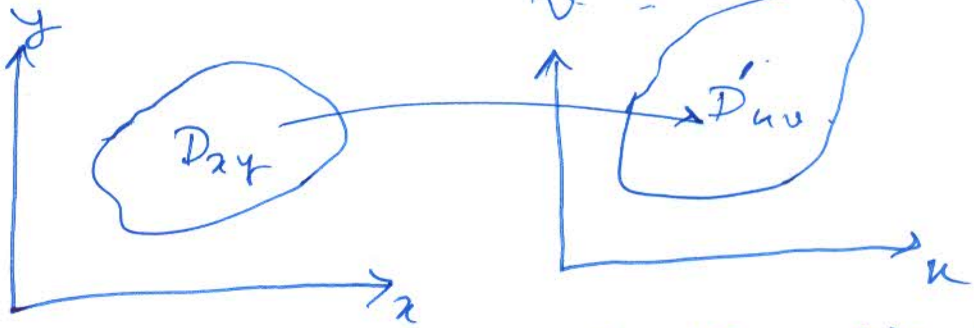
# Change of variables in double integrals

Lecture - 17
Friday
17/3/17

$$I = \int_a^b f(x)\,dx \qquad x \to t, \quad x = g(t)$$

$$= \int_{a_1}^{b'} f(g(t))\, g'(t)\, dt.$$

$$I = \iint_{D_{xy}} f(x,y)\,dx\,dy, \quad (x,y) \to (u,v).$$




$$I = \iint_{D'_{uv}} F(u,v)\,|J|\,du\,dv,$$ J = Jacobian of transformation from $(x,y)$ to $(u,v)$ plane.

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$ (magnification factor)

$$J = \|\vec{V}_1 \times \vec{V}_2\| \cdot \vec{k}$$

$$f(x,y) = F(x^2+y^2)$$

$(x,y) \to (r,\theta)$, $x = r\cos\theta$, $y = r\sin\theta$.

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r.$$

Evaluate.

**Ex 1**

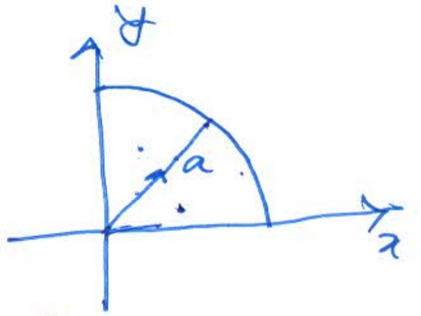
$$I = \iint_D \sqrt{a^2 - x^2 - y^2}\, dx\, dy. \quad D = \text{1st quadrant of the circle } x^2+y^2=a^2$$

$D: x^2+y^2 \le a^2$ in the 1st quad.

$x = r\cos\theta, \quad y = r\sin\theta.$

Compute $J$ (in exam) & get $J = r$.

$$\sqrt{a^2-x^2-y^2} = \sqrt{a^2-r^2}.$$




$$I = \iint_{D_{r\theta}} \sqrt{a^2-r^2}\, r\, dr\, d\theta.$$

$D_{r\theta}: \left\{ (r,\theta) : 0 \le r \le a,\ 0 \le \theta \le \frac{\pi}{2} \right\}$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r\sqrt{a^2-r^2}\, dr\, d\theta.$$


$0 \le \theta \le 2\pi$

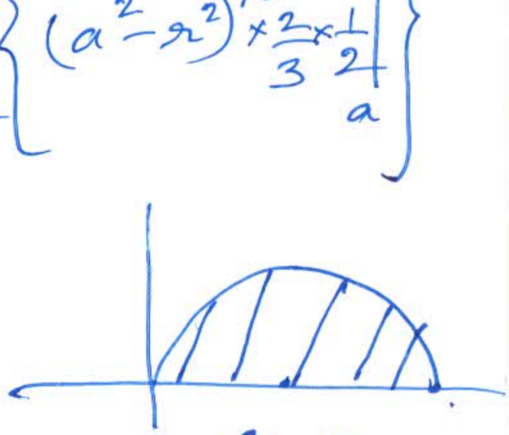

$0 \le \theta \le \pi.$

$$= \left( \int_{\theta=0}^{\pi/2} d\theta \right) \left( \int_0^a r\sqrt{a^2-r^2}\, dr \right) = \frac{\pi}{2} \left\{ (a^2-r^2)^{3/2} \times \frac{2}{3} \times \frac{1}{2} \Big|_a^0 \right\}$$

$$= \frac{\pi a^3}{6}.$$

**Ex 2.**

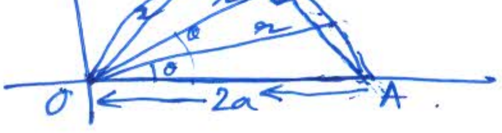
$$\int_0^{2a} \int_0^{y=\sqrt{2ax-x^2}} (x^2+y^2)\, dy\, dx.$$



$(x-a)^2 + y^2 = a^2.$

$x = r\cos\theta, \quad y = r\sin\theta.$

$J = r.$




$\frac{r}{2a} = \cos\theta$. From OPA, $\frac{OP}{OA} = \cos\theta$.

$r = 2a\cos\theta.$

$0 \le r \le 2a\cos\theta, \quad 0 \le \theta \le \frac{\pi}{2}$

$$I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta} r^2 \underbrace{|J|}_{=r}\, dr\, d\theta \qquad \sum_{n=1}^{4}\left(\sum_{m=1}^{2n}\right)$$

$$= \int_0^{\pi/2} \frac{(2a\cos\theta)^4}{4}\, d\theta = \frac{2^4 a^4}{4}\int_0^{\pi/2}\cos^4\theta\, d\theta$$

$$= \frac{\cancel{8}\,2\,a^4}{4} \times 2\int_0^{\pi/2}\cos^4\theta\, d\theta$$

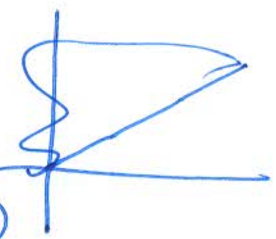
$$= 2a^4\, B\left(\frac{1}{2}, \frac{5}{2}\right) = 2a^4 \cdot \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(3)}$$

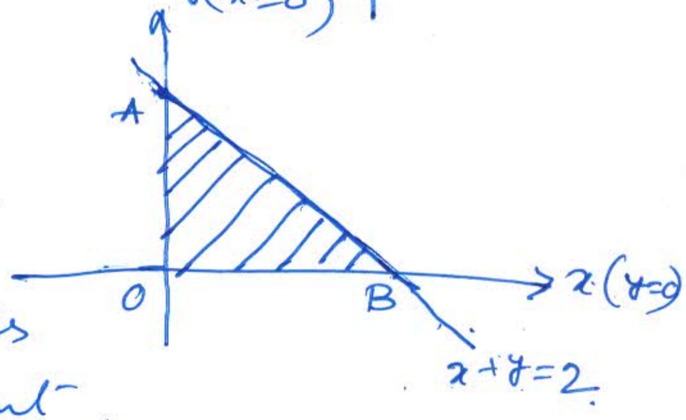
$$B(m,n) = 2\int_0^{\pi/2}\cos^{2m-1}\theta\,\sin^{2n-1}\theta\, d\theta$$

$$2m-1=4$$

$$\Gamma(n+1) = n\Gamma(n)$$

$$= \frac{3}{4}\cancel{\pi}\,a^4$$






Ex-3 $\iint e^{\frac{y-x}{y+x}}\, dx\, dy$

D = bounded by

$x+y=2$; x axis & y axis in the 1st quadrant.

$u = y+x, \quad v = y-x$. $\quad x = \frac{u-v}{2}$, $\quad y = \frac{u+v}{2}$

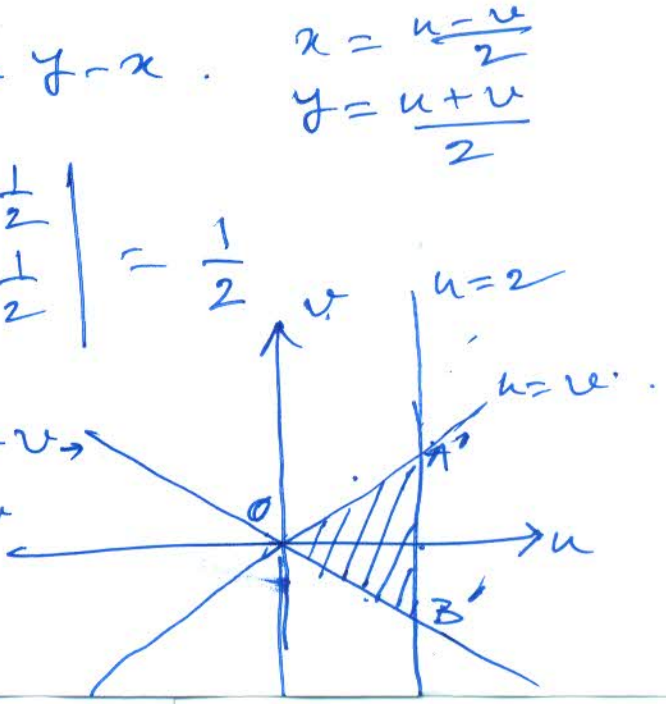
$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

$x+y=2 \Rightarrow u=2$

$y=0 \Rightarrow u=x,\ v=-x,\ u=-v$

$x=0 \Rightarrow u=y,\ v=y,\ u=v$

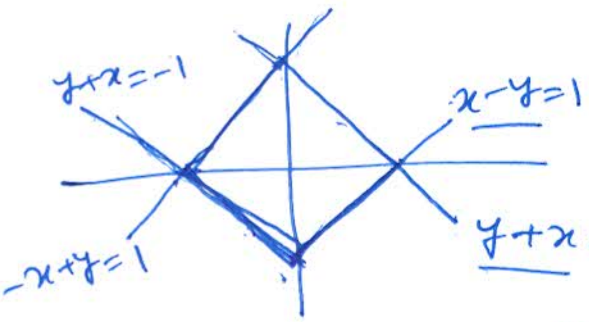
$$I = \int_{u=0}^{2}\int_{v=-u}^{u} e^{\frac{v}{u}}\,\frac{1}{2}\, du\, dv$$



$$I = \frac{1}{2}\int_{u=0}^{2}\int_{v=-u}^{u} e^{\frac{v}{u}}\,du\,dv = \left(e - \frac{1}{e}\right).$$

<u>Exercise 1</u>. Compute $\iint e^{x+y}\,dx\,dy$.

$D = \left\{ (x,y) : |x| + |y| \le 1 \right\}$.




$y + x = u$

$y - x = v$.

<u>Exercise 2</u>. $\iint x^2 y^2\,dx\,dy$.

$D$ = portion in the 1st quadrant bounded by the curves $xy=1$, $xy=2$, $y=x$, $y=4x$.

$u = xy$, $v = \frac{y}{x}$.

# Triple integrals.

$$\iiint\limits_R f(x,y,z)\,dx\,dy\,dz.$$

If $f=1$,

$$\iiint\limits_R dx\,dy\,dz = \text{volume of the solid } R.$$

$R \to$ sphere, paraboloid, cone, cylinder, tetrahedron.

1. $$\iiint (x^2+yz)\,dx\,dy\,dz.$$

$$R = \left\{ (x,y,z) \,\middle|\, 0 \le x \le 2,\ -3 \le y \le 0,\ -1 \le z \le 1 \right\}$$

$$= \int_{z=-1}^{1}\int_{y=-3}^{0}\int_{x=0}^{2} x^2\,dx\,dy\,dz + \int_{z=-1}^{1}\int_{y=-3}^{0}\int_{x=0}^{2} yz\,dx\,dy\,dz.$$

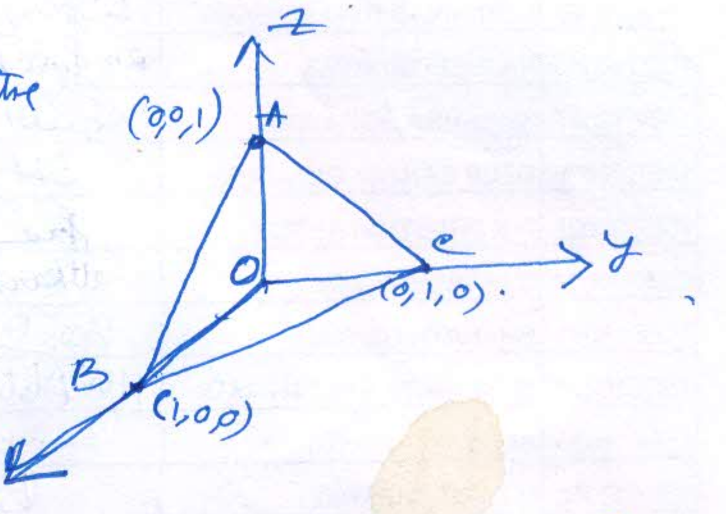
$$= \left(\int_{x=0}^{2} x^2\,dx\right)\left(\int_{y=-3}^{0} dy\right)\left(\int_{z=-1}^{1} dz\right) + \underbrace{\left(\int_{z=-1}^{1} z\,dz\right)\left(\int_{y=-3}^{0} y\,dy\right)\left(\int_{x=0}^{2} dx\right)}_{0}$$

$$= \frac{2^3}{3} \times 3 \times 2 = 16.$$

Ex 1. $\iiint z\,dx\,dy\,dz$.

R → region in the 1st octant ($x\geq 0, y\geq 0, z\geq 0$) cut by the plane $x+y+z=1$

Find intercepts of the plane $x+y+z=1$ with $x, y, z$ axes.




Tetrahedron, sides are AOB, BOC, AOC, ABC

p, q constants

$$\int_{z=p}^{q}\int_{y=g_1(z)}^{g_2(z)}\int_{x=f_1(y,z)}^{x=f_2(y,z)} f(x,y,z)\,dx\,dy\,dz = \underbrace{\iint}_{D_{yz}}\int_{x=f_1(y,z)}^{f_2(y,z)} f(x,y,z)\,dx\,dy\,dz$$

$D_{yz}$ = projection of the solid on $yz$ plane.

$$\int_{x=a}^{b}\int_{y=g_1(x)}^{y=g_2(x)}\int_{z=h_1(x,y)}^{z=h_2(x,y)} f(x,y,z)\,dz\,dy\,dx = \underbrace{\iint}_{D_{xy}}\int_{z=h_1}^{z=h_2} f(x,y,z)\,dz\,dy\,dx$$

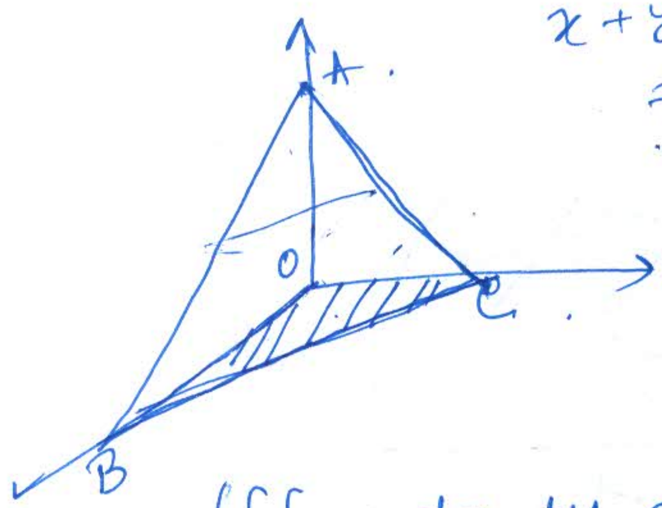
$D_{xy}$ = projection of R on $xy$ plane.

$$\int_{z=c}^{d}\int_{x=f_1(z)}^{x=f_2(z)}\int_{y=g_1(z,x)}^{y=g_2(z,x)} f(x,y,z)\,dy\,dx\,dz = \underbrace{\iint}_{D_{zx}}\int_{y=g_1}^{y=g_2} f\,dy\,dx\,dz.$$

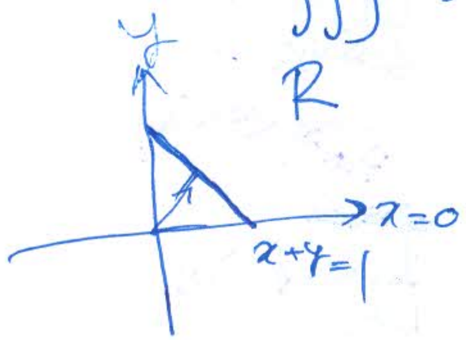
$D_{zx}$ = projection of R on $zx$-plane

$x+y+z=1$

$z=1-x-y$



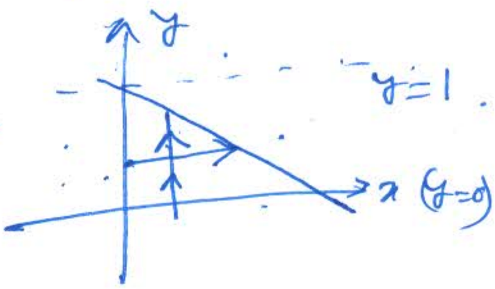

$$\iiint_R z\,dx\,dy\,dz = \iint_{D_{xy}} \int_{z=0}^{1-x-y} z\,dz\,dx\,dy.$$




$D_{xy}$ → projection of the tetrahedron on xy-plane (at $z=0$).

$D_{xy}$ is obtained by putting $z=0$ in $1-x-y=z$. $D_{xy}$: $x+y \leq 1$.

$$I = \int_{y=0}^{1}\int_{x=0}^{1-y} \left[\frac{z^2}{2}\right]_0^{1-x-y} dx\,dy$$

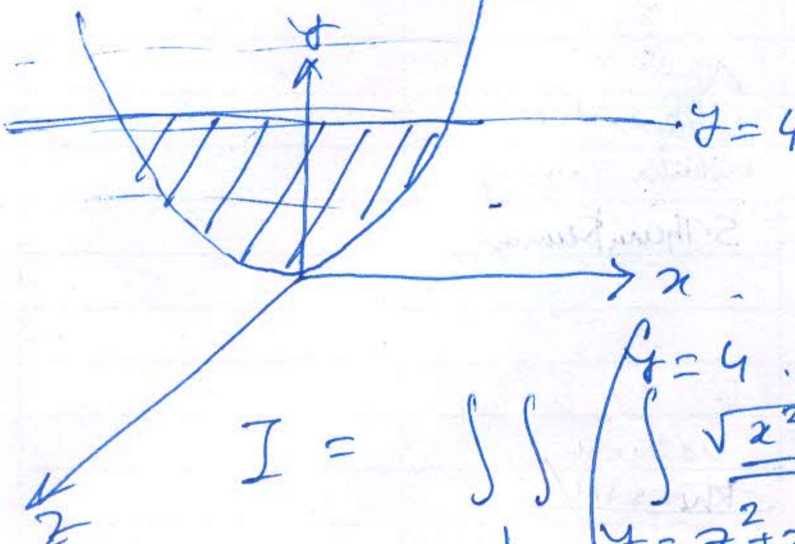



find value =

Ex. Compute

$$\iiint_R \sqrt{x^2+z^2}\, dx\, dy\, dz$$

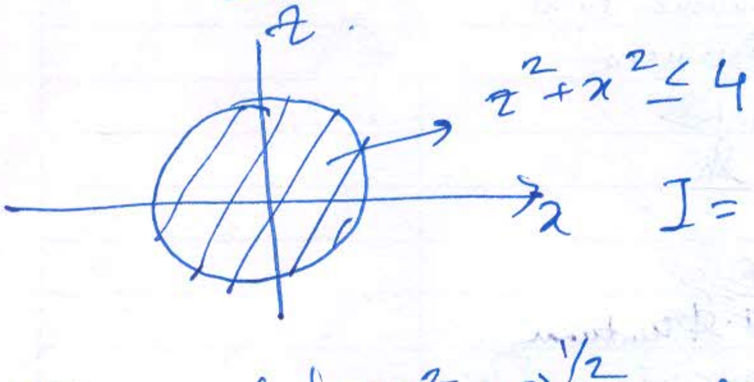
$R$ = portion of the parabolod $y = z^2 + x^2$ for $\underline{y \le 4}$.




$$I = \iint \left( \int_{y=z^2+x^2}^{y=4} \sqrt{x^2+z^2}\, dy \right) dz\, dx.$$

$D_{zx}$ = projection of the paraboloid on th $zx$ plane at the level $y=4$

$D_{zx}: z^2+x^2 \le 4$.




$$I = \iint_{D_{zx}: z^2+x^2 \le 4} \sqrt{x^2+z^2}\left(4 - z^2 + x^2\right) dz\, dx.$$

$$I = \iint_{D_{zx}} 4(z^2+x^2)^{1/2}\, dz\, dx - \iint_{D_{zx}} (z^2+x^2)^{3/2}\, dz\, dx.$$

$x = r\cos\theta,\ z = r\sin\theta,\ |J| = r.$

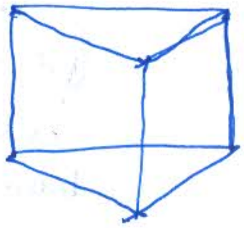
$$I = \int_{\theta=0}^{2\pi}\int_{r=0}^{2} 4r \times r\, dr\, d\theta - \int_{\theta=0}^{2\pi}\int_{r=0}^{2} r^3 \times r\, dr\, d\theta = \frac{128\pi}{15}.$$

Ex.

$\iiint z\,dx\,dy\,dz$

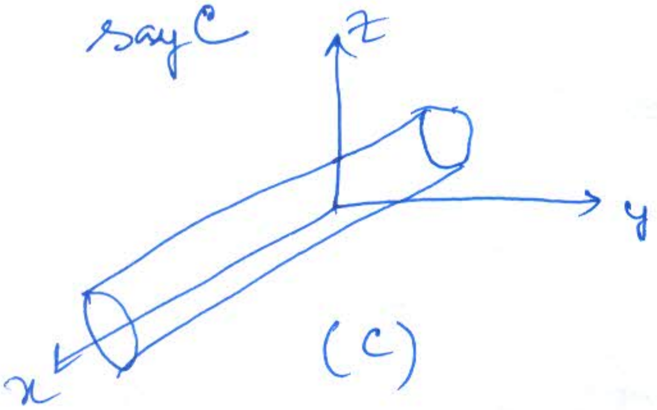
$R$ = region bounded by $x \geq 0$, $z \geq 0$, $y \geq 3x$, $y^2+z^2 \leq 9$.

$y \geq 3x$ implies an infinite wedge shaped solid ($W$, say).



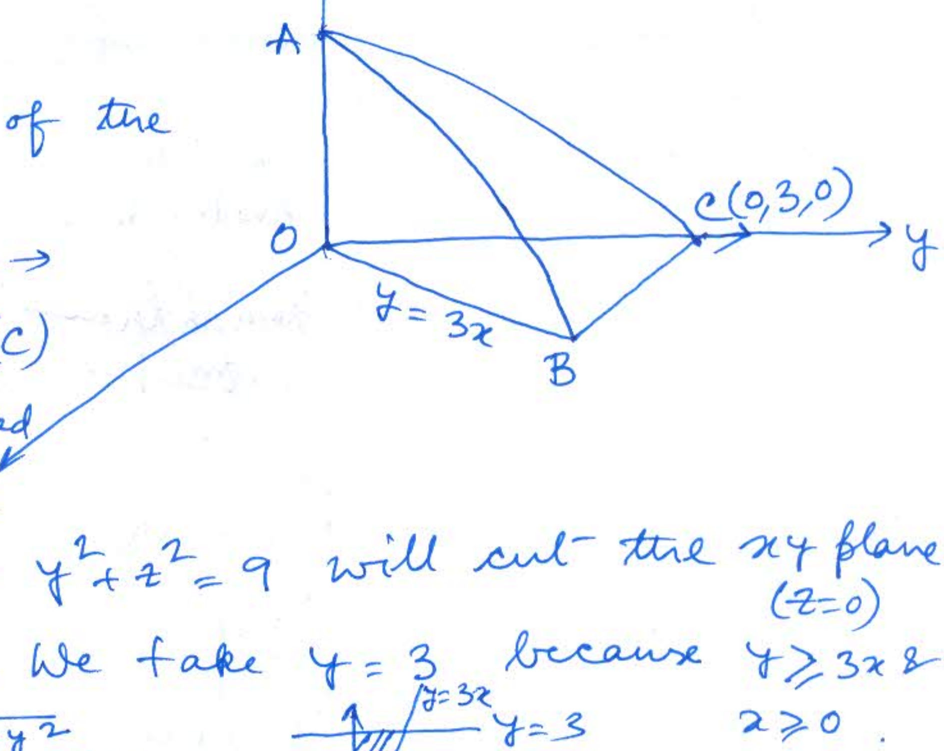
(W)

$y^2+z^2 \leq 9$ implies an infinite cylinder with axis in $x$-direction, say $C$




(C)

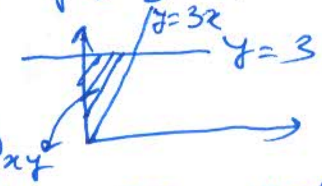
The intersection of $W$, $C$ and the regions $x \geq 0$, $z \geq 0$ is a solid region given below:




ABC → surface of the the cylinder.

AOB and AOC → $\cap$ of cylinder (C) and wedge-shaped solid (W).

The cylinder $y^2+z^2=9$ will cut the $xy$ plane ($z=0$) at $y = \pm 3$. We take $y=3$ because $y \geq 3x$ & $x \geq 0$.




$$\therefore I = \iint_{D_{xy}} \int_{z=0}^{\sqrt{9-y^2}} z\,dz$$

$D_{xy}$ ⟶ projection of the solid on $xy$-plane.

$$= \int_{x=0}^{1}\int_{y=3x}^{3} \frac{9-y^2}{2}\,dy\,dx = \int_{y=0}^{3}\int_{x=0}^{y/3} \frac{9-y^2}{2}\,dx\,dy.$$