⇒ Vector Analysis :

1) $\vec{P} \times \vec{Q} = PQ \sin\theta \; \hat{n}$ $0 \le \theta \le \pi$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \end{vmatrix}$$

2) Projection of $\vec{b}$ on $\vec{a} = \hat{a} \cdot \vec{b}$

3) $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \dfrac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{\Sigma a_1^2}\sqrt{\Sigma b_1^2}}$

4) Scalar Triple Product $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$

$$= [\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}] = -[\vec{a}\ \vec{c}\ \vec{b}]$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(a) $\vec{a}, \vec{b}, \vec{c}$ are non-zero, non-collinear, then

$\vec{a}, \vec{b}, \vec{c}$ are coplanar $\Leftrightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0$

(b) ~~Voth~~ Volume of parallelopiped $= [\vec{a}\ \vec{b}\ \vec{c}]$

(c) A, B, C, D are vertices of a tetrahedron

Volume $= \dfrac{1}{6} [\vec{AB}\ \vec{AC}\ \vec{AD}]$

5) Vector Triple Product $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

$(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$

TIP : Middle vector always have +ve coefficient.

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{vmatrix}$$

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\vec{a}$$
$$= [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$$

6) Reciprocal system of vectors :

Set of vectors $(\vec{a}, \vec{b}, \vec{c})$ & $(\vec{a}', \vec{b}', \vec{c}')$ are reciprocal

if $\vec{a}\cdot\vec{a}' = \vec{b}\cdot\vec{b}' = \vec{c}\cdot\vec{c}' = 1$

$\vec{a}'\cdot\vec{b} = \vec{a}'\cdot\vec{c} = \vec{b}'\cdot\vec{a} = \vec{b}'\cdot\vec{c} = \vec{c}'\cdot\vec{a} = \vec{c}'\cdot\vec{b} = 0$

Reciprocal iff $\vec{a}' = \dfrac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$ $\vec{b}' = \dfrac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}$ $\vec{c}' = \dfrac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$

Also $[\vec{a}\,\vec{b}\,\vec{c}] * [\vec{a}'\,\vec{b}'\,\vec{c}'] = 1$

TRICK : eg. ST $\vec{r} = (r\cdot a')a + (r\cdot b')b + (r\cdot c')c$

use $(a\times b)\times(c\times d) = [a\,c\,d]b - [b\,c\,d]a$ — (I)

$= [a\,b\,d]c - [a\,b\,c]d$ — (II)

From (I) & (II) ⇒ $[a\,b\,c]d = -[a\,c\,d]b + [b\,c\,d]a + [a\,b\,d]c$

$[a\,b\,c]\,d = [d\,b\,c]a + [d\,c\,a]b + [d\,a\,b]c$

Now put $d = r$. , we get the result.

7) Scalar field : at each pt (x, y, z) of region R in space, ∃ a unique scalar $\phi(x, y, z)$

Vector field : at each pt (x, y, z), there corresponds a vector $\vec{V}(x, y, z)$

8) vector function of scalar variables : for each $t \in S$, ∃ unique vector $\vec{f(t)}$

$$\vec{f}'(t_0) = \lim_{t\to t_0} \frac{\vec{f(t)} - \vec{f(t_0)}}{t - t_0} = \left.\frac{d\vec{f}}{dt}\right|_{t=t_0}$$

9) $\dfrac{d}{dt}(\vec{A}\cdot\vec{B}) = \vec{A}\cdot\dfrac{d\vec{B}}{dt} + \dfrac{d\vec{A}}{dt}\cdot\vec{B}$

$\dfrac{d}{dt}(\vec{A}\times\vec{B}) = \vec{A}\times\dfrac{d\vec{B}}{dt} + \dfrac{d\vec{A}}{dt}\times\vec{B}$

$\vec{A}, \vec{B}, \vec{C}$ are vector fns & ϕ is scalar fn of scalar variable 't'

$$\frac{d}{dt}(\phi\vec{A}) = \phi\frac{d\vec{A}}{dt} + \frac{d\phi}{dt}\vec{A}$$

$$\frac{d}{dt}[\vec{A}\ \vec{B}\ \vec{C}] = \left[\frac{dA}{dt}\ B\ C\right] + \left[A\ \frac{dB}{dt}\ C\right] + \left[A\ B\ \frac{dC}{dt}\right]$$

$$\frac{d}{dt}\{A\times(B\times C)\} = \frac{dA}{dt}\times(B\times C) + A\times\left(\frac{dB}{dt}\times C\right) + A\times\left(B\times\frac{dC}{dt}\right)$$

10) $\vec{F}$ is constant vector if both magnitude & direction are constant.

11) $\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$ $\quad$ $\vec{r}$ = vector fn of 't'
x, y, z = scalar fn of t

$$\therefore \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

12) $\vec{f}$ is constant $\Leftrightarrow$ $\frac{d\vec{f}}{dt} = 0$

$\vec{f}$ is of constant magnitude $\Leftrightarrow$ $\boxed{\vec{f}\cdot\frac{d\vec{f}}{dt} = 0}$

13) Do not make mistake of $\vec{a}$ vs $a = |\vec{a}|$ in differentials

$\frac{d\vec{r}}{dt}$ & $\frac{dr}{dt}$ are different expressions.

14) $\vec{f}$ has constant direction $\Leftrightarrow$ $\boxed{\vec{f}\times\frac{d\vec{f}}{dt} = 0}$

eg. $\vec{R}$ is unit vector in direction of $\vec{r}$.

ST $\vec{R}\times\frac{d\vec{R}}{dt} = \frac{1}{r^2}\vec{r}\times\frac{d\vec{r}}{dt}$ where $r = |\vec{r}|$

$$\vec{r} = r\vec{R} \Rightarrow \vec{R}\times\frac{d\vec{R}}{dt} = \frac{\vec{r}}{r}\times\frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \frac{\vec{r}}{r}\times\left[\frac{1}{r}\frac{d\vec{r}}{dt} - \frac{1}{r^2}\frac{dr}{dt}\vec{r}\right]$$

$$= \frac{1}{r^2}\vec{r}\times\frac{d\vec{r}}{dt} - \frac{1}{r^3}\frac{dr}{dt}\underbrace{\cancel{(\vec{r}\times\vec{r})}}_{0} = \frac{1}{r^2}\left(\vec{r}\times\frac{d\vec{r}}{dt}\right)$$

15) Curves can be represented as :

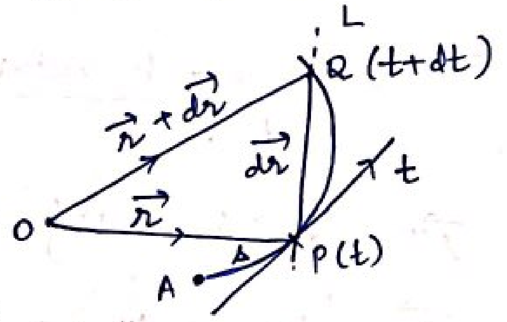
(a) intersection of 2 surface $F_1(x,y,z) = 0$ & $F_2(x,y,z) = 0$

(b) parametric form : $X = f_1(t)$ $Y = f_2(t)$ $Z = f_3(t)$

16) Straight line : $\vec{r} = \vec{a} + t\vec{b}$ & $t \in \mathbb{R}$
⇓
line through $\vec{a}$ in direction of constant vector $\vec{b}$

Plane curve : lies in a plane in space
Twisted curve : doesn't lie on a plane.
eg circular helix $\vec{r}(t) = [a\cos t, a\sin t, ct]$

17) Tangent to a curve :



tangent = limiting position of line L as Q → P.

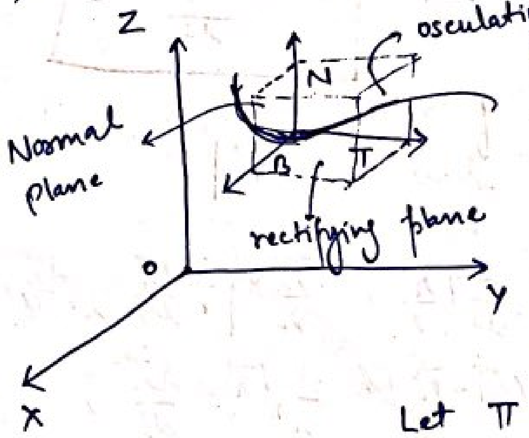
Tangent vector to curve at P(t) = $\dfrac{d\vec{r}}{dt}$

Unit tangent vector : $\vec{T} = \dfrac{d\vec{r}}{ds}$.

easy through manipulation
$|\vec{T}| = 1$ &
$\dfrac{d\vec{r}}{ds} = \dfrac{dx}{ds}\hat{i} + \dfrac{dy}{ds}\hat{j} + \dfrac{dz}{ds}\hat{k}$

Also $\dfrac{ds}{dt} = \left|\dfrac{d\vec{r}}{dt}\right|$

18) SERRET - FRENET FORMULAE :



T = unit tangent vector
N = Unit Principal Normal vector
⇓
it lies in the osculating plane

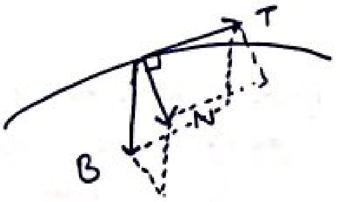
To find osculating plane, choose 2 pts near the given pt

Let Π be plane through these pts
∴ osculating Plane = $\lim_{dt \to 0} \Pi$

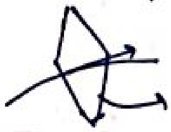
The plane turns in the OP.

$T \times N = B$; $N \times B = T$; $B \times T = N$
$T \cdot T = B \cdot B = N \cdot N = 1$ $T \cdot N = N \cdot B = B \cdot T = 0$.

(a) Normal, Osculating & Rectifying planes :



→ Plane containing N and B
Normal is $\underline{T}$



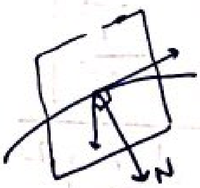
normal plane : curve is cutting through it normally.

⇒ Plane containing T and N ↪ N is the direction we are turning in ✱

osculating circle lies in this plane
distance to centre = ρ
location of centre in direction of N.

Normal is $\underline{B}$ → Binormal unit vector
we call it osculating plane

⇒ Plane containing T and B
Normal is $\underline{N}$



it is called rectifying plane

eg Find the planes at $t = \frac{\pi}{2}$ for $r(t) = \langle 3\sin t, 3\cos t, 4t \rangle$

$T = \langle \frac{3}{5}\cos t, -\frac{3}{5}\sin t, \frac{4}{5} \rangle$ $\quad \therefore T = \langle 0, -3/5, 4/5 \rangle$

$N = \langle -\sin t, -\cos t, 0 \rangle$ $\quad N = \langle -1, 0, 0 \rangle$

$B = \langle \frac{4}{5}\cos t, -\frac{4}{5}\sin t, -\frac{3}{5} \rangle$ $\quad B = \langle 0, -\frac{4}{5}, -\frac{3}{5} \rangle$

$\vec{r}\left(\frac{\pi}{2}\right) = \langle 3, 0, 2\pi \rangle$

$\therefore NP = 0(x-3) - \frac{3}{5}(y-0) + \frac{4}{5}(z-2\pi) = 0 \Rightarrow 4z - 3y = 8\pi$

$OP = 0(x-3) - \frac{4}{5}(y-0) - \frac{3}{5}(z-2\pi) = 0 \Rightarrow 4y + 3z + 6\pi = 0$

$RP = -1(x-3) = 0 \Rightarrow x = 3$

(b) Formulae : (i) $\frac{dT}{ds} = kN$ (ii) $\frac{dB}{ds} = -\tau N$

See the proof in notes.

(iii) $\frac{dN}{ds} = \tau B - kT$

$k =$ curvature of curve $\tau =$ torsion (how much curve is going out of current plane)

$\rho = \frac{1}{k} =$ radius of curvature $\sigma = \frac{1}{\tau}$ = radius of torsion

19) $\left|\frac{dT}{ds}\right| = k$ & $\left|\frac{dB}{ds}\right| = \tau$

given $\vec{r} = \vec{r}(t)$ $T = \frac{d\vec{r}}{dt} \Big/ \left|\frac{d\vec{r}}{dt}\right|$

then $\frac{d\vec{T}}{ds} = \frac{d\vec{T}}{dt} \Big/ \frac{ds}{dt}$ & $\frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right|$

$\therefore \frac{d\vec{T}}{ds}$ can be evaluated $\Rightarrow k = \left|\frac{d\vec{T}}{ds}\right|$

Also * $\boxed{\frac{d\hat{T}}{ds} = N}$ (unit vector) & $B = T \times N$

Remember : $k = \left|\frac{d\vec{r}}{ds} \times \frac{d^2\vec{r}}{ds^2}\right|$ $\tau = \dfrac{\left[\frac{d\vec{r}}{ds}\ \frac{d^2\vec{r}}{ds^2}\ \frac{d^3\vec{r}}{ds^3}\right]}{\left|\frac{d\vec{r}}{ds} \times \frac{d^2\vec{r}}{ds^2}\right|^2}$

Also $k = \dfrac{\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right|}{\left|\frac{dr}{dt}\right|^3}$ $\tau = \dfrac{\left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right]}{\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right|^2}$

Steps : (i) Find $\vec{T} = \frac{d\vec{r}}{ds}$ or $\frac{d\vec{r}}{dt} \Big/ \left|\frac{d\vec{r}}{dt}\right|$

Find $k = \left|\frac{d\vec{T}}{ds}\right| = \left|\frac{d^2\vec{r}}{ds^2}\right|$ or $\left|\frac{d^2\vec{r}}{dt^2} \Big/ \left|\frac{dr}{dt}\right|^2\right|$.

Find $\boxed{N = \frac{1}{k}\frac{d\vec{T}}{ds}}$

Find $B = T \times N$.

Find $\tau = \left|\frac{dB}{ds}\right|$

If only k & τ are asked, directly use the above formulas.

⇒ Gradient, Divergence and Curl.

1) If $\vec{r} = f(x, y, z)$, the total differential is

$$d\vec{r} = \frac{\partial \vec{r}}{\partial x} dx + \frac{\partial \vec{r}}{\partial y} dy + \frac{\partial \vec{r}}{\partial z} dz$$

2) Vector differential operator $(\nabla) = \nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

Gradient $= \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$ [where f is scalar)

Divergence $= \nabla \cdot \vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$ [where $f = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$]

Curl $= \nabla \times \vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$.

Laplacian ϕ scalar $= \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$

$\vec{F}$ vector $= \nabla^2 \vec{F} = \frac{\partial^2 \vec{F}}{\partial x^2} + \frac{\partial^2 \vec{F}}{\partial y^2} + \frac{\partial^2 \vec{F}}{\partial z^2}$

Laplace equation : $\nabla^2 \phi = 0$
such functions are called Harmonic Function.

3) Properties of gradient :

(a) $\nabla \phi = 0 \iff$ A scalar function is constant.

(b) $\nabla(fg) = f\nabla g + g\nabla f$

(c) $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

(d) $\nabla f(r) = \frac{f'(r)}{r}\vec{r}$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$
& $r = |\vec{r}|$

4) Properties of Divergence and Curl :

a) Divergence of vector pt fn is a scalar pt fn.
Curl of vector pt fn is a vector pt fn

b) ~~$\nabla\phi$~~ $\vec{\nabla}\cdot\vec{F} = \sum i\cdot\frac{\partial \vec{F}}{\partial x}$

$\vec{\nabla}\times\vec{F} = \sum i\times\frac{\partial \vec{F}}{\partial x}$

} quite handy to use in many big problems

c) Solenoidal vector : $\nabla\cdot\vec{V} = 0$

Irrotational vector : $\nabla\times\vec{V} = 0$

d) $\nabla\cdot\vec{r} = 3$ $\quad \nabla\times\vec{r} = 0$ $\quad \nabla(\vec{a}\cdot\vec{r}) = \vec{a}$

e) $\nabla\cdot\vec{A} = 0$ $\quad \nabla\times\vec{A} = 0$ $\quad \vec{A}$ is a constant vector

eg. $\text{curl}(\vec{r}\times\vec{a}) = \sum i\times\frac{\partial}{\partial x}(\vec{r}\times\vec{a}) = \sum i\times\left(\frac{\partial\vec{r}}{\partial x}\times\vec{a}\right) + \sum i\times\left(\vec{r}\times\underbrace{\frac{\partial\vec{a}}{\partial x}}_{0}\right)$

$= \sum\left\{(i\cdot\vec{a})\frac{\partial\vec{r}}{\partial x} - \left(i\cdot\frac{\partial\vec{r}}{\partial x}\right)\vec{a}\right\}$

$= \sum(i\cdot\vec{a})\frac{\partial\vec{r}}{\partial x} - \vec{a}\sum i\cdot\frac{\partial\vec{r}}{\partial x}$

$= \vec{a} - 3\vec{a} = -2\vec{a}$

f) $\nabla\cdot(\phi\vec{A}) = (\nabla\phi)\cdot\vec{A} + \phi(\nabla\cdot\vec{A})$

g) $\nabla\times(\phi\vec{A}) = (\nabla\phi)\times\vec{A} + \phi(\nabla\times\vec{A})$

h) $\nabla\cdot(\vec{A}\times\vec{B}) = B\cdot(\nabla\times\vec{A}) - A\cdot(\nabla\times\vec{B})$

i) $\nabla\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\nabla)\vec{A} - \vec{B}(\nabla\cdot\vec{A}) - (\vec{A}\cdot\nabla)\vec{B} + \vec{A}(\nabla\cdot\vec{B})$

$= (\nabla\cdot\vec{B} + \vec{B}\cdot\nabla)\vec{A} - (\nabla\cdot\vec{A} + \vec{A}\cdot\nabla)\vec{B}$

j) div grad $\phi = \nabla\cdot(\nabla\phi) = \nabla^2\phi$

k) curl of grad of $\phi = \nabla\times(\nabla\phi) = 0$

l) div curl $\phi = \nabla\cdot(\nabla\times\vec{A}) = 0$

Don't open ∇ like other vectors

go by $\sum i\cdot\frac{\partial}{\partial x}$ & $\sum i\times\frac{\partial}{\partial x}$ method.

eg Don't do $\nabla\times(\nabla\times A) = (\nabla\cdot A)\nabla - (\nabla\cdot\nabla)A$
$\Downarrow$
WRONG !!

5) Vector Integration :

a) $\int \left(2\vec{r} \cdot \frac{d\vec{r}}{dt}\right) dt = \vec{r}^{\,2} + c$; c ⇒ scalar

b) $\int \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt = \vec{r} \times \frac{d\vec{r}}{dt} + \vec{c}$; $\vec{c}$ ⇒ vector

c) $\int \left(a \times \frac{d\vec{r}}{dt}\right) dt = \vec{a} \times \vec{r} + \vec{c}$

6) Level Surface : $f(x,y,z)$ is a scalar field over R

Then LS is $f(x,y,z) = c$ (arbitrary constant)

a) Through any pt on R, there passes only 1 LS for a given scalar field.

b) ∇f is a vector normal to the surface $f(x,y,z)=c$

Proof : Let $P(x,y,z)$ be a pt with pos^n^ $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$
& $Q(x+\delta x, y+\delta y, z+\delta z)$ with $\vec{r} + \delta\vec{r}$

$\therefore \vec{PQ} = \delta x\,\hat{i} + \delta y\,\hat{j} + \delta z\,\hat{k}$

as $Q \to P$, $\vec{PQ}$ tends to tangent at P.

$\therefore d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$ lies in tangent plane

$\nabla f \cdot d\vec{r} = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = df = 0$ [$\because f = $ const^{n}]

7) Directional Derivative of f at P in direction of $\hat{a}$:

$$\frac{df}{ds} = \nabla f \cdot \hat{a}$$

Proof: Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be pos vector of P. If s denotes distance of P from some pt A in direction of $\hat{a}$, then δs is a small element in direction of $\hat{a}$. $\therefore \frac{d\vec{r}}{ds}$ is a unit vector in this direction ⇒ $\frac{d\vec{r}}{ds} = \hat{a}$

So the result follows..

⇒ If $\hat{n}$ is unit vector normal to the level surface $f(x,y,z) = c$ at $P(x,y,z)$.

$$\text{Grad } f = \frac{df}{dn}\hat{n}$$

⇒ Grad f is a vector in the direction of which the maximum value of directional derivative, ie, $\frac{df}{ds}$ occurs

8) Tangent plane: $(R - r)\cdot \nabla f = 0$

where r is the pt at which plane is drawn & ∇f is gradient at r.

$$\Rightarrow \quad (X-x)\frac{\partial f}{\partial x} + (Y-y)\frac{\partial f}{\partial y} + (Z-z)\frac{\partial f}{\partial z} = 0$$

9) Normal ~~plane~~: $(R - r)\times \nabla f = 0$

$$\frac{X-x}{\frac{\partial f}{\partial x}} = \frac{Y-y}{\frac{\partial f}{\partial y}} = \frac{Z-z}{\frac{\partial f}{\partial z}}$$

10) Invariance:

(a) Under a rotation of rectangular axes, the origin remaining the same, the ∇ (del) operator remains invariant

(b) If ϕ is scalar invariant under rotation, grad ϕ is vector invariant

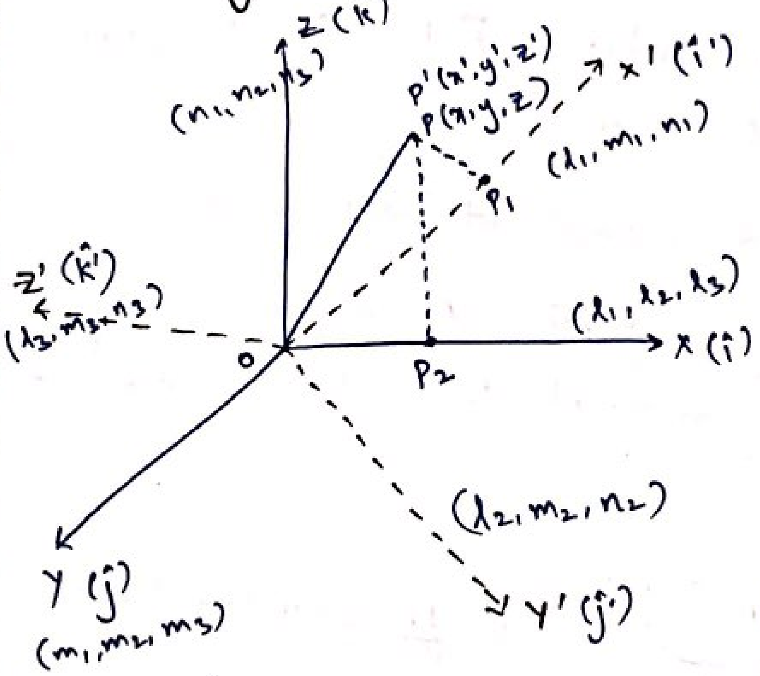
(c) If V is vector invariant under rotation, then

(i) div V is scalar invariant

(ii) curl V is vector invariant

[∵ Since they are defined at a point, ch mere change of axes shouldn't change their nature]

→ Change of axes without changing origin.



Now, $OP_1 = l_1(x-0) + m_1(y-0) + n_1(z-0)$

$x' = l_1 x + m_1 y + n_1 z$

$y' = l_2 x + m_2 y + n_2 z$

$z' = l_3 x + m_3 y + n_2 z$

$OP_2 = l_1 x' + l_2 y' + l_3 z'$

$x = l_1 x' + l_2 y' + l_3 z'$

$y = m_1 x' + m_2 y' + m_3 z'$

$z = n_1 x' + n_2 y' + n_2 z'$

⇒

	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3

$\therefore \hat{i}' = l_1 \hat{i} + m_1 \hat{j} + n_1 \hat{k}$

$\hat{j}' = l_2 \hat{i} + m_2 \hat{j} + n_2 \hat{k}$

$\hat{k}' = l_3 \hat{i} + m_3 \hat{j} + n_3 \hat{k}$

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} = \frac{\partial}{\partial x}(l_1 \hat{i}' + l_2 \hat{j}' + l_3 \hat{k}') + \frac{\partial}{\partial y}(m_1 \hat{i}' + m_2 \hat{j}' + m_3 \hat{k}') + \frac{\partial}{\partial z}(n_1 \hat{i}' + n_2 \hat{j}' + n_3 \hat{k}')$$

$$\Rightarrow \hat{i}'\left(l_1 \frac{\partial}{\partial x} + m_1 \frac{\partial}{\partial y} + n_1 \frac{\partial}{\partial z}\right) + \hat{j}'\left(l_2 \frac{\partial}{\partial x} + m_2 \frac{\partial}{\partial y} + n_2 \frac{\partial}{\partial z}\right) + \hat{k}'\left(l_3 \frac{\partial}{\partial x} + m_3 \frac{\partial}{\partial y} + n_3 \frac{\partial}{\partial z}\right)$$

$$= \hat{i}' \frac{\partial}{\partial x'} + \hat{j}' \frac{\partial}{\partial y'} + \hat{k} \frac{\partial}{\partial z'} = \nabla'$$

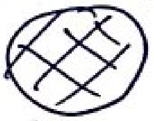
Rest results will follow

→ Green's, Gauss' & Stoke's Theorems :

1) Smooth Curve : $C : \vec{r}(t)$ where $\vec{r}(t)$ is cont. & has cont. 1^{st} order derivative $\neq \vec{0}$

2) Smooth surface : unique normal at each point.

3) Simply Connected Region : Region R in which every closed curve can be contracted to a point w/o passing out of the region.



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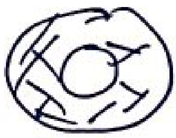


MCR.

$\hat{t} = \frac{d\vec{r}}{ds}$ [unit tangent vector

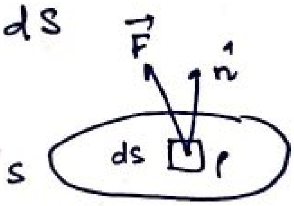
4) Line Integral : $\int_A^B \left[\vec{F} \cdot \frac{d\vec{r}}{ds}\right] ds = \int_A^B \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r}$

$= \int_C F_1 dx + F_2 dy + F_3 dz$

⇓

WD by force $\vec{F}$ in displacement along C.

5) Surface Integral : $\iint_S f(x, y, z)\, dS$



Flux : $\iint_S (\vec{F} \cdot \hat{n})\, ds$

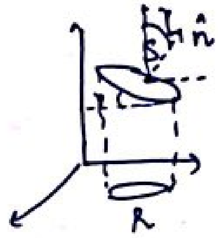
$= \iint_S \vec{F} \cdot d\vec{S}$ where $d\vec{S} = \hat{n}\, ds$

* $\hat{n}$ can be obtained using grad ϕ for level surface
* Let dc's be $(l, m, n) = (\cos\alpha, \cos\beta, \cos\gamma)$

$\therefore \iint \vec{F} \cdot \hat{n}\, ds = \iint F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma$

$= \iint_S (F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy)$

NOTE: Surface Integral can be calculated using orthogonal projection of S ~~over~~ on one of the coordinate plane BUT a line $\perp$ to that plane should intersect with the curve only once



$$ds \cos\gamma = dx\,dy$$

$$\therefore ds = \frac{dx\,dy}{|\hat{n}\cdot k|}$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \iint_R \vec{F}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot k|}$$

if we get 'z' in this integrand, it needs to be replaced by $z = h(x,y)$ using the level surface.

eg. Evaluate $\int_C xy^3\,ds$ where C is line $y = 2x$ from $(-1,-2)$ to $(1,2)$

$\vec{r} = t\hat{i} + 2t\hat{j}$ $\quad t = -1$ to 1

$$\frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right| \Rightarrow \frac{ds}{dt} = |\hat{i} + 2\hat{j}| \Rightarrow ds = \sqrt{5}\,dt$$

$$\therefore I = \int_{-1}^{1} t\cdot(2t)^3\sqrt{5}\,dt = \frac{16}{\sqrt{5}}$$

6) GREEN'S THEOREM IN PLANE:

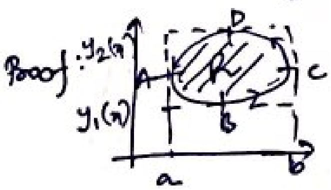
R is a region in xy-plane whose boundary is a simple closed piecewise smooth curve 'C'.

Then
$$\oint_C \vec{F}\cdot d\vec{r} = \boxed{\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy}$$

where C is traversed to keep R on its left.

Here $P(x,y)$ & $Q(x,y)$ are cont. with cont. PDs.

Proof:



$$\iint_R \frac{\partial P}{\partial y}\,dx\,dy = \int_a^b \left[\int_{y_1(x)}^{y_2(x)} \frac{\partial P}{\partial y}\,dy\right] dx = \int_a^b [P(x,y)]_{y_1}^{y_2}$$

$$= \int_a^b \{P(x,y_2) - P(x,y_1)\}\,dx = -\int_{ABC} P\,dx - \int_{CDA} P\,dx = -\oint_{ABCD} P\,dx$$

Sly we get for $\oint Q\,dy$

vector notation : $\iint_R (\nabla \times \vec{F}) \cdot \hat{k}\, dR = \oint_C \vec{F} \cdot d\vec{r}$

where $dR = dx\,dy$ and $\hat{k}$ is unit vector $\perp$ to xy-plane

Example : C is simple closed curve not enclosing origin.

ST: $\int_C \vec{F} \cdot d\vec{r} = 0$ where $\vec{F} = \dfrac{-\hat{i}y + \hat{j}x}{x^2+y^2}$

$$\int_C \vec{F} \cdot d\vec{r} = \int \underbrace{\frac{-y}{x^2+y^2}}_{M} dx + \underbrace{\frac{x}{x^2+y^2}}_{N} dy$$

This is impt

Both M, N are well-defd & continuous with cont. PD since origin is not in the region enclosed by C

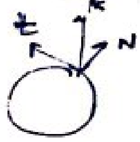
$$\therefore \int M dx + N dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint 0\, dx\,dy = 0.$$

7) Area bounded by a simple closed curve C is

$$\frac{1}{2} \oint_C (x\,dy - y\,dx)$$

Example : If $A = N\hat{i} - M\hat{j}$, ST Green's theorem may be written as $\iint_R \operatorname{div} A\, dx\,dy = \oint_c A \cdot \hat{n}\, ds$ where $\hat{n}$ is unit outward normal to C & s is the arc length

$$\iint_R \operatorname{div} A\, dx\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \oint_C M dx + N dy \text{ [by greens]}$$



$$M dx + N dy = \left((M\hat{i} + N\hat{j}) \cdot \frac{d\vec{r}}{ds}\right) ds = \left((M\hat{i} + N\hat{j}) \cdot \hat{t}\right) ds$$

$$= \left[(M\hat{i} + N\hat{j}) \cdot (\hat{k} \times \hat{n})\right] ds = \left(\left[(M\hat{i} + N\hat{j}) \times \hat{k}\right] \cdot \hat{n}\right) ds = \left((N\hat{i} - M\hat{j}) \cdot \hat{n}\right) ds$$

$$= A \cdot \hat{n}\, ds.$$

If we put $A = \nabla\phi$

we get $\iint \nabla^2 \phi\, dx\,dy = \oint \frac{\partial \phi}{\partial n} ds$

Don't forget.

$\because$ $\boxed{\nabla \phi = \frac{\partial \phi}{\partial n} \hat{n}}$

Very important.

7) GAUSS' DIVERGENCE THEOREM : (Triple to surface integral) & converse.

V is volume bounded by closed ~~sm~~ piecewise smooth surface S. , then

$$\boxed{\iiint_V (\nabla\cdot\vec{F})\, dV = \iint_S \vec{F}\cdot\hat{n}\, dS}$$

$$\Rightarrow \iiint \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz = \iint_S \left(F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy\right)$$

Example:

(a) PT: $\iiint_V \nabla\phi\, dV = \iint_S \phi\,\hat{n}\, dS$

Take $F = \phi c$ where c is a constant vector.

$\nabla\cdot\vec{F} = \nabla\cdot(\phi c) = \nabla\phi\cdot c$

$$\therefore \iiint \nabla\cdot\vec{F}\, dV = \iint_S \vec{F}\cdot\hat{n}\, ds$$

$$\Rightarrow \iiint \nabla\phi\cdot c\, dv = \iint \phi\, c\cdot\hat{n}\, ds = \iint \phi\hat{n}\cdot c\, ds$$

$$\Rightarrow \left[\iiint_V \nabla\phi\, dv - \iint \phi\hat{n}\, ds\right]\cdot c = 0$$

~~Take~~ since c is any arbitrary constant vector, result follows

(b) PT: $\iiint_V \nabla\times\vec{B}\, dV = \iint_S \hat{n}\times\vec{B}\, d\vec{S}$ Very important result & technique

Take $F = \vec{B}\times\vec{C}$ where C is constt vector & proceed as in above question.

Using this we can show: $\iint_S \vec{n}\times(\vec{a}\times\vec{r})\, ds = 2V\vec{a}$

difficult to manipulate otherwise

(c) Evaluate: $\iint_S (x+z)\,dy\,dz + (y+z)\,dz\,dx + (x+y)\,dx\,dy$

where $S: x^2+y^2+z^2=4$

$$I = \iiint_V \left[\frac{\partial}{\partial x}(x+z) + \frac{\partial}{\partial y}(y+z) + \frac{\partial}{\partial z}(x+y)\right] dx\,dy\,dz$$

$$= \iiint 2\,dx\,dy\,dz = 2V$$

(uses 2^{nd}/cartesian form of theorem. It is also impt!!)

*

(d) $F = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

Find $\iint_S (\nabla\times F)\cdot\hat{n}\,ds$ where S is the surface of sphere above xy plane.

In such questions, we use divergence theorem in a bounded volume V by completing the open surface given

Since $\nabla\cdot(\nabla\times\vec{F}) = 0$, we get

$$\iiint \nabla\cdot(\nabla\times\vec{F})\,dV = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds + \iint_{S_1} (\nabla\times\vec{F})\cdot\hat{n}\,ds$$

S_1 completes S.

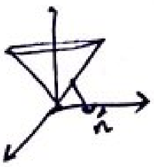
$$\therefore\ 0 = I + \iint_{S_1} (\nabla\times\vec{F})\cdot\hat{n}\,ds$$

$$\therefore\ I = -\iint_{S_1} (\nabla\times\vec{F})\cdot\hat{n}\,ds$$ which is relatively easier Integral to be calculated directly!

See example 56

Evaluate $\iint_S (x^2+y^2)\,ds$ where S is surface of cone $z^2 = 3(x^2+y^2)$ bounded by $z=0$ & $z=3$. (Ans $= 9\pi$)

Here we try to put $\iint_S A\,ds$ as $\iint F\cdot\hat{n}\,ds$



$$\hat{n} = \frac{3(x\hat{i}+y\hat{j}) - \hat{k}}{2z}$$

[Note it's not other way around because we want outward $\hat{n}$. Use ~~signs of~~ direction of $\hat{n}$ in 1^{st} octant to get it right]

we get $\vec{F} = \frac{2z}{3}(x\hat{i}+y\hat{j})$

Now apply GDT in $S + S_1'$ (plane $z=3$) $= S'$

We can easily show $\iint_{S_1} (x^2+y^2)\,ds = 0$

8> Stoke's Theorem:

S is a piecewise smooth surface bounded by a piecewise smooth simple closed curve C.

$$\boxed{\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\, dS}$$

Green's is special form of stokes where $dS = dx\,dy$ & $\hat{n} = \hat{k}$

⇒ Line integral of tangential component of $\vec{F}$ taken around C is equal to surface integral of normal component of $\nabla\times\vec{F}$ taken over any surface S which has boundary as C.

$$\oint_C F_1dx + F_2dy + F_3dz = \iint_S\left[\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\cos\gamma\right] ds$$

Examples:

(a) PT: $\oint_C \phi\, d\vec{r} = \iint_S d\vec{S}\times\nabla\phi$

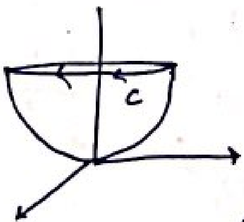
Take $\oint_C \phi\vec{B}\cdot d\vec{r} = \iint_c [\nabla\times(\phi\vec{B})]\cdot d\vec{s} = \iint \nabla\phi\times B + \phi(\overset{0}{\cancel{\nabla\times\vec{B}}})\cdot d\vec{s}$ [∵ B as constt]

$$= \iint \nabla\phi\times B\cdot d\vec{s} = \iint ds\times\nabla\phi\cdot\vec{B}$$

$$\Rightarrow \left(\oint_C \phi\, d\vec{r} - \iint ds\times\nabla\phi\right)\cdot\vec{B} = 0$$

(b) Verify SThm for $\vec{A} = 3y\hat{i} - xz\hat{j} + yz^2\hat{k}$ where S is the surface of paraboloid $2z = x^2+y^2$ bounded by $z=2$ & C is its boundary

to avoid confusion as to is this surface included, calculate the surface integral later



Here $\oint \vec{A}\cdot d\vec{r} = \oint_C (3y\,dx - 2x\,dy)$

Since ~~ea~~ surface lies below the curve, we traverse C in CW sense (very very important)

Rest result is easy

⇒ Potential & Conservative vector field :-

1) $f\,dx + g\,dy + h\,dz$ is said to be an exact differential if $\exists$ a scalar $\phi(x,y,z)$ having cont. PD st $d\phi = f\,dx + g\,dy + h\,dz$

2) $\int_C f\,dx + g\,dy + h\,dz$ is <u>independent of path</u> iff integrand is an exact differential. ie,

iff $\vec{F} = f\hat{i} + g\hat{j} + h\hat{k} = \nabla\phi$

, ie, $\underline{\text{curl } \vec{F} = 0}$

ie, F is a <u>conservative vector field</u>

ie, F is <u>irrotational</u>

In proof: if $\nabla\times F = 0$
$\Rightarrow \oint_C \vec{F}\cdot d\vec{r} = 0$
Let $\phi = \int_{P=(x_0,y_0,z_0)}^{Q(x,y,z)} \left(\vec{F}\cdot\frac{d\vec{r}}{ds}\right) ds$
$\Rightarrow \frac{d\phi}{ds} = \vec{F}\cdot\frac{d\vec{r}}{ds} \Rightarrow \nabla\phi\cdot\frac{d\vec{r}}{ds} = \vec{F}\cdot\frac{d\vec{r}}{ds}$
$\Rightarrow (\nabla\phi - \vec{F})\cdot\frac{d\vec{r}}{ds} = 0$
$\because$ It is true independent of P & Q
$\Rightarrow \nabla\phi - \vec{F} = 0 \Rightarrow F = \text{grad}\,\phi$

3) For above result to hold, the region R for which boundary is C must be simply connected.

eg $\quad F = \frac{-y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j}$

Here $\nabla\times F = 0$ But if region includes origin, we get
$$\oint_C \vec{F}\cdot d\vec{r} = 2\pi$$

4) F is Irrotational ⇒ (i) $F = \nabla\phi$ (ii) $\nabla\times\vec{F} = 0$ (iii) $\oint_C \vec{F}\cdot d\vec{r} = 0$

All these are equivalent.

5) eg check if the following is exact & if yes find potential.

$x\,dx - y\,dy - z\,dz$

Take $\vec{F} = x\hat{i} - y\hat{j} - z\hat{k}$

curl $\vec{F} = 0$ so yes exact

Let $x = \frac{\partial\phi}{\partial x} \Rightarrow \phi = \frac{x^2}{2} + f(y,z)$

Let $-y = \frac{\partial\phi}{\partial y} \Rightarrow \phi = \frac{-y^2}{2} + g(x,z)$

Let $-z = \frac{\partial\phi}{\partial z} \Rightarrow \phi = -\frac{z^2}{2} + h(x,y)$

$\Rightarrow \phi = \frac{x^2 - y^2 - z^2}{2} + C.$

⇒ Different Coordinate systems :

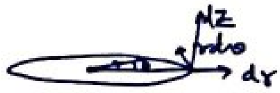
1) cylindrical : $P(x,y,z) \longrightarrow P(\rho, \phi, z)$

ϕ ↳ azimuthal angle

$x = \rho\cos\phi$ $\qquad \rho = \sqrt{x^2+y^2} \quad (0 < \rho < \infty)$

$y = \rho\sin\phi$ $\qquad \phi = \tan^{-1}(y/x) \quad (0 \le \phi \le 2\pi)$

$z = z$ $\qquad z = z \quad (-\infty < z < \infty)$



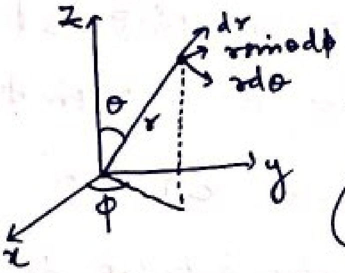
$$dV = (r\,d\theta)(dr)(dz) = r\,dr\,d\theta\,dz$$

2) spherical polar coordinates :

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

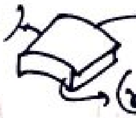


$\therefore dV = (dr)(r\,d\theta)(r\sin\theta\,d\phi)$

$= r^2\sin\theta\,dr\,d\theta\,d\phi$

↳ easy to imagine.

take 3 ⊥r directions



dS depends on the surface we are talking about

(a) r = constant ⇒ spherical surface

$$dS = (r\sin\theta\,d\phi)(r\,d\theta) = r^2\sin\theta\,d\theta\,d\phi$$

(b) θ = constant ⇒ cone

$$dS = (r\sin\theta\,d\phi)(dr) = r\sin\theta\,dr\,d\phi$$

(c) ϕ = constant ⇒ plane through origin

$$dS = r\,dr\,d\theta$$

Unit vectors : $\hat{e}_r$ = Normal to surface r = constant $= \frac{\partial\vec{r}}{\partial r} \Big/ \left|\frac{\partial\vec{r}}{\partial r}\right|$

$\hat{e}_\phi$ = Normal to surface ϕ = constant $= \frac{\partial\vec{r}}{\partial\phi} \Big/ \left|\frac{\partial\vec{r}}{\partial\phi}\right|$

$\hat{e}_\theta$ = Normal to surface θ = constant $= \frac{\partial\vec{r}}{\partial\theta} \Big/ \left|\frac{\partial\vec{r}}{\partial\theta}\right|$

$\therefore \hat{e}_r = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$ $\quad (\theta \to \theta + \frac{\pi}{2})$

$\hat{e}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$ $\qquad \hat{e}_\theta = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$

$$\vec{r} = r\hat{e}_r$$
$$\vec{dr} = \hat{e}_r dr + r\hat{e}_\theta d\theta + r\sin\theta\,\hat{e}_\phi\, d\phi$$

in cylindrical : $\vec{r} = \rho\,\hat{e}_\rho + z\hat{e}_z$

$$d\vec{r} = \hat{e}_\rho\, d\rho + \rho\, d\phi\, \hat{e}_\phi + \hat{e}_z\, dz$$

3) Ellipsoid : $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

use $x = ar\sin\theta\cos\phi$
$y = br\sin\theta\sin\phi$
$z = cr\cos\theta$

$dv = abc\, r^2\sin\theta\, dr\, d\theta\, d\phi$

Example :

Ques. Represent $A = z\hat{i} - 2x\hat{j} + y\hat{k}$ in cylindrical coordinates. Thus determine A_ρ, A_ϕ, A_z.

$r = x\hat{i} + y\hat{j} + z\hat{k} = \rho\cos\phi\,\hat{i} + \rho\sin\phi\,\hat{j} + z\hat{k}$

Tangent vectors are given by : $\frac{\partial r}{\partial \rho}$, $\frac{\partial r}{\partial \phi}$, $\frac{\partial r}{\partial z}$

$\frac{\partial r}{\partial \rho} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$ $\quad \frac{\partial r}{\partial \phi} = -\rho\sin\phi\,\hat{i} + \rho\cos\phi\,\hat{j}$ $\quad \frac{\partial r}{\partial z} = \hat{k}$

$\therefore \hat{\rho} = \frac{\partial r}{\partial \rho} \Big/ \left|\frac{\partial r}{\partial \rho}\right| = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$ $\quad \hat{\phi} = \frac{\partial r}{\partial \phi}\Big/\left|\frac{\partial r}{\partial \phi}\right| = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$

$\hat{z} = \frac{\partial r}{\partial z}\Big/\left|\frac{\partial z}{\partial z}\right| = \hat{k}$ $\quad \Rightarrow \quad \hat{i} = \cos\phi\,\hat{\rho} - \sin\phi\,\hat{\phi}$
$\hat{j} = \sin\phi\,\hat{\rho} + \cos\phi\,\hat{\phi}$
$\hat{k} = \hat{z}$

Put these values in A to get $A = A_\rho\,\hat{\rho} + A_\phi\,\hat{\phi} + A_z\,\hat{z}$