EG regret：

EG(η)投资组合更新：$w_i^{t+1} = \dfrac{w_i^t e^{\eta \frac{x_i^t}{w^t \cdot x^t}}}{Z_t}$ η为学习率

Z_t的定义为 $Z_t = \sum_{1 \le i \le N} w_i^t e^{\eta \frac{x_i^t}{w^t \cdot x^t}}$

当$\vec{u}$是投资组合，$\vec{x}$为相对价格，且有 $x_i^t \ge r > 0$，对$\eta > 0$有：

令 $\Delta_t = D_{RE}(u \| w^{t+1}) - D_{RE}(u \| w^t)$

由$D_{RE}(u \| v)$定义知 $\Delta_t = -\eta \frac{u \cdot x^t}{w^t \cdot x^t} + \log Z_t$

而 $Z_t = \sum_{1 \le i \le N} w_i^t e^{\eta \frac{x_i^t}{w^t \cdot x^t}} \le \sum_{1 \le i \le N} w_i^t (1 - (1 - e^{\frac{\eta}{w^t \cdot x^t}}) x_i^t) = 1 - (1 - e^{\frac{\eta}{w^t \cdot x^t}}) w^t \cdot x^t)$

又 $\log(1 - \alpha(1 - e^x)) \le \alpha x + \frac{x^2}{8}$ 得 $\log Z_t \le \eta + \frac{\eta^2}{8(w^t \cdot x^t)^2}$

则 $\Delta_t \le \eta(1 - \frac{u \cdot x^t}{w^t \cdot x^t}) + \frac{\eta^2}{8(w^t \cdot x^t)^2} \le -\eta \log(u \cdot x^t / w^t \cdot x^t) + \frac{\eta^2}{8(w^t \cdot x^t)^2}$

又 $1 - e^x \le -x \le r$，故对t求和可得：

$-D_{RE}(u \| w^1) \le D_{RE}(u \| w^{T+1}) - D_{RE}(u \| w^1)$

$\le \eta \sum_{t=1}^{T} (\log(w^t \cdot x^t) - \log(u \cdot x^t)) + \frac{\eta^2 T}{8r^2}$

又 $D_{RE}(u \| w^1) \le \log N$，

$LS_T^* - LS_T = \frac{1}{T}(\sum_{t=1}^{T} \log(u \cdot x^t) - \sum_{t=1}^{T} \log(w^t \cdot x^t)) \le \sqrt{\frac{\log N}{2r^2 T}}$

Regret $= \max_{t} T(LS_T^* - LS_T) = \sqrt{\frac{T \log N}{2r^2}}$

ONS regret：

首先定义 $\nabla_t = \nabla[\log(p_t \cdot r_t)] = \frac{1}{p_t \cdot r_t} r_t$

记 $\nabla^2[\log(p_t \cdot r_t)] = -\frac{1}{(p_t \cdot r_t)^2} r_t r_t^T = -\nabla_t \nabla_t^T$ ~~故~~记 $A_t = \sum_{\tau=1}^{t} \nabla_\tau \nabla_\tau^T + I_n$

令 $f_t(p) \triangleq \log(p_t r_t) + \nabla_t^T(p - p_t) - \frac{\beta}{2}[\nabla_t^T(p - p_t)]^2$

$\text{regret} = \sum_t \log(p \cdot r^t) - \log(p_t \cdot r_t)$

进行 Taylor 展开后得 $\max_p \sum_t \log(p \cdot r_t) - \log(p_t \cdot r_t) \le \max_p \sum_t f_t(p) - f_t(p_t)$

使用梯度边界 有 $f_t(p_{t+1}) - f_t(p_t) \le \nabla f_t(p_t)^T(p_{t+1} - p_t) = \nabla_t^T \Delta p_t$

记 $F_t \triangleq \sum_{\tau=1}^{t-1} f_\tau$ 则有 $\nabla F_{t+1}(p) = \sum_{\tau=1}^{t}[\nabla_\tau - \beta \nabla_\tau \nabla_\tau^T(p - p_\tau)]$

故 $\nabla F_{t+1}(p_{t+1}) - \nabla F_{t+1}(p_t) = -\beta A_t \Delta p_t$（由 A_t 定义）$= \Delta \nabla F_t(p) - \nabla_t$

则有：$-\beta A_t \Delta p_t = \Delta \nabla F_t(p_t) - \nabla_t$

左右两边同乘 $-\frac{1}{\beta}\nabla_t^T A_t^{-1}$ 得：$\nabla_t^T \Delta p_t = \frac{1}{\beta}\nabla_t^T A_t^{-1}[\Delta \nabla F_t(p_t) - \nabla_t]$

$= -\frac{1}{\beta}\nabla_t^T A_t^{-1}[\Delta \nabla F_t(p_t)] + \frac{1}{\beta}\nabla_t^T A_t^{-1}\nabla_t$

又 p_τ 使 F_τ 在 S_n 上最大，故 $\nabla F_\tau(p_\tau)^T(p - p_\tau) \le 0$

对于 $\tau = t$ 以及 $\tau = t+1$ 有 $0 \ge \nabla F_{t+1}(p_{t+1})^T(p_t - p_{t+1}) + \nabla F_t(p_t)^T(p_{t+1} - p_t)$

$= -[\Delta \nabla F_t(p_t)]^T \Delta p_t$

$= \frac{1}{\beta}[\Delta \nabla F_t(p_t)]^T A_t^{-1}[\Delta \nabla F_t(p_t) - \nabla_t]$

$= \frac{1}{\beta}[\Delta \nabla F_t(p_t)]^T A_t^{-1}[\Delta \nabla F_t(p_t)] - \frac{1}{\beta}[\Delta \nabla F_t(p_t)]^T A_t^{-1}\nabla_t$

$\ge -\frac{1}{\beta}[\Delta \nabla F_t(p_t)]^T A_t^{-1}\nabla_t$

故 $\frac{1}{\beta}\sum_{t=1}^{T} \nabla_t^T A_t^{-1} \nabla_t \le \frac{1}{\beta}\log\left[\frac{|A_T|}{|A_0|}\right] \le \frac{1}{\beta} n \log\left[\frac{nT}{\alpha^2}\right]$

即 $\sum_{t=1}^{T}[f_t(p_{t+1}) - f_t(p_t)] \le \frac{1}{\beta} n \log\left[\frac{nT}{\alpha^2}\right]$

又由 $\frac{\beta}{2}[\|p\|^2 - \|p_1\|^2] \le \frac{\beta}{2}$

则 $\text{regert(ONS)} \le \frac{1}{\beta} n \log\left[\frac{nT}{\alpha^2}\right] + \frac{\beta}{2}$