

Unit #3: Recursion, Induction, and Loop Invariants

CPSC 221: Basic Algorithms and Data Structures

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Annotated slides
from Ed's class

Unit Outline

- ▶ Thinking Recursively
- ▶ Recursion Examples
- ▶ Analyzing Recursion: Induction and Recurrences
- ▶ Analyzing Iteration: Loop Invariants
- ▶ How Computers Handle Recursion
 - ▶ Recursion and the Call Stack
 - ▶ Iteration and Explicit Stacks
 - ▶ Tail Recursion

Learning Goals

- ▶ Describe the relationship between recursion and induction.
- ▶ Prove that a program is correct using loop invariants and induction.
- ▶ Become more comfortable writing recursive algorithms.
- ▶ Convert between iterative and recursive algorithms.
- ▶ Describe how a computer implements recursion.
- ▶ Draw a recursion tree for a recursive algorithm.

Random Permutations (randomPermutation)

Problem: Permute a string so that every reordering of the string is equally likely.

cat \leftarrow 3 letters $\Rightarrow 3! = 6$ permutations

cta

act

atc

tca

tac

orange $\Rightarrow 6! = 720$ permutations

Thinking Recursively

1. DO NOT START WITH CODE. Instead, write the story of the problem, in natural language.
2. Define the problem: What should be done given a particular input?
3. Identify and solve the (usually simple) base case(s).
4. Determine how to break the problem down into smaller problems of the same kind.
5. Call the function recursively to solve the smaller problems. Assume it works. Do not think about how!
6. Use the solutions to the smaller problems to solve the original problem.

Once you have all that, write the steps of your solution as comments, and then fill in the code for each comment.

Random Permutations (random)

Problem: Permute a string so that every reordering of the string is equally likely.

orange

Idea:

1. Pick a letter to be the first letter of the output. (Every letter should be equally likely.)
2. Pick the rest of the output to be a random permutation of the remaining string (without that letter).

(It's slightly simpler if we pick a letter to be the **last** letter of the output.)

Random Permutations (rPnastma detinoRmuo)

Problem: Permute a string so that every reordering of the string is equally likely.

```
// randomly permute the first n characters of S
void permute(string &S, int n) {
    if( n > 1 ) {
        int i = rand() % n; // swap a random character of S
        char tmp = S[i];    // with the last character
        S[i] = S[n-1];
        S[n-1] = tmp;
        permute(S, n-1);    // randomly permute S[0..n-2]
    }
}
```

operate on live data in the caller (pass by reference)

orange \Rightarrow or aegn

- could be the last character, by chance

\uparrow = smaller problem

Recall that `rand() % n` returns an integer from $\{0, 1, \dots, n-1\}$ uniformly at random.

Base case? if (n == 1) or n == 0

Induction and Recursion: Twins Separated at Birth?

same principles



Induction:

Base Case

Prove for some small value(s).

Inductive Step: Break a larger case down into smaller ones that we assume work (the Induction Hypothesis).

uses



Recursion:

Base Case

Calculate for some small value(s).

Otherwise, break the problem down in terms of itself (smaller versions) and then call this function to solve the smaller versions, assuming it will work.

Proving that a Recursive Algorithm is Correct

Just follow your code's lead and use induction.

Your base case(s)? **Your code's base case(s).**

How do you break down the inductive step? **However your code breaks the problem down into smaller cases.**

Inductive hypothesis? **The recursive calls work for smaller-sized inputs.**

Proving that a Recursive Algorithm is Correct

definition (recursive):

```
// Pre: n >= 0.
```

```
// Post: returns n!
```

```
int fact(int n) {  
    if (n == 0) return 1;
```

```
    return n * fact(n-1);
```

Correct?

Does it line up with the definition?

by inductive hypothesis

non-recursive def'n

$$n! = n(n-1)(n-2) \dots (2)(1)$$

$$n! = \begin{cases} 1 & \text{if } n=0 \\ n(n-1)! & \text{if } n \geq 1 \end{cases}$$

Prove: $\text{fact}(n) = n!$

Base case: $n = 0$:

$\text{fact}(0)$ returns 1; and
 $0! = 1$, by definition

verify
Base
Case

Inductive Hypothesis:

$\text{fact}(n)$ returns $n!$ for all
 $n \leq k$

state it
assume
it works

Inductive Step: For

$n = k + 1$, the code returns

$n * \text{fact}(n-1)$. By the IH,

$\text{fact}(n-1)$ is $(n-1)!$ and

$n! = n * (n-1)!$, by

definition.

above ✓

Proving that a Recursive Algorithm is Correct

Problem: Prove that our algorithm for randomly permuting a string gives an equal chance of returning every permutation (assuming rand() works as advertised).

n! permutations

Base Case: Strings of length 1 have only one permutation.

n = 1

Induction Hypothesis: Assume that our call to permute(S, n-1) works (i.e., it randomly permutes the first $n-1$ characters of S).

assume

if true for $n=k$ (true) (works)

We choose the last letter uniformly at random from the string. To get a random permutation, we need only randomly permute the remaining letters. permute(S, n-1) does exactly that.

show, true

original to start for $n=k+1$

we need to show:

** If we assume that $P(k)$ is true, then $P(k+1)$ is also true.*

assure that all letters are accounted for

Recurrence Relations ... Already Covered

See Runtime Examples #4-6 in the Complexity lecture slides in Unit #1.

Additional Problem: Prove that binary search takes $O(\lg n)$ time.

// Search $A[i..j]$ for key.
// Return index of key or -1 if key not found.

```
int bSearch(int A[], int key, int i, int j) {
```

```
    if (j < i) return -1;
```

```
    int mid = (i + j) / 2;
```

```
    if (key < A[mid])
```

```
        return bSearch(A, key, i, mid-1);
```

```
    else if (key > A[mid])
```

```
        return bSearch(A, key, mid+1, j);
```

```
    else return mid;
```

```
}
```

start with array of size n
i.e., elements $[0] \dots [n-1]$
search for lower bound upper bound indexes

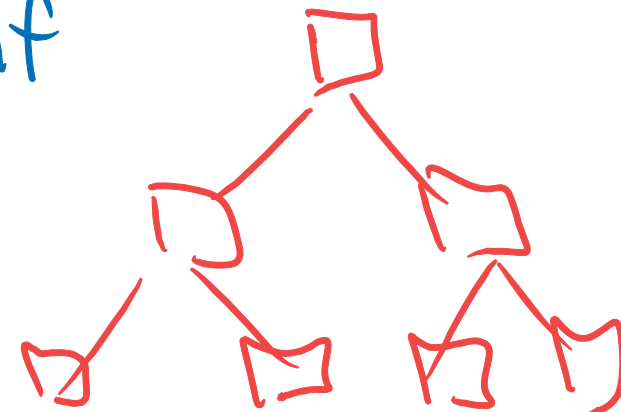
B.C.

// calc. midpoint

// key can only be in first half of A

// upper half

// if found



Binary Search Problem (Worked)

Note: Let n be the # of elements considered in the array. Thus,
 $n = j - i + 1$.

\Rightarrow $\left. \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \right\} 4 - 1 + 1 = 4$ $\left. \begin{matrix} 7 \\ 8 \\ 9 \\ 10 \end{matrix} \right\} 10 - 7 + 1 = 4$

```
int bSearch(int A[], int key, int i, int j) {  
    if (j < i) return -1;  $O(1)$  base case  
    int mid = (i + j) / 2;  $O(1)$   
    if (key < A[mid])  $O(1)$   
        return bSearch(A, key, i, mid-1);  $T(\lfloor \frac{n-1}{2} \rfloor) \leq T(\lfloor n/2 \rfloor)$   
    else if (key > A[mid])  $O(1)$   
        return bSearch(A, key, mid+1, j);  $T(\lceil \frac{n-1}{2} \rceil) = T(\lfloor n/2 \rfloor)$   
    else return mid;  $O(1)$   
}
```

Binary Search Problem (Worked, cont.)

The worst-case running time:

a constant step

$$T(n) = \begin{cases} 1 & \text{if } n = 0 \\ T(\lfloor n/2 \rfloor) + 1 & \text{if } n > 0 \end{cases}$$

Guess: *through experience*

$$T(n) = \lceil \lg(n+1) \rceil + 1$$

$\lg 7 \approx 2.65$
 $\lceil \lg 7 \rceil = 3$
 $\lceil \lg (7+1) \rceil + 1 = 3 + 1 = 4$


$\lceil \lg 4 \rceil = 2$

B.C.

$$\begin{array}{lcl} T(0) = 1 & 1 & \\ \hline T(1) = T(0) + 1 = 2 & 2 & \\ \hline T(2) = T(1) + 1 = 3 & \} & 3 \\ T(3) = T(1) + 1 = 3 & \} & \\ \hline T(4) = T(2) + 1 = 4 & \} & 4 \\ T(5) = T(2) + 1 = 4 & \} & \\ T(6) = T(3) + 1 = 4 & \} & \\ T(7) = T(3) + 1 = 4 & \} & \end{array}$$

pattern? ...
How does $T(7)$ relate to 4
 $\lceil \lg (4+1) \rceil + 1 = 3 + 1 = 4$

Binary Search Problem (Worked, cont.)

Claim $T(n) = \lceil \lg(n+1) \rceil + 1$ 

Proof (by induction on n)

Base Case: $T(0) = 1 = \lceil \lg(0+1) \rceil + 1 = 0 + 1 = 1$

Ind. Hyp.: $T(n) = \lceil \lg(n+1) \rceil + 1$ for all $n \leq k$ (for some $k \geq 0$)

Suppose true for $n=k$. Then, show true for $n=k+1$.

Ind. Step: For $n = k+1$, $T(k+1) = T(\lfloor \frac{k+1}{2} \rfloor) + 1 =$ *show*
 $T(k+1) =$

If k is even:

$$= T(\frac{k}{2}) + 1 \quad (k \text{ is even})$$

$$\stackrel{(IH)}{=} (\lceil \lg(\frac{k}{2} + 1) \rceil + 1) + 1$$

$$= \lceil \lg(2(\frac{k}{2} + 1)) \rceil + 1$$

$$= \lceil \lg(k+2) \rceil + 1$$

$$\lg(2(\frac{k}{2} + 1)) = \lg 2 + \lg(\frac{k}{2} + 1)$$

If k is odd:

$$= T(\frac{k+1}{2}) + 1 \quad (k \text{ is odd})$$

$$\stackrel{(IH)}{=} (\lceil \lg(\frac{k+1}{2} + 1) \rceil + 1) + 1$$

$$= \lceil \lg(2(\frac{k+1}{2} + 1)) \rceil + 1$$

$$= \lceil \lg(k+3) \rceil + 1$$

$$= \lceil \lg(k+2) \rceil + 1 \quad \text{because } \lceil \lg(k+2) \rceil + 1$$

because ceiling

Proving that an Algorithm with Loops is Correct

Maybe we can use the same techniques we use for proving correctness of recursion to prove correctness of loops.

We do this by stating and proving “invariants”, that is, properties that are always true (don’t vary) at particular points in the program.

watch boundary points

One way of thinking of a loop is that it starts with a true invariant and does work to keep the invariant true for the next iteration of the loop.

Insertion Sort

guarantee about a property

```
void insertionSort(int A[], int length) {  
    for (int i = 1; i < length; i++) {  
        // Invariant: the elements in A[0..i-1] are in sorted order  
        int val = A[i];  
        int j;  
        for (j = i; j > 0 && A[j-1] > val; j--)  
            A[j] = A[j-1];  
        A[j] = val;  
    }  
}
```

go backwards and swap as far as nec.

$$\textcircled{2} \text{ Big-O } T(N) = \frac{1}{2}N^2 - \frac{1}{2}N \leq \frac{1}{2}N^2$$

$$\therefore T(N) = O(N^2)$$

① # of comparisons/swaps in worst case:

$$\left. \begin{matrix} 1 \\ 2 \\ 3 \\ \vdots \\ N-2 \\ N-1 \end{matrix} \right\} \Sigma = \frac{(N-1)N}{2}$$
$$T(N) = \frac{1}{2}N^2 - \frac{1}{2}N$$

$$\textcircled{3} \text{ Big-}\Omega \quad T(N) = \frac{1}{2}N^2 - \frac{1}{2}N \geq \frac{1}{4}N^2 \quad \forall n \geq 2$$

$$\therefore T(N) \in \Omega(N^2)$$

$$\therefore \textcircled{2} \text{ \& } \textcircled{3} \Rightarrow T(N) \in \Theta(N^2)$$

Proving a Loop Invariant

Induction Variable: Number of times through the loop i

Base Case: Prove that the invariant is true before the ^{body of the} loop starts. $i=1$

Induction Hypothesis: Assume that the invariant holds just before beginning some (unspecified) iteration.

Inductive Step: Prove that the invariant holds at the end of that iteration, for the next iteration. *similar to our "assume $n=k$ is true"*

Extra Bit: Make sure that the loop will eventually end! *show that it's true for $i+1$*

We'll prove that Insertion Sort works, but the cool part is not proving that it actually works, but rather showing that the proof is a natural way to think about it working!

Insertion Sort

```
for (int i = 1; i < length; i++) {  
    // Invariant: the elements in A[0..i-1] are in sorted order  
    int val = A[i];  
    int j;  
    for (j = i; j > 0 && A[j-1] > val; j--)  
        A[j] = A[j-1];  
    A[j] = val;  
}
```

Base Case (at the start of the ($i = 1$) iteration): $A[0..0]$ only has one element; so, it's always in sorted order.

✓
trivial

Insertion Sort

```
for (int i = 1; i < length; i++) {  
    // Invariant: the elements in A[0..i-1] are in sorted order  
    int val = A[i];  
    int j;  
    for (j = i; j > 0 && A[j-1] > val; j--)  
        A[j] = A[j-1];  
    A[j] = val;  
}
```

Induction Hypothesis: At the start of iteration i of the loop,
 $A[0..i-1]$ are in sorted order.

assume

Insertion Sort

```
for (int i = 1; i < length; i++) {  
    // Invariant: the elements in A[0..i-1] are in sorted order  
    int val = A[i];  
    int j;  
    for (j = i; j > 0 && A[j-1] > val; j--)  
        A[j] = A[j-1];  
    A[j] = val;  
}
```

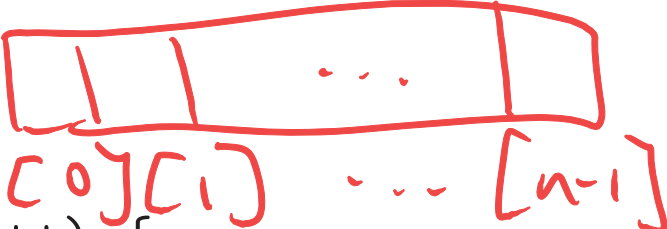
→ starts with A so far
in sorted
order
↓ argument
I.S.

Inductive Step:

new item → end the
iteration in
sorted order

- ▶ The inner loop places val = A[i] at the appropriate index $j < i$ by shifting elements of A[0..i-1] that are larger than val one position to the right.
- ▶ Since A[0..i-1] is sorted (by IH), A[0..i] ends up in sorted order and the invariant holds at the start of the next iteration ($i = i + 1$).

Insertion Sort



```
for (int i = 1; i < length; i++) {  
    // Invariant: the elements in A[0..i-1] are in sorted order  
    int val = A[i];  
    int j;  
    for (j = i; j > 0 && A[j-1] > val; j--)  
        A[j] = A[j-1];  
    A[j] = val;  
}
```

Loop Termination:

- ▶ The loop ends after $\text{length} - 1$ iterations.
- ▶ When it ends, we were about to enter the $(i = \text{length})$ iteration.
- ▶ Therefore, by the newly proven invariant, when the loop ends, $A[0..\text{length}-1]$ is in sorted order... which means A is sorted!

Practice: Proving that the Inner Loop is Correct

```
for (int i = 1; i < length; i++) {  
    // Invariant: the elements in A[0..i-1] are in sorted order  
    int val = A[i];  
    int j;  
    for (j = i; j > 0 && A[j-1] > val; j--)  
        // What's the invariant? Something like  
        // "A[0..j-1] + A[j+1..i] = the old A[0..i-1]  
        // and val is A[j+1..i]"  
        A[j] = A[j-1];  
    A[j] = val;  
}
```

Prove by induction that the inner loop operates correctly. (This may feel unrealistically easy!)

Finish the proof! (As we did for the outer loop, talk about what the invariant means when the loop ends.)

Recursion vs. Iteration

Which one is better: recursion or iteration?

Both can accomplish the same thing.

- iteration is often more efficient
- easier? often iteration
- buggier? recursion?
- resources? recursion worse because of the call stack
- intuitive/elegant?
 - often recursion (e.g., trees, Towers of Hanoi)
- faster
 - often iter.

"it depends"

Simulating a Loop with Recursion

iter

```
int i = 0;
while (i < n) {
    foo(i);
    i++;
}
```

recur

```
recFoo(0, n);
```

where recFoo is:

```
void recFoo(int i, int n) {
    if (i < n) {
        foo(i);
        recFoo(i + 1, n);
    }
}
```

Anything we can do with iteration, we can do with recursion.

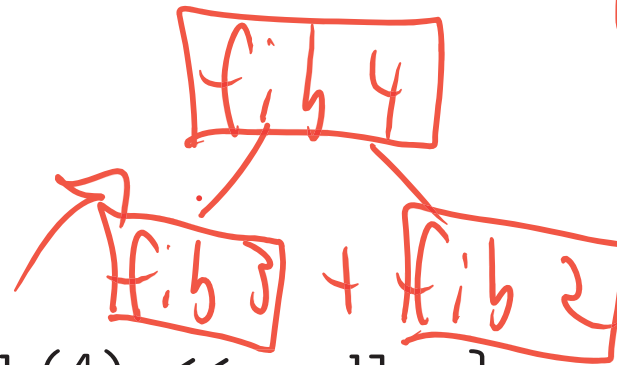
Simulating Recursion with a Stack

How does recursion work in a computer?

Each function call generates an activation record—holding local variables and the program point to return to—which is pushed on a stack (the *call stack*) that tracks the current chain of function calls.

```
int fib(int n) {  
1.   if (n <= 2) return 1;  
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
4.   return a+b;  
}
```

```
int main() { cout << fib(4) << endl; }
```

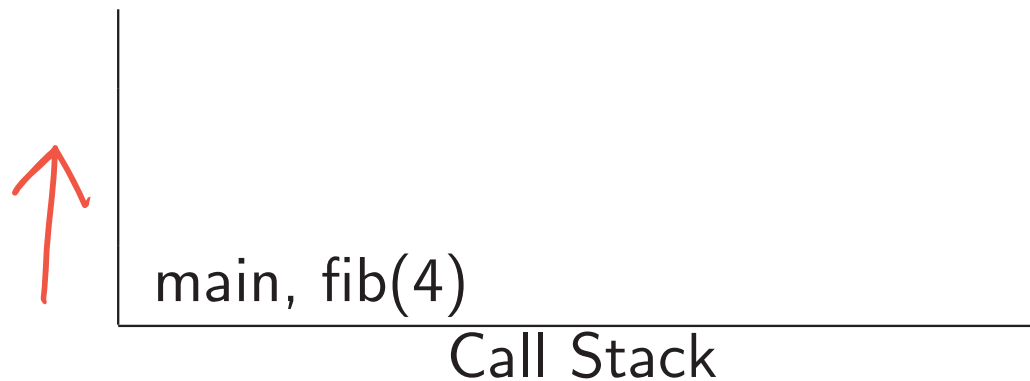


address of caller
(exact place)
program counter

Simulating Recursion with a Stack

```
int fib(int n) {  
1.   if (n <= 2) return 1;  
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
4.   return a+b;  
}
```

```
int main() { cout << fib(4) << endl; }
```

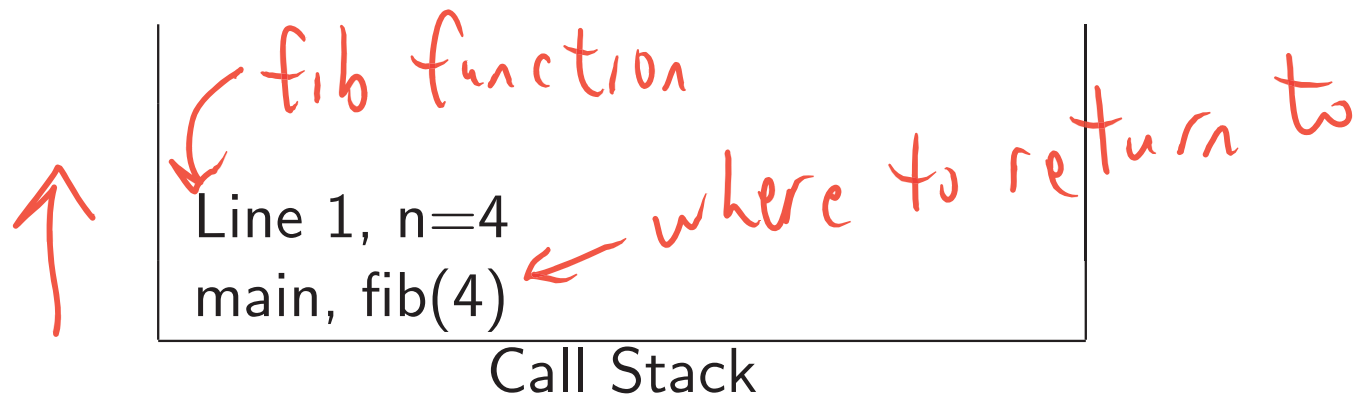


Simulating Recursion with a Stack

```
int fib(int n) {  
1.   if (n <= 2) return 1;  
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
4.   return a+b;  
}
```

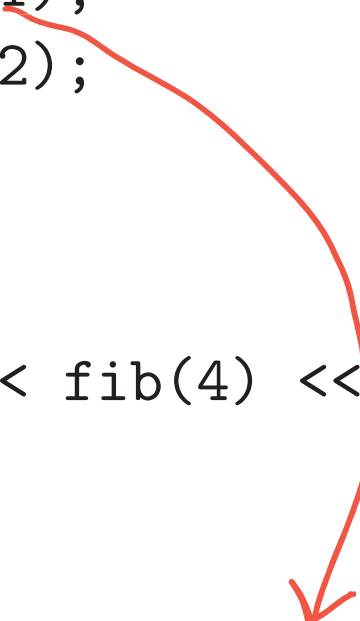


```
int main() { cout << fib(4) << endl; }
```



Simulating Recursion with a Stack

```
int fib(int n) {  
1.   if (n <= 2) return 1;  
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
4.   return a+b;  
}  
  
int main() { cout << fib(4) << endl; }
```



Line 2, n=4, a=fib(3)
main, fib(4)

Call Stack

Simulating Recursion with a Stack

```
int fib(int n) {  
1.   if (n <= 2) return 1;  
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
4.   return a+b;  
}
```



```
int main() { cout << fib(4) << endl; }
```

Line 1, n=3

Line 2, n=4, a=fib(3)

main, fib(4)

Call Stack

Simulating Recursion with a Stack

```
int fib(int n) {  
1.   if (n <= 2) return 1;  
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
4.   return a+b;  
}
```

```
int main() { cout << fib(4) << endl; }
```

Line 2, n=3, a=fib(2)
Line 2, n=4, a=fib(3)
main, fib(4)

Call Stack

Simulating Recursion with a Stack

```
int fib(int n) {  
1.   if (n <= 2) return 1;  
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
4.   return a+b;  
}
```

```
int main() { cout << fib(4) << endl; }
```

Line 1, n=2


Line 2, n=3, a=fib(2)

Line 2, n=4, a=fib(3)

main, fib(4)

Call Stack

Simulating Recursion with a Stack

```
int fib(int n) {  
1.   if (n <= 2) return 1;   
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
4.   return a+b;  
}
```

```
int main() { cout << fib(4) << endl; }
```

Line 1, n=2, <u>return 1</u>
Line 2, n=3, a=fib(2)
Line 2, n=4, a=fib(3)
main, fib(4)

Call Stack

Simulating Recursion with a Stack

```
int fib(int n) {  
1.   if (n <= 2) return 1;  
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
4.   return a+b;  
}
```

```
int main() { cout << fib(4) << endl; }
```

Line 2, n=3, a=1
Line 2, n=4, a=fib(3)
main, fib(4)

Call Stack

Simulating Recursion with a Stack

```
int fib(int n) {  
1.   if (n <= 2) return 1;  
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
4.   return a+b;  
}
```

```
int main() { cout << fib(4) << endl; }
```

Line 3, n=3, a=1, b=fib(1)
Line 2, n=4, a=fib(3)
main, fib(4)

Call Stack

Simulating Recursion with a Stack

```
int fib(int n) {  
1.   if (n <= 2) return 1;  
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
4.   return a+b;  
}  
  
int main() { cout << fib(4) << endl; }
```

Line 1, n=1
Line 3, n=3, a=1, b=fib(1)
Line 2, n=4, a=fib(3)
main, fib(4)

Call Stack

Simulating Recursion with a Stack

```
int fib(int n) {  
1.   if (n <= 2) return 1;  
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
4.   return a+b;  
}
```

```
int main() { cout << fib(4) << endl; }
```

Line 1, n=1, return 1
Line 3, n=3, a=1, b=fib(1)
Line 2, n=4, a=fib(3)
main, fib(4)

Call Stack

Simulating Recursion with a Stack

```
int fib(int n) {  
1.   if (n <= 2) return 1;  
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
4.   return a+b;  
}  
  
int main() { cout << fib(4) << endl; }
```

```
Line 3, n=3, a=1, b=1  
Line 2, n=4, a=fib(3)  
main, fib(4)
```

Call Stack

Simulating Recursion with a Stack

```
int fib(int n) {  
1.   if (n <= 2) return 1;  
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
4.   return a+b;  
}
```

```
int main() { cout << fib(4) << endl; }
```

Line 4, n=3, a=1, b=1, return 2
Line 2, n=4, a=fib(3)
main, fib(4)

Call Stack

Simulating Recursion with a Stack

```
int fib(int n) {  
1.   if (n <= 2) return 1;  
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
4.   return a+b;  
}  
  
int main() { cout << fib(4) << endl; }
```

Line 3, n=4, a=2, b=fib(2)
main, fib(4)

Call Stack

Simulating Recursion with a Stack

```
int fib(int n) {  
1.   if (n <= 2) return 1;  
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
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}  
  
int main() { cout << fib(4) << endl; }
```

Line 1, n=2 Line 3, n=4, a=2, b=fib(2) main, fib(4)

Call Stack

Simulating Recursion with a Stack

```
int fib(int n) {  
1.   if (n <= 2) return 1;  
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
4.   return a+b;  
}  
  
int main() { cout << fib(4) << endl; }
```

Line 1, n=2, return 1
Line 3, n=4, a=2, b=fib(2)
main, fib(4)

Call Stack

Simulating Recursion with a Stack

```
int fib(int n) {  
1.   if (n <= 2) return 1;  
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
4.   return a+b;  
}  
  
int main() { cout << fib(4) << endl; }
```

Line 3, n=4, a=2, b=1
main, fib(4)

Call Stack

Simulating Recursion with a Stack

```
int fib(int n) {  
1.   if (n <= 2) return 1;  
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
4.   return a+b;  
}
```

```
int main() { cout << fib(4) << endl; }
```

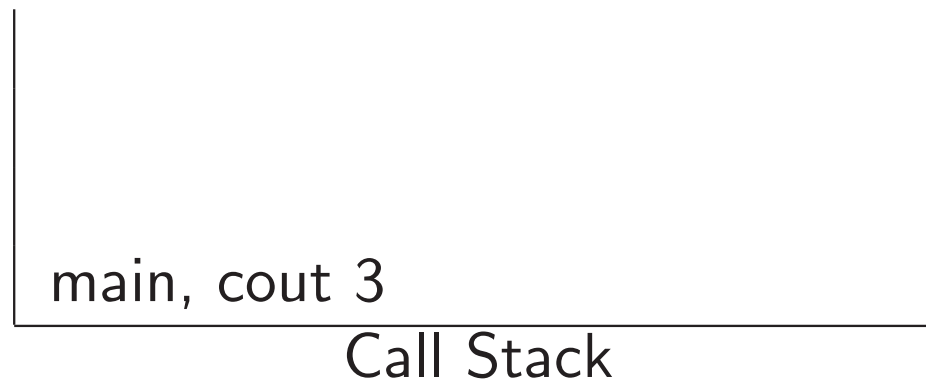
Line 4, n=4, a=2, b=1, return 3
main, fib(4)

Call Stack

Simulating Recursion with a Stack

```
int fib(int n) {  
1.   if (n <= 2) return 1;  
2.   int a = fib(n-1);  
3.   int b = fib(n-2);  
4.   return a+b;  
}
```

```
int main() { cout << fib(4) << endl; }
```



Simulating Recursion with a Stack

How do we simulate fib with a stack?

That's what our computer already does. We can sometimes do it a bit more efficiently by only storing what's really needed on the stack:

```
int fib(int n) {  
    int result = 0;  
    Stack S;  
    S.push(n);  
}
```

```
→ while( !S.is_empty() ) {
    int k = S.pop();
```

```
→ if( k <= 2 ) result++;  
   else { S.push(k - 1), S.push(k - 2); }  
}
```

```
return result;
```

}

- C++ does for a recursive program

```
main : fib(4)
```

$$n = 4$$

pop \downarrow
 $r=3$
 \uparrow


result;

Stack operations and state:

- Initial state: Stack is empty, $r = 0$.
- Push 4: Stack contains [4], $r = 0$.
- Pop: Stack is empty, $r = 0$.
- Push 3: Stack contains [3], $r = 0$.
- Push 2: Stack contains [2, 3], $r = 0$.
- Pop: Stack contains [3], $r = 1$.
- Push 1: Stack contains [1, 3], $r = 1$.
- Pop: Stack contains [3], $r = 1$.
- Push 2: Stack contains [2, 3], $r = 1$.
- Pop: Stack contains [3], $r = 2$.

Final state: Stack contains [3], $r = 2$.

Loop Invariant for Correctness

```
int fib(int n) {  
    int result = 0;  
    Stack S;  
    S.push(n);  
    while( !S.is_empty() ) {  
        // Invariant: ??   
        int k = S.pop();  
        if( k <= 2 ) result++;  
        else { S.push(k - 1); S.push(k - 2); }  
    }  
    return result;  
}
```

What is the loop invariant?

Remember for invariants:
We're at a particular
state of the
function(loop)

What is true at
this point wrt
calculation or
loop?
e.g., fact (interim calc.)
array search (...not found)

Loop Invariant for Correctness

```
int fib(int n) {  
    int result = 0;  
    Stack S;  
    S.push(n);
```

```
    while( !S.is_empty() ) {
```

→ // Invariant: $\left(\text{result} + \sum_{k \text{ on Stack}} \text{fib}_k \right) = \text{fib}_n$

```
        int k = S.pop();
```

```
        if( k <= 2 ) result++;
```

```
        else { S.push(k - 1); S.push(k - 2); }
```

```
    }
```

```
    return result;
```

```
}
```

Prove the invariant using induction.

[Base Case (zero iterations): $\left(\overset{0}{\text{result}} + \sum_{k \text{ on Stack}} \overset{\text{fib}_n}{\text{fib}_k} \right) = 0 + \text{fib}_n = \text{fib}_n$]

Loop Invariant for Correctness

```

int fib(int n) {
    int result = 0;
    Stack S;
    S.push(n);
    while( !S.is_empty() ) {
        // Invariant:
        int k = S.pop();
        → if( k <= 2 ) result++;
        else { S.push(k - 1); S.push(k - 2); }
    }
    return result;
}

```

$$\text{so far} \quad \text{to come} \quad \left(\text{result} + \sum_{k \text{ on Stack}} \text{fib}_k \right) = \boxed{\text{fib}_n}$$


$$\text{fib}_k = \underbrace{\text{fib}_{k-1}}_{\text{on stack}} + \underbrace{\text{fib}_{k-2}}_{\text{to come}}$$

Prove the invariant using induction.

Inductive Step: If $k \leq 2$, then result increases by $\text{fib}_k = 1$ and k is removed from the stack. The invariant is preserved. If $k > 2$, then k is replaced by $k - 1$ and $k - 2$ on the stack. Since $\text{fib}_k = \text{fib}_{k-1} + \text{fib}_{k-2}$, the invariant is preserved.

$$k \text{ pops off; } k-1, k-2 \text{ pushed} \Rightarrow -\text{fib}_k + \text{fib}_{k-1} + \text{fib}_{k-2} = 0$$

Loop Invariant for Correctness

```
int fib(int n) {  
    int result = 0;  
    Stack S;  
    S.push(n);  
    while( !S.is_empty() ) {  
        // Invariant:  
$$\left( \overset{\text{fib}_n}{\text{result}} + \sum_{k \text{ on Stack}}^{\overset{0}{\text{fib}_k}} \text{fib}_k \right) = \text{fib}_n$$
  
        int k = S.pop();  
        if( k <= 2 ) result++;  
        else { S.push(k - 1); S.push(k - 2); }  
    }  
    return result;  
}
```

Prove the invariant using induction.

End: When the loop terminates, the stack is empty, so $\text{result} = \text{fib}_n$.

Recursion vs. Iteration

Which one is more *elegant*: recursion or iteration?

nice
Concise
way

depends on
programmer
and type of
problem

recursion { e.g., maze,
Towers of Hanoi
trees

iteration { natural loop
searching an array

Recursion vs. Iteration

Which one is more *efficient*: recursion or iteration?

/
can be
fast, too
e.g., tail
recursive

└ often faster
└ avoids function
calls:
no additional
activation records
⇒ less space

Accidentally Making Lots of Recursive Calls; Recall ...

Recursive Fibonacci:

```
int fib(int n) {  
    if (n <= 2) return 1;  
    else      return fib(n-1) + fib(n-2);  
}
```

Lower bound analysis

$$T(n) \geq \begin{cases} b & \text{if } n = 0, 1 \\ T(n-1) + T(n-2) + c & \text{if } n > 1 \end{cases}$$

$$T(n) \geq b\varphi^{n-1}$$

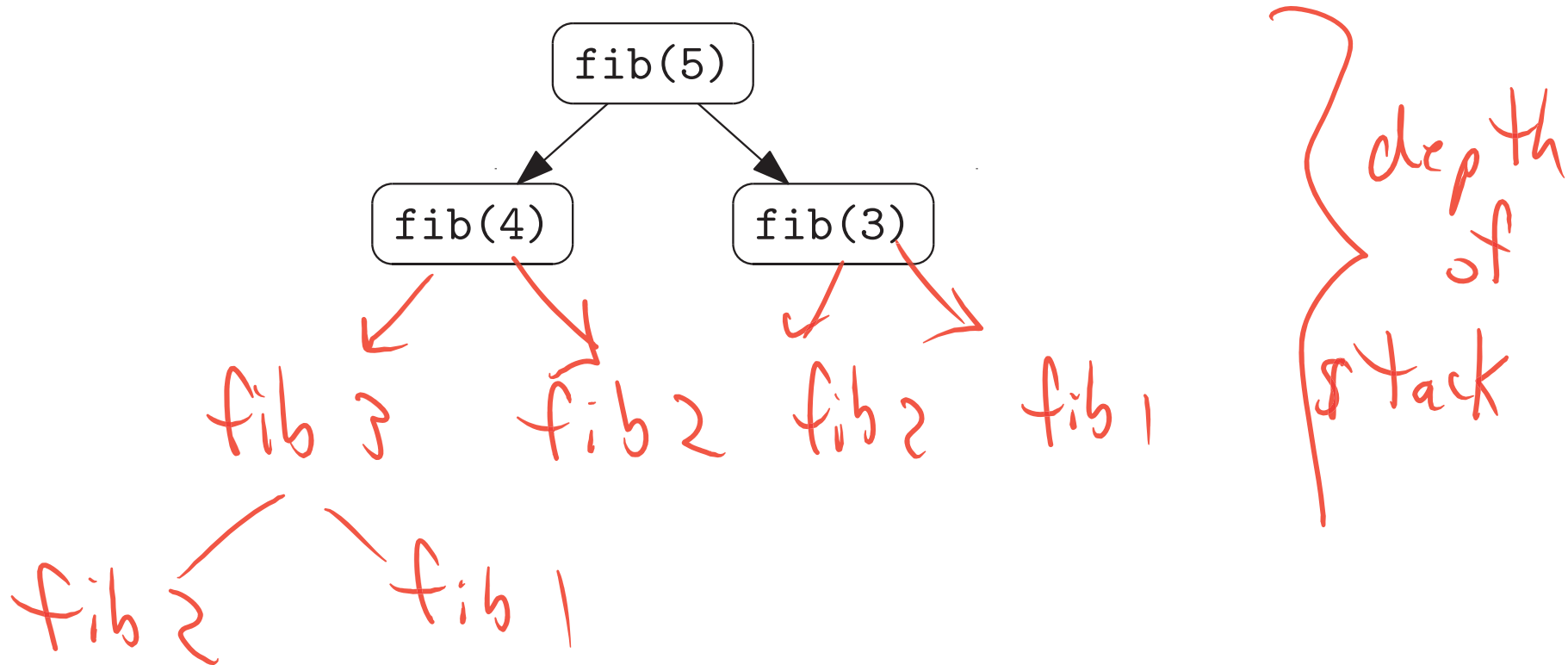
where $\varphi = (1 + \sqrt{5})/2$.

exponential
 $T(n) \in \Theta(\varphi^n)$
[done earlier]

Accidentally Making Lots of Recursive Calls; Recall ...

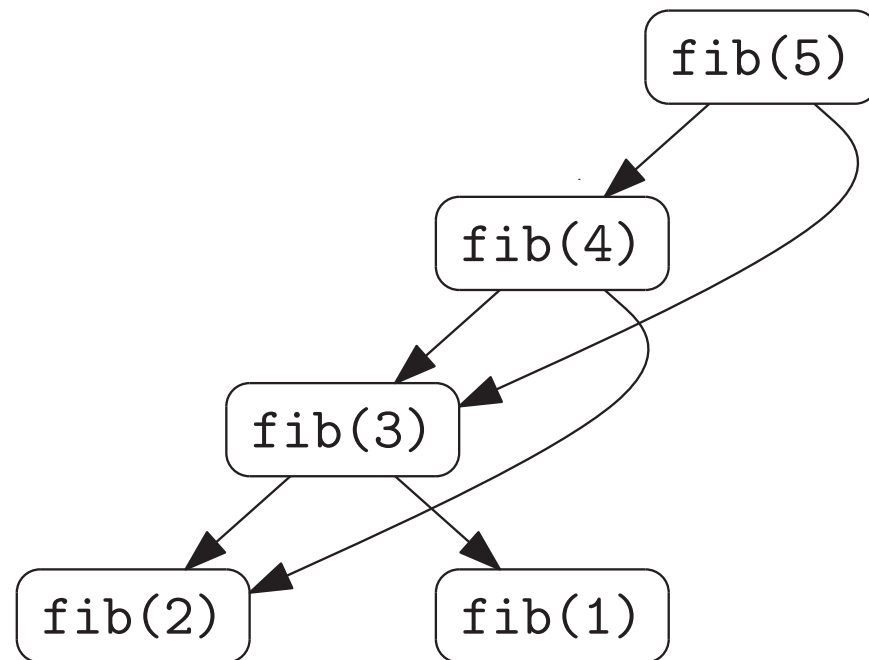
```
int fib(int n) {  
    if (n <= 2) return 1;  
    else      return fib(n-1) + fib(n-2);  
}
```

Finish the recursion tree for fib(5)...



Fixing fib with Iteration

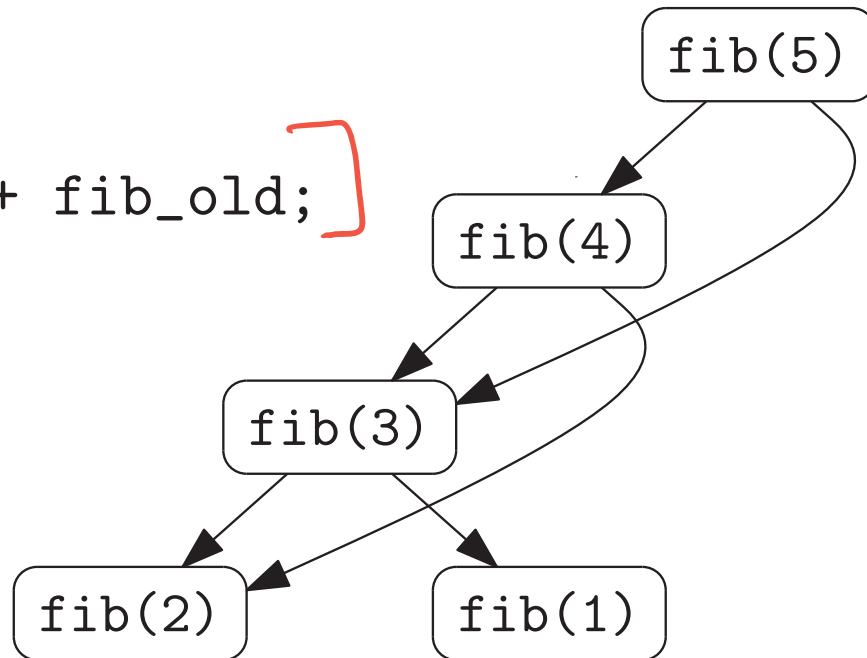
What we really want is to “share” nodes in the recursion tree:



Fixing fib with Iteration and “Dynamic Programming”

Here’s one fix that “walks up” the left side of the tree:

```
int fib_dp(int n) {  
    int fib_old = 1;  
    int fib = 1;  
    int fib_new;  
    while (n > 2) {  
        int fib_new = fib + fib_old;  
        fib_old = fib;  
        fib = fib_new;  
        --n;  
    }  
    return fib;  
}
```



Fixing Fib with Recursion and “Memoizing”

Here's another fix that stores solutions it has calculated before:

```
int* fib_solns = new int[big_enough](); // init to 0

fib_solns[1] = 1;

fib_solns[2] = 1;

int fib_memo(int n) {
    // If we don't know the answer, compute it.
    if (fib_solns[n] == 0)
        fib_solns[n] = fib_memo(n-1) + fib_memo(n-2);
    return fib_solns[n];
}
```

Recursion vs. Iteration

So, which one is more efficient: recursion or iteration?

It's probably easier to shoot yourself in the foot when you use recursion, and the call stack may carry around more memory than you actually need for storing things, but otherwise ...

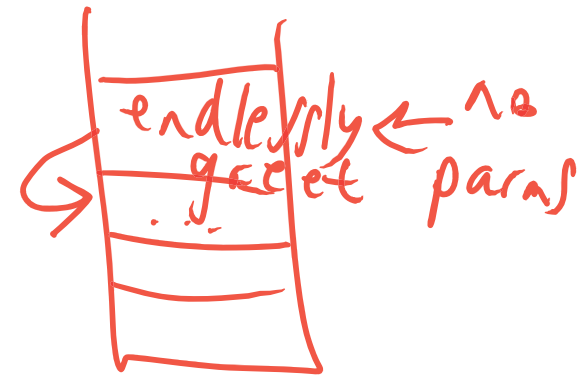
Neither is more efficient (asymptotically).



Managing the Call Stack: Tail Recursion

```
void endlesslyGreet() {  
    cout << "Hello, world!" << endl;  
    endlesslyGreet();  
}
```

nothing left
to do with
this function



← tail recursion
- no point in creating another activation

This is clearly infinite recursion. The call stack will get as deep as it can get and then bomb, right?

record that gets
added to the call
stack

But ... why have a call stack? There's no (need to) return to the caller.

Try compiling it with at least -O2 optimization and running it. It won't give a stack overflow!

Tail Recursion

A function is “tail recursive” if for any recursive call in the function, that call is the last thing the function needs to do before returning.

In that case, why bother pushing a new activation record? There's no reason to return to the caller. Just use the current record.

That's what most compilers will do.

Tail Recursive?

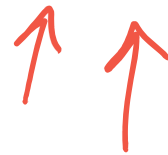
```
int fib(int n) {  
    if (n <= 2) return 1;  
    else      return fib(n-1) + fib(n-2);  
}
```

- not
tail
recursive

↑
must add results
after returning
from fib(n-2)

Tail Recursive?

```
int fact(int n) {  
    if (n == 0) return 1;  
    else      return n * fact(n - 1);  
}
```



not tail
recursive

Note: Koffman and Wolfgang (p. 416) call this tail recursion, but it really isn't.

Tail Recursive?

- another way of calculating factorial ($n!$)

```
int fact(int n) { return fact_acc(n, 1); }
```

```
int fact_acc(int b, int acc) {
```

```
    if (b == 0) return acc;
```

```
    else return fact_acc(b - 1, acc * b);
```

```
}
```

accumulate

↑ accum.value

iteration so far

to do

tail recursive

Side Note: Tail Calls

```
int fact(int n) { return fact_acc(n, 1); }
```

```
int fact_acc(int b, int acc) {  
    if (b == 0) return acc;  
    else      return fac_acc(b - 1, acc * b);  
}
```

tail recursive

Actually, we can talk about any function call being a “tail call”, even if it’s not recursive. For example, the call to `fact_acc` in `fact` is a tail call: there is no need to extend the stack.

Eliminating Tail Recursion

rewritten by compiler

```
// Search A[i..j] for key.  
// Return index of key or -1 if key not found.  
int bSearch(int A[], int key, int i, int j) {  
    while (j >= i) {  
        int mid = (i + j) / 2;  
        if (key < A[mid])  
            j = mid - 1;  
        else if (key > A[mid])  
            i = mid + 1;  
        else return mid;  
    }  
    return -1;  
}
```

original code

```
if( j < i ) return -1;  
int mid = (i + j) / 2;  
if (key < A[mid])  
    return bSearch(A, key, i, mid-1);  
else if (key > A[mid])  
    return bSearch(A, key, mid+1, j);  
else return mid;
```

start } of interval
end }