MSO203B PART A

OM SHRIVASTAVA

TOTAL POINTS

22 / 30

QUESTION 1

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+ 2 pts Obtaining $$\frac{X''(x)}{X(x)}=\frac{T'(t)}{T(t)}=\lambda$$.

✓ **+ 1 pts** *Deduction of $$X(0)=0$$.*

+ 1 pts Deduction of $$X(\pi)=0$$.

+ 3 pts Finding all eigenvalues $$\lambda_n=-n^2$$ where $$n\in\mathbb{N}$$.

+ 1 pts Deriving $$T_n(t)=e^{-n^2t}$$.

+ 2 pts Finding all eigenfunctions $$X_n(x)=A_n\sin{nx}$$.

+ 1 pts Consideration of $$\tilde{u}(x,t)=\sum_{n=1}^\infty A_n\sin{(nx)}e^{-n^2t}$$ by using law of superposition.

+ 1 pts Expression for $$A_n=\frac{2}{\pi}\int_0^\pi\sin{x}\sin{(nx)}dx$$.

+ 2 pts Calculating $$A_n=\begin{cases}1 &\text{for}\quad n=1\\0 &\text{for}\quad n\not=1\end{cases}$$, or writing directly $$\tilde{u}(x,t)=e^{-t}\sin{x}$$.

+ 14 pts Completely correct.

+ 0 pts Completely incorrect/Not attempted.

✓ **+ 12 pts** *If first eight rubrics all apply.*

QUESTION 2

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+ 0 pts Completely wrong/not attempted/other method.

✓ **+ 1 pts** *Consideration of energy function $$E(t)=\frac{1}{2}\int_{0}^ L (u_x^2+u_t^2))(x,t)dx$$*

✓ **+ 1 pts** *Derivative of energy function $$E'(t)=\int_{0}^ L (u_xu_{xt}+u_tu_{tt}))(x,t)dx$$*

+ 1 pts Mixed partial derivative equality i.e, $$u_{xt}=u_{tx}$$. So, $$E'(t)=\int_{0}^ L (u_xu_{tx}+u_tu_{tt}))(x,t)dx$$

+ 1 pts Computation of derivative of energy function$$E'(t)=0$$.

+ 1 pts Conclusion that energy function is constant $$E(t)=E(0)$$.

+ 1 pts Conclusion that energy function is 0 i.e, $$E(0)=0$$

+ 1 pts Deducing $$\int_0^L(u_x^2+u_t^2)(x,t)dx=0$$.

+ 1 pts Concluding u(x,t) is constant.

+ 1 pts Concluding $$u(x,t)=u(x,0)=0$$.

+ 9 pts Correct.

QUESTION 3

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+ 0 pts Not attempted/ fully wrong

✓ **+ 1 pts** *For computing $$a_{0} = 0$$.*

✓ **+ 2 pts** *For computing $$a_{n}= \left\{ \begin{array}{ll}
0 & \ \text{if} \ n=2m \\
\frac{-4}{n^{2} \pi} & \ \text{if} \ n=2m+1. or , \ 2(cos*

n\pi -1)/(n^2\pi)

\end{array}

\right. $$

+ 1 pts For computing $$ b_{n} = \frac{cosn

\pi}{n} - \frac{1}{n} $$

✓ **+ 2 pts** *For $$at \ x=0,   \sum_{n=1}^{\infty} a_{n}*

= \frac{f(0+)+ f(0-)}{2} = - \frac{\pi}{2}$$

✓ **+ 1 pts** *For concluding $$ \sum_{m=1}^{\infty}*

\frac{1}{(2m+1)^2} = \frac{\pi^{2}}{8} -1 $$

+ 7 pts Fully Correct

+ 0 pts Wrong answer

+ 1 *Point adjustment*

The computation of b_{n} is wrong. But in this problem, it is not required to compute b_{n}. So, I am giving you full marks.

END SEMESTER: MSO-203 (PART- A) Submitting time- 6:40 PM

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Q1 [14] Deduce the solution of the problem: $u_t - u_{xx} = 0,\ u(x,0) = \sin(x)$ on $x \in (0,\pi), u(0,t) = u(\pi,t) = 0,\ t \geq 0$, by seperation of variables method.[You may use known result on SLEVP]

$\Rightarrow$ Given :→ $U_t - U_{xx} = 0$

By separation of variable, let $u = \cancel{T(t)X(x)}\ F(t)G(x)$

$F'(t)G(x) - F(t)G''(x) = 0$

$$\frac{F'(t)}{F(t)} = \frac{G''(x)}{G(x)} = k \quad \text{, K is Constant}$$

$\Rightarrow \ln F(t) = kt + c$ — ①

$F(t) = e^{kt} \cdot C_1$

$$\frac{G''(x)}{G(x)} = k$$

let the K be λ^2.

$G''(x) = \lambda^2 G(x)$

Solution would be :→ $G(x) = Ae^{\lambda x} + Be^{-\lambda x}$

$U = F(t)G(x) = C_1 e^{kt}(Ae^{\lambda x} + Be^{-\lambda x})$

$u = e^{kt}(A'e^{\lambda x} + B'e^{-\lambda x})$

By Boundary Condition $u(0,t) = u(\pi,t) = 0$

So $u(0,t) = e^{kt}(A' + B') = 0$

$A' = -B'$

$u(\pi,t) = (A'e^{\lambda\pi} - A'e^{-\lambda\pi}) = 0$

$= e^{2\lambda\pi} = 1$

$e^{2\lambda\pi} = e^{2\pi i n}$

$\boxed{\lambda = in}$

$K = \lambda^2 = -n^2$

$u_n = A'e^{-n^2t}(e^{inx} - e^{-inx}) \Rightarrow (2A'e^{-n^2t}) \sin nx$ (here $e^{inx} - e^{-inx} = 2i\sin nx$)

By law of superposition

$u = \sum_{n=0}^{\infty} u_n$

$= \sum 2A' e^{-n^2t} \sin nx$

$\boxed{u_0 = 0}$

now using $u(x,0) = \sin x$

$u(x,0) = \sin x = \sum 2A' \sin nx$

So we can write (as Coefficient of Fourier series) $2A' = \frac{2}{\pi}\int_0^{\pi} (\sin x)(\sin nx)\,dx$

Also solving $2\sin nx \sin nx = \dfrac{\cos(n-1)x - \cos(n+1)x}{2}$

So $$\boxed{u = \sum_{n=1}^{\infty}\left(\frac{2}{\pi}\int_0^{\pi} \sin x \sin nx\,dx\right)\sin nx\, e^{-n^2t}}$$

Calculation for A'

$$A' = \frac{1}{\pi}\int_0^{\pi} \frac{\cos(n-1)x}{2} - \int_0^{\pi}\frac{\cos(n+1)x}{2}$$

$$= \frac{1}{\pi}\left[\left[\frac{\sin(n-1)x}{2}\right]_0^{\pi} - \frac{1}{2}\left[\frac{\sin(n+1)x}{2}\right]_0^{\pi}\right]$$

$= 0$

Q2 [9] Using energy method DEDUCE that trivial function is the only $C^2([0,L] \times [0,\infty])$ solution of the problem $u_{tt} - u_{xx} = 0,\ u(x,0) = u_t(x,0) = 0$ for $x \in (0,L), u(0,t) = u(L,t) = 0,\ \forall t \geq 0$.

$\Rightarrow$ Solving the wave Equation By Separation of Variable

$U = TX$

$T''X = TX''$

$$\frac{X''}{X} = \frac{T''}{T} = K$$

So let's for $K<0$ $K = -\lambda^2$

General solution of $X = A\sin\lambda x + B\cos\lambda x$

$T = C\sin\lambda t + D\cos\lambda t$

Putting Boundary Condition

$U = XT$

$u(x,0) = D(A\sin\lambda x + B\cos\lambda x) = 0$ this will give $D=0$

for ~~$\lambda=0$ we get Solution as $(A+Bx)(C+Dt)=u$~~
~~Putting the Boundary Condition~~ we get

now $u = (A\sin\lambda x + B\cos\lambda x)(C\sin\lambda t)$ ~~$u(x,0) = (C)(A+Bx) = 0$~~ $\boxed{\text{So } C=0}$

$u_t = (\lambda C\cos\lambda t)(A\sin\lambda x + B\cos\lambda x) =$
at $t=0$ it is 0

$u_t(x,0) = 0 = C\lambda(A\sin\lambda x + B\cos\lambda x) \Rightarrow \underline{\underline{C=0}}$

So ~~$u = A\sin\lambda x$~~ this is not Possible

now taking Solution as $\underline{K>0}\ \ K=\lambda^2$
$= (Ae^{\lambda x} + Be^{-\lambda x})(Ce^{\lambda t} + De^{-\lambda t})$

$\Rightarrow C+D=0$ ... By $u(x,0)=0$

By $u_t(x,0) = (C\lambda - \lambda D) = 0$
again we get $C=D=0$
So $u=0$

now For $\lambda=0$
$u = (A+Bx)(C+Dt)$
again By Boundary Conditi.
$u(x,0) = C = 0$
$u_t(x,0) = D(A+Bx) = 0$
$D=0$

For this E, only $u=0$, wave or will ~~no wave will~~ Satisfy the Condition that E = Constant in wave.

Energy method (So it's solution will only be trivial) So for all eigenvalue we set trivial Solution

$E = \int$ ~~$f(x)^2$~~ $+ s\ k|u_t|^2\,dt$

(here below is explanation that $E'=0$, $E=$ Constant By initial Condition at $E(0,0)$ we get $E=0$ hence

$E = \frac{1}{2}\int_L u_t^2 + u_x^2\,dt$

$E' = \int_0^b 2u_t u_t'\,dt + \int_0^b 2u_x u_{xt}\,dt$

for time t we get $\underline{\underline{E'=0}}$ By initial Energy of Condition hence we get $E=$ Constant
at $E(0,0)$ initial velocity is 0 as well as initial Position is 0 So $K+U=0$.

Q3 [7] Using the function $f(x) = -x$ *on* $x \in (-\pi, 0)$ and $f(x) = x - \pi$ *on* $x \in [0, \pi)$, find $\sum_{m=1}^{\infty} \frac{1}{(2m+1)^2}$

$\Rightarrow$ By writing the fourier expansion of $f(x)$.

$\Rightarrow f(x) = a_0 + \sum_{n=1} a_n\cos nx + \sum_{n=0} b_n\sin nx$

$a_0 = \int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{0} -x\,dx + \int_0^{\pi} x-\pi\,dx$

$= \frac{+\pi^2}{2} + \frac{x^2}{2} - \pi x\Big]_0^{\pi} \Rightarrow \frac{\pi^2}{2} + \left(-\frac{\pi^2}{2}\right) = 0$

$a_n = \int_{-\pi}^{\pi} f(x)\cos x\,dx = \int_{-\pi}^{0} -x\cos x\,dx + \int_0^{\pi}(x-\pi)\cos x\,dx$

~~$a_n = [-x\sin x]_{-\pi}^{0} - \int_{-\pi}^{0} -\sin x\,dx + [x\sin x]_0^{\pi} - \int_0^{\pi}\sin x\,dx - \pi\int_0^{\pi}\cos x\,dx$~~

~~$= [-\cos x]_{-\pi}^{0} - [-\cos x]_0^{\pi} - \pi[\sin x]_0^{\pi}$~~

~~$= -(-1-(-1)) + \{-1-1\}$~~

~~$= -2 - 2 = -4$~~

$f(x) = \frac{1}{\pi}\sum \frac{-4}{(2m+1)^2}\cos(2m+1)x + \sum\left(-\frac{3\pi\cos nx}{n} + \frac{\pi}{n}\right)\sin nx$

due to discontinuity at $x=0$

at $x=0$ $\hat{f}(x) = \frac{f(0^-)+f(0^+)}{2}$

$-\frac{\pi}{2} = -\frac{4}{\pi}\sum\frac{1}{(2m+1)^2}$

$\frac{+\pi^2}{8} = \sum\frac{1}{(2m+1)^2}$

last PST

final summation

Coefficient Calculation

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx$

$= \int_{-\pi}^{0} -x\cos nx + \int_0^{\pi}(x-\pi)\cos nx$

$= \left[-\frac{x\sin nx}{n}\right]_{-\pi}^{0} + \int_{-\pi}^{0}\frac{\sin nx}{n} + \left[\frac{x\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}$

$= -\frac{\cos nx}{n^2}\Big|_{-\pi}^{0} + \frac{\cos nx}{n^2}\Big|_0^{\pi} - \pi\int_0^{\pi}\cos nx$

$= \frac{-1+\cos n\pi}{n^2} + \frac{\cos n\pi - 1}{n^2}$

$= 2\left(\frac{\cos n\pi - 1}{n^2}\right) =$

When n is even $= 0$

n is odd $= \left(\frac{-4}{\pi n^2}\right) = \frac{-4}{\pi(2m+1)^2}$

Calculation of Coefficients

final Coeff

$a_0 = 0 \quad a_n = \frac{-4}{(2m+1)^2}\frac{1}{\pi}$
$b_n = -\frac{3}{n}\cos n\pi + \frac{1}{n}$

$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} f(x)\sin nx = \int_{-\pi}^{0} -x\sin nx + \int_0^{\pi}(x-\pi)\sin nx \Rightarrow\right.$
I_1

$I_1 = \frac{1}{\pi}\left[+\frac{x\cos nx}{n}\right]_{-\pi}^{0} - \int_{-\pi}^{0}\frac{\cos nx}{n^2}$

$I_1 = \left(\frac{\pi\cos n\pi}{n}\right)\frac{1}{\pi}$

$I_2 = \int_0^{\pi} x\sin nx - \pi\int_0^{\pi}\sin nx\,dx$

$= \left[-\frac{x\cos nx}{n}\right]_0^{\pi} + \int_0^{\pi}\frac{\cos nx}{n}\,dx - \pi\int_0^{\pi}\sin nx\,dx$

$= -\frac{\pi\cos n\pi}{n} - \pi\frac{\cos nx}{n}\Big]_0^{\pi} \Rightarrow -\frac{\pi\cos n\pi}{n} - \pi\left(\frac{\cos n\pi - 1}{n}\right) = \left(-\frac{2\pi\cos n\pi}{n} + \frac{\pi}{n}\right)\frac{1}{\pi}$