Date : ....

NO.1 INSTITUTE FOR IAS/IFOS EXAMINATIONS

IMS™
(INSTITUTE OF MATHEMATICAL SCIENCES)

MATHEMATICS CLASSROOM TEST

2021-22

Under the guidance *of* K. Venkanna

MATHEMATICS

MODERN ALGEBRA (CLASS TEST)

Topics : Groups, Permutations & Subgroups

Date: 20 Feb.-2021

Time: 03:00 Hours **Maximum Marks: 250**

INSTRUCTIONS

1. Write your Name & Name of the Test Centre in the appropriate space provided on the right side.
2. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
3. Candidates should attempt All Question.
4. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
5. Symbols/notations carry their usual meanings, unless otherwise indicated.
6. All questions carry equal marks.
7. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
8. All rough work should be done in the space provided and scored out finally.
9. The candidate should respect the instructions given by the invigilator.
10. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any page from this booklet.

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name:

Mobile No.

Test Centre

Email.:

I have read all the instructions and shall abide by them

Signature of the Candidate

_ _

I have verified the information filled by the candidate above

Signature of the invigilator

_ _

P.T.O.

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Total Marks

1. (i) Let G be the group of all 2 × 2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c, d are integers modulo p, p a prime number, such that $ad - bc \neq 0$. G forms a group relative to matrix multiplication. What is $o(G)$?

(ii) Let H be the subgroup of the G of part (i) defined by

$$H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \,\middle|\, ad - bc = 1 \right\}.$$

What is $o(H)$? **[15]**

2. If in the group G, $a^5 = e$, $aba^{-1} = b^2$ for some $a, b, \in G$, find $o(b)$. **[10]**

3. Let G be defined as all formal symbols $x^i y^j$, i = 0, 1, j = 0, 1, 2,...., n – 1 where we assume

$x^i y^j = x^{i'} y^{j'}$ if and only if $i = i'$, $j = j'$

$x^2 = y^n = e$, $n > 2$

$xy = y^{-1} x$.

(i) find the form of the product $(x^i y^j)(x^k y^l)(x^\alpha y^\beta)$.

(ii) Using this, prove that G is a non-abelian group of order 2n.

(iii) If n is odd, prove that the center of G is (e), while if n is even the center of G is larger than (e). [This group G is known as a dihedral group.] **[15]**

4. (i) If $a \in G$, define $N(a) = \{x \in G \mid xa = ax\}$. Show that $N(a)$ is a subgroup of G. $N(a)$ is usually called the normalizer or centralizer of a in G.

(ii) If H is a subgroup of G, then by the centralizer $C(H)$ of H we mean the set $\{x \in G | xh = hx \text{ all } h \in H\}$. Prove that $C(H)$ is a subgroup of G.

(iii) Give an example of a group G and a subgroup H such that $N(H) \neq C(H)$. Is there any containing relation between $N(H)$ and $C(H)$? **[6+6+6=18]**

5. Which of the following multiplication tables defined on the set G = {a, b, c, d} form a group ? Support your answer in each case.

(i)

o	a	b	c	d
a	a	c	d	a
b	b	b	c	d
c	c	d	a	b
d	d	a	b	c

(ii)

o	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

(ii)

o	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

(iv)

o	a	b	c	d
a	a	b	c	d
b	b	a	c	d
c	c	b	a	d
d	d	d	b	c

[10]

6. (i) Let $\beta \in S_7$ and suppose $\beta^4 = (2143567)$. Find β. What are the possibilities for β if $\beta \in S_9$?

(ii) Let $\beta = (123)(145)$. Write β^{99} in disjoint cycle form. **[7+5=12]**

7. Show that the group G of four transformations f_1, f_2, f_3, f_4 defined by $f_1(z) = z$. $f_2(z) = -z$, $f_3(z) = \frac{1}{z}, f_4(z) = -\frac{1}{z}$ with composite composition is isomorphic to the permutation group G′of degree 4 consisting of the permutation I, (a b), (c d), (a b) (c d). **[15]**

8. Prove that a non-abelian group of order 10 must have a subgroup of order 5.

[10]

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9. Let $GL(2,\mathbb{R})$ be the group of all nonsingular 2 × 2 matrices over $\mathbb{R}$. Show that is a subgroup of $GL(2,\mathbb{R})$

$$H = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \in GL(2,\mathbb{R}) \mid \text{either } a \text{ or } b \neq 0 \right\}.$$ **[10]**

10 Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 5 & 1 \end{pmatrix}$ in S_5. Find a permutation γ in S_5 such that $\alpha\gamma = \beta$. **[10]**

11. Let p be an odd prime number. Show that $G = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in Z_p \right\}$ is a non-Abelian group of order p^3, under matrix multiplication, such that each nonidentity element of G has order p.

[15]

12. If G is an infinite group, what can you say about the number of elements of order 21 in a group, Generalize. **[10]**

13. (i) Suppose a and b belong to a group, a has odd order, and $aba^{-1} = b^{-1}$. Show that $b^2 = e$.

(ii) Suppose that H is a subgroup of a group G and $|H| = 10$. If a belongs to G and a^6 belongs to H, what are the possibilities for $|a|$? **[10]**

14. If G is an infinite group, what can you say about the number elements of order 8 in the group ? Generalize. **[10]**

15. (i) Let $|G| = 33$. What are the possible orders for the elements of G ? Show that G must have an element of order 3.

(ii) Let $|G| = 8$. Show that G must have an element of order 2.

(iii) Can a group of order 55 have exactly 20 elements of order 11 ? Give a reason for your answer. **[6+6+6=18]**

16. (i) Given the permutation $x = (1\ 2)\ (3\ 4)$, $y = (5\ 6)\ (1\ 3)$ find a permutation a such that $a^{-1}\ ax = y$.

(ii) Prove that there is no permutation a such that $a^{-1}\ (1\ 2\ 3)\ a = (1\ 3)(5\ 7\ 8)$

(iii) Prove that there is no permutation a such that $a^{-1}(1\ 2)a = (3\ 4)\ (1\ 5)$

[6+6+6=18]

17. Suppose G is a group that exactly eight elements of order 3, How many subgroups of order 3 does G have ? **[10]**

18. Let $\mathbf{R}^*$ be the group of nonzero real numbers under multiplication and let H = $\{x \in \mathbf{R}^* \mid x^2 \text{ is rational}\}$. Prove that H is a subgroup of $\mathbf{R}^*$. Can the exponent 2 be replaced by any positive integer and still have H be a subgroup ? **[10]**

19. Suppose G is the group defined by the following Cayley table.

	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	1	8	7	6	5	4	3
3	3	4	5	6	7	8	1	2
4	4	3	2	1	8	7	6	5
5	5	6	7	8	1	2	3	4
6	6	5	4	3	2	1	8	7
7	7	8	1	2	3	4	5	6
8	8	7	6	5	4	3	2	1

(i) Find the centralizer of each member of G.

(ii) Find (Z(G).

(iii) Find the order of each element of G. How are these orders arithmetically related to the order or the group ? **[14]**

20. Let G be a group and $a, b \in G$, such that $ab = ba$ and $O(a)$ and $O(b)$ are relatively Prime. Then Prove that $O(ab) = O(a)\,O(b)$. **[10]**

ROUGH SPACE