Problem Set – 3

Solution 1-(e):

$$\lim_{(x,y)\to(0,0)} \frac{e^{-\frac{1}{x^2+y^2}}}{x^4+y^4}$$

If $x = r\cos\theta$ and y= $r\sin\theta$ (polar coordinate), then

$$\lim_{(x,y)\to(0,0)} \frac{e^{-\frac{1}{x^2+y^2}}}{x^4+y^4} = \lim_{r\to 0} \frac{e^{-\frac{1}{r^2}}}{r^4(\cos^4\theta + \sin^4\theta)} = \lim_{r\to 0} \frac{e^{-\frac{1}{r^2}}}{r^4(1-2\cos^2\theta\sin^2\theta)}$$

Note that

$$0 \le \frac{e^{-\frac{1}{r^2}}}{r^4(1-2\cos^2\theta\sin^2\theta)} = \frac{e^{-\frac{1}{r^2}}}{r^4\left(1-\frac{1}{2}\sin^2 2\theta\right)} \le \frac{2e^{-\frac{1}{r^2}}}{r^4}$$

Further, evaluate

$$\lim_{r\to 0} \frac{2e^{-\frac{1}{r^2}}}{r^4} = \lim_{t\to\infty} 2e^{-t^2}t^4 = \lim_{t\to\infty} \frac{2t^4}{e^{t^2}} = \lim_{t\to\infty} \frac{4t^2}{e^{t^2}} = \lim_{t\to\infty} \frac{4}{e^{t^2}} = 0$$

(using L'Hospital's rule)

Using Sandwich Theorem*, we get

$$\lim_{r\to 0} \frac{e^{-\frac{1}{r^2}}}{r^4(1-2\cos^2\theta\sin^2\theta)} = 0$$

***Sandwich Theorem (Squeeze Theorem)**

Suppose that for all $x \in [a,b]$ (except possibly at $x = x_0, x_0 \in [a,b]$) we have

$$h(x) \le f(x) \le g(x)$$

Also suppose that

$$\lim_{x\to x_0} h(x) = \lim_{x\to x_0} g(x) = L$$

Then

$$\lim_{x\to x_0} f(x) = L$$
