TARUN KAUSHIK

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MATHEMATICS

FOR UPSC CSE

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SYLLABUS

5/8 Que

PAPER-I

250 Marks

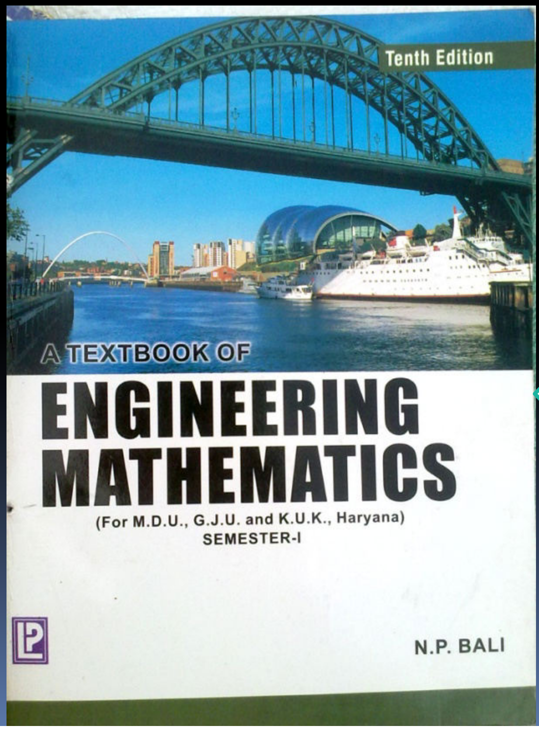
PAPER-II

5/8 Que

1. **LINEAR ALGEBRA**
2. Calculus
3. Ordinary Differential Equation
4. Analytical Geometry
5. Vector Analysis
6. Dynamics & Statics

1. Abstract Algebra
2. Real Analysis
3. **Complex Analysis**
4. **Linear Programming**
5. Partial Differential Equations
6. Numerical Analysis
7. Mechanics & Fluid Dynamics

1.



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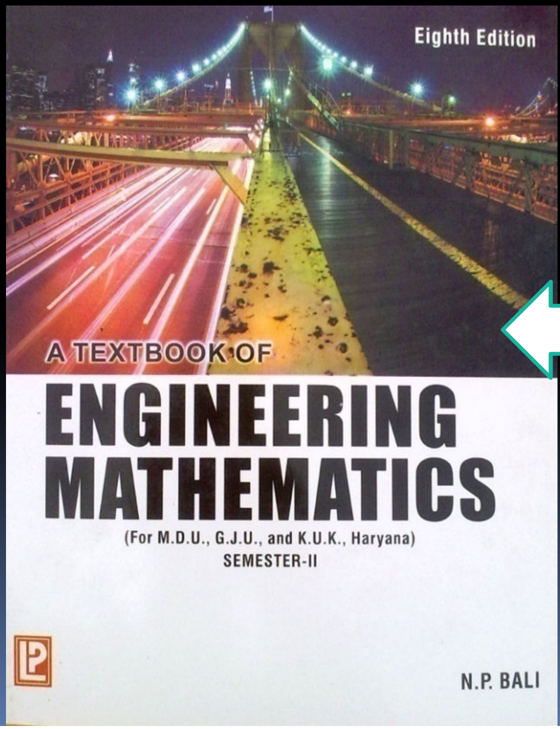
*Chapters 2, 3 and 4 are not included in the syllabus of KU, Kurukshetra.
**Chapter 13 is not included in the syllabus of MDU, Rohtak.

2.



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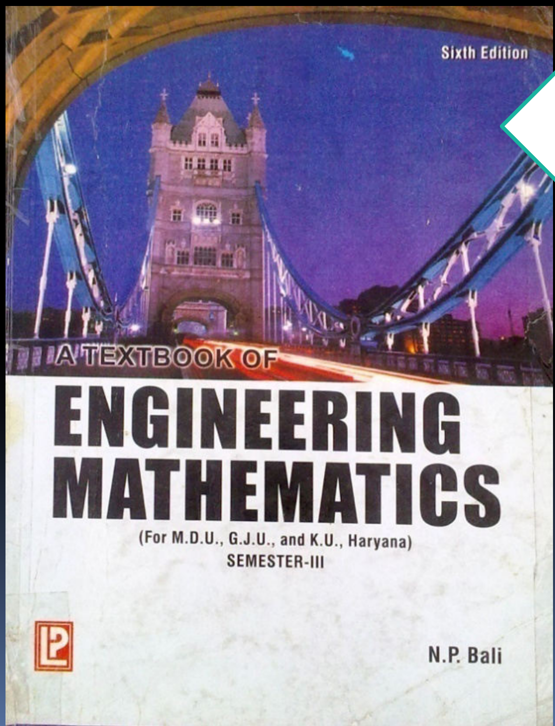
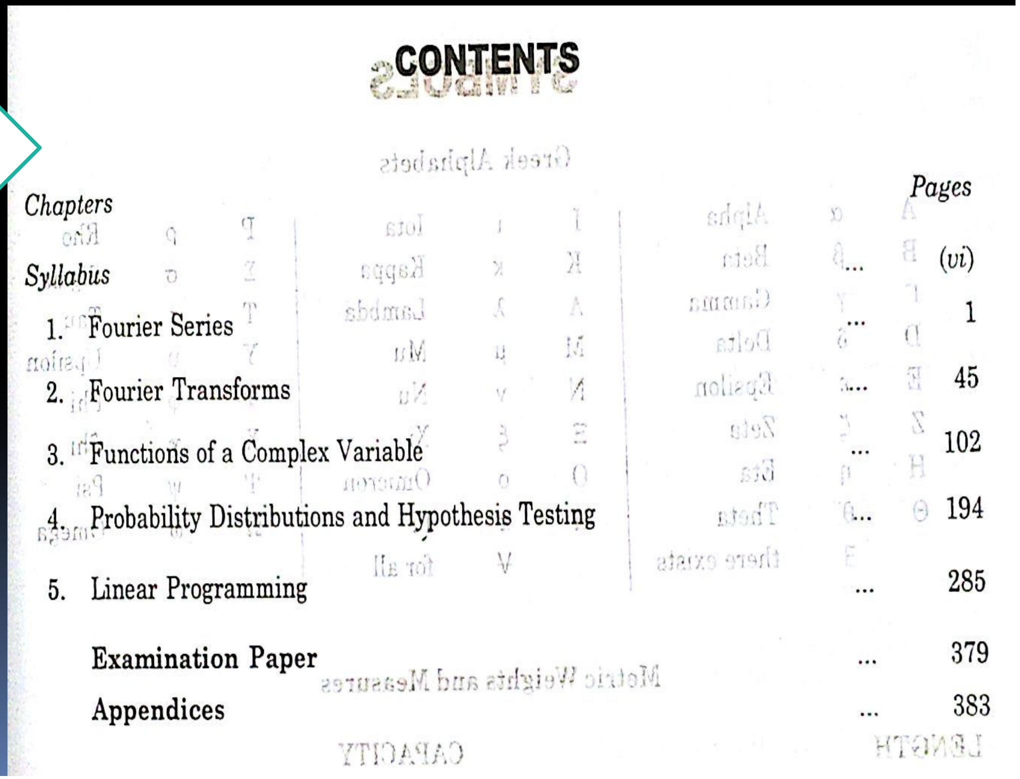
3.





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Linear Algebra Syllabus

- Vector spaces over R and C, linear dependence and independence, subspaces, bases, dimension; Linear transformations, rank and nullity,.
- Algebra of Matrices; Row and column reduction, Echelon form, congruence's and similarity; Rank of a matrix; Inverse of a matrix;
- Solution of system of linear equations; Eigen values and eigenvectors, characteristic polynomial, Cayley- Hamilton theorem,
- Symmetric, skew symmetric, Hermitian, skew-Hermitian, orthogonal and unitary matrices and their Eigen values.

Matrix.

Linear- Algebra

Dated : 10/03/2016

Linearly Independent :-

Que: (2015-Mains) The vector : (10 Marks)

$V_1 = (1,1,2,4)$; $V_2 = (2,-1,-5,2)$, $V_3 = (1,-1,-4,0)$ and $V_4 = (2,1,1,6)$ are linearly independent / ← Show that dependent.

Sol.ⁿ

$$\begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & -5 & 2 \\ 1 & -1 & -4 & 0 \\ 2 & 1 & 1 & 6 \end{bmatrix} \xleftrightarrow[R_4 \leftrightarrow R_4 - 2R_1]{R_2 \leftrightarrow R_2 - 2R_1,\ R_3 \leftrightarrow R_3 - R_1} \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 0 & -2 & -6 & -4 \\ 0 & -1 & -3 & -2 \end{bmatrix} \quad R_2 \leftrightarrow \frac{R_2}{-3},\ R_3 \leftrightarrow \frac{R_3}{-2}$$

$$\begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & -1 & -3 & -2 \end{bmatrix} \xleftrightarrow[R_4 \leftrightarrow R_4 + R_2]{R_3 \leftrightarrow R_3 - R_2} \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

⇒ Hence this shows that the vector v_1 and v_2 are linearly Independent vector & v_3 & v_4 are linearly dependent.

(mains 2015)
Que:- Reduce the following matrix to the Row Echelon form and hence find its Rank.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix}$$

Sol. $R_2 \leftrightarrow R_2 - 2R_1$, $R_3 \leftrightarrow R_3 - R_1$, $R_4 \leftrightarrow R_4 - 8R_1$

$$= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 3 & 2 & 3 \\ 0 & -15 & -10 & -15 \end{bmatrix} \quad \text{Now } R_4 \longleftrightarrow \frac{R_4}{-5}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 3 & 2 & 3 \\ 0 & 3 & 2 & 3 \end{bmatrix} \quad R_3 \longleftrightarrow R_3 - R_2,\quad R_4 \longleftrightarrow R_4 - R_2$$

$$= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \boxed{\text{Rank} = 2}$$

↑ Reduced Echelon matrix.

Que:- (Mains 2015) (12 Marks)

If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$; find A^{30}.

Sol.ⁿ

Using Cayley Hamilton theorem.

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = (1-\lambda)(\lambda^2 - 1) = 0$$

$$\lambda^2 - \lambda^3 - 1 + \lambda$$

$$A^3 - A^2 - A + I =$$

$$A^3 = A^2 + A - I \longrightarrow (1)$$

$$A^4 = A^3 + A^2 - A = A^2 + A - I + A^2 - A = 2A^2 - I$$

$$A^8 = 4A^4 + I - 4A^2 = 4(2A^2 - I) + I - 4A^2 = 4A^2 - 3I$$

$$A^{16} = 16A^4 + 9I - 24A^2 = 32A^2 - 16I + 9I - 24A^2 = 8A^2 - 7I$$

$$A^{32} = 64A^4 + 49I - 112I = 128A^2 - 64I + 49I - 112I$$

$$= 16A^2 - 15I$$

$$\Rightarrow A^{30} = \boxed{16I - 15A^{-2}}$$

$A^4 = 2A^2 - I$;	$A^{64} = 32A^2 - 31A$
$A^8 = 4A^2 - 3I$;	$A^{128} = 64A^2 - 63A$
$A^{16} = 8A^2 - 7I$;	$A^{256} = 128A^2 - 127I$
$A^{32} = 16A^2 - 15I$;	$A^{512} = 256A^2 - 255I$

$$A^3 = A^2 + A - I \Rightarrow A = I + A^{-1} - A^{-2}$$

$$\Rightarrow \boxed{A^{-2} = I + A^{-1} - A}$$

$$\rightarrow A^{30} = 16I - 15A^{-2}$$

$$= 16I - 15(I + A^{-1} - A) = 16I - 15I + 15(A - A^{-1})$$

$$A^{30} = I + 15(A - A^{-1})$$

Shortcut to calculate inverse of a matrix:-

$$\begin{array}{c|cccc} 1 & 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{array}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} = Adj(A)$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad |A| = -1$$

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$A^{30} = 15\left(\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}\right) + I$$

$$= 15\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 15 & 0 \\ 15 & 0 & 0 \\ 15 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 15 & 0 \\ 15 & 1 & 0 \\ 15 & 0 & 1 \end{bmatrix} \quad \text{Ans.}$$

Que (Mains 2015) (12 Marks)

$\{(1,0,0,0), (0,1,0,0), (1,2,0,1), (0,0,0,1)\} \leftarrow$ vector

i) Basis
ii) dimension.

Ans

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_3 - R_1} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\rightarrow R_3 \leftrightarrow R_3 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \xrightarrow{R_4 \leftrightarrow R_4 - R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis = $\{(1,0,0,0), (0,1,0,0), (0,0,0,1)\}$

Dim = 3. Ans

Que (Mains 2014) (10-Marks)

System of Linear Equation:-

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + \lambda z = \mu$$

i) No solution
ii) unique solution
iii) infinite No. of sol.

Sol^n

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}; \quad B = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}; \quad \boxed{\rho(A);\ \rho(A:B)}$$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right] \quad \begin{array}{l} R_2 \leftrightarrow R_2 - R_1 \\ R_3 \leftrightarrow R_3 - R_1 \end{array}$$

$$= \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right] \quad R_3 \leftrightarrow R_3 - R_2$$

$$= \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

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Case: I (No solution)
$\rho(A) \neq \rho(A:B)$
for this:
$\boxed{\lambda = 3, \mu \neq 10}$

ii) unique solution:
$\lambda \neq 3, \mu \neq 10$

(iii) Many solution :-
$\boxed{\lambda = 3 \text{ and } \mu = 10}$

① $\rho(A) = \rho(A:B) =$ No. of variable
$\Rightarrow$ unique solution

② $\rho(A) \neq \rho(A:B)$
$\Rightarrow$ No sol.^{n}

③ $\rho(A) = \rho(A:B) <$ No. of variable
$\Rightarrow$ Many solution

Que: (2014) (8-Marks) "Eigen Values & Eigen Vector"

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Sol.^{n}

$$|A - \lambda I| = 0 \quad ; \quad \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix}$$

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \qquad |A - \lambda I| = 0$$

$44 \times \frac{5}{2}$

$$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix}$$

$(-2-\lambda)\,[(1-\lambda)(+\lambda) + 12] \quad -2[-2\lambda - 6] \quad -3[-4 + 1 - \lambda]$

$+(2+\lambda)(\lambda - \lambda^2 + 12) \qquad +4\lambda + 12 \quad + 9 + 3\lambda$

$2\lambda - 2\lambda^2 + 24 + \lambda^2 - \lambda^3 + -12\lambda$

$= -\lambda^3 - \lambda^2 + 19\lambda + 45 = 0$

21

$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$ $\quad \lambda = -3$

$(\lambda + 3)(\lambda^2 - 2\lambda - 15) = 0$

$\lambda = -3, -3, 5$

$$\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

for $\boxed{\lambda = -3}$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}$$

$$\begin{bmatrix} +1 & +2 & +5 \\ 2 & -4 & -6 \\ -7 & 2 & -3 \end{bmatrix}$$

7 14 35

$$\begin{matrix} 1 & 2 & 5 \\ 0 & -8 & -16 \\ 0 & +16 & 32 \end{matrix}$$

$$\begin{matrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{matrix}$$

$x_1 + 2x_2 + 5x_3 = 0$; $x_2 + 2x_3 = 0$

$x_2 = -2x_3$

$\boxed{\text{JAN-July}}$

$x_1 - 4x_3 + 5x_3 = 0$ $\quad x_1 = -x_3$

$\frac{x_1}{-1} = \frac{x_2}{-2} = \frac{x_3}{+1}$

$(1, 2, -1)$ $\quad -1 \quad 2 \quad 1$

Laxmikant

☆ Orthogonal Matrix :

if $A \cdot A^T = I \Rightarrow \uparrow$

☆ Hermitian Matrix:

$\Downarrow$

$\boxed{a_{ij} = \overline{a_{ij}}}$

Complex No. Conjugate

Skew-Hermitian Matrix:-

$\boxed{a_{ij} = -\overline{a_{ij}}}$

Complex ~~Number~~ Conjugate

(12 marks)

Que :- (2015) Find Eigen values & eigen vectors of Matrix

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} = A$$

Solution $[A - \lambda I] = 0 \Rightarrow \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} = [A - \lambda I]$

$|A - \lambda I| = 0$

$\Rightarrow (1-\lambda)[(5-\lambda)(1-\lambda)-1)] - 1[(1-\lambda)-3] + 3(1-15+3\lambda)$

$= (1-\lambda)(5 - 5\lambda - \lambda + \lambda^2 - 1) - (-2-\lambda) + 3(-14+3\lambda)$

$= (1-\lambda)(4 - 6\lambda + \lambda^2) + 2 + \lambda - 42 + 9\lambda$

$= 4 - \cancel{6\lambda} + \lambda^2 - \cancel{4\lambda} + 6\lambda^2 - \lambda^3 + 2 + \cancel{\lambda} - 42 + \cancel{9\lambda}$

$= -\lambda^3 + 7\lambda^2 - 36 = 0 \Rightarrow \lambda^3 - 7\lambda^2 + 36 = 0$

by trial method $\lambda = -2$ is a factor.

$(\lambda+2)\ )\ \lambda^3 - 7\lambda^2 + 36\ (\ \lambda^2 - 9\lambda + 18$

$\lambda^3 + 2\lambda^2$

$-9\lambda^2 + 36$

$-9\lambda^2 - 18\lambda$

$18\lambda + 36$

$18\lambda + 36$

0

$\lambda^2 - 9\lambda + 18 = 0$

$(\lambda-6)(\lambda-3) = 0$

$\Rightarrow$ Eigen value $= \lambda = -2, +3, +6$ Ans

Eigen vectors:-

i) for $\lambda = -2$ $\Rightarrow [A - \lambda I] = \begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$R_3 \leftrightarrow R_3 - R_1 \Rightarrow \begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$R_2 \leftrightarrow R_2 - 3R_1$

$$\begin{bmatrix} 3 & 1 & 3 \\ 0 & 20 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow 3x_1 + x_2 + 3x_3 = 0$ & $x_2 = 0$

$3x_1 + 3x_3 = 0 \Rightarrow \cancel{3}x_1 = -\cancel{3}x_3$

$x_1 = -x_3$

vector $(1, 0, -1)$

ii) for $\lambda = 3$

$[A - \lambda I] = \begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$\begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 3 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$R_2 \leftrightarrow R_2 + 2R_1,\ R_3 \leftrightarrow R_3 - 3R_1$

$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 5 \\ 0 & -5 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$R_3 \leftrightarrow R_3 + R_2$

$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$x_1 + 2x_2 + x_3 = 0$

$x_2 + x_3 = 0$

$\boxed{x_2 = -x_3}$

$\boxed{x_1 = x_3}$

$\boxed{(1, -1, 1)}$ vector

iii) for $\lambda = 6$

$\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$R_1 \leftrightarrow R_2$

$\begin{bmatrix} 1 & -1 & 1 \\ -5 & 1 & 3 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$R_2 \leftrightarrow R_2 + 5R_1$ & $R_3 \leftrightarrow R_3 - 3R_1$

$\begin{bmatrix} 1 & -1 & 1 \\ 0 & -4 & 8 \\ 0 & 4 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$R_3 \leftrightarrow R_3 + R_2$

$\begin{bmatrix} 1 & -1 & 1 \\ 0 & -4 & 8 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$x_1 - x_2 + x_3 = 0;\ -4x_2 + 8x_3 = 0$

$\boxed{x_1 = x_3}$ $\boxed{x_2 = 2x_3}$

$(1, 2, 1)$ Ans