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MATHEMATICS FOR UPSC CSE MAINS

(LAPLACE TRANSFORM)

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Laplace Transform

Dated: 30/08/2016

①

→ L.T : Mathematical tool to convert a time domain signal into frequency domain signal.

Reason :- Since ; It is quiet easy to Analyse the signal in time domain but still why we transform it into frequency domain. Because It is easy to do the calculation in frequency domain than time domain

# Fourier Series	Fourier Transform	L.T
Periodic	Aperiodic	

$$L[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

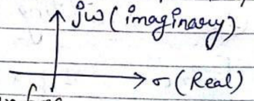
t : time

s : Complex frequency

Technical

Damping factor

$$\boxed{s = \sigma + j\omega}$$



when $\boxed{\sigma = 0}$

angular frequency ⇒ $\boxed{s = j\omega}$; s → Complex freq.

$$\Rightarrow F[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

So F.T is a special case of L.T. when the Real part of Complex freq. is Zero.

$$\boxed{L[x(t)] = \int_{-\infty}^{\infty} x(t) \cdot e^{-st}\, dt}$$

②

Some Imp. formulas :-

1. $L[1] = \frac{1}{s}$
2. $L[t] = n!/s^{n+1}$
3. $L[\sin at] = \frac{a}{s^2+a^2}$
4. $L[\cos at] = \frac{s}{s^2+a^2}$
5. $L[e^{at}] = \frac{1}{s-a}$
6. $L[\sinh at] = \frac{a}{s^2-a^2}$
7. $L[\cosh at] = \frac{s}{s^2-a^2}$

Properties :-

① Linearity Property :-

$$L[a x_1(t) + b x_2(t)] = a L[x_1(t)] + b L[x_2(t)] = a X_1(s) + b X_2(s)$$

→ "Homogenity & super-position"

② Shifting Property :-

Time : $L[x(t-t_0)] = e^{-st_0} \cdot X(s)$

Freq. : $L[e^{s_0 t} x(t)] = \cancel{X(s)}\ X(s-s_0)$

$L[e^{at} \sin bt]$ → $\frac{b}{s^2+b^2}$ → $\frac{b}{(s-a)^2+b^2}$ Ans.

③ Time Scaling :-

$x(t) \longleftrightarrow X(s)$

$x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{s}{a}\right)$

(Exp.) (Compr.)

Que:- (1) $L[t \cos t]$; (2) $L[t \sin t]$

(3) $L[t^2 \sin at]$

Que:- $L[t\, e^{-4t} \cdot \sin 3t]$

Sol.^{n} $L[t] = \frac{1}{s^2}$ $\quad L[t\, e^{+i3t}] = \frac{1}{(s-i3)^2} \times \frac{(s+i3)^2}{(s+i3)^2}$

$\left[\because e^{it} = \cos t + i \sin t\right]$

$L[t\, e^{i3t}] = \frac{(s^2-9) + i6s}{(s^2+9)^2}$

$L[t(\cos 3t + i \sin 3t)] = $ "

$\Rightarrow L[t \sin 3t] = \frac{6s}{(s^2+9)^2}$

Now $L[e^{-4t} \cdot t \cdot \sin 3t]$

$\rightarrow \frac{6s}{(s^2+9)^2}$

$= \frac{6(s+4)}{((s+4)^2+9)^2}$ Ans.

(4) Que: $f(t) = |t-1| + |t+1| + |t+2| + |t-2|;\ t \geq 0$

Sol.^{n} $t-1=0 \Rightarrow t=1$

$t+1=0 \Rightarrow t=-1$

$t+2=0 \Rightarrow t=-2$ $\Big] \times \because t>0$

$t-2=0 \Rightarrow t=2$

so interval will be $0 \leq t \leq 1$:- $f_1(t)$

$1 \leq t \leq 2$: $f_2(t)$

$t > 2$: $f_3(t)$

$f_1(t) = -(t-1) + t+1 + t+2 - (t-2) = 6$

$f_2(t) = t-1 + t+1 + t+2 - (t-2) = 2t+4$

$f_3(t) = (t-1) + (t+1) + (t+2) + (t-2) = 4t$

$\therefore L[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt$

$= \int_0^1 6 \cdot e^{-st}\, dt + \int_1^2 (2t+4) e^{-st}\, dt + \int_2^\infty 4t \cdot e^{-st}\, dt$

$= \frac{1}{s^2}\left[6s + 2e^{-s} + 2e^{-2s}\right]$

$= \boxed{\frac{2}{s^2}\left(3s + e^{-s} + e^{-2s}\right)}$ Ans.

(5)

$$L[x'(t)] = \int_0^\infty \underset{\text{II}}{x'(t)} \cdot \underset{\text{I}}{e^{-st}}\, dt$$
$$= e^{-st}x(t)\Big|_0^\infty - \int_0^\infty -se^{-st}x(t)dt$$
$$= -x(0) + sX(s)$$

(4) Differentiation :-

Time : $x(t) \longleftrightarrow X(s)$

$\frac{d}{dt}x(t) \longleftrightarrow s.X(s) - x(0)$

Freq. : $x(t) \longleftrightarrow X(s)$

$-t.x(t) \longleftrightarrow \frac{d}{ds}X(s)$

(5.) Integral :-

Time :- $x(t) \longleftrightarrow X(s)$

$\int_{-\infty}^{t} x(\tau)\,d\tau \longleftrightarrow \frac{X(s)}{s}$

Freq. :- $x(t) \longleftrightarrow X(s)$

$\frac{x(t)}{t} \longleftrightarrow \int_s^\infty X(s)\,ds$

Using differentiation

$L[x(t)] = X(s)$

$L[x'(t)] = sX(s) - x(0)$

$L[x''(t)] = s^2X(s) - sx(0) - x'(0)$

$L[x'''(t)] = s^3X(s) - s^2x(0) - sx'(0) - s''(0)$

$\vdots$

$L[x^{n}(t)] = s^{n}X(s) - s^{n-1}x(0) - s^{n-2}x'(0) \ldots\ldots$

(6) Multiplication by "t^n" :-

$L[t^n.x(t)] = (-1)^n\frac{d^n}{ds^n}X(s)$ $\quad n\in N$

Eg. $L[t.\sin^2 3t] = ?$ $\quad \sin^2 3t = \frac{1 - \cos 6t}{2}$

$L\left[\frac{1-\cos 6t}{2}\right] = \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+6^2}\right)$

$X(s) = \boxed{\frac{18}{s(s^2+36)}}$

$(-1)\frac{d}{ds}\left(\frac{18}{s(s^2+36)}\right)$

$= \frac{54(s^2+12)}{s^2(s^2+36)^2}$ Ans.

(6)

Que:- I.L.T. $X(s) = \log\frac{s+1}{s-1}$;

Sol.^{n} $X(s) = L[x(t)] = \log\frac{(s+1)}{s-1}$

$L[t.x(t)] = (-1)\frac{d}{ds}X(s)$

$= (-1)\frac{d}{ds}\left(\log\frac{s+1}{s-1}\right)$

$= (-1)\frac{d}{ds}[\log(s+1) - \log(s-1)]$

$= -\left[\frac{1}{s+1} - \frac{1}{s-1}\right] = -\left[\frac{\cancel{s}-1-\cancel{s}-1}{s^2-1}\right]$

$L[t.x(t)] = \frac{2}{s^2-1} = 2\left(\frac{1}{s^2-1}\right)$

$= 2L[\sinh t]$

$t.x(t) = 2\sinh t$

$\Rightarrow \boxed{x(t) = \frac{2\sinh t}{t}}$ Ans.

Que:- L.T. ~~x(t)~~ of $\frac{e^{-at} - e^{-bt}}{t}$

Sol.^{n} Similar like Integral property (freq.^{n})

$L[\underbrace{e^{-at} - e^{-bt}}_{x(t)}] = \underbrace{\left[\frac{1}{s+a} - \frac{1}{s+b}\right]}_{X(s)}$

$L\left(\frac{x(t)}{t}\right) \Longleftrightarrow \int_s^\infty X(s)\,ds$

$= \int_s^\infty \frac{1}{s+a} - \frac{1}{s+b}\,ds = \log\frac{s+a}{s+b}\Big|_s^\infty$

$= \log\frac{s(1+a/s)}{s(1+b/s)}\Big|_s^\infty$

$= \log 1 - \log\frac{s+a}{s+b} = \boxed{\log\frac{s+b}{s+a}}$ Ans.

7

(7) Convolution Property :-

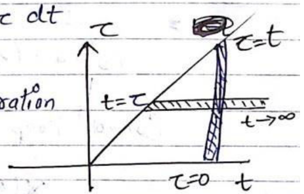
$$x(t) \longleftrightarrow X(s)$$
$$h(t) \longleftrightarrow H(s)$$
$$x(t) * h(t) \longleftrightarrow X(s) \cdot H(s)$$

Proof.

$$L[x(t) * h(t)] = \int_{-\infty}^{\infty} [x(t) * h(t)] e^{-st} dt$$

$$\left[\text{But} \quad x(t) * h(t) = \int_0^t x(\tau) h(t-\tau) d\tau \right]$$

$$= \int_0^{\infty} \int_0^t x(\tau) h(t-\tau) e^{-st} d\tau \, dt$$



on changing the Order of Integration

means :-

Original $\begin{pmatrix} 0 < \tau < t \\ 0 < t < \infty \end{pmatrix}$ L R

After changing $\begin{pmatrix} 0 < \tau < \infty \\ \tau < t < \infty \end{pmatrix}$ BT

L — R	T | B

$$= \int_0^{\infty} \int_{\tau}^{\infty} x(\tau) h(t-\tau) e^{-st} dt \, d\tau$$

Put $t - \tau = v$

$t = v + \tau$

$dt = dv$

$$= \int_0^{\infty} \int_0^{\infty} x(\tau) h(v) e^{-sv} e^{-s\tau} dv \, d\tau$$

$$= \int_0^{\infty} x(\tau) e^{-s\tau} d\tau \cdot \int_0^{\infty} h(v) e^{-sv} dv$$

$$= X(s) \cdot H(s) \quad \text{Proved.}$$

Inverse Laplace Transform :-

8

(1) $L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!}$

(2) $L^{-1}\left(\frac{1}{(s-a)^n}\right) = \frac{t^{n-1}}{(n-1)!} \times e^{at}$

Que:- $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$

Solⁿ

$$\frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{s}{(s^2+a^2)} \times \frac{s}{(s^2+b^2)} = X(s) \cdot H(s)$$

Convolⁿ Propʳ : $x(t) * h(t) = L^{-1}[X(s) \cdot H(s)]$

$$X(s) = \frac{s}{s^2+a^2} \Rightarrow x(t) = \cos at$$

$$H(s) = \frac{s}{s^2+b^2} \Rightarrow h(t) = \cos bt$$

$$x(t) * h(t) = \int_0^t x(\tau) h(t-\tau) d\tau$$

$$= \frac{1}{2} \int_0^t 2 \cos a\tau \cos b(t-\tau) d\tau$$

$$= \frac{1}{2} \int_0^t [\cos(a\tau + bt - b\tau) + \cos(a\tau - bt + b\tau)] \cdot d\tau$$

$$= \frac{1}{2} \int_0^t \cos[(a-b)\tau + bt] + \cos((a+b)\tau - bt) \, d\tau$$

$$= \frac{1}{2} \left[\frac{\sin((a-b)\tau + bt)}{a-b} + \frac{\sin((a+b)\tau - bt)}{a+b} \right]_0^t$$

$$= \frac{1}{2} \left(\frac{\sin at - \sin bt}{a-b} + \frac{\sin at + \sin bt}{a+b} \right) \text{Ans}$$

Ans = $\boxed{\frac{a \sin at - b \sin bt}{a^2 - b^2}}$

(9.)

Que:- Evaluate integral :- $\int_0^\infty \left[\frac{e^{-t}-e^{-3t}}{t}\right]dt$

$= \int_0^\infty e^{-0t}\left(\frac{e^{-t}-e^{-3t}}{t}\right)dt \Rightarrow$ indirectly means To find the L.T. of

let $x(t) = e^{-t} - e^{-3t}$

$\Rightarrow X(s) = \left(\frac{1}{s+1} - \frac{1}{s+3}\right)$

Prop. : $\frac{x(t)}{t} \xleftrightarrow{LT} \int_s^\infty X(s)\,ds$

$L\left[\frac{x(t)}{t}\right] = \int_s^\infty \left(\frac{1}{s+1} - \frac{1}{s+3}\right) ds$

$= \log(s+1) - \log(s+3)\Big|_s^\infty$

$= \log\frac{s+1}{s+3}\Big|_s^\infty = \log 1 - \log\frac{s+1}{s+3}$

$= \boxed{\log\frac{s+3}{s+1}}$ Here $\boxed{s=0}$

$\Rightarrow \int_0^\infty \frac{e^{-t}-e^{-3t}}{t}dt = \boxed{\log 3}$ Ans

Q(2015) : 6 Marks:- I.L.T.

$L^{-1}\left\{\log\left(1+\frac{1}{s^2}\right) + \frac{s}{s^2+25}e^{-\pi s}\right\}$

Sol.^{n} $\underbrace{L^{-1}\left\{\log\left(1+\frac{1}{s^2}\right)\right\}}_{I} + \underbrace{L^{-1}\left\{\frac{s}{s^2+25}e^{-\pi s}\right\}}_{II}$

I : $L[t\cdot x(t)] = -\frac{d}{ds}\left(\log(s^2+1) - \log s^2\right)$

$= -\left(\frac{2s}{s^2+1} - \frac{1}{s^2}\right) = \frac{1}{s^2} - \frac{2s}{s^2+1}$

$= L[t - 2\cos t]$

$\Rightarrow \boxed{x(t) = 1 - 2\frac{\cos t}{t}}$

(10)

II : $L^{-1}\left\{\frac{s}{s^2+25}\cdot e^{-\pi s}\right\}$

shift^n Prop^r : $x(t) \longleftrightarrow X(s)$

$x(t-t_0) \longleftrightarrow e^{-st_0}X(s)$

Here $X(s) = \frac{s}{s^2+25} \Rightarrow x(t) = \cos 5t$

But using shifting property:-

$\Rightarrow \cos 5(t-\pi) = \cos(5\pi - 5t)$

$= -\sin 5t$

so $L^{-1}\left\{\log\left(1+\frac{1}{s^2}\right) + \frac{s e^{-s\pi}}{s^2+25}\right\} = \left[1 - \frac{2\cos t}{t} - \sin 5t\right]$ Ans

Q(2015) : 6 marks : $y'' + y = t$

$y(0) = 1$

Sol.^{n} Taking L.T. both sides $y'(0) = -2$

$[s^2Y(s) - sy(0) - y'(0)] + Y(s) = \frac{1}{s^2}$

$\Rightarrow s^2Y(s) - s + 2 + Y(s) = \frac{1}{s^2}$

$\Rightarrow Y(s)(1+s^2) = \frac{1}{s^2} + s - 2$

$\Rightarrow Y(s) = \frac{1}{s^2(s^2+1)} + \frac{s}{s^2+1} - \frac{2}{s^2+1}$

$\Rightarrow y(t) = [t - \sin t]^{*} + \cos t - 2\sin t$

$= \boxed{t + \cos t - 3\sin 3t}$ Ans

* Prop.

$t\,x(t) \longleftrightarrow X(s)$ $\qquad x(t) = \sin t$

$\int_0^t x(t) \longleftrightarrow \frac{X(s)}{s}$ $\qquad \int_0^t \sin t\,dt$

$\int_0^t 1 - \cos t\,dt$ $\qquad = -\cos t\Big|_0^t$

$= t - \sin t\Big|_0^t$ $\qquad = \boxed{-\cos t + 1}$

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Q(2014): 20 Marks: Initial Value Problem

$$\frac{d^2y}{dt^2} + y = 8e^{-2t}\sin t \quad ; \quad y(0) = y'(0) = 0$$

Sol.^{n} $\quad y'' + y = 8e^{-2t}\sin t$

Taking L.T. on both sides

$$\Rightarrow [s^2Y(s) - sY(0) - Y'(0)] + Y(s) = \frac{8}{(s+2)^2+1}$$

$$\Rightarrow (s^2+1)Y(s) = \frac{8}{(s+2)^2+1} = \frac{8}{(s^2+4s+4+1)}$$

$$\Rightarrow Y(s) = \frac{8}{(s^2+4s+5)(s^2+1)}$$

$$= \frac{As+B}{s^2+4s+5} + \frac{Cs+D}{s^2+1}$$

$$\Rightarrow 8 = As^3 + Bs^2 + As + B + Cs^3 + 4Cs^2 + 5Cs + Ds^2 + 4Ds + 5D$$

Comparing coeff. of s^3, s^2, s^1, s^0; we get

ONLY CALCULATION

$A + C = 0$; $B + 4C + D = 0$; $A + 5C + 4D = 0$;

$\Rightarrow \boxed{A = -C}$ $\qquad B + 5D = 8$

$\boxed{C = -D}$

$B - 3D = 0$

$B + 5D = 8$

$-8D = -8 \Rightarrow \boxed{D = 1}$; $\boxed{C = -1}$; $\boxed{A = 1}$ & $\boxed{B = 3}$

$$\Rightarrow Y(s) = \frac{s+3}{s^2+4s+5} + \frac{-s+1}{s^2+1} = \frac{(s+2)+1}{s^2+4s+5} + \frac{-s+1}{s^2+1}$$

$$\Rightarrow Y(s) = \frac{(s+2)}{(s+2)^2+1} + \frac{1}{(s+2)^2+1} - \frac{s}{s^2+1} + \frac{1}{s^2+1}$$; Now Take I.L.T

$$\Rightarrow \boxed{y(t) = e^{-2t}\cos t + e^{-2t}\sin t - \cos t + \sin t}$$ Ans

Q: (2013); 15 Marks: Solve using L.T.

$$(D^2 + n^2)x = a\sin(nt + \alpha) \quad ; \quad a, n, \alpha \text{ Constant}$$

$$x = 0, \ \frac{dx}{dt} = 0 \text{ at } t = 0$$

Sol.^{n} Taking L.T. on both sides

$$s^2X(s) - sx(0) - x'(0) + n^2X(s) = L[a\sin(nt+\alpha)]$$

$$\Rightarrow s^2X(s) + n^2X(s) = L[a(\sin nt\cos\alpha + \cos nt\sin\alpha)]$$

$$(s^2+n^2)X(s) = \frac{an\cos\alpha}{s^2+n^2} + \frac{s\sin\alpha}{s^2+n^2}$$

$$\Rightarrow X(s) = \frac{an\cos\alpha}{(s^2+n^2)^2} + \sin\alpha\cdot\frac{s}{(s^2+n^2)^2}$$

Taking inverse L.T.

$$\because L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^3}(\sin at - at\cos at)$$

$$\because L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{2a}(t\sin at)$$

$$\Rightarrow \boxed{x(t) = \frac{(an\cos\alpha)}{2n^3}(\sin nt - nt\cos nt) + \frac{1}{2n}(t\sin nt)}$$

Ans

Q(2012); 12 Marks Solve using L.T

$$y'' + 2y' + y = e^{-t} \quad ; \quad y(0) = -1, \ y'(0) = 1$$

Sol.^{n}

$$\Rightarrow s^2Y(s) - sy(0) - y'(0) + 2[sY(s) - y(0)] + Y(s) = \frac{1}{s+1}$$

$$\Rightarrow s^2Y(s) + s - 1 + 2sY(s) + 2 + Y(s) = \frac{1}{s+1}$$

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(13.)

$$\Rightarrow (s^2+2s+1)\,Y(s) = \frac{1}{(s+1)} - (s+1)$$

$$\Rightarrow Y(s) = \frac{1}{(s+1)^3} - \frac{(s+1)}{(s+1)^2} = \frac{1}{(s+1)^3} - \frac{1}{(s+1)}$$

Taking Inverse L.T. :–

$$y(t) = e^{-t}t^2 - e^{-t} = \boxed{e^{-t}(t^2-1)}$$ Ans.

Q: (2011) 15 Marks: Obtain General Solⁿ of 2nd Order O.D.E.

$$y'' + 2y - 2y' = x + e^x \cos x$$

Solⁿ Taking L.T. on both sides; initial conditions Take zero.

$$s^2 Y(s) - 2sY(s) + 2Y(s) = \frac{1}{s} + \frac{(s-1)}{(s-1)^2+1}$$

$$(s^2-2s+2)\,Y(s) = \frac{1}{s} + \frac{(s-1)}{(s-1)^2+1}$$

$$\Rightarrow Y(s) = \frac{1}{s((s-1)^2+1)} + \frac{(s-1)}{((s-1)^2+1)^2} \leftarrow \text{Convol. Prop.}$$

Taking inverse L.T :–

$$\Rightarrow \boxed{y(x) = \frac{e^x}{2}(\sin x - \cos x) + \frac{1}{2} + \frac{e^x}{2}\cdot x\sin x}$$ Ans.

Prop. $L^{-1}\left[\frac{X(s)}{s}\right] = \int_0^t x(t)\,dt$

$$x(t) = \int_0^t e^t \sin t\,dt$$

$$= \frac{e^t}{2}(\sin t - \cos t)\Big|_0^t$$

Q: (2011) 15 Marks

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t$$

$$x(0) = 2\,;\quad \left.\frac{dx}{dt}\right|_{t=0} = -1$$

Solⁿ Take L.T. on Both sides

$$\Rightarrow s^2X(s) - s\,x(0) - x'(0) - 2sX(s) + 2x(0) + X(s) = \frac{1}{s-1}$$

(14.)

$$\Rightarrow (s^2-2s+1)\,X(s) = \frac{1}{s-1} + 4s - 5$$

$$\Rightarrow X(s) = \frac{1}{(s-1)^3} + \frac{4s}{(s-1)^2} - \frac{5}{(s-1)^2}$$

$$= \frac{1}{(s-1)^3} + 4\left(\frac{s+1-1}{(s-1)^2}\right) - \frac{5}{(s-1)^2}$$

$$= \frac{1}{(s-1)^3} + 4\left(\frac{s-1}{(s-1)^2} + \frac{1}{(s-1)^2}\right) - \frac{5}{(s-1)^2}$$

$$= \frac{1}{(s-1)^3} + \frac{4}{(s-1)} + \frac{4}{(s-1)^2} - \frac{5}{(s-1)^2}$$

$$\Rightarrow X(s) = \frac{1}{(s-1)^3} + \frac{4}{(s-1)} - \frac{1}{(s-1)^2}$$

Now Take I.L.T.

$$x(t) = e^t\frac{t^2}{2} + 4e^t - te^t$$

$$\boxed{x(t) = e^t\left(\frac{t^2}{2} - t + 4\right)}$$ Ans.

Laplace Transform of Periodic Function :-

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Let $f(t)$ be a periodic function with period T then $L[f(t)]$ is defined as

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

Que:- $f(t) = \begin{cases} 3, & 0 < t < 2 \\ 0, & 2 < t < 4 \end{cases}$

$$f(t+4) = f(t)$$

find $L[f(t)]$.

Sol.

$$L[f(t)] = \frac{1}{1-e^{-4s}} \left[\int_0^2 e^{-st}(3)\, dt + \cancel{\int_2^4 e^{-st}(0)\, dt}^{\,0} \right]$$

Formula :-

① $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}} = \begin{cases} \frac{n!}{s^{n+1}} ; & n \in N \\ \frac{n\Gamma(n)}{s^{n+1}} ; & n > 0 \end{cases}$

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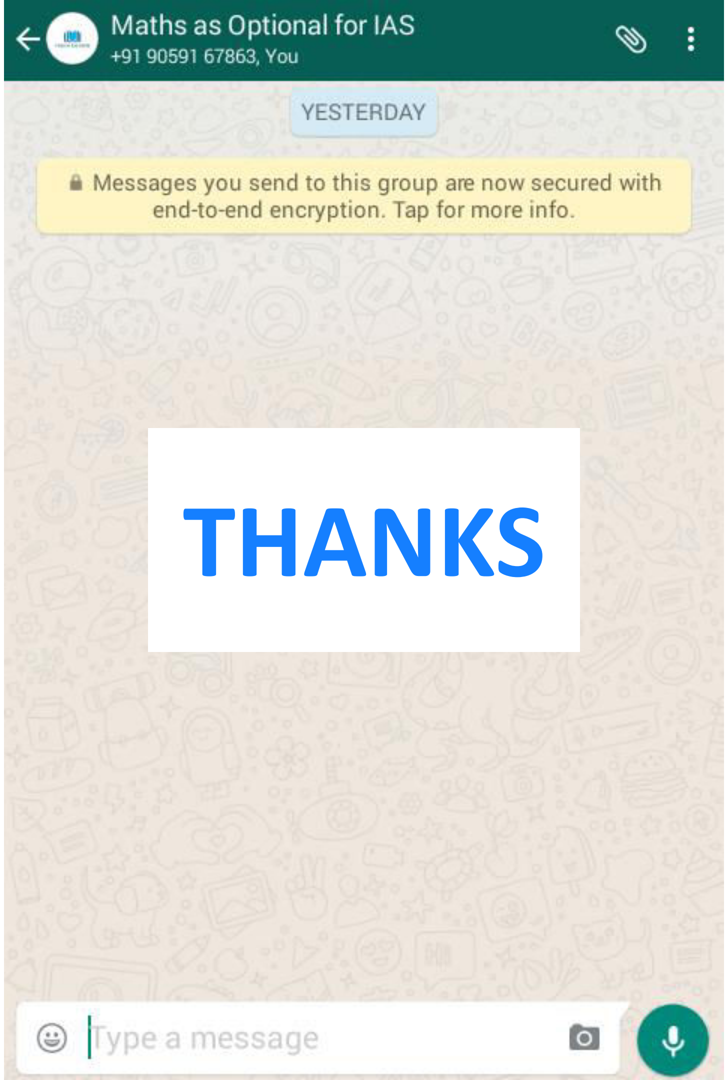

THANKS

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