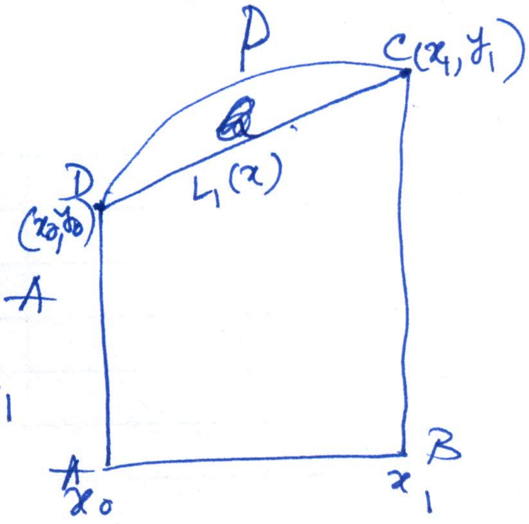
Lecture 12
Friday
24/2/17.

# Numerical Integration

## Trapezoidal Rule

$$I = \int_a^b f(x)\,dx \approx \int_a^b L_1(x)\,dx.$$

$\approx ABCPDA \qquad \approx ABCQDA$




$$L_1(x) = \frac{x - x_1}{x_0 - x_1}\,y_0 + \frac{x - x_0}{x_1 - x_0}\,y_1$$

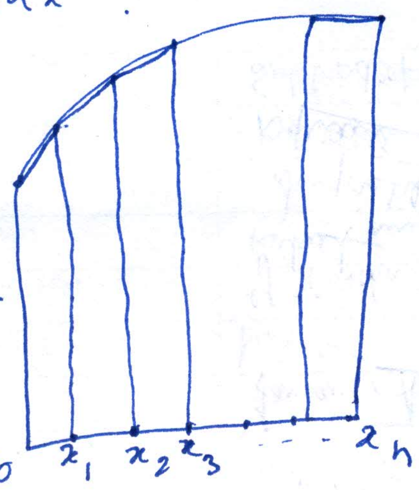
## Composite Trapezoidal Rule -

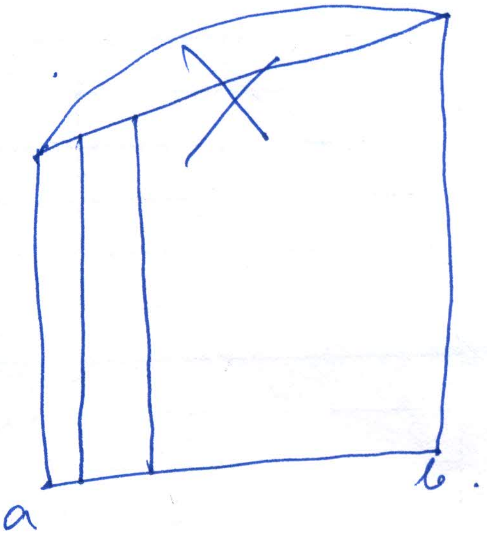
$$I = \int_a^b f(x)\,dx.$$

$n = \frac{b-a}{h}.$

$x_i - x_{i-1} = h.$

$i = 1, 2, \ldots n.$



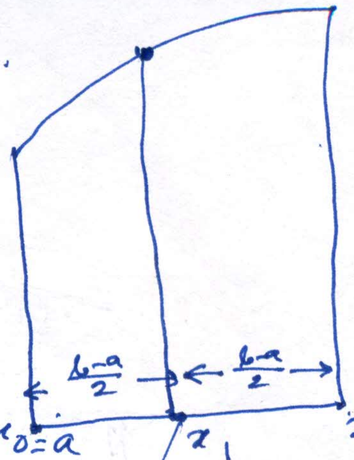




$$I^T = \frac{h}{2}\Big[f(x_0) + 2\big\{f(x_1) + f(x_2) + \cdots + f(x_{n-1})\big\} + f(x_n)\Big]$$

# Simpson's $\frac{1}{3}$ rd rule

$$\int_a^b f(x)\,dx \approx \int_a^b L_2(x)\,dx$$

Lagrange pol. of degree 2

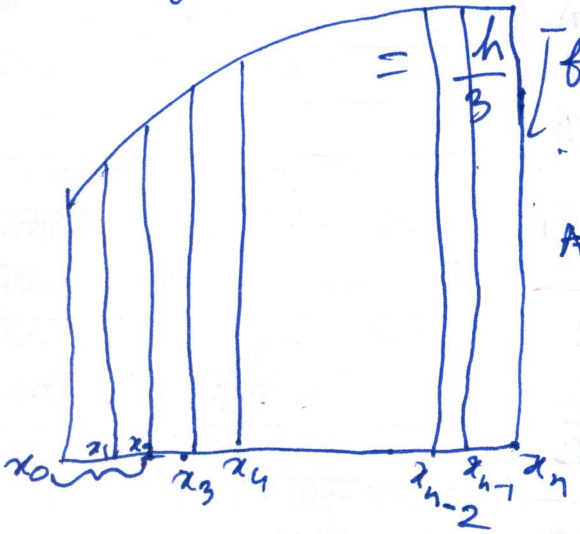



$$L_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2$$

$$x_1 = a + \frac{b-a}{2} = \frac{b+a}{2}$$

$$I^S_{simple} = \int_{a=x_0}^{b=x_2} L_2(x)\,dx = \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]; \quad h = \frac{b-a}{2}$$

$$= \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$




~~Apply~~. Apply simple Simpson's rule to $[x_0, x_2]$.

$$I^S_1 = \frac{h}{3}\left[f(x_0) + 4(x_1) + f(x_2)\right]$$

$h = \frac{b-a}{n}$. Note: $n \rightarrow$ even

Apply simple Simpson's rule to $[x_2, x_4]$; $I^S_2 = \frac{h}{3}\left[f(x_2) + 4f(x_3) + f(x_4)\right]$

" " " " " $[x_4, x_6]$; $I^S_3 = \frac{h}{3}\left[f(x_4) + 4f(x_5) + f(x_6)\right]$

⋮

" " " " " $[x_{n-2}, x_n]$; $I^S_{\frac{n}{2}} = \frac{h}{3}\left[f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$

Adding $I_j^S$ we get, composite Simpson's rule.

$$I^S_{\text{composite}} = \frac{h}{3}\left[ f(x_0) + 4\left\{ f(x_1) + f(x_3) + f(x_5) + \cdots + f(x_{n-1}) \right\} + 2\left\{ f(x_2) + f(x_4) + \cdots + f(x_{n-2}) \right\} + f(x_n) \right].$$

$h$ = length of each subinterval $= \dfrac{b-a}{n}$.

$n$ = no. of subintervals.

Ex1 Compute the integral $\int_{2.1}^{3.6} f(x)\,dx$ employing appropriate numerical integration formula taking $h = 0.3$ & using the table.

| $x$ | 2.1 | 2.4 | 2.7 | 3.0 | 3.3 | 3.6 |
|---|---|---|---|---|---|---|
| $y$ | 3.2 $f(x_0)$ | 2.7 $f(x_1)$ | 2.9 $f(x_2)$ | 3.5 $f(x_3)$ | 4.1 $f(x_4)$ | 5.2 $f(x_5)$ |

$b = 3.6$, $a = 2.1$, $h = 0.3$, $\underline{n = 5}$

$I^T$ We need to apply Trapezoidal rule, because $n = 5$ is odd.

$$\therefore I^T = \frac{h}{2}\left[ f(x_0) + 2\left\{ f(x_1) + f(x_2) + f(x_3) + f(x_4) \right\} + f(x_5) \right]$$

$$= 0.15\left[ 8.4 + 2(2.7 + 2.9 + 3.5 + 4.1) \right] = 5.22.$$

Ex. Compute $\int_{1.8}^{3.4} e^x\,dx$ using Simpson's $\frac{1}{3}$rd rule, taking $h = 0.2$.

Sol. $n = \frac{b-a}{h} = \frac{1.6}{.2} = 8$.

| | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ |
|---|---|---|---|---|---|---|---|---|---|
| X | 1.8 | 2 | 2.2 | 2.4 | 2.6 | 2.8 | 3 | 3.2 | 3.4 |
| $Y = e^x$ $f(x)=$ | 6.05 | 7.389 | 9.025 | 11.023 | 13.464 | 16.445 | 20.086 | 24.533 | 29.964 |

$$I^8 = \frac{h}{3}\Big[f(x_0) + 4\{f(x_1) + f(x_3) + f(x_5) + f(x_7)\} + 2\{f(x_2) + f(x_4) + f(x_6)\} + f(x_8)\Big]$$

$$= 23.91466$$

error = true value − computed value

$$= 23.91445 - 23.91466$$

$$= -0.00021$$

Error in Trapezoidal rule.

$$= -\frac{(b-a)h^2}{12} f''(\eta); \quad a<\eta<b.$$

$\hookrightarrow$ 2 times differentiation

Error in Simpson's $\frac{1}{3}$rd rule.

$$= -\frac{(b-a)h^4}{180} f^{(iv)}(\eta), \quad a<\eta<b.$$

$\hookrightarrow$ 4 times diff. w.r.to $\eta$

In general, $h \ll 1$ $\therefore$ in Simpson's rule absolute error is always less.

Ex-2. $\int_{1.8}^{3.4} e^x \, dx$. Measure the max. & minimum error to compute this integral by Simpson's rule taking $h=0.2$.

$$1.8 < \eta < 3.4$$

$$e^{1.8} < e^{\eta} < e^{3.4}.$$

$$E^S = -\frac{(1.6)(.2)^4}{180} e^{\eta}.$$

$$-e^{3.4} < -e^{\eta} < -e^{1.8}.$$

$$-\frac{(1.6)(.2)^4 e^{3.4}}{180} < E^S < -\frac{(1.6)(.2)^4 e^{1.8}}{180}.$$

$$-4.26 \times 10^{-4} < E^S < -8.603 \times 10^{-5}$$

Exercise. 1. Compute $\int_{1.8}^{3.4} e^x \, dx$ using Trapezoi-dal rule.

taking $h = 0.2$.

2. Find the actual error.

3. Find the max. & min errors using the formula of error.

4. Verify that actual error $\in$ (min error, max error).

## Integral Calculus

## Improper integrals

Text book : 1. Piskunov. D. V. Widder.

2. Shanti Narayan. Advanced Calculus.