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MATHEMATICS FOR UPSC CSE MAINS

ODE PART 2

"Ordinary Differential Equation"

Degree and Order :-

Eg. $$\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^5 + y = 5x$$

Here Order = 2 ; Degree = 3

Solutions of Differential equations :-

1. By variable - seprable
2. By Homogenous equation i.e. $y = vx$
3. Linear equation

i.e. $\frac{dy}{dx} + Py = Qx$

$$IF = e^{\int P dx}$$

$$\Rightarrow \boxed{y.(IF) = \int Q.(I.f)\, dx}$$

Official :-

1. Exact Differential Eqⁿ :-

$Mdx + Ndy = 0$ But $Mx + Ny \neq 0$

and $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

Sol.ⁿ $\int M dx + \int N dy = 0$

↑ "exclude terms Containing x"

2. I.F (Integrating factor) found by inspection :-

(i) $y\,dx + x\,dy = d(xy)$

(ii) $\frac{y\,dx - x\,dy}{y^2} = d(x/y)$

iii) $d(\tan^{-1} y/x) = \frac{x\,dy - y\,dx}{x^2+y^2}$

iv) $d\left(\log\left(\frac{y}{x}\right)\right) = \frac{x\,dy - y\,dx}{xy}$

v) $d(\log xy) = \frac{x\,dy + y\,dx}{xy}$

3. I.F. for a Homogeneous Equations :-

$Mdx + Ndy = 0$ But $Mx + Ny \neq 0$

then $I.F = \frac{1}{Mx + Ny}$

4. I.F. for $f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$

$I.F. = \frac{1}{Mx - Ny}$

5. for $Mdx + N\,dy = 0$

i) $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ then $I.F. = e^{\int f(x)dx}$

ii) $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y)$ then $I.F. = e^{\int f(y)dy}$

Orthogonal trajectory :-

i) Cartesian ↓ Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

ii) Polar $\frac{dr}{d\theta} = -r^2 \frac{d\theta}{dr}$

After doing differentiation w.r.t "x" or "r".

③

Linear Differential Equations :-

→ Dependent variable & its derivative ⇒ Ist degree

Operator $D = \frac{d}{dx}$

→ [Sol.^{n} ⇒ Complete Sol.^{n} = CF + PI]

$\boxed{A.E = 0}$

Working Rules :-

① All roots Real :-

i.e. $(D-m_1)(D-m_2)y = 0$

$D = m_1, m_2$

$CF = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

② Two roots real & equal :-

i.e. $D = m, m$

$CF = (c_1 + c_2 x) e^{mx}$

③ If two roots of the A.E are imaginary,

$m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$

$CF = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$

$C_1 = c_1 + c_2$ & $i(c_1 - c_2) = C_2$

④

④ If two pairs of imaginary roots be equal,

$m_1 = m_2 = \alpha + i\beta$; $m_3 = m_4 = \alpha - i\beta$

$CF = e^{\alpha x}[(c_1 x + c_2)\cos\beta x + (c_3 x + c_4)\sin\beta x)$

$\frac{1}{f(D)}$ operator :

1) $f(D)y = X \Rightarrow \boxed{y = \frac{1}{f(D)} X}$

2) $\frac{1}{D} X = \int X \, dx$

3) $\frac{1}{D-a} X = e^{ax} \int X e^{-ax} dx$

Rules for finding the particular integral :-

$P.I = \frac{1}{f(D)} X$

1.) $X = e^{ax}$

$\Rightarrow PI = \frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)}$ if $f(a) \neq 0$

Now if $f(a) = 0$ then

$\left(x \frac{e^{ax}}{f'(a)}\right) \Leftarrow f'(D)$ then put $f'(a)$, if $f'(a) \neq 0$

2.) $X = \sin(ax+b)$

$PI = \frac{1}{f(D)} \sin(ax+b)$

put $(D^2 = -a^2)$

3.) $X = x^m$

$PI = \frac{1}{f(D)} x x^m$ simplify.

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Que:- $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x e^x \sin x$

Sol.^{n}

AE: $(D^2 - 2D + 1)y = 0$

$\Rightarrow (D-1)^2 y = 0 \Rightarrow D = 1, 1$

$CF = (xC_1 + C_2)e^x$

$PI = \frac{1}{(D-1)^2} x e^x \sin x$

$= e^x \frac{1}{((D+1)-1)^2} x \sin x$

$= e^x \left(\frac{1}{D^2}(x \sin x)\right)$

$= e^x \cdot \frac{1}{D} \int x \sin x \, dx$

$= e^x \cdot \frac{1}{D}\left[x(-\cos x) - (1)(-\sin x)\right]$

$= -e^x \int x \cos x\,dx + \int -\sin x \, dx$

$= -e^x\left[x \sin x + \cancel{\cos x} + \cancel{\cos x}\right]$

$= -e^x(x \sin x + 2\cos x)$

$$\boxed{CS = (C_1 x + C_2)e^x - e^x(x \sin x + 2\cos x)}$$ Ans

(2015): 10 marks

$x \cos x \frac{dy}{dx} + y(x \sin x + \cos x) = 1$

Sol.^{n}

$\frac{dy}{dx} + y\left(\tan x + \frac{1}{x}\right) = \frac{\sec x}{x}$

$\rightarrow$ This is a linear equation of for

$\frac{dy}{dx} + Py = Q$

$\Rightarrow IF = e^{\int \tan x + \frac{1}{x} dx} = e^{\log \sec x + \log x}$

$= e^{\log x \sec x}$

$= \boxed{x \sec x}$

Sol.$^n \Rightarrow$ $y \times x \sec x = \int \cancel{x} \sec x \times \frac{\sec x}{\cancel{x}} dx$

$xy \sec x = \int \sec^2 x \, dx$

$xy \sec x = \tan x + C$

$\boxed{xy = \sin x + C \cos x}$ Ans

Q (2015) Solve this differential equation :-
10 marks

$(2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0$

Solution :- $\uparrow M$ $\qquad \uparrow N$

$\frac{\partial M}{\partial y} = 2xy^4e^y + 8xy^3e^y + 6xy^2 + 1$

$\frac{\partial N}{\partial x} = 2xy^4e^y - 2xy^2 - 3$

Now $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{\cancel{2xy^4e^y} - 2xy^2 - 3 - \cancel{2xy^4e^y} - 8xy^3e^y - 6xy^2 - 1}{y(2xy^3e^y + 2xy^2 + 1)}$

$$= \frac{-8xy^3e^y - 8xy^2 - 4}{y(2xy^3e^y + 2xy^2+1)}$$

$$= \frac{-4\,\cancel{(2xy^3e^y + 2xy^2+1)}}{y\,\cancel{(2xy^3e^y + 2xy^2+1)}} = \frac{-4}{y}$$

Now $\quad I.f = e^{-\int \frac{4}{y}dy} = e^{-4\log y}$

$$= \frac{1}{y^4}$$

Now multiply by $\left(\frac{1}{y^4}\right)$; we get

$$\Rightarrow \int\left(2x\,e^y + \frac{2x}{y} + \frac{1}{y^3}\right)dx + \int\left(x^2e^y - \frac{x^2}{y^2} - \frac{3x}{y^4}\right)dy = 0$$

Neglect terms containing x

$$\Rightarrow \boxed{x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = c} \quad \text{Ans.}$$

Q(2015) : 12 Marks :

Find the constant "a" so that $(x+y)^a$ is the I.F. of $(4x^2+2xy+6y)\,dx + (2x^2+9y+3x)\,dy = 0$ and hence solve the differential equations.

Sol.^{n} Since I.F. is given as $(x+y)^a$ so on multiplication with it ; we get a exact differential equation.

So $M = (x+y)^a\,(4x^2+2xy+6y)$ &

$N = (x+y)^a\,(2x^2+9y+3x)$

$$\frac{\partial M}{\partial y} = (x+y)^a(2x+6) + a(x+y)^{a-1}(4x^2+2xy+6y)$$

$$\frac{\partial N}{\partial x} = (x+y)^a(4x+3) + a(x+y)^{a-1}(2x^2+9y+3x)$$

$\because \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ (for Exact diff.^{r} eq.^{n})

$$\Rightarrow (x+y)^a\,[2x+6-4x-3] = a(x+y)^{a-1}(-2x^2+2xy+9y+3x)$$

$$\Rightarrow \cancel{(x+y)^a}\,\cancel{(3-2x)} = a\,\frac{\cancel{(x+y)^a}}{\cancel{(x+y)}} \times \cancel{(x+y)(3-2x)}$$

$$\Rightarrow \boxed{a = 1}$$

$$= \int (x+y)(4x^2+2xy+6y)\,dx + \int (x+y)(2x^2+9y+3x)\,dy$$

Exclude x terms

$$= \int (4x^3+2x^2y+6xy+4x^2y+2xy^2+6y^2)\,dx + \int 9y^2\,dy$$

$$= \left[x^4 + \frac{2x^3y}{3} + 3x^2y + \frac{4x^3}{3}y + x^2y^2 + 6xy^2 + 3y^3 + c\right] \text{Ans.}$$

Q(2014): 20 Marks : $x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 8y = 65\cos(\log x)$

Sol.^{n} Put $\quad x\frac{d}{dx} = D \quad$ and $x = e^z \Rightarrow z = \log x$

$$x^3\frac{d^3}{dx^3} = D(D-1)(D-2) = D^3 - 3D^2 + 2D$$

$$x^2\frac{d^2}{dx^2} = D(D-1) = D^2 - D$$

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$$\Rightarrow (D^3 - \cancel{3D^2} + \cancel{2D} + \cancel{3D^2} - \cancel{3D} + \cancel{D} + 8)y = 65\cos z$$

$$\Rightarrow (D^3 + 8)y = 65\cos z$$

$\therefore$ AE $\Rightarrow$ $D^3 + 8 = 0 \Rightarrow D^3 + 2^3 = 0$

$$(D+2)(D^2 - 2D + 4) = 0$$

$$\Rightarrow D = -2 \text{ and } D = \frac{2 \pm \sqrt{-12}}{2}$$

$$\Rightarrow D = -2 \ \& \ D = 1 \pm i\sqrt{3}$$

$$CF = c_1 e^{-2z} + (c_2 \cos\sqrt{3}z + c_3 \sin\sqrt{3}z)e^{z}$$

and $CF = c_1 x^{-2} + x\,(c_2 \cos(\sqrt{3}\log x) + c_3 \sin\sqrt{3}(\log x))$

$$PI = \frac{65}{D^3+8}\cos z = \frac{65}{D^2(D)+8}\cos z = \frac{65}{-D+8}\cos z$$

$$= \frac{65}{8-D}\cos z = \frac{65(8+D)}{8^2-D^2} = \frac{65}{8^2-(-1)}(8+D)\cos z$$

$$= \frac{\cancel{65}}{\cancel{65}}[8\cos z - \sin z] = 8\cos(\log x) - \sin(\log x)$$

Complete Sol.^{n} $= \left[c_1 x^{-2} + x[c_2 \cos(\sqrt{3}\log x) + c_3 \sin(\sqrt{3}\log x)] + 8\cos(\log x) - \sin(\log x)\right]$ Ans.

Q(2013): 13 Marks: Orthogonal Trajectory

$$r^n = a\sin n\theta \quad \text{---} \ (1)$$

Sol.^{n}

$$\cancel{n}\, r^{n-1}\frac{dr}{d\theta} = a\cancel{n}\cos n\theta \quad \text{---} \ (2)$$

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dividing eq.(1)/eq.(2) ; we get

$$\frac{r^n}{r^{n-1}\cdot\left(\frac{dr}{d\theta}\right)} = \tan n\theta \Rightarrow r = \tan n\theta \cdot \frac{dr}{d\theta}$$

$\rightarrow$ replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$

$$\Rightarrow \cancel{r} = -r^{\cancel{2}}\tan n\theta\frac{d\theta}{dr} \Rightarrow -\frac{dr}{r} = \int \tan n\theta\, d\theta$$

$$\Rightarrow -\log r = \frac{\log\sec n\theta}{n} + \log c$$

$$\Rightarrow -n\log r = \log\sec n\theta + \log c^n$$

$$\Rightarrow \log r^{-n} = \log c^n \sec n\theta$$

$$\Rightarrow r^{-n} = c^n \sec n\theta$$

$$\Rightarrow r^n = c^{-n}\cos n\theta \Rightarrow \boxed{r^n = b\cos n\theta} \text{ Ans.}$$

Q(2013): 10 Marks:- $(5x^3 + 12x^2 + 6y^2)\,dx + 6xy\,dy = 0$

Sol.^{n} $M = 5x^3 + 12x^2 + 6y^2$; $N = 6xy$

$$\frac{\partial M}{\partial y} = 12y \quad ; \quad \frac{\partial N}{\partial x} = 6y$$

Now $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \dfrac{12y - 6y}{6xy} = \dfrac{6y}{6xy} = \dfrac{1}{x}$

$$IF = e^{\int\frac{1}{x}dx} = e^{\log x} = x$$

$$= \int (5x^4 + 12x^3 + 6xy^2)\,dx + \int 6x^2 y\,dy = 0$$
exclude x term

$$= \boxed{x^5 + 3x^4 + 3x^2y^2 + C}$$ Ans

Que:- (2013) Method of variation of Parameters
10 Marks.

$$\frac{d^2y}{dx^2} + a^2 y = \sec ax$$

Sol.^{n} $(D^2+a^2)y = \sec ax \leftarrow X$

Now A.E $D^2 + a^2 = 0 \Rightarrow D = \pm ia$

so $\boxed{Cf = c_1 \underset{\uparrow y_1}{\cos ax} + c_2 \underset{\uparrow y_2}{\sin ax}}$

$$PI = -y_1 \int \frac{y_2 X}{W}\,dx + y_2 \int \frac{y_1 X}{W}\,dx$$

$$= -\cos ax \int \frac{\sin ax \cdot x \cdot \sec ax}{a}\,dx + \sin ax \int \frac{\cos ax \sec ax\,dx}{a}$$

$$\because W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & +a\cos ax \end{vmatrix} = a$$

$$= -\frac{\cos ax}{a} \int \tan ax\,dx + \frac{\sin ax}{a} \int dx$$

$$= \frac{\cos ax}{a^2} (\log \cos ax) + \frac{x}{a} \sin ax$$

Now $CS = Cf + PI$ Ans

Que (2013): 15 Marks

find the general solution of the equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \log x \sin(\log x)$$

Sol.^{n} $(\theta(\theta-1) + \theta + 1)y = z \sin z$ $\quad \because e^z = x \Rightarrow z = \log x$

$$(\theta^2 - \cancel{\theta} + \cancel{\theta} + 1)y = z \sin z$$

$$(\theta^2 + 1)y = z \sin z$$

$AE = D^2 + 1 = 0 \Rightarrow (Cf = c_1 \cos z + c_2 \sin z)$
$\Rightarrow D = \pm i$

$$PI = \frac{1}{D^2+1} z \sin z = \text{I.P. of } \frac{1}{D^2+1} z e^{iz}$$

$$= \text{I.P. of } e^{iz} \frac{1}{(D+i)^2+1} z = \text{I.P. of } e^{iz} \frac{1}{2iD\left(1+\frac{D}{2i}\right)} z$$

$$= \text{I.P. of } \frac{e^{iz}}{2i} \frac{1}{D} \left(1+\frac{D}{2i}\right)^{-1} z = \text{I.P. of } \frac{e^{iz}}{2i} \frac{1}{D}\left[\left(1-\frac{D}{2i}\right)z\right]$$

$$= \text{I.P. of } \frac{e^{iz}}{2i} \frac{1}{D}\left(z - \frac{1}{2i}\right) = \text{I.P. of } \frac{e^{iz}}{2i} \int z - \frac{1}{2i}\,dz$$

$$= \text{I.P. of } \frac{e^{iz}}{2i}\left(z^2 + \frac{iz}{2}\right) = \text{I.P. of } -\frac{e^{iz}}{2}\left(z^2 + i\frac{z}{2}\right)$$

$$= \text{I.P. of } \frac{-1}{2}(\cos z + i \sin z)\left(z^2 + i\frac{z}{2}\right)$$

$$= \frac{-z^2}{4} \cos z + \frac{z}{4} \sin z = -\frac{1}{4}(\log x)^2 \cos(\log x) + \frac{1}{4}\log x \cdot \sin(\log x)$$

$\boxed{C.S = Cf + PI}$ Ans

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Q(2012) : 20 Marks $\quad y''' - y'' = 12x^2 + 6x$

Sol.^{n} $\quad (D^3 - D^2)y = 12x^2 + 6x$

$AE = D^3 - D^2 = 0 \Rightarrow D^2(D-1) = 0$

$D = 0, 0, 1$

$CF = (c_1 x + c_2)e^{0x} + c_3 e^x$

$PI = \frac{1}{D^2(D-1)}(12x^2 + 6x)$

$= \frac{-1}{D^2}\left[(1-D)^{-1}(12x^2 + 6x)\right]$

$= \frac{-1}{D^2}\left((1 + D + D^2 + \dots)(12x^2 + 6x)\right)$

$= \frac{-1}{D^2}\left[12x^2 + 6x + 24x + 6 + 24\right]$

$= \frac{-1}{D^2}\left[12x^2 + 30x + 30\right]$

(14)

$= \frac{-1}{D}\int 12x^2 + 30x + 30\, dx = \frac{-1}{D}\left[4x^3 + 15x^2 + 30x\right]$

$= -(x^4 + 5x^3 + 15x^2)$

$CS = CF + PI = c_1 x + c_2 + c_3 e^x - (x^4 + 5x^3 + 15x^2)$

Ans.

Q (2012) : 12 Marks Orthogonal Trajectory

$$x^2+y^2 = ax$$

Sol.^{n}

$$2x + 2y\frac{dy}{dx} = a$$

$$\Rightarrow 2x + 2y\frac{dy}{dx} = \frac{x^2+y^2}{x}$$

No Need.

Now Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$$2x - 2y\frac{dx}{dy} = \frac{x^2+y^2}{x} \Rightarrow 2x^2 - 2xy\frac{dx}{dy} = x^2+y^2$$

$$\Rightarrow x^2-y^2 = 2xy\frac{dx}{dy} \Rightarrow \frac{dy}{dx} = \frac{2xy}{x^2-y^2}$$

writing this in polar coordinates :

$$x = r\cos\theta \ ; \ y = r\sin\theta$$

$$\Rightarrow x^2+y^2 = ax$$

$$\Rightarrow r^2 = ar\cos\theta \Rightarrow r = a\cos\theta$$

$$\Rightarrow \frac{dr}{d\theta} = -a\sin\theta \Rightarrow -r^2\frac{d\theta}{dr} = -r\tan\theta$$

$$\Rightarrow r\frac{d\theta}{dr} = \tan\theta \Rightarrow \int \cot\theta \, d\theta = \int \frac{dr}{r}$$

$$\Rightarrow \log\sin\theta = \log cr \Rightarrow cr = \sin\theta$$

$$\Rightarrow \boxed{r = b\sin\theta}$$

Now $r^2 = rb\sin\theta$

$$\Rightarrow \boxed{x^2+y^2 = by} \quad \text{Ans.}$$

Q (2012) : 12 Marks Solve

$$\frac{dy}{dx} = \frac{2xy\,e^{(x/y)^2}}{y^2\left(1+e^{(x/y)^2}\right) + 2x^2e^{(x/y)^2}}$$

Sol.^{n}

Put $x = vy \Rightarrow \frac{dx}{dy} = v + y\frac{dv}{dy}$

or $\boxed{v = x/y}$

$$\Rightarrow v + y\frac{dv}{dy} = \left(\frac{2ve^{v^2}}{1+e^{v^2}+2v^2e^{v^2}}\right)^{-1}$$

$$\Rightarrow y\frac{dv}{dy} = \left(\frac{2ve^{v^2}}{1+e^{v^2}+2v^2e^{v^2}}\right)^{-1} - v = \frac{1+e^{v^2}+2v^2e^{v^2}}{2ve^{v^2}} - v$$

$$\Rightarrow y\frac{dv}{dy} = \frac{1+e^{v^2}+\cancel{2v^2e^{v^2}}-\cancel{2v^2e^{v^2}}}{2ve^{v^2}}$$

$$\Rightarrow \int \frac{2ve^{v^2}}{1+e^{v^2}}\,dv = \int\frac{dy}{y}$$

Put $1+e^{v^2} = t$

$$e^{v^2}\times 2v\,dv = dt$$

$$\int\frac{dt}{t} = \int\frac{dy}{y} \Rightarrow \log t = \log cy$$

$$t = cy$$

$$\Rightarrow \boxed{1+e^{(x/y)^2} = cy} \quad \text{Ans.}$$

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y is a function of x, such that the differential coefficient $\frac{dy}{dx}$ is equal to $\cos(x+y)+\sin(x+y)$. Find out a relation between x and y, which is free from any derivative/differential.

UPSC
2013
10 Marks

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Q(2013): $\frac{dy}{dx} = \cos(x+y) + \sin(x+y)$

Sol.^{n}

Put $x+y = z \Rightarrow 1 + \frac{dy}{dx} = \frac{dz}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 1 \quad \text{——— (1)}$$

and $\frac{dy}{dx} = \cos z + \sin z$ ——— (2)

equating eq.(1) & eq.(2) : we get

$$\Rightarrow \frac{dz}{dx} - 1 = \cos z + \sin z$$

$$\Rightarrow \frac{dz}{dx} = 1 + \cos z + \sin z$$

$$\Rightarrow \frac{dz}{1+\cos z + \sin z} = dx$$

$$\Rightarrow \frac{dz}{1 + \frac{1-\tan^2\frac{z}{2}}{1+\tan^2\frac{z}{2}} + \frac{2\tan\frac{z}{2}}{1+\tan^2\frac{z}{2}}} = dx$$

$$\Rightarrow \frac{\sec^2\frac{z}{2}\,dz}{1 + \cancel{\tan^2\frac{z}{2}} + 1 - \cancel{\tan^2\frac{z}{2}} + 2\tan\frac{z}{2}} = dx$$

$$\Rightarrow \int \frac{\sec^2 z/2 \, dz}{2(1+\tan z/2)} = \int dx$$

Put $\tan\frac{z}{2} = t \Rightarrow \sec^2\frac{z}{2}\,dz = 2\,dt$

$$= \int \frac{\cancel{2}\,dt}{\cancel{2}(1+t)} = \int dx$$

$$\Rightarrow \log|1+t| = x + C$$

$$\Rightarrow \log\left|1 + \tan\frac{z}{2}\right| = x + C$$

Put $z = x+y$

$$\Rightarrow \boxed{\log\left|1 + \tan\frac{x+y}{2}\right| = x + C} \text{ Ans.}$$

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Q (2012): 12 Marks Solve

$$\frac{dy}{dx} = \frac{2xy\, e^{(x/y)^2}}{y^2\left(1+e^{(x/y)^2}\right) + 2x^2 e^{(x/y)^2}}$$

Sol.^{n}

Put $x = vy \Rightarrow \frac{dx}{dy} = v + y\frac{dv}{dy}$

or $\boxed{v = x/y}$

$$\Rightarrow v + y\frac{dv}{dy} = \left(\frac{2ve^{v^2}}{1+e^{v^2}+2v^2e^{v^2}}\right)^{-1}$$

$$\Rightarrow y\frac{dv}{dy} = \left(\frac{2ve^{v^2}}{1+e^{v^2}+2v^2e^{v^2}}\right)^{-1} - v = \frac{1+e^{v^2}+2v^2e^{v^2}}{2ve^{v^2}} - v$$

$$\Rightarrow y\frac{dv}{dy} = \frac{1+e^{v^2}+\cancel{2v^2e^{v^2}}-\cancel{2v^2e^{v^2}}}{2ve^{v^2}}$$

$$\Rightarrow \int \frac{2ve^{v^2}}{1+e^{v^2}}\,dv = \int \frac{dy}{y}$$

Put $1+e^{v^2} = t$

$e^{v^2} \times 2v\, dv = dt$

$\int \frac{dt}{t} = \int \frac{dy}{y} \Rightarrow \log t = \log cy$

$t = cy$

$$\Rightarrow \boxed{1 + e^{(x/y)^2} = cy}$$ Ans.

THANKS