Integral Calculus.

Lecture - 13
Thursday
2/3/17.

- Improper Integrals.
- Betā Gamma Function.
- Multiple integrals.
- Differentiation under integral sign.

Improper integrals

$\int_a^b f(x)\,dx \rightarrow$ proper if

1) both a, b are finite.
2) $f(x)$ is defined & bounded in $[a,b]$ & has no infinite discontinuity in $[a,b]$.

$\int_a^b f(x)\,dx \rightarrow$ improper if

either a or b or both a, b are infinite.

$f(x) = \frac{1}{x}$ has ∞ discont. at $x = 0$.

$$f(x) = \begin{cases} x, & 0 \leq x < 1 \\ 1-x, & 1 \leq x \leq 2. \end{cases}$$

$f(x)$ has a discontinuity at $x = 1$,

$\lim_{x \to 1-} f(x) = 1, \quad \lim_{x \to 1+} f(x) = 0. \quad f(1) = 0.$

proper $\Leftarrow \int_0^2 f(x)\,dx = \int_0^1 x\,dx + \int_1^2 (1-x)\,dx$

$f(x) = 1, \int_{-1}^{0} f(x)\,dx = \int_{-1}^{2} \frac{1}{x}\,dx \rightarrow$ improper.

All the improper integrals can be classified into 4 types.

Type I. $\int_a^{\infty} f(x)\,dx$; $f(x)$ is cont. in $[a,\infty)$ $\qquad \int_0^{\infty}\frac{dx}{1+x^2}$.

$f(x) \in C[a,\infty)$

Type II. $\int_{-\infty}^{b} f(x)\,dx$, $f(x) \in C[-\infty, b]$ $\qquad \int_{-\infty}^{2} e^x\,dx$.

Type III $\int_{a^+}^{b} f(x)\,dx$; $f(x) \in C(a,b]$ $\qquad \int_1^2 \frac{dx}{\sqrt{x-1}}$

$\lim_{x\to a^+} f(x)$ does not exist.

Type IV $\int_a^{b^-} f(x)\,dx$; $f(x) \in C[a,b)$ $\qquad \int_1^2 \frac{dx}{x\sqrt{x-2}}$.

$\lim_{x\to b^-} f(x)$ does not exist.

$$\int_{-\infty}^{\infty}\frac{dx}{x(x-1)} = \underbrace{\int_{-\infty}^{-1}\frac{dx}{(x-1)x}}_{\text{Type II}} + \underbrace{\int_{-1}^{0^-}\frac{dx}{x(x-1)}}_{\text{Type IV}} + \underbrace{\int_{0^+}^{1/2}\frac{dx}{x(x-1)}}_{\text{Type III}} + \underbrace{\int_{1/2}^{1^-}\frac{dx}{x(x-1)}}_{\text{Type IV}}$$

$$+ \underbrace{\int_{1^+}^{2}\frac{dx}{x(x-1)}}_{\text{Type III}} + \underbrace{\int_{2}^{\infty}\frac{dx}{x(x-1)}}_{\text{Type I}}.$$

Type I. $I = \int_a^{\infty} f(x)\,dx$ is said to be convergent to the value α, if $\lim_{B\to\infty}\int_a^B f(x)\,dx$ exists & its value $=\alpha$.

If $\lim_{B\to\infty}\int_a^B f(x)\,dx = \pm\infty$ / oscillatory, then I is said to be divergent.

Type II $\int_{-\infty}^{b} f(x)\,dx$ is said to be convergent & has the value β,

if $\lim_{A \to \infty} \int_{A}^{b} f(x)\,dx$ exists and $= \beta$.

If $\lim_{A \to \infty} \int_{A}^{b} f(x)\,dx$ doesn't exist, then, $\int_{-\infty}^{b} f(x)\,dx$ diverges.

~~Type~~ Ex-1. $I = \int_{a}^{\infty} \frac{dx}{x^p}$.

$$I = \lim_{B \to \infty} \int_{a}^{B} \frac{dx}{x^p} = \lim_{B \to \infty} \left[\frac{x^{-p+1}}{-p+1} \Bigg|_{a}^{B} \right]$$

$$= \lim_{B \to \infty} \left[\frac{B^{1-p} - a^{1-p}}{1-p} \right].$$

Case I. $p > 1$. Let, $p - 1 = m$, $m > 0$.

$$I = \lim_{B \to \infty} \left[\frac{B^{-m} - a^{-m}}{-m} \right] = \lim_{B \to \infty} \frac{1}{m} \left[\frac{1}{a^m} - \cancel{\frac{1}{B^m}} \right] = 0.$$

$$= \frac{1}{(p-1)\,a^{p-1}}. \qquad = \frac{1}{m\,a^m}$$

$\therefore$ I converges to $\frac{1}{(p-1)\,a^{p-1}}$.

Case II. $p = 1$. $I = \int_{a}^{\infty} \frac{dx}{x} = \lim_{B \to \infty} \int_{a}^{B} \frac{dx}{x}$.

$$\therefore I = \lim_{B \to \infty} \left[\ln x \Big|_{a}^{B} \right] = \lim_{B \to \infty} \left[\ln B - \ln a \right] = \infty.$$

$\therefore I = \int_{a}^{\infty} \frac{dx}{x^p}$ diverges when $p = 1$.

Case III $p < 1$, let $1-p = q$, $q > 0$.

$$I = \lim_{B\to\infty}\left[\frac{B^{1-p} - a^{1-p}}{1-p}\right] = \lim_{B\to\infty}\left[\frac{B^{q} - a^{q}}{q}\right] \to \infty$$

$\therefore\ I = \int_a^\infty \frac{dx}{x^p}$ diverges when $p < 1$.

$\therefore\ \int_a^\infty \frac{dx}{x^p}$ diverges if $p \le 1$
converges is $p > 1$

$\int^{\infty} \frac{\sin x}{x}\,dx$. proper.

$= \int_0^1 \frac{\sin x}{x}\,dx + \int_1^\infty \frac{\sin x}{x}\,dx$

$\int_a^\infty f(x)\,dx$, $a > 0$. $\cancel{\int_0^a f(x)\,dx}$

Ex-2. $I = \int_0^\infty \cos x\,dx$.

$$I = \lim_{B\to\infty} \int_0^B \cos x\,dx.$$

$$= \lim_{B\to\infty}\left[\sin x \Big|_0^B\right] = \lim_{B\to\infty} \sin B$$

does not exist.

So, $I = \int_0^\infty \cos x\,dx$ does not exist.

Ex-3. $I = \int_{-\infty}^{\infty} e^{-|x|}\,dx$.

$$= \int_{-\infty}^{0} e^{-|x|}\,dx + \int_0^\infty e^{-|x|}\,dx.$$

$$= \int_{-\infty}^{0} e^{x}\,dx + \int_0^\infty e^{-x}\,dx.$$

$$= e^x\Big|_{-\infty}^{0} + e^{-x}\Big|_{\infty}^{0}$$

$= 1 - \cancel{e^{-\infty}} = 0$
$+ 1 - \cancel{e^{-\infty}} = 0$
$= 2$.

Comparison tests.

$$\int_0^{\pi} \frac{\sin x \, dx}{1+x^3}$$

1) Inequality test.

(a) $f(x), g(x), h(x)$ are continuous in $a \le x < \infty$.

If $0 \le f(x) \le g(x)$ in $a \le x < \infty$,

then $\int_a^\infty f(x)\,dx$ converges if $\int_a^\infty g(x)\,dx$ converges.

If $0 \le h(x) \le f(x)$ in $a \le x < \infty$,

then $\int_a^\infty f(x)\,dx$ diverges if $\int_a^\infty h(x)\,dx$ diverges.

2) Limit test :

a) $f(x), g(x)$ are continuous in $a \le x < \infty$.

b) $f(x) \ge 0$, $g(x) > 0$ in $a \le x < \infty$.

Suppose $\lim_{x\to\infty} \frac{f(x)}{g(x)} = l$.

l

- l is finite $\ne 0$. $\int_a^\infty f\,dx$ & $\int_a^\infty g\,dx$ converge or diverge together.
- $l = 0$: if $\int_a^\infty g\,dx$ converges, $\int_a^\infty f\,dx$ converges.
- $l = \infty$: if $\int_a^\infty g\,dx$ diverges, $\int_a^\infty f\,dx$ diverges.

3) μ- test -

If $\lim\limits_{x\to\infty} x^{\mu} f(x) = l$ and if

l

$l \neq 0$, finite
if $\mu > 1$, $\int_a^\infty f\,dx$ converges
if $\mu \leq 1$, $\int_a^\infty f(x)$ diverges.

$l = 0$
if $\mu > 1$, $\int_a^\infty f\,dx$ converges.

$l = \infty$
if $\mu \leq 1$, $\int_a^\infty f\,dx$ diverges.

Test whether

Ex 1. $\displaystyle\int_2^\infty \frac{x^2\,dx}{\sqrt{x^7+1}}$ converges or diverges.

Way 1 $f(x) = \dfrac{x^2}{\sqrt{x^7+1}}$, $x \in [2, \infty)$.

$$\sqrt{x^7+1} > \sqrt{x^7} \Rightarrow \frac{1}{\sqrt{1+x^7}} < \frac{1}{x^{7/2}}.$$

$$\therefore \frac{x^2}{\sqrt{1+x^7}} < \frac{x^2}{x^{7/2}} = \frac{1}{x^{\frac{7}{2}-2}} = \frac{1}{x^{3/2}}.$$

$$f(x) < g(x) = \frac{1}{x^{3/2}}.$$

$\displaystyle\int_2^\infty \frac{dx}{x^{3/2}}$ converges. & so $\displaystyle\int \frac{x^2\,dx}{\sqrt{1+x^7}}$ converges.

Way 2. $$f(x) = \frac{x^2}{\sqrt{x^7+1}} = \frac{x^2 \cdot 1}{x^{7/2}\sqrt{1+\frac{1}{x^7}}}$$

$$= \frac{1}{x^{3/2}\sqrt{1+\frac{1}{x^7}}} = \underbrace{\frac{1}{x^{3/2}}}_{g(x)} \cdot \frac{1}{\sqrt{1+\frac{1}{x^7}}}.$$

$$\therefore \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x^7}}} = 1.$$

$\therefore \int_2^\infty f\,dx$ & $\int_2^\infty g\,dx$ converge / diverge together.

$\therefore \int_2^\infty \frac{dx}{x^{3/2}}$ converges, $\therefore \int_2^\infty \frac{x^2\,dx}{\sqrt{x^7+1}}$ converges.

Way 3. μ-test:

$$\lim_{x\to\infty} x^\mu f(x) = \lim_{x\to\infty} x^\mu \cdot \frac{x^2 \cdot 1}{x^{7/2}\sqrt{1+\frac{1}{x^7}}}$$

$$= \lim_{x\to\infty} \cancel{x^{3/2}} \cdot \frac{1}{\cancel{x^{3/2}}} \cdot \frac{1}{\sqrt{1+\frac{1}{x^7}}} = 1.$$

$\therefore$ $\because \mu > 1$ & $l = 1 \neq 0$,

$\therefore \int_2^\infty \frac{x^2\,dx}{\sqrt{x^7+1}}$ converges.

2. $\int_2^\infty \frac{x^3\,dx}{\sqrt{x^7+1}}$ $\quad f = \frac{x^3}{\sqrt{x^7+1}} = \frac{x^3 \cdot 1}{x^{7/2}\sqrt{1+\frac{1}{x^7}}}.$

$$\therefore f = \frac{1}{x^{1/2}\sqrt{1+\frac{1}{x^7}}}. \quad g = \frac{1}{x^{1/2}}$$

$$\frac{f}{g} = \frac{1}{\sqrt{1+\frac{1}{x^7}}} \to 1 \text{ as } x \to \infty.$$

$\therefore \int_2^\infty f\,dx$ & $\int_2^\infty g\,dx$ conv/div together.

but, $\int_2^\infty \frac{dx}{\sqrt{x}}$ diverges.

$\therefore \int_2^\infty \frac{x^3\,dx}{\sqrt{x^7+1}}$ diverges.

3. $$\int_1^\infty \frac{(x-1)\sqrt{x}\, dx}{1+x+x^3+\sin x}$$

$$f = \frac{(x-1)\sqrt{x}}{1+x+x^3+\sin x}$$

$$= \frac{\cancel{x^{3/2}}}{x^{3/2}} \cdot \frac{1-\frac{1}{x}}{\frac{1}{x^3}+\frac{1}{x^2}+1+\frac{\sin x}{x^3}}$$

$$\lim_{x\to\infty} \frac{\sin x}{x^3} = 0$$

$$-\frac{1}{x^3} < \frac{\sin x}{x^3} < \frac{1}{x^3}$$

$$g = \frac{1}{x^{3/2}}$$

$$\frac{f}{g} = \frac{1-\frac{1}{x}}{\frac{1}{x^3}+\frac{1}{x^2}+1+\frac{\sin x}{x^3}} \to 1 \text{ as } x\to\infty$$

& $\int_1^\infty \frac{dx}{x^{3/2}}$ converges $\therefore \int_1^\infty f\,dx$ converges.

Test of absolute convergence.

Thm. If $\int_a^\infty |f(x)|\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges where $f(x)$ changes sign over $[a, \infty)$.

$\int_1^\infty \frac{\cos x\,dx}{\sqrt{1+x^5}}$ $f = \frac{\cos x}{\sqrt{1+x^5}}$ changes sign over $[1,\infty)$.

but $|f| = \frac{|\cos x|}{\sqrt{1+x^5}}$ does not change sign in $[1,\infty)$

If $\int_a^\infty f\,dx$ converges that does not guarantee that $\int_a^\infty |f|\,dx$ converges.

Ex. $\int_1^\infty \frac{\cos x\,dx}{\sqrt{1+x^5}}$ converges because,

$\int_1^\infty \frac{|\cos x|}{\sqrt{1+x^5}}\,dx$ converges

$\sqrt{1+x^5} > \sqrt{x^5}$

$\therefore \frac{1}{\sqrt{1+x^5}} < \frac{1}{x^{5/2}}$

Pf. $\underbrace{\frac{|\cos x|}{\sqrt{1+x^5}}}_{f} \le \frac{1}{\sqrt{1+x^5}} < \underbrace{\frac{1}{x^{5/2}}}_{g}$.

& $\int_1^\infty \frac{dx}{x^{5/2}}$ converges.

$\therefore \int_1^\infty \frac{|\cos x|}{\sqrt{1+x^5}}\,dx$ converges. $\therefore$ by absolute convergence test,

$\int_1^\infty \frac{\cos x\,dx}{\sqrt{1+x^5}}$ converges.

Ex. $\int_1^\infty \frac{\sin x}{x}\,dx$ converges but $\int_1^\infty \left|\frac{\sin x}{x}\right| dx$ diverges.

Def. 1. If $\int_a^\infty |f|\,dx$ converges then $\int_a^\infty f\,dx$ converges, in that case $\int_a^\infty f\,dx$ is said to be absolutely convergent i.e when $\int_a^\infty f\,dx$ & $\int_a^\infty |f|\,dx$ both converge.

$\int_1^\infty \frac{\sin x}{x^p}\,dx \; (p>1)$, is absolutely convergent.

Def 2. If $\int_a^\infty f\,dx$ converges but $\int_a^\infty |f|\,dx$ diverges, then $\int_a^\infty f\,dx$ is called conditionally convergent.

Ex, $\int_1^\infty \frac{\sin x}{x}\,dx$ is conditionally convergent.

Type III, IV integrals.

D. V. Widder. - Advanced Calculus.

Apostol Calculus - Vol. I.

Problems - Piskunov, Shanti Narayan.

Engineering Math Books.

Type III, IV integrals

- $\int_{a^+}^{b} f(x)\,dx$, $f(x) \in C(a, b]$; $\lim_{x\to a^+} f(x)$ does not exist.

$\int_{a^+}^{b} f(x)\,dx$ converges to α_1,
if $\lim_{\epsilon\to 0^+} \int_{a+\epsilon}^{b} f(x)\,dx$ exists & has the value α_1.

- $\int_{a}^{b^-} f(x)\,dx$, $f(x) \in C[a, b)$, $\lim_{x\to b^-} f(x)$ does not exist.

$\int_{a}^{b^-} f(x)\,dx$ converges to β_1,
if $\lim_{\epsilon\to 0^+} \int_{a}^{b-\epsilon} f(x)\,dx$ exists & has the value β_1.

- $\int_{a}^{b} f(x)\,dx$, $f(x)$ is discontinuous (∞ discontinuity) at $x=c$.

$$I = \int_{a}^{c^-} f(x)\,dx + \int_{c^+}^{b} f(x)\,dx = I_1 + I_2$$

If both of I_1 and I_2 converge, then $\int_{a}^{b} f(x)\,dx$ is said to converge.

Tests for convergence:

Type I. $\displaystyle\int_{a^+}^{b} f(x)\,dx$.

Test 1. Inequality test (comparison test).

1. If $f(x), g(x), h(x)$ are continuous in $(a,b]$

2) $0 \le f(x) \le g(x)$ in $(a,b]$,

Then, $\displaystyle\int_{a^+}^{b} f(x)\,dx$ converges if $\displaystyle\int_{a^+}^{b} g(x)\,dx$ converges.

3) $0 \le h(x) \le f(x)$ in $(a,b]$,

Then, $\displaystyle\int_{a^+}^{b} f(x)\,dx$ diverges if $\displaystyle\int_{a^+}^{b} h(x)\,dx$ diverges.

Test 2. 1. $f(x), g(x)$ are continuous in $(a,b]$,

$f(x) \geq 0$ & $g(x) > 0$ in $(a,b]$.

If $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)} = l$.

l.

- $l \neq 0$ finite. $\displaystyle\int_{a^+}^{b} f\,dx$ & $\displaystyle\int_{a^+}^{b} g\,dx$ converge/diverge together.
- $l = 0$: $\displaystyle\int_{a^+}^{b} f\,dx$ converges if $\displaystyle\int_{a^+}^{b} g\,dx$ converges
- $l = \infty$. $\displaystyle\int_{a^+}^{b} f\,dx$ diverges if $\displaystyle\int_{a^+}^{b} g\,dx$ diverges.

Test 3. μ Test -

If $\operatorname*{Lt}_{x \to a} (x-a)^{\mu} f(x) = l$.

l.

- $l \neq 0$, finite. $\int_{a^+}^{b} f\,dx$ converges if $\mu < 1$, diverges if $\mu \geq 1$
- $l = 0$. $\int_{a^+}^{b} f\,dx$ converges if $\mu < 1$
- $l = \infty$. $\int_{a^+}^{b} f\,dx$ diverges, if $\mu \geq 1$.

$$I = \int_a^b \frac{dx}{(x-a)^{\mu}}.$$

$$= \operatorname*{Lt}_{\epsilon \to 0^+} \int_{a+\epsilon}^{b} \frac{dx}{(x-a)^{\mu}} = \operatorname*{Lt}_{\epsilon \to 0^+} \left[\frac{(x-a)^{-\mu+1}}{-\mu+1} \right]_{a+\epsilon}^{b}$$

$$= \operatorname*{Lt}_{\epsilon \to 0^+} \left[\frac{(b-a)^{1-\mu} - \epsilon^{1-\mu}}{1-\mu} \right].$$

Case 1 $\mu < 1$, $1 - \mu = \gamma$

$$\therefore I = \operatorname*{Lt}_{\epsilon \to 0^+} \left[\frac{(b-a)^{\gamma} - \epsilon^{\gamma}}{\gamma} \right] = \frac{(b-a)^{1-\mu}}{1-\mu}.$$

Case 2. $\mu = 1$. $I = \int_a^b \frac{dx}{x-a}$.

$$= \operatorname*{Lt}_{\epsilon \to 0^+} \int_{a+\epsilon}^{b} \frac{dx}{x-a} = \operatorname*{Lt}_{\epsilon \to 0^+} \Big[\ln|x-a| \Big]_{a+\epsilon}^{b}$$

$$= \operatorname*{Lt}_{\epsilon \to 0^+} \Big[\log|b-a| - \ln|\epsilon| \Big] \to \text{limit does not exist}.$$

Case 3. $\mu > 1$, ~~put~~ $\mu - 1 = \delta$, $\delta > 0$.

$$I = \lim_{\epsilon \to 0^+} \left[\frac{(b-a)^{-\delta} - \epsilon^{-\delta}}{-\delta} \right]$$

$$= \lim_{\epsilon \to 0^+} \frac{1}{\delta} \left[\frac{1}{\epsilon^{\delta}} - \frac{1}{(b-a)^{\delta}} \right] \qquad \lim_{x \to 0} \frac{1}{x^{\delta}} = \infty.$$

$\frac{1}{\epsilon^{\delta}} \to \infty$.

$\therefore$ limit does not exist.

$\displaystyle\int_{a+}^{b} \frac{dx}{(x-a)^{\mu}}$ converges if $\mu < 1$.
diverges if $\mu \geq 1$.

$\displaystyle\int_{a}^{b-} \frac{dx}{(b-x)^{\mu}}$ converges if $\mu < 1$
diverges if $\mu \geq 1$.

Cauchy - Principal value integrals

$$I = \int_{-1}^{1} \frac{dx}{x} = \int_{-1}^{0^-} \frac{dx}{x} + \int_{0^+}^{1} \frac{dx}{x}.$$

$$= \lim_{\epsilon \to 0^+} \int_{-1}^{0-\epsilon} \frac{dx}{x} + \lim_{\delta \to 0^+} \int_{0+\delta}^{1} \frac{dx}{x}.$$

$$= \lim_{\epsilon \to 0^+} \Big[\ln|x|\Big]_{-1}^{-\epsilon} + \lim_{\delta \to 0^+} \Big[\ln|x|\Big]_{\delta}^{1}.$$

$$= \lim_{\epsilon \to 0^+} \Big[\ln|-\epsilon| - \underbrace{\ln|-1|}_{=0}\Big] + \lim_{\delta \to 0^+} \Big[\underbrace{\ln|1|}_{0} - \ln|\delta|\Big].$$

$$= \lim_{\epsilon \to 0^+} \Big[\ln \epsilon\Big] - \lim_{\delta \to 0^+} \ln|\delta|.$$

$\because \lim_{\epsilon \to 0^+} \ln \epsilon$ does not exist

& $\lim_{\delta \to 0^+} \ln \delta$ " " "

$\therefore \int_{-1}^{1} \frac{dx}{x}$ " " " in ordinary sense.

But, if we take $\epsilon = \delta$.

Then, $I = \lim_{\epsilon \to 0} \ln \epsilon - \lim_{\epsilon \to 0} \ln \epsilon = 0$.

We say that the integral converges in the Cauchy sense & the value of the integral is known as Cauchy principal value.

Note. $\delta = 2\epsilon$

$$I = \lim_{\epsilon \to 0^+} \ln \frac{\epsilon}{2\epsilon} = \ln \frac{1}{2}.$$