Lecture - 7
27/1/2017

Linear Transformation

$$T : V \rightarrow W$$

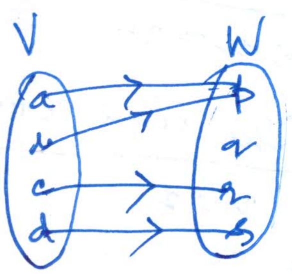
$$\ker\{T\} = \{ \underset{\sim}{v} \in V : T(\underset{\sim}{v}) = \underset{\sim}{0}_W \}$$

$$\mathrm{Im}\{T\} = \{ \underset{\sim}{w} \in W : T(\underset{\sim}{v}) = \underset{\sim}{w} \text{ for } \underset{\sim}{v} \in V \},$$

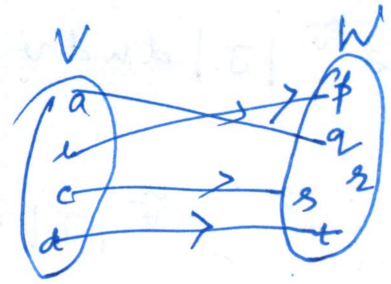
$$\dim \ker\{T\} = \text{nullity } T, \quad \dim \mathrm{Im}\{T\} = \text{rank } T$$

$$\text{rank } T + \text{nullity } T = \dim V$$

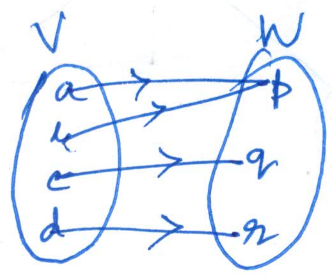
Injective & Surjective LTs.
(one to one / 1-1) (onto)



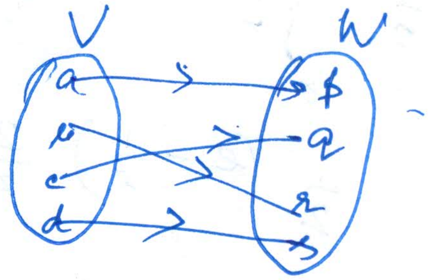
not 1-1, not onto.



1-1, not onto



not 1-1, onto.



1-1, onto.

$$T : V \rightarrow W.$$

A LT (T) is injective or one-to-one (1-1) if every distinct elements in V has distinct images in W. i.e

$$\underset{\sim}{v}_1 \neq \underset{\sim}{v}_2 \Rightarrow T(\underset{\sim}{v}_1) \neq T(\underset{\sim}{v}_2) \quad \forall \underset{\sim}{v}_1, \underset{\sim}{v}_2 \in V.$$

A LT (T) is surjective or onto if every element in W has atleast one pre-image in V. i.e none of the elements in W remains unused under the mapping T.

Ex-1 $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by $T(x,y,z) = (x,y)$

not injective because, $(2,3,1)$ & $(2,3,-4)$, say, have same image $(2,3)$.

The LT is onto, because every element in $\mathbb{R}^2$ has atleast 1 pre-image in $\mathbb{R}^3$.

Ex-2. $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(x,y,z) = (x,y,0)$

Not 1-1 because $(1,2,3)$ & $(1,2,4)$ have same image $(1,2,0)$.

Not onto, because $(3,5,2)$ has no pre-image in $\mathbb{R}^3$.

Def- A LT is bijective (isomorphism) if the LT is both 1-1 and onto.

$T: V \to W$

Thm 1. T is injective if and only if
$\ker\{T\} = \{\underset{\sim}{0}_V\}$.

Note: nullity $T = 0$

$\text{rank } T + \text{nullity } T = 0 + \text{rank } T$.

$\Rightarrow \dim V = \text{rank } T$.

Thm 2. T is surjective if and only if
$\text{rank}(T) = \dim W$.

$\text{Im}(T) = W$
$\dim \text{Im}(T) = \dim W$

for verification

Thm 3 If $\dim V > \dim W$, T is not injective.

Thm 4. If $\dim V < \dim W$, T is not surjective.

$T: \mathbb{R}^4 \to \mathbb{R}^3$ $\{Te_1, Te_2, \ldots, Te_n\}$
is not ~~so~~ injective.

Composition of linear transformation.

Let $T_1 : V \to W$, $T_2 : W \to U$ be two LT's.

$T_2 \circ T_1$ is defined as,

$$(T_2 \circ T_1)(\underset{\sim}{v}) = T_2 \circ (T_1(\underset{\sim}{v})) = T_2(\underset{\sim}{w}) = \underset{\sim}{u}$$

$\underset{\sim}{v} \in V$

$\underset{\sim}{w} \in W$

$\underset{\sim}{u} \in U$.

In general, $T_2 \circ T_1 \neq T_1 \circ T_2$

<u>Ex-1</u> Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$, $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T_1(x, y) = (y, x)$, $T_2(x, y) = (0, x)$.

Find $T_1 \circ T_2$, $T_2 \circ T_1$, T_1^2, T_2^2.

<u>Soln.</u> $(T_1 \circ T_2)(x, y) = T_1 \circ (T_2(x, y)) = T_1(0, x)$
$= (x, 0)$.

$(T_2 \circ T_1)(x, y) = T_2 \circ (T_1(x, y)) = T_2(y, x) = (0, y)$

$$T_1 \circ T_2 \neq T_2 \circ T_1$$

$T_1^2(x, y) = T_1 \circ (T_1(x, y)) = T_1(y, x) = (x, y)$

$T_1^2 = I$ (Identity mapping)

$T_2^2(x, y) = T_2 \circ (T_2(x, y)) = T_2(0, x) = (0, 0)$.

$T_2^2 = 0$ (zero mapping).

Inverse mapping (T^{-1}).

Let $T : V \to W$

Then $T^{-1} : W \to V$ is a (linear) mapping T'

such that $T \circ T' = I = T' \circ T$.

i.e $T \circ T^{-1} = I = T^{-1} \circ T$.

<u>Thm.</u> If inverse of a mapping T exists, then it is unique.

i.e if $T \circ T_1 = I = T_1 \circ T$.

& $T \circ T_2 = I = T_2 \circ T$.

then $T_1 = T_2 = T^{-1}$.

<u>Thm.</u> A LT T is non-singular (i.e T^{-1} exists) if and only if $\ker\{T\} = \{\underset{\sim}{0}_V\}$

<u>Ex.</u> Show that $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$T(x, y, z) = (2x, 4x - y, 2x + 3y - z)$

is non-singular. Find T^{-1}.

<u>Sol.</u> $\ker\{T\} = \{(x, y, z) \in \mathbb{R}^3 : T(x, y, z) = \underset{\sim}{0}_W = (0,0,0)\}$

$(x, y, z) \in \ker\{T\}$ can be determined from

$T(x, y, z) = (2x, 4x - y, 2x + 3y - z) = (0, 0, 0)$

i.e we have to solve,

$$2x = 0$$
$$4x - y = 0$$
$$2x + 3y - z = 0$$

$(0, 0, 0)$ is the only solution to the above homogeneous system.

$\therefore \ker\{T\} = \{(0,0,0)\} = \{\underset{\sim}{0}_V\}$

$\therefore$ T^{-1} exists / T is non-singular.

$T: R^3 \to R^3$

$T(x,y,z) = (u,v,w) = \big(f_1(x,y,z), f_2(x,y,z), f_3(x,y,z)\big)$

$T^{-1}(u,v,w) = (x,y,z) = \big(g_1(u,v,w), g_2(u,v,w), g_3(u,v,w)\big)$

Note, $T(x,y,z) = (u,v,w) = (2x, 4x-y, 2x+3y-z)$

$\therefore$ $u = 2x$, $v = 4x - y$, $w = 2x + 3y - z$.

$x = \frac{u}{2}$, $y = 4x - v = 2u - v$.

$z = w - 2x - 3y$
$= w - u - 3(2u - v)$

$T^{-1}(u,v,w) = \left(\frac{u}{2}, 2u - v, -7u + 3v + w\right)$

Matrix representation of a LT.

$T : V \to W$

$\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \ldots, \underset{\sim}{e}_n\} \to$ a basis of V ; $\dim V = n$

$\{\underset{\sim}{f}_1, \underset{\sim}{f}_2, \ldots, \underset{\sim}{f}_m\} \to$ a " " W $\dim W = n$.

Let $\underset{\sim}{v} \in V$, $\underset{\sim}{w} \in W$.

$$\underset{\sim}{v} = c_1 \underset{\sim}{e}_1 + c_2 \underset{\sim}{e}_2 + \cdots + c_n \underset{\sim}{e}_n$$

$$\underset{\sim}{w} = d_1 \underset{\sim}{f}_1 + d_2 \underset{\sim}{f}_2 + \cdots + d_m \underset{\sim}{f}_m.$$

$$T(\underset{\sim}{e}_1) = a_{11} \underset{\sim}{f}_1 + a_{21} \underset{\sim}{f}_2 + \cdots + a_{m1} \underset{\sim}{f}_m$$

$$T(\underset{\sim}{e}_2) = a_{12} \underset{\sim}{f}_1 + a_{22} \underset{\sim}{f}_2 + \cdots + a_{m2} \underset{\sim}{f}_m.$$

$$\vdots$$

$$T(\underset{\sim}{e}_n) = a_{1n} \underset{\sim}{f}_1 + a_{2n} \underset{\sim}{f}_2 + \cdots + a_{mn} \underset{\sim}{f}_m$$

$$\begin{pmatrix} T\underset{\sim}{e}_1 \\ T\underset{\sim}{e}_2 \\ \vdots \\ \vdots \\ T\underset{\sim}{e}_n \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & & & \\ & & & \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}}_{B_{n \times m}} \begin{pmatrix} \underset{\sim}{f}_1 \\ \underset{\sim}{f}_2 \\ \vdots \\ \vdots \\ \underset{\sim}{f}_m \end{pmatrix}.$$

$A_{m \times n} = B^T \to$ matrix of T. & is denoted as $[T]_e^f$

Ex. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T(x_1, x_2) = (x_2, -5x_1 + 13x_2, -7x_1 + 16x_2).$$

Find the matrices of T w.r.to the bases

1) $\{(1,0), (0,1)\}$ of $\mathbb{R}^2$ & $\{(1,0,0), (0,1,0), (0,0,1)\}$ of $\mathbb{R}^3$

2) $\{(3,1), (5,2)\}$ of $\mathbb{R}^2$ & $\{(1,0,-1), (-1,2,2), (0,1,2)\}$ of $\mathbb{R}^3$

(1)

$$Te_1 = T(1,0) = (0, -5, -7).$$

$$Te_2 = T(0,1) = (1, 13, 16)$$

$$Te_1 = (0,-5,-7) = 0.(1,0,0) - 5(0,1,0) - 7(0,0,1)$$

$$Te_2 = (1, 13, 16) = 1(1,0,0) + 13(0,1,0) + 16(0,0,1)$$

$$[T]_e^f = \begin{pmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{pmatrix}$$

$$Te_1 = T(3,1) = (1, -2, -5) = c_1(1,0,-1) + c_2(-1,2,2) + c_3(0,1,2)$$

$$Te_2 = T(5,2) = (2, 1, -3),$$

$$= d_1(1,0,-1) + d_2(-1,2,2) + d_3(0,1,2)$$

$$[T] = \begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \\ c_3 & d_3 \end{pmatrix}$$

check

$$\begin{pmatrix} 1 & 3 \\ 0 & 1 \\ -2 & -1 \end{pmatrix}.$$

<u>Exercise</u> The matrix of the LT $T: \mathbb{R}^3 \to \mathbb{R}^2$ relative to the bases

$\{(0,1,1), (1,0,1), (1,1,0)\}$ of $\mathbb{R}^3$ and

$\{(1,0), (1,1)\}$ of $\mathbb{R}^2$ is

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 0 \end{pmatrix}.$$

(a) Find the LT.

(b) Find the matrix of T relative to the ordered bases

$\{(1,1,0), (1,0,1), (0,1,1)\}$ of $\mathbb{R}^3$ and

$\{(1,1), (0,1)\}$ of $\mathbb{R}^2$.

<u>Ans.</u> (a) $T(x,y,z) = \left(2x + 2y + z, \ \dfrac{-x+y+3z}{2}\right)$

(b) $\begin{pmatrix} 4 & 3 & 3 \\ -4 & -2 & -1 \end{pmatrix}$