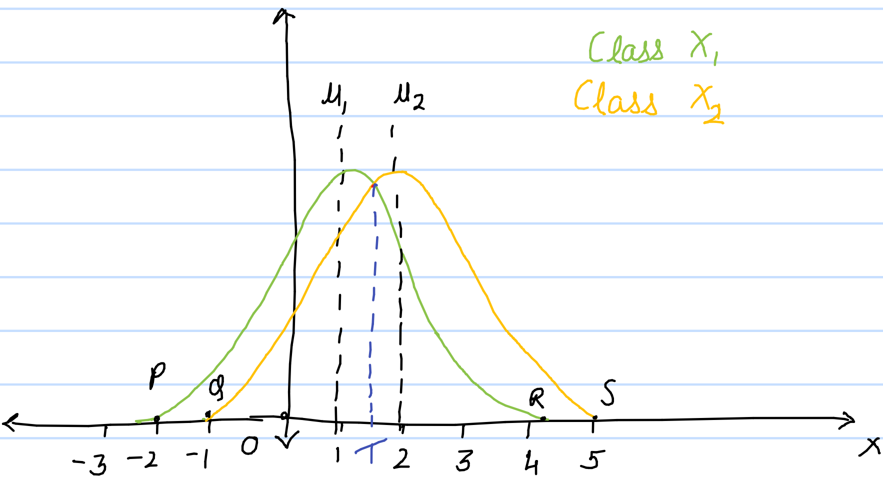
3) Theoretical limit of Gaussian Classification

Ans:- given:- two classes say X_1 & X_2

For X_1,
Mean $(\mu_1) = 1$
Variance (var) $= \sigma^2$

For X_2,
Mean $(\mu_2) = 2$
Variance $= \sigma^2$

where σ = standard deviation.



[Fig:- 3.1]

From the above figure (3.1), we can observe that two curves of classes X & X_2 intersect each other at 'T'

But from the graph it is clear the T is located at 1.5 units on x-axis.

Now, we already have an equation to calculate theoretical limit of optimal accuracy.

$$F(x) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \times e^{\frac{-(x-\mu)^2}{2\sigma^2}} \cdot dx. \quad \ldots\ldots (3.2)$$

where, σ :- Standard deviation or the spread of the curve

μ :- Mean

σ^2 :- Variance.

x :- Class.

But from the figure 3.1, it is clear that we have two classifications (x_1 & x_2). Hence, we will get two functions

$\therefore$ Lets calculate ERF(x_1) from the equation (3.2).

$$\therefore \quad F(x_1) = \int_{-\infty}^{1.5} \frac{1}{\sigma_1\sqrt{2\pi}} \times e^{\frac{-(x_1-\mu_1)^2}{2\sigma_1^2}} \cdot dx_1$$

$$\therefore \quad F(x_1) = \int_{-\infty}^{1.5} \frac{1}{\sigma_1\sqrt{2\pi}} \times e^{\frac{-(x_1-\mu_1)^2}{2\sigma_1^2}} \cdot dx_1$$

But we already have, $\mu_1 = 1$ & $\sigma_1^2 = 1$

$$\therefore \quad F(x_1) = \int_{-\infty}^{1.5} \frac{e^{\frac{-(x_1-1)^2}{2}}}{\sqrt{2} \times \sqrt{\pi}} \cdot dx_1$$

By substitution Method,

$$u = \frac{x-1}{\sqrt{2}} \longrightarrow dx = \sqrt{2}\, u$$

$$= \frac{1}{2} \int_{-\infty}^{1.5} \frac{2e^{-u^2}}{\sqrt{\pi}}\, du$$

But this is a error function.

$\therefore$ On integrating and resubstituting x, we get....

$$\therefore \left[\frac{\mathrm{erf}\left(\frac{x-1}{\sqrt{2}}\right)}{2} \right]_{-\infty}^{1.5} + C$$

Distributing limits

$$\frac{\frac{\sqrt{\pi}\,\mathrm{erf}(0.25.\sqrt{2})}{\sqrt{2}} + \frac{\sqrt{\pi}}{\sqrt{2}}}{\sqrt{2}\sqrt{\pi}}$$

$$= \frac{\mathrm{erf}(0.25 \times \sqrt{2}) + 1}{2} = \boxed{0.69146 \ldots \text{ (approx.)} \quad 3.3}$$

Now same way, we calculate it for x_2.

$$\therefore \quad F(x_2) = \int_{1.5}^{+\infty} \frac{1}{\sigma_2\sqrt{2\pi}} \times e^{\frac{-(x_2-\mu_2)^2}{2\sigma_2^2}} \cdot dx_2$$

= Same as previous calculation for $F(x_1)$.

$$= -\frac{\operatorname{erf}(0.25 \times \sqrt{2}) - 1}{2}$$

$$\boxed{= 0.3085} \quad \text{———} \quad (3.4)$$

Hence finally, we get two values

i.e. $\boxed{\begin{aligned} F(x_1) &= 0.6914 \\ F(x_2) &= 0.3085 \end{aligned}}$

From the above values we can observe clearly, that

$$\boxed{F(x_1) + F(x_2) = 1}$$