// T-8 28/10/2016

A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET

IMS™
(INSTITUTE OF MATHEMATICAL SCIENCES)

171/250

TEST SERIES (MAIN)-2016

Test Code: PAPER-II: IAS (M)/09-10-16

MATHEMATICS

by **K. VENKANNA**

Test - 08

FULL LENGTH TEST

Time: Three Hours *Maximum Marks: 250*

INSTRUCTIONS

1. This question paper-cum-answer booklet has 48 pages and has 33 PART/SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated. "
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any **THREE** of the remaining questions selecting at least **ONE** question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any page from this booklet.

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name ATUL PRAKASH.

Roll No.

Test Centre ORN

Medium ENGLISH

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

AtulPrakash

Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the invigilator

IMPORTANT NOTE:
Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

P.T.O.

DO NOT WRITE ON THIS SPACE

INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED	
1	(a)			7	
	(b)			6	
	(c)			8	32
	(d)			7	
	(e)			4	
2	(a)				
	(b)				
	(c)				
	(d)				
3	(a)				
	(b)				
	(c)				
	(d)				
4	(a)			8	
	(b)			12	33
	(c)			8	
	(d)			5	
5	(a)			6	
	(b)			8	
	(c)			2	24
	(d)			8	
	(e)				
6	(a)			8	
	(b)			9	41
	(c)			9	
	(d)			15	
7	(a)			13	
	(b)			12	41
	(c)			5	
	(d)			11	
8	(a)				
	(b)				
	(c)				
	(d)				
				Total Marks	

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SECTION – A

1. (a) In S_{10}, let $\beta = (13)(17)(265)(289)$, Find an element in S_{10} that commutes with β but is not a power of β. (10)

$(13)(17) \Rightarrow (173)$

$(265) \Rightarrow (25)(26)$

$(289) \Rightarrow (29)(28)$

$(265)(289) \Rightarrow (25)(26)(29)(28)$

$= (28965)$

$\beta = (173)(28965)$

Consider,

$(4\ 10) \equiv \alpha$.

As there is nothing common between α, β.

$\Rightarrow \alpha$ is not a power of β. [By repeated β application 4, 10 are unchanged]

$\boxed{\alpha\beta = \beta\alpha}$ Also as no common, ensures no change in output even after changing order.

1. (b) Consider the mapping from $M_2(Z)$ into Z given by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to a$. Prove or disprove that this is a ring homomorphism. (10)

For mapping ϕ to be ring homomorphism

$$\phi(A+B) = \phi(A) + \phi(B)$$
$$\phi(AB) = \phi(A).\phi(B)$$
$$\phi(\underset{M_2(Z)}{Id}) = Id_Z$$

Consider $A, B \in M_2(Z)$ $\left\{ \begin{array}{l} A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ B = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \end{array} \right.$

$\phi(A) = a \quad \phi(B) = e$

$AB = \begin{bmatrix} ae+bf & - \\ - & - \end{bmatrix} \Rightarrow \phi(AB) = ae+bf \neq$ But $\phi(A).\phi(B) = ae$.

06

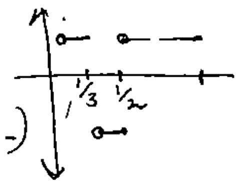
So, it is not a ring homomorphism

1. (c) A function f is [0, 1] by $f(0) = 1$,

$f(x) = (-1)^{n-1}$ when $\frac{1}{n+1} < x \le \frac{1}{n}$ $(n = 1, 2, 3, \ldots)$.

Prove that (i) f is integrable on [0, 1], (ii) $\int_0^1 f = \log(4/e)$. (10)

(i) f has infinite pts of discontinuity at $x = \frac{1}{n}$ $(n = 2, 3 \ldots)$ f =



As f is bounded in [0, 1] and the points of discontinuity $\left(x = \frac{1}{n}\right)$ have finite limit points,

Only 1 lt pt at $x = 0$

$$\boxed{\lim_{n\to\infty} \frac{1}{n} = 0}$$

So, f is integrable on [0, 1].

(ii) $\sum \int_0^1 f = \int_{1/2}^{1} + \int_{1/3}^{1/2} + \int_{1/4}^{1/3} \ldots$

08

$$= \sum_1^\infty \int_{1/(r+1)}^{1/r} (-1)^{r-1}\, dx = \sum_1^\infty (-1)^{r-1} \cdot x \left(\frac{1}{r} - \frac{1}{r+1}\right)$$

$$= \sum_1^\infty (-1)^{r-1} \cdot \frac{1}{r(r+1)} = \left(1 - \frac{1}{2}\right) - \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots$$

$$= -1 + 2\left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) \ldots\right]$$

Let Sum be γ

$$= -1 + 2\left[\left(1 + \frac{1}{2} + \cdots \frac{1}{2n}\right) - \frac{1}{2}\left(1 + \frac{1}{2} + \cdots \frac{1}{n}\right)\right]$$

Using rearrangement of series.

$$= -1 + 2\left[\gamma_{2n} + \log 2n - \frac{1}{2}\gamma_n + \frac{1}{2}\log n\right]$$

$$= \log 4 - e \Rightarrow \boxed{\log 4/e}$$

1. (d) Prove that the function $f(z) = u + iv$, where

$$f(z) = \begin{cases} \frac{x^3 - y^3}{x^2 + y^2} + i\left(\frac{x^3 + y^3}{x^2 + y^2}\right) & \text{if } (x, y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous and that Cauchy-Riemann equations are satisfied at the origin, but not differentiable at the origin **(10)**

Continuity $\Rightarrow$ $x = r\cos\theta,\ y = r\sin\theta.$

$$\lim_{r \to 0} r(\cos^3\theta - \sin^3\theta) + r(\cos\theta + \sin\theta) = 0. = f(z) \text{ at } (x,y)=(0,0)$$

Hence continuous.

C-R $\Rightarrow$.

$$u_x(0,0) = \frac{u(h,0) - u(0,0)}{h} = \frac{h^3}{h^2 \cdot h} = 1.$$

$$u_y(0,0) = \frac{u(0,k) - u(0,0)}{k} = -1$$

$$v_x(0,0) = 1 \qquad v_y(0,0) = 1.$$

$\boxed{u_x = v_y}$ & $\boxed{u_y = -v_x}$ C-R equations hold at origin.

Not differentiable

↳ Derivative along x-axis $\Rightarrow$ $\frac{f(h,0) - f(0,0)}{h}.$

$$\Rightarrow \frac{h + ih}{h} = 1+i \quad -(1)$$

↳ Derivative along y-axis $\Rightarrow$ $\frac{f(0,k) - f(0,0)}{k}$

$$= \frac{-k + ik}{k} = -1+i \quad -(2)$$

(1) $\neq$ (2) Hence not differentiable

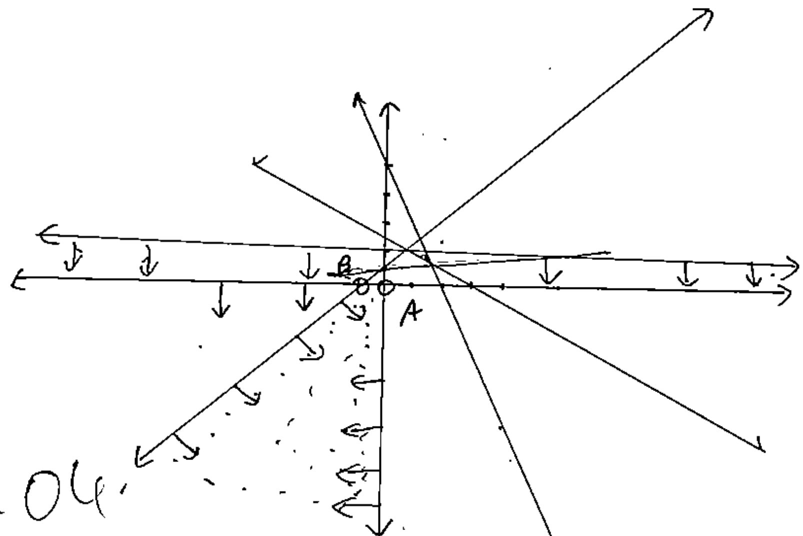
1. (e) Solve graphically the following LPP.

Maximise $Z = 3x_1 + 2x_2$

Subject to
$x_1 + 2x_2 \le 6$ — 1
$2x_1 + x_2 \le 8$ — 2
$-x_1 + x_2 \le 1$ — 3
$2x_2 \le 2$
$x_1, x_2 \ge 0$

(10)

Intersection pts ⇒



The given feasible region is shown dotted

The two corner points A, B are (0,0), ~~to~~ (-1,0)

$Z = \begin{cases} A = 0 \\ B = -3. \end{cases}$

So, max z is at A → (0,0)

Optimal z value is 0 Ans

P.T.O.

2. (a) (i) $H = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \middle| a, b, d \in \mathbf{R}, ad \neq 0 \right\}$. Is H a normal subgroup of GL $(2, \mathbf{R})$?

(ii) Construct a multiplication table for $Z_2[i]$, the ring of Gaussian integers modulo 2. Is this ring a field ? Is it an integral domain ?

(18)

2. (b) (i) Give an example of a family $\{I_n : n \in \mathbb{N}\}$ of non-empty closed intervals such that $I_1 \supset I_2 \supset I_3 \supset \ldots$ and $\bigcap_{n=1}^{\infty} I_n = \phi$.

(ii) Give an example of a family $\{I_n : n \in \mathbb{N}\}$ of bounded open intervals such that $I_1 \supset I_2 \supset I_3 \supset \ldots$ and $\bigcap_{n=1}^{\infty} I_n = \phi$.

(08)

2. **(c)** Show that the series

$$1-\frac{e^{-2x}}{2^2-1}+\frac{e^{-4x}}{4^2-1}-\frac{e^{-6x}}{6^2-1}+\ldots\ldots$$ converges uniformly for all $x \geq 0$. **(08)**

2. (d) Use the method of contour integration to prove that

$$\int_{-\pi}^{\pi} \frac{a\cos\theta}{a+\cos\theta}d\theta = 2\pi a\left\{1-\frac{a}{\sqrt{(a^2-1)}}\right\}, \text{ where } a>1.$$ (16)

3. (a) Suppose that a finite group is generated by two elements a and b (that is, every element of the group can be expressed as some product of a's and b's). Given that $a^3 = b^2 = e$ and $ba^2 = ab$, construct the Cayley table for the group. (06)

3. (b) Let $H = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \middle| a, b, c \in Q \right\}$ under matrix multiplication.

(i) Find Z(H).

(ii) Prove that Z(H) is isomorphic to Q under addition.

(iii) Prove that H/Z(H) is isomorphic to $Q \oplus Q$.

(iv) Are your proofs for parts a and b valid when Q is replaced by R ? Are they valid when Q is replaced by Z_p, where p is prime ? (12)

3. (c) A function f is defined on $(-1, 1)$ by $f(x) = x^{\alpha} \sin \frac{1}{x^{\beta}}, x \neq 0$

$= 0, x = 0.$

Prove that (i) if $0 < \beta < \alpha - 1$, f' is continuous at 0;

(ii) if $0 < \alpha - 1 \leq \beta$, f' is discontinuous at 0. **(14)**

3. (d) Find the optimal solution of the following transportation problem. **(18)**

	D_1	D_2	D_3	D_4	D_5	D_6	a_i
O_1	1	2	1	4	5	2	30
O_2	3	3	2	1	4	3	50
O_3	4	2	5	9	6	2	75
O_4	3	1	7	3	4	6	20
b_j	20	40	30	10	50	25	

4. (a) Let F be the field of integers modulo 5. Show that the polynomial $x^2 + 2x + 3$ is irreducible over F. Use this to construct a field containing 25 elements. **(12)**

$F = [0, 1, 2, 3, 4]$.

$f = x^2 + 2x + 3$

$x = 0 \Rightarrow f = 3$

$x = 1 \Rightarrow f = 6$

$x = 2 \Rightarrow f = 11$

$x = 3 \Rightarrow f = 18$

$x = 4 \Rightarrow f = 27$.

08

As $f \neq 0$ for all values of elements in F.

So, f is irreducible over F.

Let $F[x]$ be the field of polynomials over F.

Then, as f is irreducible $\langle f \rangle$ is a principal ideal.

Then, $\langle f \rangle / F[x]$ is a field.

Elements of $\frac{\langle f \rangle}{F[x]} = a_0 + a_1 x$ where $a_0, a_1 \in F = \{0,1,2,3,4\}$

So, set of polynomials $a_0 + a_1 x$ has $5 \times 5 = 25$ elements.

4. (b) Prove that the function f defined by

$f(x) = \sin\frac{1}{x} \qquad \forall x > 0$

is continuous but not uniformly continuous on R^+. (15)

On R^+,

Continuity of f:

Let $\alpha \in \mathbb{R}$, then

$\lim_{x \to \alpha^+} \sin 1/x$ ~~$= \sin 1/\alpha = $ value~~

$= \lim_{h \to 0} \sin \frac{1}{\alpha + h} = \sin 1/\alpha$

$\lim_{x \to \alpha^-} \sin 1/x = \lim_{h \to 0} \sin \frac{1}{\alpha - h} = \sin \frac{1}{\alpha}$

$\Big\}$ equals value of $\sin 1/x$ at α.

So, $\sin 1/x$ is continuous at α.

Let, x_n be $\left\{\frac{1}{n\pi}\right\}$.

y_n be $\left\{\frac{1}{n\pi+\frac{\pi}{2}}\right\}$.

So, $\forall\, n > \sqrt{\frac{1}{2\pi\varepsilon}}$ we have,

$|x_n - y_n| < \varepsilon$.

$$|x_n - y_n| = \left|\frac{1}{n\pi} - \frac{1}{n\pi+\frac{\pi}{2}}\right|$$

$$= \left|\frac{\pi/2}{n^2\pi^2 + \frac{n\pi^2}{2}}\right|.$$

$$< \left|\frac{1}{2\pi n^2}\right|$$

$$\frac{1}{2\pi n^2} < \varepsilon$$

$$\Rightarrow \boxed{n > \sqrt{\frac{1}{2\pi\varepsilon}}}$$

But, $\left|\sin\frac{1}{x_n} - \sin\frac{1}{y_n}\right|$

$$= \left|\sin n\pi - \sin\left(n\pi + \frac{\pi}{2}\right)\right|$$

$$= 1 > \frac{1}{2} (= \delta).$$

So, f is not uniformly continuous.

$\sin\frac{1}{x}$ is not uniformly continuous

4. (c) Evaluate the integral $\int_{r} \frac{z^2}{(z^2+1)(z-1)^2} dz$, where r is the circle $|z| = 2$. (10)

Poles of the given integrand are

$+i, -i, 1, 1.$

By Cauchy Integral Formula,

$$\int_{\Gamma} \frac{z^2}{(z^2+1)(z-1)^2} dz = 2\pi i \sum \text{Res at poles}$$

Res at.

$$z = 1 \Rightarrow \left\{\frac{z^2}{z^2+1}\right\}'_{at\ z=1} \Rightarrow \left[\frac{2z}{z^2+1} - \frac{2z^3}{(z^2+1)^2}\right]_{z=1}$$

$$= \left\{\frac{2}{2} - \frac{2.1}{4}\right\} = \frac{1}{2}$$

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$$z = i \Rightarrow \left\{\frac{z^2}{(z+i)(z-1)^2}\right\}_{z=i} = \left\{\frac{-1}{2i(\cancel{i^2+1}-2i)}\right\} = -\frac{1}{4}.$$

$$z = -i \Rightarrow \left\{\frac{z^2}{(z-i)(z-1)^2}\right\}_{z=-i} = \left\{\frac{-1}{+2i(2i)}\right\} = -\frac{1}{4}.$$

$$\int_{\Gamma} \frac{z^2}{(z^2+1)(z-1)^2} dz. = 2\pi i \left[\frac{1}{2} - \frac{1}{4} - \frac{1}{4}\right] = 0 \text{ Ans}$$

4. (d) Show by solving the following LPP by simplex method that the problem has an unbounded solution.

Maximise $Z = 107x_1 + x_2 + 2x_3$

subject to $14x_1 + x_2 - 6x_3 + 3x_4 = 7$

$16x_1 + x_2 - 6x_3 \le 5$

$3x_1 - x_2 - x_3 \le 0$

$x_1, x_2, x_3, x_4 \ge 0$ **(13)**

In standard form,

$$Z = 107x_1 + x_2 + 2x_3 - Mx_a.$$

$$14x_1 + x_2 - 6x_3 + 3x_4 + x_a = 7.$$

$$16x_1 + x_2 - 6x_3 + x_5 = 5.$$

$$3x_1 - x_2 - x_3 + x_6 = 0.$$

		x_1	x_2	x_3	x_4	x_5	x_6	x_a	
x_a	7	14	1	-6	3	0	0	1	-M
x_5	5	16	1	-6	0	1	0	0	0
x_6	0	3	-1	-1	0	0	1	0	0.
	-7M.	-14M-107	-M-1.	6M-2.	-3M	0	0	0.	
		↑							
x_a.	7	0	17/3	-4/3.	3	0	-14/3	1	
x_5	5	0.	19/3	-2/3	0	1	-16/3.	0.	
x_1	0	1	-1/3	-1/3	0	0	1/3	0	
	-7M.	0	$\frac{-17M-110}{3.}$	$\frac{4M-113}{3.}$	-3M	0	$\frac{14M+107}{3}$	0.	
			↑.						
	21/17.	0	1	-68	9/17.	0	-14/17	3/17.	

SECTION – B

5. (a) Solve $(y+z)p+(z+x)q=x+y$. **(10)**

$$\frac{dx}{y+z}=\frac{dy}{z+x}=\frac{dz}{x+y}$$

$$\frac{d(x-y)}{(y-x)}=\frac{d(y-z)}{(z-y)}=\frac{d(z-x)}{(x-z)} \quad -(1)$$

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$$=\frac{d(x+y+z)}{2(x+y+z)} \quad -(2)$$

From (1), (2).

$$\frac{x-y}{y-z} = c_1. \qquad \frac{z-x}{y-z} = c_2.$$

$$\boxed{\phi\left(\frac{x-y}{y-z}, \frac{z-x}{y-z}\right) = 0}$$ Ans

5. (b) Solve $(D+D'-1)(D+D'-3)(D+D')z = e^{x+y}\sin(2x+y)$ (10)

$$C.F = e^{x}\phi_1(y-x) + e^{3x}\phi_2(y-x) + \phi_3(y-x).$$

$(bD - aD' - c)$

$L\ e^{cx/b}\phi(by+ax)$

$$PI \Rightarrow \frac{1}{(D+D'-1)(D+D'-3)(D+D')} e^{x+y}\sin(2x+y).$$

$$e^{x+y}\frac{1}{(D+D'+1)(D+D'-1)(D+D'+2)}\sin(2x+y).$$

$$e^{x+y}\frac{1}{(D+D'+2)}\frac{1}{D^2+D'^2+2DD'-1}\sin(2x+y) = \frac{e^{x+y}}{(-10)}\frac{1}{D+D'+2}\sin(2x+y)$$

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$$-\frac{e^{x+y}}{810}\;\frac{(D+D'-2)}{D^2+D'^2+2DD'-4}\;\sin(2x+y)$$

$$+\frac{e^{x+y}}{810\cdot 13}\;\underbrace{(D+D'-2)\;(\sin 2x+y)}= 2\cos(2x+y)+\cos(2x+y)-2\sin(2x+y)$$

$$\boxed{\frac{e^{x+y}}{130}\left(3\cos(2x+y)-2\sin(2x+y)\right)}$$

Solution is ⇒

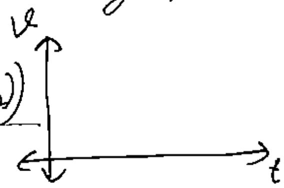
$$e^{x}\phi_1(y-x)+e^{3x}\phi_2(y-x)+\phi_3(y-x)+\frac{e^{x+y}}{130}\left(3\cos(2x+y)-2\sin(2x+y)\right)$$ Ans

5. (c) The velocities of a car (running on a straight road) at intervals of 2 minutes are given below.

Time in minutes	0	2	4	6	8	10	12
Velocity in km/hr.	0	22	30	27	18	7	0

Apply Simpson's rule to find the distance covered by the car. (10)

Distance covered is area under v-t graph.



Simpson's Rule $\Rightarrow \frac{h}{3}\left(f(x_0)+4f(x_1)+f(x_2)\right)$
(1/3)

~~h = 2~~ $h = 2/60$ hr.

So, distance covered is ⇒

$$\frac{2}{60\times 3}\left[0+0+4(22+30+27+18+7)\right]$$

$\boxed{4.622\text{ km}}$ Ans

$2(30+18)$

5. (d) A committee of three approves proposal by majority vote. Each member can vote for the proposal by pressing a button at the side of their chairs. These three buttons are connected to a light bulb. For a proposal whenever the majority of votes takes place, a light bulb is turned on. Design a circuit as simple as possible so that the current passes and the light bulb is turned on only when the proposal is approved. (10)

A	B	C	Output
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1.

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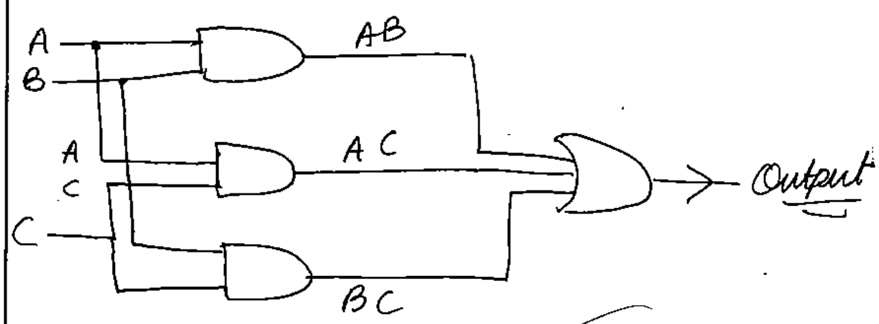
A \ BC	00	01	11	10
0			1	
1		1	1	1

Equation $\equiv BC + AB + CA$

Circuit is ⇒



5. (e) Use Hamilton's equations to find the equations of motion of a projectile in space. (10)

6. (a) Form partial differential equation by eliminating arbitrary functions f and g from $z = f(x^2 - y) + g(x^2 + y)$. **(08)**

$$\frac{\partial z}{\partial x} = f'(x^2-y).2x + g'(x^2+y).2x \quad -(1)$$

$$\frac{\partial z}{\partial y} = -f'(x^2-y) + g'(x^2+y)$$

$$\frac{\partial^2 z}{\partial y^2} = f''(x^2-y) + g''(x^2+y) \quad -(2)$$

$$\frac{\partial^2 z}{\partial x^2} = 2\left(f'(x^2-y) + g'(x^2+y)\right) + 4x^2\left(f''(x^2-y) + g''(x^2+y)\right)$$

$$\boxed{\frac{\partial^2 z}{\partial x^2} = \frac{1}{x}\frac{\partial z}{\partial x} + 4x^2 \cdot \frac{\partial^2 z}{\partial y^2}} \quad - \text{from}(1),(2)$$

Ans

6. (b) Find a surface satisfying r – 2s + t = 6 and touching the hyperbolic paraboloid z = xy along its section by the plane y = x. **(12)**

$C.F \Rightarrow \quad D^2 - 2DD' + D'^2 = 0.$

$\frac{2 \pm \sqrt{4-4}}{2} = 1.$

$1+\frac{D'}{D} \quad (1+D'$

$(D-D')^2 = 0.$

$$\boxed{\phi_1(x+y) + x\,\phi_2(x+y)}$$

$PI \Rightarrow \quad \frac{1}{(D-D')^2}\,6 = \cancel{\frac{\left(1-\frac{D'}{D}\right)^{-2}}{D^2}\,6}$

09

$= \frac{1}{D^2}\left(1 + \frac{2D'}{D} + \cancel{\ }\right) 6.$

$= \frac{1}{D}\,6x = (3x^2)$

END OF THE EXAMINATION

General Solution is ⇒

$$Z^0 = \boxed{\phi_1(x+y) + x\,\phi_2(x+y) + 3x^2} \quad -(A)$$

$$Z = xy \quad -(B)$$

As it touches the given curve, we equate the partial derivatives with respect to x, y of both ⇒

$$Z^0_x = \phi_1'(x+y) + \phi_2(x+y) + x\phi_2'(x+y) + 6x$$

$$Z^0_y = \phi_1'(x+y) + x\phi_2'(x+y)$$

$$Z_x = y \qquad Z_y = x. \qquad \text{at } \boxed{x=y}$$

$$y = \phi_1'(x+y) + \phi_2(x+y) + x\phi_2'(x+y) + 6x \quad -(1)$$

$$x = \phi_1'(x+y) + x\phi_2'(x+y) \quad -(2)$$

$$(1)-(2) \Rightarrow y - x = \phi_2(x+y) + 6x \quad \leftarrow \text{Put } y = x.$$

$$\phi_2(2x) = -3(2x) \Rightarrow \boxed{\phi_2(x+y) = -3(x+y)}$$

$$\cancel{\phi_2(x) = -3x \Rightarrow \phi_2'(x) = -3 \Rightarrow \phi_2'(x+y)}$$

$$\phi_2'(2x) = -6 \Rightarrow \boxed{\phi_2'(x+y) = -3}$$

$$x = \phi_1'(x+y) + x(-3) \Rightarrow \phi_1'(x+y) = 4x.$$

$$\text{At } x = y \Rightarrow \phi_1'(2x) = \tfrac{4}{2}(2x). \Rightarrow \cancel{\phi_1'(t) = \tfrac{7}{2}t}$$

$$\phi_1'(x+y) = 2(x+y) \Rightarrow \qquad \Rightarrow \cancel{\phi_1(t) = \tfrac{7}{4}t^2 + C}$$

$$\cancel{\Rightarrow \phi_1(x+y) = \tfrac{7}{4}(x+y)^2 + C} \qquad \boxed{\phi_1(x+y) = (x+y)^2 + C.}$$

Equate A, B to get C ⇒

$$x^2 = \cancel{\tfrac{7}{4}(2x)^2 + C + x(-6x) + 3x^2}\;(x+y)^2 + x(-3x-3y) + 3x^2 + C$$

$$x^2 = 4x^2 - 6x^2 + 3x^2 + C \Rightarrow \boxed{C = 0} \Rightarrow \boxed{\phi_1 = (x+y)^2}.$$

$z = x^2 - xy + \ldots$

P.T.O.

6. (c) Reduce $y^2(\partial^2 z/\partial y^2)+\partial^2 z/\partial x^2=0$ to canonical form **(12)**

$$\frac{\partial^2 z}{\partial x^2}+y^2\frac{\partial^2 z}{\partial y^2}=0.$$

$$R\lambda^2+S\lambda+T=0 \Rightarrow \lambda^2+y^2=0 \Rightarrow \lambda=\pm iy.$$

$$\frac{dy}{dx}=\mp iy \Rightarrow \log y=\mp ix+c \Rightarrow \cancel{y\pm c}$$

$$\Rightarrow y=ce^{\pm ix}=\cancel{c(\cos x\pm\sin x)}$$

$\cancel{u=\cos x,\ v=\sin x}$ $\cancel{\frac{\partial z}{\partial x}}$ $\boxed{u=ye^{ix}}$ $\boxed{v=ye^{-ix}}$ $\cancel{W=\begin{vmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{vmatrix}=1\neq 0}$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\,iye^{ix}-iye^{-ix}\frac{\partial z}{\partial v};\quad \frac{\partial^2 z}{\partial x^2}=-ye^{ix}z_u+iy(e^{ix})\left[\frac{\partial^2 z}{\partial u^2}iye^{ix}+\frac{\partial^2 z}{\partial u\partial v}\cdot -iye^{-ix}\right]$$

$$-ye^{ix}z_u+iye^{ix}\left[z_{uu}\,iye^{ix}-z_{uv}\,iye^{-ix}\right]-ye^{-ix}z_v$$

$$-iye^{-ix}\left[z_{vu}\,iye^{ix}-z_{vv}\,iye^{-ix}\right]$$

$$\boxed{-uz_u+(-1)u^2z_{uu}-v^2z_{vv}-vz_v=\frac{\partial^2 z}{\partial x^2}.}$$

$$\partial z/\partial y=z_u e^{ix}+z_v e^{-ix}\qquad \frac{\partial^2 z}{\partial y^2}=z_{uu}(e^{ix})^2+z_{uv}+z_{vu}+z_{vv}(e^{-ix})^2.$$

$$y^2\frac{\partial^2 z}{\partial y^2}=u^2z_{uu}+v^2z_{vv}+2y^2z_{uv}=u^2z_{uu}+v^2z_{vv}+2uvz_{uv}$$

Adding $\Rightarrow$ $\boxed{2uvz_{uv}-uz_u-vz_u=0}$

P.T.O.

6. (d) A tightly stretched elastic string of length l, with fixed end points $x = 0$ and $x = l$ is initially in the position given by $y = C\sin^3(\pi x / l)$, C being constant. It is released from the position of rest. Find the displacement $y(x, t)$. (18)

The wave equation to this motion is,

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}.$$

Assuming solution as $Y = X(x)T(t)$.

$$\Rightarrow \frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2 T(t)} = k.$$

p is real

$k = -p^2 \Rightarrow X(x)T(t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt). \quad -(1)$

$k = p^2 \Rightarrow X(x)T(t) = (C_1 e^{px} + C_2 e^{-px})(C_3 e^{cpt} + C_4 e^{-cpt}) \quad -(2)$

$k = 0 \Rightarrow X(x)T(t) = (C_1 x + C_2)(C_3 t + C_4). \quad -(3)$

Given conditions $\Rightarrow$

$Y(0,t) = Y(l,t) = 0$

$\frac{\partial Y}{\partial t}(x,0) = 0. \qquad Y(x,0) = C\sin^3\left(\frac{\pi x}{l}\right).$

(2), (3) are rejected as they do not satisfy constraints.

(1) $\Rightarrow Y(0,t) = 0 \Rightarrow C_1 = 0$

$\frac{\partial Y}{\partial t}(x,0) = 0 \Rightarrow C_4 = 0.$

P.T.O.

$Y(l, t) = 0 \Rightarrow \sin pl = 0 \Rightarrow pl = n\pi$

$p = \frac{n\pi}{l}.$

So,

$Y \equiv C_2 \sin\left(\frac{n\pi x}{l}\right) . C_3 \cos\left(\frac{cn\pi}{l} t\right).$

$Y = \sum C_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi c t}{l}\right).$

Using initial condition $\Rightarrow$

$C \sin \frac{3\pi x}{l} = \sum C_n \sin\left(\frac{n\pi x}{l}\right).$

$\Downarrow$

$\boxed{\sin 3\theta = 3\sin\theta - 4\sin^3\theta.}$

$\frac{3C}{4} \sin\frac{\pi x}{l} - \frac{C}{4} \sin \frac{3\pi x}{l} = \sum C_n \sin\left(\frac{n\pi x}{l}\right).$ $\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}.$

$n = 1 \Rightarrow C_1 = 3C/4$

$n = 3 \Rightarrow C_3 = -C/4$

$\boxed{C_n = 0 \text{ for all other } n}$

$$Y = \frac{3C}{4}\sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi c t}{l}\right) - \frac{C}{4}\sin\left(\frac{3\pi x}{l}\right)\cos\left(\frac{3\pi c t}{l}\right)$$

Ans

7. (a) Solve the following system of linear equations correct to two decimal places by Gauss-Seidel method.
$10x+2y+z=9$
$2x+20y-2z=-44$
$-2x+3y+10z=22$ (15)

$$x = \frac{9-2y-z}{10}$$

Starting with $(0,0,0)$

$$y = \frac{-44-2x+2z}{20}$$

$$z = \frac{22+2x-3y}{10}$$

x	y	z
0	0	0
0.9	-2.29	3.067
1.051	-1.94	2.992
0.988	-1.996	2.9964
0.999	-2.000	2.999

Hence, solution tends to exact values $(1, -2, 3)$ which is the required solution

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7. (b) Given $\frac{dy}{dx} = 1 + y^2$, where y = 0 when x = 0, find y(0.2) y(0.4) and y (0.6) **(14)**

Using Euler's method,

$h = 0.2$. $f \equiv 1 + y^2$

$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$

$y(0.2) = 0 + 0.2(1 + 0^2) = \boxed{0.2}$

$y(0.4) = 0.2 + 0.2(1 + 0.2^2)$

$= \boxed{0.408}$

$y(0.6) = 0.408 + 0.2(1 + 0.408^2)$

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$= \boxed{0.641}$

7. (c) Convert :
(i) 46655 given to be in the decimal system into one in base 6.
(ii) $(11110.01)_2$ into a number in the decimal system. (06)

(i)

6	46655	5
6	7775	5
6	1295	5
6	215	5
6	35	5
	5	5

$\boxed{555555}$ Ans

(In Base 6)

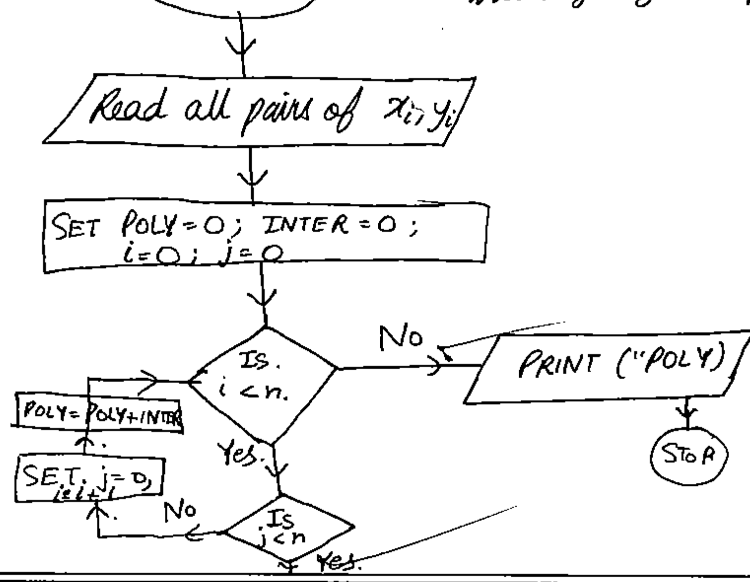
(ii) $(11110.01)_2$

$$= 0\times 1 + 1\times 2 + 1\times 2^2 + 1\times 2^3 + 1\times 2^4$$
$$+ 0\times 2^{-1} + \phi\times 2^{-2}$$
$$= 2+4+8+16+\tfrac{1}{4} = \boxed{30.25}\ \text{Ans}$$

7. (d) Draw a flow chart for Lagrange's interpolation formula (15)

(x_i, y_i) are from $(0$ to $n-1)$ for (i)

Given: (x_i, y_i) pair
To OUTPUT : Polynomial via Lagrange interpolation.

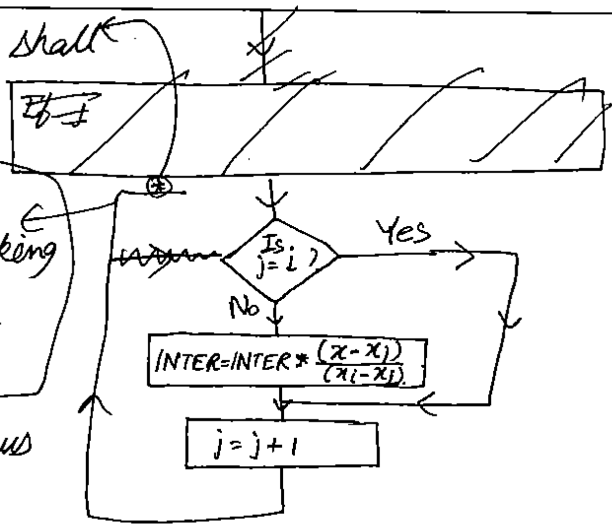


P.T.O.

This line shall join

Before the box, checking Is $j < n$

On previous page



8. (a) Two equal rods AB and BC, each of length l smoothly joined at B are suspended from A and oscillate in a vertical plane through A. Show that the periods of normal oscillations are $\frac{2\pi}{n}$, where $n^2 = \left(3 \pm \frac{6}{\sqrt{7}}\right)\frac{g}{l}$. (16)

8. (b) If n rectilinear vortices of the same strength k are symmetrically arranged along generators of a circular cylinder of radius a in an infinite liquid, prove that the vortices will move round the cylinder uniformly in time $\frac{8\pi^2a^2}{(n-1)k}$, and find the velocity at any point of the liquid. (16)

8. (c) An infinite mass of fluid acted on by a force $\mu r^{-3/2}$ per unit mass is directed to the origin. If initially the fluid is a rest and there is a cavity in the form of the sphere $r = c$ in it, show that the cavity will be filled up after an interval of time $(2/5\mu)^{1/2} c^{5/4}$. (18)

ROUGH SPACE

$x^2+y^2=0 \quad \lambda = \pm iy.$

$\frac{dy}{dx} + \pm xiy.$

$\log y \pm ix = c.$

$\frac{dy}{iy} \pm i\,dx$

$y = e^{\pm ix}$

$u \quad y - e^{ix}$

$v \quad y - e^{-ix}.$

$\frac{\partial z}{\partial x} = z_x$

P.T.O.