⇒ Complex Analysis :

1) $Re(z) = \frac{z+\bar{z}}{2} = x$ $Im(z) = \frac{z-\bar{z}}{2i} = y$

$|z|^2 = z\bar{z}$

$z = x+iy = re^{i\theta}$ $\theta = \arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$

If $\theta \in (-\pi, \pi]$ it is called principal value of argument.

$|z-z_0| = r$ ⇒ circle with centre z_0 & radius r

2) ϵ- nbd of z_0 $= \{ z \in C \mid |z-z_0| < \epsilon \} = N(z_0, \epsilon)$

Deleted nbd $=$ $0 < |z-z_0| < \epsilon$

3) Bounded set : S is bdd if $\exists\ \epsilon$ st $|z| < \epsilon\ \forall z \in S$
ie, contained in some nbd of origin

Interior Point : z_1 is interior pt of S if $\exists$ a nbd of z_1 contained completely in S.

Boundary Point : All nbds of z_1 contain some pts of 'S' & some points outside of S.

Exterior Point : Neither interior nor boundary

4) Open set : $\forall\ z \in S$, z is interior pt of S

Limit Point : z_0 is lmt pt of S if every deleted nbd of z_0 contains a pt of S.
It may not lie in S.

Closed set : A set which contains all its lmt pts.

Closure of a set : Union of a set & its limit points

5) Connected Set : any 2 pts in S can be connected by a cont. curve all of whose pts lie in S.

6) In complex analysis function can be both single valued (eg $w = z^2$) or multiple valued (eg $w = z^{1/2}$)

7) Limit of a function :

$\lim_{z \to z_0} f(z) = L$ if for any $\epsilon > 0$ $\exists\ \delta > 0$ (depending on ϵ) st $|f(z) - L| < \epsilon$ whenever $0 < |z - z_0| < \delta$

if $f(z_0) = L$, then f is continuous at z_0.

Here z can approach z_0 along any path.

8) Theorem : $f(z) = u(x,y) + i\,v(x,y)$ is cont. at $z_0 = x_0 + iy_0$ iff $u(x,y)$ & $v(x,y)$ are both cont. at (x_0, y_0)

9) Differentiability :

If $f(z)$ is single valued in D, then derivative of $f(z)$ at z_0 is defd as $f'(z) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$

To prove differentiable, we have to show that $\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$ exists irrespective of the direction we approach origin

Diff $\Rightarrow$ continuous

converse not true eg $f = |z|$ at $(0,0)$.

infact $f(z) = |z|$ is no where differentiable

⇒ (1) Cauchy Reimann Equations :

If $f(z) = u + iv$ and $f'(z)$ exists at $z = x + iy$

Then CR eq^n are satisfied : $u_x = v_y$
$u_y = -v_x$

: $\boxed{\begin{array}{l} f'(z) = u_x + i v_x \\ \text{or } f'(z) = v_y - i u_y \end{array}}$ ⇒ $\boxed{f'(z) = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}}$

CR-eq^n are necessary conditions not sufficient.

eg $f(z) = \begin{cases} \frac{xy^2}{x+y^2} & z \neq 0 \\ 0 & z = 0 \end{cases}$

CR satisfied at $(x,y) = (0,0)$ but $f(z)$ is not diff. as $\lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$ doesn't exist

NOTE: (a) If u_x, u_y, v_x, v_y have possibility of getting 0 in the denominator, then they should be calculated by definition & not analytically.

eg $f(z) = z|z|$ here $u_x = \frac{2x^2+y^2}{\sqrt{x^2+y^2}}$

But for $(0,0)$ $u_x = \lim_{h\to 0} \frac{u(h,0) - u(0,0)}{h} = 0$

(b) $y = mx$ can be used to negate existence of limit.

It doesn't ensure existence of limit since it doesn't cover all the curves.

⇒ Analytic Functions :

1) $f(z)$ is analytic at $z = z_0$ if it is differentiable everywhere in a nbd of z_0. (not all)

Analyticity is region based property

$f(z)$ analytic at every point in the complex plane is called an entire function.

If $f'(z)$ doesn't exist at finite no of pts in domain it is said to be analytic & those exceptional points are called the singularities.

Regular & Holomorphic are same as Analytic.

2) Singular Point : z_0 is singular if $f'(z_0)$ doesn't exist

3) All polynomials are entire functions & $\frac{1}{1-z}$ is analytic everywhere except $z = 1$

4) NECESSARY AND SUFFICIENT CONDITION :

~~f(z)~~ is ~~analytic~~ ~~iff~~ $f(z) = u + iv$ & u, v have $\boxed{\text{continuous PD in } D}$, then

$f(z)$ is analytic in D $\Leftrightarrow$ CR eq^{ns} are satisfied.

ie, $u_x = v_y$ &
$u_y = -v_x$

∴ Continuity of u_x, u_y, v_x, v_y is very important for Necessary & sufficient conditions eg. Show that analytic $g(z)$ is independent of $\bar{z}$

$$\frac{\partial g}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}}(g(z)) = \frac{\partial}{\partial \bar{z}}(g(x+iy))$$

$$= \frac{\partial g}{\partial x} \cdot \frac{\partial x}{\partial \bar{z}} + i\frac{\partial g}{\partial y} \cdot \frac{\partial y}{\partial \bar{z}}$$

$2x = z+\bar{z} \qquad 2yi = z-\bar{z}$

$$\therefore \frac{\partial g}{\partial \bar{z}} = \frac{\partial g}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial g}{\partial y}\left(\frac{-1}{2i}\right) = \frac{1}{2}\left[\frac{\partial g}{\partial x} + i\frac{\partial g}{\partial y}\right]$$

Since g is analytic $\frac{\partial g}{\partial z} = \frac{\partial g}{\partial x} = -i\frac{\partial g}{\partial y}$

$$\therefore \frac{\partial g}{\partial \bar{z}} = 0$$

eg Show that $f(z) = \begin{cases} e^{-z^{-4}} & z \neq 0 \\ 0 & z = 0 \end{cases}$ is not analytic at $z=0$

but CR-eq^n are satisfied. Explain this.

Ans. $z = x+iy \qquad z^{-1} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$

$$z^{-4} = \frac{(x^4+y^4-6x^2y^2) - 4ixy(x^2-y^2)}{(x^2+y^2)^4}$$

$$\therefore f(z) = u+iv = e^{-\frac{(x^4+y^4-6x^2y^2)}{(x^2+y^2)^4}} \cdot e^{4ixy\frac{(x^2-y^2)}{(x^2+y^2)^4}}$$

$$\therefore u = e^{-(\ )}\cos\left(\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}\right)$$

$$v = e^{-(\ )}\sin\left(\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}\right)$$

$$u_x = \lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0}\left(\frac{e^{-x^{-4}}}{x}\right) = \lim_{x\to 0}\left(\frac{1}{xe^{1/x^4}}\right)$$

$$= \lim_{x\to 0}\left(\frac{1}{x\left(1+\frac{1}{x^4}+\frac{1}{x^8}+\dots\right)}\right) = \lim_{x\to 0}\left(\frac{1}{x+\frac{1}{x^3}+\dots}\right) = 0$$

Sly $u_y = v_x = v_y = 0$.

Now show $f(z)$ is discontinuous by approaching along $z = re^{i\pi/4}$

5) Polar Form of Cauchy - Reimann :

$$\frac{du}{dr} = \frac{1}{r}\frac{dv}{d\theta} \qquad \frac{dv}{dr} = -\frac{1}{r}\frac{du}{d\theta}$$

[using chain rule for $u(x,y)$ & $v(x,y)$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} \qquad x = r\cos\theta \quad y = r\sin\theta$$

& $$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

These equations are not valid at origin, ie, $r = 0$

6) L' Hospital Rule :

Let $f(z)$, $g(z)$ are differentiable at z_0 with

$f(z_0) = g(z_0) = 0$

If $g'(z_0) \neq 0$ then $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$

eg $f(z)$ is analytic with continuous PDs in D that excludes origin. st $\boxed{f'(z) = e^{-i\theta}\frac{\partial f}{\partial r} = \frac{1}{iz}\frac{\partial f}{\partial \theta}}$

$\because f$ is analytic $\Rightarrow f'(z) = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$

$r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= \frac{\partial f}{\partial x}\cdot\cos\theta + \frac{\partial f}{\partial y}\cdot\sin\theta$$

$$= -i\frac{\partial f}{\partial y}\cos\theta + \frac{\partial f}{\partial y}\sin\theta = -i\frac{\partial f}{\partial y}e^{+i\theta}$$

$\therefore f'(z) = e^{-i\theta}\frac{\partial f}{\partial r}$ $\quad$ Sly do for $\frac{\partial f}{\partial \theta}$.

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7) Harmonic Functions :

cont. real value fn $u(x,y)$ is harmonic if it has cont. 1^{st} & 2^{nd} order partials s.t.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow \text{La Place Equation}$$

8) If $f(z)$ is analytic then u & v are harmonic in D

u & v are called conjugate harmonic functions.

u is conjugate harmonic of $v \Leftrightarrow v$ is conjugate harmonic of u.

This gives a NECESSARY CONDITION for a fn. to be real (or img) part of any analytic function

Proof: we $f'(z) = \frac{\partial f}{\partial x}$, $f''(z) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$, $= -i\frac{\partial}{\partial y}\left(-i\frac{\partial f}{\partial y}\right) = -\frac{\partial^2 f}{\partial y^2}$, $\nabla^2 f = 0$

9) By using CR-eq^n, ie, $u_x = v_y$ & $u_y = -v_x$ to find harmonic conjugate of a given fn.

10) Simple Methods to construct $f(z)$ given $u(x,y)$ &/or $v(x,y)$:-

(a) $f(z) = u(x,y) + i\,v(x,y)$ $\quad x = \frac{z+\bar{z}}{2}$ $\quad y = \frac{z-\bar{z}}{2i}$

$$\therefore f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + i\,v\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$$

Put $\bar{z} = z \Rightarrow x = z \quad y = 0$ $\quad \Leftarrow$ MILNE-THOMPSON METHOD

$$\therefore f(z) = u(z,0) + i\,v(z,0)$$

Also $f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$

Given: $u(x,y)$ $\quad \phi_1(x,y) = \frac{\partial u}{\partial x}$ $\quad \phi_2(x,y) = \frac{\partial u}{\partial y}$

Then: $f(z) = \int [\phi_1(z,0) - i\,\phi_2(z,0)]\,dz + C$

Given $v(x,y)$: $\varphi_1(x,y) = \frac{\partial v}{\partial y}$ $\varphi_2(x,y) = \frac{\partial v}{\partial x}$

then $f(z) = \int [\varphi_1(z,0) + i\varphi_2(z,0)]\, dz + c$

eg. $u(x,y) = e^x(x\cos y - y\sin y)$

$\phi_1 = u_x = e^x(x\cos y - y\sin y) + e^x(\cos y)$

$\phi_2 = u_y = e^x(-x\sin y - \sin y - y\cos y)$

$\therefore f'(z) = \int [e^z(z\cdot 1 - 0) + e^z(1)] - i[e^z(-z\cdot 0 - 0 - 0)]\, dz + c$

$= \int e^z(z+1)\, dz + c = ze^z + c.$

(b) If $u(x,y)$ is given $f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0) + c_i$

eg $u = \sin x \cosh y$

$u\left(\frac{z}{2}, \frac{z}{2i}\right) = \sin\left(\frac{z}{2}\right)\cosh\left(\frac{z}{2i}\right) = \sin\left(\frac{z}{2}\right)\cos\left(\frac{z}{2}\right) = \frac{\sin z}{2}$

$u(0,0) = 0$

$\therefore f(z) = \sin z + c_i$

can't use directly if u becomes undefined at $\left(\frac{z}{2}, \frac{z}{2i}\right)$. So be careful

(c) Trick when $(u-v)$ or $(u+v)$ is given

$f(z) = u + iv$

$i f(z) = -v + iu$

$\Rightarrow (1-i) f(z) = (u-v) + i(u+v) = F(z)$

$= U + iV = F(z)$

Now we have Re or Img part of $F(z)$.

Apply Milne method to get $F(z)$

$\phi_1 = \frac{\partial U}{\partial x}$ $\phi_2 = \frac{\partial U}{\partial y}$

$\therefore (1-i) f(z) = F(z) = \int \phi_1(z,0) - i\phi_2(z,0)\, dz + C$

Now $f(z)$ can be found.

⇒ Complex Integration :

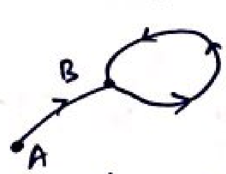
1) Different curves :

a) simple open curve

b) Simple Closed curve / Jordan curve :

c) closed but not simple :

d) Not Simple , Not Closed



∴ simple ⇒ does curve intersect with itself
closed ⇒ is initial & terminal pt same ?
ie $z(t_1) = z(t_2)$?

e) Smooth curve : $z(t) = \phi(t) + i\,\psi(t)$ has cont. derivatives in $t_1 \leq t \leq t_2$

Piecewise smooth or Contour : composed of finite no of smooth arcs.

f) Simply Connected Domain : if any simple closed curve in D can be shrunk to a pt w/o leaving D.

g) Multiply connected : any domain which is not simply connected.

h) Positive Orientation : domain lies on the left while walking on the curve.

2) If $f(z)$ is analytic at all points of $\mathbb{R}$ & $\exists$ C is a curve lying in $\mathbb{R}$, then $f(z)$ is certainly integrable along C.

if $C: \; z(t) = x(t) + i\,y(t) \qquad a \le t \le b$

then $\displaystyle\int_C f(z)\,dz = \int_a^b f(z)\,dz = \int_a^b f(z(t))\, z'(t)\, dt$

3) Parametrization of equation of a curve :

Line : $\quad C: \; z(t) = (a+bt) + i(c+dt)$

Parabola : $\quad C: \; z(t) = (a+bt) + i(c+dt^2)$

Circle $\quad C: \; z(t) = z_0 + Re^{it}$

4) Curve C can be expressed as sum of multiple curves

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \cdots + \int_{C_n} f(z)\,dz$$

where $C = C_1 + C_2 + \cdots + C_n$.

5) ARC LENGTH :

$dy^2 + dx^2 = ds^2$

$C: \; x = \phi(t) \qquad y = \psi(t) \qquad a \le t \le b$

$$L = \int_a^b \sqrt{(\phi'(t))^2 + (\psi'(t))^2}\; dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt$$

If $f(z)$ is integrable along C having length L & if $\exists$ a +ve no. M st $|f(z)| \le M$ on C

then $\left|\int f(z)\,dz\right| \le ML \quad \left[\because \left|\int f(z)\,dz\right| \le \int_C |f(z)|\,|dz| \le \int_C M\,|dz| \le ML\right.$

6) Cauchy's Fundamental Theorem :

If $f(z)$ is analytic at all points w/i & on the closed contour C, $\Rightarrow \int_c f(z)dz = 0$

7) Green's Theorem :

Let $P(x,y)$ & $Q(x,y)$ be continuous ~~partials~~ ~~in~~ ~~a~~ with continuous PDs in simply connected region R

$$\Rightarrow \int_c P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$$

[Proof using Fundamental Theorem of Calculus]

rectangular contour when C is traversed in +ve sense.

If $f(x)$ is cont. in $[a,b]$, then
$\exists$ $F(x)$ st $F'(x) = f(x)$ on $[a,b]$
with $F(b) - F(a) = \int_a^b f(x)dx$

8) Cauchy's Weak Theorem :

If $f(z)$ is analytic (with continuous derivative) in a simply connected domain D & C is a closed contour in D, then $\int_c f(z)\, dz = 0$

Proof:. simply put $f(z) = u + iv$ $dz = dx + idy$ Apply Green's theorem & ~~a~~ CR conditions ($\because$ PDs are cont.)

Corr. If C_1 & C_2 have same endpts. $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$

ie, Integral is independent of path in the domain

eg. c is $y = x^3 - 3x^2 + 4x - 1$ joining $(1,1)$ & $(2,3)$

Find $\int_c (12z^2 - 4iz)\, dz$. $\rightarrow$ directly integrate

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9) Cauchy Goursat theorem: (did away the reqt of having continuous derivative)

(I) Let $f(z)$ be continuous in D & suppose $\exists$ $F(z)$ st $F'(z) = f(z)$ in D.

Then $\int_C f(z)dz = F(z(b)) - F(z(a))$

where C is $z(t)$, $t \in [a,b]$

↳ don't confuse with FD of Calculus. There continuity is sufficient for existence of anti-derivative. Here we have merely assumed.

This is used in further theorems under Cauchy - Goursat.

(II) If $f(z)$ is analytic in simply connected domain D and C is a closed contour in D, then

$$\int_C f(z)\, dz = 0$$

(III) For multiply connected region

Let $f(z)$ be analytic in a MC domain & its boundary be C, then $\int_C f(z)dz = 0$

$\therefore$ $C = C_1 + AB + C_2 + BA$

$$\int_{AB} = -\int_{BA}$$

$\therefore$ $\int_C = \int_{C_1} + \int_{C_2} = 0$

⇒ Applications of Cauchy's Theorem :

① If $f(z)$ is analytic in domain D, then $f(z)$ will have derivatives of all orders in D.

2) Cauchy's 1^{st} Integral formula :

Let $f(z)$ be analytic in a SC domain containing simple closed curve 'C'. If z_0 is inside C, then

$$\boxed{f(z_0) = \frac{1}{2\pi i} \int \frac{f(z)}{z-z_0}\, dz}$$

Proof :

$$\int_{C-C_1} \frac{f(z)}{z-z_0} = 0$$

$$\Rightarrow \int_C \frac{f(z)}{z-z_0}\, dz = \int_{C_1} \frac{f(z)}{z-z_0}\, dz$$

$\because f(z)$ is analytic

$|f(z) - f(z_0)| < \epsilon$ whenever $|z-z_0| < \delta$

Let radius of C_1 be less than δ.

$$= \int \frac{f(z) - f(z_0) + f(z_0)}{z-z_0}\, dz$$

$$= f(z_0) \int \frac{1}{z-z_0}\, dz + \int \frac{f(z)-f(z_0)}{z-z_0}\, dz$$

$$I = \underset{f(z_0)}{2\pi i} + \underbrace{\cancel{\int \frac{f(z)-f(z_0)}{z-z_0}\, dz}}_{0 \text{ as } \epsilon \to 0}$$

$$\therefore I = 2\pi i\, f(z_0)$$

3) Cauchy's General Integral formula :

Let z_0 be inside C & let $f(z)$ be analytic in a SC domain containing C.

Then $$\boxed{f^n(z_0) = \frac{n!}{2\pi i} \int \frac{f(z)}{(z-z_0)^{n+1}}\, dz}$$

eg. $\displaystyle\int_C \frac{e^z \sin z}{(z-2)^2}\, dz \quad C : |z| = 3 \qquad I = \frac{2\pi i}{1!} f'(z_0)$

$$= 2\pi i \times \left[e^z \sin z + e^z \cos z\right]_{z=z_0}$$

$$= 2\pi i\, e^2 [\sin 2 + \cos 2]$$

eg. $\displaystyle\int_{|z|=2} \frac{z^3+3z-1}{(z-1)(z+3)}\, dz \rightarrow$ do only for $z_0 = 1$ & -3 is not inside $|z|=2$

eg $\displaystyle\int_{|z|=2} \frac{1}{z^4-1}\, dz = 0$ as $\displaystyle\frac{1}{z^4-1} = \frac{1}{(z-1)(z+1)(z-i)(z+i)}$

equal & opp. partial fraction coeffs.

4) Cauchy's Integral formula for multiply connected regions :

Let $f(z)$ be analytic in a multiply connected domain $C = C_1 \cup C_2$

If z_0 lies in C, then

$$f(z_0) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z)}{z-z_0}\,dz - \frac{1}{2\pi i} \oint_{C_2} \frac{f(z)}{z-z_0}\,dz$$

$$\because \int_{C-\Gamma} \frac{f(z)}{z-z_0} = 0$$

& $C - \Gamma = C_1 - C_2 - \Gamma$

& $\displaystyle\int_{\Gamma} \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$.

Similarly, Cauchy's General formula also remains valid inside MC Region

5) TAYLOR'S THEOREM :

Let $f(z)$ be analytic in domain D with boundary 'C'. z_0 is a pt lying inside C; then $f(z)$ can be expressed as

$$f(z) = f(z_0) + \frac{(z-z_0)\, f'(z_0)}{1!} + \frac{(z-z_0)^2 f''(z_0)}{2!} + \dots + \frac{f^n(z_0)(z-z_0)^n}{n!} + \dots$$

This series $\boxed{\text{converges for } |z-z_0| < \delta}$ where δ is the distance of z_0 from the nearest point on c.

Conversely, $f(z)$ is analytic at point z_0 iff

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \text{ in some disk } |z-z_0| \le r$$

where $$\boxed{a_n = \frac{f^n(z_0)}{n!} = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}} dz}$$

6) MORERA'S THEOREM :

Let $f(z)$ be continuous in a domain 'D' & $\int_C f(z) dz = 0$ along every simple contour contained in D.

Then $f(z)$ is analytic in D.

So its a qualified converse of Cauchy's Theorems.

$\therefore$ Continuous + integral independent of path $\Rightarrow$ Analytic

eg $\int \frac{1}{z^2} dz = 0$ But its not analytic since $\frac{1}{z^2}$ is not continuous in a closed curve containing origin

$\therefore$ continuity is needed.

7) LAURENT'S THEOREM :

Let $f(z)$ be analytic in annulus $R_1 < |z-z_0| < R_2$.

Then $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ is valid throughout the annulus.

The coefficients a_n are given by $a_n = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi$

where C is any simple closed curve in the annulus that makes a CW rotation about the point

PRINCIPAL PART : series of $-ve$ powers of $(z-z_0)$ of the Laurent Series. It converges ~~of~~ for $|z-z_0| > R_1$

ANALYTIC PART : series of $+ve$ powers of $(z-z_0)$. It converges for $|z-z_0| < R_2$

8) Important Binomial Expansions :

$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - - - -$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + - -$

$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + - -$

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + - -$

These are valid for $x < 1$ so while expanding $(1+z)$ ensure $|z| < 1$.

eg. Expand $f(z) = \dfrac{1}{(z+1)(z+3)}$ in a laurent series valid for (a) $1 < |z| < 3$ (b) $|z| > 3$ (c) $0 < |z+1| < 2$ (d) $|z| < 1$ (use partial fractions)

a) so in $1 < |z| < 3$ we expand $\dfrac{1}{1+z}$ as $\dfrac{1}{z}\left(1+\dfrac{1}{z}\right)^{-1}$ as $\dfrac{1}{|z|} < 1$

and $\dfrac{1}{z+3}$ as $\dfrac{1}{3}\left(1+\dfrac{z}{3}\right)^{-1}$ as $\dfrac{|z|}{3} < 1$

b) for $|z| > 3 > 1$ we have $\dfrac{1}{|z|} < 1$ & $\dfrac{3}{|z|} < 1$

$\therefore$ expansions are $\dfrac{1}{z}\left(1+\dfrac{1}{z}\right)^{-1}$ & $\dfrac{1}{z}\left(1+\dfrac{3}{z}\right)^{-1}$ - -

c) for $0 < |z+1| < 2$ put $z+1 = u$ & then expand, later replace u by $z+1$.

d) for $|z| < 1$ we get taylor's expansion.

eg Find Laurent expansion of $\dfrac{z^2}{z^4-1}$ for $0 < |z-i| < \sqrt{2}$

Here 1st split into partial fractions $\dfrac{A}{z-1} + \dfrac{B}{z+1} + \dfrac{C}{z-i} + \dfrac{D}{z+i}$ & expand each individually using $z - i = u$ & $\left|\dfrac{u}{1+i}\right| < 1$ $[\because |u| < \sqrt{2}]$

eg Expand $\sin z \cdot \sin\left(\dfrac{1}{z}\right)$ in a laurent series for $|z| > 0$

Directly expand sin $\Rightarrow$ $\left(z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - - -\right)\left(\dfrac{1}{z} - \dfrac{1}{3!z^3} + \dfrac{1}{5!z^5} - - -\right)$

eg. For $f(z) = \frac{2z^3+1}{z^2+z}$ find (a) Taylor series valid w/i nbd of $z=i$

(b) Laurent series valid w/i annulus $0<|z|<1$

$$f(z) = 2(z-1) + \frac{1}{z} + \frac{1}{z+1}.$$

$$= f_1(z) + f_2(z) + f_3(z)$$

(a) for $f_1(z)$ Expansion $= \cancel{\quad} f(i) + \frac{(z-i)f'(i)}{1!} + 0$

$$= 2(i-1) + (z-i)\times 2$$

for $f_2(z)$ Expansion $= \sum_{n=0}^{\infty} \frac{(-1)^n (z-i)^n}{i^{n+1}}$

for $f_3(z)$ Expansion $= \sum_{n=0}^{\infty} \frac{(-1)^n (z-i)^n}{(1+i)^{n+1}}$

(b) for $|z|<1$

$$f(z) = 2(z-1) + \frac{1}{z} + (1 - z + z^2 - z^3 + \cdots)$$

$$= \frac{1}{z} - 1 + z + z^2 - z^3 + \cdots$$

* RADIUS OF CONVERGENCE OF $\sum a_n (z-z_0)^n$: $\frac{1}{R} = \lim_{n\to\infty} \sup |a_n|^{1/n}$

$\Downarrow$

HADAMARD'S FORMULA (simple to use) $\Rightarrow$ $\boxed{\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|}$

eg for $\sum \left(1+\frac{1}{n}\right)^{n^2} z^n$ $\quad \frac{1}{R} = \lim_{n\to\infty} a_n^{1/n} = \lim \left(1+\frac{1}{n}\right)^n = e$

for $\sum \left(\frac{n!}{n^n}\right) z^n$ $\quad R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{n!\,(n+1)^n}{n^n\,(n+1)!} = \left(1+\frac{1}{n}\right)^n = e$

⇒ Singularities :

1) Singular Point : if $f(z)$ is not analytic at that point

a) Isolated Singularity : if $f(z)$ is analytic in some deleted nbd of singular point, then that is isolated singularity. eg $f(z) = \frac{\sin z}{z}$ at $z=0$

eg $\log(z)$ at $z=0$ has singularity but not isolated since every nbd of $z=0$ has $z<0$ for which $\log(z)$ is not analytic

Since in deleted nbd of z_0 $f(z)$ is analytic

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \underbrace{\sum_{n=1}^{\infty} b_n (z-z_0)^{-n}}_{\text{Principal Part}} \text{ is valid.}$$

2) Reimann's Theorem :

If $f(z)$ has isolated singularity at $z = z_0$ & is bdd in some deleted nbd of z_0 then $f(z)$ can be defined at z_0 in such a way as to be analytic at z_0.

3) Removable Singularity :

If $f(z)$ has isolated singularity at z_0, then it is removable singularity if $\lim_{z \to z_0} f(z)$ exists.

eg $f(z) = \frac{\sin z}{z}$ at $z=0$ if we define $f(0) = 1$.
$= \lim_{z \to 0} \frac{\sin z}{z}$

4) Pole of order 'n' :

If z_0 is isolated singularity for $f(z)$ & ~~lt~~ $f(z) \to \infty$ as $z \to z_0$ then $f(z)$ has pole at z_0.

Let 'n' be a +ve integer st $\lim_{z \to z_0} (z-z_0)^n f(z) = A \neq 0, \infty$.

then $f(z)$ has pole of order 'n' at z_0. $n=1 \Downarrow$ SIMPLE POLE

Essentially for pole of order n at z_0, $f(z)$ is of the form $f(z) = \dfrac{F(z)}{(z-z_0)^n}$ where $F(z)$ is analytic at z_0 & $F(z_0) \neq 0$.

If $f(z)$ has a pole of order 'k' at $z = z_0$

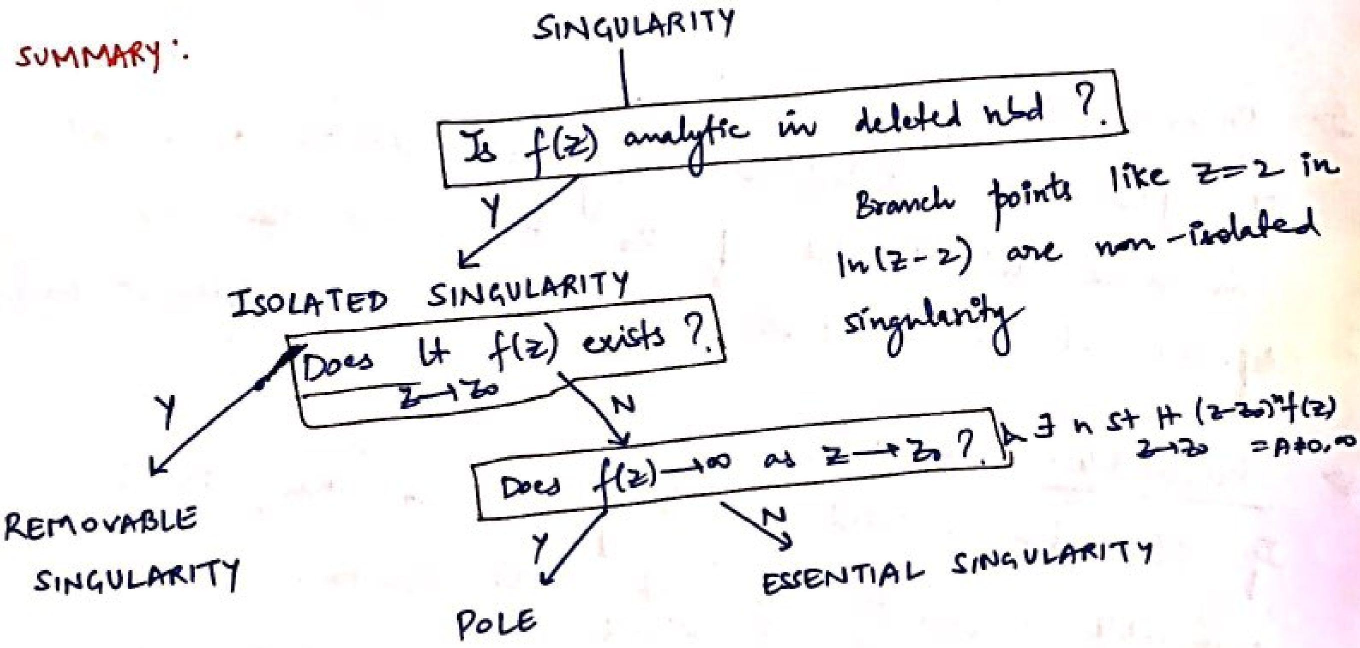
then $f(z) = \sum\limits_{n=-k}^{\infty} b_n (z-z_0)^n$

5) <u>Isolated Essential Singularity</u>:

Isolated singularity that is neither removable nor pole.

eg $e^{1/z}$ at $z=0$

SUMMARY:



6) CASORATI- WEISTRASS THEOREM:

If $f(z)$ has isolated essential singularity at $z = z_0$ then $f(z)$ comes arbitrarily close to every complex value in each deleted nbd of z_0.

ie, $\forall\ \epsilon > 0$, $\delta > 0$ & complex no w, $\exists\ z'$ st $|z'-z_0| < \delta$ & $|f(z') - w| < \epsilon$.

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7) Single valued function (ie, we can't define for specific points separately as in $\frac{\sin z}{z}$) having singularity means its neither pole or essential singularity.

8) Equivalently if we can't find $n_{(+ve)}$ st $\lim_{z \to z_0} (z-z_0)^n f(z) = A \neq 0, \infty$

then z_0 is essential singularity.

9) $\Rightarrow$ Laurent expansion has infinitely many terms in its Principal part.

9) $\boxed{\text{Singularity at } z = \infty}$

singularity of $f(z)$ at ∞ is removable, pole or essential as per the singularity of $f(1/z)$ at $z=0$.

eg $f(z) = z^2+1$ has pole of order 2 at $z=\infty$

since $f(1/z) = \frac{1}{z^2}+1$

10) Determining nature of singularity using Principal Part:

ⓐ Principal Part is zero $\Leftrightarrow$ Removable Singularity

ⓑ Principal Part is finite $\Leftrightarrow$ Pole

ⓒ Principal Part has infinite terms $\Leftrightarrow$ Essential singularity

coefficients are not evaluated by the integrals instead it is the other way around.

Use std expansions of logarithmic, trignometric & exponential functions along with binomial expansions for algebraic function.

eg Find nature & location of singularities of $f(z) = \frac{1}{z(e^z-1)}$

PT it can be expanded in the form $\frac{1}{z^2} - \frac{1}{2z} + a_0 + a_2 z^2 + \dots$

$\Rightarrow e^z - 1 = 0 \Rightarrow z = 2n\pi i$

$z = 0$ is a pole & $\lim_{z \to 0} z^2 f(z) = \lim_{z \to 0} \frac{z}{e^z - 1} = 1$

& double pole

$z = 2n\pi$ is simple pole as $\lim_{z \to 2n\pi i} \frac{(z - 2n\pi i)}{z(e^z - 1)} = \frac{1}{2n\pi i}$

Also $f(z) = \frac{1}{z(e^z-1)} = \frac{1}{z\left(z + \frac{z^2}{2!} + \dots\right)} = \frac{1}{z^2}\left(1 + \left(\frac{z}{2!} + \frac{z^2}{3!} + \dots\right)\right)^{-1}$

$= \frac{1}{z^2}\left(1 - \left(\frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} \dots\right) + \left(\frac{z^2}{4} + \frac{z^4}{36} + \frac{z^3}{6} + \dots\right)\right)$

$- \left(\frac{z^3}{8} \dots\right.$

$= \frac{1}{z^2}\left(1 - \frac{z}{2} + \frac{z^2}{12} - \dots\right)$ HP

VVI. eg types: Give singularity type at the given pt & region of convergence.

a) $\frac{e^{2z}}{(z-1)^3}$, $z = 1 \Rightarrow$ Put $z - 1 = u \Rightarrow \frac{e^2}{(z-1)^3} + \frac{2e^2}{(z-1)^2} + \dots$

$\therefore$ pole of order 3 & converges for all $z \neq 1$

b) $\frac{z - \sin z}{z^3}$, $z = 0 \Rightarrow \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \dots$

$\Rightarrow$ Removable singularity & converges for all z

c) $\frac{z}{(z+1)(z+2)}$; $z = -2 \Rightarrow$ Put $z + 2 = u \Rightarrow \frac{2}{u} - \frac{1}{u-1}$

$= \frac{2}{u} + (1-u)^{-1} = \frac{2}{u} + (1 + u + u^2 + \dots)$ [for $|u| < 1$ then only we can expand]

$\therefore$ Pole of order 1 & cgs for $0 < |u| < 1$

ie $0 < |z+2| < 1$.

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⇒ Residues :

① Residues are used for evaluating certain integrals.

From Laurent expansion we have $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$

where $a_n = \frac{1}{2\pi i} \int_c \frac{f(z)}{(z-z_0)^{n+1}} dz$

$= \frac{f^n(z_0)}{n!}$

Also $\boxed{a_{-1} = \frac{1}{2\pi i} \int_c f(z)\, dz}$

where C is a simple closed curve enclosing z_0 & contained in nbd of z_0

a_{-1} is called Residue of $f(z)$ at z_0. It is coeff of $\frac{1}{(z-z_0)}$ in Laurent expansion

& $\int_c f(z) dz = 2\pi i\, a_{-1}$

② **Residue Theorem.**

Let $f(z)$ be single valued & analytic inside & on a simple closed curve 'C' except at singularities a, b, c, .. which have residues a_{-1}, b_{-1}, c_{-1} . . .

Then $\int_c f(z)\, dz = 2\pi i\, (a_{-1} + b_{-1} + c_{-1} + \dots)$

3> If $z = z_0$ is simple pole, Residue $= \lim_{z \to z_0} (z-z_0) f(z)$

4> If $z = z_0$ is pole of order 'k', then

$$\boxed{a_{-1} = \frac{1}{(k-1)!} \lim_{z \to z_0} \frac{d^{k-1}}{dz^{k-1}} \left[(z-z_0)^k f(z) \right]}$$

DONT FORGET $\int_c$ if it is asked after finding a_{-1} !!!

5> Residue at infinity : Residue at $z=0$ in $\frac{1}{z^2} f\left(\frac{1}{z}\right)$

Residue at $z=\infty$ ⇒ $\frac{1}{2\pi i} \int_c f(z) dz$ where C is large circle containing all the finite singularities of $f(z)$ & Integral is done in Clockwise direction eg Residue of $f(z) = e^z \operatorname{cosec}^2 z$ in finite plane at all the poles

$\sin^2 z = 0 \Rightarrow z = n\pi$, each being a double pole.

$$a_{-1} = \frac{1}{(2-1)!} \lim_{z \to n\pi} \frac{d}{dz} \left(\frac{(z-n\pi)^2 e^z}{\sin^2 z} \right)$$

$$= \lim_{z \to n\pi} \frac{d}{dz} \left[e^z \cdot \left(\frac{z-n\pi}{\sin z} \right)^2 \right]$$

Now apply product rule, division rule & L'Hospital to evaluate limit we get $a_{-1} = e^{n\pi}$.

b) Residue at infinity: Residue at 0 in $\frac{1}{z^2} f(1/z)$

$$\Rightarrow \boxed{\lim_{w \to 0} \left[\frac{w f(1/w)}{w^2} \right]} = \lim_{z \to \infty} [z f(z)] \quad \left[\because \text{using pole prop } \lim_{z \to 0} z \cdot f(z) \to A \right]$$

It basically uses the fact that in extend $\mathbb{C}$ plane, sum of all residues is 0.

$\therefore$ $\sum$ Residues at finite sing. + $\sum$ Residue at $z = \infty$ = 0

$\therefore$ Residue at $z = \infty$ = $\frac{-1}{2\pi i} \int_C f(z)\, dz$ where C is very very large in ACW sense

$= \frac{1}{2\pi i} \int_{C'} f(z)\, dz$ C' is in CW sense

So residue at ∞ = residue at $w=0$ for $f(1/w)$

$$= \frac{-1}{2\pi i} \int_C f(1/w) \cdot -\frac{dw}{w^2} = \frac{1}{2\pi i} \int_C \frac{1}{w^2} f(1/w)$$

$\therefore$ Residue = $\boxed{\text{Residue at } w=0 \text{ in } F(w) = \frac{1}{w^2} f\left(\frac{1}{w}\right)}$

⇒ Evaluation of Real Finite Integrals :

1) Remember :

a) $I = \int_0^\infty f(x)\,dx$ is said to converge if $\int_0^\infty f(x)dx = \lim_{R\to\infty} \int_0^R f(x)dx$

b) $\int_{-\infty}^{\infty} f(x)dx = \lim_{R_1\to\infty} \int_{-R_1}^{0} f(x)dx + \lim_{R_2\to\infty} \int_0^{R_2} f(x)dx$

when both lmts on RHS exists, LHS is said to converge

c) Cauchy's principal value = $\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx$

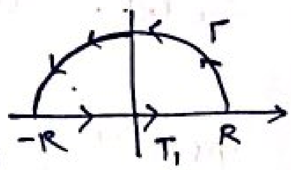
Now, if (b) cgs its value is ~~generally~~ same as (c)

But if (c) cgs (eg for $f(x)=x$, value $=0$), it might not cgs for (b) ($f(x)=x$ doesn't)

But generally we have functions where c ⇒ b, therefore we find using (c) only.

2) TYPE - I : $F(x)$ is rational function

To find $\int_{-\infty}^{\infty} F(x)dx$, consider $\int_C F(z)dz$



where C is $\Gamma + T_1$ (line joining $-R$ to R)

& Γ is semicircle $(Re^{i\theta})$

$\int_C = \int_\Gamma + \int_{T_1}$ Find $\int_C$ using residues inside C

We show $\int_\Gamma \to 0$ as $R \to \infty$

☆ RESULT : If $|F(z)| \le \frac{M}{R^k}$ for $z = Re^{i\theta}$ where $k>1$ & M are constants, then $\lim_{R\to\infty} \int_\Gamma F(z)dz = 0$ where Γ is semicircular arc of radius R

very easy as $\left|\int_\Gamma F(z)dz\right| \le \int_\Gamma |F(z)||dz| = \frac{M}{R^k}\int_\Gamma |dz| = \frac{M\pi}{R^{k-1}}$

$\therefore \lim_{R\to\infty} \frac{M\pi}{R^{k-1}} = 0$

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Remember TRICK: ST $|f(z)| \le \frac{M}{R^k}$ $k>1$ if $f(z) = \frac{1}{z^6+1}$

$$|f(z)| = \frac{1}{|z^6+1|} \le \frac{1}{|z^6|-1} = \frac{1}{R^6-1} < \frac{2}{R^6} \text{ for } R>2$$

(ie if R is large enough)

$\ll$ *

$\left(|z_1|-|z_2| \le |z_1+z_2|\right) \le |z_1|+|z_2|$

3) TYPE - II : $\int_0^{2\pi} F(\sin\theta, \cos\theta)\, d\theta$ F is rational fn of $\sin\theta$, $\cos\theta$

Put $z = e^{i\theta}$ $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$ $\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$

$I = \int_C F(z)\,dz$ where $C : |z|=1$

4) TYPE - III $\int_{-\infty}^{\infty} F(x) \cdot \begin{Bmatrix} \cos mx \\ \sin mx \end{Bmatrix} dx$ $F(x)$ is rational function

Here use $\int_C F(z)\, e^{imz}\, dz$ & pick Re or Img parts as needed. Contour remains same as Type - I.

5) TYPE - IV : They require particular case by case contours

6) TIPS / TRICKS :

a) Use odd/even fn properties & don't ignore if it is $\int_0^{\infty}$ or $\int_{-\infty}^{\infty}$.

b) Use L'Hospital rule to calculate residues eg $\lim_{z\to e^{\pi i/6}} \frac{(z-e^{\pi i/6})}{z^6+1} = \lim_{z\to e^{\pi i/6}} \frac{1}{6z^5}$

c) In type - II, when we find poles for $f(z)=0$, we get $f(z) = (z-\alpha)(z-\beta)$ (generally $f(z)$ is quad in z) To find residue do $\lim_{z\to\beta} \frac{(z-\beta)}{f(z)}$ & apply L'Hospital to avoid coefficient mistakes in factor of $f(z)$. Also only 1 of α & β lie in the contour as one of them will have $|\alpha|>1$ & from $f(z)$ we'll have $|\alpha\beta| = 1$. $\Rightarrow |\beta|<1$

d) Very impt to get $\int_0^{2\pi}$ limit in Type-2. Use. $\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$ if $f(2a-x) = f(x)$

EXAMPLES :

① TYPE - I

PT. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)^2\,(x^2+2x+2)} = \frac{7\pi}{50}$

Take $\displaystyle F(z) = \frac{z^2}{(z^2+1)^2\,(z^2+2z+2)}$

Simple poles at $z = -1+i$ & $-1-i$ — only i & $-1+i$ lies in the contour $\Gamma+\Gamma_1$

Order 2 poles at $z = +i, -i$

Find Residues at $z = -i+1$ by $\displaystyle\lim_{z \to -1+i} (z+1-i)\,F(z) = a_{-1}$

& at $z = i$ by $\displaystyle\frac{1}{(2-1)!} \lim_{z \to i} \frac{d}{dz}\Big((z-i)^2 F(z)\Big) = b_{-1}$

$\therefore \displaystyle\int_C F(z)\,dz = 2\pi i\,(a_{-1}+b_{-1}) = 7\pi/50$ (after calculation)

$\therefore \displaystyle\int_C F(z)\,dz = \int_\Gamma F(z)\,dz + \int_{-\infty}^{\infty} f(x)\,dx$

as $R \to \infty$ $\displaystyle\int_\Gamma F(z)\,dz$ can be shown to be approaching 0

② TYPE - II

ⓐ ST : $\displaystyle\int_0^{2\pi} \frac{\cos\theta}{5+4\cos\theta}\,d\theta = \frac{-\pi}{3}$ Put $z = e^{i\theta}$ & directly solve

ⓑ ST : $\displaystyle\int_0^{\pi} \frac{a\,d\theta}{a^2+\sin^2\theta} = \frac{\pi}{\sqrt{1+a^2}}$ Need limits from 0 to 2π.

So convert $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

$\Downarrow$

$\therefore$ Put $2\theta = \phi$

$\displaystyle\int_0^{2\pi} \frac{a\,d\phi}{2a^2+1-\cos\phi}$ Now put $z = e^{i\phi}$ & solve.

ⓒ Problems with $\cos n\theta$, $\sin n\theta$ get z^n in denominator making calculation at high order pole very difficult. Either use expansions to get coeff of $1/z$ as the residue in such cases or better use $\cos n\theta = \mathrm{Re}(e^{in\theta})$ & $\sin n\theta = \mathrm{Img}(e^{in\theta})$

eg PT: $\displaystyle\int_0^{2\pi} \frac{\cos^2 3\theta \, d\theta}{1-2p\cos 2\theta + p^2} = \frac{\pi(1-p+p^2)}{1-p}$ $\quad 0<p<1$

Either (a) put $z = e^{i\theta}$ & for z^5 in denominator get residue by expansion & coeff of $1/z$

or (b) $\displaystyle I = \frac{1}{2}\int_0^{2\pi} \frac{1+\cos 6\theta \, d\theta}{1-2p\cos 2\theta + p^2} = \mathrm{Re}\left(\frac{1}{2}\int_0^{2\pi} \frac{1+e^{i6\theta}}{1-2p\cos 2\theta + p^2} d\theta\right)$

Now put $e^{i\theta} = z$

$$I = \mathrm{Re}\left(\frac{1}{2}\oint_C \frac{1+z^6}{1-p\left(\frac{z^4+1}{z^2}\right)+p^2} \times \frac{dz}{iz}\right)$$

Now it's straight forward.

(d) PT: $\displaystyle\int_0^{\pi} \frac{\cos 2\theta}{1-2a\cos\theta + a^2} = \frac{\pi a^2}{1-a^2}$ $\quad |a|<1$

$f(\theta) = \dfrac{\cos 2\theta}{1-2a\cos\theta + a^2}$ $\quad f(2\pi-\theta) = f(\theta)$

$\displaystyle\therefore \int_0^{2\pi} \frac{\cos 2\theta \, d\theta}{1-2a\cos\theta+a^2} = 2\int_0^{\pi} \frac{\cos 2\theta \, d\theta}{1-2a\cos\theta+a^2}$

$\displaystyle\therefore I = \frac{1}{2}\int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2}$ Now put $z = e^{i\theta}$ & solve

(e) PT: $\displaystyle\int_0^{2\pi} e^{\cos\theta}\cos(n\theta - \sin\theta)\, d\theta = \frac{2\pi}{n!}$

$\displaystyle I = \mathrm{Re}\int_0^{2\pi} e^{\cos\theta} \cdot e^{-i(n\theta-\sin\theta)} d\theta$ it is imp^t, else we'll get $e^{e^{-i\theta}} = e^{1/z}$ which has essential singularity

$\displaystyle\therefore I = \int_0^{2\pi} e^{-ni\theta} \cdot e^{e^{i\theta}} d\theta = \oint \frac{e^z \, dz}{i z^{n+1}}$.

Residue at $z=0$ $\displaystyle\lim_{z\to 0} \frac{1}{(n+1-1)!} \frac{d^n}{dz^n}(e^z) = \frac{1}{n!}$ ....

③ Type – III

Use Jordan's Lemma Directly : $\lim_{R\to\infty} \int_\Gamma e^{imz} f(z)\,dz = 0$ as $f(z) \to 0$ when $z \to \infty$ $(m>0)$

a) If $|F(z)| \le \frac{M}{R^k}$ for $z = Re^{i\theta}$ where $k>0$ & M are constants

PT $\boxed{\lim_{R\to\infty} \int_\Gamma e^{imz} F(z)\,dz = 0}$ where Γ is semicircle of radius R & $m > 0$.

$$\left|\int e^{imz} F(z)\,dz\right| = \left|\int_0^\pi e^{imRe^{i\theta}} F(Re^{i\theta})\, Ri e^{i\theta}\, d\theta\right|$$

$$= \left|\int_0^\pi e^{imR\cos\theta - mR\sin\theta} F(Re^{i\theta})\, Ri e^{i\theta}\, d\theta\right|$$

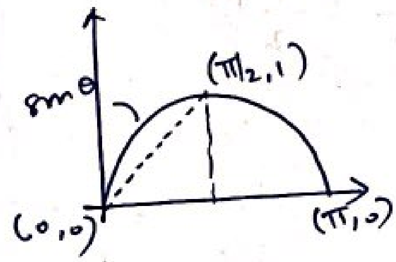
$$\le \frac{M}{R^k} \cdot R \int_0^\pi e^{-mR\sin\theta}\, d\theta \qquad \because |e^{imR\cos\theta}| = 1$$

$$= \frac{2M}{R^{k-1}} \int_0^{\pi/2} e^{-mR\sin\theta}\, d\theta \qquad \left[\because \int_0^\pi f(x)dx = 2\int_0^{\pi/2} f(x)dx \text{ if } f(x) = f(\pi - x)\right]$$

Consider :⇒ $f(\theta) = \frac{\sin\theta}{\theta}$ $\quad f'(\theta) = \cancel{\frac{\theta\cos\theta - \sin\theta}{\theta^2}}\ \frac{\theta\cos\theta - \sin\theta}{\theta^2}$

↳ it goes from 1 to $\frac{2}{\pi}$

$$= \frac{\underset{(\text{decreasing})}{\cos\theta} - \frac{\sin\theta}{\theta}}{\theta}$$



$\sin\theta \ge \frac{\theta \cdot 2}{\pi}$ $\quad \forall\, \theta \in \left[0, \frac{\pi}{2}\right]$

Directly quote as JORDAN'S INEQUALITY ⇠ $\frac{\sin\theta}{\theta} \ge \frac{2}{\pi}$ ⇒ $-\sin\theta \le \frac{-2\theta}{\pi}$

$$-mR\sin\theta \le \frac{-2mR\theta}{\pi}$$

$$\therefore I \le \frac{2M}{R^{k-1}} \int_0^{\pi/2} e^{-\frac{2mR\theta}{\pi}}\, d\theta = \frac{2M}{R^{k-1}} \times \frac{\pi}{-2mR} \left[e^{-\frac{2mR\theta}{\pi}}\right]_0^{\pi/2}$$

$$= \frac{M\pi}{mR^k}\left[1 - e^{-mR}\right]$$

$$\lim_{R\to\infty} I = 0$$

This proves I over arc is 0 as $R \to \infty$. So we can get $\int_{-\infty}^{\infty} F(z)\,dz$ by finding $\int_C F(z)\,dz$

eg. Evaluate $\int_0^\infty \frac{x \sin x}{x^2+a^2}\,dx \quad a>0$

$$\int_0^R \frac{x \sin x}{x^2+a^2} = \frac{1}{2}\int_{-R}^{R} \frac{x \sin x}{x^2+a^2}\,dx$$

Consider. $I = \int_C \frac{z \sin z}{z^2+a^2}\,dz$ for $C: \{ z = Re^{i\theta}$

$$\therefore I = \text{Img} \int_C \frac{z e^{iz}}{z^2+a^2}\,dz$$

Pole at $z = ai$ only

Residue at $z = ai \Rightarrow \lim_{z\to ai} \frac{z e^{iz}}{z+ai} = \frac{(ai)e^{-a}}{2ai} = \frac{e^{-a}}{2}$

$$\therefore I = \text{Img}\left(2\pi i \times \frac{e^{-a}}{2}\right) = \text{Img}\left(\pi e^{-a} i\right) = \pi e^{-a}$$

$$\lim_{R\to\infty} \int_0^R \frac{x \sin x}{x^2+a^2} = \frac{1}{2} \lim_{R\to\infty} \int_{-R}^{R} \frac{x \sin x}{x^2+a^2} = \frac{1}{2}\pi e^{-a}$$

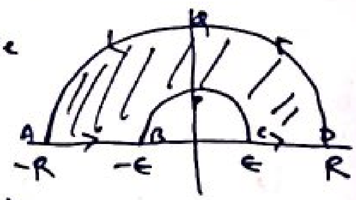
④ Type- IV Miscellaneous:

eg ST: $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$

$$= \text{Img} \int_0^\infty \frac{e^{iz}}{z}\,dz$$

We can't take Type-I contour as it contains origin which is a pole for e^{iz}/z

so we take



$C = AB + BPC + CD + DQA$

$\int_C f(z)\,dz = 0$ as it is analytic

$$0 = \int_\epsilon^R \frac{e^{iz}}{z}\,dz + \int_0^\pi \frac{e^{iz}}{z}\,dz + \int_{-R}^{-\epsilon} \frac{e^{iz}}{z}\,dz + \int_\pi^0 \frac{e^{iz}}{z}\,dz$$

$$0 = \int_\epsilon^R \frac{e^{iz} - e^{-iz}}{z}\,dz + \int_0^\pi \frac{e^{iRe^{i\theta}}}{Re^{i\theta}} \cdot iRe^{i\theta}\,d\theta + \int_\pi^0 \frac{e^{i\epsilon e^{i\theta}}\, i\epsilon e^{i\theta}\,d\theta}{\epsilon e^{i\theta}}$$

$$0 = \int_{\epsilon}^{R} \frac{e^{iz}-e^{-iz}}{z}\,dz + \int_{0}^{\pi} i e^{iRe^{i\theta}}\,d\theta + \int_{\pi}^{0} i e^{i\epsilon e^{i\theta}}\,d\theta$$

$$\Rightarrow \quad 0 = 2\int_{\epsilon}^{R} i\,\frac{\sin z}{z}\,dz + i\int_{0}^{\pi} e^{iRe^{i\theta}}\,d\theta - i\int_{0}^{\pi} e^{i\epsilon e^{i\theta}}\,d\theta$$

$$\Rightarrow \quad 2\int_{\epsilon}^{R} \frac{\sin z}{z}\,dz = \int_{0}^{\pi} e^{iRe^{i\theta}}\,d\theta - \int_{0}^{\pi} e^{i\epsilon e^{i\theta}}\,d\theta$$

$$\left|\int_{0}^{\pi} e^{iRe^{i\theta}}\,d\theta\right| \le \int_{0}^{\pi} \left|e^{iR\cos\theta - R\sin\theta}\right| d\theta \le \int_{0}^{\pi} e^{-R\sin\theta}\,d\theta$$

$$\sin\theta > \frac{2\theta}{\pi} \quad \Rightarrow \quad -R\sin\theta < -\frac{2R\theta}{\pi}$$

$$\le 2\int_{0}^{\pi/2} e^{-R\sin\theta}\,d\theta \le 2\int_{0}^{\pi/2} e^{-2R\theta/\pi}\,d\theta = \frac{2\pi}{2R}\left[1-e^{-R}\right]$$

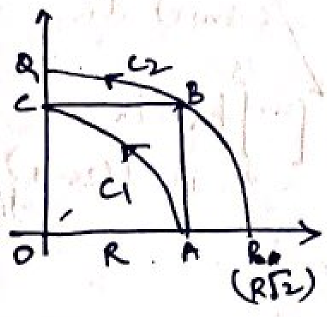
$\rightarrow 0$ as $R \rightarrow \infty$

$$\therefore \quad 2\int_{\epsilon}^{\infty} \frac{\sin z}{z}\,dz = \int_{0}^{\pi} e^{i\epsilon e^{i\theta}}\,d\theta$$

$$\epsilon \rightarrow 0 \quad \Rightarrow \quad 2I = \int_{0}^{\pi} d\theta = \pi \quad \Rightarrow \quad I = \pi/2$$

eg ST: $\int_{0}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}/2$ using contour integration

Let $I = \int_{0}^{R} e^{-x^2}\,dx \qquad I \cdot I = I^2 = \int_{0}^{R}\int_{0}^{R} e^{-(x^2+y^2)}\,dx\,dy$



This is equivalent to integrating along the side of a square of side R.

$$\int_{C_1} \le \int_{OABC} \le \int_{C_2}$$

$$\int_{0}^{\pi/2}\int_{0}^{R} e^{-r^2}\cdot r\,dr\,d\theta \le I^2 \le \int_{0}^{\pi/2}\int_{0}^{R\sqrt{2}} e^{-r^2}\,dr\,d\theta$$

$$\int_{0}^{\pi/2}\int_{0}^{R} e^{-r^2}\,r\,dr\,d\theta = \frac{1}{2}\int_{0}^{\pi/2}\int_{0}^{R^2} e^{-t}\,dt\,d\theta = \frac{1}{2}\times\frac{\pi}{2}\times\left[1-e^{-R^2}\right] = \frac{\pi}{4}\left[1-e^{-R^2}\right] = I_1$$

Sly $\displaystyle\int_{0}^{\pi/2}\int_{0}^{R\sqrt{2}} e^{-r^2}\,r\,dr\,d\theta = \frac{\pi}{4}\left[1-e^{-2R^2}\right] = I_2$

$I_1 \le I^2 \le I_2$ as $R \rightarrow \infty$ $\quad I_1 = I_2 \rightarrow \pi/4$

$\therefore \quad I^2 = \pi/4 \quad \Rightarrow \quad I = \sqrt{\pi}/2$

eg ST: $e^{\frac{1}{2}c\left(z-\frac{1}{z}\right)} = \sum_{n=-\infty}^{\infty} a_n z^n$ where $a_n = \frac{1}{2\pi}\int_0^{2\pi}\cos(n\theta - c\sin\theta)d\theta$

$e^{\frac{1}{2}c\left(z-\frac{1}{z}\right)}$ has singularity at $z=0$ & $z=\infty$, ie, it is analytic in the ring $r \le z \le R$ where r is small & R is large. Hence it can be written as a Laurent's series in the form:

$$e^{\frac{1}{2}c\left(z-\frac{1}{z}\right)} = \sum_{n=-\infty}^{\infty} a_n z^n \quad \text{where } a_n = \frac{1}{2\pi i}\int_c \frac{f(z)}{z^{n+1}}\,dz$$

$$\therefore a_n = \frac{1}{2\pi i}\int_c \frac{e^{\frac{1}{2}c(z-z^{-1})}}{z^{n+1}}\,dz$$

where c is any circle with centre as origin.

Let us take Radius $=1$ $\therefore c: |z|=1$ $\therefore z = e^{i\theta}$

$$\therefore a_n = \frac{1}{2\pi i}\int_0^{2\pi} \frac{e^{ic\sin\theta}}{e^{i(n+1)\theta}}\cdot ie^{i\theta}d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{-i(n\theta - c\sin\theta)}d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - c\sin\theta) - i\sin(n\theta - c\sin\theta)d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\cos(n\theta - c\sin\theta) - \frac{i}{2\pi}\int_0^{2\pi}\sin(n\theta - c\sin\theta)\,d\theta$$

0 $\left[\because \int_0^{2a} f(x)dx = 0 \text{ if } f(2a-x) = -f(x)\right]$

HP
=

Impt Example :

$\Rightarrow$ $f(z)$ has double pole at $z=0$ with residue 2 & simple pole at $z=1$ with residue 2, is analytic at all other finite points of the plane & is bounded at $|z|\to\infty$. If $f(2)=5$ $f(-1)=2$. Find $f(z)$.

Given $z=0$ is double pole & $z=1$ is simple pole

$$f(z)= \sum_{n=0}^{\infty} a_n z^n + \frac{2}{z-1} + \frac{2}{z} + \frac{m_1}{z^2}$$

As $f(z)$ is bdd when $|z|\to\infty$, $\exists$ a +ve constant M st $|f(z)|\le M$ $\forall$ z. $\Rightarrow$ $f(z)$ has no singularity at $z=\infty$

Hence $f\left(\frac{1}{w}\right)$ has no singularity at $w=0$.

So the principal part of $f(1/w)$ expansion will have 0 terms

$$f(1/w)= \sum_{n=0}^{\infty} a_n \frac{1}{w^n} + \frac{2w}{1-w} + 2w + m_1 w^2$$

$$= a_0 + \sum_{n=1}^{\infty} a_n w^{-n} - 2 + \frac{2}{1-w} + 2w + m_1 w^2$$

$\therefore$ $a_n = 0$ $\forall$ $n \ge 1$

$$\therefore f(z) = a_0 + \frac{2}{z-1} + \frac{2}{z} + \frac{m_1}{z^2}$$

$f(2) = a_0 + 2 + 1 + \frac{m_1}{4} = 5$ $\Rightarrow$ $a_0 + \frac{m_1}{4} = 2$

$f(-1) = 2 = a_0 - 1 - 2 + m_1 = 2$ $\Rightarrow$ $a_0 + m_1 = 5$

$\therefore$ $\frac{3m_1}{4} = 3 \Rightarrow m_1 = 4$

$a_0 = 1$

$$\therefore f(z) = 1 + \frac{2}{z-1} + \frac{2}{z} + \frac{4}{z^2} = \frac{z^3 - z^2 + 2z^2 + 2z^2 - 2z + 4z - 4}{z^2(z-1)}$$

$$= \frac{z^3 + 3z^2 + 2z - 4}{z^2(z-1)}$$

⇒ Miscellaneous Theorems :

1) If $|f(z)|$ is constant in a region where $f(z)$ is analytic ⇒ $f(z)$ is constant.

Easy: Use CR conditions & $u^2+v^2=c^2 \Rightarrow u u_x + v v_x = 0$
& $u u_y + v v_y = 0$

2) Logarithm of a complex no is multivalued function.

$$\text{Log } z = 2in\pi + \log z$$

$$\text{Log}(x+iy) = 2in\pi + \log(x+iy)$$

$$= \log\sqrt{x^2+y^2} + i\left(2n\pi + \tan^{-1}\frac{y}{x}\right)$$

3) CAUCHY's INEQUALITY: Directly use for $a_n \le \frac{M}{r^n}$ quoting Cauchy's Inequality

Let $f(z)$ be analytic inside & on C having centre z_0 & radius r. If $|f(z)| \le M$ then $|f^n(z_0)| \le \frac{M \cdot n!}{r^n}$

Easy: CIF: $f^n(z_0) = \frac{n!}{2\pi i}\int \frac{f(z)}{(z-z_0)^{n+1}} dz$

$$\therefore |f^n(z_0)| \le \frac{n!}{2\pi}\int \frac{|f(z)|\,|dz|}{|z-z_0|^{n+1}} = \frac{n!}{2\pi} \times \frac{M}{r^{n+1}} \times 2\pi r = \frac{Mn!}{r^n}$$

4) LIOUVILLE'S THEOREM :

If for all z in entire complex plane i) $f(z)$ is analytic
ii) $f(z)$ is bdd

then $f(z)$ must be constant. [In (3) do $R \to \infty$]

Corr. A non-constant entire function comes arbitrarily close to every complex no.

i.e., $f(z)$ is non-constt entire function. Given any complex no a, $\exists$ a sequence $\{z_n\}$ st $f(z_n) \to a$

5) Fundamental Theorem on Algebra :

Every non-constant polynomial has atleast one zero(root)

Every non polynomial of degree 'n' has exactly n zeroes.

6) Suppose $f(z)$ is analytic in the disk $|z-z_0| < R$ & that $\{z_n\}$ is a sequence of distinct points converging to z_0. If $f(z_n)=0$ for each n, then $f(z) \equiv 0$ everywhere in disk $|z-z_0|<R$

Proof: By Taylor's : $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$

$$= a_0 + \sum_{k=1}^{\infty} a_k (z-z_0)^k$$

$$f(z_n) = 0 = a_0 + \sum_{k=1}^{n} a_k (z_n - z_0)^k$$

$$0 = \lim_{k\to\infty} a_0 + \sum_{k=1}^{n} a_k (z_n - z_0)^k = a_0 \quad \left[\because \begin{array}{l} z_n \to z_0 \text{ as } n \to \infty \\ \Rightarrow z_n - z_0 \to 0 \end{array}\right]$$

$\therefore \quad a_0 = 0$

Now $f(z) = (z-z_0)\left[a_1 + \sum_{k=2}^{\infty} a_k (z-z_0)^{k-1}\right]$

we get $a_1 = 0$ sly.

$\therefore \quad a_0 = a_1 = \dots = 0$

$\therefore f(z) \equiv 0$

7) Identity Theorem: Suppose $\{z_n\}$ is a sequence of points having a limit point in the domain D. If $f(z)$ & $g(z)$ are analytic in D with $f(z_n) = g(z_n)$ for each 'n', then $f(z) \equiv g(z)$ throughout D.

Proof. Take $h(z) = f(z) - g(z)$ & apply above theorem (6)

Here: Limit Point has to be inside the analytic domain

8) Gauss Mean Value Theorem:

Let $f(z)$ be analytic inside and on a disk $|z-z_0| \le R$

Then $f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) \, d\theta$ [$\because$ easy using CIF & putting $z = z_0 + re^{i\theta}$]

, ie, value at centre = average of values on circumference

9) Maximum - Modulus Theorem: (1st form):

If $f(z)$ is analytic in domain D, then $|f(z)|$ cannot attain maximum in D, unless $f(z)$ is constant.

Proof: In GMVT: $|f(z_0)| \le \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + r_1 e^{i\theta})| |d\theta|$

where $|f(z_0 + r_1 e^{i\theta})| \underset{\text{(strictly)}}{<} |f(z_0)|$ [$\because$ assuming $|f(z)|$ is not constant & attains maxima at z_0]

$\Rightarrow |f(z_0)| < |f(z_0)|$ # [$\therefore$ can't attain maxima if $|f(z)|$ is not constant]

If $|f(z)|$ is constt, then by 1st theorem (in misc.) $f(z)$ is constant

2nd form: If $f(z)$ is analytic in bdd. D & continuous on its closure, then $|f(z)|$ attains maximum at boundary.

Furthermore, $|f(z)|$ doesn't attain maximum at any interior point unless its constant.

Above results hold for minimum of $|f(z)|$ also if $f(z) \neq 0$ in D.

10) Argument Theorem:

Let $f(z)$ be analytic inside & on a simple closed curve C except for a pole $z = \alpha$ of order 'p' inside C.

Also suppose $f(z)$ has only one zero $z = \beta$ of order 'n' in C.

Then $\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz = n - p$

Proof. Easy: $f(z) = \frac{(z-\beta)^n g(z)}{(z-\alpha)^p}$ take log & proceed.

$g(z)$ has no zeroes ($\neq 0$) & no poles in C.

$\oint_C \frac{g'(z)}{g(z)} dz = 0$.

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11) Rouche's Theorem :

If $f(z)$ & $g(z)$ are analytic inside and on a simple closed curve C & if $|g(z)| < |f(z)|$ on C, then $f(z) + g(z)$ & $f(z)$ have same no of zeroes inside C.

Proof: Let $g(z) = F(z) f(z)$. & N_1, N_2 be zeroes of $f+g$ & f respectively : Then

$$N_1 = \frac{1}{2\pi i}\oint_c \frac{f'+g'}{f+g}\,dz \qquad N_2 = \frac{1}{2\pi i}\oint_c \frac{f'}{f}\,dz$$

$$N_1 - N_2 = \frac{1}{2\pi i}\oint_c \frac{F'}{1+F}\,dz = \frac{1}{2\pi i}\oint_c F'(1-F+F^2-F^3+\dots)\,dz$$

$= 0$

↳ could do this because $|g| < |f|$ $\Rightarrow \left|\frac{g}{f}\right| = |F| < 1$

eg. Prove that all roots of $z^7 - 5z^3 + 12 = 0$ lie b/w $|z| = 1$ & $|z| = 2$

Consider $C_1 : |z| = 1$. Let $f(z) = 12$ & $g(z) = z^7 - 5z^3$

$|g(z)| = |z^7 - 5z^3| \le |z|^7 + 5|z|^3 \le 12 = f(z)$

$\therefore$ $f(z)$ & $f(z)+g(z)$ has same roots in $|z| = 1$

But $f(z) = 12$ has no roots in $|z| = 1$.

Now take $C_2 : |z| = 2$. Let $f(z) = z^7$ $g(z) = -5z^3 + 12$

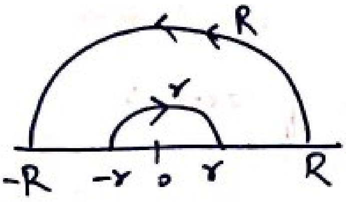
$|g(z)| \le 5|z|^3 + 12 = 52 \le z^7 = f(z)$

$\therefore$ $f(z)$ & $f(z)+g(z)$ has same no of roots in $|z| = 2$

$f(z) = z^7$ has all its roots inside C_2.

$\therefore$ All roots are in $1 < |z| < 2$ [$\because f+g$ has only 7 roots as its deg is 7]

Important. In questions with the contour having poles at real-axis.



$$\int_C = \int_{-R}^{-r} + \int_{\gamma} + \int_{r}^{R} + \int_{R} \quad \text{—— (1)}$$

Now if $\int_R f(z)\,dz = \int_R e^{imz} g(z)\,dz$

then $I \to 0$ as $R \to 0$ if $\lim_{z\to\infty} g(z) = 0$

(Use as JORDAN'S LEMMA DIRECTLY)

For $\int_r f(z)\,dz$: If $\lim_{z\to 0} z f(z) = l$

$\boxed{\text{then } \lim_{r\to 0} I = i \times (0-\pi) \times l}$ → use directly as well

as do $r \to 0$, $R \to \infty$ directly in (1)

Ques Let $f(z) = \dfrac{a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}}{b_0 + b_1 z + \cdots + b_n z^n}$, $b_n \neq 0$

Assume that zeroes of denominator are simple. Show that the sum of the residues ~~at~~ of $f(z)$ at its poles $= \dfrac{a_{n-1}}{b_n}$

In the extended complex plane, sum of all residues $= 0$

$\therefore$ (Σ residue at poles) + Residue at $\infty = 0$

$\therefore$ $S = -$ Residue at ∞ [for $f(z)$] $= -\oint_C f(z)$

$= -$[Residue at $z=0$ for $f\left(\frac{1}{z}\right)$]

$= -\frac{1}{2\pi i}\int_C f\left(\frac{1}{z}\right) \times \left[-\frac{1}{z^2}dz\right] = \frac{1}{2\pi i}\int \frac{1}{z^2} f\left(\frac{1}{z}\right) dz$

$\therefore$ $S =$ Residue at $z=0$ for $\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2}\left[\dfrac{a_0 + \frac{a_1}{z} + \cdots + \frac{a_{n-1}}{z^{n-1}}}{b_0 + \frac{b_1}{z} + \cdots + \frac{b_n}{z^n}}\right]$

$= \frac{1}{z}\left[\dfrac{a_{n-1} + a_{n-2} z + \cdots + a_0 z^{n-1}}{b_n + b_{n-1} z + \cdots + b_0 z^n}\right]$

$\therefore$ Residue : $\lim_{z\to 0} z \cdot \left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] = \lim_{z\to 0} \dfrac{a_{n-1} + \cdots + a_0 z^{n-1}}{b_n + \cdots + b_0 z^n} = \dfrac{a_{n-1}}{b_n}$