Unit #7: B$^+$-Trees

CPSC 221: Basic Algorithms and Data Structures

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2016W2

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Annotated Slides from Ed's Class

Unit Outline

- Minimizing disk I/Os
- B^+-Tree properties
- Implementing B^+-Tree insert and delete
- Some final thoughts on B^+-Trees

Learning Goals

- Describe the structure, navigation and time complexity of a B^+-Tree.
- Insert and delete keys from a B^+-Tree.
- Relate M, L, the number of nodes, and the height of a B^+-Tree.
- Compare and contrast B^+-Trees with other data structures.
- Justify why the number of I/Os becomes a more appropriate complexity measure (than the number of CPU operations) when dealing with large datasets and their indexing structures (e.g., B^+-Trees).
- Explain the difference between a B-Tree and a B^+-Tree

Memory Hierarchy

latency = wait time

Why worry about the number of disk I/Os?

Size		Access Time
hundreds of bytes	CPU registers	<1 cycle
tens of kilobytes	Cache memory L1	a few cycles
	L2	
megabytes	L3	tens of cycles
gigabytes	Main memory	hundreds of cycles
terabytes	Disk	millions of cycles

arm: head

RPM

$\frac{1}{2}$ spin on average

access time: seek time, rotational delay, transfer

Time Cost: Processor to Disk

Processor

- Operates at a few GHz (gigahertz = billion cycles per second)
- Several instructions per cycle
- Average time per instruction < 1 ns (1 nanosecond $= 10^{-9}$ seconds)

page is the smallest unit of transfer between disk and main memory

Disk (HDD = Hard (spinning) Disk Drive)

- Seek time $\approx$ 10 ms (1 millisecond $= 10^{-3}$ seconds)
- Note: Solid State Drives (SSDs) have "seek time" $\approx$ 0.03 ms

wait time

main memory = RAM = buffer pool

Result: $\approx$ 10 million instructions for each disk read!
Hold on... How long does it take to read a 1 TB (1 terabyte $= 10^{12}$ bytes) disk? 1 TB $\times$ 10 ms $=$ 10 billion seconds $>$ 300 years? !
What's wrong? Each disk read/write moves more than a byte (e.g., 4 KB, 8 KB, ... block sizes). Continuous HDD disk access is about the same speed as on an SSD.

Memory Blocks

Each memory access to a slower level of the hierarchy fetches a block of data.

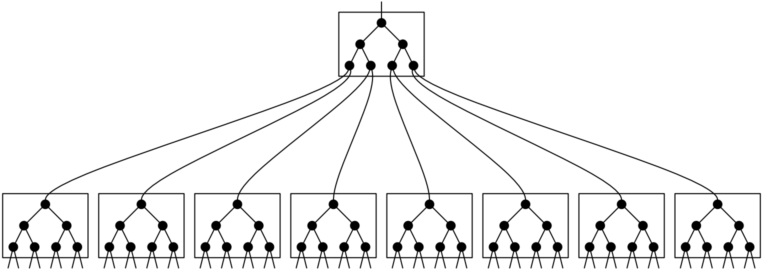
Block Size		Block Name
	CPU	
a few bytes	$\updownarrow$	word
	Cache	
10s bytes	$\updownarrow$	cache line
	Main memory	
a few kilobytes	$\updownarrow$	page
	Disk	

A block is the contents of consecutive memory locations.
So random access between levels of the hierarchy is very slow.

Chopping Trees into Blocks

Idea

Store data for many adjacent nodes in consecutive memory locations.



Result

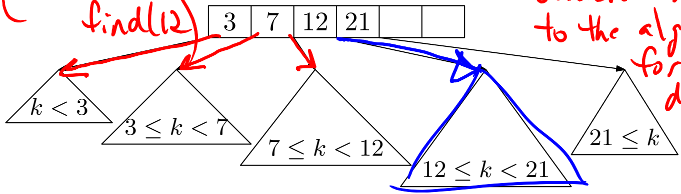
One memory block access provides keys to determine many (more than two) search directions.

M-ary Search Tree

root

Assume unique keys
- small changes to the algorithm for duplicate

find(12)



M-ary tree property

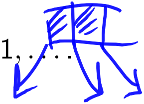
- Each node has $\leq M$ children.

max. # of children = max. fanout

Result: Complete M-ary tree with n nodes has height $\Theta(\log_M n)$

Search tree property

- Each node has $\leq M - 1$ search keys: $k_1 < k_2 < k_3 \ldots$
- All keys k in ith subtree obey $k_i \leq k < k_{i+1}$ for $i = 0, 1, \ldots$



Disk I/O's (runtime) for `find`:

B^+-Trees

B^+-Trees of order M are specialized M-ary search trees:

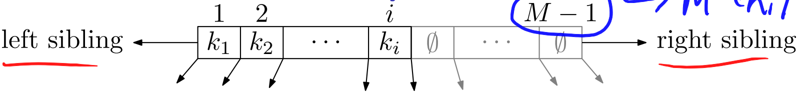
- ALL leaves are at the same depth!
- Internal nodes have between $\lceil M/2 \rceil$ and M children. half-full rule
- Keys and values are stored only in the leaves. Search keys in internal nodes only direct traffic. B-Trees store (key, value) pairs at internal nodes.
- Leaves hold between $\lceil L/2 \rceil$ and L (key, value) pairs. half-full rule
- The root is special. If it is an internal page, it has between 2 and M children. If it is a leaf page, it holds at most L (key, value) pairs.

Result

- Height is $\Theta(\log_M n)$
- `insert`, `delete`, and `find` operations visit $\Theta(\log_M n)$ nodes.
- M and L are chosen so that each (full) node fills one page of memory. Each node visit (e.g., disk I/O operation) retrieves about $M/2$ to M keys or $L/2$ to L (key, value) pairs at a time.

B^+-Tree Nodes

- Internal node with i search keys



⇒ M children

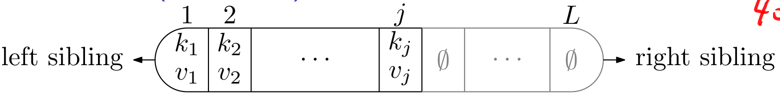
- ▸ $i+1$ subtree pointers
- ▸ parent and left & right sibling pointers

node = page

e.g., 4K

4096 bytes

- Leaf with j (key, value) pairs



- ▸ parent and left & right sibling pointers
- ▸ values may be pointers to disk records

Q9

Each node may hold a different number of items.

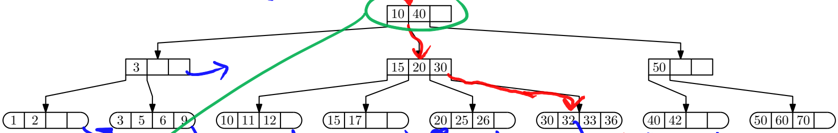
Example B$^+$-Tree with $M = 4$ and $L = 4$

aka fanout

max. # of children for an internal node

max # of (key, value) pairs

32?



4

great for range search

enlarged

L

"data values" are child pointers in internal nodes

Values in leaf nodes are not shown.

Parent

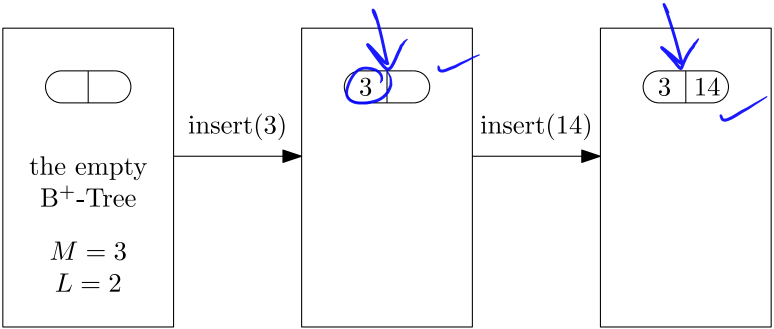
curr 32

if more than 3 keys in general

M = 4 here (red box + sibling & parent pointers)

∴ M − 1 = 3 keys, but 4 child pointers (for internal nodes).

Making a B^+-Tree



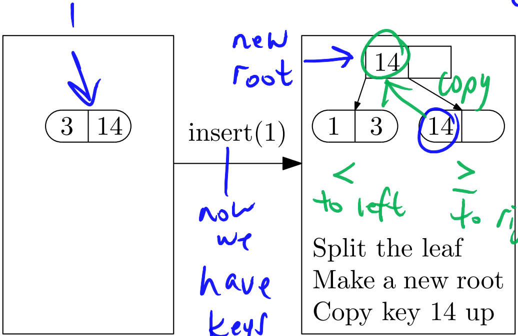
The root is a leaf.

What happens when we now insert(1)?

Splitting the Root

L = 2 (max. 2 data entries)

(max. 2 key-value pairs)



new root

copy

< to left

≥ to right

now we have keys 1, 3, 14 ⇒ 2 nodes (two go left, one goes right i.e., $\frac{1}{2}$ to each node)

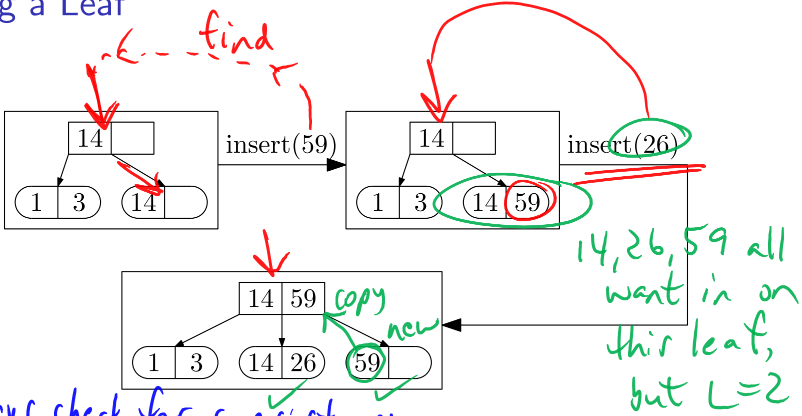
Too many keys for one leaf!

So, make a new leaf and create a parent (the new root) for both.

Why is key 14 duplicated?

- leaves contain all the (key, value) pairs
- internal nodes simply direct traffic

Splitting a Leaf



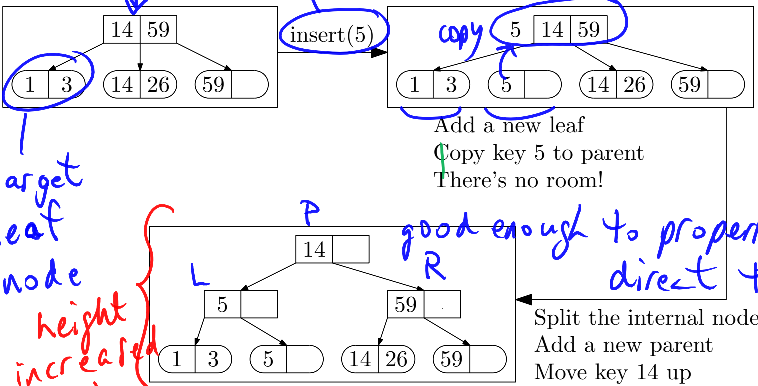
always check for consistency

insert(26) causes too many keys for the (14 | 59) leaf to store.

So, make a new leaf and **copy** ~~the "middle" key~~ (the smallest key in the new leaf (holding the larger keys) up to the common parent.

Propagating Splits

split



target leaf node

good enough to properly direct traffic

h=2 height increased by 1

#levels =3

insert(5) causes too many keys for $\boxed{1 \mid 3}$ leaf

Copy up key 5 causes too many keys for $\boxed{14 \mid 59}$ node

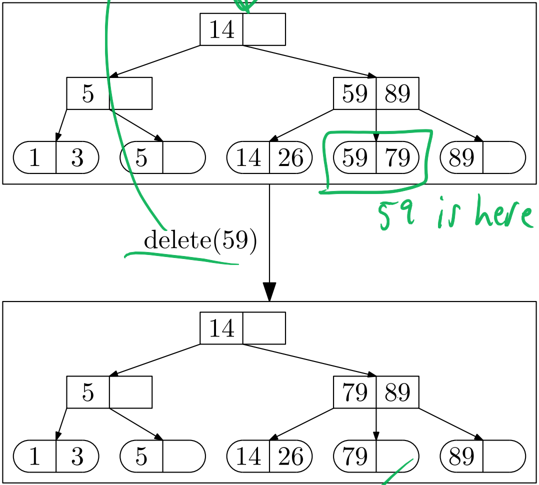
So, make a new internal node and **move up** the middle key.

Insertion Algorithm

1. Insert (key, value) pair in the target leaf page. If there's room, you're done.
2. If the leaf now has $L+1$ pairs: // overflow
   - Split the leaf into two leaves:
     - Original holds the $\lceil (L+1)/2 \rceil$ small key pairs
     - New one holds the $\lfloor (L+1)/2 \rfloor$ large key pairs
   - **Copy** smallest key in new leaf ~~(the middle key)~~ up to parent
3. If an internal node now has M keys: // overflow
   - Split the node into two nodes:
     - Original holds the $\lceil (M-1)/2 \rceil$ small keys
     - New one holds the $\lfloor (M-1)/2 \rfloor$ large keys
   - If root, hang the new nodes under a new root. Done.
   - **Move** the remaining middle key up to parent & go to 3

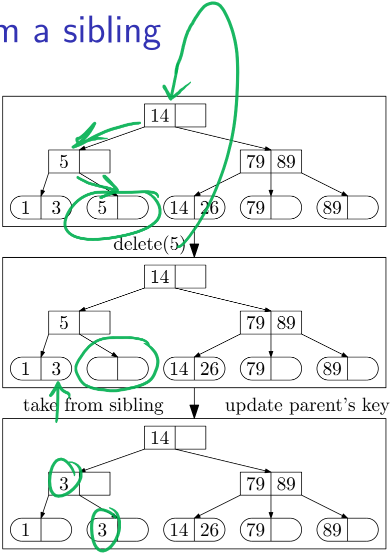
Delete

deletions not on exam



done

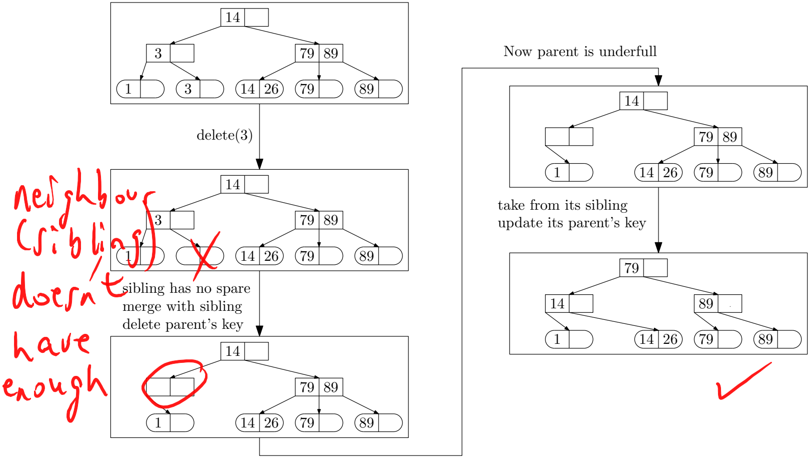
Delete: Take from a sibling



less than ½ full

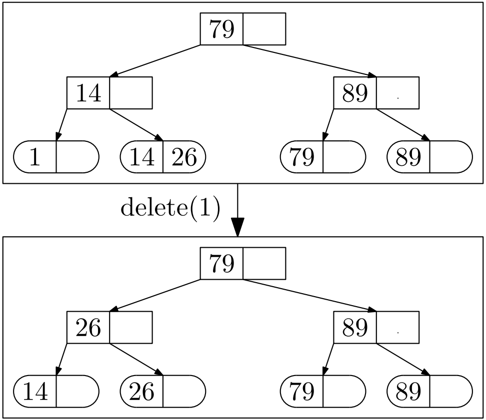
Take 3 from (1 | 3). It has enough items that it can spare one.
Update the parent's search key. This is not optional.

Delete: Merge

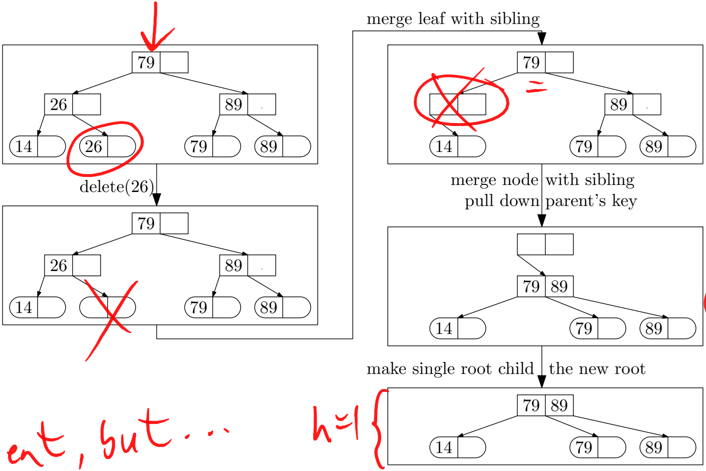


WARNING: A leaf is underfull if it holds fewer than $\lceil L/2 \rceil$ items.
For $L > 2$, an underfull leaf is not empty!

Delete: Take from a sibling



Deleting the Root



one less key in parent, but...

h=1

height is one less

The root only gets deleted when it has just one subtree (no matter how big M is).

Deletion Algorithm

1. Remove the (key, value) pair from the correct leaf.
2. If the leaf now has $\lceil L/2 \rceil - 1$ items: // underflow
   - If a sibling has a spare item then take it (smallest from right sibling or largest from left sibling) & update parent's key
   - Else merge with a sibling & **delete** parent's key
3. If internal non-root node now has $\lceil M/2 \rceil - 2$ keys: // underflow
   - If a sibling has a spare child then take it (leftmost from right sibling or rightmost from left sibling) & update parent's key
   - Else merge with a sibling & **pull down** parent's key & go to 3
4. If the root now has only one child, make that child the new root.

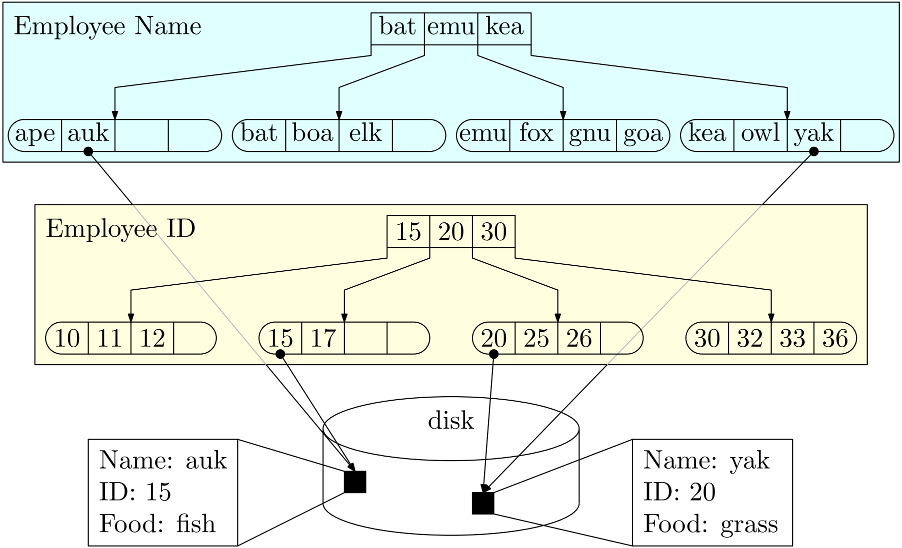
Note: Merge never creates a node with too many items. Why?

Thinking about B^+-Trees

- Deletion is fast if the leaf doesn't underflow or if we can take a (key, value) pair from a sibling. Merging and propagation take more time.
- Insertion is fast if the leaf doesn't overflow (could we give to a sibling?) Splitting and propagation take more time.
- Propagation is rare if M and L are large. (Why?)
- Repeated insertions and deletions can cause thrashing.
- If $M = L = 128$, then a B^+-Tree of height 4 will store at least 30,000,000 items.
- Range queries (i.e., `findBetween(key1, key2)`) are fast thanks to the sibling pointers in the leaves.

B$^+$-Trees in Practice

Multiple B$^+$-Trees can **index** the same data records. This is good.



A Tree by Any Other Name...

- B-Trees with $M = 3$ are called 2-3 trees
- B-Trees with $M = 4$ are called 2-3-4 trees