1 Introduction

Integrands of kernels with spherical Bessel functions occur in MRI applications [6]. More general kernels resulting in products of Bessel functions of the first or second kind with real valued order i.e. $a, b = n + 1/2$ could emerge in future imaging technologies. Also, magnetic induction tomography imaging [1] involve integrands of complex valued kernels with Bessel functions. In this note, our goal is to test the effectiveness of the `IIPBF` [5] toolbox with real-valued order and complex-valued kernels.

2 Test Cases

As Bessel functions of the first kind with fractional order have already been dealt with, case 6 in Ratnanather et al. [5]:

$$\int_0^\infty xK_0(xu)J_0(\rho x)J_0(\tau x)dx = 1/\left(\left(u^2+\rho^2+\tau^2\right)^2 - 4\rho^2\tau^2\right)^{1/2} \qquad (1)$$

can be generalised for complex-valued u i.e. $u = \alpha + \imath\beta$ where $(\alpha, \beta) \in \{(10, 2), (2, 2), (2, 10)\}$.

Next to test for Bessel functions of the second kind with fractional order [2, see 6.633.3] consider for $u = 10, b = \frac{1}{2}$ or $\frac{1}{3}$:

$$\int_0^\infty x^{2b+1}e^{-ux^2}J_b(x)Y_b(x)dx = -\frac{1}{2\sqrt{\pi}}u^{-(3b+1)/2}e^{-1/2u}W_{b/2,b/2}\left(\frac{1}{u}\right) \qquad \text{Re } u > 0, \text{Re } b > -1/2 \quad (2)$$

where W is the Whittaker function. For non-integer $b \neq 0$ consider additional variants from McPhedran et al. [4, see Eqs. 31, 38-41]:

$$\begin{aligned}
\int_0^\infty xe^{-ux^2}J_b(x)Y_b(x)dx &= \frac{e^{-1/2u}}{2\pi u}[\pi\cot(\pi b)I_b(1/2u) + bh_{-1,b}(-1/2u)] && (3)\\
\int_0^\infty xe^{-ux^2}J_b(x)J_{-b}(x)dx &= -\frac{e^{-1/2u}}{2\pi u}[b\sin(\pi b)h_{-1,b}(-1/2u)] && (4)\\
\int_0^\infty xe^{-ux^2}J_b(x)Y_{-b}(x)dx &= \frac{e^{-1/2u}}{2\pi u}[\pi\csc(\pi b)I_b(1/2u) + b\cos(b\pi)h_{-1,b}(-1/2u)] && (5)\\
\int_0^\infty xe^{-ux^2}Y_b(x)Y_{-b}(x)dx &= \frac{e^{-1/2u}}{2\pi u}\frac{1}{\sin^2(b\pi)}[\pi\cos(\pi b)(I_b(1/2u) + I_{-b}(1/2u)) \\
&\quad +(1+\cos^2(b\pi))b\sin(b\pi)h_{-1,b}(-1/2u)] && (6)\\
\int_0^\infty xe^{-ux^2}Y_b(x)Y_b(x)dx &= \frac{e^{-1/2u}}{2\pi u}[\pi\cot^2(\pi b)I_b(1/2u) + \pi(1+\cot^2(b\pi))I_{-b}(1/2u) \\
&\quad +2b\cot(b\pi)h_{-1,b}(-1/2u)] && (7)
\end{aligned}$$

where $h_{-1,b}$ is an associated Bessel function that is a converging infinite series and conveniently expressed as a hypergeometric function (see appendix):

$$h_{-1,b}(-1/2u) = \frac{e^{-1/2u}}{b^2} {}_2F_2(1, 1/2; 1-b, 1+b; -1/u).$$

For integer b, we have from McPhedran et al. [4, see Eqs. 7 and 33]

$$\int_0^\infty xe^{-ux^2} J_0(x)Y_0(x)dx = -\frac{e^{-1/2u}}{2\pi u} K_0(1/2u) \tag{8}$$

$$\int_0^\infty xe^{-ux^2} J_b(x)Y_b(x)dx = -\frac{e^{-1/2u}}{2\pi u} [(-1)^b K_b(1/2u) - bH_{-1,b}(-1/2u)] \tag{9}$$

where $H_{-1,b}$ is a second associated Bessel function that is a terminating series and conveniently expressed as a hypergeometric function (see appendix):

$$H_{-1,b}(-1/2u) = -\frac{e^{1/2u}}{1/2u} {}_3F_1(1, 1+b, 1-b; 3/2; u).$$

A Associated Bessel Functions

McPhedran et al. [4] used series forms for the first and second associated Bessel functions originally derived by Luke [3, see pages 108-109]. The series form for the first associated Bessel function is:

$$h_{-1,b}(-1/2u) = \frac{e^{1/2u}\sqrt{\pi}}{b\sin(\pi b)} \sum_{k=0}^{\infty} \frac{(-1/u)^k \Gamma(k+1/2)}{\Gamma(k-b+1)\Gamma(k+b+1)}. \tag{10}$$

The original series form from Luke is:

$$h_{\mu,b}(z) = -\frac{e^{-z}z^{\mu+1}\Gamma(\mu+b+1)\Gamma(\mu-b+1)}{\Gamma(\mu+3/2)} \sum_{k=0}^{\infty} \frac{(2z)^k \Gamma(\mu+k+3/2)}{\Gamma(\mu-b+k+2)\Gamma(\mu+b+k+2)} \tag{11}$$

$$= -\frac{e^{-z}z^{\mu+1}}{(\mu-b+1)(\mu+b+1)} {}_2F_2(1, \mu+3/2; \mu-b+2, \mu+b+2; 2z). \tag{12}$$

With $z = -1/2u$, $\mu = -1$, $\Gamma(b)\Gamma(-b) = -\pi/b\sin(\pi b)$ and $\Gamma(1/2) = \sqrt{\pi}$, it is evident that Eq. 10 is easily obtained from Eq. 11. Note there should have been a minus sign in the original definition by Luke.

The second associated Bessel function, $H_{\mu,b}$, is:

$$H_{\mu,b}(z) = -\frac{e^{-z}z^{\mu}}{2\mu+1} \left[1 + \frac{\mu^2-b^2}{(2\mu-1)z} + \frac{(\mu^2-b^2)((\mu-1)^2-b^2)}{(2\mu-1)(2\mu-3)z^2} + \dots\right] \tag{13}$$

$$= -\frac{e^{-z}z^{\mu}}{2\mu+1} {}_3F_1(1, -\mu+b, -\mu-b; 1/2-\mu; -1/2z). \tag{14}$$

References

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