ASSIGNMENT 8 SOLUTIONS.

sol ① Half wave symmetry $x(t) = -x(t - T/2)$

FS coefficients

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(n\Omega_0 t)\,dt \quad \text{where } \Omega_0 = \frac{2\pi}{T}$$

$$= \frac{2}{T}\left[\int_{-T/2}^{0} x(t)\cos(n\Omega_0 t)\,dt + \int_{0}^{T/2} x(t)\cos(n\Omega_0 t)\,dt\right]$$

$$= \frac{2}{T}\left[I_1 + I_2\right]$$

making the subsitution $\tau = t + T/2$ in I_1 gives

$$I_1 = \int_0^{T/2} x(\tau - T/2)\cos(n\Omega_0(\tau - T/2))\,d\tau$$

$$= -\int_0^{T/2} x(\tau)\cos(n\Omega_0\tau - n\pi)\,d\tau$$

$$= -\int_0^{T/2} x(\tau)(-1)^n \cos n\Omega_0\tau\,d\tau$$

$$= -(-1)^n \int_0^{T/2} x(t)\cos(n\Omega_0 t)\,dt$$

we can write

$$a_n = \frac{2}{T}(1 - (-1)^n)\int_0^{T/2} x(t)\cos(n\Omega_0 t)\,dt$$

from this expression we find that $a_n = 0$ whenever n is even. In fact, we have

$$a_n = \begin{cases} \frac{4}{T}\int_0^{T/2} x(t)\cos(n\Omega_0 t)\,dt, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

A similar derivation leads to

$$b_n = \begin{cases} \frac{4}{T}\int_0^{T/2} x(t)\sin(n\Omega_0 t)\,dt, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

(2) $x(t) \leftrightarrow X(\omega)$, $y(t) \longleftrightarrow Y(\omega)$

$$r_{xy}(t) = \int_{-\infty}^{\infty} x(\tau) y^*(t+\tau) d\tau \quad , \quad r_{xy}(t) \longleftrightarrow R_{xy}(\omega)$$

ie, $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$, $Y(\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt$.

Similarly, $R_{xy}(\omega) = \int_{-\infty}^{\infty} r_{xy}(t) e^{-j\omega t} dt$

ie, $R_{xy}(\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} \left[\int_{-\infty}^{\infty} x(\tau) y^*(t+\tau) d\tau \right] dt$

$$= \int_{-\infty}^{\infty} e^{-j\omega t} \left[\int_{-\infty}^{\infty} x(\tau) y^*(t+\tau) dt \right] d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} y^*(t+\tau) e^{-j\omega t} dt \right] d\tau$$

Now, using the property of Fourier transform that

if $\boxed{\begin{array}{l} x(t) \leftrightarrow X(\omega) \\ x(t-t_0) \leftrightarrow X(\omega) e^{-j\omega t_0} \end{array}}$

we get,

$$R_{xy}(\omega) = \int_{-\infty}^{\infty} x(\tau) . Y^*(-\omega) e^{-j\omega\tau} d\tau$$

$$= Y(\omega) \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau = Y^*(-\omega) . X(\omega)$$

Hence, $R_{XY}(\omega) = F\left[\int_{-\infty}^{\infty} x(\tau) y^*(t+\tau) d\tau \right] = X(\omega) \cdot Y^*(-\omega)$

Q3) ⇒

a) No, Not every LTI system have an inverse

Ex:- 1) $y(t) = x(t-4)$ it is a LTI system.

Now, if we give different input it will produce different output.

i/p	$\delta(t)$	$-\delta(t)$	$u(t)$	$-u(t)$
o/p	$\delta(t-4)$	$-\delta(t-4)$	$u(t-4)$	$-u(t-4)$

Hence, the system is invertible. and the inverse system gives $y(t+4)$.

2) $y(t) = \frac{d}{dt} x(t)$ is a LTI system.

different i/p gives

i/p	2	5	10
o/p	0	0	0

Hence, for different i/p it gives same output therefore, the system doesn't have inverse.

→ b) To have inverse of system, it must give different output for different input.

Hence, it should have one one onto mapping between input & output.

c) Given, $h(t) * h'(t) = \delta(t)$. Take F.T. on both side.

$\Rightarrow H(\omega) \cdot H'(\omega) = 1$.

$\Rightarrow \boxed{H(\omega) = 1/H'(\omega)}$

4. Let $x(t)$ have a support of (a,b). Then with no loss of generality we can write

$$x(t) = x(t)\,p(t) \quad \text{where}$$

$$p(t) = \begin{cases} 1 : & |t| < \max(|a|,|b|) \\ 0 : & \text{otherwise} \end{cases}$$

Then $X(\omega) = \frac{1}{2\pi} X(\omega) * P(\omega)$

$P(\omega)$, the transform of a rectangular pulse, has an infinite support as it is of the form $(\sin x)/x$. Since $P(\omega)$ is convolved with $X(\omega)$, to get $X(\omega)$ the result, $X(\omega)$ will have infinite support.

To prove the vice versa result, express $X(\omega)$ similarly as a product of a symmetric rectangular freq domain pulse with $X(\omega)$ and then apply the modulation property discussed in class.

5.

$$H(\omega) = \frac{5j\omega+7}{(j\omega+5)(-3j^2\omega^2-2j\omega+2)}$$

$$\boxed{\text{Inverse, } H^{-1}(\omega): \quad \frac{(j\omega+5)(-3j^2\omega^2-2j\omega+2)}{5j\omega+7}}$$

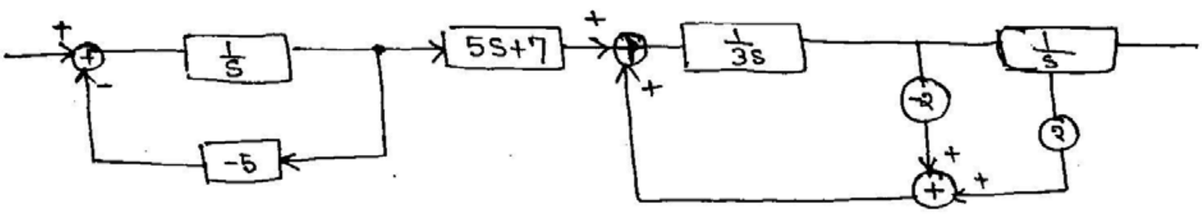
Let $j\omega = s$

$$H(s) = \frac{5s+7}{(s+5)(3s^2-2s+2)}$$

Cascade Form:

$$H(s) = \left(\frac{5s+7}{s+5}\right) \cdot \frac{1}{(3s^2-2s+2)}$$

Since $3s^2-2s+2$ has Imaginary roots we can not make it in more simpler realizable form ($D = -20$)



$\boxed{\frac{1}{s}} \longrightarrow \boxed{\int}$

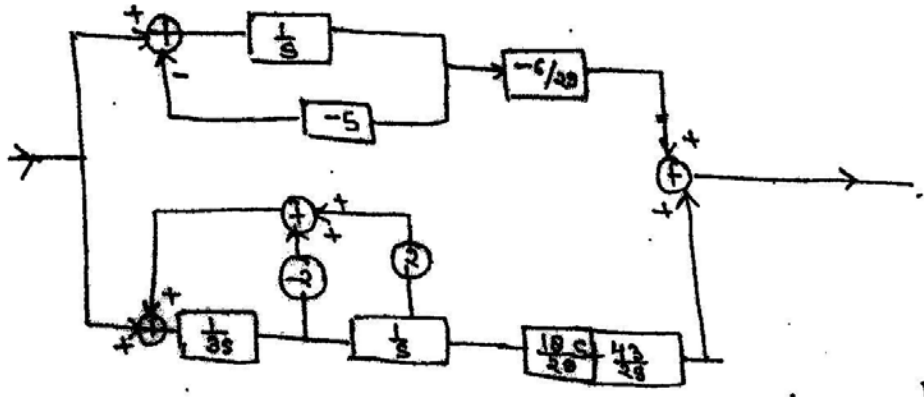
$\boxed{s} \longrightarrow \boxed{\frac{d}{dt}}$

Parallel Form

$$H(s) = \frac{5s+7}{(s+5)(3s^2-2s+2)}$$

$$= \frac{A}{s+5} + \frac{Bs+C}{3s^2-2s+2}$$

$$\Rightarrow \frac{-6/29}{s+5} + \left(\frac{18}{29}\right)s + (43/29)$$



$\frac{1}{s} \rightarrow \boxed{\int}$

$s \rightarrow \boxed{\frac{d}{dt}}$

$$G(w) = \frac{16 + (jw)^2}{(jw)(jw+4)(jw-4)}$$

Put $jw = s$

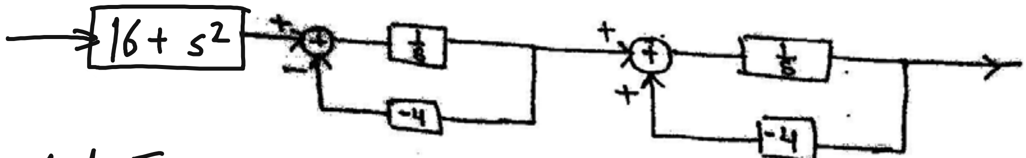
Then $G(s) = \dfrac{16 + s^2}{s(s+4)(s-4)}$

Inverse $G^{-1}(w)$:

$$\frac{jw(jw+4)(jw-4)}{(16 - w^2)}$$

Cascade Form:

$$G(s) = \frac{(16+s^2)}{(s)} \cdot \frac{1}{(s+4)} \cdot \frac{1}{(s-4)}$$



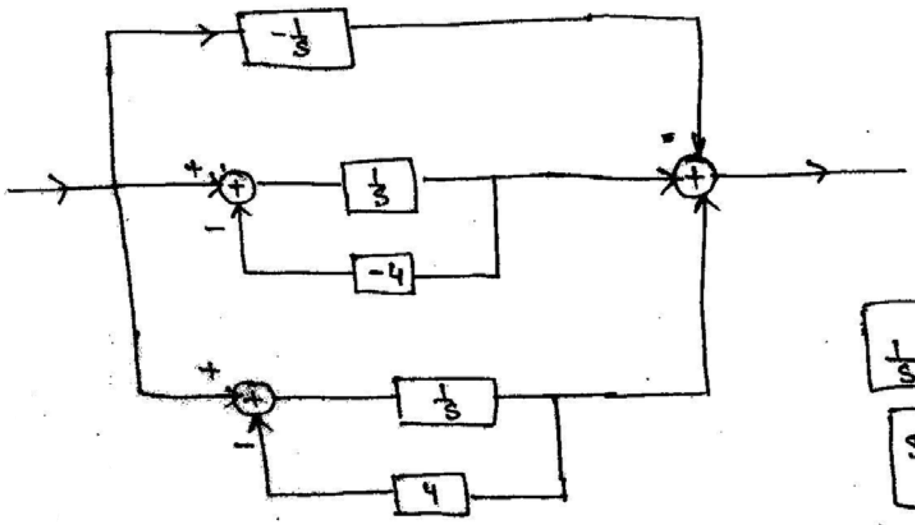
Parallel Form:

$\frac{1}{s} \rightarrow \boxed{\int}$

$s \rightarrow \boxed{\frac{d}{dt}}$

$$G(s) = \frac{A}{s} + \frac{B}{s+4} + \frac{C}{s-4}$$

$$= \frac{-1}{s} + \frac{1}{s+4} + \frac{1}{s-4}$$



$$\boxed{\frac{1}{s} \longrightarrow \int}$$

$$\boxed{s \longrightarrow \frac{d}{dt}}$$

Sol^n (6)

$$x[n] \xleftrightarrow{DFT} X(K)$$

$$y[n] = X(n) \xleftrightarrow{DFT} Y(K)$$

we know, $$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$$

and $$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{j\frac{2\pi}{N}kn}$$

$$N x(n) = \sum_{K=0}^{N-1} X(K) e^{j\frac{2\pi}{N}kn}$$

$$N x(-n) = \sum_{K=0}^{N-1} X(K) e^{-j\frac{2\pi}{N}kn}$$

Exchanging the role of K and n we get

$$N x(-K) = \sum_{n=0}^{N-1} X(n) e^{-j\frac{2\pi}{N}kn}.$$

$$N x(-K) = Y(K)$$

$$\boxed{Y(n) = N x(-n)}$$

7 (a) $\boxed{x[n] = (-1)^n}$ $\therefore X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k n}{N}}$

$$\Rightarrow X[k] = \sum_{n=0}^{N-1} (-1)^n e^{-j\frac{2\pi k n}{N}} = \sum_{n=0}^{N-1} \left(-e^{-j\frac{2\pi k}{N}}\right)^n$$

$$X[k] = 1 - \left(e^{-j\frac{2\pi k}{N}}\right) + \left(e^{-j\frac{2\pi k}{N}}\right)^2 + \cdots \left(-e^{-j\frac{2\pi k}{N}}\right)^{N-1}$$

$$\text{or, } X[k] = \frac{1 - (-1)^N \exp\left(-j\frac{2\pi k}{\cancel{N}}\right)\cdot\cancel{N}}{1 + e^{-j\frac{2\pi k}{N}}}$$

$$\boxed{X[k] = \frac{1 - (-1)^N \exp(-j2\pi k)}{1 + \exp\left(-\frac{j2\pi k}{N}\right)}} \quad \text{for } 0 \le k < N$$

(b) $x[n] = 1 + (-1)^n = x_1[n] + x_2[n]$.

$x[n] \leftrightarrow X[k]$, By linearity property,

$$X[k] = X_1[k] + X_2[k]$$

$x_1[n] = 1 \qquad 0 \le n < N$

$X_1[k] = N\delta(k)$

and, $X_2[k]$ as found in the previous problem

$\therefore$ so, $x[n] = 1 + (-1)^n \longleftrightarrow X[k]$

Where,
$$\boxed{X[k] = N\delta(k) + \frac{1 - (-1)^N \exp(-j2\pi k)}{1 + \exp\left(-\frac{j2\pi k}{N}\right)}}$$

for $0 \le k < N$

(7) (c) $x[n] = j^n \quad 0 \le n < N$

$\therefore X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k n}{N}}$

$= \sum_{n=0}^{N-1} j^n e^{-j\left(\frac{2\pi k}{N}\right)n} = \sum_{n=0}^{N-1} \left(j e^{-j\frac{2\pi k}{N}}\right)^n$

$$= \frac{1 - \left(j e^{-j\frac{2\pi k}{N}}\right)^N}{1 - j e^{-j\frac{2\pi k}{N}}}$$

or, $\boxed{X[k] = \frac{1 - \left(j e^{-j\frac{2\pi k}{N}}\right)^N}{1 - j e^{-j\frac{2\pi k}{N}}}}$ for $0 \le k < N$

(d) $x[n] = 1, 0, \ldots 0$

or, $x[n] = \delta[n] \quad 0 \le n < N$

$X[k] = \sum_{n=0}^{N-1} x[n] \exp\left(-j\frac{2\pi k n}{N}\right)$

$= \sum_{n=0}^{N-1} \delta[n] \exp\left(-j\frac{2\pi k n}{N}\right)$

$= 1$

$\therefore \boxed{X[k] = 1 \quad 0 \le k \le N-1}$

Q8) ⇒ Given, $x(n)$ is N - length sequence and $x'[n]$ is 2N - length sequence.

Then, DFTs of above signals are,

$$\therefore X[K] = \sum_{n=0}^{(N-1)} x[n] \cdot e^{-j2\pi nk/N}$$

$$\therefore X[k] = x[0] + x[1] e^{-j2\pi K/N} + \cdots\cdots + x[N-1] \cdot e^{-j2\pi k(N-1)/N}$$

$$\therefore X[0] = x[0] + x[1] + \cdots\cdots + x[N-1].$$

$$\therefore X[1] = x[0] + x[1] e^{-j2\pi/N} + \cdots\cdots + x[N-1] e^{-j2\pi(N-1)/N}$$

$$\vdots$$

$$X[N-1] = x[0] + x[1] \cdot e^{-j2\pi(N-1)/N} + \cdots + x[N-1] \cdot e^{-j2\pi(N-1)^2/N}$$

And $X[K]$ is periodic with period N.

Similarly,

$$\therefore X'[K] = x[0] + x[1] \cdot e^{-j2\pi k/2N} + \cdots\cdots + x[N-1] e^{-j2\pi k(N-1)/2N}$$

$$\therefore X'[0] = x[0] + x[1] + \cdots\cdots + x[N-1]$$

$$\therefore X'[1] = x[0] + x[1] e^{-j2\pi/2N} + \cdots\cdots + x[N-1] e^{-j2\pi(N-1)/2N}$$

$$\vdots$$

$$\therefore X'[2N-1] = x[0] + x[1] \cdot e^{-j2\pi(2N-1)/2N} + \cdots\cdots + x[N-1] e^{-j\frac{2\pi(N-1)(2N-1)}{2N}}$$

Hence every member of $X[K]$ is found to be on even values of K in $X'[K]$ and values at odd value of K in $X'[K]$ are to be evaluated seperately their no such relation with old 'N members'. (no shortcuts)

9. $y[n] = x[n] - \alpha y[n-1]$.

Assuming $Y(\Omega)$ exists, we have

$$Y(\Omega) = X(\Omega) - \alpha Y(\Omega) e^{-j\Omega}.$$

So $$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1}{1-\alpha e^{-j\Omega}}.$$

$Y(\Omega)$ will exist whenever $|h[n]|$ is summable.

$h[n]$ is found as follows.

$$y[n] = x[n] - \alpha[x[n-1] - \alpha[x[n-2] \cdots] \cdots]]$$

Setting $x[n] = \delta[n]$, we have

$$h[n] = \sum_{n=0}^{\infty} (-1)^n \alpha^n$$

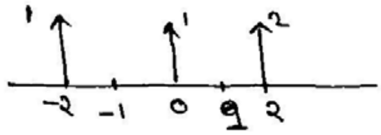
This geometric series is summable only for $|\alpha| < 1$
But this resulted because we solved the equation forward in time. Solving backward in time would that summability happens for $|\alpha| > 1$.

Summability of the impulse response implies that $y[n]$ is bounded. $Y(\Omega)$ will exist when both $|h[n]|$ <u>and</u> $|x[n]|$ are individually summable, as this ensures that $|y[n]|$ is summable.

(16) $x[n] \longleftrightarrow X(\Omega)$

(a) $\ldots 0, x[-2], 0, \ldots x[0], 0, x[2], \ldots$

$p(n) =$



$$y[n] = x[n] \cdot p[n]$$

$$Y(\Omega) = X(\Omega) \circledast P(\Omega)$$

$$p[n] = \frac{1}{2}\left(1 + (-1)^n\right)$$

$$P(\Omega) = \frac{1}{2}\left[2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - 2\pi l) + 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - \pi - 2\pi l)\right]$$

$$P(\Omega) = \pi \sum_{l=-\infty}^{\infty} \delta(\omega - \pi l)$$

$$Y(\Omega) = X(\Omega) \circledast P(\Omega)$$

$$= \frac{1}{2\pi} \int_{2\pi} X(\omega)\, P(\Omega - \omega)\, d\omega$$

$$= \frac{1}{2\pi}\, \pi \int_{2\pi} X(\Omega - \omega) \sum_{l=-\infty}^{\infty} \delta(\omega - 2\pi l)\, d\omega$$

take $\omega \in [0, 2\pi)$

$$= \frac{1}{2} \sum_{l=-\infty}^{\infty} X(\Omega - \pi l)$$

$$\boxed{Y(\Omega) = \frac{1}{2}\left[X(\Omega) + X(\Omega - \pi)\right]}$$ Ans.

(b) $0, x[-1], 0, x[1], \ldots$

same as part (a) with

$$p[n] = \frac{1}{2}[1 - (-1)^n]$$

$$P(\Omega) = \pi \sum_{l=-\infty}^{\infty} \delta(\omega - 2\pi l) - \pi \sum_{l=-\infty}^{\infty} \delta(\omega - 2\pi l - \pi)$$

$$Y(\Omega) = X(\Omega) \otimes P(\Omega)$$

$$Y(\Omega) = \frac{1}{2\pi} \pi \int_{2\pi} X(\Omega - \omega) \left[\sum_{l=-\infty}^{\infty} \delta(\omega - 2\pi l) - \sum_{l=-\infty}^{\infty} \delta(\omega - \pi - 2\pi l) \right] d\omega$$

$$\boxed{Y(\Omega) = \frac{1}{2}[X(\Omega) - X(\Omega - \pi)]}$$ Ans

(c) odd members inverted

$$Y(\omega) = Y_a(\omega) - Y_b(\omega)$$

using part (a) & (b)

$$\boxed{Y(\omega) = X(\Omega - \pi)}$$ Ans.

(d) even members inverted

$$Y(\omega) = Y_b(\omega) - Y_a(\omega)$$

From part (a) & (b)

Ans $$\boxed{Y(\omega) = -X(\Omega - \pi)}$$