MATHEMATICS

UPSC

CIVIL SERVICES MAINS

Linear Programming (LPP)

n eqn
$$\begin{cases} x_1 + x_2 + x_3 + \cdots\ x_m \le 2 \\ \vdots \\ \underbrace{2x_1 - 3x_2 + 4x_3 + \cdots \qquad \le 3}_{m = \text{variables}} \end{cases}$$

→ Optimal solution = Most Authentic solution
it can occur more than once.

→ Basic solution : Remove enquality & put $(m-n)$ equal to zero & solve.

Terminology
- → Basic Solution
- → Basic Variable
- → Non- " "
- → Basic feasible solution
- → Degenerate solution
- → Non- " ".

- → Slack variable.
- → Surplus "

Example:- Consider the following LPP:

Max. $Z = 5x + 8y$

$$x + y \le 4 \quad \text{--- (1)}$$
$$2x + y \le 6 \quad \text{--- (2)}$$
$$x, y \ge 0$$

Sol. No. of eq. = 2

Add slack variables i.e. $x + y + S_1 = 4$
$2x + y + S_2 = 6$

Now, No. of eqn = 2.
" " variable = 4 $\qquad (4-2) = 2$ variable

Put them equal to zero ← which are Non-basic variables

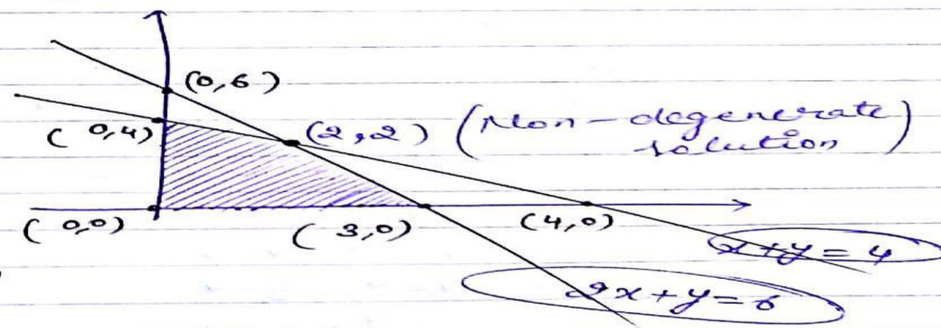
No. of Basic solution	Non-Basic variables (each = 0)	Basic variable	Basic solution
1	S_1, S_2	x, y	$x = 2, y = 2$ ✓
2	x, S_1	y, S_2	$y = 4, S_2 = 2$ ✓
3	x, S_2	y, S_1	$y = 6, S_1 = -2$
4	y, S_1	x, S_2	$x = 4, S_2 = -2$
5	y, S_2	x, S_1	$x = 3, S_1 = 1$ ✓
6	x, y	S_1, S_2	$S_1 = 4, S_2 = 6$ ✓

→ Check result = Basic feasible solution.
(2, 2), (0, 4), (3, 0), (0, 0)

→ Degenerate solution : if one or more zero is present in B.F.S. i.e.
$\boxed{(0,4), (3,0), (0,0)}$

→ Non-degenerate solution :- if all +ve variables are present. in B.F.S.
i.e. (2,2) is

All solution lies on X-Y-axis ⇒ Degen.



→ Graphical Method
→ Simplex Method
→ Dual-Simplex Method
• Duality
→ Transportation.

(Mains 2015) (20 Marks)

Que:- $\max. Z = x_1 + 2x_2 - 3x_3 + 4x_4$

$$x_1 + x_2 + 2x_3 + 3x_4 = 12$$
$$x_2 + 2x_3 + x_4 = 8$$

i) Using definition, find its all basic solutions. Which of these are degenerate Basic feasible solution and which are non-degenerate basic feasible solution.

Ans

No. of solution	Non-Basic variable	Basic variable	Basic solution
1	x_1, x_2	x_3, x_4	✓ $x_3 = 3, x_4 = 2$
2	x_1, x_3	x_2, x_4	✓ $x_2 = 6, x_4 = 2$
3	x_1, x_4	x_3, x_2	No solution
4	x_2, x_3	x_1, x_4	$x_1 = -12, x_4 = 8$
5	x_2, x_4	x_1, x_3	✓ $x_1 = 4, x_3 = 4$
6	x_3, x_4	x_2, x_1	✓ $x_1 = 4, x_2 = 8$

→ Checked sign solution = feasible solution and are Non-degenerate solution.

→ There is No degenerate ".

(ii) Max. (Z) at $x_3 = 3, x_4 = 2$ $= -9 + 8 = -1$

" $x_2 = 6, x_4 = 2$ $= 12 + 8 = 20$ ✓

" $x_1 = -12, x_4 = 8$ $= -12 + 32 = 20$ — optimal but Not feasible.

" $x_1 = 4, x_3 = 4$ $= 4 - 12 = -8$

" $x_1 = 4, x_2 = 8$ $= 4 + 16 = 20$ ✓

} optimal solution But feasible

Ans

Graphical Method :-

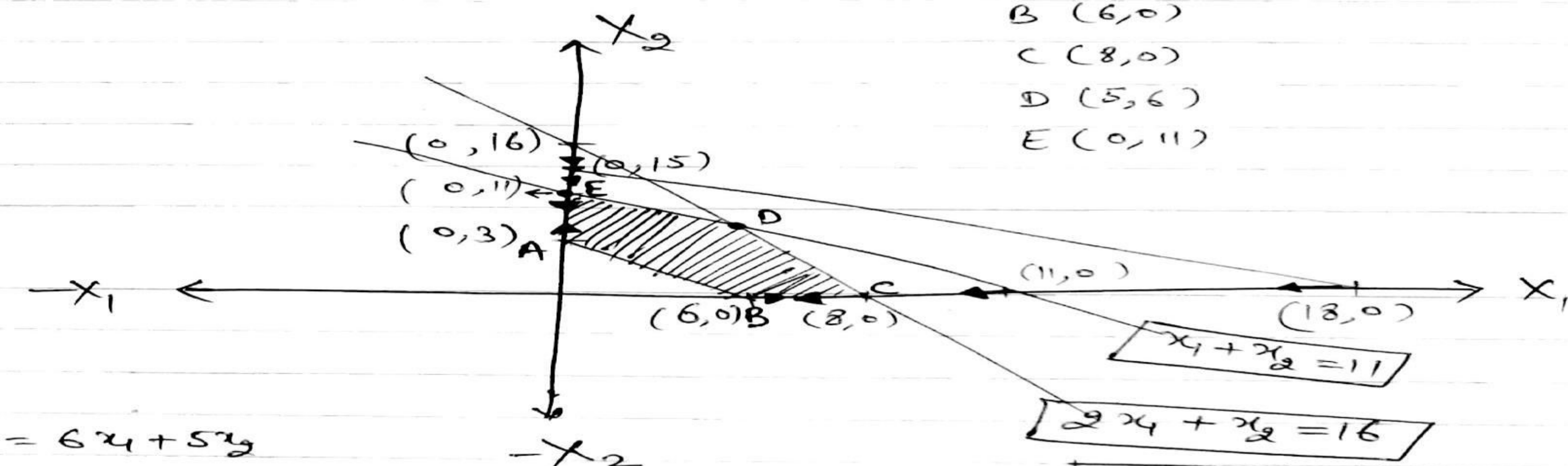
Que: (Mains 2014) Max. $z = 6x_1 + 5x_2$

10 Marks

$$2x_1 + x_2 \le 16 \quad \text{——— (1)}$$
$$x_1 + x_2 < 11 \quad \text{——— (2)}$$
$$x_1 + 2x_2 \ge 6 \quad \text{——— (3)}$$
$$5x_1 + 6x_2 \le 90 \quad \text{——— (4)}$$
$$x_1, x_2 \ge 0$$

A (0,3)
B (6,0)
C (8,0)
D (5,6)
E (0,11)

Sol.



$\boxed{x_1 = 5}$ & $\boxed{x_2 = 6}$

Max. $Z = 6x_1 + 5x_2$

$Z|_{A(0,3)} = 15$

$Z|_{B(6,0)} = 36$

$Z|_{C(8,0)} = 48$

$Z|_{D(5,6)} = 30 + 30 = \boxed{60}$ — Max Z.

$Z|_{E(0,11)} = 55$

So $\boxed{\text{Max } Z = 60 \text{ at } x_1 = 5,\ x_2 = 6}$

Ans.

Simplex Method :-

MAINS - 2015 ; 20 Marks

Que :- Solve LPP by using simplex method :-

$$\text{Max. } Z = 2x_1 - 4x_2 + 5x_3$$

$$x_1 + 4x_2 - 2x_3 \le 2 \quad \text{——— (1)}$$

$$-x_1 + 2x_2 + 3x_3 \le 1 \quad \text{——— (2)}$$

$$x_1, x_2, x_3 \ge 0$$

Sol. Add two slack variable in eq. (1) & eq. (2) as follows:

$$x_1 + 4x_2 - 2x_3 + S_1 = 2$$
$$-x_1 + 2x_2 + 3x_3 + S_2 = 1$$

$$\text{Max. } Z = 2x_1 - 4x_2 + 5x_3 + 0S_1 + 0S_2$$

		C_j	2	-4	5	0	0	
C_B	Basis	Solution	x_1	x_2	x_3	S_1	S_2	Ratio
0	S_1	2	1	4	-2	1	0	$\frac{2}{-2} = -1$
0	S_2	1	-1	2	3	0	1	$\frac{1}{3} = \frac{1}{3}$ Min. Ratio in +ve
		Z_j	0	0	0	0	0	
		$C_j = c_j - Z_j$	2	-4	$\boxed{5}$ ↑	0	0	

Highlight Most +ve No.

Now (New) $R_2 \longleftrightarrow \frac{1}{3} R_2(\text{old})$

$R_1 \longleftrightarrow R_1 + 2R_2(\text{new})$

$\frac{8}{3}$	$\frac{1}{3}$	$\frac{16}{3}$	0	1	$\frac{2}{3}$
$\frac{1}{3}$	$\frac{-1}{3}$	$\frac{2}{3}$	1	0	$\frac{1}{3}$ ← R_2(New)
$\frac{2}{3}$	$\frac{-2}{3}$	$\frac{4}{3}$	2	0	$\frac{2}{3}$
+ 2	1	4	-2	1	0

		C_j	2	-4	5	0	0	
C_B	Basis	Sol.^{n}	x_1	x_2	x_3	S_1	S_2	Ratio
0	S_1	$\frac{8}{3}$	$\boxed{\frac{1}{3}}$	$\frac{16}{3}$	0	1	$\frac{2}{3}$	8 →
5	x_3	$\frac{1}{3}$	$\frac{-1}{3}$	$\frac{2}{3}$	1	0	$\frac{1}{3}$	-1
		Z_j	$-5/3$	$10/3$	5	0	$5/3$	
		C_j	↑ $1/3$	$-22/3$	0	0	$-5/3$	

$R_1 \longleftrightarrow 3R_1\ (old)$

$\longrightarrow \quad 8 \quad 1 \quad 16 \quad 0 \quad 3 \quad 2$

$R_2 \longleftrightarrow R_1\ (old) + R_2\ (old)$

$\longrightarrow \quad 3 \quad 0 \quad 6 \quad 1 \quad 1 \quad 1$

		c_j	2	−4	5	0	0
C_B	Basis	sol^n	x_1	x_2	x_3	S_1	S_2
2	x_1	8	1	16	0	3	2
5	x_3	3	0	6	1	1	1
		z_j	2	62	5	11	9
		$C_j = c_j - z_j$	0	−66	0	−11	−9

Since all the elements in the $C_j = c_j - z_j$ are −ve
Hence optimal solution is obtained.

$\Rightarrow \quad x_1 = 8, \quad x_3 = 3, \quad x_2 = 0, \quad S_1 = 0, \quad S_2 = 0$

$\Rightarrow \quad \text{Max } Z = 2(8) - 4(0) + 5(3)$
$= 16 + 15 = 31$ Ans.

How to write the dual of the given question:

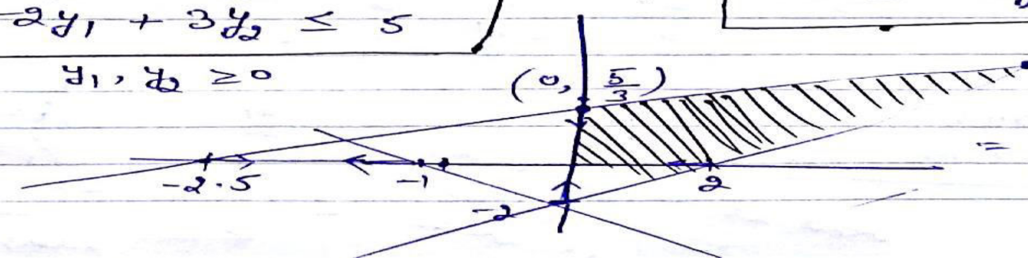
$$\begin{bmatrix} 1 & 4 & -2 \\ -1 & 2 & 3 \end{bmatrix}' \longleftrightarrow \begin{bmatrix} 1 & -1 \\ 4 & 2 \\ -2 & 3 \end{bmatrix}$$

$$\boxed{\begin{aligned} y_1 - y_2 &\le 2 \\ 4y_1 + 2y_2 &\le -4 \\ -2y_1 + 3y_2 &\le 5 \end{aligned}}$$

$y_1, y_2 \ge 0$

and $\boxed{\text{Max. } Z = 2y_1 + y_2}$

Ans.



(11, 9)

$= 2(11) + 9$

$= 22 + 9 = 31$

Ans.

Que:- Mains (2014) $\boxed{\text{20 Marks}}$

Using Simplex method solve :-

$$\text{Max } Z = 30x_1 + 24x_2$$

$$5x_1 + 4x_2 \le 200$$

$$x_1 \le 32 \qquad x_1, x_2 \ge 0$$

$$x_2 \le 40$$

Sol.

$$5x_1 + 4x_2 + S_1 = 200$$

$$x_1 + S_2 = 32$$

$$x_2 + S_3 = 40$$

		C_j	30	24	0	0	0	
C_B	Basis	Solⁿ	x_1	x_2	S_1	S_2	S_3	Min. Ratio
0	S_1	200	5	1	1	0	0	40
0	S_2	32	1	0	0	1	0	32 →
0	S_3	40	0	1	0	0	1	∞
		Z_j	0	0	0	0	0	
		C_j	30 ↑	24	0	0	0	

Operation: $R_1 \longleftrightarrow R_1 - 5R_2$

		C_j	30	24	0	0	0	
C_B	Basis	Solⁿ	x_1	x_2	S_1	S_2	S_3	Min. Ratio
0	S_1	40	0	1	1	-5	0	40
30	x_1	32	1	0	0	1	0	∞
0	S_3	40	0	1	0	0	1	40
		Z_j	30	0	0	30	0	
		C_j	0	24 ↑	0	-30	0	

Tie

Operation:

$R_1 \longleftrightarrow R_1 - R_3$

	S_1	S_3
S_1	$1/1 = 1$	$0/1 = 0$
S_3	$0/1 = 0$	$1/1 = 1$

L ⟶ R column wise.

And it is found in S_3 i.e. 2nd Row

<1.

		c_j	30	24	0	0	0
C_B	Basis	Sol.	x_1	x_2	S_1	S_2	S_3
0	S_1	0	0	0	1	-5	-1
30	x_1	32	1	0	0	1	0
24	x_2	40	0	1	0	0	1
		Z_j	30	24	0	30	24
		$C_j = c_j - Z_j$	0	0	0	-30	-24

Hence all the elements of $(C_j = c_j - Z_j)$ are $-ve$, we get optimal solution.

And the optimal solution is :

$$x_1 = 32 \quad \text{and} \quad x_2 = 40 \quad \text{and} \quad \boxed{S_1 = S_2 = S_3 = 0}$$

$$\Rightarrow \quad \text{Max. } Z = 30(32) + 24(40) = 960 + 960 = \boxed{1920}$$ Ans.

Dual-Simplex Method :-

Que:- Max. $Z = -2x_1 - x_2$; $3x_1 + x_2 \geq 3$
$4x_1 + 3x_2 \geq 6$; $x_1, x_2 \geq 0$
$x_1 + 2x_2 \geq 3$

Sol^n In this case we will make this "greater than" inequality into "less than" inequality. And after that we will add "surplus variable"

$$-3x_1 - x_2 + S_1 = -3$$
$$-4x_1 - 3x_2 + S_2 = -6$$
$$-x_1 - 2x_2 + S_3 = -3$$

		c_j	-2	-1	0	0	0
C_B	Basis	sol.^{n}	x_1	x_2	S_1	S_2	S_3
0	S_1	-3	-3	-1	1	0	0
0	S_2	-6	-4	-3	0	1	0
0	S_3	-3	-1	-2	0	0	1
		Z_j	0	0	0	0	0
		C_j	-2	-1	0	0	0

$$\left[\frac{-2}{-4} = \frac{1}{2} = 0.5 \qquad \frac{-1}{-3} = \frac{1}{3} = 0.3 \checkmark \right]$$

$R_2 \longleftrightarrow \frac{-1}{3} R_2(\text{old})$: Operation

(26.)

$R_1 \longleftrightarrow R_1 + R_2$, $R_3 \longleftrightarrow R_3 + 2R_2$.

		C_j	-2	-1	0	0	0
C_B	Basis	solution	x_1	x_2	S_1	S_2	S_3
0	S_1	-1	$\boxed{-5/3}$	0	1	$-1/3$	0
-1	x_2	2	$4/3$	1	0	$-1/3$	0
0	S_3	1	$5/3$	0	0	$-2/3$	1
		Z_j	$-4/3$	-1	0	$1/3$	0
		C_j	$-2/3 \uparrow$	0	0	$-1/3$	0

$\dfrac{\frac{-2}{3}}{\frac{-5}{3}} = \dfrac{2}{5}$ MIN

$\frac{-1}{3} / -1/3 = 1$

$R_1 \longleftrightarrow R_1^{(old)} \times \frac{-3}{5}$, $R_2 \longleftrightarrow R_2 - \frac{4}{3} R_1$, $R_3 \longleftrightarrow R_3 - \frac{5}{3} R_1$

		C_j	-2	-1	0	0	0
C_B	Basis	sol.^{n}	x_1	x_2	S_1	S_2	S_3
-2	x_1	$3/5$	1	0	$-3/5$	$1/5$	0
-1	x_2	$6/5$	0	0	$4/5$	$-3/5$	0
0	S_3	0	0	0	1	-1	1

Since all the elements in sol.^{n} column are +ve, hence we get optimal solution and No further iteration is required.

$\Rightarrow \quad x_1 = \frac{3}{5}$ and $x_2 = \frac{6}{5}$ and $S_1 = S_2 = S_3 = 0$

$\Rightarrow \quad$ Max. $Z = -2x_1 - x_2 \Big|_{x_1 = \frac{3}{5},\ x_2 = \frac{6}{5}}$

$= -2\left(\frac{3}{5}\right) - \frac{6}{5} = \frac{-6}{5} - \frac{6}{5} = \frac{-12}{5}$ Ans.

THANKS