

$$I_1 = \iint_{ABCD} \vec{F} \cdot d\vec{S}$$

$$= \iint \vec{F} \cdot \vec{n} dS = \iint \vec{F} \cdot \vec{\nabla} \phi dz dx$$

Lecture-22  
Thursday  
6/4/17

$$\phi = y - \sqrt{9-z^2}$$

$$= 0$$

$$\phi = y - \sqrt{9-z^2}$$

$$\vec{\nabla} \phi = \hat{j} + \frac{z}{\sqrt{9-z^2}} \hat{k}$$

$$\vec{n} = \frac{\vec{\nabla} \phi}{\|\vec{\nabla} \phi\|} \|\vec{\nabla} \phi\| dz dx$$

$$\iint (2x^2 y \hat{i} - y^2 \hat{j} + 4z^2 x \hat{k}) \left( 0 \hat{i} + \hat{j} + \frac{z}{\sqrt{9-z^2}} \hat{k} \right) dz dx$$

$$= \iint \left( -y^2 + \frac{4z^3 x}{\sqrt{9-z^2}} \right) dz dx \quad \text{on } y^2+z^2=9 \quad \text{or } y^2=9-z^2$$

$$= \iint \left( z^2 - 9 + \frac{4z^3 x}{\sqrt{9-z^2}} \right) dz dx$$

$D_{zx}$  : rectangle

$$= \int_{z=0}^3 \int_{x=0}^2 \left( z^2 - 9 + \frac{4z^3 x}{\sqrt{9-z^2}} \right) dz dx = 108$$

... To be completed on 6/4/17

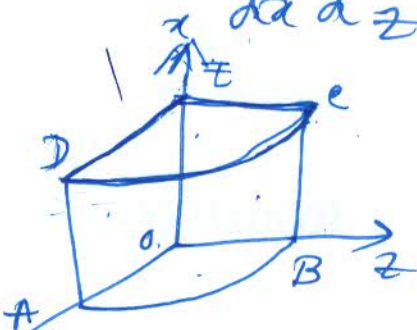
$$I_2 = \iint \vec{F} \cdot d\vec{S} = \iint \vec{F} \cdot \vec{n} dS$$

$$S = \text{EDAO} = \iint_{D_{xy}} (2x^2y\hat{i} - y^2\hat{j} + 4z^2x\hat{k}) \cdot (-\hat{k}) dx dy$$

$$= \int_{y=0}^3 \int_{x=0}^2 -4z^2x \Big|_{z=0} dx dy = 0$$

$$I_3 = \iint_{OBCE} \vec{F} \cdot \vec{n} dS = \iint (2x^2y\hat{i} - y^2\hat{j} + 4z^2x\hat{k}) \cdot (-\hat{j}) dx dz$$

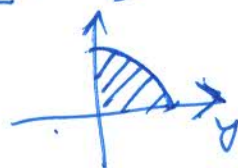
$$= \int_{z=0}^3 \int_{x=0}^2 y^2 dx dz \Big|_{y=0} = 0$$



$$I_4 = \iint_{CED} \vec{F} \cdot \vec{n} dS = \iint (2x^2y\hat{i} - y^2\hat{j} + 4z^2x\hat{k}) \cdot \hat{i} dy dz$$

CED.

$$= \iint_{x=2} 2x^2y dy dz \Big|_{y^2+z^2 \leq 9}$$



$$= \int_{y^2+z^2 \leq 9} 8y dy dz$$

$$y = r \cos \theta$$

$$z = r \sin \theta$$

$$= \int_{r=0}^3 \int_{\theta=0}^{\pi/2} 8r \sin \theta r dr d\theta = 72$$

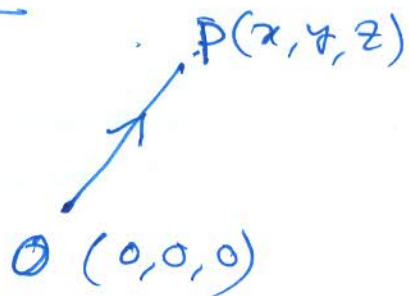
$$I_5 = \iint_{AOB} \vec{F} \cdot \vec{n} dS = \iint (\hat{i} - y^2\hat{j} - \hat{k}) \cdot (-\hat{i}) dy dz$$

$$= \iint_{z=0} -2x^2y dy dz = 0$$

$$\therefore I_1 + I_2 + I_3 + I_4 + I_5 = 108 + 72 = 180$$

## Vector Calculus

$\vec{OP}$  = Position vector of  $P$ .  
( $O \rightarrow$  origin)



$$= (x-0)\hat{i} + (y-0)\hat{j} + (z-0)\hat{k}$$

$$\vec{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$\hat{i}, \hat{j}, \hat{k} \rightarrow$  unit vectors along +ve  $x$  direction, +ve  $y$  dir., +ve  $z$  dir. respectively.

$$|\vec{OP}| = \sqrt{x^2 + y^2 + z^2} = r = |\vec{r}|$$

$$||\vec{r}|| = |\vec{r}| = \text{magnitude of vector } \vec{r}$$

$$|\hat{i}| = 1, |\hat{j}| = 1 = |\hat{k}| = 1.$$

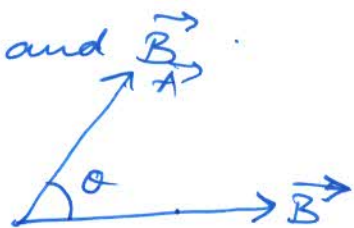
### Products -

$\vec{A} \cdot \vec{B}$  = dot product of  $\vec{A}$  and  $\vec{B}$ .

$$= |\vec{A}| \cdot |\vec{B}| \cos \theta,$$

$$\cos \theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}| |\vec{B}|}$$

$$0 \leq \theta \leq \pi$$



$$\theta = \cos^{-1} \left( \frac{\vec{A} \cdot \vec{B}}{|\vec{A}| \cdot |\vec{B}|} \right)$$

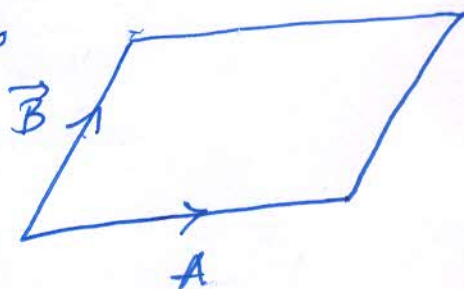
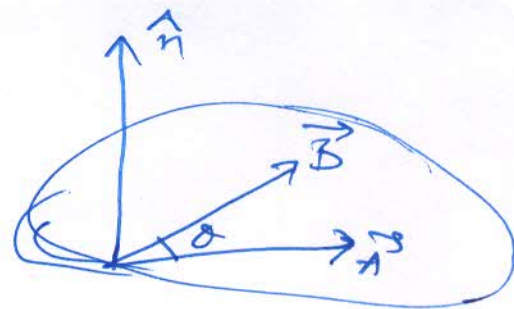
$$\vec{A} \times \vec{B} = |\vec{A}| \cdot |\vec{B}| \cdot \hat{n} \sin \theta$$

$$|\vec{A} \times \vec{B}| = |\vec{A}| \cdot |\vec{B}| \cdot |\hat{n}| \sin \theta = 1$$

$$= |\vec{A}| \cdot |\vec{B}| \cdot \sin \theta$$

= area of the parallelo-

- gram whose one side is  $\vec{A}$ , other side is  $\vec{B}$ .



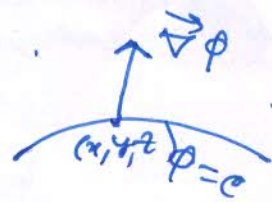
Vector differential operator / nabla operator

$$\vec{\nabla} = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k}$$

$$\vec{\nabla} \phi(x, y, z) = \left( \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \phi$$

Gradient of  $\phi = \vec{\nabla} \phi = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k}$

Geometrically  $\vec{\nabla} \phi$  is the normal to the surface  $\phi(x, y, z) = c$  at the point  $(x, y, z)$



$$\vec{A} = (A_1, A_2, A_3)$$

$$\vec{\nabla} \cdot \vec{A} \quad \text{or} \quad \vec{\nabla} \times \vec{A}$$

↓  
divergence of  $\vec{A}$       ↓  
Curl of  $\vec{A}$

$$\vec{\nabla} \cdot \vec{A} = \left( \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot (A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k})$$

$$\left[ \vec{A} \cdot \vec{B} = (A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}) (B_1 \hat{i} + B_2 \hat{j} + B_3 \hat{k}) \right]$$

$$= A_1 B_1 + A_2 B_2 + A_3 B_3$$

$$\Rightarrow = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} = \text{div. } \vec{A} = \vec{\nabla} \cdot \vec{A}$$

If  $\text{div } \vec{A} = 0$ ,  $\vec{A}$  is said to be solenoidal.

Curl of A.

$$\vec{\nabla} \times \vec{A}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$\left[ \vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \right]$$

$$= \left( \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) \hat{i} + \left( \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) \hat{j} + \left( \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \hat{k}$$

Directional derivative of  $\phi$ .

You know  $\frac{\partial \phi}{\partial x}$  or  $\frac{\partial \phi}{\partial y}$  or  $\frac{\partial \phi}{\partial z}$

$$\vec{A} = A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}$$

Directional derivative  
in the direction of  $\vec{A}$  of  $\phi(x, y, z)$

$$= \vec{\nabla} \phi \cdot \frac{\vec{A}}{|\vec{A}|}$$

Directional derivative in the direction of  
(D.D.) of  $\phi$   $x$  axis

Any vector along  $x$  axis  $= \vec{A} = a \hat{i}$

$\therefore$  D.D. of  $\phi$  along  $\vec{A} = \vec{\nabla} \phi \cdot \frac{\vec{A}}{|\vec{A}|}$

$$= \left( \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k} \right) \left( \frac{a \hat{i}}{|a|} \right)$$

$$= \left( \frac{\partial \phi}{\partial x} \hat{i} + \dots \right) (\pm \hat{i})$$

$$= - \frac{\partial \phi}{\partial x} / \frac{\partial \phi}{\partial x}$$

Ex 1 Find the gradient of  $f = 2x^2y - xy^2$   
at the point  $(2, 1, 1)$  & also find the unit-  
normal to the surface  $2x^2y - xy^2 = c$  at  
 $(2, 1, 1)$

$$\vec{\nabla} f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k}$$

$$= -y^2 \hat{i} + (2x^2 - 2xy) \hat{j} + 4xy \hat{k}$$

$$(\vec{\nabla} f)_{(2,1,1)} = -\hat{i} - 2\hat{j} + 4\hat{k}$$

$$\text{unit-normal} = \frac{-\hat{i} - 2\hat{j} + 4\hat{k}}{\sqrt{(-1)^2 + (-2)^2 + (4)^2}}$$

$$\text{Sol. } \frac{-8\hat{i} + 4\hat{j} + 12\hat{k}}{\sqrt{14}} = -\frac{1}{\sqrt{21}} \hat{i} - \frac{2}{\sqrt{21}} \hat{j} + \frac{4}{\sqrt{21}} \hat{k}$$

Ex 2. Find the gradient of  $f = 4x^2y + z^3$  at the  
point  $(1, -1, 2)$ . Find the unit-normal to the  
surface  $4x^2y + z^3 = c$  at  $(1, -1, 2)$ . 6

2. If  $\phi = 2x^2y - xy^2$ , find the D.D. of  $\phi$  at  $(2, 1, 1)$  in the direction of  $\vec{A} = 3\hat{i} + 6\hat{j} + 2\hat{k}$ .

Sol. dir. der. of  $\phi$  at  $(2, 1, 1)$  along  $\vec{A}$

$$= \vec{\nabla} \phi \cdot \frac{\vec{A}}{|\vec{A}|}$$

$$\vec{\nabla} \phi = -y^2\hat{i} + (2x^2 - 2xy)\hat{j} + 4yz\hat{k}$$

$$\therefore (\vec{\nabla} \phi)_{(2, 1, 1)} = -\hat{i} - 2\hat{j} + 4\hat{k}$$

$$\rightarrow = (-\hat{i} - 2\hat{j} + 4\hat{k}) \cdot \frac{(3\hat{i} + 6\hat{j} + 2\hat{k})}{\sqrt{3^2 + 6^2 + 2^2}}$$

$$= \frac{1}{7} (-\hat{i} - 2\hat{j} + 4\hat{k}) \cdot (3\hat{i} + 6\hat{j} + 2\hat{k})$$

$$= \frac{1}{7} (-3 - 12 + 8) = -1$$

Thm. Dir. derivative of  $\phi$  is maximum along the ~~vec~~ the direction of  $\text{grad } \phi$  & the magnitude is  $|\vec{\nabla} \phi|$ .

[In this case  $\vec{A} = \vec{\nabla} \phi$ .

$$\begin{aligned} \therefore \left| \text{D.D. of } \phi \text{ along } \vec{A} (= \vec{\nabla} \phi) \right| &= |\vec{\nabla} \phi| \cdot \left| \frac{\vec{\nabla} \phi}{|\vec{\nabla} \phi|} \right| \\ &= |\vec{\nabla} \phi| \cdot \frac{|\vec{\nabla} \phi|}{|\vec{\nabla} \phi|} = |\vec{\nabla} \phi|. \end{aligned}$$

Ex-4 Find the D.D. of  $f = xy^2 + yz^3$  at the pt.  $(2, -1, 1)$  in the direction of the vector  $\hat{i} + 2\hat{j} + 2\hat{k}$ . Find the max. value of the D.D. of  $f$  at  $(2, -1, 1)$ .

$$-\frac{11}{3}, \sqrt{19}.$$

Ex-5 Find the angle between the surfaces  $x^2 + y^2 + z^2 = 9$  &  $z = x^2 + y^2 - 3$  at  $(2, -1, 2)$ .

Hint, angle between the surfaces at  $(2, -1, 2)$

= angle between the normals to the surfaces at  $(2, -1, 2)$

$$= \angle \text{ between } \vec{\nabla}\phi \text{ \& } \vec{\nabla}\psi \text{ at } (2, -1, 2) \quad \left| \begin{array}{l} \phi = x^2 + y^2 + z^2 = 9 \\ \psi = x^2 + y^2 - z = 3 \end{array} \right.$$

$$\theta = \cos^{-1} \left( \frac{\vec{\nabla}\phi \cdot \vec{\nabla}\psi}{|\vec{\nabla}\phi| \cdot |\vec{\nabla}\psi|} \right)$$

$$= \cos^{-1} \left( \frac{8}{(\sqrt{21})3} \right) = 54.4^\circ.$$

## Conservative vector field -

$$\vec{F} = e^y \hat{i} + x e^y \hat{j} + (z+1) e^z \hat{k}$$

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^y & x e^y & (z+1) e^z \end{vmatrix} = \vec{0}$$

If  $\vec{F} = \vec{\nabla} \phi$ , where  $\phi$  is some scalar function  $\phi(x, y, z)$ . then  $\vec{F}$  is called a conservative vector field.

$$\vec{v} = 2\hat{i} + 3\hat{j} + 4\hat{k} \rightarrow \text{vector}$$

$$\vec{F}(x, y, z) = (2x + 3y)\hat{i} + (3z - 4yx)\hat{j} + x^2 z^2 \hat{k}$$

$\rightarrow$  vector field

$\phi(x, y, z) \rightarrow$  scalar "

Thm. A vector field is conservative if and only if  ~~$\vec{F} = \vec{\nabla} \phi$~~   $\text{curl } \vec{F} = \vec{0}$ .

Def. If  $\text{curl } \vec{F} = 0$ ,  $\vec{F}$  is called irrotational.

Thm.  $\vec{F}$  can be derived from a scalar potential  $\phi(x, y, z)$  as  $\vec{F} = \vec{\nabla} \phi$  if and only if  $\vec{F}$  is irrotational.

Ex  $\vec{F} = e^y \hat{i} + x e^y \hat{j} + (z+1)e^z \hat{k}$ .

We've seen  $\text{Curl } \vec{F} = 0$ .

$\therefore \vec{F}$  must be such that  $\vec{F} = \vec{\nabla} \phi$ ,  
for some  $\phi = \phi(x, y, z)$

Let  $\vec{F} = (F_1, F_2, F_3)$ .

If  $\vec{F} = \vec{\nabla} \phi$ .

$$\Rightarrow F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k} = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k}.$$

$$\therefore \frac{\partial \phi}{\partial x} = F_1, \quad \frac{\partial \phi}{\partial y} = F_2, \quad \frac{\partial \phi}{\partial z} = F_3$$

$$\frac{\partial \phi}{\partial x} = e^y \rightarrow (1) \quad \frac{\partial \phi}{\partial y} = x e^y \rightarrow (2), \quad \frac{\partial \phi}{\partial z} = (z+1)e^z \rightarrow (3)$$

Integrating w.r. to  $x$ ,

$$\phi = \int e^y dx + f(y, z)$$

$$\phi = x e^y + f(y, z) \rightarrow (4)$$

Diff. (4) w.r. to  $y$  & get

$$\frac{\partial \phi}{\partial y} = x e^y + \frac{\partial f(y, z)}{\partial y} \rightarrow (5)$$

Compare (2) & (5). This gives

$$x e^y = x e^y + \frac{\partial f}{\partial y}(y, z)$$

$$\therefore \frac{\partial f(y, z)}{\partial y} = 0$$

Integrating  $f(y, z) = f_1(z)$

$$\therefore \phi = x e^y + f_1(z) \rightarrow (6)$$

Differentiating (6) w.r. to  $z$ ,

$$\frac{\partial \phi}{\partial z} = \frac{d}{dz} f_1(z) = f_1'(z) \rightarrow (7)$$

Comparing (3) & (7), get -

$$\frac{d f_1(z)}{dz} = (z+1) e^z$$

$$f_1(z) = \int (z+1) e^z dz + C$$

$$= (z+1) \cdot e^z - \int e^z dz + C$$

$$= z e^z + C$$

$$\therefore \phi(x, y, z) = x e^y + z e^z + C$$

2. ~~Ques~~ Show that  $\text{curl } \vec{F} = \vec{0}$

$$\text{where } \vec{F} = (x^2 + yz) \hat{i} + (y^2 + zx) \hat{j} + (z^2 + xy) \hat{k}$$

Hence find the corresponding potential function  $\phi$ .

$$\text{Ans } \phi = \frac{x^3 + y^3 + z^3}{3} + xyz + C$$