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DATE:

A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET

IMS™

INSTITUTE OF MATHEMATICAL SCIENCES

163/250

MAINS TEST SERIES-18

JUNE-2018 TO SEPT.-2018

Under the guidance *of* K. Venkanna

MATHEMATICS

PAPER - 2 : FULL SYLLABUS

TEST CODE: TEST-06: IAS(M)/22-JULY.-2018

Time: Three Hours

Maximum Marks: 250

INSTRUCTIONS

1. This question paper-cum-answer booklet has 50 pages and has 34 PART/SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated. "
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any THREE of the remaining questions selecting at least ONE question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any page from this booklet.

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name Kanishak Kataria

Roll No. 1133664

Test Centre Jaipur

Medium English

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

KK

Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the Invigilator

IMPORTANT NOTE:
Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

P.T.O.

SECTION – A

1. (a) Union of two subgroups is a subgroup iff one of them is contained in the other. [10]

Q.NO.	Marks obtained
1(a)	→ 08
1(b)	→ 04
1(d)	→ 08
1(e)	→ 08
2(b)	→ 05
2(c)	→ 05
2(d)	→ 12
5(a)	→ 08
5(b)	→ 05
5(c)	→ 07
5(d)	→ 08
5(e)	→ 08
7(a)	→ 13
7(b)	→ 08
7(c)	→ 08
7(d)	→ 12
8(a)	→ 08
8(b)	→ 13
8(c)	→ 15

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H and K are subgroups of G.

To prove: $H \cup K$ is a subgroup $\iff H \subseteq K$ or $K \subseteq H$

(i) Given $H \subseteq K$ or $K \subseteq H$

$\therefore\ H \cup K = H \quad [\because H \subseteq K \text{ or } K \subseteq H]$

$\Rightarrow\ H \cup K$ is a subgroup [$\because$ H is subgroup]

(ii) Given $H \cup K$ is a subgroup

Let $H \not\subseteq K$ and $K \not\subseteq H$ —— ①

$\therefore\ \exists\ a \in H$ such that $a \notin K$ & $\exists\ b \in K$, such that $b \notin H$ — ②

Now $a \in H,\ b \in K \Rightarrow a, b \in H \cup K$ & $a^{-1} \in H$, $b^{-1} \in K$

$\therefore\ a^{-1}, b^{-1} \in H \cup K$ [$\because H \cup K$ is a subgroup]

Also $ab \in H \cup K$

$\Rightarrow\ ab \in H$ — ③ or $ab \in K$ — ④

if $ab \in H$, then as $a^{-1} \in H \Rightarrow a^{-1}(ab) \in H$
$\Rightarrow b \in H$ — ⑤

if $ab \in K$, then as $b^{-1} \in K \Rightarrow (ab)(b^{-1}) \in K$
$a(bb^{-1}) \in K$
$a \in K$ — ⑥

Both ⑤ & ⑥ contradict ②. $\therefore$ assumption ① is wrong

$\Rightarrow$ Either $H \subseteq K$ or $K \subseteq H$ → Hence proved

08

1. (b) Let R be the ring of 3×3 matrices over reals. Show that $S = \left\{ \begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix} \middle| x \text{ real} \right\}$,

is a subring of R and has unity different from unity of R. [10]

$S = \left\{ \begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix} \middle| x \in \mathbb{R} \right\}$.

unity of $S = a$ st $a \times s = s \ \forall\, s \in S$

Consider $a = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$

<04

$$Sa = \begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix} \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix} = S$$

$\therefore$ a it is a unity which is different from unity of R which is $I_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

1. (c) Prove that every infinite bounded subset of real numbers has a limit point. **[10]**

Let $S \subseteq \mathbb{R}$ where S is an Infinite set

Now S is bounded $\Rightarrow$ $\forall\, x \in S$ $m \le |x| \le M$ for some $m, M \in \mathbb{R}$

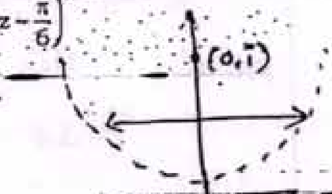
Limit point of $S = \lambda$ st for given $\epsilon > 0$ $\exists\, x \in S$ such that $l - \epsilon < x < l + \epsilon$.

1. (d) Use Canchy's theorem/Cauchy integral formule evaluate

(i) $\int_c \frac{z-1}{(z+1)^2(z-2)} dz$ where $C : |z-i| = 2$ (ii) $\int_c \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} dz$ where C is the circle $|z| = 1$

[10]

(i) $\int_c \frac{z-1}{(z+1)^2(z-2)} dz$ $C: |z-i| = 2$



Points to check: $z = -1$ & $z = 2$ $|-1-i| = \sqrt{1+1} = \sqrt{2} < 2$

$|2-i| = \sqrt{4+1} = \sqrt{5} > 2$

$\therefore$ $\boxed{\text{only } (z+1)}$ is the singularity point

Cauchy's formula:

$$f^n(z_0) = \frac{n!}{2\pi i} \int_c \frac{f(z)dz}{(z-z_0)^{n+1}}$$

$\Rightarrow \int_c \frac{z-1}{(z+1)^2(z-2)} dz = \int_c \frac{\frac{z-1}{z-2}}{(z+1)^2} dz$

$= \frac{2\pi i}{1!} \times f'(-1)$ $f(z) = \frac{z-1}{z-2} = 1 + \frac{1}{z-2}$ $\therefore f'(z) = \frac{-1}{(z-2)^2}$

$\therefore$ Integral $= \dfrac{2\pi i}{1} \times \left[\dfrac{-1}{(-1-2)^2}\right] = \boxed{-\dfrac{2\pi i}{9}}$

(ii) $\displaystyle\int_c \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} dz \qquad c = |z| = 1 \qquad z = \frac{\pi}{6}$ is the point of singularity

$\therefore I = \dfrac{2\pi i}{2!} \times f''\left(\dfrac{\pi}{6}\right)$ where $f(z) = \sin^6 z$

$\therefore f'(z) = 6 \sin^5 z \cos z$

$f''(z) = 6\left[5 \sin^4 z \cos^2 z - \sin^6 z\right]$

$\therefore f''\left(\frac{\pi}{6}\right) = 6\left[\sin^4\left(\frac{\pi}{6}\right)\cos^2\left(\frac{\pi}{6}\right) \times 5 - \sin^6\left(\frac{\pi}{6}\right)\right]$

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$= 6\left[\frac{1}{16} \times \frac{3}{4} \times 5 - \frac{1}{64}\right] = \cancel{6}^{3} \times \frac{\cancel{14}^{7}}{\cancel{64}_{16}} = \frac{21}{16}.$

$\therefore I = \dfrac{2\pi i}{2} \times \dfrac{21}{16} = \boxed{\dfrac{21\pi i}{16}}$

1. (e) Write the dual of the following problem.

Min. $z = x_1 + x_2 + x_3$, subject to the constraints :

$x_1 - 3x_2 + 4x_3 = 5$, $x_1 - 2x_2 \le 3$, $2x_2 - x_3 \ge 4$; $x_1, x_3 \ge 0$ and x_2 is unrestricted.

[10]

Primal: $\quad$ Min $z = x_1 + x_2 + x_3$

st: $\quad x_1 - 3x_2 + 4x_3 = 5$

$x_1 - 2x_2 \le 3$

$2x_2 - x_3 \ge 4$

$x_1, x_3 \ge 0 \quad x_2$ is unrestricted

$\Rightarrow$ ~~Min $z = x_1 + x_2 + x_3$~~

~~st $x_1 - 3x_2 + 4x_3 \ge 5$~~ , ~~$x_1 - 3x_2 + 4x_3 \le 5$~~

~~$-x_1 + 2x_2 \ge -3$~~ $\quad$ Put $x_2 = x_2' - x_2''$

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where $x_2', x_2'' \ge 0$

and convert all the constraints into '$\ge$' type.

at a po... ... satisfies ...

Primal: $\min z = x_1 + x_2' - x_2'' + x_3$

st $x_1 - 3x_2' + 3x_2'' + 4x_3 \geq 5$

$-x_1 + 3x_2' - 3x_2'' - 4x_3 \geq -5$

$-x_1 + 2x_2' - 2x_2'' \geq -3$

$2x_2' - 2x_2'' - x_3 \geq 4$

$x_1, x_2', x_2'', x_3 \geq 0$

Dual: $\max W = 5y_1 - 5y_2 - 3y_3 + 4y_4$

st- $y_1 - y_2 - y_3 + \cancel{2y_4} \leq 1$

$-3y_1 + 3y_2 + 2y_3 + 2y_4 \leq 1$

$3y_1 - 3y_2 - 2y_3 - 2y_4 \leq -1$

$4y_1 - 4y_2 - y_4 \leq 1$

$\Rightarrow$ Dual: $\max W = 5y_5 - 3y_3 + 4y_4$

st: $y_5 - y_3 \leq 1$ $\quad 4y_5 - y_4 \leq 1$ $\quad y_3, y_4 \geq 0$

$-3y_5 + 2y_3 + 2y_4 = 1$ $\quad y_5$ unrestricted

2. (a) Let H be a subgroup of a group G. Then $W = \bigcap_{g \in G} gHg^{-1}$ is a normal subgroup of G. **[15]**

H is a subgroup of G.

$W = \bigcap_{g \in G} gHg^{-1}$

consider $g_1 \in G$

& the subset $g_1 H g_1^{-1} = G_1$

$e \in H \Rightarrow g_1 e g_1^{-1} \in g_1 H g_1^{-1}$

$\Rightarrow e \in g_1 H g_1^{-1}$ $\therefore G_1$ is not empty.

Let $a \in G_1 \Rightarrow a \in G$

2. (b) If $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ and f is continuous at a point of $\mathbb{R}$, prove that f is uniformly continuous on $\mathbb{R}$. **[15]**

$f(x+y) = f(x)+f(y) \quad \forall\ x, y \in \mathbb{R}$

$\therefore$ f is continuous at some point $c \in \mathbb{R}$.

$f(0+0) = f(0)+f(0) \Rightarrow f(0)=0$.

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~~Also~~ $\lim_{x \to c} f(x) = f(c)$.

f is UC if $|f(x)-f(y)| < \epsilon$ whenever $|x-y| < \delta$

$f(x+(-x)) = f(x)+f(-x) \Rightarrow f(x) = -f(x)$

$\therefore f(x) = -f(x)$

$|f(x)-f(y)| = |f(x)+f(-y)| = |f(x-y)|$

$\Rightarrow |f(x-y)| < \epsilon$

$\Rightarrow$ whenever $|x-y| < \epsilon$

$\Rightarrow f(x)$ is uniformly continuous

2. (c) The integral function $f(z)$ satisfies everywhere the inequality $|f(z)| \le A|z|^k$ where A and k are positive constants. Prove that $f(z)$ is a polynomial of degree not exceeding k. [06]

$|f(z)| \le A|z|^k \quad A, k \geq 0$ constants

By cauchy's theorem $f^n(z_0) = \frac{n!}{2\pi i}\int \frac{f(z)}{(z-z_0)^{n+1}} dz$

$$|f^n(0)| = \frac{n!}{2\pi i}\left| \int \frac{f(z)\,dz}{z^{n+1}} \right| \le \frac{n!}{2\pi i}\int \frac{|f(z)|dz}{|z|^{n+1}}$$

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$$\le \int A|z|^{k-n+1} dz$$

$$\le \frac{A \times \cancel{2\pi} z}{|z|^{n+1-k}} \longrightarrow 0 \text{ as } |z| \to \infty$$

$\therefore \lim_{|z|\to\infty} |f^n(0)| = 0$ when $n+1-k > 0$

$\Rightarrow \quad f^n(0) = 0$ for $n > k$

Now by Taylor's theorem

$$f(z) = f(0) + z f'(0) + \frac{z^2 f''(0)}{2!} + \dots + \frac{z^n f^n(0)}{n!}$$

whose terms vanish for $n > k$

as $f^n(0) = 0$ for $n > k$

$\therefore f(z)$ is of maximum degree 'k'

P.T.O.

2. (d) Prove that

$$\int_0^{2\pi} \frac{\cos^2 3\theta}{1-2p\cos 2\theta + p^2}\, d\theta = \pi\frac{1-p+p^2}{1-p}, 0<p<1.$$ [14]

$$I = \int_0^{2\pi} \frac{\cos^2 3\theta}{1-2p\cos 2\theta+p^2}\, d\theta \quad \text{Put } z = e^{i\theta}, \quad \frac{dz}{iz} = d\theta$$

$$\therefore I = \int_0^{2\pi} \frac{1+\cos 6\theta}{2(1-2p\cos 2\theta+p^2)}\, d\theta = \text{Re}\int_0^{2\pi} \frac{1+e^{i6\theta}}{2(1-2p\cos 2\theta+p^2)}\, d\theta \quad (1)$$

$$\text{Consider } I' = \int_0^{2\pi} \frac{1+e^{i6\theta}\, d\theta}{2(1-2p\cos 2\theta+p^2)} = \int_C \frac{1+z^6}{2\left(1-p\left(\frac{z^4+1}{z^2}\right)+p^2\right)} \times \frac{dz}{iz}$$

$$= \int_C \frac{1+z^6 \quad dz}{2i(p^2z+z-pz^2-p)} = \frac{-1}{2i}\int \frac{z(1+z^6)\, dz}{(pz^4-(p^2+1)z^2+p)}$$

$$= \frac{-1}{2i}\int_C \frac{z(1+z^6)\, dz}{(pz^2-1)(z^2-p)}$$ only the pole $z=\sqrt{p}, -\sqrt{p}$ lies in $C: |z|=1$ as $0<p<1$

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~~Residue at $z=p \Rightarrow \lim_{z\to p} \frac{(z-p)(1+z^6)z}{(pz^2-1)(z-p)(z+p)} = \frac{p^6+1}{p^2-1}$~~

~~$\therefore \int_C \frac{1+z^6}{(pz-1)(z-p)}\, dz = 2\pi i \times \text{Residue} = 2\pi i \times \frac{p^6+1}{p^2-1}$~~

~~$\therefore I' = \frac{-1}{2i} \times 2\pi i \times \frac{(p^2)^3+1^3}{p^2-1} = -\pi(p$~~

$$\text{Residue at } z=\sqrt{p} \Rightarrow \lim_{z\to\sqrt{p}} \frac{(z-\sqrt{p})\, z(1+z^6)}{(pz^2-1)(z-\sqrt{p})(z+\sqrt{p})} = \frac{p^3+1}{2(p^2-1)}$$

$$\text{Residue at } z=-\sqrt{p} \Rightarrow \lim_{z\to-\sqrt{p}} \frac{(z+\sqrt{p})(z)(1+z^6)}{(pz^2-1)(z+\sqrt{p})(z-\sqrt{p})} = \frac{p^3+1}{2(p^2-1)}$$

P.T.O.

$$\therefore I' = \frac{-1}{2i} \times 2\pi i \times \left[\frac{p^3+1}{2(p^2-1)} + \frac{p^3+1}{2(p^2-1)} \right]$$

$$= \pi \frac{\cancel{(p+1)}(1-p+p^2)}{(1-p)\cancel{(1+p)}}$$

$$I' = \frac{\pi(1-p+p^2)}{1-p}$$

$$\therefore I = \text{Real part of } I' = \pi \frac{(1-p+p^2)}{1-p} \quad [\because \text{from } ①]$$

$\therefore$ Hence proved.

3. (a) (i) If in a ring R, with unity, $(xy)^2 = x^2 y^2$ for all $x, y \in R$ then show that R is commutative.

(ii) Show that the ring R of real valued continuous functions on [0, 1] has zero divisors. [9 + 9 = 18]

SECTION – B

5. (a) Find the general integral of the partial differential equation $(2xy - 1)\,p + (z - 2x^2)\,q = 2(x - yz)$ and also the particular integral which passes through the line $x = 1, y = 0$. **[10]**

PDE: $(2xy-1)\,p + (z-2x^2)\,q = 2(x-yz)$

Auxillary Equations: $\dfrac{dx}{2xy-1} = \dfrac{dy}{z-2x^2} = \dfrac{dz}{2(x-yz)}$

Multipliers: $(2x, 2y, 1) = \dfrac{2x\,dx + 2y\,dy + dz}{2x(2xy-1) + 2y(z-2x^2) + 2(x-yz)}$

$\Rightarrow \dfrac{d(x^2+y^2+z)}{0} \Rightarrow$ integrate $\Rightarrow \boxed{x^2+y^2+z = c_1}$

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Multipliers: $(z, 1, x) \Rightarrow \dfrac{z\,dx + dy + x\,dz}{2xyz - z + z - 2x^2 + 2x^2 - 2xyz}$

$\Rightarrow \dfrac{d(y+xz)}{0} \Rightarrow \boxed{xz+y = c_2}$

$\therefore$ $\boxed{\text{General Solution: } f(x^2+y^2+z,\ xz+y) = 0}$

Line $x=1,\ y=0 \Rightarrow$ $c_1 = 1^2+0^2+z = z$

$c_2 = 1(z)+0 = z$

$\Rightarrow c_1 = c_2$

$\Rightarrow x^2+y^2+z = xz+y$

$\therefore$ $\boxed{\text{Particular Integral} \Rightarrow x^2+y^2+xz+z-y=0}$

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5. (b) Find complete integral of $(x^2 - y^2) pq - xy(p^2 - q^2) = 1$. [10]

PDE: $(x^2-y^2)pq - xy(p^2-q^2) = 1$

$f_x = 2pqx - (p^2-q^2)y$; $f_y = -2pqy - (p^2-q^2)x$;

$f_z = 0$; $f_p = (x^2-y^2)q - 2pxy$; $f_q = (x^2-y^2)p + 2qxy$

Charpits Auxillary equations: $\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dp}{f_x+pf_z} = \frac{dq}{f_y+qf_z} = \frac{dz}{-pf_p-qf_q}$

$$\frac{dx}{x(py-qx)+y(px-qy)} = \frac{dy}{x(px-qy)+y(py-qx)}$$

$$= \frac{dp}{p(qx-py)+q(px+qy)} = \frac{dq}{-[p(px+qy)+q(py-qx)]}$$

$$\frac{x\,dx - y\,dy}{(x^2-y^2)(py-qx)} = \frac{p\,dp + q\,dq}{(p^2+q^2)(qx-py)}$$

$$\Rightarrow \frac{d(x^2-y^2)}{x^2-y^2} + \frac{d(p^2+q^2)}{p^2+q^2} = 0$$

$$\Rightarrow \boxed{(x^2-y^2)(p^2+q^2) = c_1}$$

-05

9?

5. (c) Given that f(0) = 1, f(1) = 3, f(3) = 55, find the unique polynomial of degree 2 or less, which fits the given data. find the bound on the error. [10]

x	0	1	3
$f(x)$	1	3	55

using Lagrange interpolation : $f(x) = \frac{(x-1)(x-3)}{(0-1)(0-3)}(1) + \frac{(x)(x-3)}{(1)(1-3)}(3) + \frac{(x)(x-1)}{(3)(3-1)}(55)$

$\Rightarrow f(x) = \sum_{i} \prod_{j\neq i} \frac{(x-x_j)}{x_i-x_j} f(x_i)$

$= \frac{(x^2-4x+3)}{3} - \frac{3(x^2-3x)}{2} + \frac{55(x^2-x)}{6}$

$= \frac{2x^2 - 8x + 6 - 9x^2 + \cancel{27x} + 55x^2 - 55x}{6}$

$= \frac{48x^2 - 36x + 6}{6} = 8x^2 - 6x + 1$

$\therefore \boxed{\text{Polynomial} = 8x^2 - 6x + 1}$

Error bound is given by $\frac{\pi_{n+1}(x)}{(n+1)!} f^{n+1}(x)$

$= \frac{(x)(x-1)(x-3)}{3!}$

07

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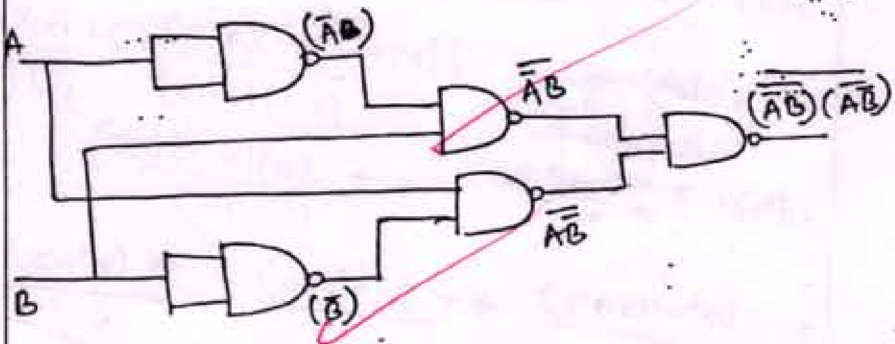
5\. (d) (i) Implement $Y = \bar{A}B + A\bar{B}$ using NAND gates only

(ii) Find the hexadecimal equivalent of the decimal number $(587632)_{10}$. **[10]**

$$Y = \bar{A}B + A\bar{B} = \overline{\overline{(\bar{A}B)}} + \overline{\overline{(A\bar{B})}} = \overline{(\overline{\bar{A}B})(\overline{A\bar{B}})}$$



$$\overline{(\overline{\bar{A}B})(\overline{A\bar{B}})} = \overline{\overline{\bar{A}B}} + \overline{\overline{A\bar{B}}} = \bar{A}B + A\bar{B}$$

$(\because \overline{xy} = \bar{x} + \bar{y})$ $\qquad \bar{\bar{x}} = x$

-08

(ii) $(587632)_{10} \Rightarrow$

16	587632	0 = 0	(lsb)
16	36727	7 = 7	
16	2295	7 = 7	
16	143	15 = F	
16	8	8 = 8	(msb)
	0		

$\therefore$ Hexadecimal = $\boxed{8F770}$

5. (e) Prove that the necessary and sufficient condition that vortex lines may be at right angles to the streamlines are $u, v, w = \mu\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$, where μ and ϕ are functions of x, y, z, t. [10]

Velocity $= u\hat{i} + v\hat{j} + w\hat{k}$

Vortex lines are given by $\frac{dx}{\Omega_x} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_z} = \mu$

where $\Omega_x = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$ $\Omega_y = \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$

$\Omega_z = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

Streamlines are given by $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$

They are perpendicular if $u\Omega_x + v\Omega_y + w\,d\Omega_z = 0$

$\Rightarrow \frac{u\,dx}{\mu} + \frac{v\,dy}{\mu} + w\frac{dz}{\mu} = d\phi$ (an exact differential) — ①

$= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = \Rightarrow$ — ②

$\Rightarrow$ on comparing ① & ② : $u, v, w = \mu\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$

Also if it is given that $u, v, w = \mu\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$

Then $u\Omega_x + v\Omega_y + w\Omega_z$ is an exact differential

$\Rightarrow$ streamlines & vortex lines are perpendicular

Hence proved.

08

P.T.O.

7. (a) Reduce the equation yr + (x + y) s + xt = 0 to canonical form and hence find its general solution. [15]

$$yr + (x+y)s + xt = 0 \quad — (1)$$

$$Rr + Ss + Tt = 0 \Rightarrow R = y \quad S = x+y \quad T = x$$

$$R\lambda^2 + S\lambda + T = 0 \Rightarrow y\lambda^2 + (x+y)\lambda + x = 0$$

$$(y\lambda + x)(\lambda + 1) = 0 \Rightarrow \lambda = -\frac{x}{y}, \ \lambda = -1$$

$$\therefore \frac{dy}{dx} - \frac{x}{y} = 0 \quad \boxed{u = x^2 - y^2} \quad \frac{dy}{dx} - 1 = 0 \Rightarrow \boxed{v = x - y}$$

$$\frac{J(u,v)}{(x,y)} = \begin{vmatrix} 2x & -2y \\ 1 & -1 \end{vmatrix} = -2x + 2y \neq 0$$ $\therefore$ they are independent

$$\Rightarrow p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = 2p'x + q'$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = -2p'y - q'$$

$$r = \frac{\partial p}{\partial x} = \frac{\partial}{\partial x}(2p'x + q') = 2x\frac{\partial p'}{\partial x} + 2p' + \frac{\partial}{\partial x}(q')$$

$$= 2x[2xr' + s'] + 2p' + [2xs' + t']$$

$$s = \frac{\partial q}{\partial x} = \frac{\partial}{\partial x}(-2p'y - q') = -2y\frac{\partial p'}{\partial x} - \frac{\partial q'}{\partial x}$$

$$= -2y[2xr' + s'] - [2xs' + t']$$

$$t = \frac{\partial q}{\partial y} = \frac{\partial}{\partial y}[-2p'y - q'] = -2p' - 2y\frac{\partial p'}{\partial y} - \frac{\partial q'}{\partial y}$$

$$= -2p' - 2y[-2yr' - s'] - [-2ys' - t']$$

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Put in ①: $4x^2y\,r' + 2xys' + 2yp' + 2xys' + yt' - 4x^3y\,r'$

$-4xy^2r' - 2xys' - 2y^2s' - 2x^2s' - 2xys' - xt' - yt'$

$-2p'x + 4xy^2r' + 2xys' + 2xys' + xt'$

$= \quad -2p'(x+y) - 2s'(x^2+y^2-2xy) = 0$

$\Rightarrow \quad p'(x+y) + s'(x-y)^2 = 0$

$\Rightarrow \quad p'(x^2-y^2) + s'(x-y)^3 = 0$

$\Rightarrow \quad \boxed{p'u + s'v^3 = 0} \Rightarrow \boxed{u\dfrac{\partial z}{\partial u} + v^3\dfrac{\partial^2 z}{\partial u \partial v} = 0}$

Let $\dfrac{\partial z}{\partial u} = m \Rightarrow \quad um + v^3\dfrac{\partial m}{\partial v} = 0.$

P.T.O.

7. (b) Find the inverse of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$$

by Gauss-Jordan method. [10]

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 - 4R_1$

$R_3 \to R_3 - 3R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 0 & 2 & 0 & -3 & 0 & 1 \end{array}\right]$$

$R_1 \to R_1 + R_2$

$R_3 \to R_3 + 2R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -4 & -3 & 1 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 0 & 0 & -10 & -11 & 2 & 1 \end{array}\right]$$

$R_2 \to R_2 \times -1$

$R_3 \to R_3 \times -1/10$

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$$\left[\begin{array}{ccc|ccc} 1 & 0 & -4 & -3 & 1 & 0 \\ 0 & 1 & 5 & 4 & -1 & 0 \\ 0 & 0 & 1 & 11/10 & -2/10 & -1/10 \end{array}\right]$$

$\xrightarrow[R_2 \to R_2 - 5R_3]{R_1 \to R_1 + 4R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 14/10 & 2/10 & -4/10 \\ 0 & 1 & 0 & -15/10 & 0 & 1/2 \\ 0 & 0 & 1 & 11/10 & -2/10 & -1/10 \end{array}\right]$

$= [I \mid A^{-1}]$

$$\therefore \boxed{A^{-1} = \begin{bmatrix} 14/10 & 2/10 & -4/10 \\ -15/10 & 0 & 1/2 \\ 11/10 & -2/10 & -1/10 \end{bmatrix}}$$

7. (c) The velocity of a train which starts from rest is given in the following table. The time is in minutes and velocity is in km/hour.

t	2	4	6	8	10	12	14	16	18	20
v	16	28.8	40	46.4	51.2	32.0	17.6	8	3.2	0

Estimate approximately the total distance run in 30 minutes by using composite simpson's $\frac{1}{3}$ rule. [10]

Since the velocity is 0 after t = 20 minutes, it covers the same distance in 20 minutes as in 30 minutes.

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$\therefore$ Distance $= \int v \, dt = \frac{1}{3}\left[(y_0 + y_{10}) + 2(\text{sum even ordinates}) + 4(\text{sum odd ordinates})\right]$

$= \frac{1}{3}\left[(0+0) + 2(28.8 + 46.4 + 32 + 8 + 0) + 4(16 + 40 + 51.2 + 17.6 + 3.2)\right]$

We have taken $v=0$ at $t=0$ as y_0
& $v=0$ at $t=30$ as y_n

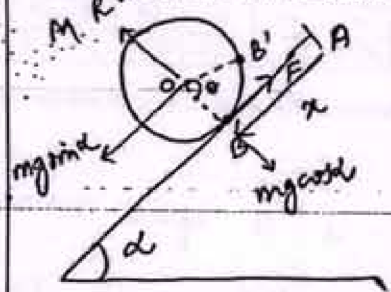
$\therefore$ Distance $= \frac{1}{3}\left[0 + 2\times 115.2 + 4\times 128\right]$

$= \frac{1}{3}\times \frac{3712}{5} \times \frac{km}{hour} \times$ minute.

$= \frac{3712}{15} \times \frac{km}{hour} \times \frac{hour}{60}$

Distance $\boxed{= 4.1244 \text{ kms}}$

7. (d) A sphere of radius a and mass M rolls down a rough plane inclined at an angle α to the horizontal.
If x be the distance of the point of contact of the sphere from a fixed point on the plane, find the acceleration by using Hamilton's equations. [15]



- Suppose it has rolled down distance $AB = x$
Point on sphere moves from B to B' : $\boxed{x = a\theta}$ — ①

Kinetic Energy $T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2$ $\boxed{I = \frac{2}{5}ma^2 \text{ for sphere}}$

$\therefore T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\times\frac{2}{5}\times ma^2 \times \frac{\dot{x}^2}{a^2}$ [using ①]

$\boxed{T = \frac{7}{10}m\dot{x}^2}$ — ② Also $\boxed{V = -W = -mg\sin\alpha\, x}$ — ③

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$\therefore L = T - V = \frac{7}{10} m\dot{x}^2 + mgx \sin\alpha$

$\therefore p_x = \frac{\partial L}{\partial \dot{x}} = \frac{7}{10} \times 2m\dot{x} = \boxed{\frac{7}{5} m\dot{x} = p_x}$ — (4)

$\therefore$ Hamiltonian $= p_x \dot{x} - L = \frac{7}{10} m\dot{x}^2 - mgx \sin\alpha$

$= \frac{7}{10} \times m \times \frac{25 p_x^2}{49 m^2} - mgx \sin\alpha$ [using (4)]

$= \frac{5 p_x^2}{14m} - mgx \sin\alpha$

Now: $\frac{\partial H}{\partial p_x} = \dot{x}$ & $\frac{\partial H}{\partial x} = -\dot{p}_x$ [Hamiltonian Equations]

$\Rightarrow \frac{5}{7} \frac{p_x}{m} = \dot{x}$ $\quad -mg \sin\alpha = -\dot{p}_x$

$\therefore mg \sin\alpha = \frac{\partial}{\partial t}\left(\frac{7m\dot{x}}{5} \right)$

$\Rightarrow mg \sin\alpha = \frac{7m\ddot{x}}{5}$

$\Rightarrow \boxed{\ddot{x} = \frac{5}{7} g \sin\alpha}$ $\Rightarrow$ accelaration

8. (a) The ends A and B of a rod 20 cm long have the temperature at 30° and 80° until steady state prevails. The temperatures of the ends are changed to 40° and 60° respectively. Find the temperature distribution in the rod at time t. [18]

A ——— B

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}$$ ↳ Heat equation

at steady state: $\frac{\partial u}{\partial t} = 0$

$\Rightarrow \frac{d^2 u}{\partial x^2} = 0 \Rightarrow u = c_1 x + c_2$

at $x=0$ $u=30$
at $x=20$ $u=80$
⇓
$30 = 0 + c_2$
$80 = 20c_1 + c_2$

$\therefore$ Initial temperature distribution:

$$u(x,0) = \frac{5x}{2} + 30$$

Also $u(0,t) = 40$ $u(20,t) = 60$ $\Rightarrow$ Initial conditions

Now let $U(x,t) = X(x)T(t) \Rightarrow \frac{X''}{X} = \frac{T'}{kT} = \lambda$

(i) $\lambda = 0 \Rightarrow X'' = 0 \Rightarrow X(x) = a_1 x + a_2$

$X(0) = 40 \Rightarrow 40 = a_2 \Rightarrow a_2 = 40$

$X(20) = 60 \Rightarrow 60 = 20a_1 + a_2 \Rightarrow a_1 = 1$

$\therefore X(x) = x + 40$

$T' = 0 \Rightarrow T(t) = \text{constant} = t_0$

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(ii) $\lambda = \mu^2 \Rightarrow X'' = \mu^2 X \Rightarrow X(x) = b_1 e^{\mu x} + b_2 e^{-\mu x}$

$T' = \mu^2 T \Rightarrow T(t) = e^{\mu^2 t} \rightarrow$ not feasible as $T(t) \rightarrow \infty$ as $t \rightarrow \infty$

(iii) $\lambda = -\mu^2 \Rightarrow X'' = -\mu^2 X \Rightarrow X(x) = c_1 \cos \mu x + c_2 \sin \mu x$

P.T.O.

not complete answer

8. (b) Solve the initial value problem

$u' = -2tu^2, u(0) = 1$

whith h = 0.2 on the interval [0, 0.4]. Use the fourth order classical Runge-Kutta method. compare with the exact solution. [15]

$u' = -2tu^2 \quad u(0) = 1 \qquad h = 0.2$

$k_1 = hf(t_0, u_0) = 0$

$k_2 = hf\left(t_0 + \frac{h}{2}, u_0 + \frac{k_1}{2}\right) = -0.04$

$k_3 = hf\left(t_0 + \frac{h}{2}, u_0 + \frac{k_2}{2}\right) = -0.038416$

$k_4 = hf\left(t_0 + h, u_0 + k_3\right) = -0.073972$

$u(0.2) = u(0) + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} = 0.961533$

$k_1' = hf(t_{0.2}, u_{0.2}) = -0.073964$

$k_2' = hf\left(t_{0.2} + \frac{h}{2}, u_{0.2} + \frac{k_1'}{2}\right) = -0.102575$

$k_3' = hf\left(t_{0.2} + \frac{h}{2}, u_{0.2} + \frac{k_2'}{2}\right) = -0.099425$

$k_4' = hf\left(t_{0.2} + h, u_{0.2} + k_3\right) = -0.118917$

$u(0.4) = u(0.2) + \frac{k_1' + 2(k_2' + k_3') + k_4'}{6} = \boxed{0.8620524}$

Direct solution $= \frac{du}{-u^2} = +2t\,dt \Rightarrow \frac{-1}{u} = -t^2 + c$

$\frac{1}{1} = 0 + c \Rightarrow c = 1 \quad \therefore \quad u = \frac{1}{1+t^2}$

at $t = 0.4 \quad u = \frac{1}{1+0.4^2} = \boxed{0.86206896}$

$\therefore$ Error $= 0.000016$ matches upto 4th decimal.

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P.T.O.

8. (c) Prove that liquid motion is possible when velocity at (x, y, z) is given by

$u = \frac{3x^2 - r^2}{r^5}, v = \frac{3xy}{r^5}, w = \frac{3xz}{r^5}$, where $r^2 = x^2 + y^2 + z^2$ and the stream lines are the intersection of the surfaces, $(x^2 + y^2 + z^2)^3 = c(y^2 + z^2)^2$, by the planes passing through Ox. Is this irrotational?

[17]

$u = \frac{3x^2 - r^2}{r^5}$ $v = \frac{3xy}{r^5}$ $w = \frac{3xz}{r^5}$

Motion is possible if $\nabla \cdot q = 0 \Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

$\frac{\partial u}{\partial x} = \left(6x - 2r\frac{\partial r}{\partial x}\right)\frac{1}{r^5} - \frac{(3x^2 - r^2) \times 5}{r^6}\frac{\partial r}{\partial x} = \frac{9x}{r^5} - \frac{15x^3}{r^7}$

Similarly $\frac{\partial v}{\partial y} = \frac{3x}{r^5} - \frac{15xy^2}{r^7}$ $\frac{\partial w}{\partial z} = \frac{3x}{r^5} - \frac{15xz^2}{r^7}$

$\therefore \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{15x}{r^5} - \frac{15x}{r^7}(x^2 + y^2 + z^2) = 0$ [$\because$ it is possible fluid motion]

Streamlines are given by $\frac{dx}{u} = \frac{dy}{v} = \frac{dw}{z}$

$\Rightarrow \frac{dx}{3x^2 - r^2} = \frac{dy}{3xy} = \frac{dz}{3xz}$

$= \frac{x\,dx + y\,dy + z\,dz}{3x(x^2+y^2+z^2) - xr^2} = \frac{d(x^2+y^2+z^2)}{x \times 2(x^2+y^2+z^2)} = \cancel{dy}$

$\therefore \frac{d(x^2+y^2+z^2)}{2(x^2+y^2+z^2)} = \frac{dy}{3y} = \frac{dz}{3z} = \frac{d(y^2+z^2)}{3(y^2+z^2)}$

$\Rightarrow \boxed{(x^2+y^2+z^2)^3 = c(y^2+z^2)^2}$ Hence proved

[upon integrating]

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Irrotational motion if $\nabla \times q = 0$

$$\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{vmatrix} = i\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + j\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$+ k\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \nabla \times q$$

upon putting the values of the partial differentials

for $u = \frac{3x^2 - r^2}{r^5}$ $v = \frac{3xy}{r^5}$ $w = \frac{3xz}{r^5}$

we get $\nabla \times q = 0$ $\therefore$ $\boxed{\text{It is Irrotational}}$

END OF THE EXAMINATION