

$$I = \iint_{ABCD} \vec{F} \cdot d\vec{S}$$

$$= \iint \vec{F} \cdot \vec{n} dS = \iint \vec{F} \cdot \vec{\nabla} \phi dz dx$$

$$\phi \equiv y - \sqrt{9-z^2}$$

$$\phi = y - \sqrt{9-z^2}$$

$$\vec{\nabla} \phi = \hat{i} + \frac{z}{\sqrt{9-z^2}} \hat{k}$$

$$\vec{n} = \frac{\vec{\nabla} \phi}{\|\vec{\nabla} \phi\|} \quad \|\vec{\nabla} \phi\| dz dx = 0$$

$$\iint \left(2x^2 y \hat{i} - y^2 \hat{j} + 4z^2 x \hat{k} \right) \left(\hat{i} + \hat{j} + \frac{z}{\sqrt{9-z^2}} \hat{k} \right) dz dx$$

$$= \iint \left(-y^2 + \frac{4z^3 x}{\sqrt{9-z^2}} \right) dz dx \quad \text{on } y^2 + z^2 = 9 \quad \cancel{y^2 + z^2 = 9}$$

$$= \iint \left(z^2 - 9 + \frac{4z^3 x}{\sqrt{9-z^2}} \right) dz dx$$

D_{xz} : rectangle

$$= \int_0^3 \int_0^2 \left(z^2 - 9 + \frac{4z^3 x}{\sqrt{9-z^2}} \right) dz dx$$

$$z=0 \quad x=0$$

$$= 108.$$

~~To be completed on 6/4/17~~

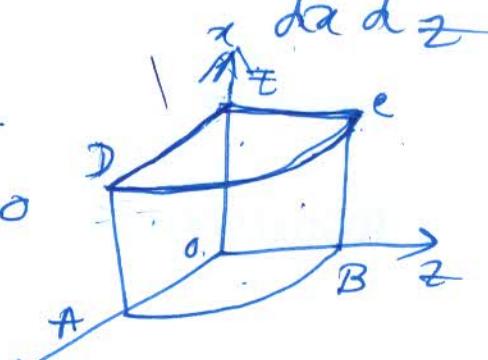
$$I_2 = \iint \vec{F} \cdot d\vec{s} = \iint \vec{F} \cdot \vec{n} ds.$$

$$\begin{aligned} S &= \text{Area } EDAO \\ &= \iint_D (2x^2y\hat{i} - y^2\hat{j} + 4z^2\hat{k})(-\hat{k}) dx dy \\ &= \int_{y=0}^3 \int_{x=0}^2 -4z^2 x \Big|_{z=0} dx dy = 0. \end{aligned}$$

$$I_3 = \iint_O \vec{F} \cdot \vec{n} \cdot ds = \iint_O (2x^2y\hat{i} - y^2\hat{j} + 4z^2\hat{k})(-\hat{i}).$$

OBCE.

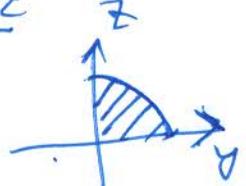
$$= \iint_{z=0}^3 \int_{x=0}^2 y^2 \Big|_{y=0} dx dz = 0$$



$$I_4 = \iint_C \vec{F} \cdot \vec{n} \cdot ds = \iint_C (2x^2y\hat{i} - y^2\hat{j} + 4z^2\hat{k}) \hat{i} dy dz$$

CED.

$$= \iint_{y^2+z^2 \leq 9} 2x^2 y \Big|_{x=2} dy dz$$



$$= \iint_{y^2+z^2 \leq 9} 8y dy dz$$

$$y = r \cos \theta, \\ z = r \sin \theta$$

$$= \int_{r=0}^3 \int_{\theta=0}^{\pi/2} 8r \sin \theta r dr d\theta = 72.$$

$$r=0, \theta=0.$$

$$I_5 = \iint_A \vec{F} \cdot \vec{n} \cdot ds = \iint_A (-\hat{i} - y^2\hat{j} - \hat{k}) \cdot \hat{i} dy dz$$

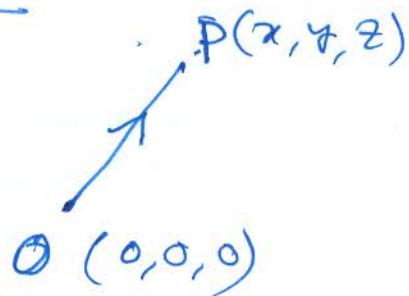
$$= \iint_{z=0}^3 -2x^2y \Big|_{x=0} dy dz = 0$$

$$\therefore I_1 + I_2 + I_3 + I_4 + I_5 = 108 + 72 = 180,$$

Vector Calculus

\vec{OP} = Position vector of P .

($O \rightarrow$ origin)



$$= (x-0)\hat{i} + (y-0)\hat{j} + (z-0)\hat{k}$$

$$\vec{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$\hat{i}, \hat{j}, \hat{k} \rightarrow$ unit-vectors along +ve x direction, +ve y dir., +ve z dir. respectively.

$$|\vec{OP}| = \sqrt{x^2 + y^2 + z^2} = r = |\vec{r}|$$

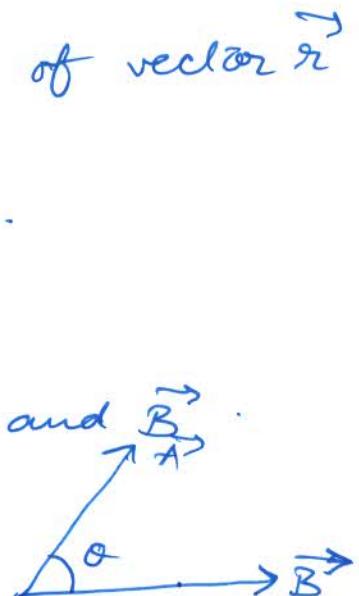
$$||\vec{r}|| = |\vec{r}| = \text{magnitude of vector } \vec{r}$$

$$|\hat{i}| = 1, |\hat{j}| = 1 = |\hat{k}| = 1.$$

Products -

$\vec{A} \cdot \vec{B}$ = dot product of \vec{A} and \vec{B}

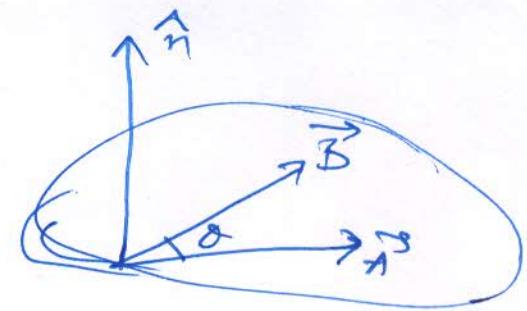
$$= |\vec{A}| \cdot |\vec{B}| \cos \theta,$$



$$\cos \theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}| |\vec{B}|} \quad 0 \leq \theta \leq \pi$$

$$\theta = \cos^{-1} \left(\frac{\vec{A} \cdot \vec{B}}{|\vec{A}| |\vec{B}|} \right)$$

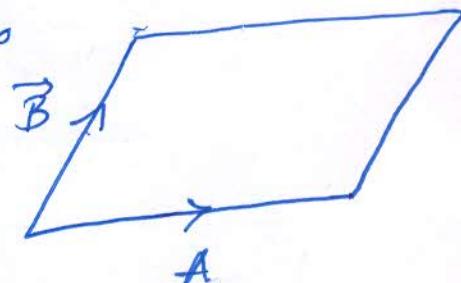
$$\vec{A} \times \vec{B} = |\vec{A}| \cdot |\vec{B}| \hat{n} \sin\theta.$$



$$|\vec{A} \times \vec{B}| = |\vec{A}| \cdot |\vec{B}| \cdot |\hat{n}| \sin\theta$$

$$= |\vec{A}| \cdot |\vec{B}| \cdot \sin\theta.$$

= area of the parallelogram whose one side is \vec{A} , other side is \vec{B} .



Vector differential operator / nabla operator

$$\vec{\nabla} = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k}$$

$$\vec{\nabla} \phi(x, y, z) = \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \phi.$$

Gradient of ϕ = $\vec{\nabla} \phi$ = $\frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k}$.

Geometrically $\vec{\nabla} \phi$ is the normal to the surface $\phi(x, y, z) = c$ at the point (x, y, z)

$$\vec{A} = (A_1, A_2, A_3)$$

$$\vec{\nabla} \cdot \vec{A} \text{ or } \vec{\nabla} \times \vec{A}$$

\downarrow divergence of \vec{A} \downarrow curl of \vec{A} .

$$\vec{\nabla} \cdot \vec{A} = \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot (A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k})$$

$$\begin{aligned} \vec{A} \cdot \vec{B} &= (A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}) (B_1 \hat{i} + B_2 \hat{j} + B_3 \hat{k}) \\ &= A_1 B_1 + A_2 B_2 + A_3 B_3 \\ &= \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} = \text{div. } \vec{A} = \vec{\nabla} \cdot \vec{A} \end{aligned}$$

If $\vec{\nabla} \cdot \vec{A} = 0$, \vec{A} is said to be solenoidal.

Curl of A :

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) \hat{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) \hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \hat{k}$$

Directional derivative of ϕ .

You know $\frac{\partial \phi}{\partial x}$ or $\frac{\partial \phi}{\partial y}$ or $\frac{\partial \phi}{\partial z}$

$$\vec{A} = A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}$$

Directional derivative in the direction of \vec{A} of $\phi(x, y, z)$

$$= \vec{\nabla} \phi \cdot \frac{\vec{A}}{|\vec{A}|}$$

Directional derivative in the direction of
(D.D.) of ϕ along x axis

Any vector along x axis = $\vec{A} = \cancel{\text{a}} \vec{i}$

$$\therefore \text{D.D. of } \phi \text{ along } \vec{A} = \vec{\nabla} \phi \cdot \frac{\vec{A}}{|\vec{A}|}$$

$$= \left(\frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k} \right) \left(\frac{\vec{a} \hat{i}}{|\vec{a}|} \right)$$

$$= \left(\frac{\partial \phi}{\partial x} \hat{i} + \dots \right) (\pm \hat{i})$$

$$= - \frac{\partial \phi}{\partial x} / \frac{\partial \phi}{\partial x}$$

Ex 1. Find the gradient of $f = 2z^2y - xy^2$
at the point $(2, 1, 1)$ & also find the unit
normal to the surface $2z^2y - xy^2 = c$ at

$$\vec{\nabla} f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k} \quad (2, 1, 1)$$

$$= -y^2 \hat{i} + (2z^2 - 2xy) \hat{j} + 4zy \hat{k}$$

$$(\vec{\nabla} f)_{(2, 1, 1)} = -\hat{i} - 2\hat{j} + 4\hat{k}$$

$$\text{unit-normal} = \frac{-\hat{i} - 2\hat{j} + 4\hat{k}}{\sqrt{(-1)^2 + (-2)^2 + (4)^2}}$$

$$\text{Sol. } -8\hat{i} + 4\hat{j} + 12\hat{k} \\ -\frac{2\hat{i}}{\sqrt{14}} + \frac{\hat{j}}{\sqrt{14}} + \frac{4\hat{k}}{\sqrt{14}} = -\frac{1}{\sqrt{21}} \hat{i} - \frac{2}{\sqrt{21}} \hat{j} + \frac{4}{\sqrt{21}} \hat{k}$$

Ex 2. Find the gradient of $f = 4x^2y + z^3$ at the
point $(1, -1, 2)$. Find the unit-normal to the
surface $4x^2y + z^3 = c$ at $(1, -1, 2)$.

Q. If $\phi = 2z^2y - xy^2$, find the D. D.
of ϕ at $(2, 1, 1)$ in the direction of

$$\vec{A} = 3\hat{i} + 6\hat{j} + 2\hat{k}$$

Sol. dir. der. of ϕ at $(2, 1, 1)$ along \vec{A}

$$= \vec{\nabla} \phi \cdot \frac{\vec{A}}{|\vec{A}|}$$

$$\vec{\nabla} \phi = -y^2\hat{i} + (2z^2 - 2xy)\hat{j} + 4yz\hat{k}$$

$$\therefore (\vec{\nabla} \phi)_{(2, 1, 1)} = -\hat{i} - 2\hat{j} + 4\hat{k}$$

$$= (-\hat{i} - 2\hat{j} + 4\hat{k}) \cdot \frac{(3\hat{i} + 6\hat{j} + 2\hat{k})}{\sqrt{3^2 + 6^2 + 2^2}}$$

$$= \frac{1}{7} (-\hat{i} - 2\hat{j} + 4\hat{k}) \cdot (3\hat{i} + 6\hat{j} + 2\hat{k})$$

$$= \frac{1}{7} (-3 - 12 + 8) = -1$$

Thm. Dir. derivative of ϕ is maximum
along the ~~good~~ the direction of $\text{grad } \phi$.
& the magnitude is $|\vec{\nabla} \phi|$.

[In this case $\vec{A} = \vec{\nabla} \phi$.

$$\begin{aligned} \therefore |\text{D. D. of } \phi \text{ along } \vec{A} (\in \vec{\nabla} \phi)| &= |\vec{\nabla} \phi| \cdot \left| \frac{\vec{\nabla} \phi}{|\vec{\nabla} \phi|} \right| \\ &= |\vec{\nabla} \phi| \cdot \frac{|\vec{\nabla} \phi|}{|\vec{\nabla} \phi|} = |\vec{\nabla} \phi|. \end{aligned}$$

Ex4 find the D. D. of $f = xy^2 + yz^3$ at the pt. $(2, -1, 1)$ in the direction of the vector $\hat{i} + 2\hat{j} + 2\hat{k}$. Find the max. value of the D. D. of f at $(2, -1, 1)$.

$$-\frac{11}{3}, \sqrt{19}.$$

Ex-5 Find the angle between the surfaces

$$x^2 + y^2 + z^2 = 9 \text{ & } z = x^2 + y^2 - 3 \text{ at } (2, -1, 2)$$

Hint. angle between the surfaces $=$ angle between the normals to the surfaces at $(2, -1, 2)$

$$\begin{aligned} & \text{to the surfaces at } (2, -1, 2) \\ &= \angle \text{ between } \vec{\nabla} \phi \text{ & } \vec{\nabla} \psi \quad \left| \begin{array}{l} \phi = x^2 + y^2 + z^2 = 9 \\ \psi = x^2 + y^2 - z = 3 \end{array} \right. \\ & \qquad \qquad \qquad \text{at } (2, -1, 2) \end{aligned}$$

$$\theta = \cos^{-1} \left(\frac{\vec{\nabla} \phi \cdot \vec{\nabla} \psi}{|\vec{\nabla} \phi| \cdot |\vec{\nabla} \psi|} \right)$$

$$= \cos^{-1} \left(\frac{8}{(\sqrt{21})3} \right) = 54.4^\circ.$$

Conservative vector field -

$$\vec{F} = e^y \hat{i} + x e^y \hat{j} + (z+1) e^z \hat{k}$$

Curl \vec{F} =
$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^y & x e^y & (z+1) e^z \end{vmatrix} = \vec{0}$$

If $\vec{F} = \vec{\nabla} \phi$, where ϕ is some scalar function $\phi(x, y, z)$. then \vec{F} is called a conservative vector field.

$$\vec{v} = 2\hat{i} + 3\hat{j} + 4\hat{k} \rightarrow \text{vector}$$

$$\vec{F}(x, y, z) = (2x+3y)\hat{i} + (3z-4x)\hat{j} + x^2 z^2 \hat{k} \rightarrow \text{vector field}$$

$$\phi(x, y, z) \rightarrow \text{scalar "}$$

Thm. A vector field is conservative if and only if ~~$\vec{F} = \vec{\nabla} \phi$~~ , $\text{curl } \vec{F} = \vec{0}$.

Def. If $\text{curl } \vec{F} = 0$, \vec{F} is called irrotational.

Thm. \vec{F} can be derived from a scalar potential $\phi(x, y, z)$ as $\vec{F} = \vec{\nabla} \phi$ if and only if \vec{F} is irrotational.

$$\underline{\text{Ex}} \quad \vec{F} = e^y \hat{i} + xe^y \hat{j} + (z+1)e^z \hat{k}$$

We've seen $\cdot \operatorname{Curl} \vec{F} = 0$.

$\therefore \vec{F}$ must be such that $\vec{F} = \vec{\nabla} \phi$,
for some $\phi = \phi(x, y, z)$

$$\text{let } \vec{F} = (F_1, F_2, F_3)$$

$$\text{If } \vec{F} = \vec{\nabla} \phi$$

$$\Rightarrow F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k} = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k}$$

$$\therefore \frac{\partial \phi}{\partial x} = F_1, \quad \frac{\partial \phi}{\partial y} = F_2, \quad \frac{\partial \phi}{\partial z} = F_3$$

$$\frac{\partial \phi}{\partial x} = e^y \xrightarrow{(1)} \quad \frac{\partial \phi}{\partial y} = xe^y \xrightarrow{(2)},$$

$$\text{Integrating w.r.t } x, \quad \frac{\partial \phi}{\partial z} = (z+1)e^z \xrightarrow{(3)}$$

$$\phi = \int e^y dx + f(y, z) \xrightarrow{(3)}$$

$$\phi = xe^y + f(y, z) \xrightarrow{(4)}$$

Diff. (4) w.r.t y & get -

$$\frac{\partial \phi}{\partial y} = xe^y + \frac{\partial f}{\partial y}(y, z) \xrightarrow{(5)}$$

Compare (2) & (5). This gives .

$$xe^y = xe^y + \frac{\partial f}{\partial y}(y, z)$$

$$\therefore \frac{\partial f}{\partial y}(y, z) = 0$$

$$\text{Integrating } f(y, z) = f_1(z)$$

$$\therefore \phi = xe^y + f_1(z) \xrightarrow{10} (6)$$

Differentiating (6) w.r.t. z ,

$$\frac{\partial \phi}{\partial z} = \frac{d}{dz} f_1(z) = f_1'(z) \rightarrow (7)$$

Comparing (3) & (7), get —

$$\frac{d f_1(z)}{d z} = (z+1)e^z$$

$$f_1(z) = \int (z+1)e^z dz + C.$$

$$= (z+1) \cdot e^z - \int e^z dz + C.$$

$$= ze^z + C.$$

$$\therefore \phi(x, y, z) = xe^y + ze^z + C.$$

2. ~~Show~~ Show that $\text{curl } \vec{F} = \vec{0}$

$$\text{where } \vec{F} = (x^2 + yz)\hat{i} + (y^2 + zx)\hat{j} + (z^2 + xy)\hat{k}$$

Hence find the corresponding potential function ϕ .

$$\text{Ans: } \phi = \frac{x^3 + y^3 + z^3}{3} + xyz + C$$