⇒ Partial Differential Equations :

1) Order : order of the highest order derivative

Degree : degree of the HOD.

2) Linear PDE : (i) dept variable z & its PD's occur in 1st degree only

(ii) No product of dependent variable or PDs.

Semi-Linear : $P(x,y)\dfrac{\partial z}{\partial x} + Q(x,y)\dfrac{\partial z}{\partial y} = R(x,y,z) + z^2$

Quasi-Linear : $P(x,y,z)\dfrac{\partial z}{\partial x} + Q(x,y,z)\dfrac{\partial z}{\partial y} = R(x,y,z)$

3) $\dfrac{\partial z}{\partial x} = p$ $\quad \dfrac{\partial z}{\partial y} = q \quad \dfrac{\partial^2 z}{\partial x^2} = r \quad \dfrac{\partial^2 z}{\partial x \partial y} = s \quad \dfrac{\partial^2 z}{\partial y^2} = t$

4) Formulation of PDE :

(i) Elimination of arbitrary constants

$f(x,y,z,a,b) = 0$ where a & b are constants

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x} = 0 \quad \text{—(I)} \qquad \& \qquad \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y} = 0 \quad \text{—(II)}$$

Eliminate using (I) & (II).

NOTE: If (no of arbitrary constants = no of indept variables) then PDE is of 1st order only.

eg. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ a, b, c are constants.

we get $\dfrac{x}{a^2} + \dfrac{zp}{c^2} = 0$ & $\dfrac{y}{b^2} + \dfrac{zq}{c^2} = 0$ ⇒ $c^2 = \dfrac{-a^2pz}{x}$ or $\dfrac{-b^2qz}{y}$

(Put in (I)) or (Put in (II))

$\Downarrow$

$\dfrac{1}{a^2} + \dfrac{1}{c^2}p^2 + \dfrac{z}{c^2}r = 0$ —(I) $\quad \dfrac{1}{b^2} + \dfrac{1}{c^2}q^2 + \dfrac{z}{c^2}t = 0$ —(II) $\quad -\dfrac{pz}{x} + p^2 + zr = 0$ $\quad$ similarly another PDE

$\Downarrow$

$$xz\frac{\partial^2 z}{\partial x^2} + x\left(\frac{\partial z}{\partial x}\right)^2 - \frac{z\partial z}{\partial x} = 0$$

since constants > ...

(ii) Elimination of arbitrary function :

u & v are functions of x, y, z and they have a relation $f(u,v)=0$

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right)=0$$

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right)=0$$

compare the $\frac{\partial f}{\partial u} \Big/ -\frac{\partial f}{\partial v}$ ratios from both

we get $P\cdot p+Q\cdot q=R$ where $P=\frac{\partial(u,v)}{\partial(y,z)}$

$Q=\frac{\partial(u,v)}{\partial(z,x)}$

This is a linear PDE.

$R=\frac{\partial(u,v)}{\partial(x,y)}$

For more than 1 function, higher order may be there and it might not be Linear PDE

Example: ST PDE for all cones with vertex at origin is $px+qy=z$.

cone: $ax^2+by^2+cz^2+2hxy+2fyz+2gzx=0$

$\Rightarrow (ax+hy+gz)+(gx+fy+cz)p=0$ — (i)

$(hx+by+fz)+(gx+fy+cz)q=0$ — (ii)

Do $\underbrace{(i)\times x+(ii)\times y}_{\text{Remember such manipulations}} \Rightarrow (gx+fy+cz)(px+qy)+(ax^2+by^2+2hxy+gxz+fzy)=0$

$\Rightarrow (gx+fy+cz)(px+qy)+(-cz^2-gxz-fzy)=0$.

$\Rightarrow (gx+fy+cz)(px+qy-z)=0$

$\Rightarrow px+qy=z$

Example : $\phi(x+y+z,\ x^2+y^2-z^2)=0$

$$\frac{\partial\phi}{\partial u}(1+p)+\frac{\partial\phi}{\partial v}(2x-2zp)=0$$
& $$\frac{\partial\phi}{\partial u}(1+q)+\frac{\partial\phi}{\partial v}(2y-2zq)=0$$

$$\Rightarrow \frac{1+p}{x-zp}=\frac{1+q}{y-zq}$$

$$\Rightarrow y-\cancel{pzq}+py-zq = x-zp-\cancel{zpq}+xq$$

$$\Rightarrow \quad p(y+z)-q(x+z)=x-y.$$

(iii) Equations solvable by direct integration:

eg $$\frac{\partial^3 z}{\partial x^2\partial y}+18xy^2+\sin(2x-y)=0$$

$\Rightarrow$ Integrate twice wrt x keeping y constt & then wrt y

$$\Rightarrow \frac{\partial z}{\partial y}+3x^3y^2-\frac{1}{4}\sin(2x-y)=xf(y)+g(y)$$

$$\Rightarrow z+x^3y^3-\frac{1}{4}\cos(2x-y)=x\underbrace{\int f(y)dy}_{u(y)}+\underbrace{\int g(y)dy}_{v(y)}+h(x)$$

5) Order 1 PDE can be classified as

Linear

$P(x,y)p+Q(x,y)q$
$=R(x,y)z+S(x,y)$
(linear in p,q,z)

Semi-Linear

$P(x,y)p+Q(x,y)q$
$=R(x,y,z)$
(linear in p,q
& p,q have only
x,y coeffs)

Quasilinear

$P(x,y,z)p+$
$Q(x,y,z)q=R(x,y,z)$
(linear in p,q)

Non-linear

If not
in others

6) Lagrange's Linear Equation : $Pp+Qq=R$ $\quad P,Q,R$ are fns of (x,y,z)

The general solution is $f(u,v)=0$ where u & v are fns of (x,y,z)

Eliminating this arbitrary 'f' gives the above PDE.

Here u & v are solutions of $\underbrace{\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}}$ — ①

This is called the Auxillary / Subsidiary Equation

7) Solving the simultaneous equations : $\frac{dx}{P} = \frac{dy}{Q}$; $\frac{dx}{P} = \frac{dz}{R}$; $\frac{dy}{Q} = \frac{dz}{R}$

① Directly integrating if it is variable separable
② If one of them can be found directly, then use it to find the other one
③ using Componendo - Dividendo :

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l_1 dx + m_1 dy + n_1 dz}{l_1 P + m_1 Q + n_1 R} = \frac{l_2 dx + m_2 dy + n_2 dz}{l_2 P + m_2 Q + n_2 R}$$

now if $l_1 dx + m_1 dy + n_1 dz = 0$ great (try to get Den^r = 0)

else try $\frac{l_1 dx + m_1 dy + n_1 dz}{l_1 P + m_1 Q + n_1 R} = d\phi$ & then proceed.

Try to remember some smart manipulations.

eg. $(x-y)p + (x+y)q = 2xz$

$$\frac{dx}{x-y} = \frac{dy}{x+y} = \frac{dz}{2xz}$$

choose $(1, 1, \frac{1}{z}) \Rightarrow dx + dy + \frac{dz}{z} = 0$

$\Rightarrow x + y + \log z = c_1$

$$= \frac{x\,dx + y\,dy}{x^2+y^2} = \cancel{\frac{x\,dx - y\,dy}{}} \quad (x+y)\,dx = (x-y)\,dy$$

$$= x\,dx + y\,dy + y\,dx - x\,dy = 0$$

$$\Rightarrow \frac{1}{2}\frac{d(x^2+y^2)}{x^2+y^2} + \frac{1}{x^2+y^2}\,d\left(\frac{y}{x}\right) = 0$$

$$\Rightarrow \frac{\log(x^2+y^2)}{2} + \tan^{-1}\left(\frac{y}{x}\right) = c$$

$(x^2 - yz)p + (y^2 - zx)q = (z^2 - xy)$

$$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy} = \frac{x\,dx + y\,dy + z\,dz}{x^3+y^3+z^3-3xyz} = \frac{dx-dy}{(x-y)(x+y+z)} = \frac{dy-dz}{(y-z)(x+y+z)}$$

$$= \frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx}$$

$\Rightarrow \frac{(x-y)}{(y-z)} = c_1$

$$\frac{1}{2}\frac{d(x^2+y^2+z^2)}{(x+y+z)} = d(x+y+z)$$

$\Rightarrow x^2+y^2+z^2 = (x+y+z)^2 + c_2$

$[x^3+y^3+z^3 - 3abc = x^2+y^2+z^2-xy-yz-zx]$

$$\cos(x+y)\,p + \sin(x+y)\,q = z$$

$$\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z} = \frac{d(x+y)}{\cos(x+y)+\sin(x+y)} = \frac{d(x-y)}{\cos(x+y)-\sin(x+y)}$$

Last $-2 \Rightarrow$ $$d(x-y) = \frac{\cos(x+y)-\sin(x+y)}{\cos(x+y)+\sin(x+y)}\, d(x+y)$$

$$\Rightarrow (x-y) = \log|\cos(x+y)+\sin(x+y)| + c_1$$

$$\Rightarrow \cos(x+y)+\sin(x+y) = c_1' e^{x-y}.$$

Also $$\frac{dz}{z} = \frac{d(x+y)}{\cos(x+y)+\sin(x+y)} = \frac{dt}{\cos t+\sin t} \qquad t = x+y.$$

$$\frac{dz}{z} = \frac{1}{\sqrt{2}}\sec\left(t-\frac{\pi}{4}\right)dt \Rightarrow z = \frac{1}{\sqrt{2}}\log\left|\sec\left(t-\frac{\pi}{4}\right)+\tan\left(t-\frac{\pi}{4}\right)\right|$$

$$\Rightarrow \log z = \frac{1\log}{\sqrt{2}}\left|\sec\left(x+y-\frac{\pi}{4}\right)+\tan\left(x+y-\frac{\pi}{4}\right)\right|.$$

$$\Rightarrow z^{\sqrt{2}} = \sec\left(x+y-\frac{\pi}{4}\right)+\tan\left(x+y-\frac{\pi}{4}\right)c_2$$

Q) This method can be generalised for > 2 independent variables PDE also.

$$P_1p_1 + P_2p_2 + \dots + P_np_n = R$$

$$\Rightarrow \frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \dots = \frac{dx_n}{P_n} = \frac{dz}{R}$$

& GS is $f(u_1, u_2, \dots, u_n) = 0.$

eg. $x_1^3 + x_2^3 + x_3^3 = 1$ when $z=0$ then soln of $(s-x_1)P_1 + (s-x_2)P_2 + (s-x_3)P_3 = s-z$ can be given as $s^3[(x_1-z)^3 + (x_2-z)^3 + (x_3-z)^3]^4 = (x_1+x_2+x_3-3z)^4$. where $s = x_1+x_2+x_3+z$

Write lagrange & obtain: $$\frac{d(x_1+x_2+x_3-3z)}{-(x_1+x_2+x_3-3z)} = \frac{d(s)}{3s} \Rightarrow (x_1+x_2+x_3+z)(x_1+x_2+x_3-3z)^3 = a$$

$a = (x_1+x_2+x_3)^4$ [$\because z=0$]

Also $$\frac{d(x_1-z)^3}{-3(x_1-z)^3} = \frac{d(x_2-z)^3}{-3(x_2-z)^3} = \frac{d(x_3-z)^3}{-3(x_3-z)^3} = \frac{d[(x_1-z)^3+(x_2-z)^3+(x_3-z)^3]}{-3[(x_1-z)^3+(x_2-z)^3+(x_3-z)^3]} = \frac{d(\Sigma x_i - 3z)}{-(x_i-3z)}$$

$\Rightarrow \Sigma(x_i-z)^3 = b(\Sigma x_i - 3z)^3 \Rightarrow b = 1/(x_1+x_2+x_3)^3$ [$\because z=0$]

No do manipulations from here! $$\left[(x_1+x_2+x_3+z)(x_1+x_2+x_3-3z)^3 = (x_1+x_2+x_3)^4 \text{ \& } [(x_1-z)^3+(x_2-z)^3+(x_3-z)^3] = \frac{(x_1+x_2+x_3-3z)^3}{(x_1+x_2+x_3)^3}\right.$$

9) Integral Surfaces : through a given curve :

(a) Say curve is in parametric form then $u_1(x(t), y(t), z(t)) = c_1$ & $u_2(x(t), y(t), z(t)) = c_2$

Eliminate 't' to get relation in c_1, c_2 & then put $u_1 = c_1$ & $u_2 = c_2$

(b) Curve is given as $\phi(x,y,z) = 0$ & $\psi(x,y,z) = 0$

Eliminate x, y, z from u_1, u_2, ϕ & ψ & then obtain relation by c_1 & c_2

eg. $(2xy-1)p + (z-2x^2)q = 2(x-yz)$ passing through $x=1, y=0$

$$\frac{dx}{2xy-1} = \frac{dy}{z-2x^2} = \frac{dz}{2(x-yz)}$$

choose $(x, y, \frac{1}{2}) \Rightarrow x^2+y^2+z = c_1$

& $(z, 1, x) \Rightarrow y + xz = c_2$

$\Rightarrow c_1 - c_2 = 1$

$\therefore (x^2+y^2+z) - (y+xz) = 1$.

10) Surfaces orthogonal to a given system of surfaces :

Given system is $f(x,y,z) = c$

Normal DR's at (x,y,z) to f are $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

Let the ⊥ surface be $z = \phi(x,y)$

Normal DR's are $\left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, -1\right)$ i.e. $(p, q, -1)$

$\therefore \quad \frac{\partial f}{\partial x} p + \frac{\partial f}{\partial y} q - \frac{\partial f}{\partial z} = 0 \Rightarrow$ Lagrange form

$$\Rightarrow \frac{dx}{\frac{\partial f}{\partial x}} = \frac{dy}{\frac{\partial f}{\partial y}} = \frac{dz}{\frac{\partial f}{\partial z}}$$

eg. $f = z(x+y) = c(3z+1) \quad \therefore f = \frac{z(x+y)}{(3z+1)} = c$

$$\therefore \frac{dx}{\frac{z}{3z+1}} = \frac{dy}{\frac{z}{3z+1}} = \frac{dz}{\frac{(x+y)}{(3z+1)^2}} \Rightarrow \begin{array}{l} x - y = c_1 \\ \frac{(x+y)^2}{2} = 2z^3 + z^2 + c_2 \end{array}$$

⇒ Non-Linear Equations :

1) $f(x,y,z,a,b) = 0 \longrightarrow F(x,y,z,p,q)$ upon elimination of a & b.

Complete solution : contains as many constants as the no of independent variables. eg. given F we get f as complete solⁿ.

Particular Integral : giving specific values to a & b.

Singular Integral : eliminating a & b b/w f, $\frac{\partial f}{\partial a}$ & $\frac{\partial f}{\partial b}$.

General Integral : if $b = \phi(a)$, then we have a subfamily of solution. The envelope of this subfamily obtained by eliminating 'a' from $f(x,y,z,a,\phi(a)) = 0$ & $\frac{\partial f}{\partial a} = 0$ is the general integral.

2) CHARPIT'S METHOD : (1st order PDE) :

Charpit's Auxillary Equations :

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{d\phi}{0}$$

Working :

a) Get in the standard format $f(x,y,z,p,q) = 0$ & form the aux. eqⁿ.

b) Select 2 simplest ~~solution~~ relation to get p & q in terms of (x,y,z) [atleast one of p,q HAS TO BE there].

Sometimes if you get 'p', directly put in f to get 'q' instead of solving another relation.

c) Put in $dz = p\,dx + q\,dy$ to get complete integral.

Example: $p^2x(x-1)+q^2y(y-1)+2pqxy-2pxz-2qyz+z^2=0$

Simplified Aux eq^n give $\frac{dp}{-p^2}=\frac{dq}{-q^2}=\frac{\frac{1}{p}dp}{-p}=\frac{\frac{1}{q}dq}{-q}$

$$\Rightarrow \frac{\frac{1}{p}dp-\frac{1}{q}dq}{-p+q} \qquad \text{Also} \quad \frac{\frac{1}{x}dx-\frac{1}{y}dy}{2(p-q)}$$

Remember this.

$$\Rightarrow \frac{1}{2}\left(\frac{1}{x}dx-\frac{1}{y}dy\right)+\frac{1}{p}dp-\frac{1}{q}dq=0$$

$$\Rightarrow \frac{1}{2}\log\left(\frac{x}{y}\right)+\log\left(\frac{p}{q}\right)=\log a.$$

$$\therefore \left(\frac{x}{y}\right)^{1/2}=\frac{qa}{p} \Rightarrow p=qa\left(\frac{y}{x}\right)^{1/2}$$

Also PDE $\equiv (px+qy-z)^2=p^2x+q^2y$

$$\Rightarrow px+qy-z=\sqrt{p^2x+q^2y}$$

$$\Rightarrow q\left[y+a(xy)^{1/2}-(a^2+1)^{1/2}y^{1/2}\right]=z$$

$$\Rightarrow q=\frac{z}{y+a(xy)^{1/2}-(1+a^2)^{1/2}y^{1/2}}$$

$$\therefore dz=pdx+qdy \Rightarrow dz=\frac{a\,z\,dx}{x^{1/2}\left[y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\right]}+\frac{z\,dy}{y^{1/2}\left[y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\right]}$$

$$\Rightarrow \frac{dz}{z}=\frac{ay^{1/2}dx+x^{1/2}dy}{(xy)^{1/2}\left(y^{1/2}+a^{1/2}\mp(1+a^2)^{1/2}\right)}$$

$$\Rightarrow \log z=2\log\{y^{1/2}+ax^{1/2}\mp(1+a^2)^{1/2}\}+\log b$$

$$\Rightarrow z=b\left[y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\right]^2$$

Find CI, SI, GI for $(p^2+q^2)y=qz$ $\qquad f_x=0 \quad f_y=p^2+q^2 \quad f_z=-q$ $\quad f_p=2py \quad f_q=2qy-z$

$$\therefore \frac{dx}{-2py}=\frac{dy}{q}=\frac{dp}{-pq}=\frac{dq}{p^2}=\frac{dz}{-2p^2y+qz-2q^2y}$$

$$\Rightarrow p^2+q^2=a^2 \Rightarrow q=\frac{a^2y}{z} \quad p=a\sqrt{\left(1-\frac{a^2y^2}{z^2}\right)} \Rightarrow dz=pdx+qdy$$

$$\therefore dz=a\sqrt{\left[1-\frac{a^2y^2}{z^2}\right]}dx+\frac{a^2y\,dy}{z} \Rightarrow \frac{1}{2}d(z^2-a^2y^2)=a\,dx\sqrt{z^2-a^2y^2} \Rightarrow (z^2-a^2y^2)=(ax$$

CI: $(z^2 - a^2y^2) = (ax+b)^2$ $\therefore$ $\phi = a^2x^2 + b^2 + 2abx + a^2y^2 - z^2$

$\frac{\partial \phi}{\partial a} = 2ax^2 + 2bx + 2ay = 0$

$= ax^2 + bx + ay = 0$

$\frac{\partial \phi}{\partial b} = 2b + 2ax = 0$

$b + ax = 0$

$\Rightarrow a = b = 0.$

$\therefore z = 0$ is the SI.

GI: $\psi : z^2 - a^2y^2 = (ax + \phi(a))^2$

$\therefore \frac{\partial \psi}{\partial a} = -2ay^2 = 2(ax+\phi(a))(x+\phi'(a))$.

Eliminate a from ψ & $\frac{\partial \psi}{\partial a}$ to get GI.

3) Some special forms of f:

ⓐ $f(p,q) = 0$

$\frac{dp}{0} = \frac{dq}{0} = \ldots$

$\Rightarrow p = a, \ f(a,q) = 0$

& $q = \phi(a)$

$\therefore z = ax + \phi(a)y + b$

Try if pz, px, etc. are multiplied.

Sometimes subs^n is needed to reduce to $f(p,q)=0$ form. Very very useful.

eg. $z^2p^2y + 6zpxy + 2zqx^2 + 4x^2y = 0$

Put $xdx = dX$, $ydy = dY$

$zdz = dZ$

ⓑ $f(z,p,q) = 0$

$\frac{dp}{p} = \frac{dq}{q} = \ldots \Rightarrow p = aq$

$\Rightarrow dz = p\, d(ax+y)$

$dZ = p\, dX \quad X = ax+y$

$\therefore p = \frac{dZ}{dX} \quad q = \frac{1}{a}\frac{dZ}{dX}$

$\therefore f\left(Z, \frac{dZ}{dX}, \frac{1}{a}\frac{dZ}{dX}\right) = 0.$

ⓒ Separable $f_1(x,p) = f_2(y,q)$

$\frac{dx}{\frac{\partial f_1}{\partial p}} = \frac{dp}{-\frac{\partial f_1}{\partial x}} \Rightarrow \frac{\partial f_1}{\partial p}dp + \frac{\partial f_1}{\partial x}dx = 0$

$\Rightarrow df_1 = 0 \Rightarrow f_1 = \text{const} = a$

$\therefore f_1 = f_2 = a$, solve to get

$p = F_1(x,a) \quad q = F_2(y,a)$

then $z = \int F_1(x,a)\,dx + \int F_2(y,a)\,dy + b$.

ⓓ Clairaut Eq^n: $z = px + qy + f(p,q)$

$\frac{dp}{0} = \frac{dq}{0} = \ldots \quad p = a \quad q = b$

$\therefore z = ax + by + f(a,b)$.

eg. $4xyz = pq + 2px^2y + 2qxy^2$

$z = \frac{\partial z}{2x\partial x}\cdot\frac{\partial z}{2y\partial y} + \frac{\partial z\, x^2}{2x\partial x} + \frac{\partial z\, y^2}{2y\partial y}$

$z = PX + QY + PQ \quad X = x^2 \quad Y = y^2$

$\therefore z = ax^2 + by^2 + ab \Rightarrow \begin{matrix} x^2 + b = 0 \\ y^2 + a = 0 \end{matrix}$

$\therefore z = -x^2y^2$ is SI

4) Solution through given curve: CRAM!!!

We get the CI: $f(x,y,z,a,b) = 0$ & the curve C (parametrised by t).

The pts of intersection are $f(x(t), y(t), z(t), a, b) = 0$. This should have equal roots, ie, $b^2 - 4ac = 0$ or f & $\frac{\partial f}{\partial t}$ having common root. Using which we get relation $\psi(a,b) = 0$. Use it to get one-parameter subsystem of the CI. Its envelope is the reqd sol^n [Do b^2-4ac again on this subsystem] example: CI of PDE: $(p^2+q^2)x = pz$ & deduce the solⁿ through $x=0$, $z^2=4y$

We get CI as $z^2 = a^2x^2+(ay+b)^2$ — ①

$\therefore$ C is $(0, t^2, 2t)$

$\therefore \quad 4t^2 = 0+(at^2+b)^2 \Rightarrow a^2t^4+(2ab-4)t^2+b^2=0$

equal roots $\equiv (2ab-4)^2-4(a^2b^2)=0 \Rightarrow 4-4ab=0 \Rightarrow ab=1$

$\therefore \quad z^2 = a^2x^2+\left(ay+\frac{1}{a}\right)^2$ is the one parameter sub-system of ①

$\Rightarrow \quad (x^2+y^2)a^4+(2y-z^2)a^2+1=0$

whose envelope we get by $(2y-z^2)^2-4(x^2+y^2)=0$

$\Rightarrow \quad z^2 = 2y-2\sqrt{x^2+y^2}$

5) Deriving one CI from another:

Suppose wkt $f(x,y,z,a,b)=0$ — ① is CI for $F(x,y,z,p,q)=0$ — ②

& we want to show $g(x,y,z,h,k)=0$ — ③ is also CI.

choose on ③, a curve C in whose eqⁿ the constants h & k appear as independent parameters, then find the envelope of one parameter sub-family of ① touching C.

Since this solⁿ contains 2 arbitrary constants, its a CI

eg ST $xpq+yq^2=1$ has CI (a) $(z+b)^2 = 4(ax+y)$

(b) $kx(z+h) = k^2y+x^2$

& deduce (b) from (a)

Write aux. eqⁿ & solve twice to get (a) & (b) $\Rightarrow$ straightforward

Now choose $C = y=0$, $x=k(z+h)$ on (b). Get the relation b/w a, b for a PI touching C, ie, $(z+b)^2 = 4ak(z+h)$ & do $b^2-4ac=0$ for quadⁿ in z.

We get $b = ak+h$.

Put this in (a) to get one-parameter sub-family. Now again eliminate (a) from quad. in 'a' $[z+(h+ak)]^2 = 4(ax+y)$

on Doing $D=0$ we get the cond (b). HP

6) Complete integral satisfying other condition such as circumscribing a given surface:

* 2 surfaces circumscribe each other if they ~~tough~~ touch along a curve. PDE is $F(x,y,z,p,q)=0$ — ①

We want CI $f(x,y,z,a,b)$ to circumscribe $\psi(x,y,z)=0$

$f(x,y,z,a,b)=0$ — ② (CI) & $\psi(x,y,z)=0$ — ③ is the surface

* ② touches ④ iff ②, ③ & $\frac{f_x}{\psi_x}=\frac{f_y}{\psi_y}=\frac{f_z}{\psi_z}$ are consistent

Use these 3 to eliminate x,y,z & get relation in $\xi(a,b)$

* Using $\xi(a,b)$ get a one-parameter subsystem of ② whose envelope is the required surface.

eg ST only integral surface of the equation $2q(z-px-qy)=1+q^2$ which is circumscribed about the paraboloid $2x=y^2+z^2$ is the enveloping cylinder which touches it along its section by plane $y+1=0$

$F(x,y,z,p,q) = z=px+qy+\frac{1+q^2}{2q}$

$\therefore f(x,y,z,a,b) = z=ax+by+\frac{1+b^2}{2b}$ [$\because$ clairaut form]

$\psi(x,y,z) = 2x-y^2-z^2$ $\quad\therefore \frac{a}{2}=\frac{b}{-2y}=\frac{-1}{-2z}$ $\Rightarrow z=\frac{1}{a}$, $y=\frac{-b}{a}$.

$\therefore 2x-\frac{b^2}{a^2}-\frac{1}{a^2}=0 \Rightarrow x=\frac{b^2+1}{2a^2}$

$\therefore \frac{1}{a}=\frac{b^2+1}{2a}-\frac{b^2}{a}+\frac{1+b^2}{2b}$ $\quad \frac{1+b^2}{2a}=\frac{1+b^2}{2b} \Rightarrow \boxed{b=a}$

$\therefore$ one parameter form: $z=ax+ay+\frac{1+a^2}{2a} \Rightarrow 2a^2(x+y)+1+a^2-2az=0$

$\Rightarrow 4z^2-4[2(x+y)+1]1 \Rightarrow \boxed{z^2=2(x+y)+1}$

This is the reqd circumscribing surface.

Since it touches $\psi \Rightarrow$ $2(x+y)+1=2x-y^2$

$\Rightarrow y^2+2y+1=0$

$\Rightarrow (y+1)^2=0$

$\Rightarrow y+1=0.$

⇒ **Jacobi's Method** :

Solving PDE with 3 (or) more than 3 Independent variables :

eg $F(x_1, x_2, x_3, p_1, p_2, p_3) = 0$ — ① | w/o variable 'z' |

Jacobi's Aux eqn : $\dfrac{dx_1}{-\frac{\partial f}{\partial p_1}} = \dfrac{dx_2}{-\frac{\partial f}{\partial p_2}} = \dfrac{dx_3}{-\frac{\partial f}{\partial p_3}} = \dfrac{dp_1}{\frac{\partial f}{\partial x_1}} = \dfrac{dp_2}{\frac{\partial f}{\partial x_2}} = \dfrac{dp_3}{\frac{\partial f}{\partial x_3}}$

Solving gives 2 additional eqn. $F_1(x_1, x_2, x_3, p_1, p_2, p_3) = a_1$ ②
$F_2(x_1, x_2, x_3, p_1, p_2, p_3) = a_2$ ③

Verify that $(F_1, F_2) = \sum_{r=1}^{3} \dfrac{\partial(F_1, F_2)}{\partial(x_r, p_r)} = 0$ — ④

If ④ is satisfied, solve ①, ②, ③ get p_1, p_2, p_3 & put in $dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$.

For 4 independent variables : verify $(F_1, F_2) = 0$, $(F_2, F_3) = 0$, $(F_3, F_1) = 0$ separately.

eg $p_1^3 + p_2^2 + p_3 = 1$

JAE : $\dfrac{\partial x_1}{-3p_1^2} = \dfrac{\partial x_2}{-2p_2} = \dfrac{\partial x_3}{-1} = \dfrac{\partial p_1}{0} = \dfrac{\partial p_2}{0} = \dfrac{\partial p_3}{0}$

$\therefore \quad p_1 = a_1 \quad p_2 = a_2 \qquad \therefore F_1 = p_1 - a_1 \quad F_2 = p_2 - a_2$

$(F_1, F_2) = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0$

$\therefore \quad p_1 = a_1 \quad p_2 = a_2$ & put in PDE ⇒ $p_3 = 1 - a_1^3 - a_2^2$

$\therefore$ ~~dz=p~~ $dz = a_1 dx_1 + a_2 dx_2 + (1 - a_1^3 - a_2^2) dx_3$

$z = a_1 x_1 + a_2 x_2 + (1 - a_1^3 - a_2^2) x_3 + a_3$

eg $p_1 p_2 p_3 = z^3 x_1 x_2 x_3$

We have to remove z, so put $Z = \ln z \quad X_i = \dfrac{x_i^2}{2}$

So PDE becomes $P_1 P_2 P_3 = 1$ whose soln is $Z = a_1 X_1 + a_2 X_2 + \dfrac{1}{a_1 a_2} X_3$

$\therefore \ln z = \dfrac{a_1 x_1^2}{2} + \dfrac{a_2 x_2^2}{2} + \dfrac{x_3^2}{2a_1 a_2} + a_3$.

⇒ Cauchy's Method of Characteristics for non-linear PDE:

1) Given $f(x, y, z, p, q) = 0$ we wish to find integral surface touching a given curve $x = f_1(\lambda)$ $y = f_2(\lambda)$ $z = f_3(\lambda)$
λ is parameter

2) We first want to find initial values x_0, y_0, z_0, p_0, q_0, to They are needed to determine the arbitrary constants from the solution of characteristics strip PDES.

Take $x_0 = f_1(\lambda)$ $y_0 = f_2(\lambda)$, $z_0 = f_3(\lambda)$, then p_0, q_0 are determined using $f_3'(\lambda) = p_0 f_1'(\lambda) + q_0 f_2'(\lambda)$
& $f(x_0, y_0, z_0, p_0, q_0) = 0$

3) Once we have initial values, then solve following 5 equations:

$$\frac{dp}{dt} = -f_x - pf_z \qquad \frac{dq}{dt} = -f_y - qf_z$$ [∵ -ves of Charpit denrs]

$$\frac{dx}{dt} = +f_p \qquad \frac{dy}{dt} = +f_q \qquad \frac{dz}{dt} = pf_p + qf_q$$

4) We find 5-6 solutions by combining different equations. with each solution we get an arbitrary constant. It is determined using x_0, y_0, z_0, p_0, q_0 found in (2).
Sometimes we get t as part of the above function: initial value t_0 is entirely our own choice for convenience.

5) So we get $x = \phi_1(\lambda, t)$ $y = \phi_2(\lambda, t)$ $z = \phi_3(\lambda, t)$

Eliminate λ, t to get relation b/w x, y, z.

Example: $z = \frac{1}{2}(p^2+q^2) + (p-x)(q-y)$ passing through x-axis

Parametric eqn of c: $(x = \lambda,\ y = 0,\ z = 0)$

Let initial values be $x_0 = \lambda$ $y_0 = 0$, $z_0 = 0$

then $f_3'(\lambda) = p_0 f_1'(\lambda) + q_0 f_2'(\lambda) \Rightarrow 0 = p_0 + 0 \Rightarrow \boxed{p_0 = 0}$

& $0 = \frac{1}{2}(p_0^2+q_0^2) + (p_0-\lambda)(q_0-0) \Rightarrow \boxed{q_0 = 2\lambda}$

Now $f_x = y-q$ $f_y = x-p$ $f_z = -1$ $f_p = p+q-y$ $f_q = p+q-x$

$\therefore \frac{dp}{dt} = p+q-y$ $\frac{dq}{dt} = p+q-x$ $\frac{dz}{dt} = (p+q)^2 - py - qx$

$\frac{dx}{dt} = p+q-y$ $\frac{dy}{dt} = p+q-x$

$\therefore$ $\left.\begin{array}{l} p-x = c_1 \\ \& \ q-y = c_2 \end{array}\right\}$ use $p_0 = 0 = y_0$, $q = 2\lambda$ $x = \lambda$ $\Rightarrow$ $c_1 = -\lambda$, $c_2 = 2\lambda$ $\Rightarrow$ $x = p+\lambda$, $y = q-2\lambda$ — (1)

Also $\left.\begin{array}{l} \frac{d(p+q-x)}{dt} = p+q-x \\ \frac{d(p+q-y)}{dt} = p+q-y \end{array}\right\} \Rightarrow$ $p+q-x = c_3e^t$, $p+q-y = c_4e^t$ $\Rightarrow$ $\lambda = c_3e^{t_0}$, $2\lambda = c_4e^{t_0}$ $\Rightarrow$ $c_3 = \lambda$, $c_4 = 2\lambda$ take $t_0 = 0$

$\therefore$ $p+q-x = \lambda e^t$ & $p+q-y = 2\lambda e^t$

Using (1), $q = \lambda(1+e^t)$ $p = 2\lambda(e^t-1)$

$\therefore \boxed{x = \lambda(2e^t-1)}$ $\boxed{y = \lambda(e^t-1)}$

Also $\frac{dz}{dt} = p[p+q-y] + q[p+q-x] = 4\lambda^2(e^{2t}-e^t) + \lambda^2(e^{2t}+e^t)$
$= 5\lambda^2e^{2t} - 3\lambda^2e^t$

$\therefore z = \frac{5}{2}\lambda^2e^{2t} - 3\lambda^2e^t + c_5$

$0 = \frac{5\lambda^2}{2} - 3\lambda^2 + c_5 \Rightarrow c_5 = \lambda^2/2$

$\therefore \boxed{z = \frac{\lambda^2}{2}\left[5e^{2t} - 6e^t + 1\right]}$

Now $\Rightarrow$ $e^t = \frac{y+\lambda}{\lambda} \Rightarrow x = \lambda\left[\frac{2(y+\lambda)}{\lambda} - 1\right] \Rightarrow \lambda = x-2y$ & $e^t = \frac{x-y}{x-2y}$.

$\therefore z = \frac{(x-2y)^2}{2}\left[5\left(\frac{x-y}{x-2y}\right)^2 - 6\left(\frac{x-y}{x-2y}\right) + 1\right]$

⇒ Linear PDE with constant coefficients :

1) Linear Homogenous : $\left[D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \dots + A_n D'^n\right] z = f(x,y)$

Complementary Function : $f(D, D') z = f(x,y)$.

then CF is sol^n to $f(D, D')z = 0$

(a) Replace $D \rightarrow m$ & $D' \rightarrow 1$ & get the Aux eq^n $m^n + A_1 m^{n-1} + A_2 m^{n-2} + \dots + A_n = 0$ & solve for 'm'

(i) Distinct roots $m_1, m_2, \dots m_n$

then $CF = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \dots + \phi_n(y + m_n x)$

(ii) Repeated roots $m = m'$ (n times)

then $CF = \phi_1(y + mx) + x\,\phi_2(y + mx) + \dots + x^{n-1}\phi_n(y + mx)$

Note all ϕ_is are diff^t.

(iii) If D^m is a factor, then

$CF = \phi_1(y) + x\,\phi_2(y) + \dots + x^{m-1}\phi_m(y)$

(iv) If D'^m is a factor, then

$CF = \phi_1(x) + y\,\phi_2(x) + \dots + y^{m-1}\phi_m(x)$.

[OR]

(b) Factorise $F(D, D')$ as $(bD - aD')$ factors,

then for $(bD - aD') \equiv \phi(by + ax)$ is the part of CF

Easy: remember $\left(D - \frac{a}{b} D'\right)$ has $m = \frac{a}{b}$

$\therefore \phi(y + mx) = \phi(by + ax)$

Q) Particular Integral : $PI = \frac{1}{F(D,D')} f(x,y)$

$\frac{1}{D} \equiv$ partial integration wrt x $\quad \frac{1}{D'} \equiv$ wrt y.

(i) $F(D,D')z = \phi(ax+by)$ where $F(D,D')$ is homogeneous with degree 'n'

(a) $F(a,b) \neq 0$

then $PI = \boxed{\frac{1}{F(a,b)} \underbrace{\iiiint \cdots \int \phi(v)\, dv \cdot dv \cdot dv \cdots}_{n \text{ times}}}$

(b) $F(a,b) = 0 \Rightarrow (bD - aD')$ is a factor

then $\frac{1}{(bD-aD')^n} \phi(ax+by) = \boxed{\frac{x^n}{b^n n!} \phi(ax+by)}$

(ii) $F(D,D')z = x^m y^n$ or some algebraic function of x^m, y^n

Expand $\frac{1}{F(D,D')}$ as $\boxed{\frac{D'}{D}}$ if $\underline{n<m}$ or $\boxed{\frac{D}{D'}}$ if $\underline{m<n}$

Basically try to differentiate smaller power

Example : $r + 5s + 6t = (y-2x)^{-1}$

$= (D+2D')(D+3D')z = (y-2x)^{-1} \Rightarrow z_c = \phi_1(y-2x) + \phi_2(y-3x)$

$z_p = \frac{1}{D+2D'}\left[\frac{1}{D+3D'}(y-2x)^{-1}\right] = \frac{1}{D+2D'} \log(y-2x).$

$= \frac{x^1}{1! \cdot 1^1} \cdot \log(y-2x) = x \log(y-2x)$

$(D^2 - 6DD' + 9D'^2)z = 12x^2 + 36xy \Rightarrow z_c = \phi_1(y+3x) + x\phi_2(y+3x)$

$z_p = \frac{1}{(D-3D')^2}(12x^2+36xy) = \frac{1}{D^2}\left[1 - \frac{3D'}{D}\right]^{-2}[12x^2+36xy]$

$= \frac{1}{D^2}\left[1 + \frac{6D'}{D} + \frac{27D'^2}{D^2} \cdots\right][12x^2+36xy] = \frac{1}{D^2}\left[12x^2 + 36xy + 108x^2\right] = \frac{1}{D}\left[4x^3 + 12x^2y + 36x^3\right]$

$= x^4 + 6x^3y + \cancel{108}\,9x^4 - 10x^4 + 6x^3y$

3) General method : $\frac{1}{(D-mD')}\phi(x,y) = \int \phi(x, a-mx)\,dx$

After integration put $a = y+mx$.

$\boxed{\text{use for sin, cos functions}}$ repeatedly.

Example : $(D^2 + DD' - 6D'^2)z = y\sin x \Rightarrow (D+3D')(D-2D')z = y\sin x$

$z_c = \phi_1(y-3x) + \phi_2(y+2x)$

$z_p = \frac{1}{(D+3D')(D-2D')} y\sin x = \frac{1}{D+3D'}\int (a-2x)\sin x\,dx$

$= \frac{1}{D+3D'}\left[-\cos x(a-2x) + 2\cos x\right]$

might have error $\Rightarrow$ $= \frac{1}{D+3D'}\left[2\cos y - y\cos x\right] = \int 2\cos(a+3x) - (a+3x)\cos x\,dx$

$= \frac{2}{3}\sin(a+3x) - (a+3x)\sin x - 3\cos x$

just proceed like this

4) Non-homogenous Linear PDE with constant coefficients :

CF:

(a) $F(D,D')$ can be resolved as linear factors.

if $F(D,D') = [(D-m_1D'-k_1)(D-m_2D'-k_2)\cdots(D-m_nD'-k_n)]z$

$CF = e^{k_1x}\phi_1(y+m_1x) + e^{k_2x}\phi_2(y+m_2x) + \cdots + e^{k_nx}\phi_n(y+m_nx)$

Repeated factor $(D-mD'-k)$

$CF = e^{kx}[\phi_1(y+mx) + x\phi_2(y+mx) + \cdots + x^{r-1}\phi_{r-1}(y+mx)]$

(b) If factor is $(\alpha D + \beta D' + \gamma)z = 0 \Rightarrow$ CF part is $e^{-\frac{\gamma}{\alpha}x}[\phi(\alpha y - \beta x)]$

(b) If can't factorise into linear factors :

Use hit & trial (mostly e^{ax+by})

eg $(2D^4 - 3D^2D' + D'^2)z = 0 \Rightarrow (2D^2-D')(D^2-D')z = 0$.

let $z = Ae^{hx+ky} \Rightarrow Ah^2e^{hx+ky} - Ake^{hx+ky} \Rightarrow h^2 = k$. [using $2D^2-D'$]

$\Rightarrow z = \Sigma Ae^{hx+h^2y}$

using $D^2 - D' \Rightarrow z = \Sigma Be^{h'x+2h'^2y} \Rightarrow GS = z = \Sigma Ae^{hx+h^2y} + \Sigma Be^{h'x+2h'^2y}$.

Particular Integral: (similar to ODE):

(a) $\dfrac{1}{F(D,D')}e^{ax+by} = \dfrac{1}{F(a,b)}e^{ax+by}$ $[F(a,b)\neq 0]$

(b) $\dfrac{1}{F(D,D')}\sin(ax+by)$ Put $D^2=-a^2$ $D'^2=-b^2$ $DD'=-ab$

(c) $f(x,y)=x^m y^n$ use binomial expansion

(d) $f(x,y)=e^{ax+by}V$ $\Rightarrow \dfrac{1}{F(D,D')}f(x,y) = e^{ax+by}\dfrac{1}{F(D+a,D'+b)}V$.

5) Reducible to Linear PDE with constant coefficients:

if $A_0 x^n \dfrac{\partial^n z}{\partial x^n} + A_1 x^{n-1}y\dfrac{\partial^n z}{\partial x^{n-1}\partial y} + \cdots + A_n\dfrac{\partial^n z}{\partial y^n} = f(x,y)$

Put $x=e^{u}$ $y=e^{v}$ (like ODE)

This will reduce $x^m y^n \dfrac{\partial^{m+n}z}{\partial x^m \partial y^n} = D(D-1)\cdots(D-(m-1))\cdot D'(D'-1)\cdots(D'-(n-1))$

eg. $x^2 r - y^2 t = xy$ $e^X = x$ $e^Y = y$

$\Rightarrow [D(D-1) - D'(D'-1)]z = e^{X+Y}$

$\Rightarrow (D-D')(D+D'-1)z = e^{X+Y}$

$z = \phi_1(X+Y) + \phi_2(X+Y)e^X + Xe^{X+Y}$

$z = \psi_1(xy) + x\,\psi_2\left(\frac{y}{x}\right) + (\log x)\,xy$.

6) Finding surface satisfying some conditions & PDE.

Miscellaneous Example: Surface is drawn satisfying $r+t=0$ & touching $x^2+z^2=1$ along its section by $y=0$. Find its equation in the form $z^2(x^2+z^2-1) = y^2(x^2+z^2)$

$(D^2+D'^2)z=0 \Rightarrow z = \phi_1(y+ix) + \phi_2(y-ix)$

$p = \frac{\partial z}{\partial x} = i\,[\phi_1'(y+ix) - \phi_2'(y-ix)]$

$q = \frac{\partial z}{\partial y} = \phi_1'(y+ix) + \phi_2'(y-ix)$

Also $x^2+z^2=1 \Rightarrow z = \sqrt{1-x^2} \quad \therefore p = \frac{-x}{\sqrt{1-x^2}} \quad q=0$

Since they touch, equate p's & q's $\Rightarrow \phi_1'(y+ix) + \phi_2'(y-ix) = 0$

$\Rightarrow \quad p = 2i\,\phi_1'(y+ix)$ where $y=0$.

$\therefore \phi_1'(ix) = \frac{xi}{2\sqrt{1+(ix)^2}} \qquad \phi_2(-ix) = \frac{-ix}{2\sqrt{1+(-ix)^2}}$

$\therefore \phi_1(u) = \frac{1}{2}\sqrt{1+u^2} + c_1 \qquad \phi_2(v) = \frac{1}{2}\sqrt{1+v^2} + c_2$

$\therefore z = \frac{1}{2}\{\sqrt{1+(y+ix)^2} + \sqrt{1+(y-ix)^2}\} + c. \quad$ — (1)

Also 2 values of z should be same from $x^2+z^2=1$ & (1) when $y=0$

$\Rightarrow \sqrt{1-x^2} = \sqrt{1-x^2} + c \Rightarrow c=0$.

$\therefore 2z = \sqrt{1+(y+ix)^2} + \sqrt{1+(y-ix)^2}$ [which can be written in the desired form]

7) CANONICAL FORM :

Let PDE be $Rr + Ss + Tt + f(x,y,z,p,q) = 0$

This can be converted into canonical form by substitutions

$x \rightarrow u = u(x,y) \qquad y \rightarrow v = v(x,y) \qquad$ [u, v are independent]

To find u & v :-

(a) Solve $R\lambda^2 + S\lambda + T = 0 \;\; \begin{matrix}\nearrow \lambda_1 \\ \searrow \lambda_2\end{matrix}$ & solve $\frac{dy}{dx} + \lambda_1 = 0$ & $\frac{dy}{dx} + \lambda_2 = 0$

(b) Let solutions be $f_1(x,y) = c_1$ & $f_2(x,y) = c_2$. Then put $u = f_1(x,y)$ & $v = f_2(x,y)$

$u = xy$, then take $v = x$

$J\frac{(u,v)}{(x,y)} \neq 0$ & Hence independent

(c) Find p, q, r, s, t in ~~form~~ terms of u & v & rewrite the PDE. We get the canonical form

(d) In case of equal roots, choose any 'v' independent of 'u'. Also in case of complex roots $u = \alpha + i\beta$ & $v = \alpha - i\beta$ choose α, β as new coordinates

Example: $\frac{\partial^2 z}{\partial x^2} + x^2\frac{\partial^2 z}{\partial y^2} = 0 \Rightarrow r + x^2 t = 0$

$\therefore \; R\lambda^2 + S\lambda + T = 0 \Rightarrow \lambda^2 + x^2 = 0 \Rightarrow \lambda = \pm ix$

$\therefore \; \frac{dy}{dx} \pm ix = 0 \Rightarrow y \pm i\frac{x^2}{2} = u$

$u = y + i\frac{x^2}{2} \qquad v = y - \frac{ix^2}{2}$

Use $\alpha = y$ & $\beta = \frac{x^2}{2}$ as the transformation

p', q', r', s', t' are wrt α, β

Now $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\cdot\frac{\partial \beta}{\partial x} = xq'$

$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\cdot\frac{\partial \beta}{\partial y} = p'$

DO THE HARD WORK, ELSE ERROR CHANCES

$r = \frac{\partial p}{\partial x} = \frac{\partial}{\partial x}(xq') = q' + x\frac{\partial q'}{\partial x} = q' + x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \beta}\right) = q' + x\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x}$

$= q' + x^2 t'$

$t = \frac{\partial q}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial \alpha}\right) = \frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right)\cdot\frac{\partial \alpha}{\partial y} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha}\right)\cdot\frac{\partial \beta}{\partial y} = r'$

$\therefore \; r + x^2 t = q' + x^2 t' + x^2 r' = 0$

$q' + (r' + t')(2\beta) = 0.$

$\Rightarrow \frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \frac{-1}{2\beta}\frac{\partial z}{\partial \beta} \Rightarrow$ canonical form

8> Product Method or Variable separation method:

Assume $z = X(x)\,Y(y)$ as the solution

eg. $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0 \Rightarrow X''Y - 2X'Y + XY' = 0 \Rightarrow \frac{X'' - 2X'}{X} = \frac{-Y'}{Y} = a$

$\Rightarrow X'' - 2X' - aX = 0 \Rightarrow X = c_1 e^{(1+\sqrt{1+a})x} + c_2 e^{(1-\sqrt{1+a})x}$

$Y' + aY = 0 \Rightarrow Y = c_3 e^{-ay}$

$\Rightarrow z = XY = k_1 e^{(1+\sqrt{1+a})x - ay} + k_2 e^{(1-\sqrt{1+a})x - ay} \Rightarrow$ sol^n

This is used in Wave, heat & laplace equations!

→ Applications of PDE :

1) Standard Equations :

1-D wave equation : $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$

1-D heat flow equation : $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

2-D heat eq^n : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

2-D wave eq^n : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$

Laplace's eq^n in 3D : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$.

2) Fourier Sine series : F-expansion of $f(x)$ in $0<x<l$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{where } b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$n = 1, 2, 3 \ldots$

Fourier cosine series :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \quad \text{where } a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$n = 1, 2, 3 \ldots$

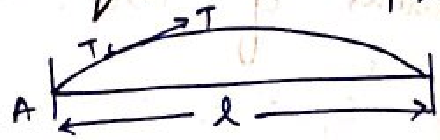
Double Fourier sine series : $f(x)$ expansion in $0 \le x \le a$ $0 \le y \le b$

$$f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$A_{mn} = \frac{4}{ab}\int_{x=0}^{a}\int_{y=0}^{b} f(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dx\,dy$$

$m = 1, 2, 3 \ldots ; n = 1, 2, 3 \ldots$

3) Stretched elastic string - wave equation : $T >>> w$



$$\Rightarrow \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

$$y(x,t) = X(x)\,T(t) \Rightarrow XT'' = c^2 X'' T \Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T}$$

initial condition : $y(0,t) = y(l,t) = 0 \ \forall t$ & $\frac{dy}{dt}(x,0) = 0 \ \forall x$

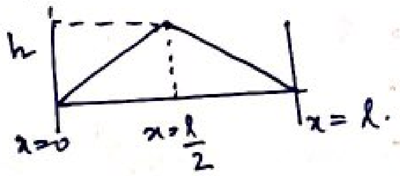
$$\Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = a$$

- $a = 0 \Rightarrow X = a_1 x + a_2$ $\quad T = a_3 t + a_4$
- $a = \lambda^2 \Rightarrow X = b_1 e^{\lambda x} + b_2 e^{-\lambda x}$ $\quad T = b_3 e^{\lambda ct} + b_4 e^{-\lambda ct}$
- $a = -\lambda^2 \Rightarrow X = c_1 \cos \lambda x + c_2 \sin \lambda x$ $\quad T = c_3 \cos \lambda ct + c_4 \sin \lambda ct$

Among these 3, choose the one satisfying the initial conditions :

Std solⁿ (for above initial values) $\Rightarrow$ $\boxed{y(x,t) = \sum_{n=1}^{\infty} \left(E_n \cos \frac{n\pi ct}{l} + F_n \sin \frac{n\pi ct}{l}\right) \sin\left(\frac{n\pi x}{l}\right)}$

Example: A taut string of length l has its end $x=0$ & $x=l$ fixed. Mid point is taken to a small height 'h' & released from rest at $t=0$. Find $y(x,t)$.



IV $\Rightarrow$ $y(0,t) = y(l,t) = 0 \quad \forall t$

$$\therefore y(x,t) = \sum_{n=1}^{\infty} \left(E_n \cos \frac{n\pi ct}{l} + F_n \sin \frac{n\pi ct}{l}\right) \sin\left(\frac{n\pi x}{l}\right)$$

Also $\frac{dy}{dt}(x,0) = 0 \ \forall x$ $\ast$ $\frac{dy}{dt} = \sum_{n=1}^{\infty} \left(-E_n \times \frac{n\pi c}{l} \sin \frac{n\pi ct}{l} + F_n \cos \frac{n\pi ct}{l} \times \frac{n\pi c}{l}\right) \sin\left(\frac{n\pi x}{l}\right) = 0$

$$\Rightarrow F_n = 0$$

initially at rest

$$\therefore \boxed{y(x,t) = \sum_{n=1}^{\infty} E_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}}$$

Now $y(x,0) = g(x) = \begin{cases} \frac{2hx}{l} & x < l/2 \\ -\frac{2h}{l}(l-x) & x > \frac{l}{2} \end{cases} = \sum_{n=1}^{\infty} E_n \sin \frac{n\pi x}{l}.$

$$\therefore E_n = \frac{2}{l}\int_0^l g(x) \sin\left(\frac{n\pi x}{l}\right) dx = \frac{2}{l}\left[\int_0^{l/2} \frac{2hx}{l} \sin\left(\frac{n\pi x}{l}\right) dx + \int_{l/2}^{l} -\frac{2h}{l}(l-x) \sin\left(\frac{n\pi x}{l}\right) dx\right]$$

This gives E_n.

4) One-Dimensional heat equation :

$u(x,t)$ is temperature function, then $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$

Take $u(x,t) = X(x)\,T(t)$. $\Rightarrow XT' = k X'' T \Rightarrow \frac{T'}{kT} = \frac{X''}{X} = a$

$a = 0 \Rightarrow u = (a_1 x + a_2)(a_3)$

$a = -\lambda^2 \Rightarrow u = (c_1 \cos \lambda x + c_2 \sin \lambda x)\, c_3 e^{-k\lambda^2 t}$ → generally we choose this

$a = \lambda^2 \Rightarrow u = (b_1 e^{\lambda x} + b_2 e^{-\lambda x})\, b_3 e^{k\lambda^2 t}$

as $e^{-k\lambda^2 t}$ ensure $u \to 0$ as $t \to \infty$.

Now discard as per the initial conditions

* Situations : ① Both ends at zero & initial temperature is given

$u(0,t) = u(a,t) = 0 \quad \forall t$ & $u(x,0) = f(x)$.

$$\Rightarrow \boxed{u(x,t) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{a}\right) e^{-c_n^2 t}}$$ where $c_n^2 = \frac{n^2\pi^2 k}{a^2}$

& $E_n = \frac{2}{a}\int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx$

Example : $u(0,t) = 2$ & $u(1,t) = 3$
& $u(x,0) = x(1-x)$ Find $u(x,t)$.

Consider $g(x) = c_1 x + c_2 \Rightarrow g(0) = 2 \quad g(1) = 3 \Rightarrow c_1 = 1 \quad c_2 = 2$

$\therefore g(x) = x + 2$

Consider $v(x,t) = u(x,t) - g(x)$ $\therefore v(0,t) = 0 \quad v(1,t) = 0$

very handy to do this !! & $v(x,0) = x(1-x) - x - 2$
$= -(x^2 + 2)$.

Now proceed as normally.

② Both ends are insulated and initial temperature is given

INSULATION $\Rightarrow \left.\frac{\partial u}{\partial x}\right|_{x=x_0} = 0 \quad \therefore u_x(0,t) = u_x(a,t) = 0 \;\forall t$
& $u(x,0) = f(x)$.

$$\Rightarrow \boxed{u(x,t) = \frac{E_0}{2} + \sum_{n=1}^{\infty} E_n \cos\left(\frac{n\pi x}{a}\right) e^{-k\frac{n^2\pi^2}{a^2}t}}$$

$E_0 = \frac{2}{a}\int_0^a f(x)\, dx$

$E_n = \frac{2}{a}\int_0^a f(x) \cos\left(\frac{n\pi x}{a}\right) dx$

NOTE : while derivation we need to consider $\lambda = 0$ case otherwise $\frac{E_0}{2}$ will be missed !!

③ One end is insulated & other at constant temperature :

eg $u(x,0) = 1-x \quad 0 < x < 1 \qquad u(0,t) = 10 \qquad u_x(1,t) = 0$.

Consider $y(x,t) = u(x,t) - 10 \Rightarrow y(x,0) = -(x+9) \quad y(0,t) = 0 \quad y_x(1,t) = 0$

Now solve we get $u(x,t) = 10 + \sum_{n=1}^{\infty} E_n \sin \frac{(2n-1)}{2}\pi x \, e^{-c_n^2 t}$

(4) Insulated rod, steady state achieved & some action...

STEADY STATE = $\frac{\partial u}{\partial t} = 0$ PDE: $\frac{\partial^2 u}{\partial x^2} = \frac{1}{K}\frac{\partial u}{\partial t}$ Du...

For steady state, $\frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow u = c_1 x + c_2$

Temp at ends is given $\Rightarrow$ $a = c_1(0) + c_2$ & $b = c_1(L) + c_2$

$\Rightarrow$ $C_2 = a$ $c_1 = \frac{b-a}{l}$.

$\therefore u(x,0) = f(x) = \frac{(b-a)x}{l} + a$.

Also given $u(0,t) = u(L,t) = 0 \quad \forall \ t \geq 0$.

Now proceed like in case (1).

5) 2D - Heat eq^n ~~+ Laplace equation~~ : in steady state

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{K}\frac{\partial u}{\partial t}$$

take $u(x,y,t) = X(x)\,Y(y)\,T(t)$.

$$\Rightarrow \frac{X''}{X} + \frac{Y''}{Y} = \frac{1}{K}\frac{T'}{T}$$

Take $\frac{X''}{X} = -m^2$ $\frac{Y''}{Y} = -n^2$ $\frac{1}{K}\frac{T'}{T} = -(m^2+n^2)$

$\therefore u(x,y,t) = \sum\sum X_m(x)\,Y_n(y)\,T_{mn}(t)$

$X_m(x) = A_m \cos mx + B_m \sin mx$

$Y_n(y) = A_n \cos ny + B_n \sin ny$

$T_{mn}(t) = F_{mn}\, e^{-(m^2+n^2)K}$.

so for a plate $0 \leq x \leq a$, $0 \leq y \leq b$ with edges kept at $0°C$ & $u(x,y,0) = f(x,y)$

$\therefore u(0,y,t) = u(a,y,t) = u(x,0,t) = u(x,b,t) = 0$

$\therefore u_{mn}(x,y,t) = F_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, e^{-\lambda_{mn}^2 t}$

$$F_{mn} = \frac{4}{ab}\iint f(x,y) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dx\, dy$$

Que Find CI for $(x+y)(p+q)^2 + (x-y)(p-q)^2 = 1$

Put $x+y = X^2$ & $x-y = Y^2$ $\Rightarrow$ ~~$p+q = 2X$~~

$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial x} = \frac{1}{2X}\frac{\partial z}{\partial X} + \frac{1}{2Y}\frac{\partial z}{\partial Y}$

$q = \frac{1}{2X}\frac{\partial z}{\partial X} - \frac{1}{2Y}\frac{\partial z}{\partial Y}$ $\Rightarrow$ $p+q = \frac{1}{X}\frac{\partial z}{\partial X}$ $p-q = \frac{1}{Y}\frac{\partial z}{\partial Y}$

$\therefore$ eq^n $\Rightarrow$ $P^2 + Q^2 = 1$ $\Rightarrow$ $Z = aX + \sqrt{1-a^2}\,Y + b$

$z = a\sqrt{x+y} + \sqrt{1-a^2}\sqrt{x-y} + b.$