MSO203B PART B

OM SHRIVASTAVA

TOTAL POINTS

25 / 30

QUESTION 1

1 Question 1 6 / 10

+ **10 pts** Correct

✓ **+ 1 pts** *$$\dot{x}(t)=a(z(t)), x(0)=s$$*

✓ **+ 1 pts** *$$\dot{y}(t)=1, y(0)=0$$*

✓ **+ 1 pts** *$$\dot{z}(t)=0, z(0)=h (s)$$*

✓ **+ 1 pts** *$$z(t)=h(s)$$*

✓ **+ 1 pts** *$$y(t)=t$$*

✓ **+ 1 pts** *$$x(t)=a(h(s)) t + s$$*

+ **4 pts** $$s \mapsto \dfrac {1}{a(h(s)}$$ is an increasing function, or $$s\mapsto a(h(s))$$ is a decreasing or non increasing function.

+ **0 pts** Totally wrong

QUESTION 2

2 Question 2 10 / 10

+ **0 pts** Completely wrong/ Not attempted.

+ **3 pts** Substitute $$u(t)=e^\frac{t^2}{2}v(t)$$

+ **2 pts** For finding the Normal form $$v''(t)+(2a+1-t^2)v(t)=0$$

+ **3 pts** Compare this normal form with $$v''(t)=0$$ and conclude $$v $$ can have at most finitely many zeroes.

+ **2 pts** Since $$u(t)=e^\frac{t^2}{2}v(t)$$ so $$u(t)$$ can have at most finitely many zeroes.

✓ **+ 10 pts** *Completely Correct.*

QUESTION 3

3 Question 3 9 / 10

✓ **+ 1 pts** *$$B^2- 4AC >0$$, the problem is hyperbolic.*

+ **1 pts** $$y = 3x + c_1 , y = \frac{x}{3} + c_2$$

✓ **+ 1 pts** *$$\zeta(x,y) = y - 3x$$*

✓ **+ 1 pts** *$$\eta(x,y) = y - \frac{x}{3}$$.*

For rubrics 3 and 4, one may use this expression or some constant multiple of this ($$\zeta, \eta$$ is not unique)

Introducing new variables $$\bar{A}, \bar{B}, \bar{C}, \bar{D}, \bar{E}, \bar{F}$$, where $$\bar{A}$$= coefficient of $$v_{\zeta \zeta},\bar{B}$$= coefficient of $$v_{\zeta \eta}, \bar{C}$$= coefficient of $$v_{\eta \eta}, \bar{D}$$= coefficient of $$v_{\zeta}, \bar{E}$$= coefficient of $$v_{\eta}, \bar{F}$$= coefficient of $$v, G= \bar{G}$$.

✓ **+ 1 pts** *$$\bar{A} = 0 = \bar{C}$$*

✓ **+ 1 pts** *$$\bar{B} = -\frac{64}{3}$$.*

✓ **+ 1 pts** *$$\bar{D} = \bar{E} = \bar{F} = \bar{G} = 0$$.*

✓ **+ 1 pts** *The canonical form is $$ v_{\zeta \eta} = 0$$. ** If some one is directly deduce the canonical form by using correct calculations, give him/her full 4 marks(for rubrics 5-8). ***

✓ **+ 2 pts** *General solution is $$v(\zeta, \eta) =*

f(\zeta) + g(\eta)$$ $$u(x,y) = f(y - 3x) + g(y - \frac{x}{3})$$.

+ 10 pts Completely correct

+ 0 pts Incomplete / Completely wrong

gradescope

END SEMESTER: MSO-203 (PART- B) Submitting time: 7:45 PM

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Q1 [10] Let $a, h : \mathbb{R} \to \{t > 0\}$. Find a condition on the function a which ensures that the projected characteristics of the problem $a(u)u_x + u_y = 0, u(x, 0) = h(x)$ intersects at some points on the set $\{y > 0\}$ but do not intersects at any point of the set $\{y < 0\}$.

$\Rightarrow$ By given Condition $\dot{x}(t) = a(u)$ — i $x(0) = s$

$\dot{y}(t) = 1$ — ii $y(0) = 0$

$\dot{u} = \dot{z}(t) = 0$ — iii $z(x(0), y(0)) = h(s)$

t → x-axis

By (ii) $y = t + c$

yet By $y(0) = 0$, we get $y = t$

By (i) $x(t) = a(u)t + c$

By $x(0) = s$

$x = a(u)t + s$

By (iii) $Z = k$ ... k is Constant

$\boxed{Z = h(s)}$ also we know

$Z = K = h(s) = h(x)$

~~$x(t) = a(h(x))t + s$~~

Put $t = y$

$\boxed{x = a(h(x))y + s}$ ~~---- this is the Projected Characteristics~~

no space final answer :→

$x = a(h(s))t + s$

if $a\,h(s)$ is linear function of s, P.C. will intersect if $a h(s)$ is constant function it will be family of Parallel line. hence for $y<0$ Z should be Constant so ~~h has Constant~~ let $a h(s)$ be Constant and for $y>0$ let $a h(s)$ be linear function of S or of form ms, here m is any integer

For the P.C. to do not intersect at ~~$y<0$~~ for $y<0$ $a(h(x))$ ~~should be~~ so ~~$x = a(h)$~~ Constant

Solving ~~(i) $\frac{dx}{dt} = a(h(x))$~~ $a(h(x))$ is some function of x

$\int \frac{dx}{a(h(s))} = t + c$

$\frac{x}{a(h(s))} = t + c$

$\int \frac{dx}{a(h(x))} = t + c$ $dx = a(h(x))\,dt$

$\int \frac{dx}{a(h(x))} = t + C$

$x = a h(s) t + s$, if $a h(s) = ms$ ... m is some integer then $x = (mt+1)s$ so here P.C intersect hence for $y>0$

Clearly it can be seen that if a is Constant function we get a line so for $y<0$ a should be a Constant function to ensure P.C do not intersect.

For $y>0$ $a(u)$ or $a(h(x))$ should not be a linear function.

By initial Condition $\boxed{\frac{s}{a h(s)} = C}$, at $t=0 \Rightarrow \frac{x}{a h(s)} = t + \frac{s}{a h(s)} \Rightarrow \boxed{x = a h(s) t + s}$

if Const it will give no intersection

if $a h(s)$ is some not linear function then the curve will intersect for $y>0$.

Q2 [10] Show that for $a \in [0, \infty)$, any nontrivial solution of $u''(t) - 2tu'(t) + 2au(t) = 0$, $t \in (\sqrt{2a+1}, \infty)$ has atmost finitely many zeros.

$\Rightarrow$ $u''(t) - 2t u'(t) + 2a u(t) = 0$ Converting to normal form.

~~Let's Compare it with~~ let's Put $u = V(t) W(t)$

$(V(t) W(t))'' - 2t(VW)' + 2a VW = 0$

$\Rightarrow$ $(V'W + W'V)' - 2t(V'W + W'V) + 2aVW = 0$

$= V''W + W''V + 2V'W' - 2tV'W - 2tW'V + 2aVW = 0$

= writing ODE in V

$W V'' + (2W' - 2tW)V' + (2aW - 2tW' + W'')V = 0$

Put V' Coeff $= 0$

So $2W' = 2tW \Rightarrow \frac{W'}{W} = t$ — (i)

$\ln w = \frac{t^2}{2}$

$W = e^{\frac{t^2}{2} + c} \rightarrow W' = (t e^{\frac{t^2}{2}+c})$

$\Rightarrow$ ~~$(e^{\frac{t^2}{2}+c}) V'' + 2a e^{\frac{t^2}{2}+c}$~~ $2t$

$W' = t e^c e^{\frac{t^2}{2}}$

$W'' = e^c \left[e^{\frac{t^2}{2}} + t^2 e^{\frac{t^2}{2}}\right]$

$\frac{W''}{W} = (t^2 + 1)$ — (ii)

we know $W \neq 0$

So $V'' + \left(2a - \frac{2tW'}{W} + \frac{W''}{W}\right)V = 0$

From (i) and (ii) eliminating W in terms of t.

$V'' + (2a - 2t^2 + (t^2+1))V = 0$

So we have normal form.

$V'' + (-t^2 + 1 + 2a)V = 0$

$q_1 = -[t^2 - (2a+1)]$ — iii

For $t > \sqrt{2a+1}$

$q_1 < 0$ — Can be seen from (iii)

at most it can be Two zeros. Since V'' solution $at+b$ will have one root at most)

Comparing with $V'' = 0$

also $q_1 < q_2$

So For $q_2 = 0$ $V'' + 0V' = 0$ solution $(at+b)$ --- will have finite root.

So By S.L.C thm. we have that $V'' + q_1 V$ will have finitely zeros at most.

Strum Liouville theorem tells that for $v'' + q_1 v = 0$ — (iii)

and $v'' + q_2 v = 0$ — (iv) if $q_1 > q_2$ then between

2 roots of ~~q_2~~ ~~(iv)~~ (iv) there will lie on root of (iii).

normal form: $\to$ $v'' + (-t^2 + 1 + 2a)v = 0$

Q3 [10] Calssify and reduce the equation $3u_{xx} + 10u_{xy} + 3u_{yy} = 0$ to its cannonical form. Hence find its general solution.

$\Rightarrow$ $3u_{xx} + 10u_{xy} + 3u_{yy} = 0$

$\Rightarrow$ ~~let~~ so $A = 3,\ B = 10,\ C = 3$

Hence to Classify it $B^2 - 4AC$, we get $(10)^2 - 4 \cdot 9$

$= 64 > 0$

Hence $B^2 - 4AC > 0$ so it's Hyperbolic

now By method of Characteristics and Changing of variable.

$$\frac{dy}{dx} = \frac{+10 \pm \sqrt{64}}{2 \cdot 3} = \frac{10 \pm 8}{6} = 3, \frac{1}{3}$$

$\hookrightarrow \frac{B^2 - 4AC}{2A}$

~~$y = 3x$~~ so we get $y - 3x$ and $3y - x$ as the suitable variable.

let $\xi = y - 3x$ $\quad \eta = 3y - x$

$\xi_x = -3$ $\quad \eta_x = -1$

$\xi_{xx} = 0$ $\quad \eta_y = 3$

$\xi_y = 1$ $\quad \eta_{xx} = 0$

$\xi_{yy} = 0$ $\quad \eta_{yy} = 0$

$3u_{xx} + 10u_x u_y + 3u_{yy} = 0$

for some ~~V~~ $V(\xi, \eta)$ take $u(x,y) = V(\xi,\eta)$

~~u_x~~ $u_x = V_\xi \xi_x + V_\eta \eta_x \Rightarrow u_x = -3V_\xi - V_\eta$

$u_{xx} = -3(V_{\xi\xi}\xi_x + V_{\xi\eta}\eta_x) - [(V_{\eta\xi}\xi_x) + V_{\eta\eta}\eta_x]$

~~$u_{xy} = (V_\xi)_y \xi_x + V_\xi \xi_{xy} + (V_\eta)_y \eta_x + (V_\eta)\eta_{xy}$~~

~~$= (V_{\xi\xi}\xi_y + V_{\xi\eta}\eta_y)\xi_x + V_\xi \xi_{xy} + (V_{\eta\xi}\xi_y + V_{\eta\eta}\eta_y)\eta_x + (V_\eta)\eta_{xy}$~~

$u_{xx} = -3$

~~$u_{xx} = (V_\xi)_x \xi_x + V_\xi \xi_{xx} + (V_\eta)_x \eta_x + V_\eta \eta_{xx}$~~

~~$= (V_{\xi\xi}\xi_x + V_{\xi\eta}\eta_x)\xi_x + V_\xi \xi_{xx} + (V_{\eta\xi}\xi_x + V_{\eta\eta}\eta_x)\eta_x + V_\eta \eta_{xx}$~~

~~$= V_{\xi\xi}(\xi_x)^2 + V_{\xi\eta}\xi_x\eta_x + V_\xi \xi_{xx} + V_{\eta\xi}\xi_x\eta_x + V_{\eta\eta}(\eta_x)^2 + V_\eta \eta_{xx}$~~

$(V_\xi)_x = \partial V_\xi$ ~~$\frac{\partial \xi}{\partial \xi} + \frac{\partial V_\xi}{\partial \eta}\frac{\partial \eta}{\partial x}$~~

$-3[-3V_{\xi\xi} - V_{\xi\eta}]$

$u_{xx} = $ ~~$-3(V_{\xi\xi}3)$~~

$=$ ~~$V_{\xi\xi}$~~

Simply $\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$ $\quad \bar{B} = 2A\xi_x\eta_x + 2C\xi_y\eta_y + B(\xi_x\eta_y + \eta_x\xi_y)$

$\bar{C} =$ ~~$A\xi_x^2 + B\xi_x$~~ $A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$ $\quad \bar{D} = 0, \bar{F} = 0; \bar{f}$

$\bar{A} = 3(9) + 10(-3) + 3(1)$ $\quad$ due to mixed derivative $\xi_{xx} = 0$, $\xi_{xx} = 0$

$\bar{A} = 0$

$\bar{B} =$ ~~20~~ $6(-3)(-1) + 2(3)(3) + 10(-10)$

$\bar{B} = -64$ $\quad \bar{C} = 3(-1)^2 + 10(-3) + 3(3)^2$

$\bar{A}V_{\xi\xi} + \bar{B}V_{\xi\eta} + \bar{C}V_{\eta\eta} + \bar{D}V_\xi + \bar{F}V_\eta + F^V = G$ — general form.

$-64V_{\xi\eta} = 0$

$V_{\xi\eta} = 0$ ... Canonical form so $u = f(\xi) + g(\eta)$

$\to$ ~~$u_\xi = f(\eta) \to u_\xi = f(\eta)$~~

$u = f(y - 3x) + g(3y - x)$ ... general solution.