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MATHEMATICS FOR UPSC CSE MAINS

(CONTOUR INTEGRAL)

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"Contour Integral"

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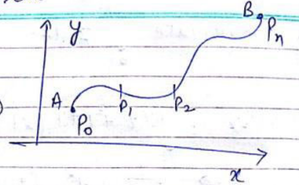
$z = x + iy$

$f(z) = u(x,y) + i\,v(x,y)$

↑ continuous.



Que:- $\int_0^{2+i} (\bar{z})^2\, dz$ along: a) The Real axis to 2 and then vertically to $2+i$

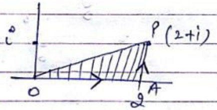
b) along the line $2y = x$.

Sol.ⁿ $z = x + iy$

$\bar{z} = x - iy \Rightarrow (\bar{z})^2 = (x - iy)^2 = x^2 + i^2y^2 - (2xy)i$

a) along Real Axis to 2 & vertically to $2+i$

$\int_{0\,(OA)}^{2} x^2\, dx$ $\quad\because$ Along OA $\;\cancel{y=0}$, $dy = 0$ $\quad\because dz = dx + i\,dy$



$I_{OA} = \frac{x^3}{3}\Big|_0^2 = 8/3$

along AP: $x = 2 \Rightarrow dx = 0$

$\Rightarrow dz = i\,dy$

$= i\int_0^1 (4 + y^2 - i4y)\,dy = i\left[4y + \frac{y^3}{3} - i\frac{4y^2}{2}\right]\Big|_0^1$

$= i\left(4 + \frac{1}{3} - i2\right) = i\left(\frac{13}{3} - 2i\right) = \frac{13}{3}i + 2$

$\Rightarrow I = I_{OA} + I_{AP} = \frac{8}{3} + \frac{11}{3}i + 2 = \frac{14}{3} + \frac{11}{3}i$ Ans.

Q. Evaluate $\int_0^{4+2i} \bar{z}\, dz$ along $z = t^2 + it$

$\Rightarrow dz = (2t + i)\,dt$

$\Rightarrow = \int \overline{(t^2 + it)}\,(2t + i)\,dt$

$\Rightarrow = \int (t^2 - it)(2t + i)\,dt$

$= \int (2t^3 + it^2 - 2it^2 + t)\,dt$

Now limits:-

as $0 < z < 4 + 2i$

$0 < t^2 + it < 4 + 2i$

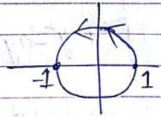
$\Rightarrow \boxed{t = 2}$

$\Rightarrow \int_0^2 \cancel{2t^3 - it^2}\,(2t^3 - it^2 + t)\,dt$

$= \frac{t^4}{2} - i\frac{t^3}{3} + \frac{t^2}{2}\Big|_0^2 = 8 - \frac{8}{3}i + 2$

$= \left(10 - i\frac{8}{3}\right)$ Ans.

Que:- Evaluate $\oint_C (z - z^2)\, dz$; where C is the upper Half of the circle $|z| = 1$. What is the value of this integral if C is lower half of the above.



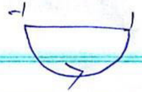
Sol. Upper Half :-

Anticlock wise:

$= \int_1^{-1} (z - z^2)\,dz = \int_{-1}^{1} z^2 - z\, dz = \frac{z^3}{3} - \frac{z^2}{2}\Big|_{-1}^{1}$

$= \left(\frac{1}{3} - \frac{1}{2}\right) - \left(\frac{-1}{3} - \frac{1}{2}\right) = \frac{2}{3}$

* LOWER HALF :-



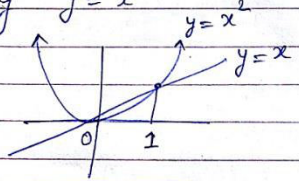
Anticlock

$$= \int_{-1}^{1} (z - z^2)\, dz$$

$$= \frac{z^2}{2} - \frac{z^3}{3} \Big|_{-1}^{1} = -2/3$$

Que:- $\int_{0}^{1+i} (x^2 - iy)\, dz$ along $y = x^2$

Sol.ⁿ



$y = x^2,$

$dy = 2x\, dx$

$\Rightarrow \quad dz = dx + i\,dy$

$dz = dx + i\, 2x\,dx = \boxed{(1 + i2x)\,dx}$

And limits $\Rightarrow \quad 0 \, x < 1$

$$= \int_0^1 (x^2 - i x^2)\,(1 + 2ix)\, dx$$

$$= (1-i) \int_0^1 (x^2 + i2x^3)\, dx$$

$$= (1-i) \left(\frac{x^3}{3} + i\frac{x^4}{2} \Big|_0^1 \right) = (1-i)\left(\frac{1}{3} + \frac{i}{2}\right)$$

$$= \left(\frac{5}{6} + \frac{1}{6} i\right) \text{ Ans}$$

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THANKS