Interpolation.

Lecture-10
7/2/17
6:00 - 7:30 p.m

Absolute error = | true value - approximate value |.

$xe^x = 2$. 0.3292 (computed / approx. value)

true value 0.32925

abs. error = $|0.32925 - 0.3292|$

= 0.00005.

relative error = $\dfrac{\text{abs. error}}{\text{true value}}$.

percentage error = relative error × 100.

bisection method:

Find $f(x) = 0$. $x_n = \dfrac{a_{n-1} + b_{n-1}}{2}$

$\left|\dfrac{b-a}{2^n}\right| \le \epsilon$. root lies in $[a, b]$.
error = ϵ.
$n = ?$

If a root of $f(x) = 0$ lies in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, using bisection method, how many iterations are needed, so that the error is at most 10^{-4}?

$$\left|\frac{\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)}{2^n}\right| \le 10^{-4}. \quad \text{or}, \quad \frac{\pi}{2^n} \le 10^{-4}.$$

$$\Rightarrow 2^n \geq \pi 10^4.$$

$$\Rightarrow n \log_e 2 \geq \log_e(\pi 10^4)$$

$$\Rightarrow n \geq \frac{\log_e(\pi 10^4)}{\log_e 2} = 14.93$$

$\therefore n = 15$ (minimum 15 iterations are required)

Interpolation

Scarborough (book) (theory).

Practice problems → Engineering Maths.

Taneja / Babu Ram / Ramana.

X	0·1	0·4	0·6	0·8	1	1·2	1·3
Y	10	14	16	19	27	32	36

Find Y at $x = 0·5$ → interpolation.

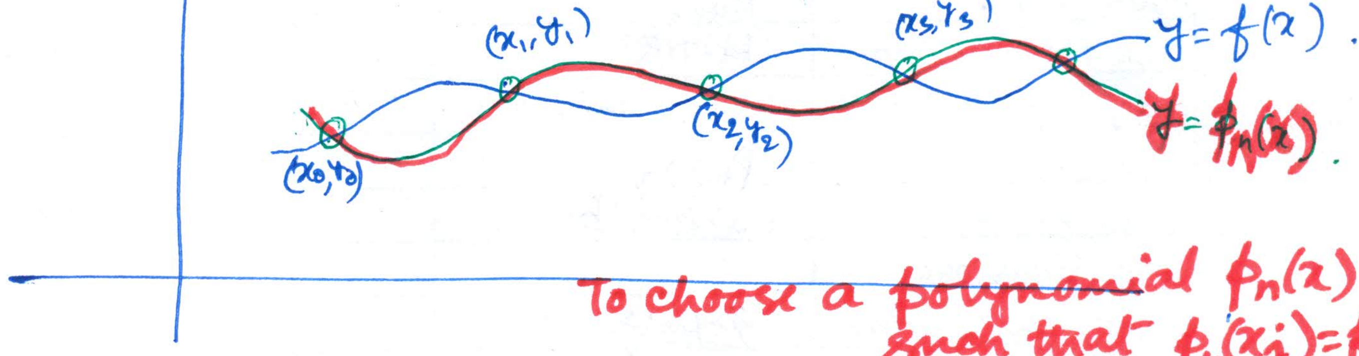
" Y at $x = 0·05 / 1·4$ → extrapolation.

The table corresponds to functional values of $y = f(x)$ at diff. x.

We don't know what is f.

To approximate f through some other function, say polynomial / finite trigonometric series etc.

Polynomial interpolation



To choose a polynomial $p_n(x)$ such that $p_n(x_j) = f(x_j)$, $j = 0, 1, 2, \ldots, n$

X	x_0	x_1	x_2	. . .	x_n
$Y = f(x)$	y_0	y_1	y_2		y_n

Find

$p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

such that $p_n(x_j) = f(x_j)$; $j = 0, 1, 2, \to n$.

1) Newton's forward difference polynomial.

2) " backward " polynomial.

3) Lagrange's polynomial.

Finite Difference

(x values are equidistant)

X	0.2	0.5	0.8	1.1	1.4
Y = f(x)	0	-5	19	-21	16

forward difference operator. Δ

$x_i - x_{i-1} = h$.

x	x_0	x_1	. . . .	x_n
y = f(x)	y_0	y_1		y_n

$\Delta f(x) = f(x+h) - f(x)$

$\Delta f(x_0) = f(x_0 + h) - f(x_0)$.

or. $\Delta y_0 = f(x_1) - f(x_0) = y_1 - y_0$

$\Delta y_2 = y_3 - y_2$

$\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0$.

$\Delta^2 y_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$.

$\Delta^3 y_0 = \Delta(\Delta^2 y_0) = \Delta(y_2 - 2y_1 + y_0)$

$= \Delta y_2 - 2\Delta y_1 + \Delta y_0 = (y_3 - y_2) - 2(y_2 - y_1) + (y_1 - y_0)$

$= y_3 - 3y_2 + 3y_1 - y_0$.

$\Delta^3 y_r = y_{r+3} - 3y_{r+2} + 3y_{r+1} - y_r$

Find $\Delta^r y_k$

Define shift operator E.

$$E f(x) = f(x+h).$$

$$E y_r = y_{r+1}$$

$$\Delta f(x) = f(x+h) - f(x) = E f(x) - I f(x)$$

Identity operator.

$$\Delta f(x) = (E - I) f(x).$$

$$\Delta = E - I.$$

$$\Delta^r y_k = (E-I)^r y_k = \sum_{j=0}^{r} \binom{r}{j} E^{r-j} (-I)^j y_k.$$

$$= \sum_{j=0}^{r} \binom{r}{j} E^{r-j} (-1)^j I^j y_k.$$

$$= \sum_{j=0}^{r} \binom{r}{j} E^{r-j} (-1)^j y_k.$$

$$\Delta^r y_k = \sum_{j=0}^{r} \binom{r}{j} (-1)^j y_{k+r-j}$$

$$E y_r = y_{r+1}$$

$$E^2 y_r = E(E y_r) = E y_{r+1} = y_{r+2}$$

Backward difference operator ∇

$$\nabla f(x) = f(x) - f(x-h)$$

$$\nabla y_5 = y_5 - y_4 \quad | \quad \Delta y_5 = y_6 - y_5$$

$$\nabla f(x) = I f(x) - E^{-1} f(x)$$

$$\nabla = I - E^{-1}.$$

operate E on both sides.

$$\nabla E = E - E^{-1}E = E - I = \Delta$$

$$\boxed{\begin{matrix} \nabla E = \Delta \\ \Delta E^{-1} = \nabla \end{matrix}}$$

$$\Delta^r y_k = (\nabla E)^r y_k = \nabla^r E^r y_k = \nabla^r y_{k+r}.$$

$$\boxed{\Delta^r y_k = \nabla^r y_{k+r}}$$

$$\nabla^r y_k = (\Delta E^{-1})^r y_k = \Delta^r E^{-r} y_k = \Delta^r y_{k-r}.$$

$$\boxed{\nabla^r y_k = \Delta^r y_{k-r}}$$

Properties of Δ.

1) $\Delta(c) = 0$ $\quad$ 2) $\Delta(c f(x)) = c\Delta f(x)$

$c \to$ constant.

3) $\Delta(f \pm g) = \Delta f \pm \Delta g$.

4) $\Delta^r p_n(x) = \begin{cases} \text{a polynomial of degree } n-r \text{ if } n > r. \\ 0 \quad \text{if } n < r. \\ \text{a constant} = a_n n! h^n. \end{cases}$

$$p_n(x) = a_0 + a_1 x + \cdots + a_n x^n.$$

$$p_1(x) = a_0 + a_1 x.$$

$$\Delta p_1(x) = \cancel{a_0} + a_1(\cancel{x} + h) - \cancel{a_0} - \cancel{a_1 x}.$$

$$= a_1 h = \text{const}.$$

$$\Delta^2 p_1(x) = a_1 h - a_1 h = 0.$$

(can be Proved by mathematical induction.)

$$\Delta^n p_n(x) = a_n n! h^n.$$

Newton's forward difference polynomial

Given

x	x_0	x_1	- - - -	x_n
y	y_0	y_1		y_n

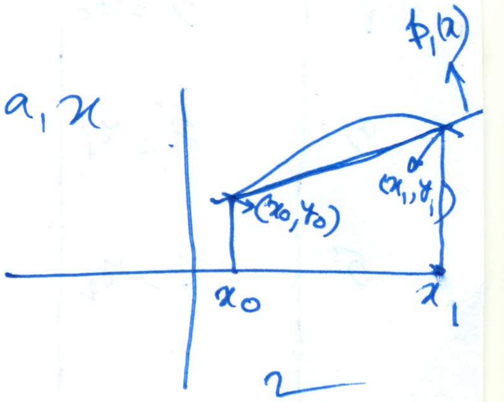
To construct $p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

such that $p_n(x_j) = y_j$; $j = 0, 1, 2, \ldots, n$.

x	x_0	x_1
y	y_0	y_1

$p_1(x) \equiv a_0 + a_1 x$

linear interpolation.



x	x_0	x_1	x_2
y	y_0	y_1	y_2

$p_2(x) = a_0 + a_1 x + a_2 x^2$

quadratic interpolation.

- given a set of n points $(x_0, y_0), (x_1, y_1)$ $\ldots (x_n, y_n)$, the interpolating polynomial are all the same.

$$p_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) + \cdots + a_n(x - x_0)(x - x_1) \cdots (x - x_{n-1}) \longrightarrow (A)$$

Use.

$p_n(x_j) = y_j$; $j = 0, 1, 2, \ldots, n$.

Put $x = x_0$ on both sides of (A).

① $\phi_n(x_0) = y_0 = a_0$

$x_i - x_{i-1} = h$.

$x_1 - x_0 = h$

$\boxed{x_i = x_0 + ih}$

Put $x = x_1$, $\phi_n(x_1) = y_1 = a_0 + a_1(x_1 - x_0)$

$y_1 = a_0 + a_1 h$.

or, $y_1 = y_0 + a_1 h$.

$$a_1 = \frac{y_1 - y_0}{h} = \frac{\Delta y_0}{h}.$$

$x = x_2$, $\phi_n(x_2) = y_2 = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$.

or, $y_2 = y_0 + \frac{\Delta y_0}{\cancel{h}} \times 2\cancel{h} + a_2 \times 2h \times h$.

or, $a_2 \times 2h^2 = y_2 - y_0 - 2(y_1 - y_0)$.

$= y_2 - 2y_1 + y_0 = \Delta^2 y_0$.

$$\therefore a_2 = \frac{\Delta^2 y_0}{2h^2} = \frac{\Delta^2 y_0}{2!\, h^2}.$$

$$a_3 = \frac{\Delta^3 y_0}{3!\, h^3} \; \ldots\ldots \; a_n = \frac{\Delta^n y_0}{n!\, h^n}.$$

$$\phi_n(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!\, h^2}(x - x_0)(x - x_1)$$

$$+ \; \ldots \; + \frac{\Delta^n y_0}{n!\, h^n}(x - x_0)(x - x_1) \ldots (x - x_n)$$

$x \to u$

$x - x_0 = uh$. $x - x_k = x - x_0 + x_0 - x_k$

$= uh - kh$

$= (u - k)h$.

$x_k = x_0 + kh$

$$p_n(x) = P_n(u) = y_0 + \frac{\Delta y_0}{\cancel{h}} \times u\cancel{h} + \frac{\Delta^2 y_0}{2!\,\cancel{h^2}} u\cancel{h}(u-1)\cancel{h}$$

$$+ \cdots + \frac{\Delta^n y_0}{n!\,h^n} uh(u-1)h \cdots (u-n+1)h.$$

$$= y_0 + \Delta y_0 . u + \frac{\Delta^2 y_0}{2!} u(u-1)$$

$$+ \cdots + \frac{\Delta^n y_0}{n!} u(u-1) \cdots (u-n+1).$$

Newton's Backward difference polynomial.

$$q_n(x) = b_n + b_{n-1}(x - x_n) + b_{n-2}(x-x_n)(x-x_{n-1})$$

$$+ \cdots \quad b_0(x-x_n)(x-x_{n-1}) \cdots (x-x_1) \rightarrow (B)$$

Put $x = x_n$, $x = x_{n-1}$, $x = x_{n-2}$ ... in (B).

$$q_n(x_n) = y_n = b_n$$

$$q_n(x_{n-1}) = y_{n-1} = b_n + b_{n-1}(\underbrace{x_{n-1} - x_n})$$

$$= b_n + b_{n-1} \times (-h)$$

$$y_{n-1} = y_n - h b_{n-1} \Rightarrow b_{n-1} = \frac{y_n - y_{n-1}}{h}.$$

$$\therefore b_{n-1} = \frac{\nabla y_n}{h}.$$

$$q_n(x_{n-2}) = b_n + b_{n-1}(x_{n-2} - x_n) + b_{n-2}(x_{n-2} - x_n)(x_{n-2} - x_{n-1})$$

$$= b_n + b_{n-1} \times -2h + b_{n-2}(-2h)(-h).$$

$$b_{n-2} = \frac{\nabla^2 y_n}{2!\,h^2}.$$

$$q_n(x) = y_n + \frac{\nabla y_n}{h}(x-x_n) + \frac{\nabla^2 y_n}{2!\,h^2}(x-x_n)(x-x_{n-1})$$

$$+ \cdots + \frac{\nabla^n y_n}{n!\,h^n}(x-x_n)\cdots\cdots(x-x_1).$$

$x \to v.$

$x - x_n = vh.$

$x - x_{n-1} = x - x_n + x_n - x_{n-1} = vh + h.$
$= (v+1)h.$

$x - x_{n-2} = x - x_n + x_n - x_{n-2} = (v+2)h.$

$$\therefore q_n(x) = Q_n(v) = y_n + \nabla y_n \cdot v + \frac{\nabla^2 y_n}{2!} v(v+1)$$

$$+ \frac{\nabla^3 y_n}{3!} v(v+1)(v+2) \cdots\cdots + \frac{\nabla^n y_n}{n!} v(v+1)\cdots(v+n-1).$$