EE 200

Assignment & Solutions

1. Since $x(t)$ is real, $x(t) = x^*(t)$ so that

$$\sum_k x_k e^{jkw_0t} = \sum_k x_k^* e^{-jkw_0t}.$$

Replacing k by $-k$ in the summation, we have equivalently

$$x(t) = \sum_{k=-\infty}^{+\infty} a_{-k}^* e^{jk\omega_0 t}$$

which, by comparison with eq. (4.13), requires that $a_k = a_{-k}^*$ or equivalently that

$$a_k^* = a_{-k} \tag{4.17}$$

On the other hand, when $x(t)$ is purely imaginary, $x^*(t) = -x(t)$, so that $\sum_k x_k e^{jkw_0t} = -\sum_k x_k^* e^{-jkw_0t}$

From this, arguing as above we conclude $x_k^* = -x_k$

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2. To derive the alternative forms of the Fourier series, we first rearrange the summation in eq. (4.13) as

$$x(t) = a_0 + \sum_{k=1}^{\infty} [a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t}]$$

Using (4.17), this becomes

$$x(t) = a_0 + \sum_{k=1}^{\infty} [a_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega_0 t}]$$

Since the two terms inside the summation are complex conjugates of each other, this can be expressed as

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\mathcal{R}e\{a_k e^{jk\omega_0 t}\} \tag{4.18}$$

If a_k is expressed in polar form as†

$$a_k = A_k e^{j\theta_k}$$

then eq. (4.18) becomes

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\mathcal{R}e\{A_k e^{j(k\omega_0 t + \theta_k)}\}$$

That is,

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k) \tag{4.19}$$

Equation (4.19) is one commonly encountered form for the Fourier series of real periodic signals in continuous time. Another form is obtained by writing a_k in rectangular form as

$$a_k = B_k + jC_k$$

where B_k and C_k are both real. With this expression for a_k, eq. (4.18) takes the form

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} [B_k \cos k\omega_0 t - C_k \sin k\omega_0 t] \tag{4.20}$$

3 . Continuous-time case. Note that given T, $e^{j\left(\frac{2\pi k}{T}\right)t}$ is of a period that is harmonically related to T for all $-\infty < k < \infty$. So there are infinitely many

Discrete-time case: Given N, we see that $e^{jk\frac{2\pi}{N}n}$ repeats every N samples for every k; $-\infty < k < \infty$. But all these are not distinct. For example if we take $k = N$, we get $e^{j\frac{2\pi}{N}\cdot Nn} = e^{j0n}$. Thus the same exponential results for $k=N$ as for $k=0$. More generally, $e^{jk\frac{2\pi}{N}n} = e^{j(k+N)\left(\frac{2\pi}{N}\right)n}$ for all k. Hence, there are only exactly N different periodic complex exponentials with periods that are harmonic with N, namely $e^{jk\left(\frac{2\pi}{N}\right)n}$; $k = 0, 1, \ldots, N-1$.

4 $e^{jw_1 t} \cdot e^{jw_2 t} = e^{j(w_1 + w_2)t}$ which has a fundamental period of $w_1 + w_2$. If 2 generally periodic signals are given, $x_1(t) = x_1(t+T_1)$ and $x_2(t) = x_2(t+T_2)$ then while they will have many FS components in general, their fundamental frequencies will remain $2\pi/T_1$, $2\pi/T_2$. and thus $x_1(t)\cdot x_2(t)$ will have a fundamental period given by $2\pi / \left(\frac{2\pi}{T_1} + \frac{2\pi}{T_2}\right)$ i.e. $\frac{1}{T} = \frac{1}{T_1} + \frac{1}{T_2}$.

5. (a) Not periodic. If for some integers m, n, $mT_0 = nT$, then only will $y(t)$ be periodic with a period nT.

(b) Not true. Only when the input $x(t)$ is given shifts $\tau = nT$ for integer n, will an equal shift appear in the output $y(t)$

(c) H is linear, because $[\alpha x(t) + \beta x'(t)]p(t) = \alpha x(t)p(t) + \beta x'(t)p(t)$. It is memoryless, because $y(t)$ either depends on $x(nT)$ (at $t = nT$) or is zero (at $t \neq nT$). Therefore, it also causal. Finally, it is not stable because impulse trains are unbounded.

6. (a) When $T' \neq T$ it is not causal, because for certain values of n, $nT' < mT$ and yet $y(nT') = x(mT)$. However, when $T = T'$, it immediately becomes memoryless & causal as $T' = T \Rightarrow m = n$.

(b) For given T, T', the system is linear as both cases of $w(nT')$ are linear functions of $x(nT)$. It possesses memory because whenever $nT' \neq mT$, the input needs to be stored. Not causal, as already discussed in (a) above. It will be time invariant only for input shifts $\tau = mT$. Stability of the BIBO kind cannot be considered as both input & output are (unbounded) impulse trains.

7. If $x(t) = x(t-T)$, then $x(\alpha t) = x(\alpha(t - T/\alpha))$ i.e, it has a period of T/α. Thus, for $0 < \alpha < 1$. the FS coeffs. will be scaled by α and the fundamental frequency for $x'(t) = x(\alpha t)$ will be $\alpha \omega_0 = \omega_0'$. So, $x_k' = \alpha x_k$. Same holds for $\alpha > 1$.