Eigen value & eigenvectors

$A\underset{\sim}{v} = \lambda \underset{\sim}{v}$; $A \to n \times n$

$\underset{\sim}{v} \neq \underset{\sim}{0}$.

Ex-1

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Problem discussed on 13/1/17.

$$|\lambda I - A| = \begin{vmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{vmatrix} = (\lambda-2)^2(\lambda-1).$$

$|\lambda I - A| = 0$.

$\therefore \lambda = 2, 2, 1$ (e-values of A).

To find e-vectors corresponding to $\lambda = 2$.

Solve, $(\lambda I - A)_{\lambda=2}\underset{\sim}{v} = \underset{\sim}{0}$ $(\underset{\sim}{v} \neq \underset{\sim}{0})$.

$$\Rightarrow \begin{pmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$$

$2x_1 + 2x_3 = 0$.

$\therefore x_3 = c$, $x_2 = b$ (arbit.)

$\therefore (x_1, x_2, x_3) = (-c, b, c) = c\underbrace{(-1, 0, 1)}_{\underset{\sim}{v}_1} + b\underbrace{(0, 0, 1)}_{\underset{\sim}{v}_2}$.

$\because (x_1, x_2, x_3)$ is an e-vector both of b & c cannot be zero

e-space corresponding to $\lambda = 2$

$= E_{\lambda=2} = \{(-c, b, c) : b, c \in \mathbb{R}\}$

Ch. $\underset{\sim}{v}_1, \underset{\sim}{v}_2$ are l.i. $\therefore \underset{\sim}{v}_1, \underset{\sim}{v}_2$ form a basis for $E_{\lambda=2}$.

Note dimension of $E_{\lambda=2} = 2$ $\therefore \{(-1, 0, 1), (0, 0, 1)\}$ forms a basis for $E_{\lambda=2}$.

To find e-vectors corresponding to $\lambda = 1$

Solve, $(\lambda I - A)_{\lambda=1}\underset{\sim}{v} = \underset{\sim}{0}$ $(\underset{\sim}{v} \neq \underset{\sim}{0})$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$x_1 + 2x_3 = 0$. $\quad x_3 = c$, $x_2 = c$, $x_1 = -2c$.

$-x_2 + x_3 = 0$. $\quad (x_1, x_2, x_3) = c(-2, 1, 1)$.

$E_{\lambda=1} = \{(-2c, c, c) : c \in \mathbb{R}\}$

Dimension of $E_{\lambda=1} = 1$ $\quad \therefore (-2, 1, 1)$ forms a basis for $E_{\lambda=2}$.

Lecture - 5
19/1/2017.

<u>Definition</u>. <u>Algebraic multiplicity</u> of an eigenvalue λ is the multiplicity of λ as a root of the ch. equation.
(characteristic)

$\because$ in the given example roots of the ch. equation are $\lambda = 2, 2, 1$

$\therefore$ algebraic multiplicity of 2 is $a_{\lambda=2} = 2$.

" " " 1 is $a_{\lambda=1} = 1$.

<u>Geometric multiplicity</u> (g_λ) of λ is the dimension of e-space corr. to λ.
(Eigen)

geometric multiplicity of $\lambda = 2$ is 2

" " " $\lambda = 1$ is 1.

<u>Thm.</u> $a_\lambda \geq g_\lambda$.

The difference $a_\lambda - g_\lambda$ ~~when~~ is called <u>defect</u> of λ.

<u>Ex.</u> $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. $|\lambda I - A| = 0$.

$\begin{vmatrix} \lambda & -1 \\ 0 & \lambda \end{vmatrix} = 0$ $\quad \lambda^2 = 0 \Rightarrow \lambda = 0, 0$.

$a_{\lambda=0} = 2$.

$$\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$0 \cdot x_1 - x_2 = 0$.
$x_2 = 0$.

$(x_1, x_2) = (a, 0) = a(1, 0)$

$E_{\lambda=0} = \{(x_1, x_2) = (a, 0) ; a \in \mathbb{R}\}$.

$(1, 0) \rightarrow$ basis for $E_{\lambda=0}$. dim of $E_{\lambda=0} = 1$

$\therefore g_{\lambda=0} = 1$.

$a_{\lambda=0} = 2$, $g_{\lambda=0} = 1$. $\therefore$ defect of $\lambda = 0 = a_\lambda - g_\lambda$
$= 1$.

Exercise. Find the e-values & e-vectors of the matrix $A = \begin{bmatrix} 2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$.

Also find the g_λ, a_λ for each λ.

Thm. Let λ be an e-value of a matrix A.

(1). λ^k is an e-value of A^k.

(2) $\frac{1}{\lambda}$ is an e-value of A^{-1}.

(3) $\lambda - p$ " " " " $A - pI$.

Thm. AB & BA have same e-values.

Thm. sum of e-values of A = Trace of A.

product of e-values of A = det. A.

$\boxed{\text{Cayley-Hamilton theorem}}$

Every square matrix is a root of its characteristic equation.

$A \to n \times n$ matrix, Ch. eq. of A looks like

$$a_0 \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \cdots + a_n = 0$$

Then $a_0 A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_n I_n = \underset{\sim}{0}$.

Example. Verify Cayley-Hamilton thm for $A = \begin{bmatrix} 1 & 0 \\ 2 & -2 \end{bmatrix}$. Hence find A^{-1}.

Solution. $|xI - A| = 0$. (ch. equation)

$$\begin{pmatrix} x-1 & 0 \\ -2 & x+2 \end{pmatrix} = 0 \Rightarrow (x-1)(x+2) = 0$$
$$\Rightarrow x^2 + x - 2 = 0$$

$$A^2 = \begin{pmatrix} 1 & 0 \\ -2 & 4 \end{pmatrix} \qquad A^2 + A - 2I = \underset{\sim}{0} \longrightarrow (*)$$
$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

To find A^{-1}, multiply both sides of $(*)$ by A^{-1}

$$A + I - 2A^{-1} = \underset{\sim}{0}. \Rightarrow A^{-1} = \frac{1}{2}(A + I).$$

$$A^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & -\frac{1}{2} \end{pmatrix}$$

Exercise. Verify C-H theorem for

$$A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$ Hence find A^{-1}.

~~Solution~~

Similar Matrices.

$A \to n \times n$ matrix.

A square matrix $B_{n\times n}$ is said to be similar to the matrix A, if there exists a non-singular matrix P, such that

$$B = P^{-1}AP.$$

Note, $PBP^{-1} = PP^{-1}APP^{-1} = I_n A I_n = A.$

$$\therefore A = PBP^{-1} = (P^{-1})^{-1} B (P^{-1})$$
$$= Q^{-1}BQ, \quad Q = P^{-1}.$$

$\therefore \Rightarrow A$ is similar to B. & we say. A & B are similar matrices.

<u>Thm.</u> Two similar matrices have same rank, determinant, e-values, ch. equation.

<u>Definition.</u> A square matrix $A_{n\times n}$ is said to be <u>diagonalizable</u>, if it is similar to a diagonal matrix D. i.e if there exists a non-singular matrix P, such that

$$D = P^{-1}AP.$$

Theorems helpful for diagonalization

Thm. Distinct e-vectors corresponding to distinct e-values are linearly independent.

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} \qquad \lambda = 2, 2, 1$$

$E_{\lambda=2} = \{(-c, b, c)\}$. $\quad (-c, b, c) = b(0,1,0) + c(-1,0,1)$

$E_{\lambda=1} = \{(-2c, c, c)\}$. $\quad (-2c, c, c) = c(-2,1,1)$.

$(0,1,0) \in E_{\lambda=2}$, $(-1,0,1) \in E_{\lambda=2}$, $(-2,1,1) \in E_{\lambda=1}$.

$(0,1,0)$ & $(-2,1,1)$ are l.i.

$\therefore (-1,0,1)$ & $(-2,1,1)$ are l.i.

Also, $(0,1,0), (-1,0,1)$ being basis vectors are l.i. So, $(0,1,0), (-1,0,1), (-2,1,1)$ are l.i.

Thm. If $\underset{\text{(geometric multiplicity)}}{g_\lambda} = \underset{\text{(algebraic multiplicity)}}{a_\lambda}$ for each e-value of λ, ~~then~~ of a matrix A, then A is diagonalizable.

Thm. If an $n \times n$ matrix A has n distinct e-values, then it is similar to a diagonal matrix, whose elements are the e-values of A. Suppose $2, -7, 5$ are the e-values of A. Then A is similar to

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & 5 \end{pmatrix} \Big/ \begin{pmatrix} -7 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

~~To~~ 20/1/17 → 3-4 (Friday).

27/1/17 → 3-4 (Friday)

3/2/17 → 3-5 (Friday)

No class on 9/2 & 10/2

8/2 → 5'15 p.m (revision,

– 7:00 p.m. query

(approx)

How to diagonalize a square matrix A.

steps. 1. Find e-values of A & e-vectors.

2. Find g_λ of each e-value.

3. Check whether $g_\lambda = a_\lambda$ for each λ.

If no, stop & say 'A' is not diagonalizable.

4. Else, proceed as follows:

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} \qquad |\lambda I - A| = 0$$

$\Rightarrow \lambda = 2, 2, 1.$

$a_{\lambda=2} = 2, \quad a_{\lambda=1} = 1$

$E_{\lambda=2} = \{(-c, b, c)\}$ $\qquad g_{\lambda=2} = 2.$

Basis vectors → $(-1, 0, 1), (0, 1, 0)$

$E_{\lambda=1} = \{(-2c, c, c)\}$ $\qquad g_{\lambda=1} = 1.$

Basis vector → $(-2, 1, 1)$.

$\therefore$ $a_{\lambda=2} = g_{\lambda=2}$

$a_{\lambda=1} = g_{\lambda=1}$ $\Rightarrow$ A is diagonalizable.

Form ~~the~~ a matrix P with the help of basis vectors.

$$P = \begin{pmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

[Other forms of P -

$$\begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} -2 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Find $P^{-1}AP$

go to p.9.

P is a matrix whose columns are the basis ~~e~~ vectors corresponding to the e-values.

[Gauss - Jordan elimination method.

Gauss - elimination:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \longleftrightarrow \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & a_{32}' & a_{33}' \end{pmatrix} \xrightarrow[\text{from Row 3}]{\text{Eliminate } x_2} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & 0 & a_{33}' \end{pmatrix}$$

Gauss - Jordan elimination.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \longleftrightarrow \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & a_{32}' & a_{33}' \end{pmatrix} \xrightarrow[\text{Row 1, Row 3}]{\text{Eliminate } x_2 \text{ from}} \begin{pmatrix} a_{11}' & 0 & a_{13}' \\ 0 & a_{22}' & a_{23}' \\ 0 & 0 & a_{33}'' \end{pmatrix}$$

$$\xrightarrow[x_3 \text{ from Row 1 \& Row 2}]{\text{Eliminate}} \begin{pmatrix} a_{11}'' & 0 & 0 \\ 0 & a_{22}'' & 0 \\ 0 & 0 & a_{33}'' \end{pmatrix}.$$

Equivalent syst. of equations:

$$a_{11}''x_1 = b_1''$$
$$a_{22}''x_2 = b_2''$$
$$a_{33}''x_3 = b_3''.$$

Form the Block Matrix.

$$\left(\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right) \xrightarrow[\text{-Jordan}]{\text{Apply. Gauss-}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & b_1 & b_2 & b_3 \\ 0 & 1 & 0 & b_4 & b_5 & b_6 \\ 0 & 0 & 1 & b_7 & b_8 & b_9 \end{array}\right)$$

A^{-1}.

$$(A \mid I_n) \xrightarrow[G-J]{\text{Apply}} (I_n \mid A^{-1})$$

Identity matrix.

$$P = \begin{pmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Form $(P \mid I)$

$$= \begin{array}{c} \begin{matrix} \lambda=2 & \lambda=2 & \lambda=1 & & & \end{matrix} \\ \left(\begin{array}{ccc|ccc} -1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \end{array} \xrightarrow{R_3 \to R_3 + R_1} \left(\begin{array}{ccc|ccc} -1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_2 \to R_2 + R_3]{R_1 \to R_1 - 2R_3} \left(\begin{array}{ccc|ccc} -1 & 0 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3 \to -R_3]{R_1 \to -R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & -1 \end{array}\right) \therefore P^{-1} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{pmatrix}$$

Compute

$$P^{-1} A P = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P = \begin{array}{c} \begin{matrix} \lambda=1 & \lambda=2 & \lambda=2 \end{matrix} \\ \begin{pmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \end{array} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Symmetric, Skew-symmetric, orthogonal matrices (real entries)

Hermitian, Skew-hermitian, unitary matrics (complex entries).

Symmetric matrices. A is symmetric if $A = (a_{ij})$,

$a_{ji} = a_{ij}$ i.e if $A = A^T$

A is skew-symmetric if $A = -A^T$.
i.e $a_{ji} = -a_{ij}$.

A is orthogonal if $A^{-1} = A^T$ i.e $A^TA = I$
or, $AA^T = I$.

$$A = \begin{pmatrix} 4 & 3 & -2 \\ 3 & 6 & -9 \\ -2 & -9 & 8 \end{pmatrix} \qquad \begin{pmatrix} 0 & -5 & 3 \\ 5 & 0 & 6 \\ -3 & -6 & 0 \end{pmatrix}$$

symmetric matrix.

skew-symmetric matrix.
$a_{ji} = -a_{ji}$
$a_{ii} = -a_{ii}$ (put $i=j$)
$\Rightarrow 2a_{ii} = 0$.

Check

$$A = \begin{pmatrix} 2/3 & 1/3 & 2/3 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/3\sqrt{2} & -4/3\sqrt{2} & 1/3\sqrt{2} \end{pmatrix}$$

is orthogonal.

Check $AA^T = I$.

~~Herm~~ conjugate transpose of a matrix A (whose elements are complex no.s)

$= A^* = \bar{A}^T$

Hermitian matrix $A = A^* = \bar{A}^T$.

Skew- " matrix $A = -A^* = -\bar{A}^T$.

Unitary matrix. if $A^{-1} = A^* = \bar{A}^T$.

$$A = \begin{pmatrix} 2 & 1+i & 6i \\ 1-i & 0 & -3+4i \\ -6i & -3-4i & 4 \end{pmatrix} \qquad A = \begin{pmatrix} i & 2+2i & 4+i \\ -2+2i & 0 & -6-3i \\ -4+i & 6-3i & 3i \end{pmatrix}$$

$-3i$

Hermitian.

(Note. diagonal elements of Herm. matrix → real)

" " " skew-Herm " → purely imaginary / zero.

prove (Hint). $c_{jk} = \bar{c}_{kj}$ (Hermitian)

$a_{jk} + i b_{jk} = a_{kj} - i b_{kj}$.

<u>Thm.</u> E-values of Hermitian (& symmetric) matrices are real.

E-values of Skew-Hermitian (& skew-symmetric) matrices are purely imaginary or zero.

E-values of Unitary (& orthogonal) matrices have modulus equal to 1.

[orthogonal, $\lambda = +1/-1$. $|\lambda| = 1$. $AA^T = I$, $|AA^T| = I$. $|A|^2 = 1$.

Unitary. $\lambda = \frac{1}{2} + \frac{i\sqrt{3}}{2}$, $\lambda = \frac{1}{2} - \frac{i\sqrt{3}}{2}$. $\frac{1}{4} + \frac{3}{4}$ 11