IIPBF - a Matlab toolbox for computing infinite integrals of products of Bessel functions of the 1st and 2nd kind

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Introduction

$$\int_0^\infty f(x) B_{a,b,\rho,\tau}(x)dx \text{ where } B_{a,b,\rho,\tau}(x) = \begin{cases} J_a(\rho x) J_b(\tau x) \\ J_a(\rho x) Y_b(\tau x) \\ Y_a(\rho x) Y_b(\tau x) \end{cases}$$

occurs in elasticity, biophysics, electrodynamics, geology, ...

- `MATLAB` toolbox[1] for a, b: non-negative integer; $f(x)$: real valued, smooth and monotonic as $x \to \infty$
- adaptation of ISE - integration, summation, extrapolation - (Longman, 1956) for $J_a(\rho x) J_b(\tau x)$ (Lucas, 1995)
- extended[1] to $J_a(\rho x) Y_b(\tau x)$ and $Y_a(\rho x) Y_b(\tau x)$ for a fluid jet on a planar wall[2] with applications in cochlear physiology

[1] Ratnanather JT, Kim JH, Zhang S, Davis AMJ, Lucas SK (2013) Algorithm XXX: IIPBF, a MATLAB toolbox for infinite integral of products of two Bessel functions, ACM TOMS, in press

[2] Davis AMJ, Kim JH, Ceritoglu C, Ratnanather JT (2012) A Stokesian analysis of a submerged viscous fluid jet impinging on a planar wall. J. Fluid Mech. 712:531-551

New Applications

- kernels with spherical Bessel functions occur in MRI applications (Hosseinbor et al., 2013)
- general kernels with Bessel functions in image registration for biomedical shape analysis (Micheli and Glaunès, 2013)
- tomography imaging (Dodd and Deeds, 1968) involve complex valued kernels with Bessel functions
- oscillating jet on a planar wall involves complex valued kernels[3] with applications in cochlear mechanics
- Goal: effectiveness of `IIPBF` for Bessel functions with real-valued order and complex valued kernels

[3] Davis AM, Kim JH, Gunter GM, Ratnanather JT (2013) The Stokesian flow field of an oscillatory submerged viscous jet impinging on a planar wall. Proc. Roy. Soc. Lond. A, in press

Algorithm

- Adaptation of `FORTRAN77` algorithm (Lucas, 1995)

  Integration: `MATLAB` conversion of quadrature routines in `SLATEC` (Barrowes, 2009)

  Summation: accelerates convergence by decomposing product as a sum of high and low frequency components and subdivision via zeros of components

  Extrapolation: ε-algorithm (`dqelg`) from `QUADPACK` and mW transform (Sidi, 1988, 2012) to deal with oscillatory components and acceleration

- `dqage` (definite integrals), `dqagie` (infinite integrals), and `dqelg` (ε-algorithm) from `SLATEC`
- Adaptive Gauss-Kronrod quadrature - `quadgk` - (Shampine, 2008) instead of `dqk15` and `dqk15i`

$$B_{a,b,\rho,\tau}(x) = h_1(x; a, b, \rho, \tau) + h_2(x; a, b, \rho, \tau)$$

$J_a(\rho x) J_b(\tau x)$

$$h_1(x; a, b, \rho, \tau) = \frac{1}{2}(J_a(\rho x)J_b(\tau x) - Y_a(\rho x)Y_b(\tau x)) \sim \frac{1}{\pi\sqrt{\rho\tau}x}\cos\left((\rho+\tau)x - \frac{(a+b+1)\pi}{2}\right)$$

$$h_2(x; a, b, \rho, \tau) = \frac{1}{2}(J_a(\rho x)J_b(\tau x) + Y_a(\rho x)Y_b(\tau x)) \sim \frac{1}{\pi\sqrt{\rho\tau}x}\cos\left((\rho-\tau)x - \frac{(a-b)\pi}{2}\right)$$

$J_a(\rho x) Y_b(\tau x)$

$$h_1(x; a, b, \rho, \tau) = \frac{1}{2}(J_a(\rho x)Y_b(\tau x) + Y_a(\rho x)J_b(\tau x)) \sim \frac{1}{\pi\sqrt{\rho\tau}x}\sin\left((\rho+\tau)x - \frac{(a+b+1)\pi}{2}\right)$$

$$h_2(x; a, b, \rho, \tau) = \frac{1}{2}(J_a(\rho x)Y_b(\tau x) - Y_a(\rho x)J_b(\tau x)) \sim -\frac{1}{\pi\sqrt{\rho\tau}x}\sin\left((\rho-\tau)x - \frac{(a-b)\pi}{2}\right)$$

$Y_a(\rho x) Y_b(\tau x)$

$$h_1(x; a, b, \rho, \tau) = -\frac{1}{2}(J_a(\rho x)J_b(\tau x) - Y_a(\rho x)Y_b(\tau x)) \sim -\frac{1}{\pi\sqrt{\rho\tau}x}\cos\left((\rho+\tau)x - \frac{(a+b+1)\pi}{2}\right)$$

$$h_2(x; a, b, \rho, \tau) = \frac{1}{2}(J_a(\rho x)J_b(\tau x) + Y_a(\rho x)Y_b(\tau x)) \sim \frac{1}{\pi\sqrt{\rho\tau}x}\cos\left((\rho-\tau)x - \frac{(a-b)\pi}{2}\right)$$

Zeros of h_1 and h_2 for mW transform and ε-algorithm

- ISE applied to $f(x)h_1(x)$ and $f(x)h_2(x)$
- $h_{1,1}$ and $h_{2,1}$: 1st zeros of $h_1(x)$ and $h_2(x)$ respectively
- $Y_a(\rho x)$ and $Y_b(\tau x)$ are singular near 0 so use approximate zeros (Olver et al., 2010)
- Two scenarios
  - $\rho = \tau$: split $[0, \infty)$ into $[0, ymax]$ and $[ymax, \infty)$ used for $f(x)B_{a,b,\rho,\tau}(x)$ and $f(x)h_1(x) + f(x)h_2(x)$ respectively

    h_1: $ymax$ from 1st zero of Y_a or Y_b and asymptotic form used for subsequent zeros

    h_2: non-oscillating and monotonic decreasing so zeros not used
  - $\rho \neq \tau$: split $[0, \infty)$ into $[0, ymin]$, $[ymin, ymax]$ and $[ymax, \infty)$ with $ymin = min(h_{1,1}, h_{2,1})$ and $ymax = max(h_{1,1}, h_{2,1})$ where $h_{1,1}$ and $h_{2,1}$ are obtained from the 1st zeros of Y_a or Y_b

    h_1: asymptotic form used for subsequent zeros

    h_2: zeros after $h_{2,1}$ obtained via stepwise increments $\pi / |\rho - \tau|$

mW transform: Generalized Richardson Extrapolation Process

Following Sidi (1988, 2012), begin with $S_0 = \int_a^{x_0} f(x)h_1(x)dx$, $T_0 = \int_{x_0}^{x_1} f(x)h_1(x)dx$ and iterate until $W_t < \mathrm{TOL}$:

$$\begin{aligned} S_{t+1} &= S_t + T_t = \int_a^{x_{t+1}} f(x)h_1(x)dx \\ T_{t+1} &= \int_{x_{t+1}}^{x_{t+2}} f(x)h_1(x)dx \\ M_{t+1,-1} &= S_{t+1}/T_{t+1}, N_{t+1,-1} = 1/T_{t+1} \end{aligned}$$

and for $s = t, t-1, \ldots, 0$

$$\begin{aligned} M_{s,t-s} &= (M_{s,t-s-1} - M_{s+1,t-s-1})/(1/x_s - 1/x_{t+1}) \\ N_{s,t-s} &= (N_{s,t-s-1} - N_{s+1,t-s-1})/(1/(x_s - 1/x_{t+1}) \\ W_t &= M_{0,t}/N_{0,t} \end{aligned}$$

ε-algorithm: Aitken, Euler, Padé, Shanks, Stieltjes, Wynn

- nonlinear transformation of a slowly converging sequence which identifies and removes oscillatory transients
- given partial sum $\{A_n\}_{n=0}^{\infty}$, $\varepsilon_n^{(-1)} = 0$ and $\varepsilon_n^{(0)} = A_n$:

$$\varepsilon_n^{(p)} = \varepsilon_n^{(p-2)} + \left[\varepsilon_{n+1}^{(p-1)} - \varepsilon_n^{(p-1)}\right]^{-1}$$

- $\varepsilon^{(2k)}$ is the kth Shanks' transform of the sequence $\{A_n\}$
- approximate zeros permits ignorance of early poor results before h_2 settles to a simple oscillation

Modifications

- $|\rho/\tau - 1| < 10^{-10}$: h_2 is almost monotone so standard infinite integration
- $|\rho/\tau| > 10^2$ or $|\tau/\rho| > 10^2$: asymptotic forms of h_1 and h_2 have almost the same frequency so use ε-algorithm
- small values of ρ or τ can cause the 1st zeros of Y_a and Y_b to be large so the minimum of these zeros is used when the ratio of the larger zero to the smaller one is too large; otherwise the maximum is used as recommended (Lucas, 1995)

Flow Chart

if $|\rho/\tau - 1| < 10^{-10}$ **then**

$\cdots$

else if $|\rho/\tau| > 10^2$ *or* $|\tau/\rho| > 10^2$ **then**

$\cdots$

else

Calculate $h_{1,1}$ and $h_{2,1}$;

$I_2 = \int_{h_{1,1}}^{\infty} f(x)h_1(x; a, b, \rho, \tau)dx$ using mW transform;

$I_3 = \int_{h_{2,1}}^{\infty} f(x)h_2(x; a, b, \rho, \tau)dx$ using ε-algorithm;

if $h_{1,1} < h_{2,1}$ **then**

$I_1 = \int_{0}^{h_{1,1}} f(x)B_{a,b,\rho,\tau}(x)dx$ using `quadgk`;

$I_4 = \int_{h_{1,1}}^{h_{2,1}} f(x)h_2(x; a, b, \rho, \tau)dx$ using `quadgk`

else

$I_1 = \int_{0}^{h_{2,1}} f(x)B_{a,b,\rho,\tau}(x)dx$ using `quadgk`;

$I_4 = \int_{h_{2,1}}^{h_{1,1}} f(x)h_1(x; a, b, \rho, \tau)dx$ using `quadgk`

end

end

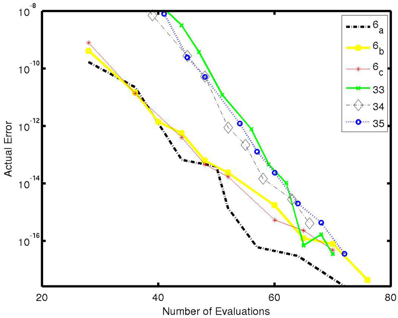
$I_1 + I_2 + I_3 + I_4$

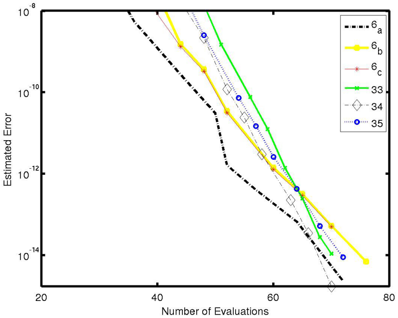
Test Cases

- 32 test cases in `IIPBF` including 23 already tested[1]
- so consider complex-valued kernel for one of the the 23 cases plus 3 new ones from Gradshteyn and Ryzhik (2007)

Case	Integrand	Value
6	$xK_0(xc)J_0(\rho x)J_0(\tau x)$	$1/\left(\left(c^2+\rho^2+\tau^2\right)^2-4\rho^2\tau^2\right)^{1/2}$
33	$\frac{x}{x^2+c^2}J_a(\rho x)J_a(\tau x)$	$I_a(\tau c)K_a(\rho c)$ for $0<\tau<\rho$ and $I_a(\rho c)K_a(\tau c)$ for $0<\rho<\tau$ $\operatorname{Re} c>0, \operatorname{Re} a>-1$
34	$x^{-c}J_a(\rho x)J_b(\tau x)$	$\dfrac{\tau^a\Gamma\left(\frac{a+b-c+1}{2}\right)}{2^c\rho^{b-c+1}\Gamma\left(\frac{a-b+c+1}{2}\right)\Gamma(a+1)}$ $\times F\left(\frac{a+b-c+1}{2}, \frac{-a+b-c+1}{2}; b+1; \frac{\tau^2}{\rho^2}\right)$ $\operatorname{Re}(a+b-c+1)>0, \operatorname{Re} c>-1, 0<\tau<\rho$
35	$x^{-c}J_a(\rho x)Y_b(\tau x)$	$\dfrac{2}{\pi}\sin\left(\dfrac{\pi(a-b-c)}{2}\right)\dfrac{\rho^a\Gamma(\frac{1}{2}-\frac{c}{2}+\frac{b}{2}+\frac{a}{2})\Gamma(\frac{1}{2}-\frac{c}{2}-\frac{b}{2}+\frac{a}{2})}{2^{c+1}\Gamma(a+1)\tau^{-c+a+1}}$ $\times F(\frac{1}{2}-\frac{c}{2}+\frac{b}{2}+\frac{a}{2}, \frac{1}{2}-\frac{c}{2}-\frac{b}{2}+\frac{a}{2}; a+1; \frac{\rho^2}{\tau^2})$ $\operatorname{Re}(a\pm b-c+1)>0, \operatorname{Re} c>-1, \tau<\rho$

Actual and Estimated Errors: $xK_0(xc)J_0(\rho x)J_0(\tau x)$, $\frac{x}{x^2+c^2}J_a(\rho x)J_b(\tau x)$, $x^{-c}J_a(\rho x)J_b(\tau x)$, $x^{-c}J_a(\rho x)Y_b(\tau x)$





- number of function evaluations of h_1 and h_2 based on relative error tolerances from 10^{-4} to 10^{-14}
- for case 6, $\rho = 0.1, \tau = 100$ and $c = \alpha + \imath\beta$ where $(\alpha, \beta) \in \{(10, 2), (2, 2), (2, 10)\}$
- for cases 33, 34 and 35: $\{c, a, b, \rho, \tau\} = \{2, 1, 1, 2, 1\}$, $\{2, 1, 2, 2, 1\}$ and $\{1, 2, 1, 1, 2\}$ respectively

Comparison with BESSELINT (Van Deun and Cools, 2008)

- for $f(x)\Pi_{i=1}^{k} J_{a_i}(\rho_i x)$ where $f(x) = x^s e^{-ux}/(t^2 + x^2)$
- Case 33: $x^{-c} J_a(\rho x) J_b(\tau x)$
- Case 34: $x^{-c} J_a(\rho x) Y_b(\tau x)$

Case	Time (secs)		Estimated Error		Absolute
	BESSELINT	IIPBF	BESSELINT	IIPBF	Difference
33	0.1099	0.1834	1.85 x10^{-12}	1.16 x10^{-15}	2.19 x10^{-16}
34	0.0950	0.1805	4.73 x10^{-15}	1.43 x10^{-15}	9.58 x10^{-16}

- bottleneck in both is due to the infinite integration routine
- MATLAB version 7.14.0.739 (R2012a) on an Intel(R) Xeon(R) CPU E31290 at 3.60GHz which is a 64bit machine

(ρ, τ) parameter tests for Case 35: $x^{-c} J_a(\rho x) Y_b(\tau x)$

$a = 2.5$, $b = 1.5$, $c = 1$

		ρ						
		0.001	0.01	0.1	1	10	100	1000
τ	0.0011	3.18 x10^{-13}	-	-	-	-	-	-
	0.011	4.12 x10^{-13}	3.18 x10^{-13}	-	-	-	-	-
	0.11	1.71 x10^{-13}	4.12 x10^{-13}	3.18 x10^{-13}	-	-	-	-
	1.1	2.91 x10^{-11}	1.99 x10^{-13}	4.12 x10^{-13}	3.18 x10^{-13}	-	-	-
	11	0.00	7.28 x10^{-12}	1.71 x10^{-13}	4.12 x10^{-13}	3.18 x10^{-13}	-	-
	101	0.00	0.00	7.28 x10^{-12}	3.69 x10^{-13}	7.12 x10^{-13}	1.46 x10^{-13}	-
	1001	0.00	0.00	0.00	1.46 x10^{-11}	3.13 x10^{-13}	8.36 x10^{-12}	2.52 x10^{-13}

$a = 4.2$, $b = 0.3$, $c = 2.5$

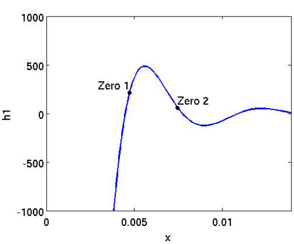
		ρ						
		0.001	0.01	0.1	1	10	100	1000
τ	0.0011	1.35 x10^{-12}	-	-	-	-	-	-
	0.011	1.48 x10^{-12}	6.64 x10^{-13}	-	-	-	-	-
	0.11	1.08 x10^{-11}	1.42 x10^{-12}	1.12 x10^{-12}	-	-	-	-
	1.1	2.21 x10^{-16}	1.13 x10^{-10}	1.11 x10^{-12}	2.11 x10^{-13}	-	-	-
	11	4.41 x10^{-19}	6.98 x10^{-15}	3.84 x10^{-9}	5.80 x10^{-13}	2.01 x10^{-13}	-	-
	101	1.11 x10^{-21}	1.75 x10^{-17}	2.78 x10^{-13}	5.52 x10^{-8}	5.04 x10^{-13}	2.47 x10^{-13}	-
	1001	2.26 x10^{-24}	3.59 x10^{-20}	5.68 x10^{-16}	9.01 x10^{-12}	1.43 x10^{-7}	6.36 x10^{-13}	3.27 x10^{-13}

1st zeros of h_1

Case 35: $x^{-c} J_a(\rho x) Y_b(\tau x)$ for $\rho = 10$, $\tau = 1001$

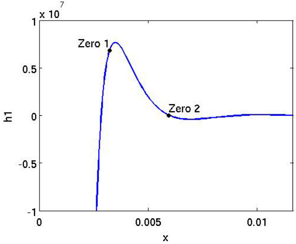
$c = 1$; $a = 2.5$; $b = 1.5$

Error: 3.13×10^{-13}



$c = 2.5$; $a = 4.2$; $b = 0.3$

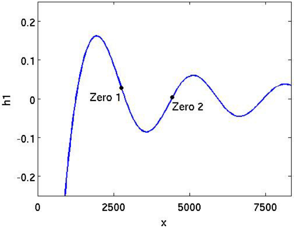
Error: 1.43×10^{-7}



Case 23: $x^{b-a+1} J_a(\rho x) J_b(\tau x)/(x^2 + c^2)$ for $\rho = 0.001$, $\tau = 0.0011$

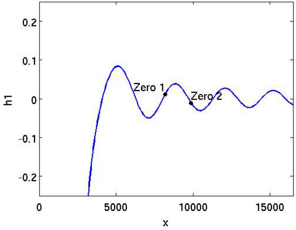
$c = 2$; $a = 1$; $b = 1$

Error: 3.70×10^{-14}



$c = 2$; $a = 3$; $b = 5$

Error: 1.04×10^{-7}



Discussion

- `quadgk` enabled error estimates with significantly less function evaluations than before which is not surprising (Gonnet, 2012) but its structure makes it difficult to vectorize mW transform
- product decomposition avoids loss of precision when mW transform is applied to the product (Sidi, 2012)
- `IIPBF` works for a wide range of ρ and τ with higher orders but for extreme values results should be checked
- $|a - b| > 5$ works (Lucas, 1995) but problems when the Bessel functions become very small (Van Deun and Cools, 2008)
- warnings when the maximum number of intervals in `quadgk` is exceeded due to either the first zero or the spacing between the first two zeros or both becoming too big
- wild idea: best of both `IIPBF` and `BESSELINT` in `Chebfun`?
- `http://www.cis.jhu.edu/software/iipbf/`

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