Treewidth, Applications, and some Recent Developments

Chandra Chekuri

Univ. of Illinois, Urbana-Champaign

Goals of Tutorial

- Give an intuitive understanding of **treewidth** and **tree decompositions**
- Describe some algorithmic applications
- Describe some recent developments

Graphs

Powerful modeling tool

Numerous applications

However, many natural problems are *intractable*

Question:

- What graph properties allow tractability?
- How can they be leveraged in applications?

Graph Properties/Parameters

- Sparsity
- Connectivity
- Topological properties (planarity, genus, …)
- Spectral properties (expansion, …)
- …

Graph Properties/Parameters

- Sparsity
- Connectivity
- Topological properties (planarity, genus, …)
- Spectral properties (expansion, …)
- …
- *Decomposability*

Tree decompositions and Treewidth

Studied by [Halin'76]

Again by [Robertson & Seymour'84] as part of their seminal graph minor project

In ML tree decompositions related to **junction trees**

Tree decompositions and Treewidth

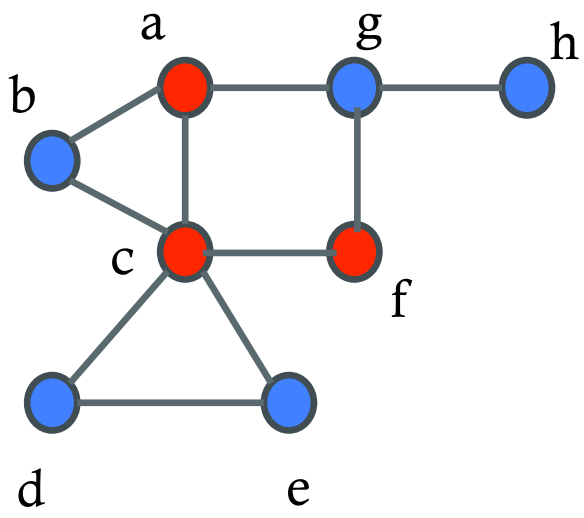
- key to graph minor theory of Robertson & Seymour
- many algorithmic applications

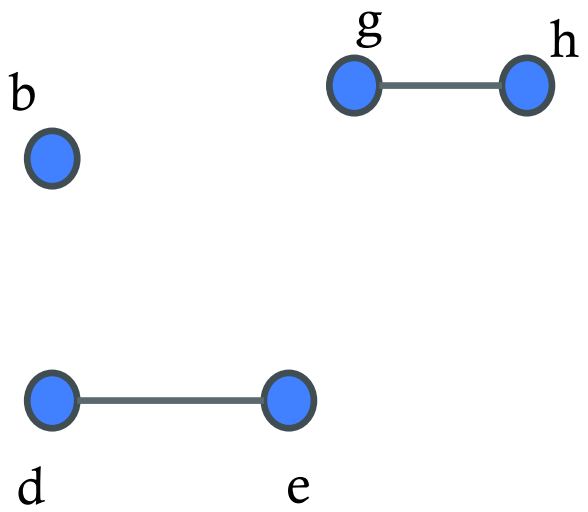
Message:

algorithms and structural understanding intertwined

Separator

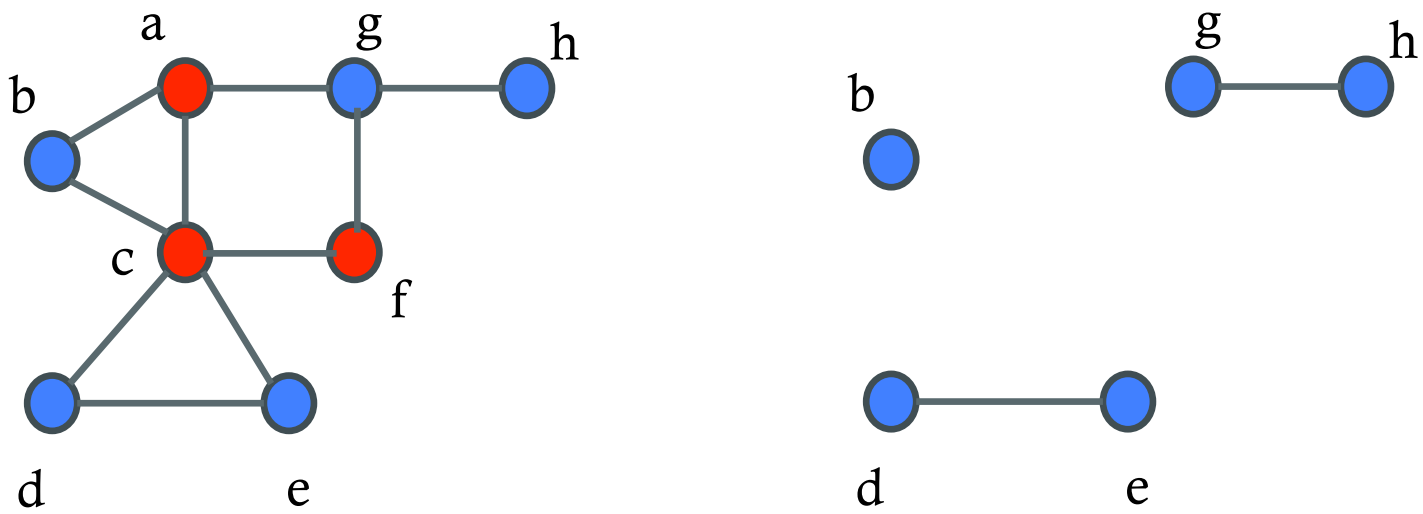
Given G = (V,E), $S \subset V$ is a *vertex separator* if G – S has at least two connected components





Balanced Separator

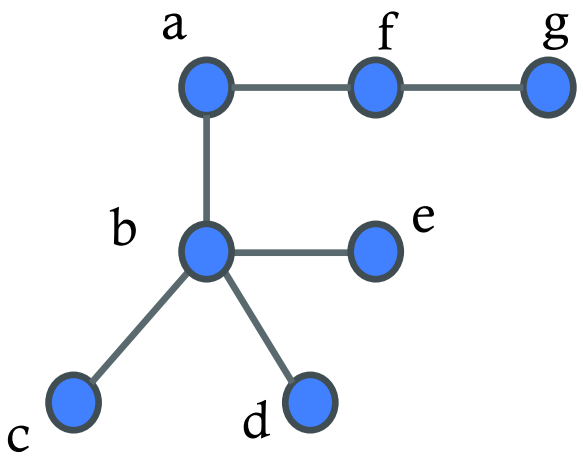
Given $G = (V,E)$, $S \subset V$ is a *balanced* vertex separator if every component of $G - S$ has $\leq (2/3)\,|V|$ vertices



Trees

Easy exercise: Every tree T has a vertex v s.t v is a balanced separator

Recursive decomposition via separators of size one

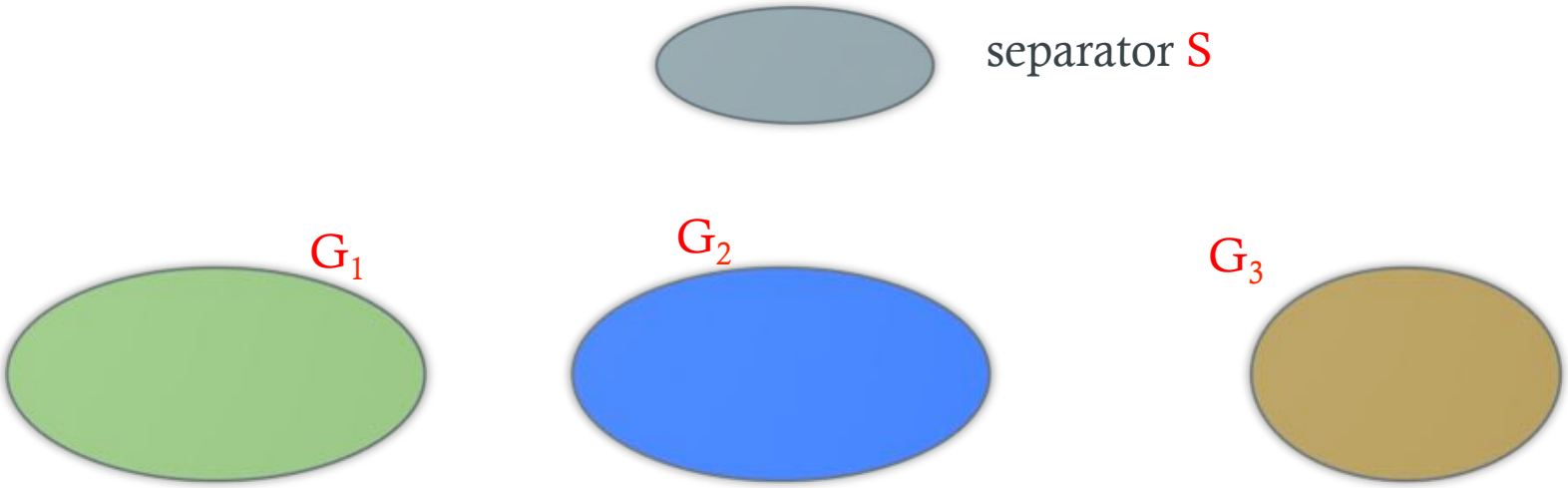


Planar Separator Theorem

[Lipton-Tarjan'79]

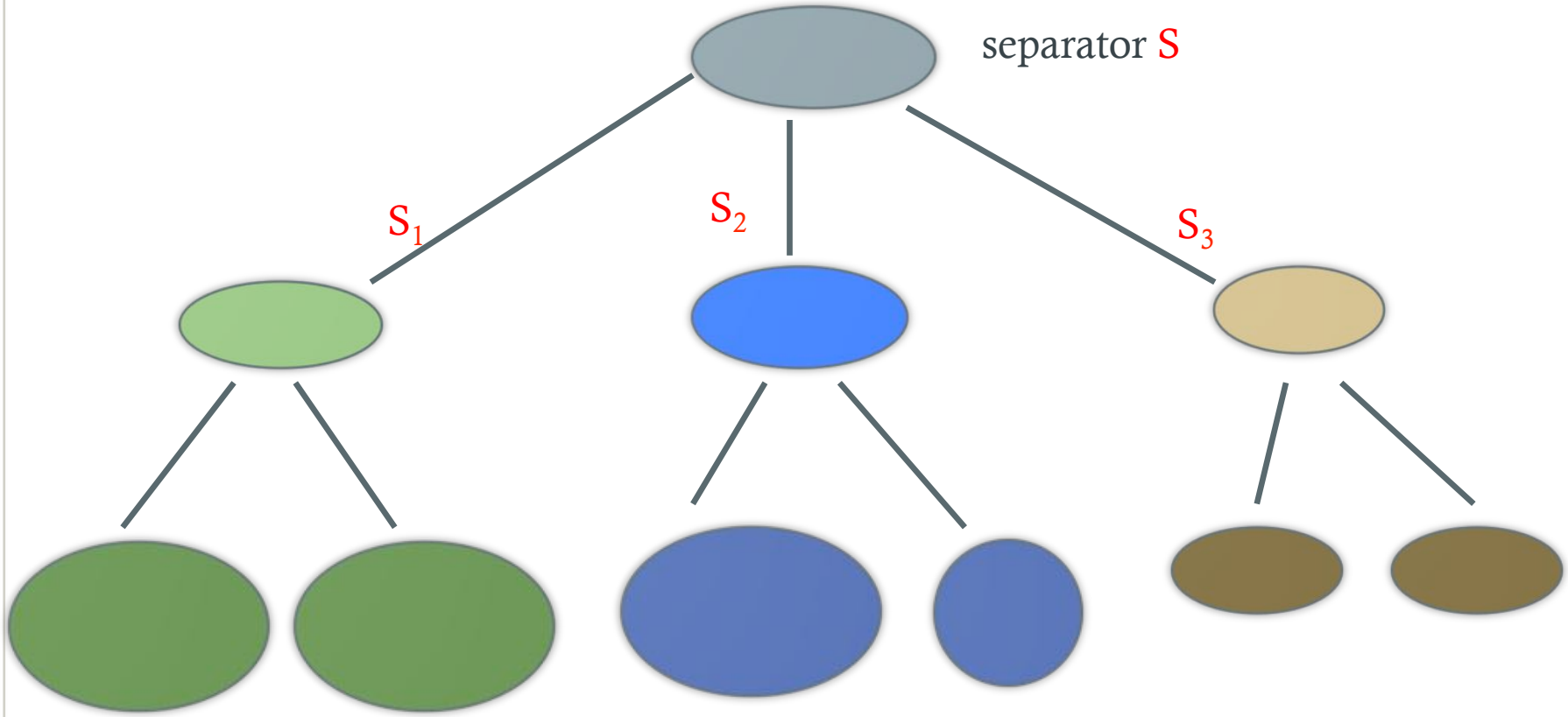
Every n vertex *planar graph* has a balanced separator of size $O(\sqrt{n})$

Recursive Decomposition



components of G - S

Recursive Decomposition



Planar Separator Theorem

[Lipton-Tarjan'79]

Every n vertex *planar graph* has a balance separator of size $O(\sqrt{n})$

Many applications via recursive decomposition

Treewidth

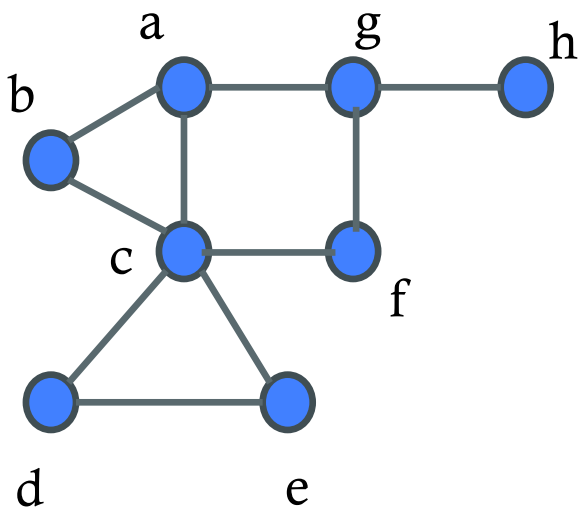
Informal: treewidth(G) $\leq$ k implies G can be *recursively decomposed* via "balanced" separators of size k

(A measure tailored for a given graph)

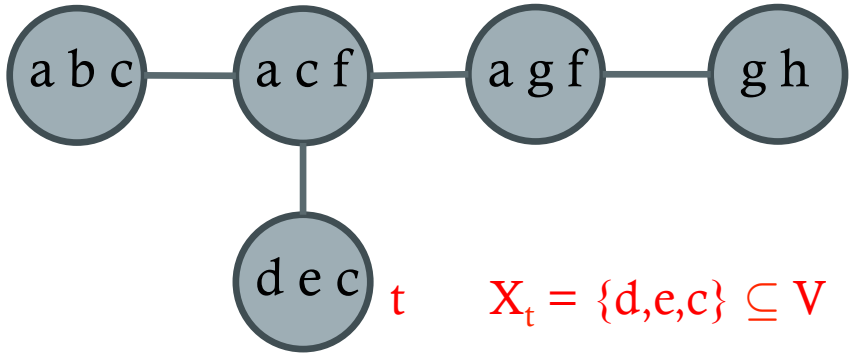
Formal definition a bit technical

Tree Decomposition

G=(V,E)



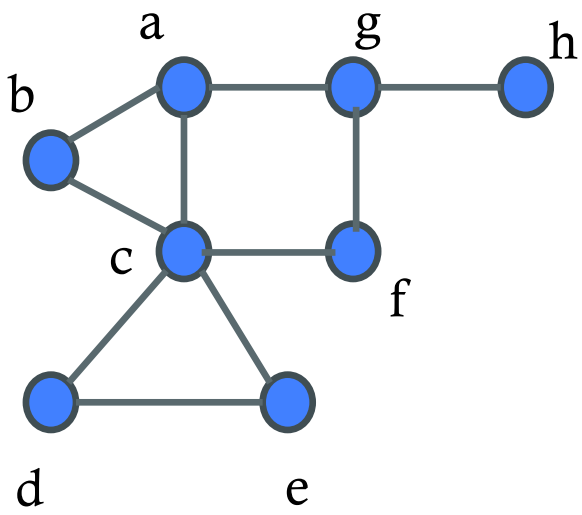
$T=(V_T, E_T)$



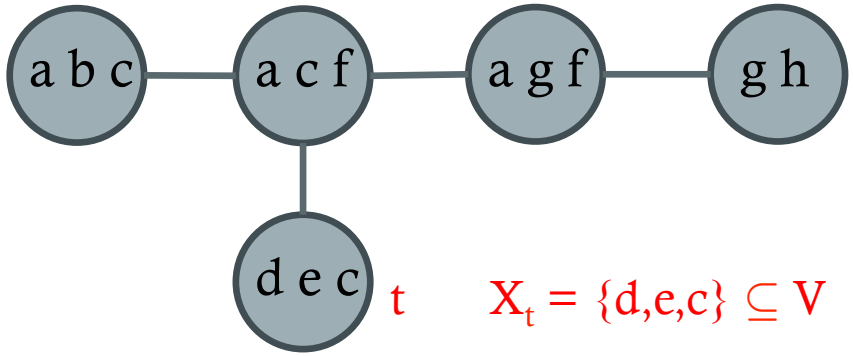
$X_t = \{d,e,c\} \subseteq V$

Tree Decomposition

G=(V,E)



$T=(V_T, E_T)$

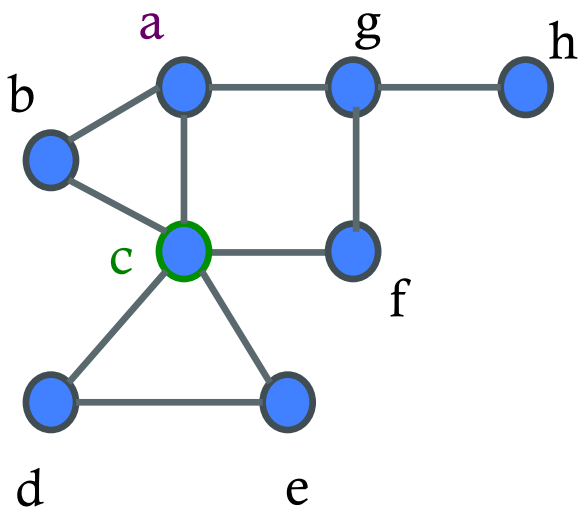


$X_t = \{d,e,c\} \subseteq V$

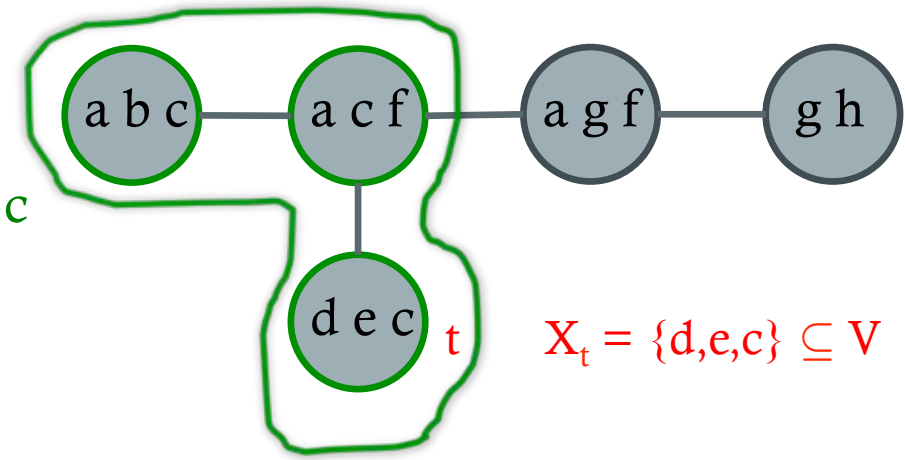
- $\cup_t X_t = V$
- For each $v \in V$, $\{ t \mid v \in X_t \}$ is sub-tree of T
- For each edge $uv \in E$, exists t such that $u,v \in X_t$

Tree Decomposition

G=(V,E)



$T=(V_T, E_T)$

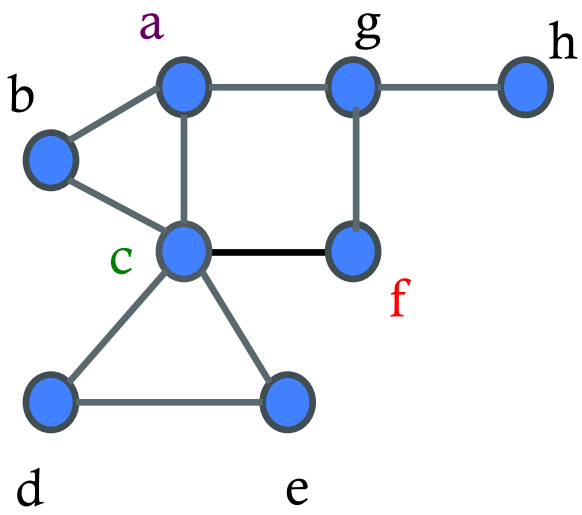


$X_t = \{d,e,c\} \subseteq V$

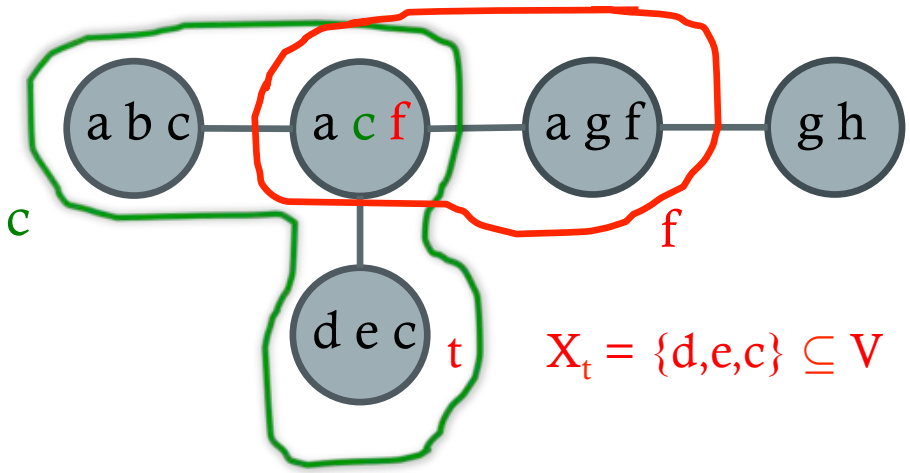
- $\cup_t X_t = V$
- *For each* $v \in V$, $\{ t \mid v \in X_t \}$ *is sub-tree of* T
- For each edge $uv \in E$, exists t such that $u,v \in X_t$

Tree Decomposition

G=(V,E)



$T=(V_T, E_T)$

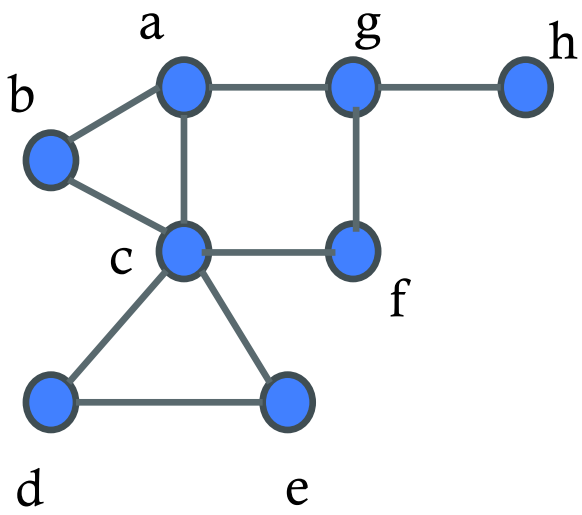


$X_t = \{d,e,c\} \subseteq V$

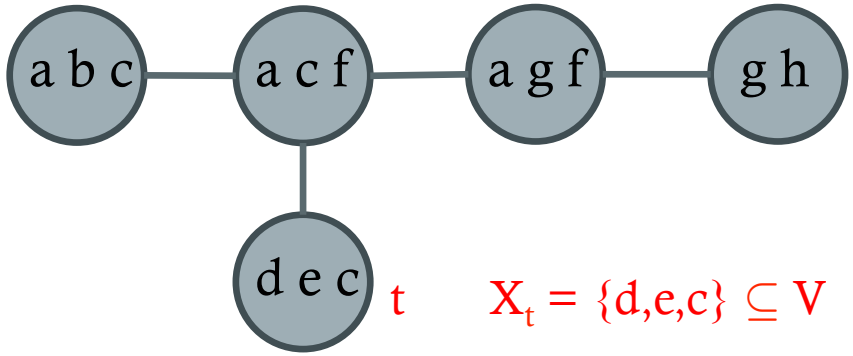
- $\cup_t X_t = V$
- For each $v \in V$, $\{ t \mid v \in X_t \}$ is sub-tree of T
- *For each edge $uv \in E$, exists t such that $u,v \in X_t$*

Treewidth

G=(V,E)



$T=(V_T, E_T)$



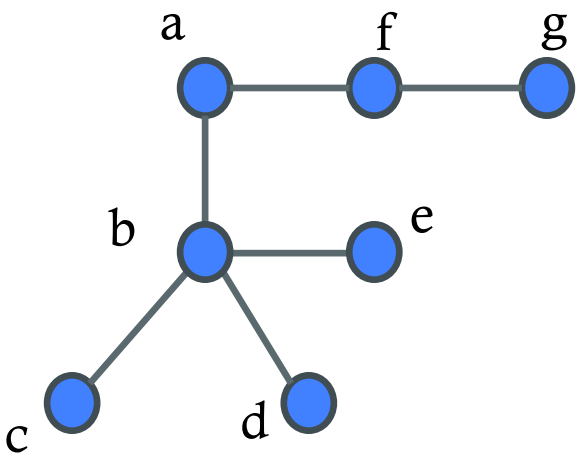
$X_t = \{d,e,c\} \subseteq V$

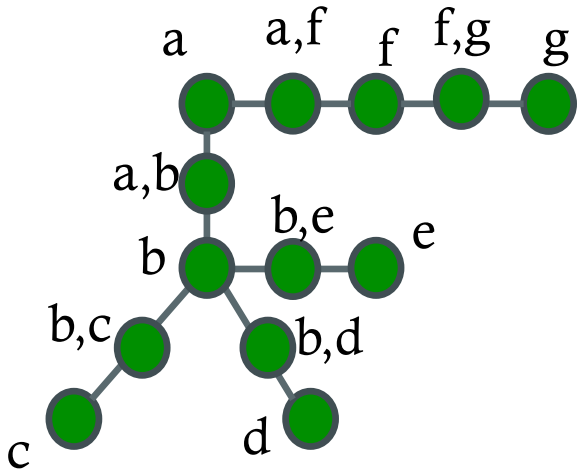
Width of decomposition := $\max_t |X_t|$

tw(G) = (min width of tree decomp for G) – 1

Example: tree

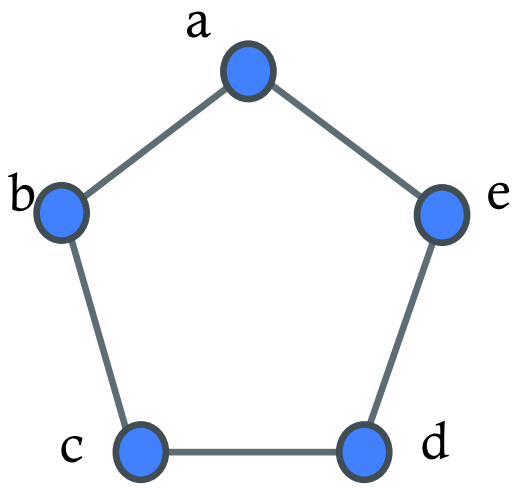
tw(Tree) = 1

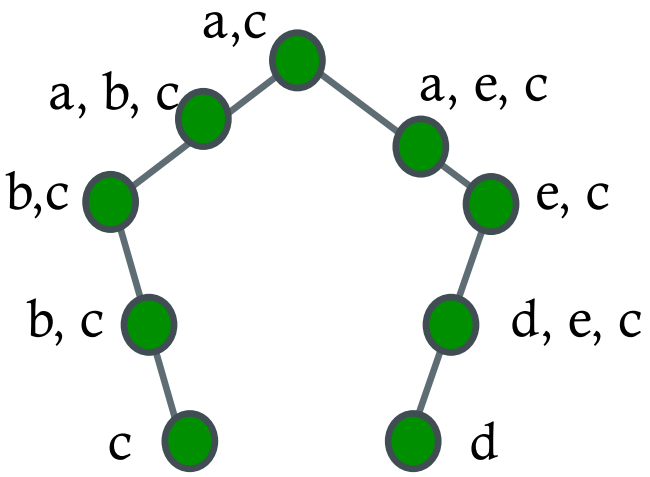




Example: cycle

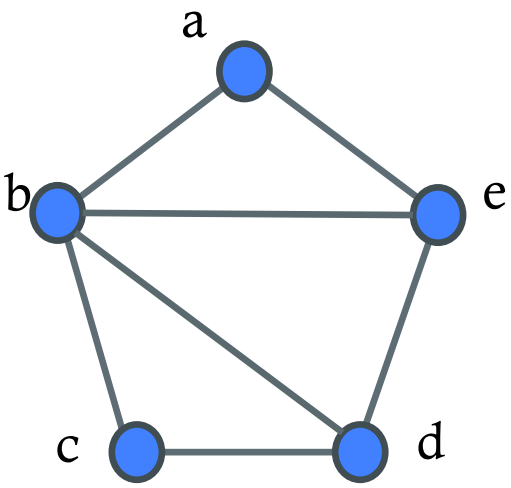
tw(Cycle) = 2



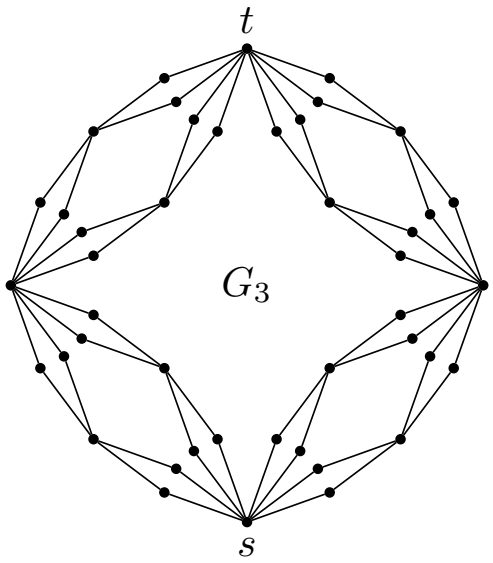


Example: series-parallel

$tw(G) \leq 2 \Leftrightarrow G$ is series-parallel (a sub-class of planar graphs)



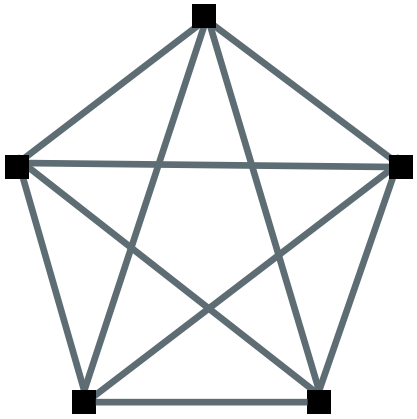
Outerplanar graph

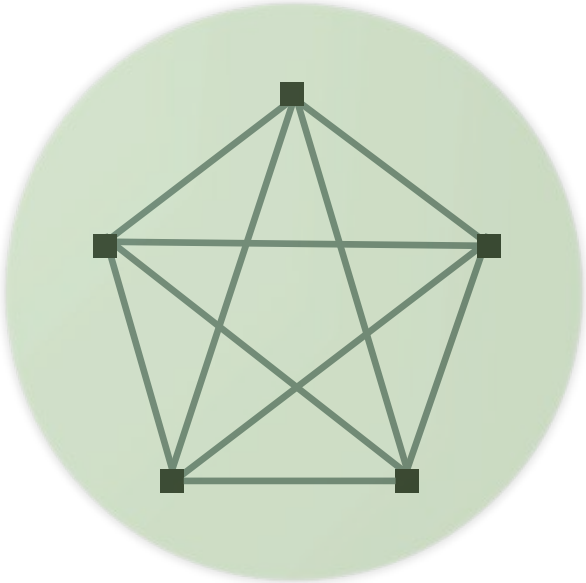


Diamond graph. Figure from Serge Gasper's paper

Example: clique

$tw(K_n) = n-1$



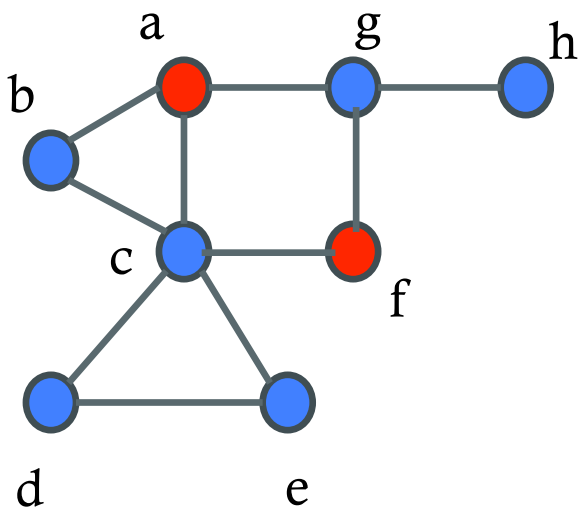


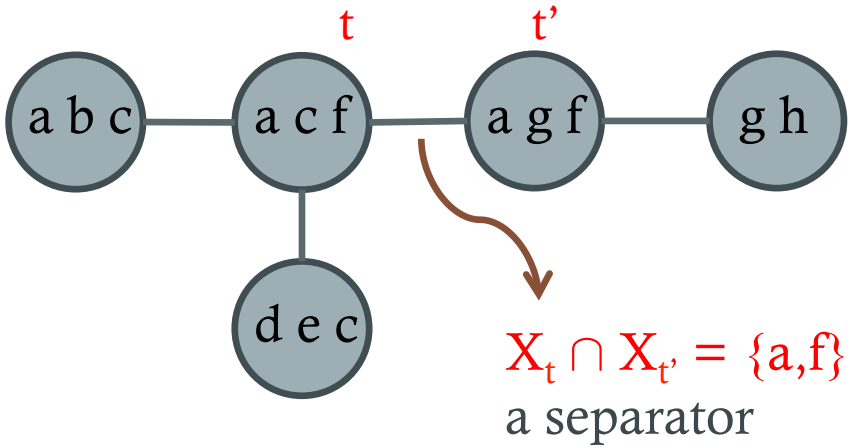
Treewidth and separators

$tw(G) \leq k$ implies G can be *recursively decomposed* via "balanced" separators of size k

Approximate converse also holds: If there is a subgraph H of G with no balanced separator of size k then $tw(G) \geq k/c$

Treewidth and separators





For every edge (t,t') in tree decomposition $X_t \cap X_{t'}$ is a separator of G

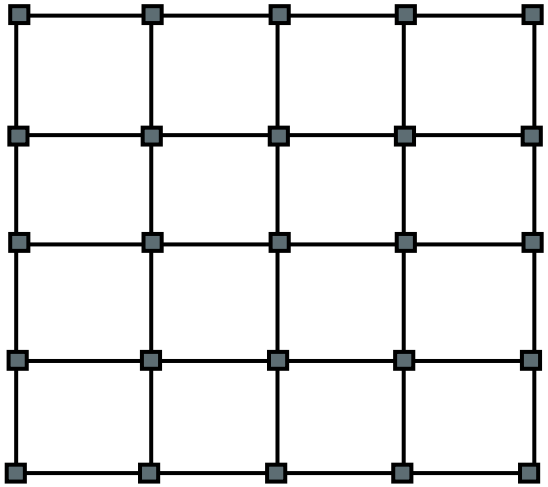
Recursive decomposition

$tw(G) \leq k$ implies G can be *recursively decomposed* via "balanced" separators of size k

- $tw(G) \leq k$ implies G has a balanced separator S of size k
- Recursively decompose graphs in $G - S$
  - $tw(H) \leq tw(G)$ for any subgraph H of G

Example: grid

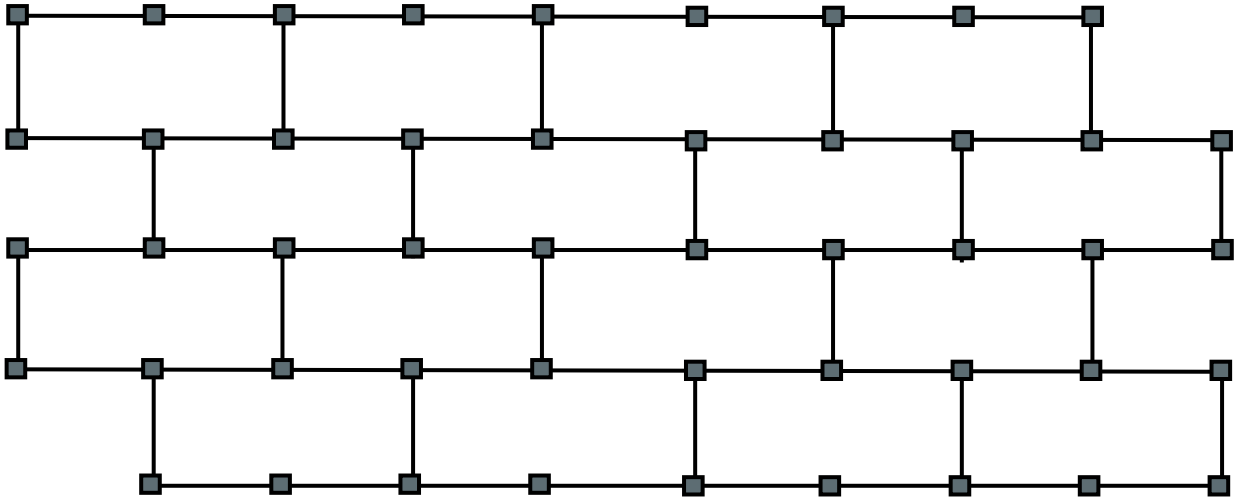
- k x k grid: tw(G) = k-1



- tw(G) = $O(n^{1/2})$ for any planar G (via [Lipton-Tarjan])

Example: wall

- k wall: $tw(G) = \Theta(k)$



wall is *degree 3* planar graph

Example: random graph

Random d-regular graph: $tw(G) = \Theta(n)$ with high prob

Recall treewidth of complete graph is $n-1$

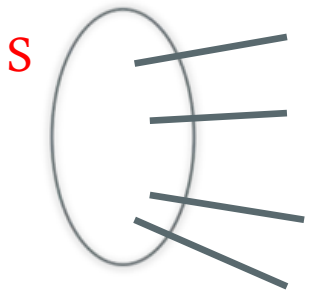
Reason for large treewidth:

random graph is an *expander* whp

balanced separators in expanders have size $\Omega(n)$

Example: expander

Graph $G=(V,E)$ is an *expander* if $|\delta(S)| \geq |S|$ for every $S \subset V$, $|S| \leq n/2$,



Degree 3 expanders exist

Treewidth and Sparsity

- Small treewidth implies sparsity
  - $tw(G) \leq k$ implies *average degree* is $O(k)$
- Converse does not hold
  - Degree 3 wall has treewidth $\Omega(\sqrt{n})$
  - Degree 3 expander has treewidth $\Omega(n)$

Complexity of Treewidth

[Arnborg-Corneil-Proskurowski'87]

Given G, k checking if $tw(G) \leq k$ is NP-Complete

[Bodleander'93]

$O(k^{k^3} n)$ time algorithm to check if $tw(G) \leq k$
(for fixed k, *linear time*)

[Bodleander et al' 2013]

$O(c^k n)$ time 5-approximation

Complexity of Treewidth

α-approx. for node separators implies $O(\alpha)$-approx. for treewidth

[Feige-Hajiaghayi-Lee'05]

Polynomial time algorithm to output tree decomposition of width

$$\leq c\ tw(G)\ \sqrt{\log tw(G)}$$

Applications of Treewidth

- Graph Theory
- Polynomial-time algorithms for problems on graphs/ structures with *bounded*/*fixed* treewidth
- Fixed parameter tractability
- Approximation algorithms

Treewidth "template" for applications

- If G has **"small"** (constant) treewidth, solve problem via dynamic programming.
- If G has **"large"** treewidth use *structure*, in particular, obstructions such as grids
  - Answer is clear from obstruction

    or
  - "Reduce" problem in some fashion and recurse

Outline

- **Topic I**: Leveraging small treewidth
  - dynamic programming based algorithms
  - reducing to small treewidth
- **Topic II**: Interplay of small and large treewidth
  - fixed parameter intractability
- **Topic III**: Large treewidth for approximation
  - disjoint paths and recent developments on structure

Algorithms for bounded/small treewidth graphs

Dynamic programming based algorithms for trees extends naturally to bounded treewidth graphs

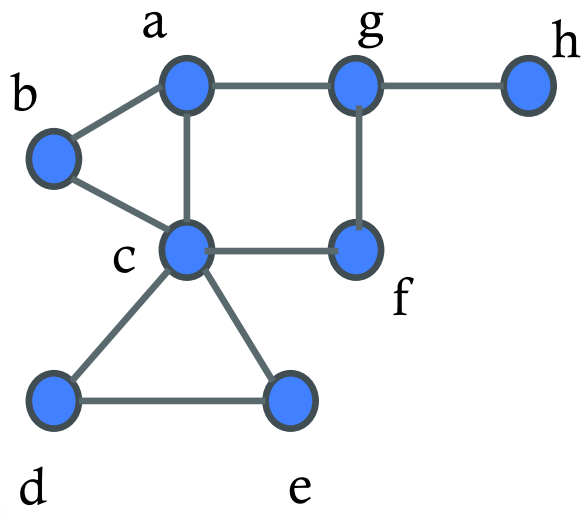
Consequence:

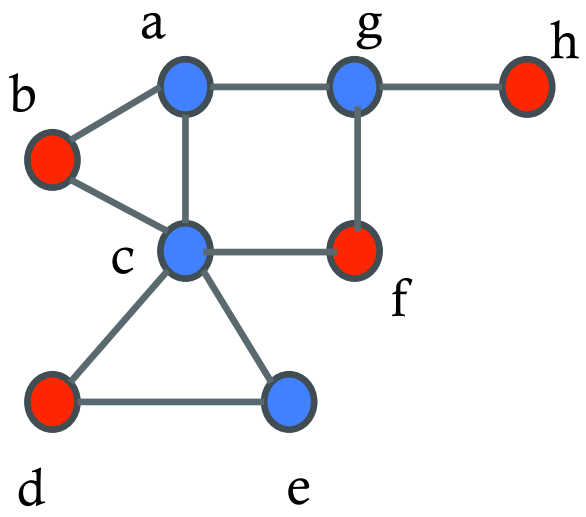
Many hard problems can be solved efficiently in graphs of small treewidth

Maximum (Weight) Independent Set Problem

Max (Weight) Independent Set Problem (MWIS):

Given graph G=(V,E) and weights w: V $\rightarrow$ R output max w(S) such that S $\subset$ V is an *independent set*





Maximum (Weight) Independent Set Problem

Negative results:

MIS is NP-Hard (even in planar graphs)

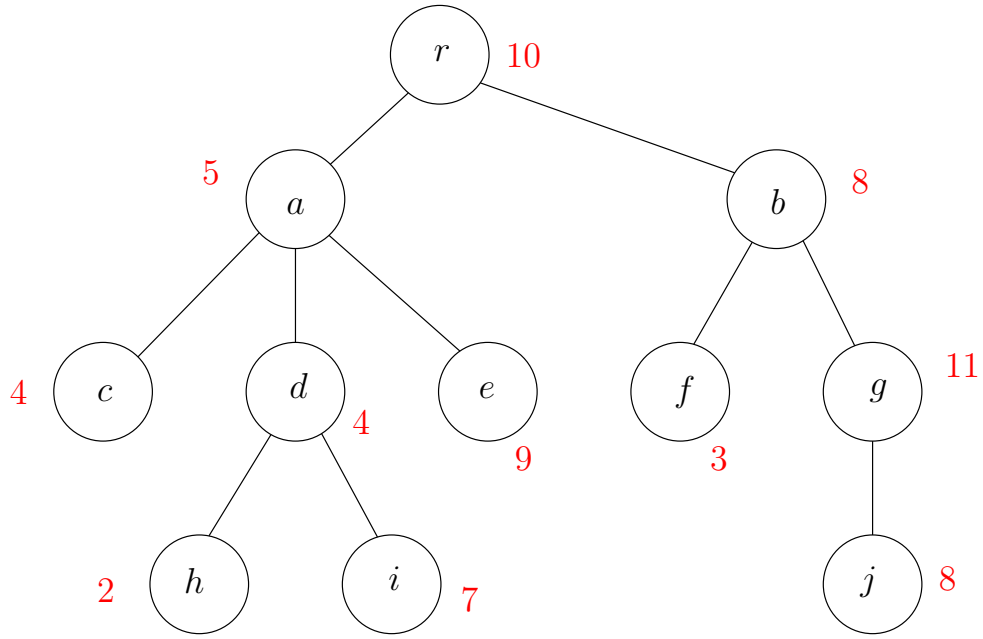
MIS is very hard *even to approximate* in general graphs

Some positive results:

MIS is poly-time solvable in bounded treewidth graphs

For every $\epsilon > 0$ a $(1-\epsilon)$-approximation in planar graphs

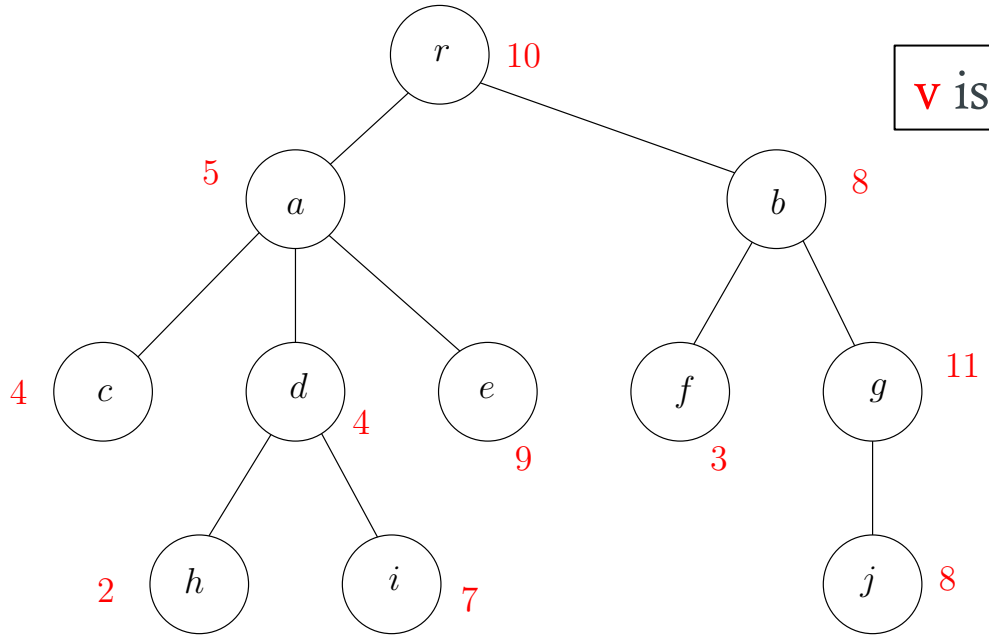
MWIS in Trees



T_v: subtree of T rooted at node v

$OPT(v)$: optimum value of MWIS in T_v

MWIS in Trees

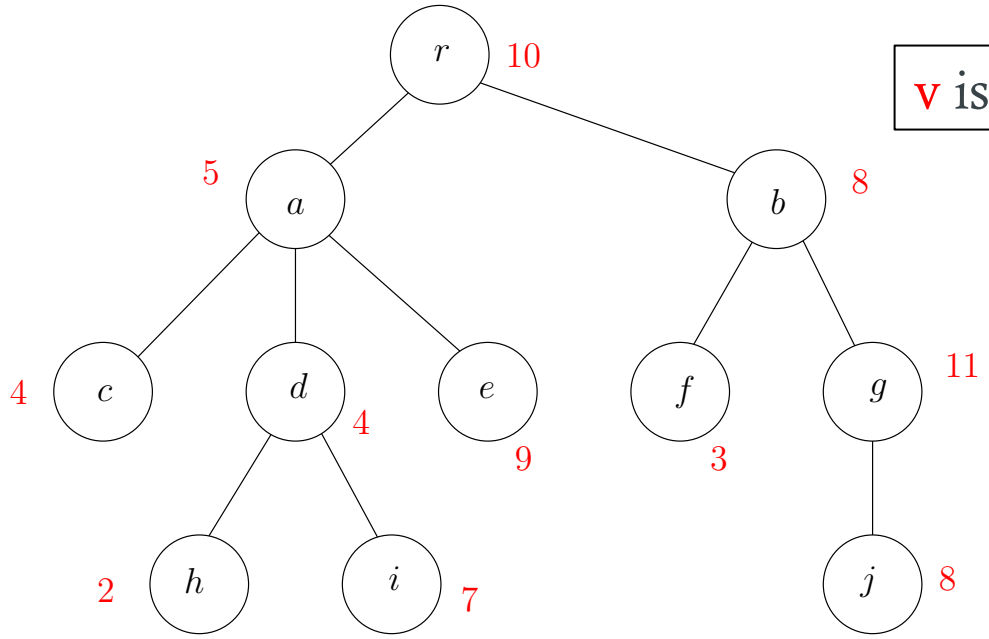


v is a separator for T_v

OPT(v, 1): optimum value of MWIS in T_v that includes v
OPT(v, 0): optimum value of MWIS in T_v that does NOT include v

OPT(v) = max { OPT(v, 1), OPT(v, 0) }

MWIS in Trees

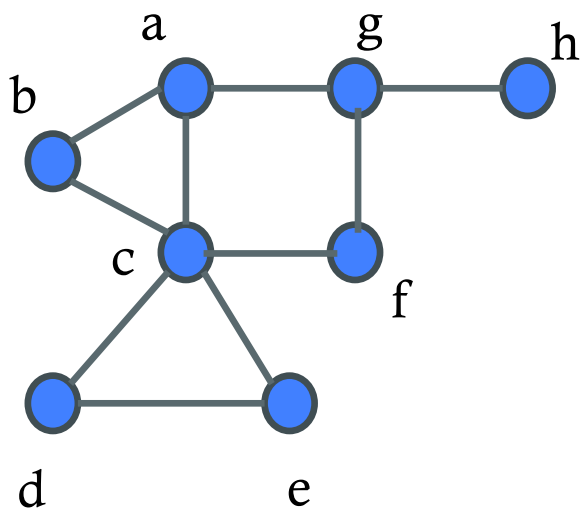


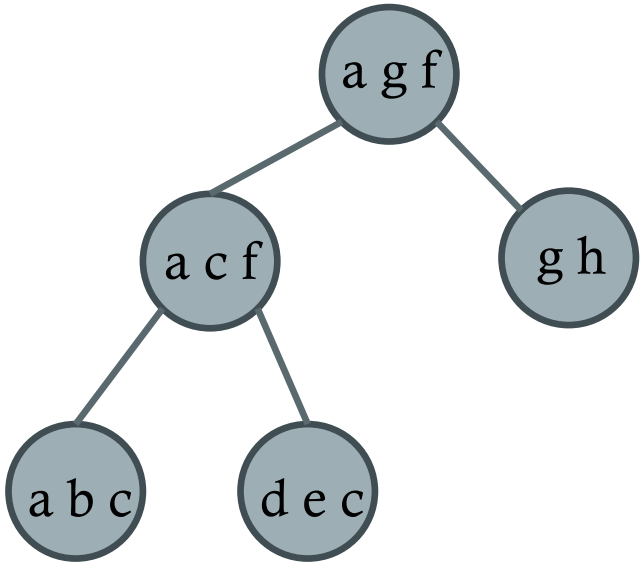
v is a separator for T_v

$OPT(v, 1) = w(v) + \sum_{u \text{ child of } v} OPT(u, 0)$

$OPT(v, 0) = \sum_{u \text{ child of } v} OPT(u)$

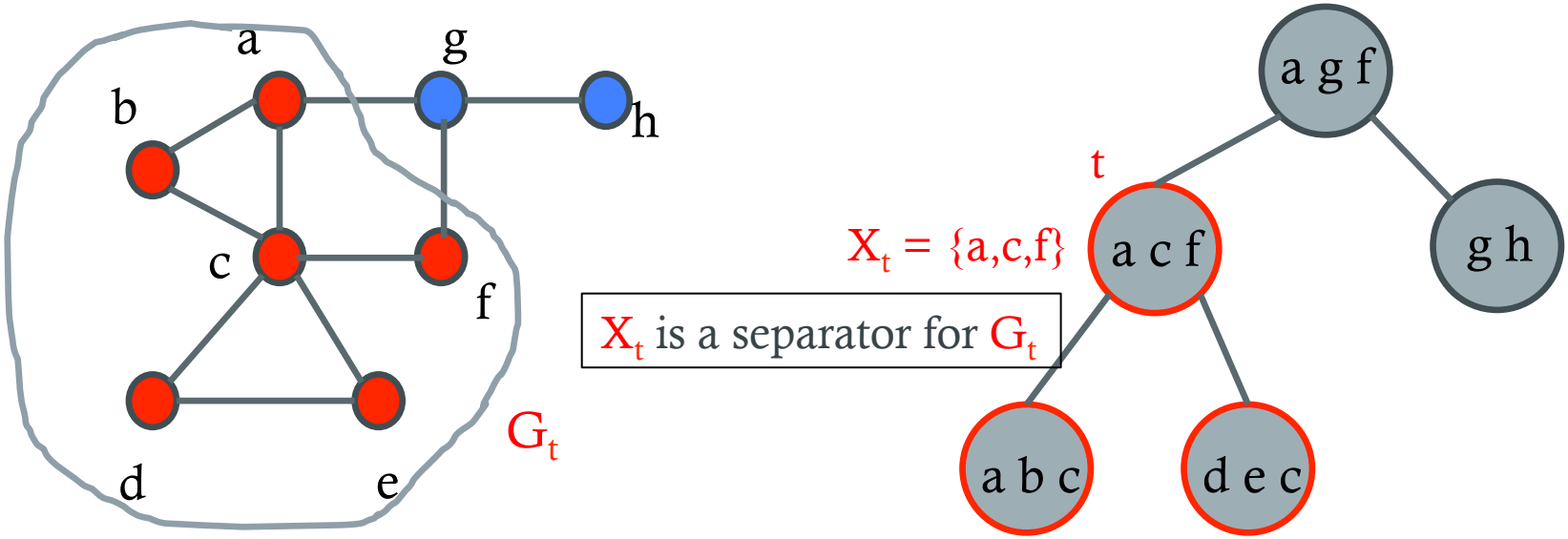
MWIS and Tree Decompositions





Dynamic programming over tree decomposition

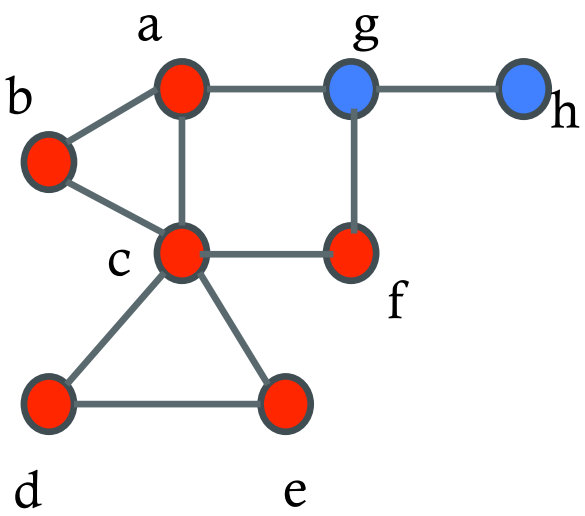
MWIS and Tree Decompositions

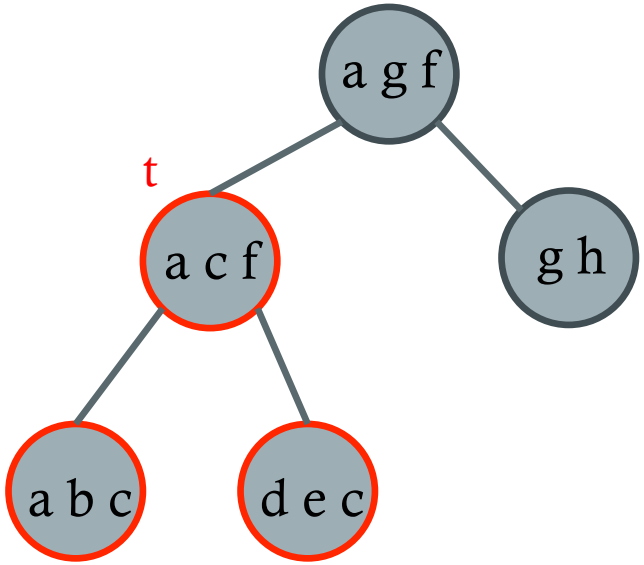


For t in T, G_t is subgraph of G induced by nodes in bags of T_t

X_t nodes in bag at t

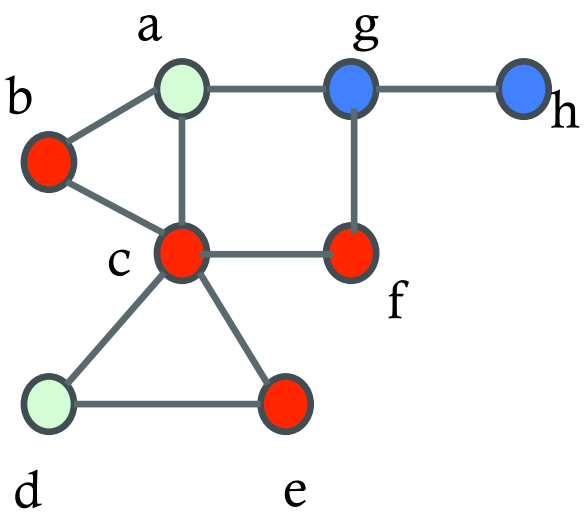
MWIS and Tree Decompositions



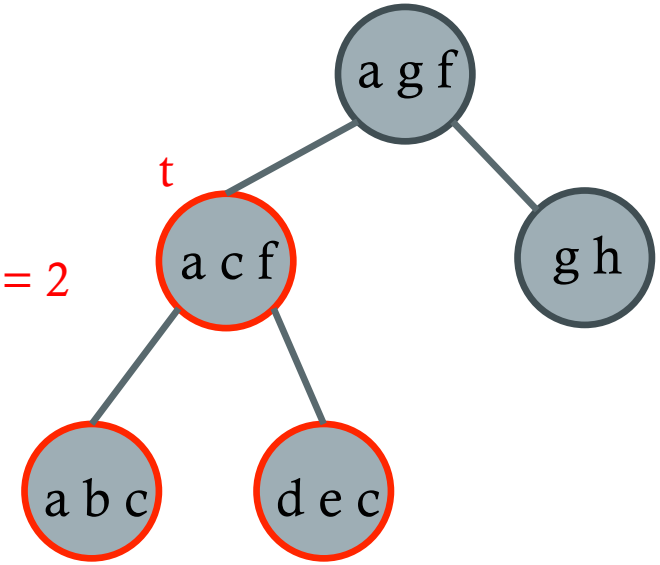


OPT(t, S): value of MWIS in G_t among indep sets I s.t $I \cap X_t = S$

MWIS and Tree Decompositions



OPT(t, {a}) = 2



OPT(t, S): value of MWIS in G_t among indep sets I s.t $I \cap X_t = S$

MWIS and Tree Decompositions

- OPT(t, S): max MWIS among independent sets I s.t $I \cap X_t = S$
- # of values to compute at each node is $\leq 2^{k+1}$ where k is width of decomposition
- Can compute all values from leaves to root in $O(k\ 2^{k+1}\ N)$ time where N is # of nodes in T

MWIS and Tree Decompositions

Consequence:

Given tree decomposition of width k for a graph G on n nodes MWIS can be computed in $O(k\, 2^{k+1}\, n)$ time

Polynomial-time for any *fixed* k

$2^{O(\sqrt{n})}$ time algorithm for planar graphs (can also be seen via the planar separator theorem)

Application: SAT

SAT: Is given CNF formula ϕ satisfiable?

#SAT: Count the # of satisfying assignments to ϕ

ϕ is a conjunction of clauses

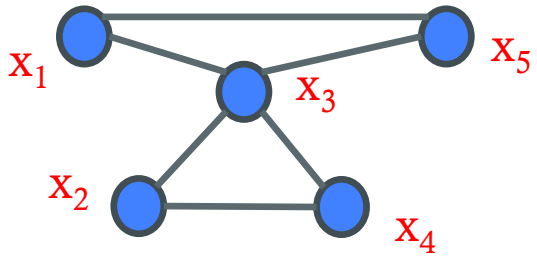
$(x_1 \vee x'_3 \vee x_5)\,(x_4 \vee x'_5)\,(x_2 \vee x'_3 \vee x_4 \vee x'_5)\,(x'_1 \vee x_4)$

Primal Graph

Given ϕ create graph $G_p(\phi)$

- one vertex per variable
- edge between two variables if they occur in a clause

$(x_1 \vee x'_3 \vee x_5)\ (x_2 \vee x'_3 \vee x'_4)\ (x'_2 \vee x_4)\ (x_3 \vee x'_5)$

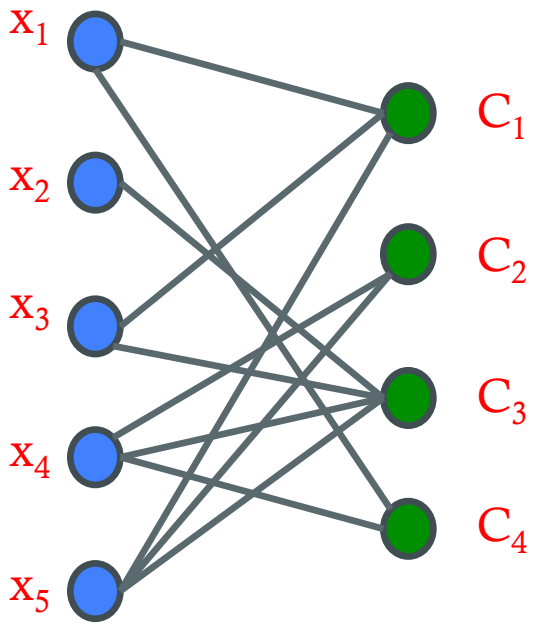


Incidence Graph

$(x_1 \vee x'_3 \vee x_5)\,(x_4 \vee x'_5)\,(x_2 \vee x'_3 \vee x_4 \vee x'_5)\,(x'_1 \vee x_4)$

Bipartite graph $G_i(\phi)$

- one vertex for each variable
- one vertex for each clause
- edge from variable to clause if variable occurs in clause



SAT

$O(c^k \ size(\phi))$ time algorithm for SAT and #SAT where $k = tw(G_p(\phi))$ **or** $k = tw(G_i(\phi))$

Question: which graph is better to use?

SAT

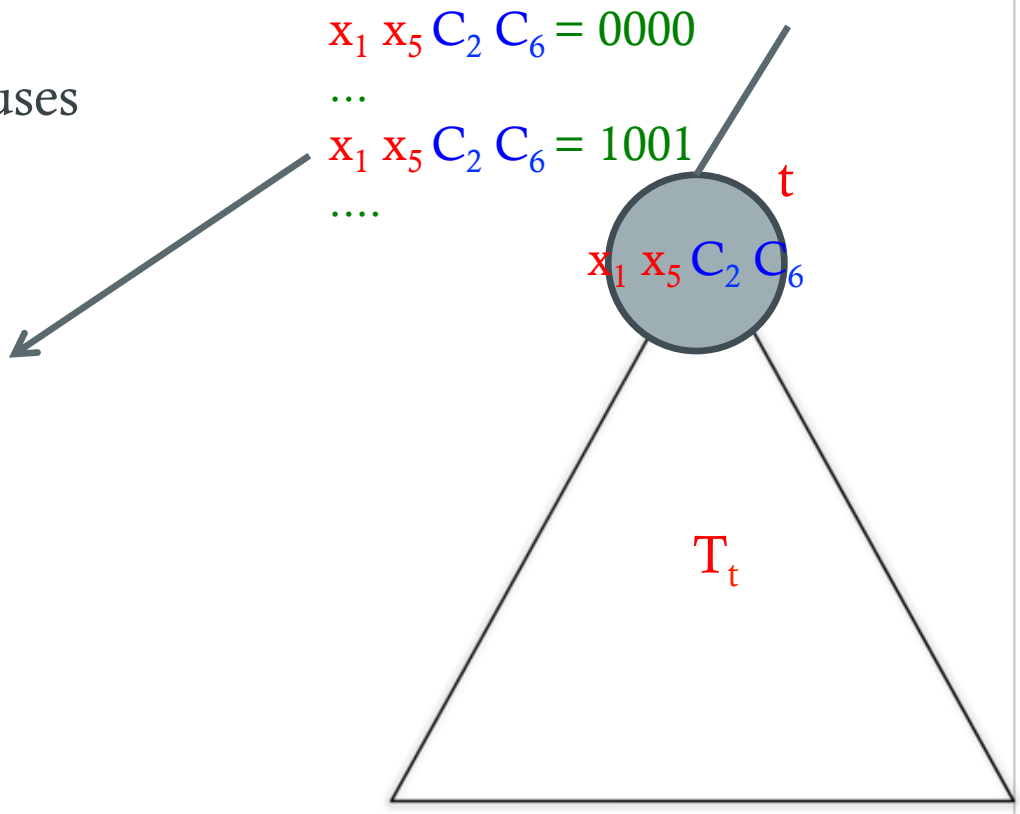
$O(c^k \ size(\phi))$ time algorithm for SAT and #SAT where $k = tw(G_p(\phi))$ **or** $k = tw(G_i(\phi))$

Question: which graph is better to use?

- $tw(G_i(\phi)) \leq tw(G_p(\phi)) + 1$
- Simple examples: $tw(G_i(\phi))=1$ and $tw(G_p(\phi)) = n-1$

Dynamic Prog for SAT

Bag X_t contains variables & clauses



Is there an extension of $x_1\ x_5 = 10$ to variable in T_t s.t

- all clauses *properly contained* in T_t are satisfied
- C_6 in X_t **is** satisfied
- C_2 in X_t **may not** be satisfied

Application: Graphical Models

Inference in Graphical Models:

- Bayesian networks (directed acyclic graphs)
- Markov random fields (undirected)

Small treewidth of underlying graphs implies efficient algorithm via dynamic programming (and variants of belief propagation)

Many NP-Hard problems can be solved in poly-time on graphs of bounded treewidth

- minimum dominating set
- chromatic number
- Hamilton cycle/TSP
- minimum cost Steiner tree
- …

Question: which problems can be solved?

Courcelle's Theorem

A meta-algorithmic result via logic:

[Courcelle'90]

Any property ϕ of graphs expressible in $EMSO_2$ logic can be checked in time $f(|\phi|, k)\, n$ on an n node graph G given a tree decomposition of width k for G. Here f is some computable function.

Various extensions of above for optimization/counting and related problems.

Summary

Graph/Structure has small/bounded treewidth

implies

efficient/poly-time algorithm for many intractable problems

Next: leveraging bounded treewidth graphs for more general graphs

MWIS in Planar Graphs

MWIS is exactly solvable in bounded treewidth graphs

Can we extend ideas to broader class of graphs?

Approximation Algorithm

Approximation algorithm for optimization problem Π

- a worst-case polynomial time algorithm
- gives a worst-case guarantee on the output of solution
  - $\mathcal{A}(I)$ – value of solution output by $\mathcal{A}$ on instance I
  - $OPT(I)$ – value of an optimum solution for I
  - For maximization: $\mathcal{A}(I) \geq \alpha\, OPT(I)$ *for all* I
  - α – the approximation ratio of $\mathcal{A}$

MWIS in Planar Graphs

[Baker'94]

There is a *polynomial-time approximation scheme* (PTAS) for MWIS in **planar graphs**. Given $\epsilon > 0$,

- Algorithm runs in $2^{O(1/\epsilon)}$ poly(n) time
- Gives a $(1-\epsilon)$ approximation to MWIS

Decomposing Planar Graphs

[Baker'94]

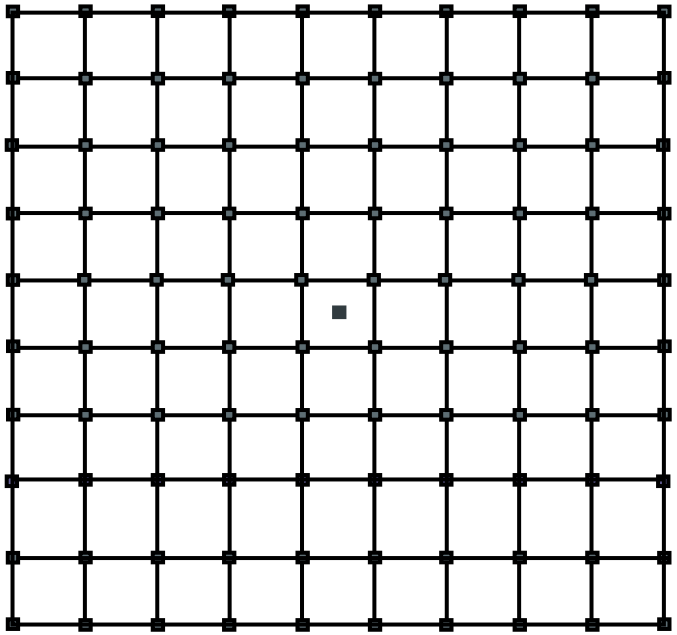
$G=(V,E)$ planar graph, h any non-negative integer

Can efficiently partition V into $V_1, \ldots, V_h$ such that for *any* $1 \leq i \leq h$

- $G_i = G - V_i$ has treewdith at most $O(h)$

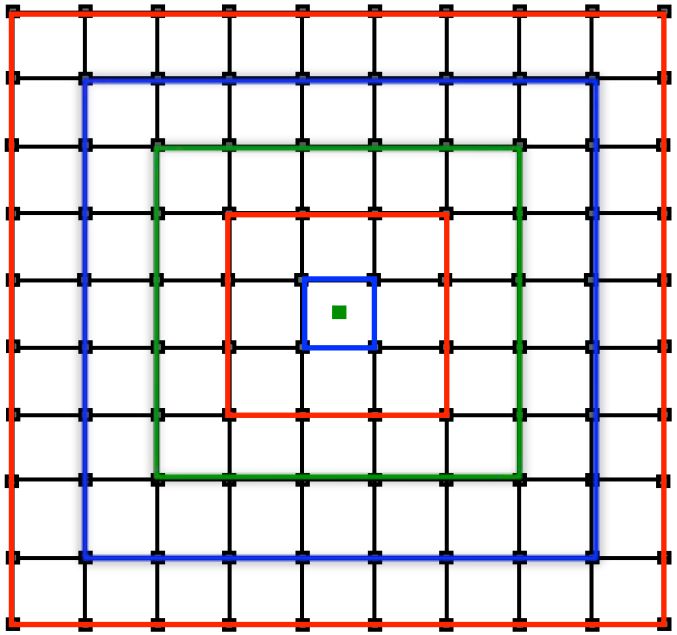
Baker's Decomposition

h = 3



Baker's Decomposition

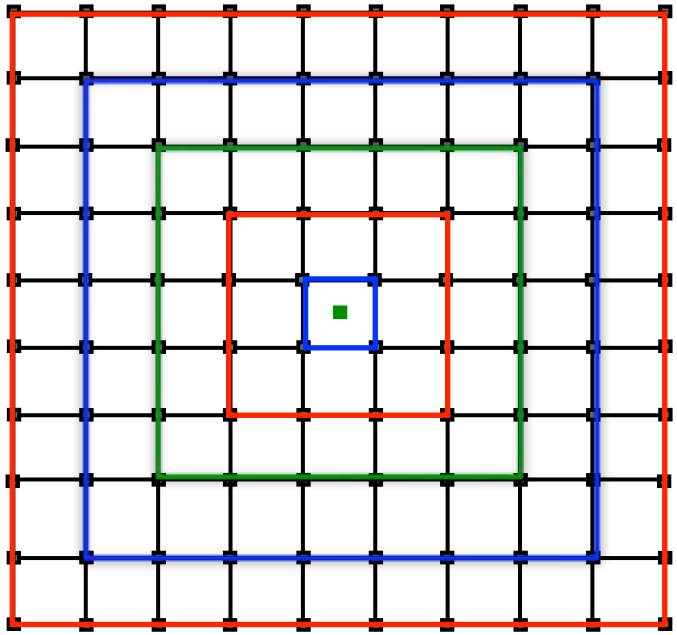
h = 3



Baker's Decomposition

Removing any color leaves disconnected graphs each of which is a grid-strips of h-1 layers

Such a graph is (h-1)-**outerplanar** and has treewidth $\leq 3h$



Decomposition to PTAS

Can efficiently partition V into $V_1, \ldots, V_h$ such that for any $1 \leq i \leq h$, $G_i = G - V_i$ has treewdith at most $O(h)$

1. Choose partition for $h = 1/\epsilon$
2. for $i = 1$ to h do
   - Find optimum solution S_i in $G_i = G - V_i$
3. Output S, best of $S_1, S_2, \ldots, S_h$

PTAS

Claim: Algorithm runs in time $O(2^{O(h)} n)$

Claim: Output S satisfies $|S| \geq (1-1/h)\,OPT$

Some j such that $OPT(G - V_j) \geq (1 - 1/h)OPT$

Algorithm find's optimum solution for each j

Power of Baker: PTASes Galore

Baker's ideas and techniques have been generalized and extended to obtain PTASes:

- for H-minor free graphs for any fixed H substantially generalizing results for planar graphs
- graphs of bounded "local treewidth"
- *large* number of optimization problems

Summary

Bounded treewidth results can be leveraged to provide algorithms/heuristics for *much larger and useful* classes of graphs

Outline

- **Topic I**: Leveraging small treewidth
  - dynamic programming based algorithms
  - reducing to small treewidth
- **Topic II**: Interplay of small and large treewidth
  - fixed parameter intractability
- **Topic III**: Large treewidth for approximation
  - disjoint paths and recent developments on structure

Small to Large Treewidth

Important applications require a fine/deep understanding of structure of **large** treewidth graphs

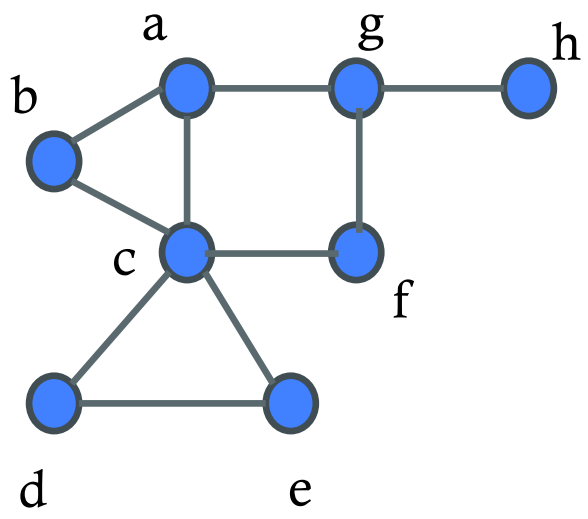
Robertson-Seymour theory provides many powerful tools

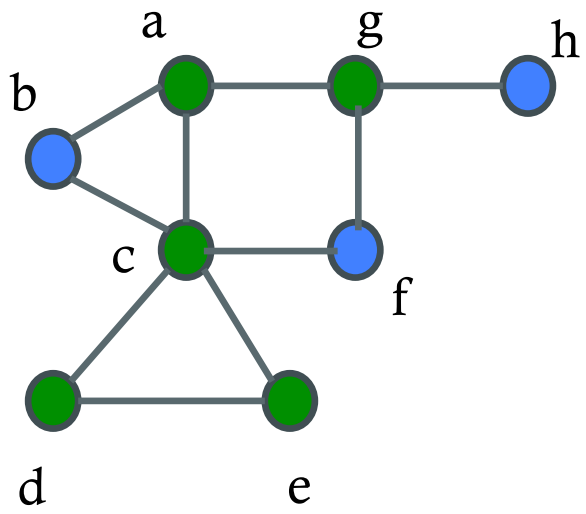
Fixed Parameter Tractability

Vertex Cover

Vertex Cover: Given G=(V,E), k does G have a vertex cover of size $\leq$ k?

$S \subseteq V$ is a vertex cover if S covers all edges





Vertex Cover

Vertex Cover: Given $G=(V,E)$, k does G have a vertex cover of size $\leq k$?

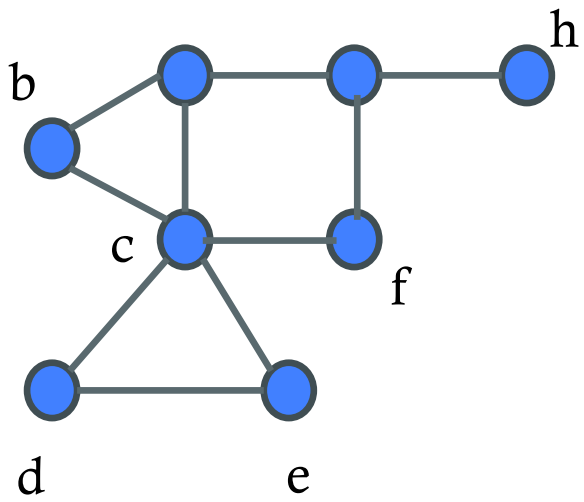
NP-Complete if k is part of input

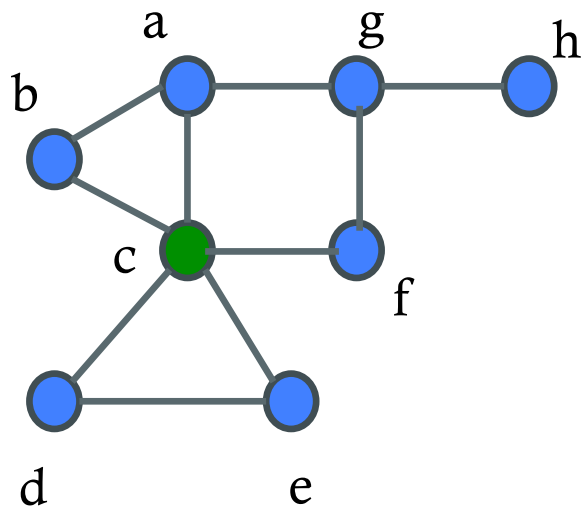
Fact: There is an algorithm that runs in $c^k \text{ poly}(n)$ for Vertex Cover where c is some fixed constant

Feedback Vertex Set

Feedback Vertex Set: Given G=(V,E), k does G have a feedback vertex set (FVS) of size $\leq$ k?

$S \subseteq V$ is a FVS if G – S has no cycles (S kills all cycles)





Feedback Vertex Set

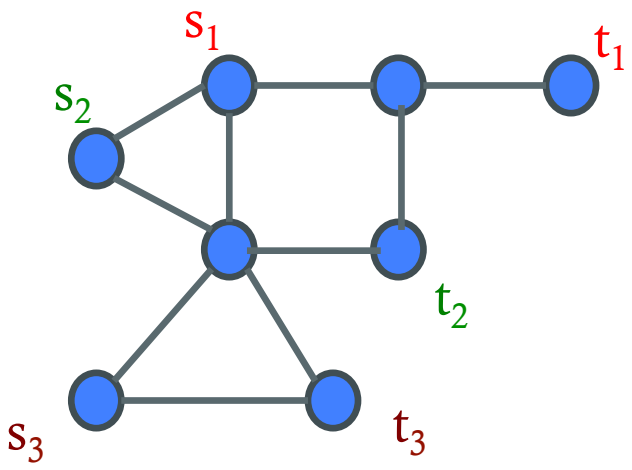
Feedback Vertex Set: Given $G=(V,E)$, k does G have a feedback vertex set (FVS) of size $\leq k$?

NP-Complete if k is part of input

Fact: There is an algorithm that runs in c^k poly(n) for FVS where c is some fixed constant

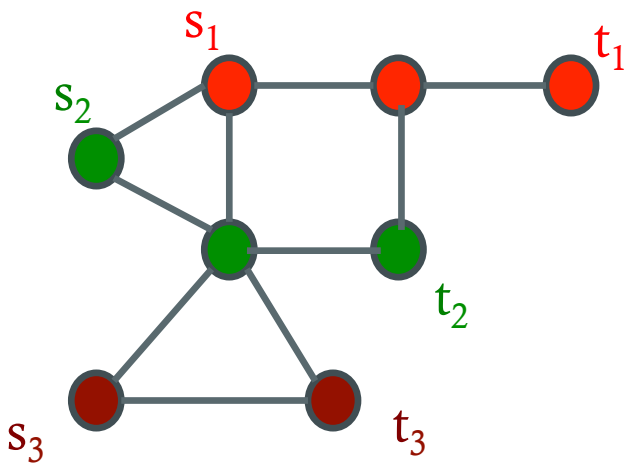
Disjoint Paths Problem

Given $G=(V,E)$ and pairs $(s_1,t_1),\ldots,(s_k,t_k)$ are there **disjoint** paths connecting given pairs



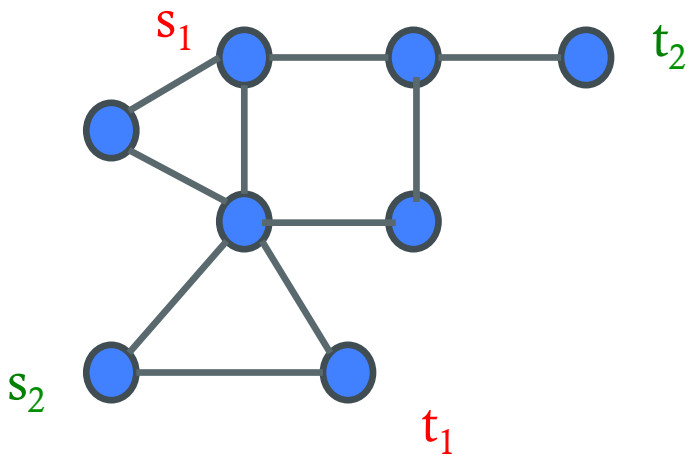
Disjoint Paths Problem

Given $G=(V,E)$ and pairs $(s_1,t_1),\ldots,(s_k,t_k)$ are there **disjoint** paths connecting given pairs



Disjoint Paths Problem

Given $G=(V,E)$ and pairs $(s_1,t_1),\ldots,(s_k,t_k)$ are there **disjoint** paths connecting given pairs



Disjoint Paths Problem

Given $G=(V,E)$ and pairs $(s_1,t_1),\ldots,(s_k,t_k)$ are there **disjoint** paths connecting given pairs

$k = 1$ Is there a path from s to t? Easy

$k = 2$ NP-Complete in directed graphs! [FHW'80]

NP-Complete if k is part of input in undir graphs

[Robertson-Seymour] Poly-time solvable for any fixed k in $O(n^3)$ time in *undirected* graphs

Fixed Parameter Tractability

Fixed Parameter Tractable: has algorithm with run-time

$$f(k)\,\mathrm{poly}(n)$$

where k is **parameter** size and n is instance size

- Many different parameterizations possible for a problem
- Choice depends on application

FPT and Treewidth

Several FPT algorithms can be obtained via treewidth

Generic paradigm:

1. If tw(G) is small use exact algorithm via dynamic programming
2. If tw(G) is large use "structure" of G

FPT and Treewidth

FPT algorithms for Vertex Cover and FVS

1. If $tw(G) \leq g(k)$ solve in $c^{g(k)}\ poly(n)$ time
2. If $tw(G) > g(k)$ answer NO

Caveat: not the most efficient FPT algorithms for these problems

FPT and Treewidth

FPT algorithms for Vertex Cover and FVS

1. If $tw(G) \leq g(k)$ solve in $c^{g(k)}$ poly(n) time
2. If $tw(G) > g(k)$ answer NO

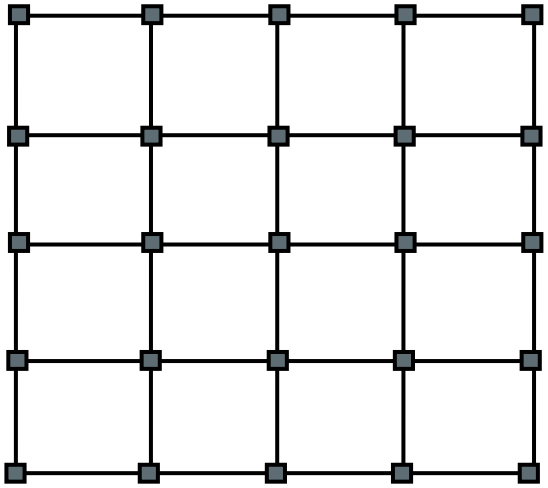
Need to show correctness

Structure of graphs with "large" treewidth

What can we say about a graph with "large" treewidth?

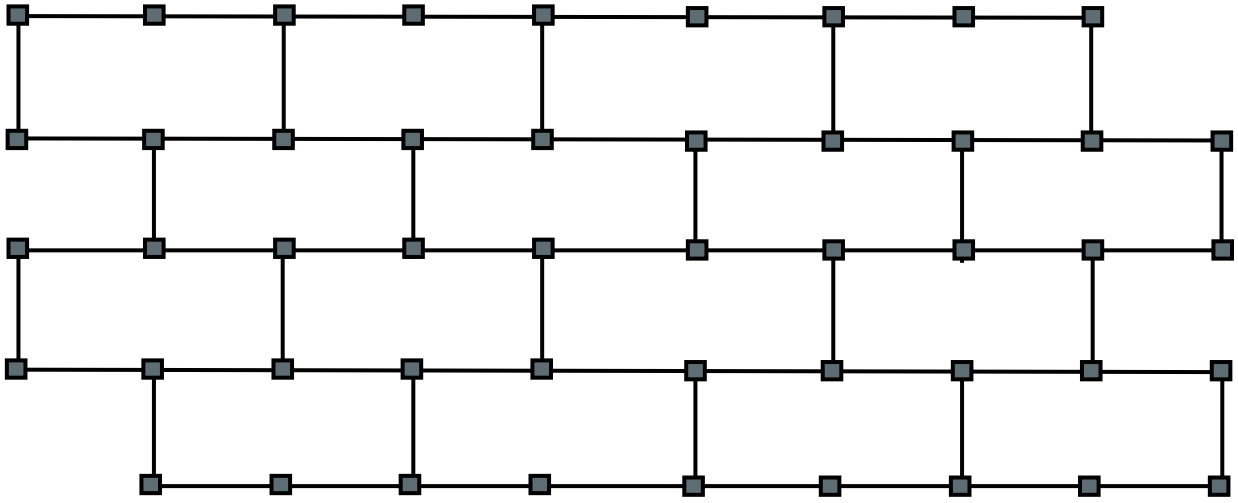
Robertson-Seymour Grid-Minor Theorem

Theorem: There exists f such that $tw(G) \geq f(k)$ implies G contains a grid of size k as a **minor**



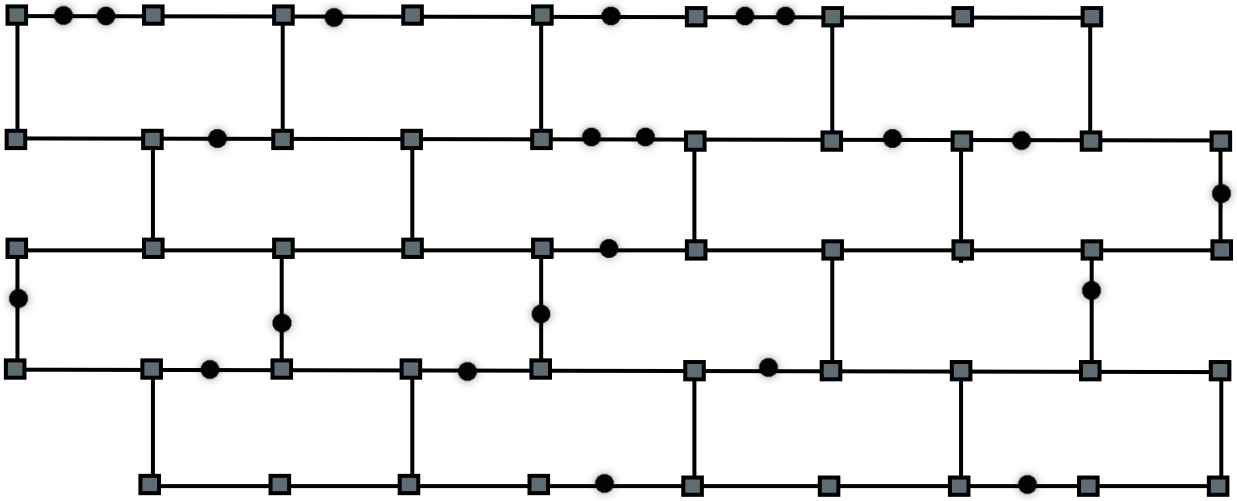
Robertson-Seymour Grid-Minor Theorem

Theorem: There exists f such that $tw(G) \geq f(k)$ implies G contains the *subdivision* of a wall of size k as a subgraph



Robertson-Seymour Grid-Minor Theorem

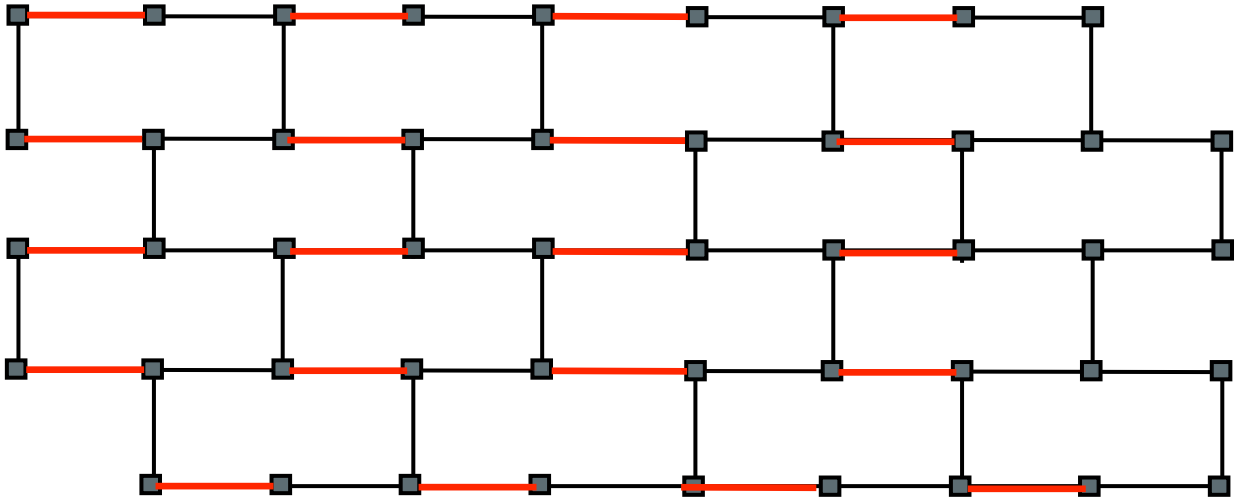
Theorem: There exists f such that $tw(G) \geq f(k)$ implies G contains the *subdivision* of a wall of size k as a subgraph



Back to FPT for VC

Fact: Vertex Cover of k wall is $\Omega(k^2)$

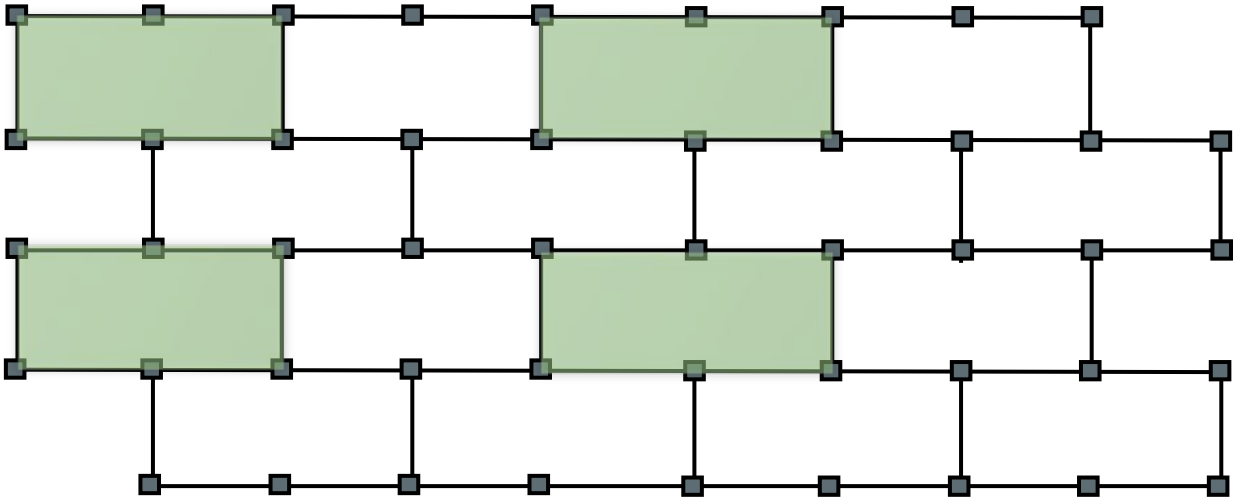
k wall has a matching of size $\Omega(k^2)$



Back to FPT for FVS

Fact: FVS of k wall is $\Omega(k^2)$

k wall has $\Omega(k^2)$ *disjoint* cycles



FPT and Treewidth

FPT algorithms for Vertex Cover and FVS

1. If $tw(G) \leq g(k)$ solve in $c^{g(k)}\ poly(n)$ time
2. If $tw(G) > g(k)$ answer NO

$g(k) = f(c\sqrt{k})$ for appropriate constant c suffices

RS Disjoint Path Algorithm

1. If $tw(G) \leq f(k)$ use dynamic programming
2. Else
   - G has "large" treewidth. Use heavy machinery of graph minor structure theory to find in polynomial time an "**irrelevant vertex**" v
   - Pairs routable in G *iff* they are routable in G – v
   - Recurse on G - v

RS Disjoint Paths Algorithm

1. If $tw(G) \leq f(k)$ use dynamic programming
2. Else
   - G has "large" treewidth. Use heavy machinery of graph minor structure theory to find in polynomial time an "**irrelevant vertex**" v
   - Pairs routable in G *iff* they are routable in G – v
   - Recurse on G – v

Algorithm/proof requires full power of graph minor machinery. No other algorithmic approach known yet

Summary

Important applications require a fine/deep understanding of structure of large treewidth graphs

Robertson-Seymour theory provides many powerful tools

Quantitative bounds are weak, proofs are hard & long

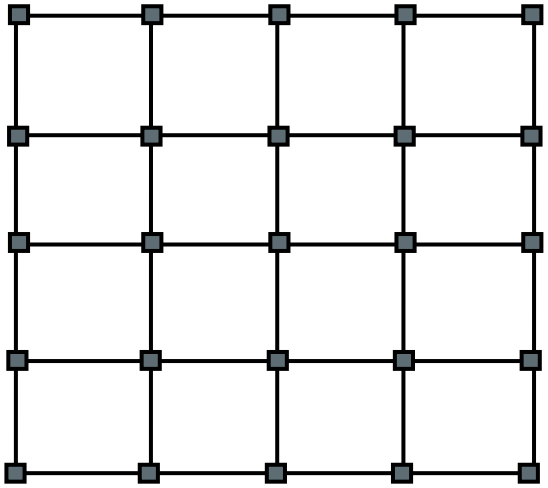
Substantial (ongoing) work on improving bounds, simplifying proofs, and algorithmic applications

Outline

- **Topic I**: Leveraging small treewidth
  - dynamic programming based algorithms
  - reducing to small treewidth
- **Topic II**: Interplay of small and large treewidth
  - fixed parameter intractability
- **Topic III**: Large treewidth for approximation
  - disjoint paths and recent developments on structure

Robertson-Seymour Grid-Minor Theorem

Theorem: There exists f such that $tw(G) \geq f(k)$ implies G contains as a **minor** a grid of size k



Bounds for Grid Minor Theorem

[Robertson-Seymour]: f is "enormous"

[Robertson-Seymour-Thomas]: $f(k) \leq 2^{c\,k^5}$

[Leaf-Seymour,Kawarabayashi-Kobayashi'12]:

$$f(k) \leq 2^{c\,k^2 \log k}$$

[Robertson-Seymour-Thomas]: If G is planar $f(k) \leq 6k$

Recent Improvement

[C-Chuzhoy'13]

Theorem: $tw(G) \geq k^{98+o(1)}$ implies that G has a grid-minor of size $k \times k$. Also a poly-time algorithm.

First polynomial relationship between treewidth and grid-minor size

Recent Improvement

[C-Chuzhoy'13]

Theorem: $tw(G) \geq k^{98+o(1)}$ implies that G has a grid-minor of size $k \times k$. Also a poly-time algorithm.

Previously: $tw(G) = h$ implies grid of size $< \sqrt{\log h}$

Now: $tw(G) = h$ implies grid of size $h^{1/98}$

Limit: $tw(G) = h$ cannot get grid of size $> \sqrt{(h/\log h)}$

Other Results on Structure of Large Treewidth Graphs

[C-Chuzhoy]

- Large routing structures in large treewidth graphs
  - applications to approximating disjoint paths problems
- Treewidth decomposition theorems
  - applications to fixed parameter tractability
  - applications to Erdos-Posa type theorems
- Treewidth sparsification

Improvements

- parameters in various applications improve from "exponential" to "polynomial" (in some cases to near linear)
- hardness results conditional on poly-sized grid-minor are now "unconditional"
- several technical tools of potential future use

Treewidth and Routing

Disjoint paths problem:

Given $G=(V,E)$ and pairs $(s_1,t_1),\ldots,(s_k,t_k)$ are there **disjoint** paths connecting given pairs

Optimization version: maximize # of pairs routed

NP-Hard when k is part of input even on trees

Can we approximate well?

Multicommodity Flow Relaxation

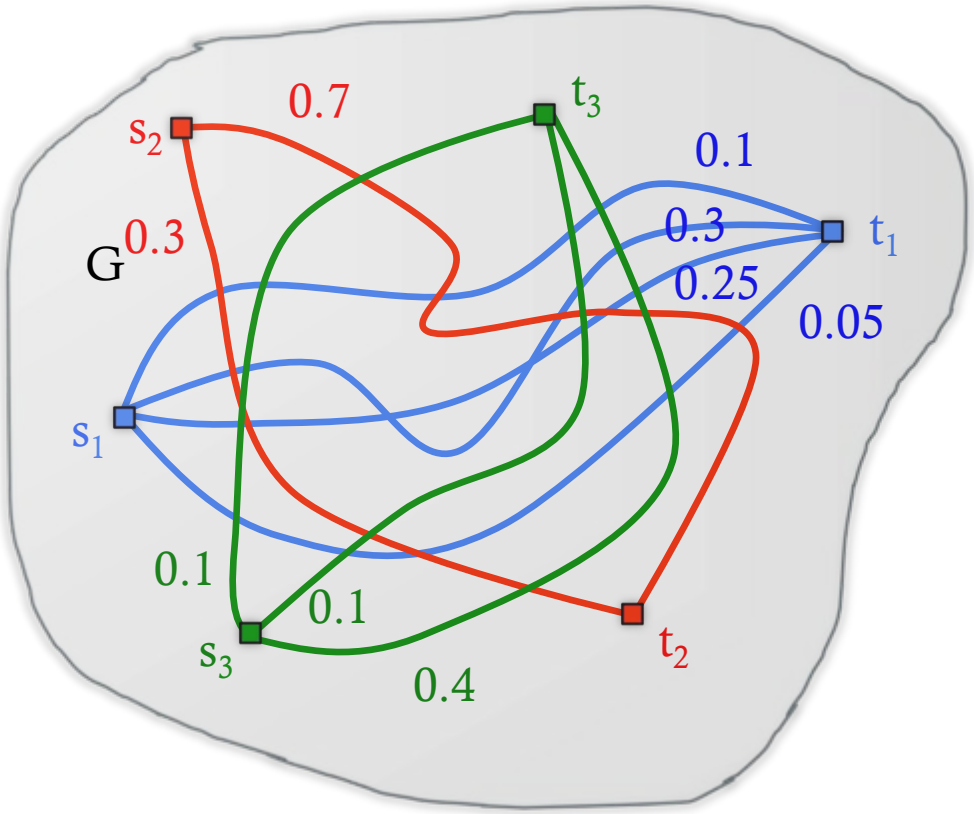
variable x_i for each pair $s_i t_i$

max $\sum x_i$ s.t

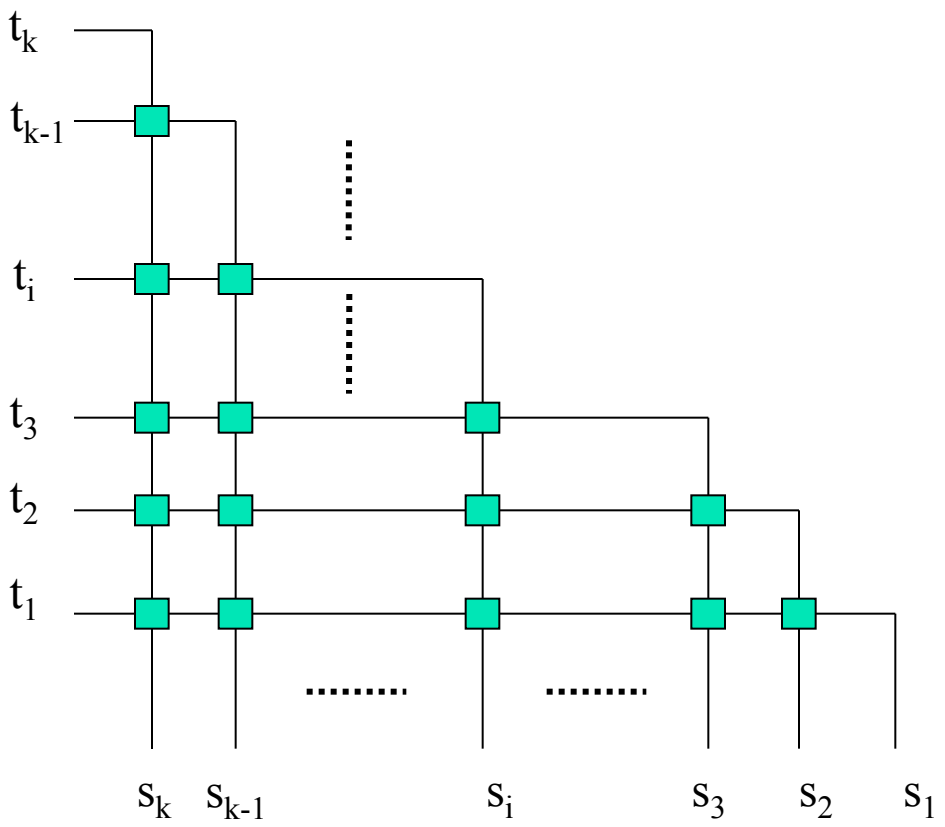
G supports multicomm. flow of x_i for pair $s_i t_i$

$0 \leq x_i \leq 1$

Multicommodity Flow Relaxation



Integrality Gap



[GVY]
$\Omega(n^{1/2})$ gap

Routing with Congestion

Can we route many pairs if we allow 2 paths per node?

Can we route many pairs if capacity of each node is 2?

"many pairs" compared to OPT_{LP} the value of flow

Routing with Congestion

Can we route many pairs if we allow 2 paths per node?

Can we route many pairs if capacity of each node is 2?

Question finally resolved in the affirmative!

Reduction to Treewidth Question

[C-Khanna-Shepherd'05]

If treewidth(G) = k does G have a "routing structure" of size comparable to k?

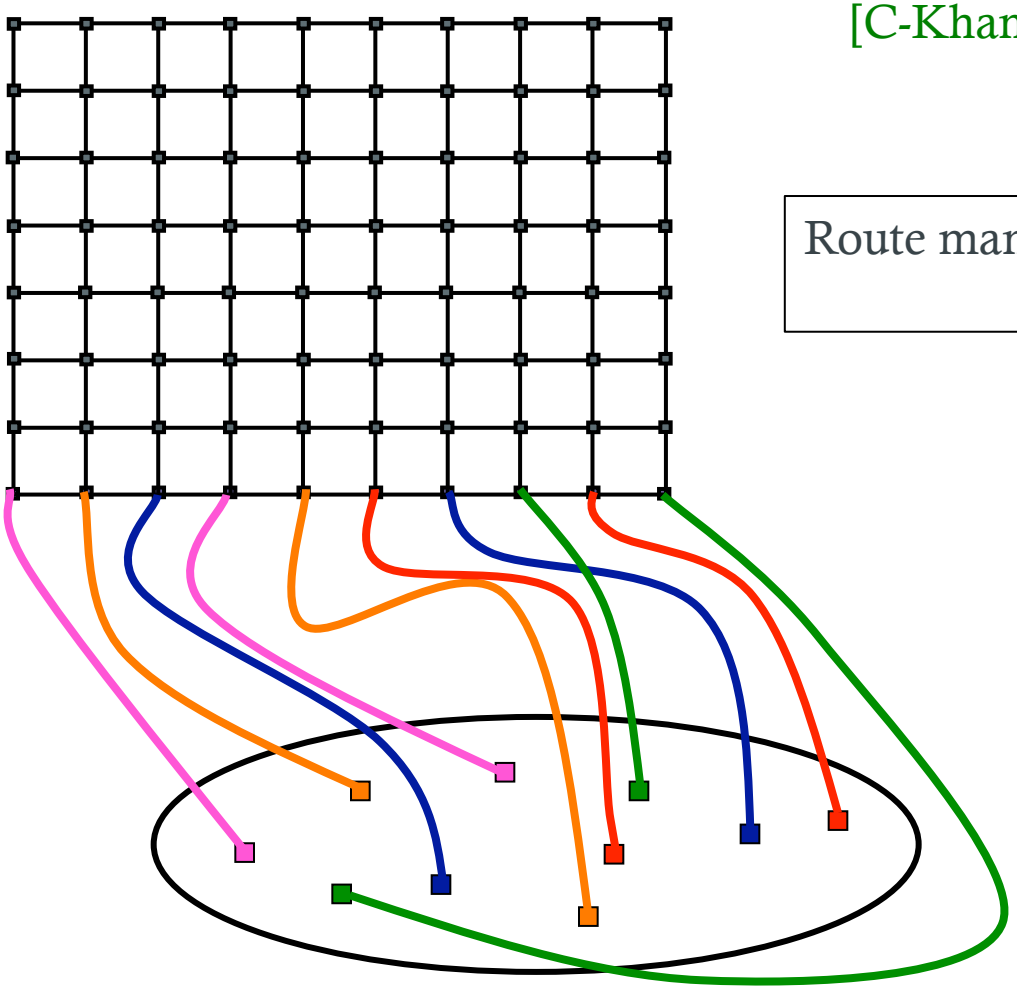
In particular $\Omega(k/\text{polylog}(k))$?

Treewidth and Routing

Question: If $tw(G) = k$ does G have a large routing structure?

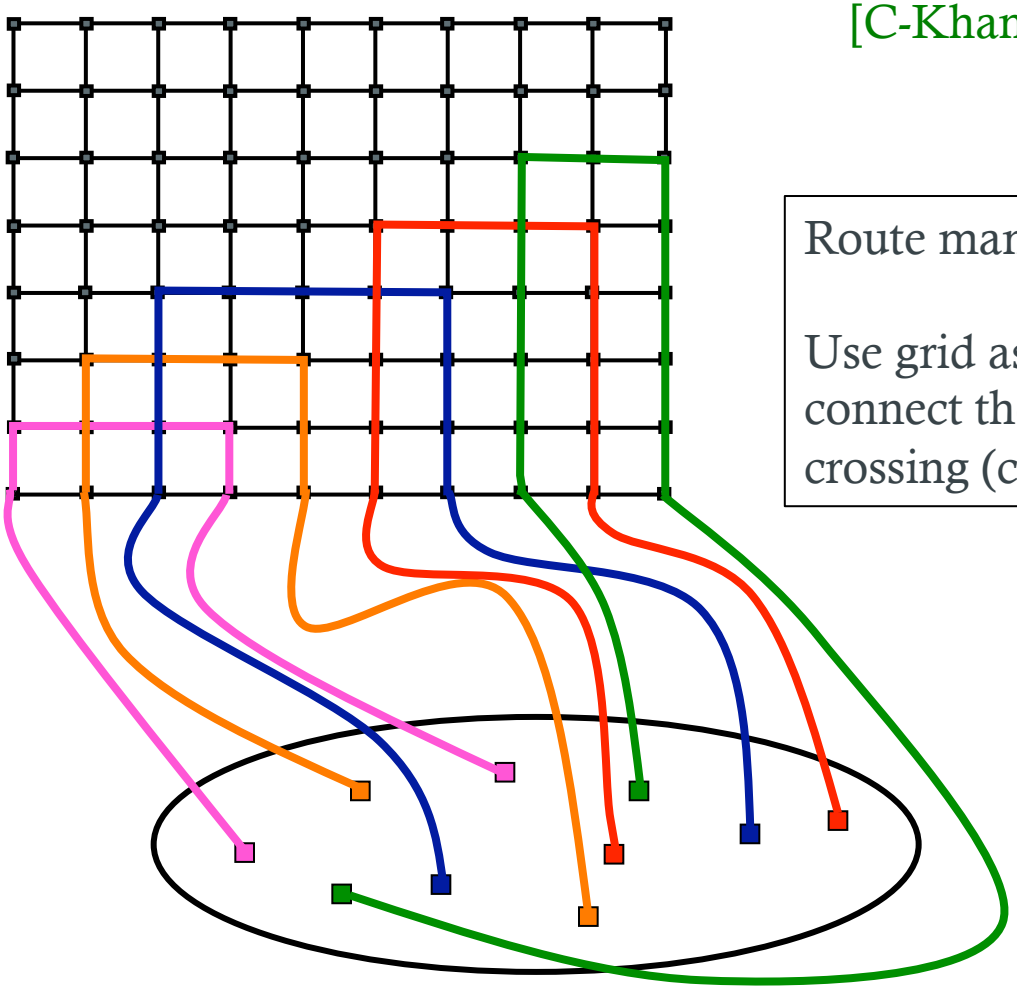
[Robertson-Seymour-Thomas] If $tw(G) = k$ and G is *planar* then G has a grid-minor of size $\Omega(k)$

Grid minors are good routing structures.



[C-Khanna-Shepherd'05]

Route many pairs to the grid



[C-Khanna-Shepherd'05]

Route many pairs to the grid

Use grid as a "switch" to connect the pairs with one crossing (congestion 2)

Treewidth and Routing

[Rao-Zhou'08] Idea for general graphs:

"Embed" an *expander* using cut-matching game of [Khandekar-Rao-Vazirani'05]

Treewidth and Routing

[Chuzhoy'11,Chuzhoy-Li'12]

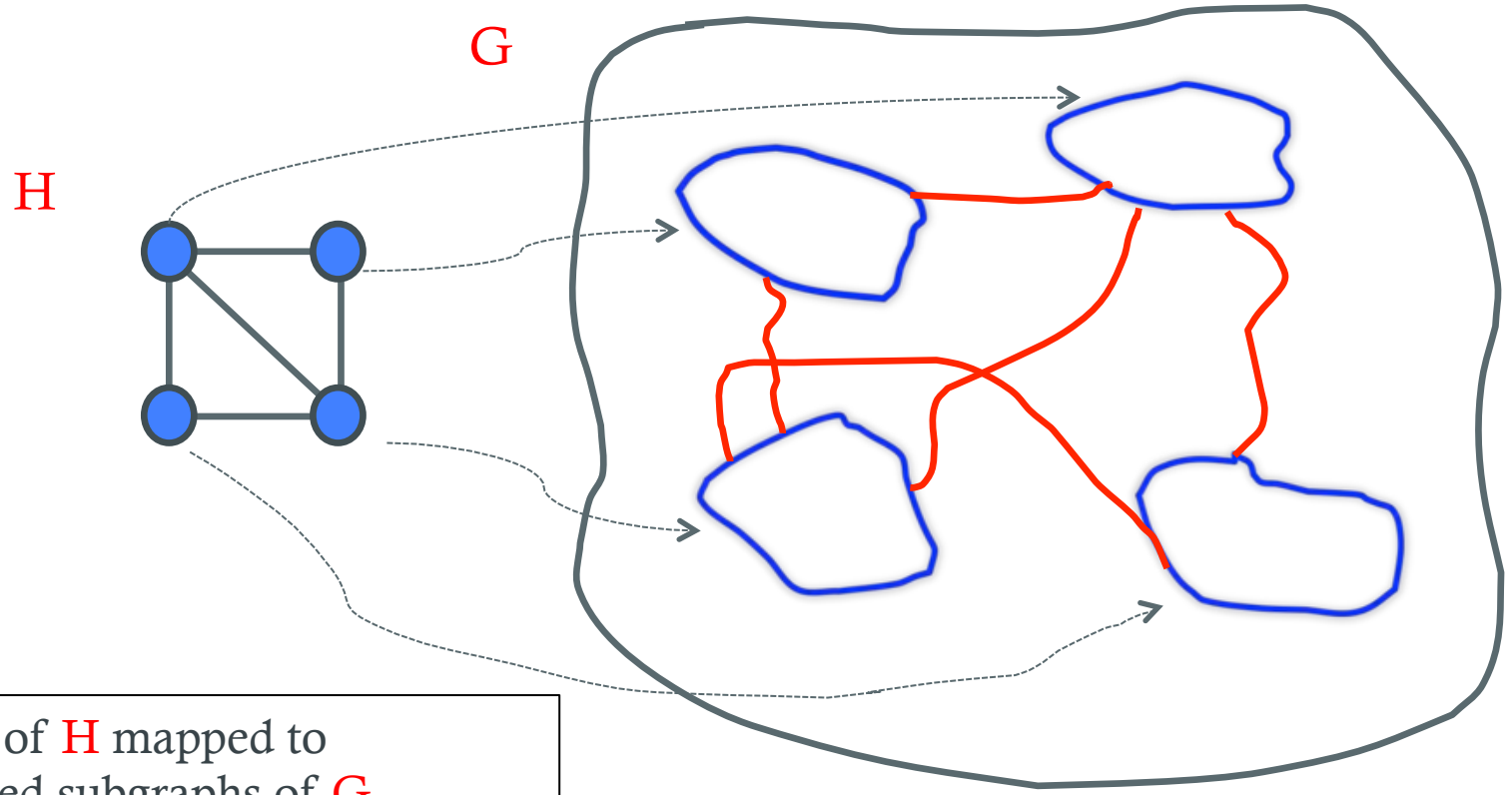
If $tw(G) \geq k$ then there is an expander of size $k/polylog(k)$ that can be "embedded" into G with edge congestion 2

[C-Ene'13]

If $tw(G) \geq k$ then there is an expander of size $k/polylog(k)$ that can be "embedded" into G with node congestion 51

[C-Chuzhoy'14] *improve node congestion to 2*

Embedding H into G



vertices of H mapped to connected subgraphs of G

edges of H mapped to paths in G

congestion defined by overlap of paths/subgraphs

Treewidth and Routing

Bottom line:

- can route $OPT_{LP}/polylog(k)$ pairs
- congestion 2
- polynomial-time algorithm

Resolves a long standing open problem by understanding the structure of "large" treewidth graphs

Treewidth and Routing

- Routing work motivated graph theoretic question
- Needed very good quantitative parameters in some sense (size of routing structure vs treewidth)
- But could relax requirements in another sense (congestion)
- Led to several other improvements including the grid-minor theorem

One Last Application

SAT

SAT: a fundamental problem in theory and practice

Canonical **hard** problem in theory

SAT Solvers: can solve many extremely large instances

Explanation?

Easy Cases of SAT

Several easy cases of SAT

- From Schaefer's dichotomy theorem (2-SAT, Horn-SAT …)
- Bounded treewidth instances

Can a SAT instance be "reduced" to a known easy class?

Backdoors to SAT

[Williams-Gomes-Selman'03]

A SAT formula ϕ has a "backdoor" if it has a "small" set of *variables* that make it easy to solve

Strong backdoor: S is a strong backdoor if for *every* assignment **a** to S the formula $\phi_{S \leftarrow a}$ is easy

$\phi_{S \leftarrow a}$ obtained by assigning **a** to S in ϕ and simplifying

Backdoors to SAT

Question: Given ϕ and k can we check if ϕ has a strong backdoor S such that $|S| \leq k$?

Backdoors to SAT

Question: Given ϕ and k can we check if ϕ has a strong backdoor S such that $|S| \leq k$?

Suppose we could do above efficiently. Then

Algorithm for SAT:

- Find strong backdoor S
- For each assignment **a** to variables in S use known algorithm for "easy" formula $\phi_{S \leftarrow a}$

Backdoors to SAT

Question: Given ϕ and k can we check if ϕ has strong backdoor S such that $|S| \leq k$?

[Gaspers-Szeider'12,'13, Fomin etal '14]

Algorithm with run-time

$$f(k, t)\ |\phi|$$

to test if ϕ has strong backdoor S of size at most k s.t $G_i(\phi_{S \leftarrow a})$ has treewidth at most t

Conclusion

- Treewidth & tree decomposition are a powerful way to understand graphs and related structures
- Closely connected to separators and recursive decomposability
- Many theoretical and conceptual applications
- Some practical successes
- Hope for more in the future

Thank You!