Iwasawa generalizing Theorem 2.3).

[illegible]

[illegible] [illegible], [illegible] [illegible] [illegible]

[01] [illegible], [illegible] [illegible] l'[illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible], [illegible]. [illegible] [illegible] [illegible] [illegible] [illegible]

[illegible] [illegible], [illegible], [illegible] [illegible] [illegible] [illegible], [illegible] [illegible], [illegible]–[illegible] .

[illegible] [illegible], [illegible], [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible], [illegible] [illegible] [illegible] [illegible], [illegible]–[illegible].

[illegible] [illegible], [illegible] [illegible], [illegible].[illegible] [illegible], [illegible] [illegible], [illegible].

[illegible] [illegible], [illegible] [illegible] [illegible] [illegible] [illegible] [illegible]

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