MATHEMATICS

UPSC

CIVIL SERVICES MAINS

Complex Analysis:

1. Analytic functions
2. Cauchy-Riemann equations
3. Cauchy's theorem
4. Cauchy's integral formula
5. Power series representation of an analytic function
    1. Taylor's series
    2. Singularities
    3. Laurent's series
6. Cauchy's residue theorem
7. Contour integration

Chapter No. 1 : "COMPLEX ANALYSIS"

Complex No. = Real + imaginary

i.e. $f(z) = \underset{\text{Real}}{u(x,y)} + i\,\underset{\text{Imaginary.}}{v(x,y)}$

Analytic function : If a single-valued function $f(z)$ possess a unique derivative at every point of region R, then $f(z)$ is called A.F.

- Analytic / Holomorphic / Regular function.

$\boxed{\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}}$ $\boxed{\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}}$ $\Rightarrow$ CR Eq^n

Proof :- $f(z) = u + iv$

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta(z)}$$

$$= \lim_{\delta z \to 0} \frac{(u+\delta u) + i(v+\delta v) - (u+iv)}{\delta z}$$

$$= \lim_{\delta z \to 0} \left(\frac{\delta u}{\delta z} + i\frac{\delta v}{\delta z} \right)$$ $\boxed{\delta z = \delta x + i\delta y}$

Case I
line parallel to x-axis
$\delta y = 0$
$\boxed{\delta z = \delta x}$

Case II
line parallel to Y-axis
$\delta x = 0$
$\boxed{\delta z = i\delta y}$

Case I $\Rightarrow \lim_{\delta x \to 0} \left(\frac{\delta u}{\delta x} + i \frac{\delta v}{\delta x} \right)$ ——— (1)

Case II $\Rightarrow \lim_{\delta y \to 0} \left(\frac{\delta u}{i\,\delta y} + \frac{i\,\delta v}{i\,\delta y} \right) = \lim_{\delta y \to 0} \left(-\frac{\delta u}{\delta y} + \frac{\delta v}{\delta y} \right)$ ——— (2)

Comparing Real & Imaginary Part :

$$\boxed{\frac{\delta u}{\delta x} = +\frac{\delta v}{\delta y}} \quad \& \quad \frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \; , \; \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

(Mains 2014)

Que:- $f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$, $z \neq 0$: $f(0) = 0$

Prove that this function is not analytic function though it satisfy CR - eq^n

Sol^n

$$f(z) = \left(\frac{x^3 - y^3}{x^2+y^2} \right) + i \left(\frac{x^3+y^3}{x^2+y^2} \right)$$

- $f(z)$ is Continuous at $z = 0$.

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = 1$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = -1$$

$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = 1$$

$$\frac{\partial v}{\partial y} = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = 1$$

$\longrightarrow$ CR-equation satisfied.

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z}$$

$$= \lim_{z \to 0} \frac{(x^3 - y^3) + i(x^3 + y^3) - 0}{(x^2+y^2)(x+iy)}$$

Put $\boxed{y = x}$

$$= \lim_{x \to 0} \frac{i\, \cancel{2}\, x^{\cancel{3}}}{\cancel{x^2}\,(2x)(1+i)} = \frac{1}{2}\left(\frac{i}{1+i}\right)$$

$$= \frac{1+i}{2}$$

when $y = 0$ (along x-axis)

$$\Rightarrow f'(z) = f'(0) = \lim_{x \to 0} \frac{x^3 + i x^3}{x^3}$$

$$= 1+i$$

both are unequal $\Rightarrow$ $f'(0)$ does not exist.

Que:- (Mains 2015)

$[\, v = \log(x^2+y^2) + x + y \,]$ Given.

i) Harmonic function.

ii) u

iii) $f(z)$

Sol.

$$\frac{\partial v}{\partial x} = \frac{2x}{(x^2+y^2)} + 1$$

$$\frac{\partial v}{\partial y} = \frac{2y}{x^2+y^2} + 1$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{(x^2+y^2)(2) - 2x(2x)}{(x^2+y^2)^2} \quad \text{--- (1)}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{(x^2+y^2)(2) - 2y(2y)}{(x^2+y^2)^2} \quad \text{--- (2)}$$

(1) + (2) = 0

$\Rightarrow$ Harmonic func.

By CR-Eq.^{n}

(ii) $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{2y}{x^2+y^2} + 1$$

$$\int \frac{\partial u}{\partial x} = \int\left(\frac{2y}{x^2+y^2} + 1\right)$$

$$u = \frac{2\cancel{y}}{\cancel{y}} \tan^{-1}\frac{x}{y} + x + \phi(y)$$

$$\frac{\partial u}{\partial y} = \frac{-2x}{x^2+y^2} + \phi'(y) = -\frac{\partial v}{\partial x} = -\left(\frac{2x}{x^2+y^2} + 1\right)$$

$$\phi'(y) = -1 \Rightarrow \boxed{\phi(y) = -y}$$

$$\Rightarrow \boxed{u = 2\tan^{-1}\frac{x}{y} + x - y}$$

(iii) $$f(z) = u + iv \Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\Big|_{\text{w.r.t. } x}$$

$$f'(z) = \left(\frac{2y}{x^2+y^2} + 1\right) + i\left(\frac{2x}{x^2+y^2} + 1\right)\Big|_{x=z;\ y=0}$$

$$f'(z) = 1 + i\left(\frac{2}{z} + 1\right)$$

$$f(z) = \int f'(z) = \int\left(1 + i\left(\frac{2}{z} + 1\right)\right) = z + 2i\log z + iz$$

$$f(z) = \boxed{z + i(2\log z + z)}$$ Ans

Cauchy Residue theorem :-

$$\oint_C \frac{f(z)}{(z-a)^n}$$

$C : |z| = b \Rightarrow$ $b > a$

$a \longrightarrow$ Pole

$n \longrightarrow$ No. of Pole.

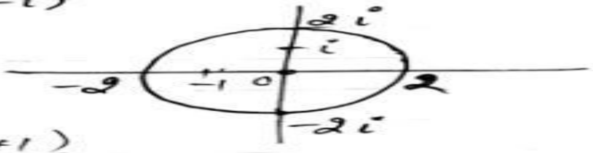
then Residue at Pole $z = a$, will be

$$= \lim_{z \to a} \frac{d^{n-1}\left[f(z) \times (z-a)^n\right]}{dz^{n-1}} \times \frac{1}{(n-1)!}$$

Que:- (Mains Dec-2015)

$$\oint_C \frac{(e^z+1)}{z(z+1)(z-i)^2}\,dz \; ; \; C : |z| = 2$$

All poles $z = 0, -1, i$ lies inside the circle.



- Residue at $z=0$ $= \lim_{z \to 0} \cancel{z} \times \frac{(e^z+1)}{\cancel{z}(z+1)(z-i)^2}$

$= -2$

- Residue at $z = -1$ $= \lim_{z \to -1} \cancel{(z+1)} \times \frac{(e^z+1)}{z\cancel{(z+1)}(z-i)^2} = \frac{-1}{2}\left(1 + \frac{1}{e}\right)$

- Residue at $z = i \Rightarrow$ Multipole.

$$= \lim_{z \to i} \frac{1}{1!} \times \frac{d}{dz}\left[\frac{\cancel{(z-i)^2} \times (e^z+1)}{z(z+1)\cancel{(z-i)^2}}\right] = \frac{-1}{2}(1+i)(1+e^i)$$

$$\Rightarrow \oint_C \frac{e^z+1}{z(z+1)(z-i)^2}\,dz = 2\pi i \times (\text{sum of all poles's residue})$$

$$= 2\pi i \times \left(-2 - \frac{1}{2}\left(1+\frac{1}{e}\right) - \frac{(1+i)(1+e^i)}{2}\right)$$

$$= \boxed{-\pi i\left(6 + \frac{1}{e} + e^i(1+i)\right)}$$ Ans.

Cauchy's Integral formula :-

$$\oint_C \frac{f(z)}{(z-a)}\,dz = 2\pi i \times f(a)$$ ✓.

if pole lies with in the Circle/Curve.

if Not lies then integration $= 0$.

Que:- (Mains 2013) Evaluate using Cauchy int. formula

$$\oint_C \frac{e^{3z}}{(z+1)^4}\,dz \qquad C: |z| = 2.$$

15 Marks

Mains 2013

Sol. multipole: at $z = -1$ = 4 pole.

$$\oint \frac{e^{3z}}{(z+1)^4}\,dz = \frac{2\pi i \times f'''(a)}{3!}$$

$$f(z) = e^{3z}$$
$$f'(z) = 3e^{3z}$$
$$f''(z) = 9e^{3z}$$
$$f'''(z) = 27e^{3z} \Rightarrow f'''(-1) = 27e^{-3}$$

$$\oint \frac{e^{3z}}{(z+1)^4}\,dz = 2\pi i \times \frac{27}{3!} e^{-3} = \boxed{\frac{9\pi i}{e^3}}$$ Ans.

Que:- $$\oint_C \frac{e^z}{(z+1)(z+2)}\,dz \qquad C: |z| = 2.5$$

Sol.

$z = -1$; $z = -2$

Both the pole lies within the circle.

$$f_1(z) = \left(\frac{e^2/(z+1)}{(z+2)}\right) + \left(\frac{e^2/(z+2)}{(z+1)}\right) = f_2(z)$$

$$f_1(a) = f_1(-2) = \frac{e^{-2}}{-3} \quad \text{and} \quad f_2(-1) = \frac{e^{-1}}{1} =$$

$$f_1(-2) = -e^{-2} \qquad\qquad f_2(-1) = e^{-1}$$

$$\oint \frac{e^2}{(z+1)(z+2)}\,dz = \boxed{(e^{-1} - e^{-2})\, 2\pi i}$$

Power Series :-

- Taylor Series $\longrightarrow$ greater / smaller
- Laurentz Series $\longrightarrow$ when Ring formation
- Singularity
- Mclaurine Series

Taylor Series :-

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2 f''(a)}{2!} + \cdots\cdots + \frac{(z-a)^{n-1}}{(n-1)!} f^{n-1}(a)$$

Que: (Mains 2015) $\quad f(z) = \dfrac{(2z-3)}{z^2-3z+2}\,dz \quad : \quad C: |z| = 2.$

point ~~$|z|$~~ = $z=0$

Sol.ⁿ

$$f(z) = \frac{2z-3}{z^2-3z+2} = \frac{(2z-3)}{(z-1)(z-2)} = \frac{1}{(z-1)} + \frac{1}{(z-2)}$$

$f(z) = \dfrac{1}{(z-1)} + \dfrac{1}{(z-2)}$; Now Cases are

a) $|z| < 1$

b) $1 < |z| < 2$

c) $|z| > 2$.

(8) $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \cdots$

2000 X 5.. ₹10,000..

<a> when $|z|<1$ $\Rightarrow$ $f(z) = \frac{1}{z-1} + \frac{1}{z-2}$

$$= \frac{-1}{1-z} - \frac{1}{2(1-\frac{z}{2})}$$

Taylor's Series

$$= -(1-z)^{-1} - \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= -\left(1+z+\frac{z^2}{2}+\frac{z^3}{3}+\cdots\right) - \frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\cdots\right)$$

Ans.

<b> when $1<|z|<2$ $\Rightarrow$ $\boxed{|z|>1}$ $\left(\frac{1}{z}<1\right)$; $|z|<2$

$$f(z) = \frac{1}{z-1} + \frac{1}{z-2} = \frac{1}{z(1-\frac{1}{z})} + \frac{-1/2}{(1-\frac{z}{2})}$$

$$= \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right) - \frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\cdots\right)$$ Ans.

<c> when $|z|>2$; $|z|>2 \Rightarrow \frac{1}{|z|}<1$

$$f(z) = \frac{1}{(z-1)} + \frac{1}{(z-2)} = \frac{1}{z(1-\frac{1}{z})} + \frac{1}{z(1-2/z)}$$

$$= \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right) + \frac{1}{z}\left(1+\frac{2}{z}+\frac{4}{z^2}+\cdots\right)$$

Ans.

Que :- Expand in Laurentz series

$$f(z) = \frac{1}{z^2(z-1)} \text{ about } z=0,\ z=1$$

$$\frac{1}{z^2(z-1)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z-1} = \frac{Az(z-1)+B(z-1)+Cz^2}{z^2(z-1)}$$

$$1 = Az^2 - Az + Bz - B + Cz^2$$

$$1 = (A+C)z^2 + z(-A+B) - B$$ $B = -1$

$-A+B=0$

$$f(z) = \frac{-1}{z} - \frac{1}{z^2} + \frac{1}{(z-1)} = -\frac{1}{z} - \frac{1}{z^2} + -(1-z)^{-1}$$

$| \; |z|<1$

$$= -\frac{1}{z} - \frac{1}{z^2} - (1+z+z^2+\ldots\ldots)$$

$$= \cancel{-\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\ldots\ldots\right)} \qquad -|\, |z|>1$$

Put $\quad z-1 = t$

$$\Rightarrow \quad f(z) = \frac{1}{(t+1)^2 t} = \frac{1}{t}(1+t)^{-2}$$

$$= \frac{1}{t}\left(1-2t+3t^2-4t^3+\ldots\ldots\right) \text{ Ans}$$

(Mains 2015) Evaluate $\displaystyle\int_0^{\pi} \frac{d\theta}{\left(1+\frac{1}{2}\cos\theta\right)^2}$ using Residue

Sol. This is the solution / application of Residue theorem.

Polar form $\longrightarrow$ Cartesian form.

$$z = e^{i\theta} \quad \Rightarrow \quad dz = e^{i\theta} \cdot i \cdot d\theta$$

$$\cos\theta = \frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right) \qquad \Rightarrow \quad d\theta = \frac{dz}{iz}$$

$$= \frac{1}{2}\left(z+\frac{1}{z}\right) = \frac{z^2+1}{2z}$$

$$\int \frac{1}{\left(1+\frac{1}{2}\left(\frac{z^2+1}{2z}\right)\right)^2} \times \frac{dz}{iz} = \frac{(4z)^2}{\left[4z+(1+z^2)\right]^2} \times \frac{dz}{iz}$$

$$= \int \frac{4z}{(z^2+4z+1)^2}\, dz$$

$$z^2+4z+1 = 0$$

$$z = \frac{-4 \pm \sqrt{16-4}}{2}$$

$$= \frac{-4 \pm 2\sqrt{3}}{2} = \boxed{-2 \pm \sqrt{3}}$$

$$\frac{4z}{(z+2-\sqrt{3})^2 (z+2+\sqrt{3})^2}$$

$$z \to -2+\sqrt{3} \;\; \frac{d}{dz}\left(\cancel{(z+2-\sqrt{3})^2} \times \frac{4z}{\ldots}\right)$$

$$\frac{(z+2+\sqrt{3})^2 4 - 4z\times 2(z+2+\sqrt{3})}{(z+2+\sqrt{3})^4}$$

$$\lim_{z\to -2+\sqrt{3}} \frac{4(z+2+\sqrt{3}) - 8z}{(z+2+\sqrt{3})^3} = \frac{4(-2+\sqrt{3}+2+\sqrt{3}) - 8(-2+\sqrt{3})}{(-2+\sqrt{3}+2+\sqrt{3})^3}$$

$$= \frac{4(2\sqrt{3}) + 16 - \cancel{8\sqrt{3}}}{(2\sqrt{3})^3} = \frac{16}{8\times 3\sqrt{3}} = \frac{2}{3\sqrt{3}}$$

$$2\pi i \times \frac{2}{3\sqrt{3}} = \frac{\sqrt{3}\,4}{9}\pi i \quad \text{Ans}$$

$\boxed{\text{Mains 2013}}$

Que:- $$I = \int_0^{\pi} \sin^4\theta \, d\theta$$ $\boxed{15 \text{ Marks}}$

$$= \frac{1}{2}\int_0^{2\pi} \left[\frac{1}{2i}\left(z - \frac{1}{z}\right)\right]^4 \times \frac{dz}{iz}$$

$$= \frac{1}{16\times 2}\left(z - \frac{1}{z}\right)^4 \frac{dz}{iz} = \frac{1}{32i}\int_0^{2\pi} \frac{(z^2-1)^4}{z^5}\, dz$$

$$= \frac{1}{16}\left[z^4 A\ 4(z)^3 \left(\frac{A}{z}\right) + 12 z\right.$$

Pole is at $\boxed{z=0}$; No. of pole $= 5$

$\Rightarrow$ residue at $z=0 = \lim_{z\to 0} \frac{1}{4!} \times \frac{d^4}{dz^4}\left[z^5 \times \frac{(z^2-1)^4}{z^5}\right]$

$$= \lim_{z\to 0} \frac{1}{4!} \frac{d^4}{dz^4}\left((z^2-1)^4\right)$$

$(z^2-1)^4 = z^8 - 4z^6 + 6z^4 - 4z^2 + 1$

$f'(z) = 8z^7 - 24z^5 + 24z^3 - 8z$

$f^{iv}(0) = 144$

$$= \frac{1}{4!} \times 144 = \frac{144}{24} = 6$$

$$= \frac{1}{32i} \times 6 \times 2\pi i = \frac{3}{8}\pi \quad \text{Ans}$$

(Mains 2012) $\boxed{15 \text{ Marks}}$

Que Expand $f(z) = \dfrac{1}{(z+1)(z+3)}$ Laurent's Series.

i> $1 < |z| < 3$ ii) $|z| > 3$

iii> $0 < |z+1| < 2$ iv) $|z| < 1$

Sol. $f(z) = \dfrac{1/2}{(z+1)} - \dfrac{1/2}{(z+3)}$

i) Along $1 < |z| < 3$; $\Rightarrow$ $|z| > 1$; $|z| < 3$

$\frac{1}{|z|} < 1$

$$f(z) = \frac{1}{2z\left(1+\frac{1}{z}\right)} - \frac{1}{3\times 2\left(1+\frac{z}{3}\right)}$$

$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \cdots\right) - \frac{1}{6}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right)$$

Ans.

ii) Along $|z| > 3$

$\Rightarrow \frac{3}{|z|} < 1$ and $\frac{1}{|z|} <$

$$f(z) = \frac{1}{2z\left(1+\frac{1}{z}\right)} - \frac{1}{2z\left(1+\frac{3}{z}\right)} = \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{2z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= \frac{1}{2z}\left[\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right) + \left(1 - \frac{3}{z} + \frac{3^2}{z^2} - \cdots\right)\right]$$

Ans.

iii) $0 < |z+1| < 2$; $f(z) = \dfrac{1}{(z+1)(z+3)}$

$$\Rightarrow f(z) = \frac{1}{2(z+1)} - \frac{1}{2(z+1+2)}$$

Let $(z+1) = t$

$0 < t < 2$

$$= \frac{1}{2t} - \frac{1}{2(t+1)}$$

$$= \frac{1}{2t} - \frac{1}{2}(1+t)^{-1} \text{ Ans.} \longleftarrow t < 1$$

$$= \frac{1}{2t} - \frac{1}{2t}\left(1+\frac{1}{t}\right)^{-1} \text{ Ans.} \longleftarrow 1 < t < 2$$

iv) Along $|z|<1$

$$f(z) = \frac{1/2}{(z+1)} - \frac{1/2}{(z+3)}$$

$$= \frac{1}{2}(1+z)^{-1} - \frac{1}{2\times 3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2}\left(1 - z + z^2 - z^3 + \cdots\cdots\right) - \frac{1}{6}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \cdots\cdots\right)$$

Ans.

Que: (Mains 2012) $\boxed{\text{Marks} = 15}$

$$I = \int_0^{2\pi} \frac{d\theta}{1 - 2a\cos\theta + a^2} \; ; \quad a^2 < 1$$

Sol.

$$z = e^{i\theta} \Rightarrow d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

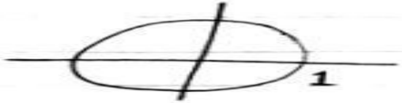
$$= \oint_C \frac{1}{1 - 2a \times \frac{1}{2}\left(z + \frac{1}{z}\right) + a^2} \times \frac{dz}{iz}$$

$$= \oint_C \frac{1}{1 - \frac{a(z^2+1)}{z} + a^2} \times \frac{dz}{iz} = \frac{1}{i}\oint_C \frac{dz}{z - az^2 - a + a^2 z}$$

$$= -i\oint_C \frac{dz}{-az^2 + (1+a^2)z - a} = i\oint_C \frac{dz}{az^2 - (1+a^2)z + a}$$

$$= i\oint_C \frac{dz}{(az-1)(z-a)} \qquad C: |z| < 1$$

Pole: $z = \frac{1}{a}$ and $z = a$



$\because a^2 < 1 \Rightarrow a < 1$

$\Rightarrow$ only $z = a$ lies within the circle

$$= i\oint \frac{1/(az-1)}{(z-a)}\, dz$$

Residue at $z = a$ $= \lim_{z \to a} \left[(z-a) \times \frac{i}{(az-1)(z-a)} \right]$

$= \frac{i}{a^2-1}$

$$\oint_C \frac{i\,dz}{(az-1)(z-a)} = 2\pi i \times \left(\frac{i}{a^2-1} \right) = \frac{2\pi}{1-a^2} \text{ Ans.}$$

Que:- Laurents series for the function:-

$f(z) = \frac{1}{1-z^2}$; with centre $(z=1)$

Sol.

$(z-1) = t$

$f(z) = \frac{1}{(1-z)(1+z)} = \frac{1/2}{(1-z)} + \frac{1/2}{(1+z)}$

$z + 1 - 1 + 1$

$= z-1+2$

$= (t+2)$

$= \frac{-1}{2t} + \frac{1}{2(t+2)}$

$= \frac{-1}{2t} + \frac{1}{2 \times 2 \left(1+\frac{t}{2}\right)}$

$= \frac{-1}{2t} + \frac{1}{4}\left(1+\frac{t}{2}\right)^{-1} = \frac{-1}{2t} + \frac{1}{4}\left(1 - \frac{t}{2} + \frac{t^2}{4} - \frac{t^3}{8} + \dots\right)$ Ans.

Que:- (Mains 2011) $u = 2x - x^3 + 3xy^2$

i) Harmonic

ii) $v(x,y)$

iii) $f(z)$

Sol.ⁿ $u = 2x - x^3 + 3xy^2$

$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2 - 3x^2 + 3y^2$ $\qquad \frac{\partial^2 u}{\partial x^2} = -6x$

$-\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = 6xy$ $\qquad \frac{\partial^2 u}{\partial y^2} = 6x$

$+ \quad = 0$

$\Rightarrow$ H.f.

$\frac{\partial v}{\partial y} = 2 - 3x^2 + 3y^2$

→ integrate w.r.t. "y"

$$v = 2y - 3x^2y + y^3 + \phi(x)$$

→ Now, differentiate w.r.t. x

$$\frac{\partial v}{\partial x} = -6xy + \phi'(x) = -\frac{\partial u}{\partial y} = -(+6xy)$$

$$\Rightarrow \cancel{-6xy} + \phi'(x) = \cancel{-6xy}$$

$$\Rightarrow \phi'(x) = 0 \Rightarrow \phi(x) = C \quad (\text{constant})$$

$$\boxed{v = 2y - 3x^2y + y^2 + C}$$

iii) Now : $f(z) = u + iv$

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ | diff^r w.r.t. "x".

$$\Rightarrow f'(z) = (2 - 3x^2 + 3y^2) - i\,6xy \Big|_{x=z \;\&\; y=0}$$

$$\Rightarrow f'(z) = 2 - 3z^2$$

→ integrate w.r.t. z.

$$\boxed{f(z) = 2z - z^3 + iC} \text{ Ans.}$$

Que:- $\oint_c (z - z^2)\,dz$ c: upper half of $|z| = 1$

$$z = re^{i\theta}$$

Anti clockwise

$$= \int_0^{\pi} (re^{i\theta} - r^2 e^{2i\theta})\, ire^{i\theta}\, d\theta$$

$$= ir\int_0^{\pi} [re^{2i\theta} - re^{3i\theta}]\, d\theta$$

$$= ir\left[\frac{r\,e^{2i\theta}}{2i}\cancel{\times 2\pi} - \frac{r^2 e^{3i\theta}}{3i}\right]\Bigg|_0^{\pi}$$

$$= ir\left[\frac{-r^2(-1)}{3i} + \frac{r^2}{3i}\right] = \frac{2i\,r^3}{3i} = \frac{2}{3}$$

Part : B $\quad z = re^{i\theta}$ where $r=1$

$z = e^{i\theta}$

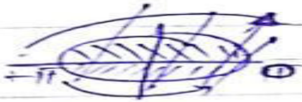
$\Rightarrow dz = ie^{i\theta}d\theta$

$$\int_{\pi}^{2\pi} i(e^{i\theta} - e^{2i\theta})e^{i\theta}\,d\theta$$

$$= i\int_{\pi}^{2\pi}(e^{2i\theta} - e^{3i\theta})\,d\theta = i\left[\frac{e^{2i\theta}}{2i} - \frac{e^{3i\theta}}{3i}\right]\Bigg|_{\pi}^{2\pi}$$

$$= \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{2} + \frac{1}{3}\right) = \cancel{\frac{1}{2}} - \frac{1}{3} - \cancel{\frac{1}{2}} - \frac{1}{3} = \frac{-2}{3} \text{ Ans}$$

Que: $\oint_C \frac{1}{z}\,dz = -\pi i$ or πi $\qquad C : |z| = 2$



Sol.^{n} $\Rightarrow \quad z = re^{i\theta} \Rightarrow z = 2e^{i\theta}$

$dz = 2ie^{i\theta}$

$$= \int_{\pi}^{0} \frac{e^{-i\theta}}{2}\,2\cdot ie^{i\theta} = i\int_{\pi}^{0} d\theta$$

$$= \int_{\pi}^{0} i\,\frac{e^{-i\theta}}{2}\times 2e^{i\theta}\,d\theta = i\int_{\pi}^{0} d\theta = i[\theta]\Big|_{\pi}^{0} = i(0-\pi)$$

$= \boxed{-\pi i}$ Ans.

$$\rightarrow \int_{\pi}^{2\pi} i\cdot\frac{e^{-i\theta}}{2}\times 2e^{i\theta}\,d\theta = i\theta\Big|_{\pi}^{2\pi}$$

$= i[2\pi - \pi] = \pi i$ Ans

Que:-
(10.) $\int_{1-i}^{2+3i} (z^2+z)\,dz$ along the line joining $(1,-1)$ and $(2,3)$.

$$(y-1) = 4(x+1)$$
$$y = 4x+4+1 = 4x+5$$
$$\boxed{dy = 4dx}$$

$z = x+iy$

$$= \int_1^2 \left[(x^2+iy)^2 + (4x+iy)\right](dx+i\,dy)$$
$$= \int_1^2 \left((x+i4x+5i)^2 + [x+i(4x+5)]\right)(dx+i4dx)$$
$=$

Most Que:- $\int_C \frac{3z^2+7z+1}{(z-\alpha)}\,dz$. $c: x^2+y^2=4$

· Find the value of $f(3)$, $f'(1-i)$, $f''(1-i)$

Solution:-
$$\phi(\alpha) = 3\alpha^2+7\alpha+1 \qquad f(3)$$
$$\boxed{f(3) = 0} \quad \because \text{ 3 lies out the circle.}$$

$\cancel{f'(\alpha) = 2(3\alpha)+7}$
$\cancel{f''(\alpha) = 6}$

so $\rightarrow$ $\cancel{f(3) = 10}$
$\rightarrow$ $\cancel{f'(1-i) = 2(1-i)+7 = 13-6i}$
$\rightarrow$ $\cancel{f''(1-i) = 6}$ Ans.

$$f(\alpha) = 2\pi i \times \phi(\alpha)$$
$$f(\alpha) = 2\pi i \times (3\alpha^2+7\alpha+1)$$

$f'(\alpha) = 2\pi i\,[6\alpha + 7]$

$f''(\alpha) = 12\pi i$ — Ans:

$$f'(\alpha)\Big|_{\alpha=(1-i)} = 2\pi i\,[6(1-i)+7)] = (2\pi i(6-6i+7))$$

$$= 2\pi i(13-6i)$$

$$= \boxed{2\pi(6+13i)} \text{ Ans:}$$

Que. $f(z) = u+iv$ is analytic function.

$$u - v = \frac{e^y - \cos x + \sin x}{\cosh y - \cos x}$$

Sol.

$f(z) = u+iv$

$i f(z) = iu - v$ $\Rightarrow$ $f(z) + i f(z) = \underline{\underline{(u-v)}} + i\underline{\underline{(u+v)}}$

$$\boxed{F(z) = U + iV}$$

$$U = \frac{e^y - \cos x + \sin x}{\cosh y - \cos x}$$

$$\frac{\partial U}{\partial x} = \frac{(\cosh y - \cos x)(\sin x - \cos x) - (e^y - \cos x + \sin x)\sin x}{(\cosh y - \cos x)^2} = \frac{\partial V}{\partial y}$$

$$\frac{\partial U}{\partial y} = \frac{(\cosh y - \cos x)e^y - (e^y - \cos x + \sin x)(\sinh y)}{(\cosh y - \cos x)^2} = -\frac{\partial V}{\partial x}$$

$F(z) = U + iV \Rightarrow F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial y}\Big|$ $x=z$ and $y=0$

$$F(z) = \frac{\cancel{(1-\cos z)}(-\cos z)}{(1-\cos z)^{\cancel{2}}} - i\frac{\cancel{(1-\cos z)}}{(1-\cos z)^{\cancel{2}}} = -\left[\frac{i+\cos z}{1-\cos z}\right]$$

$$= 1 - \frac{1}{1-\cos z} - \frac{i}{1-\cos z} = \int\left(1 - (1+i)\times\frac{1}{2\sec^2 z/2}\right)dz$$

$$= z - \frac{(1+i)}{\cancel{2}}\times \cot\frac{z}{2}\times \cancel{2} = \boxed{z - (1+i)\cot z/2} \text{ Ans}$$

THANKS