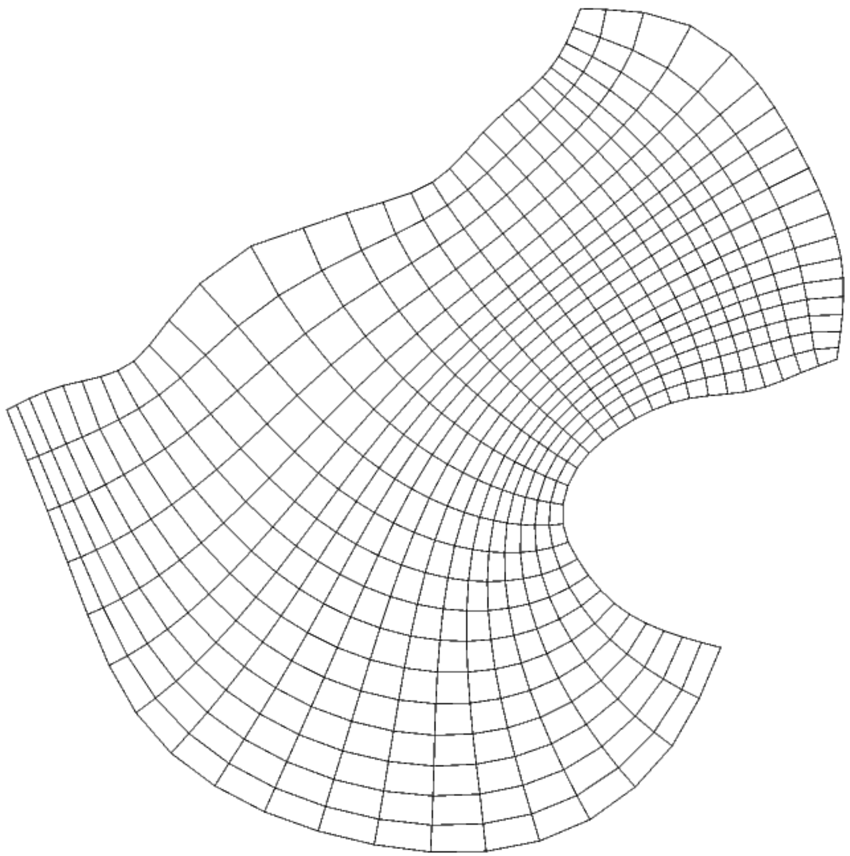

Curvilinear Operators

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Fall 2020

Before we start…

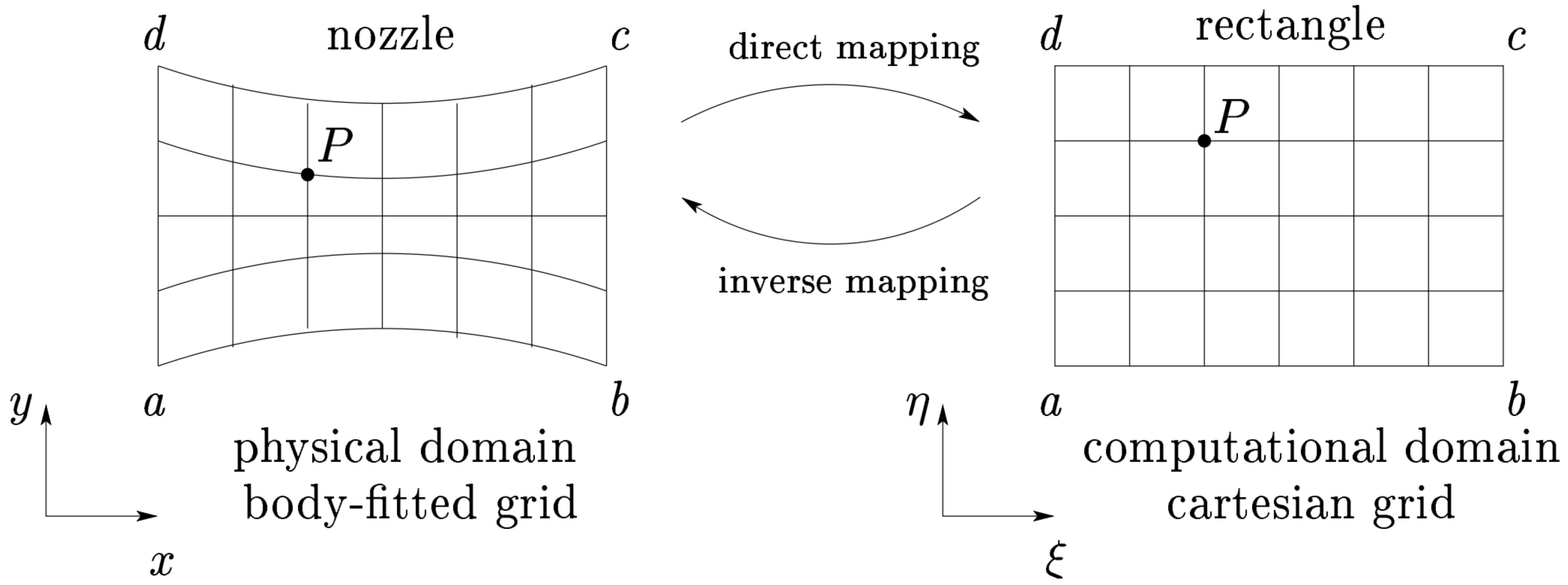
- Mimetic operators for non-uniform, logically rectangular grids were already implemented into MOLE (1D, 2D and 3D)

- In 1D, non-uniform and curvilinear operators are identical

$$f = f(x), \qquad a < x < b$$

$$x = x(\xi), \qquad 0 < \xi < 1$$

$$J = (x_\xi) = \left(\frac{dx}{d_\xi}\right)$$



The original PDE must be rewritten in terms of (ξ, η) instead of (x, y) and discretized in the computational domain rather than the physical domain.

$$\underbrace{\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}}_{\text{Difficult to compute}} \longrightarrow \underbrace{\frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}}_{\text{Easy to compute}}$$

Direct mapping: $\xi = \xi(x, y), \qquad \eta = \eta(x, y)$

Inverse mapping: $x = x(\xi, \eta), \qquad y = y(\xi, \eta)$

By applying the chain rule:

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \eta}$$

Known metrics

In matrix form:

$$\begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}}_{J} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix}$$

$$\boxed{\begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}^{-1}} \begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix}$$

Unknown metrics

$$\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}^{-1} = \frac{1}{\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}} \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{bmatrix}$$

$$\frac{1}{|J|} \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix}$$

$$\frac{\partial u}{\partial x} = \frac{1}{|J|} \left[\frac{\partial y}{\partial \eta}\frac{\partial u}{\partial \xi} - \frac{\partial y}{\partial \xi}\frac{\partial u}{\partial \eta} \right]$$

$$\frac{\partial u}{\partial y} = \frac{1}{|J|} \left[\frac{\partial x}{\partial \xi}\frac{\partial u}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial u}{\partial \xi} \right]$$

“The function of good software is to make the complex appear to be simple”

-Grady Booch