MSO203B Part A QUIZ MSO203B

SIDDHARTH GARG

TOTAL POINTS

13 / 21

QUESTION 1

1 Question 1 8 / 9

+ **0 pts** Completely wrong/ Not Attempted.
+ **1 pts** For stating all eigen values are real.

✓ **+ 1 pts** *For $$\lambda=0$$, solution is $$u(t)=At+B.$$*

✓ **+ 1 pts** *For showing $$\lambda=0$$ can't be an eigen value.*

✓ **+ 1 pts** *For $$\lambda>0$$, solution is $$u(t)=Ae^{\psi t}+Be^{-\psi t}.$$*

✓ **+ 1 pts** *For concluding $$\lambda>0 $$ can't be an eigen value.*

✓ **+ 1 pts** *For $$\lambda=-\psi^2$$, general solution is $$u(t) = Acos(\psi t)+Bsin(\psi t)$$.*

✓ **+ 1 pts** *u(0)=0 $$\implies$$ A=0.*

✓ **+ 1 pts** *Eigen pair is $$(-(2n+1)^2\pi^2/4, Bsin((2n+1)\pi t/2)). $$ 1 mark for writing first component correctly.*

✓ **+ 1 pts** *For writing the second component correctly.*

+ **9 pts** Completely right.

QUESTION 2

2 Question 2 0 / 6

✓ **+ 0 pts** *Completely wrong/ not attempted.*

+ **1 pts** WLOG we may assume $$x_n\uparrow \xi$$.
+ **1 pts** If $$\xi\in(-1,1)$$ is a limit point of $$A$$ then there exists sequence $$\{x_n\}\subset A $$ such that $$x_n\rightarrow \xi$$.
+ **1 pts** By Rolle's theorem there exists $$\xi_n\in(x_n,x_{n+1})$$ s.t $$u'(\xi_n)=0$$
+ **1 pts** By continuity $$u(\xi)=0$$.
+ **1 pts** By Continuity $$u'(\xi)=0$$
+ **1 pts** By the uniqueness of the second order ODE with $$u(\xi)=u'(\xi)=0$$ we have u=0.
+ **6 pts** Completely correct.
+ **2 pts** (**Alternative solution**): Reduced the problem to normal form $$y''+(e^{\frac{x^2}{2}}-\frac{x^2}{4}-\frac{1}{2})y=0$$.
+ **2 pts** As $$(e^{\frac{x^2}{2}}-\frac{x^2}{4}-\frac{1}{2})\leq3$$ so Compare with $$y''+3y=0$$.
+ **2 pts** Solution of $$y''+3y=0$$ has only finitely many zeroes So by Sturm Comparison theorem A is finite.

QUESTION 3

3 Question 3 5 / 6

✓ **+ 5 pts** *If everything correct , except condition for Parseval's identity not mentioned.*

+ **1 pts** Computation of $$a_0$$.
+ **1 pts** Computation of $$a_n$$ for $$n>0$$.
+ **1 pts** Computation of $$b_n$$ for $$n>0$$.
+ **1 pts** Mentioning Parseval's identity (THE

FORMULA).

+ 1 pts Condition for applicability of Parseval's identity.

+ 1 pts Correct evaluation of the given sum.

+ 6 pts Totally correct.

+ 0 pts Not attempted/Completely incorrect.

Click here to replace this description.

+ 0 pts Click here to replace this description.

QUIZ: MSO-203 (PART- A) Time - 40 Minutes

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Q1 **9-marks** Find all eigenpairs of the following problem: $u'' = \lambda u$ on $(0,1)$, $u(0) = u'(1) = 0$.

Sol. $u'' = \lambda u \Rightarrow u'' - \lambda u = 0$

Case I: $\lambda < 0$, i.e. $\lambda = -\mu^2$; $\mu \neq 0$

$\Rightarrow u'' + \mu^2 u = 0$

$\therefore u = A\sin(\mu t) + B\cos(\mu t)$

$u(0) = 0$

$\Rightarrow A\sin(0) + B\cos(0) = 0$

$\Rightarrow \boxed{B=0}$ — ①

$u'(t) = A\mu\cos(\mu t) - B\mu\sin(\mu t)$

$\Rightarrow u'(1) = 0$

$\Rightarrow A\mu\cos(\mu) - B\mu\sin(\mu) = 0$

$\Rightarrow A\mu\cos(\mu) = 0$ [From ① $B=0$]

$\Rightarrow A = 0$ or $\cos\mu = 0$

If $A=0$

we have $u \equiv 0$ but $u \not\equiv 0$.

$\therefore \cos\mu = 0$

$\Rightarrow \mu = n\pi + \frac{\pi}{2}$; $n = 0,1,2,\dots,-1,-2,\dots$

$\therefore \lambda = -\mu^2 = -(n\pi + \frac{\pi}{2})^2 \ \forall n \in \mathbb{Z}$

~~Hence,~~ $u = A\sin((n+\frac{1}{2})\pi t)$

Case II: $\lambda = 0$

$\Rightarrow u'' = 0$

$\Rightarrow u = At + B$

$u(0) = 0$

$\Rightarrow \boxed{B=0}$

Also $u'(t) = A$

$\Rightarrow u'(1) = \boxed{A=0}$

$\therefore u \equiv 0$

Hence, $\lambda = 0$ is not a eigenvalue.

Case III: $\lambda > 0$: $\lambda = \mu^2$, $\mu \neq 0$

$\Rightarrow u'' - \mu^2 u = 0$

$\Rightarrow u = Ae^{\mu t} + Be^{-\mu t}$

$u(0) = 0 \Rightarrow A + B = 0$ — ②

$u'(t) = A\mu e^{\mu t} - B\mu e^{-\mu t}$

$u'(1) = A\mu e^{\mu} - B\mu e^{-\mu} = 0$

$\Rightarrow Ae^{\mu} - Be^{-\mu} = 0$ — ③

From ② and ③

$$\begin{pmatrix} 1 & 1 \\ e^{\mu} & -e^{-\mu} \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\det\begin{pmatrix} 1 & 1 \\ e^{\mu} & -e^{-\mu} \end{pmatrix} = -e^{-\mu} - e^{\mu} \neq 0$$

$\Rightarrow \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow A = 0$ and $B = 0$

$\therefore \boxed{u \equiv 0}$

$\therefore \lambda > 0$ is also not an eigen value.

Hence, all eigen pairs are $\left(-(n\pi + \frac{\pi}{2})^2, A\sin((n+\frac{1}{2})\pi t)\right)$; $n = \dots,-3,-2,-1,0,1,2,\dots$

Q2b **6-marks** Let u be a non trivial solution of the problem $u''(t) + tu'(t) + e^{t^2}u(t) = 0$ in $(-1,1)$. Let $\mathcal{A} = \{x \in (-1,1) \mid u(x) = 0\}$. Prove that $\mathcal{A}$ does not admit any limit point in $(-1,1)$.

Sol. $u''(t) + tu'(t) + e^{t^2}u(t) = 0$

Converting it in the form $u'' + qu = 0$

We know, $q \geq 1$

Consider $w'' + 1w = 0$

$\Rightarrow w = \sin(t)$ is a solution

From Sturm comparison theorem,

there are ~~fini~~ infinitely many solutions of $u''+qu=0$

$\Rightarrow$ The element A which ~~is an infinite~~ set

let set A has a limit point in $(-1,1)$

i.e. $x \to x_0$.

$\Rightarrow$ $u(x)=0$ has finite number of roots in $(-1,1)$

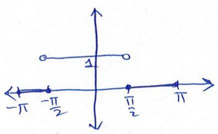
But $u(x)=0$ has infinite number of roots in $(-1,1)$

Hence, this is a contradiction.

$\therefore$ A has no limit point in $(-1,1)$

Q3a **6- marks** Consider the function $f(x)=1$ on $(-\frac{\pi}{2},\frac{\pi}{2})$, and $f(x)=0$ on $[-\pi,-\frac{\pi}{2}]\cup[\frac{\pi}{2},\pi]$. Calculate its Fourier coefficients. Hence evaluate the following sum: $\sum_{n=1}^{\infty}\frac{\sin^2(\frac{n\pi}{2})}{n^2}$.

Sol. $f(x) = \begin{cases} 1 & (-\pi/2, \pi/2) \\ 0 & [-\pi,-\pi/2]\cup[\frac{\pi}{2},\pi] \end{cases}$



Since, all the postulates of main theorem are satisfied here, we can write

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx)$$

where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,\ n>0, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$a_0 = \frac{1}{2\pi}\left(\int_{-\pi}^{-\pi/2} f(x)\,dx + \int_{-\frac{\pi}{2}}^{\pi/2} f(x)\,dx + \int_{\pi/2}^{\pi} f(x)\,dx\right) = \frac{1}{2\pi}\left(\int_{-\pi/2}^{\pi/2} dx\right) = \frac{1}{2\pi}\times\pi = \frac{1}{2}$$

$$a_n = \frac{1}{\pi}\left(\int_{-\pi}^{-\pi/2} f(x)\cos nx\,dx + \int_{-\pi/2}^{\pi/2} f(x)\cos nx\,dx + \int_{\pi/2}^{\pi} f(x)\cos nx\,dx\right)$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\cos nx\,dx = \left(\frac{1}{\pi}\right)\left(\frac{\sin nx}{n}\Big|_{-\pi/2}^{\pi/2}\right) = \frac{2\sin(\frac{n\pi}{2})}{\pi\, n}$$

$$b_n = \frac{1}{\pi}\left(\int_{-\pi}^{-\pi/2} f(x)\sin nx\,dx + \int_{-\pi/2}^{\pi/2} f(x)\sin nx\,dx + \int_{\pi/2}^{\pi} f(x)\sin nx\,dx\right)$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\sin nx\,dx = 0$$

$$\therefore\ f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos nx}{n}\sin(\tfrac{n\pi}{2})$$

From Parseval's Identity

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = 2a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2$$

$$\cancel{\Rightarrow \frac{1}{\pi}\left(\int_{-\pi/2}^{\pi/2} dx\right)^2 = 2\left(\frac{1}{2}\right)^2 + \sum_{n=1}^{\infty}\left(\frac{2}{\pi n}\right)^2}$$

$$\cancel{\Rightarrow \frac{1}{\pi}\times\pi = \frac{2}{4} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}}$$

$$\cancel{\Rightarrow \frac{1}{2} = 4}$$

$$\Rightarrow \frac{1}{\pi}\left(\int_{-\pi/2}^{\pi/2} dx\right) = 2\left(\frac{1}{2}\right)^2 + \sum_{n=1}^{\infty}\left(\frac{2}{\pi}\right)^2\left(\frac{\sin(\frac{n\pi}{2})}{n}\right)^2$$

$$\Rightarrow \frac{\pi}{\pi} = \frac{2}{4} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin^2(\frac{n\pi}{2})}{n^2}$$

$$\Rightarrow \left(1-\frac{1}{2}\right)\frac{\pi^2}{4} = \sum_{n=1}^{\infty}\frac{\sin^2(\frac{n\pi}{2})}{n^2}$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty}\frac{\sin^2(\frac{n\pi}{2})}{n^2} = \frac{\pi^2}{8}}$$