

Multiple integrals

Lecture - 18

Thursday

23/3/17

Change in variables in
triple integrals

$$(x, y, z) \longrightarrow (u, v, w)$$

$$\iiint_{R_{xyz}} f(x, y, z) dx dy dz$$

$$= \iiint_{R'_{uvw}} F(x, y, z) |J| du dv dw$$

$$\checkmark (x, y, z) \rightarrow (r, \theta, z)$$

(cylindrical
polar coordinates)

$$\odot, (x, y, z) \rightarrow (r, \theta, \phi)$$

(spherical polar
coordinates)

$$\odot, (x, y, z) \rightarrow (u, v, w)$$

(general coordinates)

$$\downarrow J =$$

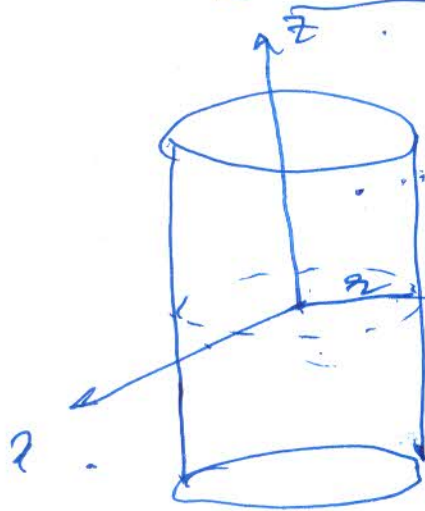
$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Transformation to cylindrical polar coordinates

$$(x, y, z) \rightarrow (r, \theta, z)$$

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z$$

$$0 \leq r \leq a \quad (a = \text{radius of cylinder}),$$

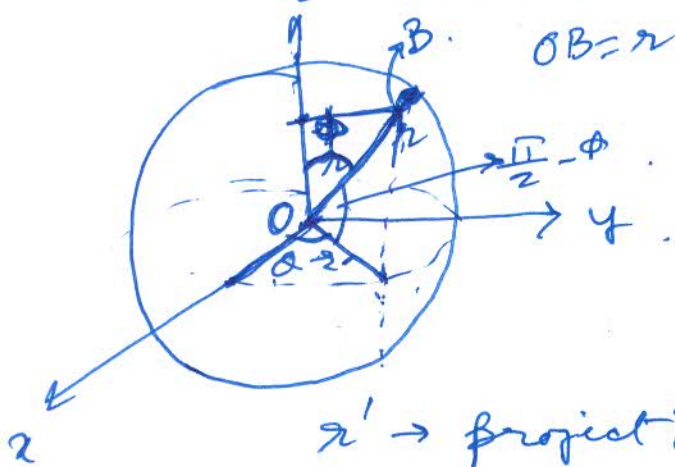


$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{vmatrix} \quad \begin{matrix} 0 \leq \theta \leq 2\pi \\ z_1 \leq z \leq z_2 \end{matrix}$$

$$J = \begin{vmatrix} \cos \theta & \sin \theta & 0 \\ -r \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

Transformation to spherical polar coordinates

$$(x, y, z) \rightarrow (r, \theta, \phi)$$



$$x = r' \cos \theta = r \sin \phi \cos \theta$$

$$y = r' \sin \theta = r \sin \phi \sin \theta$$

$$z = r \cos \phi = r \cos \phi$$

$$0 \leq r \leq a, \quad a \rightarrow \text{radius of sphere}$$

$$0 \leq \phi \leq \pi \quad \text{or} \quad -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$$

$$r' \rightarrow \text{projection of } r \text{ on } xy\text{-plane}$$

$$= r \cos \left(\frac{\pi}{2} - \phi \right) = r \sin \phi$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin \phi \cos \theta & \sin \phi \sin \theta & \cos \phi \\ -r \sin \phi \cos \theta & r \sin \phi \sin \theta & 0 \\ r \cos \phi \cos \theta & r \cos \phi \sin \theta & -r \sin \phi \end{vmatrix} = -r^2 \sin \phi$$

$$|J| = |-r^2 \sin \phi| = |-1| \cdot r^2 \cdot |\sin \phi| = r^2 |\sin \phi|$$

In general,

Use cylindrical polar, if

$$f(x, y, z) = f_1(x^2 + y^2) f_2(z)$$

Use spherical polar, if

$$f(x, y, z) = f_1(x^2 + y^2 + z^2)$$

Ex 1 Evaluate

$$\iiint (10 - x^2 - y^2 - z^2) dx dy dz$$

R : a sphere of radius 3

$$x = r \sin \phi \cos \theta, \quad y = r \sin \phi \sin \theta, \quad z = r \cos \phi$$
$$\Rightarrow x^2 + y^2 + z^2 = r^2$$
$$0 \leq r \leq 3, \quad 0 \leq \phi \leq \pi, \quad 0 \leq \theta \leq 2\pi$$

$$J = -r^2 \sin \phi$$

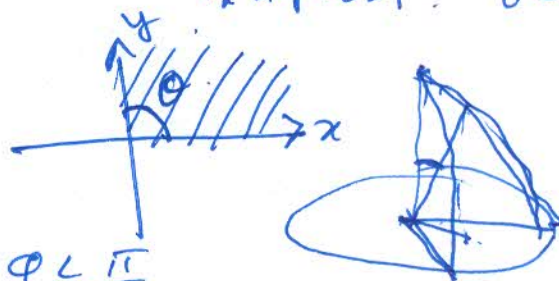
(Determine in rough)

$$I = \int_{\phi=0}^{\pi} \int_{\theta=0}^{2\pi} \int_{r=0}^3 (10 - r^2) r^2 \sin \phi dr d\theta d\phi$$

$$= \left(\int_{r=0}^3 (10 - r^2) r^2 dr \right) \left(\int_{\phi=0}^{\pi} |\sin \phi| d\phi \right) \left(\int_{\theta=0}^{2\pi} d\theta \right)$$
$$= \left[\frac{10r^3}{3} - \frac{r^5}{5} \right]_0^3 \left[\cos \phi \right]_{\pi}^0 \times 2\pi = \frac{828}{5} \pi$$

2.
$$\iiint_D \frac{dx dy dz}{\sqrt{1-x^2-y^2-z^2}}$$

D = portion of the sphere in the 1st octant.
 $x^2+y^2+z^2=1$



for 1st octant,

$$0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \phi \leq \frac{\pi}{2}$$

$$0 \leq r \leq 1$$

$$= \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \int_{r=0}^1 \frac{r^2 |\sin \phi| dr d\theta d\phi}{\sqrt{1-r^2}}$$

$$= \left(\int_0^1 \frac{r^2 dr}{\sqrt{1-r^2}} \right) \times \left(\int_{\phi=0}^{\pi/2} \sin \phi d\phi \right) \times \left(\int_{\theta=0}^{\pi/2} d\theta \right)$$

$$= \int_{t=0}^{\pi/2} \frac{\sin^2 t \cdot \cos t dt}{\sqrt{1-\sin^2 t}} \cdot \left(\cos \phi \right)_{\pi/2}^0 \times \frac{\pi}{2}$$

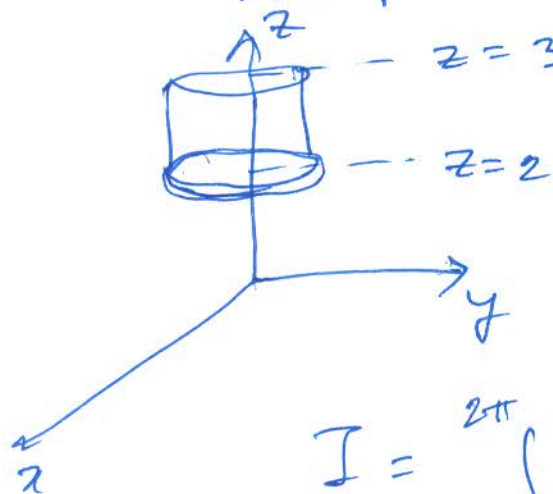
$$= \left(\frac{1}{2} \int_0^{\pi/2} 2 \sin^2 t dt \right) \frac{\pi}{2} = \frac{\pi}{4} \int_0^{\pi/2} (1 - \cos 2t) dt$$

$$= \frac{\pi}{4} \times \frac{\pi}{2} = \frac{\pi^2}{8}$$

3.

$$\iiint z(x^2 + y^2) dx dy dz$$

$R =$ portion of the cylinder $x^2 + y^2 = 1$ between $z = 2$ and $z = 3$.



$$x = r \cos \theta, \quad y = r \sin \theta, \\ x^2 + y^2 = r^2 \quad z = z$$

$$J = r \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi, \quad 2 \leq z \leq 3$$

$$I = \int_0^{2\pi} \int_2^3 \int_0^1 z r^2 \times |J| dr d\theta dz$$

$$= \left(\int_0^1 r^3 dr \right) \left(\int_2^3 z dz \right) \left(\int_0^{2\pi} d\theta \right)$$

$$= \frac{1}{4} \times \frac{3^2 - 2^2}{2} \times 2\pi = \frac{5\pi}{4}$$

4. Evaluate the integral using cylindrical polar coordinates.

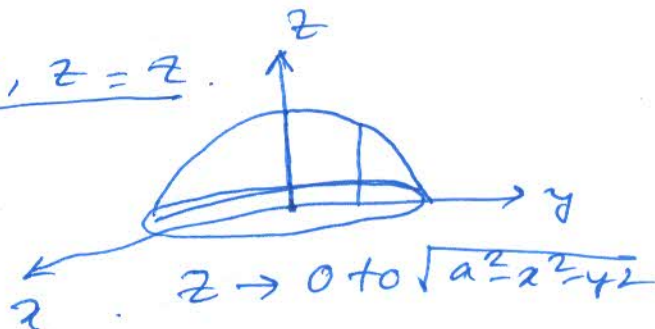
$$\iiint z dx dy dz$$

$R =$ upper hemisphere $x^2 + y^2 + z^2 = a^2$

Sol.

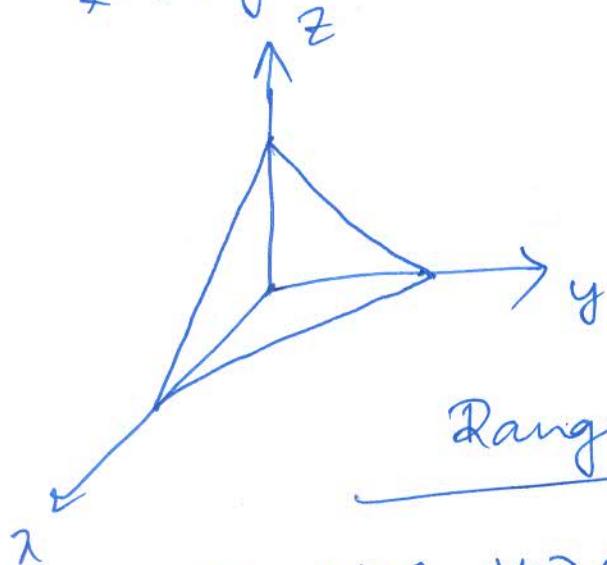
$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z$$

$$I = \int_0^{2\pi} \int_0^{a\sqrt{a^2 - z^2}} \int_0^{\sqrt{a^2 - r^2}} z r dr dz d\theta = \frac{\pi a^4}{4}$$



Ex 5 Evaluate $\iiint \left[\frac{1-x-y-z}{x+y} \right]^{1/2} dx dy dz$.

z -region bounded by the planes $\underline{x \geq 0, y \geq 0, z \geq 0, x+y+z \leq 1}$.



$$x+y+z=u \rightarrow (1)$$

$$y+z=uv \rightarrow (2)$$

$$z=uvw \rightarrow (3)$$

Ranges of u, v, w .

$$\because x \geq 0, y \geq 0, z \geq 0 \text{ \& } u = x+y+z \therefore u \geq 0$$

$$\because x+y+z \leq 1 \therefore u \leq 1 \text{ i.e. } 0 \leq u \leq 1$$

From (1) & (2)

$$v = \frac{y+z}{x+y+z}$$

$$v \geq 0$$

$$y+z \leq x+y+z$$

$$\because x \geq 0 \therefore x+y+z \geq y+z \therefore y \text{ \& } z \text{ are also } \geq 0$$

$$v = \frac{y+z}{x+y+z} \leq 1 \quad 0 \leq v \leq 1$$

From (2) & (3),

$$w = \frac{z}{y+z}$$

$$\because y \geq 0, z \geq 0$$

$$\therefore w \geq 0$$

$$\because z \geq 0, y \geq 0 \therefore y+z \geq z$$

$$w = \frac{z}{y+z} \leq 1, \quad 0 \leq w \leq 1$$

Find x, y, z as functions of u, v, w .

$$x = u - uv$$

$$y = uv - uvw$$

$$z = uvw$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1-v & v-vw & vw \\ -u & u-uw & uw \\ 0 & -uv & uv \end{vmatrix} = u^2 v.$$

$$I = \int_0^1 \int_0^1 \int_0^1 \frac{(1-u)^{1/2} \cancel{u^2} \underline{v}}{\underline{u^{1/2}(1-v)^{1/2}} \underline{u^{1/2}v^{1/2}(1-w)^{1/2}} \underline{u^{1/2}v^{1/2}w^{1/2}}} du dv dw.$$

$$= \int_0^1 \int_0^1 \int_0^1 u^{1/2}(1-u)^{1/2} (1-v)^{1/2} w^{-1/2} (1-w)^{1/2} du dv dw.$$

$$= \left\{ \int_{u=0}^1 \frac{u^{1/2}(1-u)^{1/2}}{u^{m-1}(1-u)^{n-1}} du \right\} \left\{ \int_{v=0}^1 \frac{(1-v)^{1/2}}{v^0} dv \right\} \left\{ \int_{w=0}^1 \frac{w^{-1/2}(1-w)^{1/2}}{dw} \right\}$$

$$= B\left(\frac{3}{2}, \frac{3}{2}\right) B\left(1, \frac{1}{2}\right) B\left(\frac{1}{2}, \frac{1}{2}\right)$$

$$= \frac{\Gamma\left(\frac{3}{2}\right) \cancel{\Gamma\left(\frac{3}{2}\right)}}{\Gamma(3)} \cdot \frac{\Gamma(1) \Gamma\left(\frac{1}{2}\right)}{\cancel{\Gamma\left(\frac{3}{2}\right)}} \cdot \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma(1)}$$

$$\Gamma\left(\frac{1}{2}+1\right) = \frac{\frac{1}{2} \Gamma\left(\frac{1}{2}\right)}{2 \Gamma(2)} \cdot \sqrt{\pi} \cdot (\sqrt{\pi})^2 = \frac{\pi^2}{4}.$$

Application of multiple integrals -

- ① Evaluation of area . $\iint_{D_{xy}} dx dy = \text{area of } D_{xy}$
- ② Evaluation of volume - $\iiint_R dx dy dz = \text{volume of } R$
- ③ " " " surface area .

Area -

$$\iint_{D_{xy}} dx dy = \text{area of } D_{xy}$$

D_{xy}

- ① (Using double integrals) Find the area of the region bounded between the line $y = x + 2$ & the parabola $y = x^2$

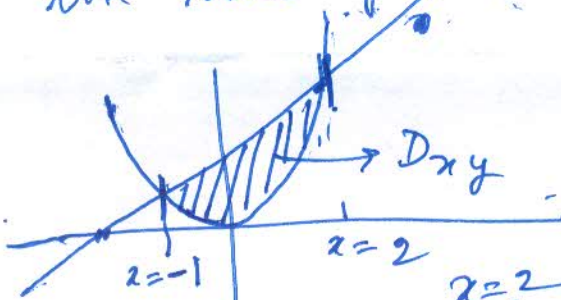
$$y = x + 2$$

$$y = x^2$$

$$x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

$$x = -1, 2$$

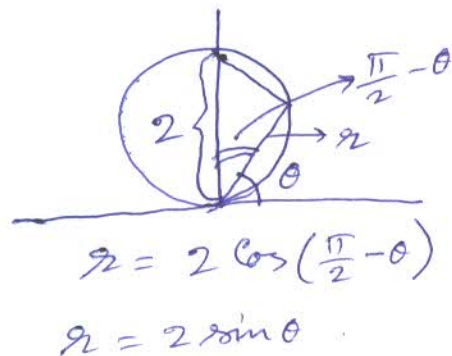
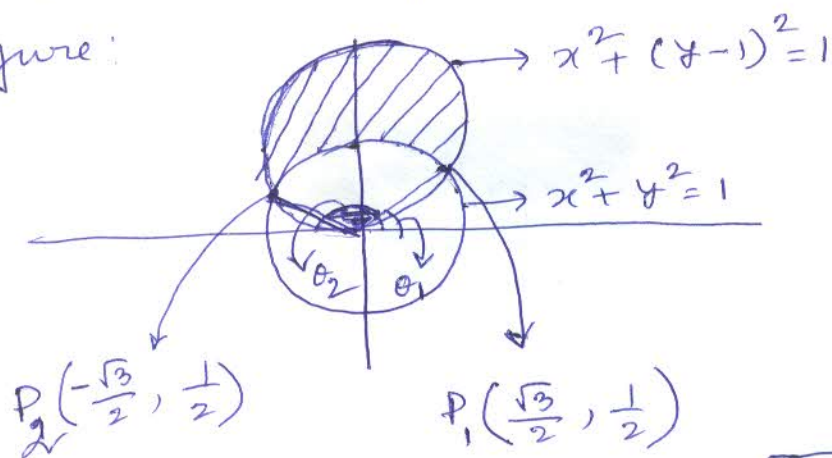


$$y = x^2$$

$$I = \iint_{D_{xy}} dx dy = \int_{x=-1}^{x=2} \int_{y=x^2}^{y=x+2} dy dx$$

$$= \int_{x=-1}^{x=2} (x+2-x^2) dx = 4.5 \text{ sq. units}$$

2. Find the area of the portion shown in the figure:



$$\theta_1 \text{ at } P_1 = \tan^{-1} \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{\pi}{6}$$

$$\theta_2 \text{ at } P_2 = \tan^{-1} \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{5\pi}{6}$$

$$\iint_{D_{xy}} dx dy = \int_{\theta=\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_{r=1}^{2\sin\theta} r dr d\theta$$

$$= \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$

Exercise Find the area bounded by $y=x$, $y=3x$, $y+x=4$ using the transformation $2x=u-v$, $2y=u+v$.

Ans 2 sq. units.