Q. Is the converse of Euler's Theorem for three variables true? Justify your answer

Soln The converse of Euler's Theorem for three variables is true i.e. if f is a real valued differentiable functions on three variables x, y, z on $S \subseteq \mathbb{R}^3$ and

$$x f_x + y f_y + z f_z = n f \text{ on } S.$$

Then, f is a homogeneous function of degree n.

Let $tx = u$, $ty = v$, $tz = w$ then for a particular x, y, z for which $(u, v, w) \in S$, we have -

$$\frac{d}{dt}\left(f(tx, ty, tz)\right) = \frac{d}{dt}\left(f(u, v, w)\right)$$

$$= f_u \cdot u_t + f_v \cdot v_t + f_w \cdot w_t$$

$$= x \frac{\partial f}{\partial u} + y \frac{\partial f}{\partial v} + z \frac{\partial f}{\partial w}$$

$$= t^{-1}\left[u \frac{\partial f}{\partial u} + v \frac{\partial f}{\partial v} + w \frac{\partial f}{\partial w}\right]$$

$$= t^{-1} n f(u, v)$$

$$\therefore \quad \frac{d}{dt} f(u,v,w) = t^{-1} n f(u,v,w)$$

$$\Rightarrow \quad \frac{d\left(f(u,v,w)\right)}{f(u,v,w)} = n t^{-1} dt$$

Integrating :

$$\log f(u,v,w) = n \log t + \log A$$

Now, for $t=1$, we have $f(x,y,z) = A$

$$\therefore \quad f(u,v,w) = t^n f(x,y,z)$$

or, $$f(tx, ty, tz) = t^n f(x,y,z)$$

So, f is a homogeneous function of degree in x, y, z.