# Ordinary Differential Equations

## Formulations :

1.> $y = Ae^{ax} + Be^{bx} \Rightarrow y'' - (a+b)y' + aby = 0$

2.> $y = Ae^{ax} + Be^{bx} + Ce^{cx} \Rightarrow y''' - (a+b+c)y'' + (ab+bc+ca)y' - abc\,y = 0$

## Solutions :

* Order = 1 , Degree = 1.

1.> Variable Separation : All $y$ and $dy$ one one side & all $x$ and $dx$ on other side

2.> Reduction to variable separable form :

$$\frac{dy}{dx} = f(ax+by+c) \Rightarrow \text{Put } z = ax+by+c$$

3.> $$\frac{dy}{dx} = \frac{(ax+by)+c}{m(ax+by)+c_1} \Rightarrow \text{Put } z = ax+by$$

4.> Homogenous D.E. $\Rightarrow \frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$

where $f$, $g$ are homogenous and of same degree

Put $y = tx$

5.> Reduction to Homogenous form $\Rightarrow \frac{dy}{dx} = \frac{ax+by+c}{a_1x+b_1y+c_1}$

* $\frac{a}{a_1} \neq \frac{b}{b_1}$ Put $x = X+h$ , $y = Y+k$

$$h = \frac{bc_1 - b_1c}{ab_1 - a_1b} , \quad k = -\frac{(ac_1 - a_1c)}{ab_1 - a_1b}$$

$* \quad \frac{a}{a_1} = \frac{b}{b_1} \Rightarrow \frac{dy}{dx} = \frac{(ax+by)+c}{m(ax+by)+c_1}$

5.) Exact Differential Equation ⇒ $M(x,y)\,dx + N(x,y)\,dy = 0$

exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$G.S = \int_{y\text{-constt}} M\,dx + \int (\text{terms of } N \text{ not having } x)\,dy = c$$

Integrating Factor :

To make $M dx + N dy = 0$ exact

1.) $M dx + N dy = 0$ is homogenous & $Mx + Ny \neq 0$

$$I.F. = \frac{1}{Mx+Ny}$$

2.) $M = y\,f_1(x,y)$ ; $N = x\,f_2(x,y)$ ; $Mx - Ny \neq 0$

not $f(x,y)$ but $f(xy)$

$$I.F = \frac{1}{Mx - Ny}$$

3.) $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$ or constt $(K)$

$$I.F. = e^{\int f(x)\,dx} \text{ or } e^{\int k\,dx}$$

4.) $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = f(y)$ or $K$

$$I.F = e^{\int f(y)\,dy} \text{ or } e^{\int K\,dy}$$

5) If $M dx + N dy = 0$ can be put as $x^{\alpha} y^{\beta}(my\,dx + nx\,dy) + x^{\alpha_1} y^{\beta_1}(m_1 y\,dx + n_1 x\,dy) = 0$ where $\alpha_i, \beta_i, m, n$ are constants, then $x^h y^k$ is an IF which can be found using conditions for exact differential.

→ Linear Equations with constant coefficients :

$$y = y_c + y_p$$

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1}\frac{dy}{dx} + a_n y = Q$$

↳ $a_1, \dots, a_n$ are constts.

Homogenous if $Q = 0$ ⇓

$$f(D)\, y = Q \quad \text{where} \quad f(D) = D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n$$

Complementary Function :

Auxilliary equation :

$$AE = \quad m^n + a_1 m^{n-1} + \dots + a_{n-1} m + a_n = 0$$

Cases :

(i) Roots are real and distinct : $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$

(ii) 2 roots are equal, other distinct $y = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$

(iii) All roots are equal : $y = (c_1 + c_2 x + \dots + c_n x^{n-1}) e^{mx}$

(iv) complex roots : $\alpha \pm i\beta$ : $y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

(v) repeated complex roots : $y = e^{\alpha x}\big((A+Bx)\cos\beta x + (C+Dx)\sin\beta x\big)$

(vi) $\alpha \pm \sqrt{\beta}$, surd roots : $y = e^{\alpha x}\left[c_1 \cosh\sqrt{\beta}\,x + c_2 \sinh\sqrt{\beta}\,x\right]$

(vii) Repeated surd roots : $y = e^{\alpha x}\left[(c_1 + c_2 x)\cosh\sqrt{\beta}x + (c_3 + c_4 x)\sinh\sqrt{\beta}x\right]$

Particular Integral :

$$y = \frac{1}{f(D)} Q(x)$$

Cases

(1) $Q(x) = e^{ax}$ , $f(a) \neq 0$ ⇒ $y = \frac{1}{f(a)} e^{ax}$

(2) $Q = \sin(ax)$ or $\cos(ax)$ , $f(-a^2) \neq 0$ ⇒ $\frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax$

$$\frac{1}{f(D^2)} \cos ax = \frac{1}{f(-a^2)} \cos ax$$

$+ma_1 dy)$
$=0$

case (iii) : $Q = x^m$ or polynomial of degree $m$ s.t $m \geq 0$ & $m \in I$

$$\frac{1}{f(D)} x^m \Rightarrow \frac{1}{[1 \pm F(D)]} x^m \Rightarrow [1 \pm F(D)]^{-1} x^m$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$$
$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$$
$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 \dots$$
$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3$$

Take out the common lowest degree term from $f(D)$ to make it of the form $[1 \pm F(D)]$

case (iv) : $Q = e^{ax} \cdot v(x)$

$$\frac{1}{f(D)} e^{ax} \cdot v(x) = e^{ax} \cdot \frac{1}{f(D+a)} v(x)$$

case (v) : $Q = e^{ax}$, $f(a) = 0$

$$\frac{1}{f(D)} e^{ax} = e^{ax} \cdot \frac{1}{f(D+a)}$$

$$\frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax}$$

factorize $f(D)$, first operate on $e^{ax}$ by the factor which does not vanish by putting 'a' for D

case (vi) : $\frac{1}{D^2+a^2} \sin ax = \frac{-x \cos ax}{2a}$ if $f(-a^2) = 0$

$\frac{1}{D^2+a^2} \cos ax = \frac{x \sin ax}{2a}$ if $f(-a^2) = 0$

case (vii) : $Q = xv$ $v = f(x)$

$$\frac{1}{f(D)} (xv) = x \frac{1}{f(D)} v - \frac{f'(D)}{[f(D)]^2} v$$

case (viii) : $Q = f(x) = Q(x)$

(i) $$\frac{1}{D-\alpha} Q = e^{\alpha x}\int e^{-\alpha x} Q\,dx$$

(ii) $$\frac{1}{D+\alpha} Q = e^{-\alpha x}\int e^{\alpha x} Q\,dx$$

## Cauchy - Euler Equations :

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_2 x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_{n-1} x\frac{dy}{dx} + a_n y = X$$

where $a_1, a_2, \ldots, a_n$ are constants and $X$ is a function of $x$.

$\Rightarrow$ Put $x = e^z$

$$\Big[[D_1(D_1-1)(D_1-2)\cdots(D_1-(n-1))] + a_1[D_1(D_1-1)\cdots(D_1-(n-2))] + \cdots + a_n\Big]y = e^z$$

where $D_1 = \frac{d}{dz}$

If $sol^n$ is $y = F(z) \Rightarrow F(\log x)$

(i) $X = x^m$ : $\frac{1}{f(D_1)} x^m = \frac{1}{f(m)} x^m$ if $f(m) \neq 0$

(ii) $$\frac{1}{D_1-\alpha} f(x) = x^{\alpha}\int x^{-\alpha-1}\, f(x)\,dx$$

$$\frac{1}{D_1+\alpha} f(x) = x^{-\alpha}\int x^{\alpha-1} f(x)\,dx$$

Beneficial when $X$ has terms like $\sin x$, $\cos x$.
When $X = x^m$ or $\log x$, better put $x = e^z$ & then solve

## Legendre's Linear Equations

$$(a+bx)^n \frac{d^n y}{dx^n} + A_1 (a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + A_{n-1}(a+bx)\frac{dy}{dx} + A_n y = Q(x)$$

$\Rightarrow$ Put $a+bx = e^z$

$$\left[b^n D_1(D_1-1)\cdots(D_1-(n-1)) + b^{n-1} D_1(D_1-1)\cdots(D_1-(n-2))A_1 + \cdots + bD_1 A_{n-1} + A_n\right]y = Q\frac{(e^z-a)}{b}$$

LDE of 2nd order with variable coefficients :

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = R(x)$$

(1) Change dependent variable if part of CF is known :

$y = u(x)$ is a known solⁿ of CF

Put $y = uv$ as the general solution

$$v = \int \left[\left(\frac{1}{u^2} e^{-\int P(x)dx} \int \left(Ru\, e^{\int P(x)dx}\right) dx + \frac{c\,e^{-\int P(x)dx}}{u^2}\right)\right] dx + c_2$$

(2) Find one integral solution in CF by inspection :

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0$$

cases :

1) $a^2 + aP + Q = 0 \Rightarrow u = e^{ax}$

2) $1 + P + Q = 0 \Rightarrow u = e^{x}$

3) $1 - P + Q = 0 \Rightarrow u = e^{-x}$

4) $m(m-1) + Pmx + Qx^2 = 0 \Rightarrow u = x^m$

5) $P + Qx = 0 \Rightarrow u = x$

6) $2 + 2Px + Qx^2 = 0 \Rightarrow u = x^2$

for $v \Rightarrow \dfrac{d^2v}{dx^2} + \left(P + \dfrac{2}{u}\dfrac{du}{dx}\right)\dfrac{dv}{dx} = \dfrac{R}{u}$

(3) Changing the dependant variable and removing first derivative

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Q = R \Rightarrow \frac{d^2v}{dx^2} + Iv = S \quad \left[\text{Normal form}\right]$$

$$u = e^{-1/2\int P(x)\,dx} ; \quad I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} ; \quad S = \frac{R}{u}$$

Get $v$ from normal form and $y = uv$.

(4) Changing the independent variable

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

Let $z = f(x)$

Transform to : $\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} ; \quad Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} ; \quad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

case (i) : $P_1 = 0 \Rightarrow z = \int e^{-\int P\,dx}\,dx$

(ii) $Q_1 = \pm a^2 \Rightarrow a\int dz = \int \sqrt{\pm Q}\,dx$

Method of Variation of parameters :

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = R(x)$$

Get the HE : $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$

If $y_c = c_1 u(x) + c_2 v(x)$

Then , $y_p = Au(x) + Bv(x)$ where

$$A = -\int \frac{vR}{uv' - u'v}\,dx \qquad B = \int \frac{uR}{uv' - u'v}\,dx$$

⇒ Differential Equations of Ist order but not of first degree :

$$\left(\frac{dy}{dx}\right)^n + A_1\left(\frac{dy}{dx}\right)^{n-1} + \cdots + A_{n-1}\frac{dy}{dx} + A_n y = 0$$

$$\Rightarrow \quad p^n + A_1 p^{n-1} + \cdots + A_{n-1}p + A_n y = 0$$

where $p = \dfrac{dy}{dx}$

① Solvable for p :

$$\Rightarrow [p - f_1(x,y)][p_2 - f_2(x,y)] \cdots [p - f_n(x,y)] = 0$$

② Solvable for x :

$$x = f(y, p)$$

$$\Rightarrow \frac{dx}{dy} = F\left(y, p, \frac{dp}{dy}\right) = \frac{1}{p}$$

③ Solvable for y :

$$y = f(x, p)$$

$$\frac{dy}{dx} = F\left(x, p, \frac{dp}{dx}\right) = p$$

④ Clairaut's Equation :

$$y = xp + f(p)$$

$$\Rightarrow \quad y = xc + f(c)$$

→ Singular Solutions

Curve which is touched by a family of curves is called the envelope of the family of curves. eg. Circle for family of tangents

Singular solution is the envelope of family of curves given by general solution.

D.E ⇒ $f(x, y, p) = 0$

G.S ⇒ $\phi(x, y, c) = 0$

this is equivalent to discriminant = 0 if the eqⁿ is quadratic in f.

p - discriminant - eliminate p b/w $f(x, y, p) = 0$ & $\frac{df}{dp} = 0$

c - discriminant - eliminate c b/w $\phi(x, y, c) = 0$ & $\frac{d\phi}{dc} = 0$

$E(x, y)$ is a singular solution if it is a factor of both disc and $E(x, y) = 0$ satisfies $f(x, y, p) = 0$

Extraneous loci not satisfying the DE

T = Tac - locus

N = Node - locus

C = cusp - locus

Extraneous loci -

p - disc = $ET^2C$

c - disc = $EN^2C^3$

Mention this when geometric significance is asked.

→ Orthogonal Trajectories -

1) Eliminate c from $f(x, y, c) = 0$ to get $F(x, y, \frac{dy}{dx}) = 0$

Replace $\frac{dy}{dx}$ by $\frac{1}{-\frac{dy}{dx}}$ and solve

2) Polar coordinates: Replace $\frac{dr}{d\theta}$ by $\frac{-r^2}{\frac{dr}{d\theta}}$

→ Oblique Trajectory -

Replace $\frac{dy}{dx}$ by $\frac{p + \tan\alpha}{1 - p\tan\alpha}$

→ La place Transform -

$$L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\, dt$$

If $F(t)$ is a function which is piecewise continuous on every finite interval in the range $t \geq 0$ and is of exponential order as $t \longrightarrow \infty$, then $L\{F(t)\}$ exists for all $p > a$.

(i) 1st translation or shifting theorem

$$L\{F(t)\} = f(p) \;;\quad L\{e^{at} F(t)\} = f(p-a)$$

(ii) 2nd translation or shifting theorem -

$$L\{F(t)\} = f(p) \qquad G(t) = \begin{cases} F(t-a) & , t > a \\ 0 & \phantom{,} t < a \end{cases}$$

$$L\{G(t)\} = e^{-ap} f(p)$$

(iii) Change of scale property :

$$L\{F(t)\} = f(p) \qquad L\{F(at)\} = \frac{1}{a} f\left(p/a\right)$$

(iv) Derivatives : Split if $F'(t)$ is discontinuous in interval

$$L\{F'(t)\} = p\, L\{F(t)\} - F(0) \qquad \text{(Integration by parts)}$$

$$L\{F^n(t)\} = p^n L\{F(t)\} - p^{n-1} F(0) - p^{n-2} F'(0) - \dots - F^{n-1}(0)$$

(v) Integrals :

$$L\left\{\int_0^t F(x)\, dx\right\} = \frac{1}{p} L\{F(t)\}$$

(vi) Multiplication by $t^n$ :

$$L\{t^n F(t)\} = (-1)^n \frac{d^n f(p)}{dp^n}$$

vii) Division by t :

$$L\{F(t)\} = f(p)$$

$$\Rightarrow \quad L\left\{\frac{1}{t}F(t)\right\} = \int_p^\infty f(x)\,dx \qquad \text{provided } \lim_{t\to 0}\left\{\frac{1}{t}F(t)\right\} \text{ exists}$$

→ Inverse La Place Transform :

i) Change of scale :

$$L^{-1}\{f(ap)\} = \frac{1}{a}F(t/a)$$

ii) Derivatives :

$$L^{-1}\{f(p)\} = F(t) \qquad L^{-1}\{f^n(p)\} = (-1)^n t^n F(t)$$

iii) Integrals :

$$L^{-1}\{f(p)\} = F(t) \qquad L^{-1}\left\{\int_p^\infty f(x)\,dx\right\} = \frac{F(t)}{t}$$

iv) Multiplication by powers of p :

$$L^{-1}\{f(p)\} = F(t) \;\&\; F(0)=0 \;\Rightarrow\; L^{-1}\{pF(p)\} = F'(t)$$

v) Division by p :

$$L^{-1}\left\{\frac{f(p)}{p}\right\} = \int_0^t F(x)\,dx \qquad \left[\, F(t) \text{ is class A } \& \lim_{t\to 0}\frac{F(t)}{t} \text{ exists}\right]$$

# ⇒ ODE 2.0 :

1) Order : order of the highest order derivative

Degree : Degree of highest order derivative when all degrees are made free of radicals & fractions.

2) Linear Differential Equation :

(a) $y$ & all its derivatives have only first degree ; and

(b) No product of dependent variables (or) derivatives

$$\frac{d^n y}{dx^n} + P_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + P_{n-1}(x)\frac{dy}{dx} + P_n(x)y = Q(x)$$

3) Types of solutions :

(a) General solution : # of arbitrary constants = order of DE

(b) Particular solution : by giving particular values to the constants in general solution

(c) Singular solution : doesn't contain any arbitrary constant & can't be obtd by giving values to constants in general solution.

Generally, it is the envelope of general solution.

4) While formulating DEs, differentiate as many times as the no of constants present & then eliminate the constants!

Don't do more !! eg $y = a\cos(x+3) \Rightarrow y' + y\tan(x+3) = 0$
don't do $y'' + ay = 0$ !!

You can also put in matrix form directly.

eg $y = Ae^x + Be^{2x} \Rightarrow \begin{vmatrix} y & -e^x & -e^{2x} \\ y' & -e^x & -2e^{2x} \\ y'' & -e^x & -4e^{2x} \end{vmatrix} = 0$. is the DE

5) Solution to $1^{st}$ order $1^{st}$ degree :

a) Variable separation

b) Reduction to variable separation

3) $\dfrac{dy}{dx} = \dfrac{(ax+by)+c}{m(ax+by)+c_1}$ Put $(ax+by) = z$

4) Homogenous DE : $y = tx$

5) Reduction to homogenous

6) Exact differential equation

Examples :

a) $\dfrac{dy}{dx} = \cos(x+y) + \sin(x+y)$ Put $x+y=t$

$\dfrac{dy}{dx} = \dfrac{dt}{dx} - 1 \Rightarrow \dfrac{dt}{dx} = 1+\cos t + \sin t$

$= 2\cos\dfrac{t}{2}\left(\sin\dfrac{t}{2} + \cos\dfrac{t}{2}\right)$

$= 2\cos^2\dfrac{t}{2}\left(1+\tan\dfrac{t}{2}\right)$

$\therefore \dfrac{dt}{\left(1+\tan\frac{t}{2}\right)}\left(\dfrac{1}{2}\right)\sec^2\dfrac{t}{2} = dx$

$\Rightarrow \log\left(1+\tan\left(\dfrac{t}{2}\right)\right) = x + c$

$\Rightarrow \log\left[1+\tan\left(\dfrac{x+y}{2}\right)\right] = x+c.$

b) Some exact differentials :

(i)* $d\left(\tan^{-1}\dfrac{y}{x}\right) = \dfrac{x\,dy - y\,dx}{x^2+y^2}$

(ii)* $d\left[\log\sqrt{x^2+y^2}\right] = \dfrac{x\,dx + y\,dy}{\sqrt{x^2+y^2}}$

(iii) $d\left(\dfrac{y^2}{x}\right) = \dfrac{2xy\,dy - y^2dx}{x^2}$

(iv) $d\left(\dfrac{-1}{xy}\right) = \dfrac{x\,dy + y\,dx}{x^2y^2}$

(v) $d\left(\dfrac{e^y}{x}\right) = \dfrac{xe^y dy - e^y dx}{x^2}$

(vi) $d\left(\log\left|\dfrac{y}{x}\right|\right) = \dfrac{x\,dy - y\,dx}{xy}$

(vii) $d(\log|xy|) = \dfrac{x\,dy + y\,dx}{xy}$

(viii) $d\left(\dfrac{y^2}{x^2}\right) = \dfrac{2x^2y\,dy - 2xy^2dx}{x^4}$

c) Sometimes instead of finding IF, directly manipulating to get exact differentials is handy.

(i) $\underline{y \sin 2x \, dx = (1+y^2+\cos^2 x)\, dy}$

$\cos^2 x \, dy - y \sin 2x \, dx + (1+y^2)\, dy = 0$

$d(y\cos^2 x) + (1+y^2)\, dy = 0$

$\Rightarrow \quad y\cos^2 x + y + \frac{y^3}{3} = C$

(ii) $\underline{y(2x^2y + e^x)\, dx - (e^x + y^3)\, dy = 0}$

$y e^x dx - e^x dy + 2y^2x^2 dx - y^3 dy = 0$

$\frac{y e^x dx - e^x dy}{y^2} + 2x^2 dx - y\, dy = 0$

$\Rightarrow \quad d\left(\frac{e^x}{y}\right) + 2x^2 dx - y\, dy = 0$

$\Rightarrow \quad \frac{e^x}{y} + \frac{2x^3}{3} - \frac{y^2}{2} = C$

(iii) $x\,dy = \left[y + x\cos^2\left(\frac{y}{x}\right)\right] dx$

$\frac{x\,dy - y\,dx}{x^2} = \frac{x\cos^2\left(\frac{y}{x}\right)}{x^2} dx$

$\Rightarrow \quad d\left(\frac{y}{x}\right) = \frac{1}{x}\cos^2\left(\frac{y}{x}\right) dx$

$\sec^2\left(\frac{y}{x}\right) d\left(\frac{y}{x}\right) = \frac{dx}{x}$

$\tan\left(\frac{y}{x}\right) = \ln x + c$

(iv) $(xy^2 + 2x^2y^3)\, dx + (x^2y - x^3y^2)\, dy = 0$

$xy(y\,dx + x\,dy) + x^2y^2(2y\,dx - x\,dy) = 0$ divide by $(xy)^3$

$\Rightarrow \quad \frac{y\,dx + x\,dy}{(xy)^2} + \frac{2\,dx}{x} - \frac{dy}{y} = 0$

$\Rightarrow \quad \frac{-1}{xy} + 2\ln x - \ln y = \log c \Rightarrow \frac{x^2}{yc} = e^{\frac{1}{xy}}$

(v) $(y^2+yz)\,dx + (xz+z^2)\,dy + (y^2-xy)\,dz = 0$

TIP: Since $y$ is more common, we try to divide by $y^2$ in manipulatio

$\Rightarrow \quad y^2dx + z^2dy + yz\,dx + y^2dz + x(z\,dy - y\,dz) = 0$

$\Rightarrow \quad (dx+dz) + \dfrac{z^2}{y^2}dy + \dfrac{z}{y}dx - x\dfrac{(y\,dz - z\,dy)}{y^2} = 0$

$\Rightarrow \quad d(x+z) + \dfrac{z^2}{y^2}dy + \dfrac{z}{y}dx - x\,d\left(\dfrac{z}{y}\right) = 0$

$\Rightarrow \quad d(x+z) + \dfrac{z^2}{y^2}dy + \dfrac{z^2}{y^2}\,d\left(\dfrac{x}{\left(\frac{z}{y}\right)}\right) = 0$

$\Rightarrow \quad d(x+z) + \dfrac{z^2}{y^2}\left[d\left(y+\dfrac{xy}{z}\right)\right] = 0$

$\Rightarrow \quad d(x+z) + \dfrac{z^2}{y^2}\left[d\left(\left(\dfrac{y}{z}\right)(x+z)\right)\right] = 0$

Take $x+z = X \qquad \dfrac{y}{z} = Y$

$\Rightarrow \quad dX + \dfrac{d(XY)}{Y^2} \Rightarrow \dfrac{dX}{X^2} + \dfrac{d(XY)}{X^2Y^2} = 0$

$\Rightarrow \quad \dfrac{-1}{X} - \dfrac{1}{XY} = c$

$\Rightarrow \quad \dfrac{1}{x+z} + \dfrac{z}{(x+z)y} = c$

$\Rightarrow \quad \dfrac{y+z}{y(x+z)} = c$

6) LINEAR DIFFERENTIAL EQUATION: OF $1^{st}$ ORDER:

(a) Type-1: $\dfrac{dy}{dx} + P(x)\,y = Q(x)$

IF $= e^{\int P(x)\,dx}$ $\qquad$ Soln: $y e^{\int P(x)dx} = \left[\int Q(x)e^{\int P(x)dx}dx\right] + C$

$\Rightarrow \quad y(IF) = \int Q(IF)\,dx + C.$

(b) Type-2 : $\frac{dx}{dy} + P(y)x = Q(y)$

$IF = e^{\int P(y)\,dy}$ Solⁿ : $x(IF) = \int Q(y)\cdot(IF)\cdot dy + c$

eg $(x+y+1)\frac{dy}{dx} = 1 \Rightarrow \frac{dx}{dy} - x = 1+y$ Now easy

(c) Reducible to linear :

$f'(y)\frac{dy}{dx} + P\,f(y) = Q$ Put $f(y) = t$

$\Rightarrow \frac{dt}{dx} + Pt = Q$

(d) Bernoulli's Equation :

$\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n$

$\Rightarrow \frac{1}{y^n}\frac{dy}{dx} + \frac{P(x)}{y^{n-1}} = Q(x)$ Put $\frac{1}{y^{n-1}} = t$

we get : $\frac{dt}{dx} + (1-n)P(x)\,t = Q(x)$

7) Proofs for different IFs for $M\,dx + N\,dy = 0$ :-

$M\,dx + N\,dy = \frac{1}{2}\left[(Mx+Ny)\left(\frac{dx}{x}+\frac{dy}{y}\right) + (Mx-Ny)\left(\frac{dx}{x}-\frac{dy}{y}\right)\right]$

Remember this trick $\Rightarrow M\,dx + N\,dy = \frac{1}{2}\left[(Mx+Ny)\,d(\log(xy)) + (Mx-Ny)\,d\left(\log\left(\frac{x}{y}\right)\right)\right]$

(a) If M & N are homogenous $\frac{Mx-Ny}{Mx+Ny} = f\left(\frac{x}{y}\right) \equiv F\left(\log\left(\frac{x}{y}\right)\right)$

which makes $\frac{M\,dx+N\,dy}{Mx+Ny}$ exact !

(b) If $M = y f_1(xy)$ & $N = x f_2(xy)$ $\frac{Mx+Ny}{Mx-Ny} = \frac{f_1+f_2}{f_1-f_2} = f(xy) \equiv F(\log xy)$

which makes $\frac{M\,dx+N\,dy}{Mx-Ny}$ exact

8) LDE with constant coefficients:

$$f(D)\,y = Q(x) \Rightarrow y = y_c + y_p$$

$y_c$ using $f(D)y = 0$ & Aux eqⁿ

$y_p$ using $y_p = \frac{1}{f(D)} Q(x)$ using special forms.

(a) for $Q(x) = e^{ax}$ & $f(a) = 0$

we can have 3 approaches:

(i) $$\frac{1}{f(D)} e^{ax} = e^{ax} \cdot \frac{1}{f(D+a)} \cdot 1$$

(ii) If $f(D) = g(D)\,(D-a)^r$

$$\frac{1}{f(D)} e^{ax} = \frac{1}{(D-a)^r\, g(D)} \cdot e^{ax} = \frac{1}{g(a)} \cdot \frac{1}{(D-a)^r} e^{ax}$$

1st consume this

$$= \frac{e^{ax}}{g(a)} \cdot \frac{1}{D^r} \cdot 1 = \frac{e^{ax} \cdot x^r}{g(a)\, r!}$$

(iii) Keep on differentiating $f(D)$ till 'a' is a solution, then for remaining term put $D = a$

For every differentiation multiply numʳ by $x$ & increment factorial in Denʳ.

eg $$y_p = \frac{1}{D^3 + 3D^2 + 3D + 1} e^{-x} = \frac{x}{3D^2 + 6D + 3} (e^{-x})$$
$$= \frac{x^2}{6D + 6} (e^{-x})$$
$$= \frac{x^3}{6} (e^{-x})$$

Just a hack
Better go for (i) & (ii)

(b) $(D^2 + a^2)y = \sin ax \Rightarrow$ such results can be obtained using Img part of $(D^2 + a^2)y = e^{iax}$

(c) $Q = xV(x)$ can be remembered with the help of integration by parts

$$\frac{1}{f(D)} Q = \frac{1}{f(D)} xV = x \frac{1}{f(D)} V - \frac{f'(D)}{[f(D)]^2} V$$ [but $f(D)=D$ we get by parts]

Ex: $(D^2+1)y = x^2 \sin 2x$

$$y_p = \frac{1}{D^2+1} \cdot x^2 \sin 2x = \text{Img}\left[e^{i2x} \frac{1}{(D+2i)^2+1} x^2\right]$$

Now proceed ahead using above formula of $\frac{1}{f(D)} xV$ will be tedious.

Ex. $(D^2+a^2)y = \sec ax$

[If you use $\frac{1}{D-\alpha} f(x) = e^{\alpha x}\int e^{-\alpha x} f(x)$ then better do partial fractions instead of sequential application]

$$y_c = c_1 \cos ax + c_2 \sin ax$$

$$y_p = \frac{1}{(D+ai)(D-ai)} \sec ax$$

$$= \frac{1}{2ai}\left[\frac{1}{D-ai} \sec ax - \frac{1}{D+ai} \sec ax\right]$$

$$= \frac{1}{2ai}\left[e^{aix}\int e^{-aix} \sec ax \, dx - e^{-aix}\int e^{aix} \sec ax \, dx\right]$$

$$= \frac{1}{2ai}\left[e^{aix}\int\left(\frac{\cos ax - i\sin ax}{\cos ax}\right) dx - e^{-aix}\int\frac{(\cos ax + i \sin ax)}{\cos ax} dx\right]$$

$$= \frac{1}{2ai}\left[e^{aix}\left(x + \frac{i}{a}\log(\cos ax)\right) - e^{-aix}\left(x - \frac{i}{a}\log(\cos ax)\right)\right]$$

$$= \frac{1}{2ai}\left[x\left(e^{aix} - e^{-aix}\right) + \frac{i}{a}\log(\cos ax)\left(e^{aix} + e^{-aix}\right)\right]$$

$$= \frac{1}{\cancel{2}a\cancel{i}}\left[\cancel{2}\cancel{i}x \sin ax + \frac{\cancel{i}}{a}\log(\cos ax)\,\cancel{2}\cos ax\right]$$

$$= \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax \log(\cos ax)$$

Very important to follow this to avoid getting stuck in ugly integrations. This is less error prone

9) NOTE :

Don't do algebraic operations on factors of $f(D)$ when they are in denominator, in order to avoid unnecessary terms. Apply them sequentially.

eg $\frac{1}{D(D+1)}x = \frac{1}{D}\left(1-D+D^2+D^3..\right)x = \frac{1}{D}(x-1) = \frac{x^2}{2}-x$

⇓ if you do

$\frac{1}{D}\left(1-D+D^2+\ldots\right)x = \left(\frac{1}{D}-1+D..\right)x = \frac{x^2}{2}-x+1$

DONT DO THIS.

⇓ unnecessarily there as it is included in $y_c = c_1 + c_2 e^{-x}$

10) Cauchy Euler : $x^n \frac{d^n y}{dx^n} + x^{n-1} a_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_n y = X$ Put $x = e^z$

Legendre equation : $(a+bx)^n \frac{d^n y}{dx^n} + (a+bx)^{n-1} A_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + A_n y = X$

Details in older notes. Put $a+bx = e^z$

11) 2nd Order LDE with Variable coefficients :

4 Methods : $y''(x) + P(x)\,y'(x) + Q(x)\,y = R(x)$

a) change $y$ if part of CF is known Put $y = uv$

b) Changing $x$, $y$ & removing $y'(x)$. (Get into canonical form)

c) changing $x$

d) Variation of parameters.

NOTE : All methods should have DE in Standard form as $(D^2+PD+Q)y = R$, then only proceed

(a) Change 'y' if part of CF is known

$u(x)$ could be given or try using inspection for $e^{ax}$ (if $a^2 + aP + Q = 0$) or $x^m$ (if $m(m-1) + Pmx + Qx^2 = 0$)

Put $y = uv$.

we get $v$ by :- $\boxed{\frac{d^2v}{dx^2} + \left(P + \frac{2u'}{u}\right)\frac{dv}{dx} = \frac{R}{u}}$

Put $\frac{dv}{dx} = t$ to get 1st order LDE

(b) Change 'y' & reduce to Canonical / Normal form :

Put $y = uv$ & get in $(\ )\frac{d^2v}{dx^2} + (\ )\frac{dv}{dx} + (\ )v = R$ form

Now make this 0 to get 'u'.

If we can't find u as $x^m$, $e^{ax}$ directly, then use this

We get $\boxed{u = e^{-\frac{1}{2}\int P dx}}$

& $\boxed{\frac{d^2V}{dx^2} + IV = S}$ $\boxed{I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} \quad \& \ S = \frac{R}{u}}$

(c) Change 'x' & make $P_1 = 0$ : or $Q_1 = \pm a^2$ :

Put $x = f(z)$ $\Rightarrow$ $y'' + P_1 y' + Q_1 y = R_1$

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} \qquad Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} \qquad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

generally gives $P_1$ constant

$P_1 = 0 \Rightarrow \boxed{z = \int e^{-\int P dx} dx}$ or $\boxed{Q_1 = \pm a^2} \Rightarrow a\int dz = \int \sqrt{\pm Q}\, dx$

go for this if Q has some perfect square

sign as per sign of 'Q'

(d) Variation of parameters :

Get the solⁿ $y_c = c_1 u(x) + c_2 v(x)$ for $y'' + Py' + Q = 0$

Let $W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = uv' - u'v$

Then $\boxed{y_p = Au + Bv}$ where $\boxed{A = \int \frac{-vR}{W} \ \& \ B = \int \frac{uR}{W}}$

This is inferior to (a) as both solⁿ for $y_c$ are needed

127 First order but Degree $\neq 1$ :- (i.e, not LDE)

i.e, $p^n + A_1 p^{n-1} + \cdots + A_{n-1} p + A_n y = 0$  $\left(p = \frac{dy}{dx}\right)$

Again 4 methods :

a) solvable for 'p' :

$[p - f_1(x,y)][p - f_2(x,y)] \cdots [p - f_n(x,y)] = 0$

$\Rightarrow F_1(x, y, c_1) = 0 \quad F_2(x, y, c_2) = 0 \cdots F_n(x, y, c_n) = 0$

$c_1 = c_2 = \cdots = c_n = c$  [$\because$ only 1 constant as order = 1]

$\Rightarrow \boxed{F_1(x, y, c) \cdot F_2(x, y, c) \cdots F_n(x, y, c) = 0} \Rightarrow$ GS

TRICK : $xy^2(p^2+2) = 2py^3 + x^2$  (add $x^2yp$ on both sides & proceed)
OR form quadratic in p & find its root !!

b) solvable for 'x' :

$x = f(y, p)$ (I) $\Rightarrow \frac{dx}{dy} = F\left(y, p, \frac{dp}{dy}\right) \Rightarrow \boxed{\frac{1}{p} = F\left(y, p, \frac{dp}{dy}\right)}$ → gives $\phi$ as sol$^n$

$\phi(y, p, c) = 0$ (II)  Eliminate 'p' b/w (I) & (II)

If not possible, $x = f_1(p, c)$ & $y = f_2(p, c)$ is the sol

NOTE : Ignore singular solutions while solving $\frac{1}{p} = F\left(y, p, \frac{dp}{dy}\right)$

c) Solvable for 'y' :

$y = f(x, p) \Rightarrow \frac{dy}{dx} = F\left(x, p, \frac{dp}{dx}\right) \rightarrow \boxed{p = F\left(x, p, \frac{dp}{dx}\right)}$ gives $\phi$ as sol$^n$

d) Clairaut's Equation :

$y = px + f(p) \Rightarrow \boxed{y = cx + f(c)}$ is the solution

e) Reduction to Clairaut's form :

Mug up!!

TRICKS :

1) $y/x$ is involved, then put $y = tx$

$(xp-y)^2 = (x^2-y^2)\sin^{-1}\left(\frac{y}{x}\right)$ Put $\frac{y}{x} = \sin t$

2) If $e^{by}(a-bp) = f(pbe^{ax-by})$ Put $Y = e^{by}$ $X = e^{ax}$

3) If $y = apx + \underline{y^{n-1}p^n}$ then put $Y = y^2$

4) If $(x^2+y^2)^{n/2}$ go for $x = r\cos\theta$ $y = r\sin\theta$

$(xp-y)^2 = a(1+p^2)(x^2+y^2)^{3/2}$

5) $(x^2+y^2)(1+p)^2 - 2(x+y)(1+p)(x+py) + (x+py)^2 = 0$

Put $x+y = u$ $x^2+y^2 = v$

13) Singular Solutions - Refer earlier notes.

Example: $4p^2 x(x-a)(x-b) = [3x^2 - 2x(a+b) + ab]^2$

Here we would like to factorise RHS into $(x-b)$ & $(x-a)$ terms but we cannot

But if we have: $x^3 - x^2(a+b) + xab$, then we can

as $\Rightarrow (x)(x-a)(x-b)$

Also note RHS $= \left[\frac{d}{dx}\left(x^3 - x^2(a+b) + xab\right)\right]^2$

$$\therefore \quad 2p\sqrt{x(x-a)(x-b)} = \frac{d}{dx}\left(x(x-a)(x-b)\right)$$

$$\Rightarrow \quad dy = \frac{d(x(x-a)(x-b))}{2\sqrt{x(x-a)(x-b)}}$$

$$\Rightarrow \quad y + c = \sqrt{x(x-a)(x-b)}$$

$$\Rightarrow \quad (y+c)^2 = x(x-a)(x-b)$$

$p$-disc: $\quad 0 - 16x(x-a)(x-b)(3x^2 - 2x(a+b) + ab)^2 = 0 \quad \equiv ET^2C$

$c$-disc: $\quad c^2 + 2cy + y^2 - x(x-a)(x-b) = 0$

$$\Rightarrow \quad 4y^2 - 4y^2 + 4x(x-a)(x-b) = 0 \qquad \equiv EN^2C^3$$

$$\Rightarrow \quad 4x(x-a)(x-b) = 0$$

$\therefore \quad E = x(x-a)(x-b)$ & $T = [3x^2 - 2x(a+b) + ab]$

Show all these 3 as separately

14) Orthogonal Trajectories - Refer earlier notes.

15) La Place Transform :

$$L = \int_0^\infty e^{-pt} F(t)\, dt$$

Laplace exists if (i) $F(t)$ is piecewise continuous on $t \geq 0$
(ii) $|F(t)| \leq Me^{at}$ for all $t \geq 0$ for some $a, M$

$L\{F(t)\}$ exists for $t > a$

They are only sufficient conditions, not necessary

16) Standard Laplace transforms :

(a) $L\{1\} = \frac{1}{p}$ $(p > 0)$ (b) $L\{t^n\} = \frac{n!}{p^{(n+1)}}$ $n \in \mathbb{N}$

(c) $L\{t^n\} = \frac{\Gamma(n+1)}{p^{n+1}}$ $-1 < n < 0$ or $n > 0$

$\Gamma(n+1) = n\,\Gamma(n)$

$\Gamma(1/2) = \sqrt{\pi}$

Here $F(t) = t^n$ is exponential order & not a class A function, still integral exists

Put $pt = x$ in Integral. We get gamma integral.

(d) $L\{e^{at}\} = \frac{1}{p-a}$ $p > a$

(e) $L\{\cos at\} = \frac{p}{p^2+a^2}$, $p > 0$; $L\{\cosh at\} = \frac{p}{p^2-a^2}$ $|p| > |a|$

$L\{\sin at\} = \frac{a}{p^2+a^2}$; $p > 0$; $L\{\sinh at\} = \frac{a}{p^2-a^2}$ $p > |a|$

17) Other theorems, look at old notes.

Remember : If $F(t)$ has different limits at $t = a$, then

$$L\{F'(t)\} = p f(p) - F(0) - e^{-ap}\left(F(a+) - F(a-)\right)$$

18> Initial & Final Value Theorem :

$$\lim_{t\to 0} F(t) = \lim_{p\to\infty} p\,L\{F(t)\} = \lim_{p\to\infty} p f(p)$$

$$\lim_{t\to \infty} F(t) = \lim_{p\to 0} p\,L\{F(t)\} = \lim_{p\to 0} p f(p)$$

Remember : Leibnitz Rule : $\frac{d}{dx}\int_A^B F(x,t)\,dt = \int_A^B \frac{\partial}{\partial x}\{F(x,t)\}\,dt + F(x,t)\Big|_{x=B}\frac{dB}{dx} - F(x,t)\Big|_{x=A}\frac{dA}{dx}$

A, B are functions of 'x' or constant.

$$L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2p^{3/2}}e^{-1/4p}$$

[expand $\sin\sqrt{t}$ & take $L\{\}$ term by terms, we get expansion of $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$]

Example : PT: If $L\{F(t), t\to p\} = f(p)$

then $L\{\int_0^t \frac{F(u)}{u}du,\ t\to p\} = \frac{1}{p}\int_p^\infty f(y)\,dy$

WKT: $L\{\int_0^t F(u)\,du\} = \frac{1}{p}f(p)$ — ①

Let $G(t) = \frac{F(t)}{t} \Rightarrow L\{G(t)\} = \int_p^\infty f(y)\,dy$

$\Rightarrow L\{\int_0^t G(u)\,du\} = \frac{1}{p}\int_p^\infty f(y)\,dy$ [using ①]

$\Rightarrow L\{\int_0^t \frac{F(u)}{u}du\} = \frac{1}{p}\int_p^\infty f(y)\,dy$

20> $\int_0^\infty \frac{F(t)}{t}dt = \int_0^\infty f(x)\,dx$ if the limit exists.

using this we can show $\int_0^\infty \frac{\sin t}{t}dt = \frac{\pi}{2}$

21) $\int_0^\infty F(t)\,dt = f(0)$, if the integral is convergent i.e, $\lim_{p\to 0} f(p)$.

22) Inverse Laplace:

Remember the difference: $L^{-1}\left\{\frac{p}{p^2+a^2}\right\} = \cos at$

but $L^{-1}\left\{\frac{1}{p^2+a^2}\right\} = \frac{1}{a}\sin(at)$

→ Dont forget

Other formulas, look at old notes.

23) Convolution:

$$F * G = \int_0^t F(x)\,G(t-x)\,dx$$

Let $L\{F(t)\} = f(p)$ & $L\{G(t)\} = g(p)$

then $L^{-1}\{f(p)\,g(p)\} = F * G$

or $$\boxed{L^{-1}\{f(p)\cdot g(p)\} = \int_0^t F(x)\,G(t-x)\,dx}$$

Proof using change of order of integration

eg $L^{-1}\left\{\frac{p}{(p^2+a^2)^2}\right\} = L^{-1}\left\{\frac{1}{p^2+a^2}\cdot\frac{p}{p^2+a^2}\right\}$

$f(p) = \frac{1}{p^2+a^2}$ $\therefore F(t) = \frac{\sin(at)}{a}$

$g(p) = \frac{p}{p^2+a^2}$ $G(t) = \cos(at)$

$\therefore L^{-1}\{f(p)\cdot g(p)\} = \int_0^t \frac{\sin(ax)}{a}\cos(at-ax)\,dx = \int_0^t \frac{\cos(at)\sin(a(t-x))}{a}\,dx$

$= \int_0^t \frac{\cos ax\,[\sin at\cos ax - \cos at\sin ax]}{a}\,dx$

$= \frac{\sin at}{a}\int_0^t \frac{(1+\cos 2ax)}{2}\,dx - \frac{\cos at}{a}\int_0^t \frac{\sin 2ax}{2}\,dx$

$= \frac{\sin at}{2a}\left[\left(x + \frac{\sin 2ax}{2a}\right)\right]_0^t + \frac{\cos at}{2a}\left[\frac{\cos 2ax}{2a}\right]_0^t = \frac{t\sin at}{2a}$