TARUN KAUSHIK

MATHEMATICS
For
UPSC CSE MAINS

Topic: ANALYTIC GEOMETRY PART -7 (UPSC QUESTIONS)

Scanned by CamScanner

Q 4(d) 2016 15 Marks

Find the Locus of the point of intersection of three mutually perpendicular tangent planes to the conicoid $ax^2+by^2+cz^2=1$.

Solⁿ

Conicoid $\Rightarrow \quad ax^2+by^2+cz^2=1$ ——— (1)

let one of the mutually ⊥ tangent plane to eq.(1) is

$$l_1x+m_1y+n_1z=p \quad \text{----} (2)$$

Acc. to the condition of tangency :

$$\frac{l_1^2}{a}+\frac{m_1^2}{b}+\frac{n_1^2}{c}=p^2$$

$$\Rightarrow \quad p=\sqrt{\frac{l_1^2}{a}+\frac{m_1^2}{b}+\frac{n_1^2}{c}}$$

so eqⁿ (2) becomes.

$$l_1x+m_1y+n_1z=\sqrt{\frac{l_1^2}{a}+\frac{m_1^2}{b}+\frac{n_1^2}{c}} \quad \text{———} (3)$$

Similliarly other two mutually ⊥ tangents planes eqⁿ :

$$l_2x+m_2y+n_2z=\sqrt{\frac{l_2^2}{a}+\frac{m_2^2}{b}+\frac{n_2^2}{c}} \quad \text{———} (4)$$

&

$$l_3x+m_3y+n_3z=\sqrt{\frac{l_3^2}{a}+\frac{m_3^2}{b}+\frac{n_3^2}{c}} \quad \text{———} (5)$$

$\Rightarrow$ Squaring & adding eqⁿ (3), (4) + (5); we get

$$\Rightarrow x^2(l_1^2+l_2^2+l_3^2)+y^2(m_1^2+m_2^2+m_3^2)+z^2(n_1^2+n_2^2+n_3^2)+$$
$$2xy(l_1m_1+l_2m_2+l_3m_3)+2yz(n_1m_1+n_2m_2+n_3m_3)+$$
$$2xz(l_1n_1+l_2n_2+l_3n_3)=\frac{1}{a}(l_1^2+l_2^2+l_3^2)+\frac{1}{b}(m_1^2+m_2^2+m_3^2)$$
$$+\frac{1}{c}(n_1^2+n_2^2+n_3^2)$$

$\Rightarrow \boxed{x^2+y^2+z^2=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$ Ans.

$\because \boxed{l_1^2+l_2^2+l_3^2=1}$

& $\boxed{l_1m_1=0}$

Tarun Kaushik

Scanned by CamScanner

Que:- 1(d) Find the equation of sphere which passes through the circle $x^2+y^2=4$; $z=0$ and is cut by the plane $x+2y+2z=0$ in a circle of radius 3.

10 Marks

Sol.^{n} Given circles $x^2+y^2=4$ & $z=0$

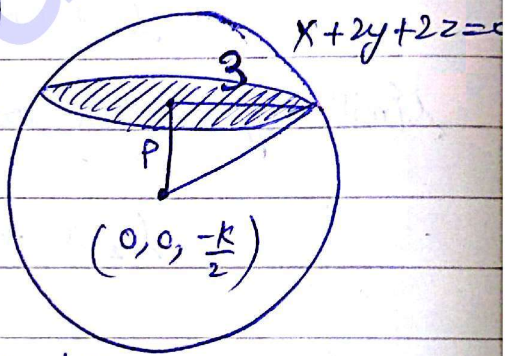
$$\because \quad z=0 \; ; \quad x^2+y^2+z^2-4=0$$

So Required sphere eq.^{n} is intersection of the above two.

$$\Rightarrow \boxed{x^2+y^2+z^2-4+Kz=0}$$

So Centre $\left(0,0,-\frac{K}{2}\right)$

Radius $= R = \sqrt{\frac{K^2}{4}+4}$



P i.e. $\perp$ distance from Origin to foot of $\perp$ to Plane is given by

$$P = \left|\frac{0+0-K}{\sqrt{1+4+4}}\right| = \left|\frac{-K}{3}\right| = \frac{K}{3}$$

$$\because \quad R^2-P^2=3^2$$

$$\Rightarrow \quad \frac{K^2}{4}+4-\frac{K^2}{9}=9 \Rightarrow \frac{5K^2}{36}=5 \Rightarrow K^2=36$$

$$\Rightarrow \boxed{K=\pm 6}$$

So Req. Eq^n of sphere is $\boxed{x^2+y^2+z^2\pm 6z-4=0}$

Ans.

Tarun Kaushik

Scanned by CamScanner

Q: 1(e) find the shortest distance b/w the lines

10 Marks $\quad \frac{x-1}{2}=\frac{y-2}{4}=\frac{z-3}{1}$ and $y-mx=z=0$.

for what values of m will be the two lines intersects.

Sol.^{n}

$$\frac{x-1}{2}=\frac{y-2}{4}=\frac{z-3}{1} \text{ ——— Line 1}$$

$$y-mx=0\,;\ z=0 \Rightarrow y=mx\,;\ z=0 \Rightarrow \frac{x}{1}=\frac{y}{m}=\frac{z}{0} \text{ -- Line 2}$$

$$\vec{a_1}=\hat{i}+2\hat{j}+3\hat{k} \qquad \vec{b_1}=2\hat{i}+4\hat{j}+\hat{k}$$

$$\vec{a_2}=0\hat{i}+0\hat{j}+0\hat{k} \qquad \vec{b_2}=\hat{i}+m\hat{j}+0\hat{k}$$

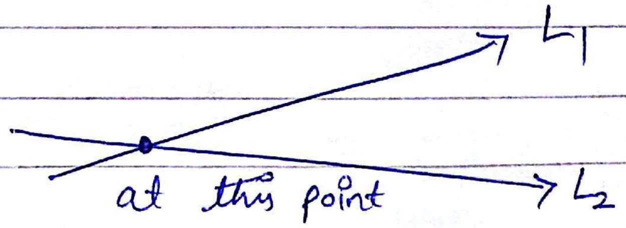
$$\vec{a_2}-\vec{a_1}=-\hat{i}-2\hat{j}-3\hat{k} \qquad \vec{b_1}\times\vec{b_2}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 4 & 1\\ 1 & m & 0\end{vmatrix}$$

$$= -m\hat{i}+\hat{j}+(2m-4)\hat{k} \Leftarrow \vec{b_1}\times\vec{b_2}=\hat{i}(-m)-\hat{j}(-1)+\hat{k}(2m-4)$$

$$|\vec{b_1}\times\vec{b_2}|=\sqrt{m^2+1+4m^2+16-16m}$$

$$\boxed{S.D.=\frac{m-2-6m+12}{\sqrt{5m^2-16m+17}}} = \boxed{\frac{-5m+10}{\sqrt{5m^2-16m+17}}} \text{ Ans.}$$

Now



$$S.D.=0=\frac{-5m+10}{\sqrt{5m^2-16m+17}}$$

at this point distance will be zero.

$$\Rightarrow -5m+10=0 \Rightarrow \boxed{m=2} \text{ Ans.}$$

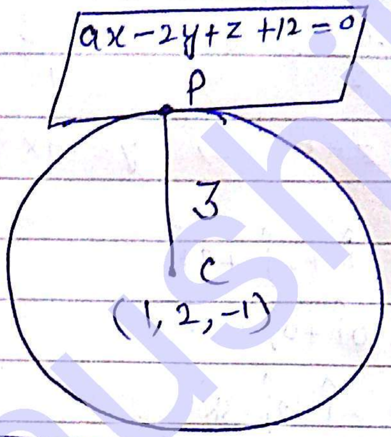
Tarun Kaushik

Scanned by CamScanner

Q 2015
10 Marks

For what +ve values of a, the plane $ax-2y+z+12=0$ touches the sphere $x^2+y^2+z^2-2x-4y+2z-3=0$ and hence find the point of contact.

Sol^n Centre $= (1, 2, -1)$



$$r=\sqrt{1^2+2^2+1^2+3}=\sqrt{9}=3$$

Now $\perp$ distance from the centre of sphere to plane is given by

$$\boxed{r=\frac{a(1)-2(2)+(-1)+12}{\sqrt{a^2+(-2)^2+(1)^2}}}$$

& $r=3$

$$\Rightarrow \quad 9=\frac{(a+7)^2}{a^2+5} \Rightarrow 9a^2+45=a^2+14a+49$$

$$\Rightarrow 8a^2-14a-4=0 \Rightarrow 4a^2-7a-2=0$$

$$a=\frac{7\pm\sqrt{49+32}}{8}=\frac{7\pm 9}{8}$$

(i) Ans $\boxed{a=2}$ $\Leftarrow$ $a=\frac{16}{8}, \frac{-2}{8}$ ✗

Plane Eq^n $\Rightarrow$ $2x-2y+z+12=0$ ——— (1)

Eq^n of Line PC $\Rightarrow$ $\frac{x-1}{2}=\frac{y-2}{-2}=\frac{z+1}{1}=r$

$(2r+1, -2r+2, r-1)$

Put generalized point in eq. (1)

$$\Rightarrow 4r+2+4r-4+r-1+12=0 \Rightarrow 9r+9=0$$

$$\boxed{r=-1}$$

$\Rightarrow$ $(-1, 4, -2)$ point of contact Ans (ii)

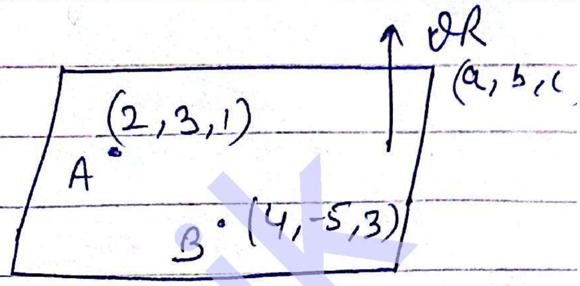
Tarun Kaushik

Scanned by CamScanner

Q(2015) Obtain the eq^n of the plane passing through the points $(2,3,1)$ & $(4,-5,3)$ parallel to x-axis.
6 Marks

Sol^n

Let Direction Ratio's are
$\langle a, b, c \rangle$



then eq^n of Plane at A point is given by :

$$a(x-2) + b(y-3) + c(z-1) = 0 \quad \text{———} \quad (1)$$

Since point B is lying on the plane, so it will satisfy the equation of Plane, so we get;

$$a(4-2) + b(-5-3) + c(3-1) = 0$$

$$2a - 8b + 2c = 0 \quad \text{———} \quad (2)$$

Now DR's of a line parallel to x axis is
$\langle 1, 0, 0 \rangle$

$\Rightarrow$ DR's of line & ~~given~~ Req. Plane will be $\perp$.

$$\Rightarrow \quad a(1) + b(0) + c(0) = 0$$

$$\Rightarrow \boxed{a = 0} \quad \Rightarrow \quad 2c = 8b$$

$$\boxed{c = 4b}$$

Put these values in eq^n (1); we get

$$0(x-2) + b(y-3) + 4b(z-1) = 0$$

$$\Rightarrow \quad b(y-3) + 4b(z-1) = 0$$

$$\Rightarrow \quad b(y-3+4z-4) = 0$$

$$\Rightarrow \quad (y-3+4z-4) = 0$$

$$\Rightarrow \quad \boxed{y + 4z - 7 = 0}$$ Req. Eq^n of Plane.

Tarun Kaushik

Scanned by CamScanner