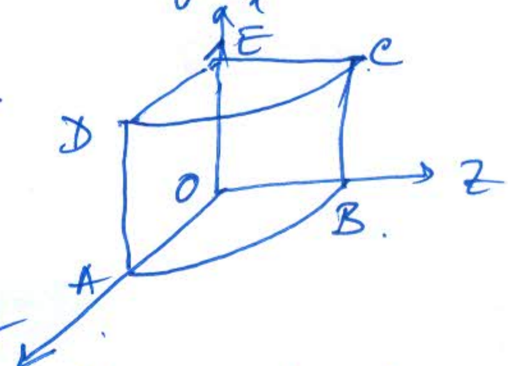
Lecture 24
13/4/17
Thursday

Stoke's theorem, Gauss divergence Theorem.

Gauss Divergence theorem

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_V \operatorname{div} \vec{F}\, dV$$

where S is a closed surface which enclose volume V.



Ex. of Lecture 21

$$\iint_S \vec{F}\cdot d\vec{S}$$

$$\vec{F} = 2x^2y\,\hat{i} - y^2\,\hat{j} + 4z^2x\,\hat{k}$$

S is bounded by $y^2+z^2=9$ & the plane $x=2$ in the 1st octant.

value of surf. integral $= 180$.

According to Gauss divergence theorem,

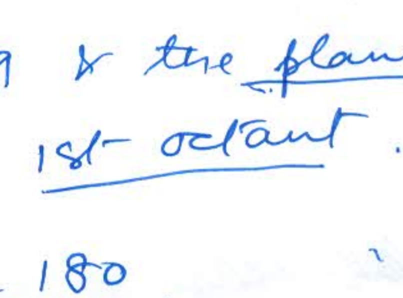
$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_V \operatorname{div}\vec{F}\cdot dV$$

$\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$.

$\operatorname{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$.

$$\operatorname{div}\vec{F} = 4xy - 2y + 8zx$$

$$\iiint_V (4xy - 2y + 8zx)\, dV$$



$$= \int_{y=0}^{3}\int_{x=0}^{2}\int_{z=0}^{\sqrt{9-y^2}} (4xy - 2y + 8zx)\, dz\, dx\, dy.$$

$$= \int_{y=0}^{3} \int_{x=0}^{2} \Big[(4xy - 2y) z + 4x z^2 \Big]_{0}^{\sqrt{9-y^2}} \, dx \, dy$$

$$= \int_{y=0}^{3} \int_{x=0}^{2} (4xy - 2y)\sqrt{9-y^2} + 4x\,(9-y^2) \, dx \, dy$$

$$= \left(\int_{x=0}^{2} 4x \, dx \right) \left(\int_{y=0}^{3} y\sqrt{9-y^2} \, dy \right) - \left(\int_{x=0}^{2} dx \right) \left(\int_{y=0}^{3} 2y\sqrt{9-y^2} \, dy \right)$$

$$+ \left(\int_{x=0}^{2} 4x \, dx \right) \left(\int_{y=0}^{3} (9-y^2) \, dy \right)$$

$$\int_{y=0}^{3} y\sqrt{9-y^2} \, dy = \cancel{\sqrt{9-y^2}}$$

$$= \left[(9-y^2)^{3/2} \times \frac{1}{3} \right]_{3}^{0} = \frac{1}{3} \times 27 = 9 \qquad 9^{3/2}.$$

$$\int_{x=0}^{2} 4x \, dx = 2 \times \left[x^2 \right]_{0}^{2} = 8$$

$$= 8 \times 9 - 2 \times 2 \times 9$$

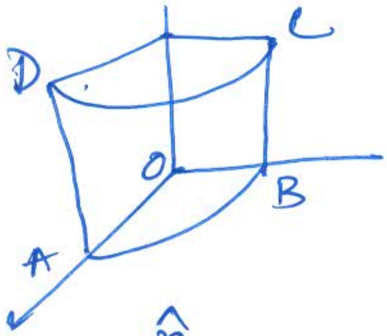
$$+ 8 \left[9y - \frac{y^3}{3} \right]_{0}^{3} \rightarrow = (27 - 9)$$

$$= 72 - 36 + 144 = 180$$

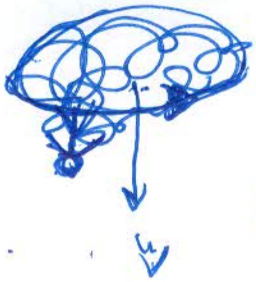
Stokes theorem.

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{Curl}\,\vec{F}\, d\vec{S}.$$

S is some open surface bounded by the curve C taken is +ve direction.



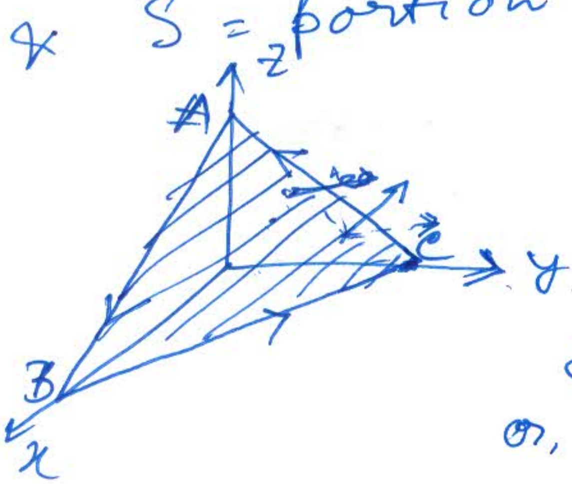
Here ABCD is an open surface bounded by the curves AB, BC, CD, DA.



Anton's book Calculus.

Ex. Verify stokes theorem for $\vec{F} = (x-y)\hat{i} + (y-z)\hat{j} + (z-x)\hat{k}$.

& S = portion of the plane in 1st octant $x+y+z=1$



ABC is the surface (open) & is bounded by AB, BC, CA

$\phi = x+y+z-1=0$, ~~[illegible]~~

or, $\psi = 1-x-y-z=0$

$\vec{\nabla}\phi = \hat{i}+\hat{j}+\hat{k}$

$\vec{\nabla}\psi = -\hat{i}-\hat{j}-\hat{k}$

$$\iint_S \text{Curl}\,\vec{F}\cdot d\vec{S} \qquad \text{Curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-y & y-z & z-x \end{vmatrix} = \hat{i}+\hat{j}+\hat{k}$$

$$= \iint_S (\hat{i}+\hat{j}+\hat{k})\cdot\hat{n}\, dS.$$

$$\hat{n}\, dS = \frac{\vec{\nabla}\phi}{\|\vec{\nabla}\phi\|} \times \|\vec{\nabla}\phi\|\, d..d..$$

$$= \iint_S (\hat{i}+\hat{j}+\hat{k})\cdot(\hat{i}+\hat{j}+\hat{k})\, dx\,dy.$$

$z = 1-x-y$. $\quad = \vec{\nabla}\phi\, dx\, dy$

$\phi = x+y+z-1 = 0$. $\quad z = \phi(x,y)$

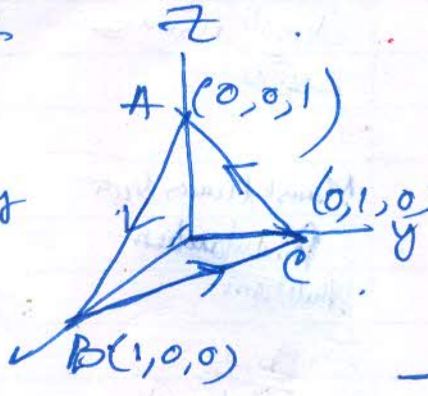
$y = \phi(z,x)$

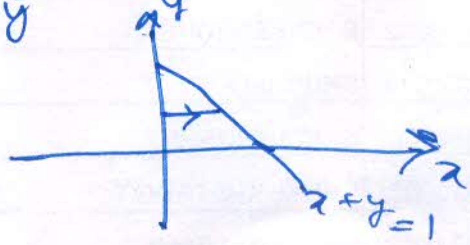
$\hat{n}\, dS = \cancel{\vec{\nabla}\phi\, dy\, dx}$

$= \vec{\nabla}g\, dz\, dx$

$$= 3\iint_{x+y\le 1} dx\,dy$$

$$= 3\int_{y=0}^{1}\int_{x=0}^{1-y} dx\,dy$$





$$= \frac{3}{2}.$$

$$\oint \vec{F}\cdot d\vec{r} = \int_A^B \vec{F}\cdot d\vec{r} + \int_B^C \vec{F}\cdot d\vec{r} + \int_C^A \vec{F}\cdot d\vec{r}.$$

AB: $\dfrac{x-0}{1-0} = \dfrac{y-0}{0-0} = \dfrac{z-1}{0-1} = t$.

$x = t,\ y = 0,\ z = 1-t,\ 0 \le t \le 1$

$$\int_A^B (x-y)\,dx + (y-z)\underbrace{dy}_{=0} + (z-x)\,dz.$$

$$\int_{t=0}^{1} t\,dt + (1-2t)\,dt = \frac{1}{2}.$$

Similarly check

$$\therefore \int_B^C \vec{F}\cdot d\vec{r} = \frac{1}{2} = \int_C^A \vec{F}\cdot d\vec{r}.$$

$$\therefore \oint \vec{F}\cdot d\vec{r} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$$

Stoke's thm for 2D.

$$\vec{F} = F_1\hat{i} + F_2\hat{j} + 0.\hat{k}.$$

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & 0 \end{vmatrix}.$$

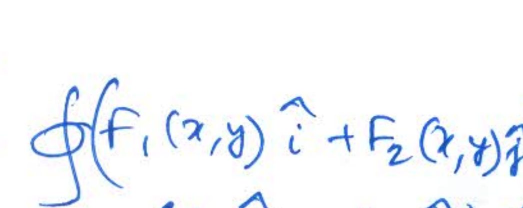
$$= -\frac{\partial F_2}{\partial z}\hat{i} + \frac{\partial F_1}{\partial z}\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}.$$

$\vec{F} = \vec{F}(x,y)$ so that $F_1 \equiv F_1(x,y)$,

$F_2 \equiv F_2(x,y)$

$$\therefore \frac{\partial}{\partial z} F_i(x,y) = 0$$

$$= \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$$



$$\oint_C \vec{F}\cdot d\vec{r} \qquad \iint_S \text{Curl } \vec{F}\cdot\hat{n}\, dS.$$

$$= \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}.\hat{k}\, dx\, dy.$$

$$= \oint_C \big(F_1(x,y)\hat{i} + F_2(x,y)\hat{j}\big)(dx\,\hat{i} + dy\,\hat{j}) = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy.$$

$$= \oint_C (F_1\, dx + F_2\, dy)$$

According to Stoke's thm.

$$\oint_C F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

which is Green's thm

This is why Stoke's thm is referred as Green's thm in space. //