

DATE: _____

A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET



206/250

MAINS TEST SERIES-18

JUNE-2018 TO SEPT-2018

Under the guidance of K. Venkanna

MATHEMATICS

PAPER - 1 : FULL SYLLABUS

TEST CODE: TEST-05: IAS(M)/08-JULY-2018

Time: Three Hours

Maximum Marks: 250

INSTRUCTIONS

1. This question paper-cum-answer booklet has 52 pages and has 32 PART/SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated.
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any **THREE** of the remaining questions selecting at least **ONE** question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any page from this booklet.

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name K. VARUN REDDYRoll No. 6314286Test Centre OLD RAJENDRA NAGARMedium ENGLISH

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Varun Reddy

Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the invigilator

IMPORTANT NOTE:

Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

$$+\frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$$

$$-\frac{3}{4} + \frac{3}{2} = +\frac{1}{4} + \frac{1}{2}$$

$$\frac{1+2}{4} = \frac{3}{4} = \frac{1}{4} - \frac{1}{2}$$

P.T.O.

**DO NOT WRITE ON
THIS SPACE**

INDEX TABLE

QUESTION	No.	PAGENO.	MAX. MARKS	MARKS OBTAINED
1	(a)			04
	(b)			09
	(c)			08
	(d)			08
	(e)			08
				37
2	(a)			
	(b)			
	(c)			
	(d)			
3	(a)			13
	(b)			18
	(c)			13
	(d)			
				44
4	(a)			
	(b)			
	(c)			
	(d)			
5	(a)			08
	(b)			08
	(c)			08
	(d)			08
	(e)			08
				40
6	(a)			11
	(b)			08
	(c)			08
	(d)			15
				42
7	(a)			
	(b)			
	(c)			
	(d)			
8	(a)			13
	(b)			13
	(c)			07
	(d)			10
Total Marks				206/250

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SECTION - A

1. (a) Let $W_1 = \langle [2 \ 0 \ 3 \ 1 \ 1]^T, [1 \ 0 \ 2 \ 1 \ 1]^T, [2 \ 0 \ 3 \ 1 \ 3]^T \rangle$
 and $W_2 = \langle [2 \ 1 \ 1 \ 0 \ 1]^T, [3 \ 2 \ 3 \ 2 \ 3]^T, [1 \ 1 \ 1 \ 1 \ 1]^T \rangle$
 be subspaces of \mathbb{R}^5 . Find a basis for $W_1 + W_2$ and a basis for $W_1 \cap W_2$. [10]

To find basis for $W_1 + W_2$

We know, $W_1 + W_2 = L(W_1 \cup W_2)$ (is a subspace)

\Rightarrow consider the matrix

$$A = \begin{bmatrix} 2 & 0 & 3 & 1 & 1 \\ 1 & 0 & 2 & 1 & 1 \\ 2 & 0 & 3 & 1 & 3 \\ 2 & 1 & 1 & 0 & 1 \\ 3 & 2 & 3 & 2 & 3 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 3 & 1 & 1 \\ 0 & 0 & 1/2 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & -2 & -1 & 0 \\ 0 & 2 & -3/2 & 1/2 & 3/2 \\ 0 & 1 & -1/2 & 1/2 & 1/2 \end{bmatrix} \begin{array}{l} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - R_1 \\ R_4 \rightarrow R_4 - R_1 \\ R_5 \rightarrow R_5 - 3R_1 \\ R_6 \rightarrow R_6 - R_1 \end{array}$$

$$\sim \begin{bmatrix} 2 & 0 & 3 & 1 & 1 \\ 0 & 2 & -3/2 & 1/2 & 3/2 \\ 0 & 1 & -1/2 & 1/2 & 1/2 \\ 0 & 1 & -2 & -1 & 0 \\ 0 & 0 & 1/2 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{array}{l} R_2 \leftrightarrow R_5 \\ R_3 \leftrightarrow R_6 \end{array} \sim \begin{bmatrix} 2 & 0 & 3 & 1 & 1 \\ 0 & 2 & -3/2 & 1/2 & 3/2 \\ 0 & 0 & 1/4 & 1/4 & -1/4 \\ 0 & 0 & -5/4 & -5/4 & -3/4 \\ 0 & 0 & 1/2 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{array}{l} R_3 \rightarrow R_3 - R_2 \\ R_4 \rightarrow R_4 - R_2 \end{array}$$

$$\sim \begin{bmatrix} 2 & 0 & 3 & 1 & 1 \\ 0 & 4 & -3 & 1 & 3 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & -5 & -5 & -3 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{array}{l} R_2 \rightarrow 2R_2 \\ R_3 \rightarrow 4R_3 \\ R_4 \rightarrow 4R_4 \\ R_5 \rightarrow 2R_5 \end{array} \sim \begin{bmatrix} 2 & 0 & 3 & 1 & 1 \\ 0 & 4 & -3 & 1 & 3 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & -8 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{array}{l} R_4 \rightarrow R_4 + 5R_3 \\ R_5 \rightarrow R_5 - R_3 \end{array}$$

Thus we get basis $\{ (2 \ 0 \ 3 \ 1 \ 1)^T; (0 \ 4 \ -3 \ 1 \ 3)^T; (0 \ 0 \ 1 \ 1 \ -1)^T; (0 \ 0 \ 0 \ 0 \ 2)^T \}$
 or $W_1 + W_2$

basis for w_1, w_2

$$\left. \begin{aligned} w_1 &= x(20311) + y(10211) + z(20313) \\ w_2 &= a(21101) + b(32323) + c(11111) \end{aligned} \right\} \text{equating}$$

Equating the two we get relation in x, y, z, a, b, c which gives the basis for w_1, w_2

Hence the result.

1. (b) Find the characteristic polynomial of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & i & 0 & 0 \\ 2 & \frac{1}{2} & -i & 0 \\ \frac{1}{3} & -i & \pi & -1 \end{bmatrix}$$

Diagonalise this matrix, if possible.

[10]

characteristic polynomial is $|A - \lambda I| = 0$

Given $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & i & 0 & 0 \\ 2 & \frac{1}{2} & -i & 0 \\ \frac{1}{3} & -i & \pi & -1 \end{bmatrix}$ (lower triangular matrix)

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ -1 & i-\lambda & 0 & 0 \\ 2 & \frac{1}{2} & -i-\lambda & 0 \\ \frac{1}{3} & -i & \pi & -1-\lambda \end{vmatrix} = (1-\lambda)(i-\lambda)(-i-\lambda)(-1-\lambda) \\ &= (\lambda^2 - 1)(\lambda^2 - i^2) \\ &= (\lambda^2 - 1)(\lambda^2 + 1) \\ &= (\lambda^4 - 1) \end{aligned}$$

(2-1)
(4+1)

now characteristic polynomial is $|A - \lambda I| = 0$

$$\Rightarrow \lambda^4 - 1 = 0 \Rightarrow \lambda = 1, -1, i, -i$$

Eigen roots are $\{1, -1, i, -i\}$ } 4 distinct eigen roots
 \Rightarrow matrix is diagonalizable

A is similar to diagonal matrix

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}$$

\Rightarrow For each Eigen root, we find the corresponding Eigen vector

which gives the matrix P
 which diagonalizes

$$P^{-1}AP = D$$

if we take roots

we get $1, i, -i, -1$

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

1. (c) Find the values of a and b in order that

$$\lim_{x \rightarrow 0} \frac{x(1+a \cos x) - b \sin x}{x^3} \text{ may be equal to } 1.$$

[10]

Given

$$A = \lim_{x \rightarrow 0} \frac{x(1+a \cos x) - b \sin x}{x^3} = 1 \quad \text{0/0 form}$$

By L-Hospital rule - differentiate

$$A = \lim_{x \rightarrow 0} \frac{(1+a \cos x) + x(-a \sin x) - b \cos x}{3x^2} = 1$$

to continue further numerator should be "0" at $x=0$

$$\Rightarrow \underline{1+a-b=0} \Rightarrow \boxed{b-a=1} \quad \text{--- (1)}$$

Again applying L-Hospital rule

$$A = \lim_{x \rightarrow 0} \frac{-a \sin x - a \sin x - a x \cos x + b \sin x}{6x} = 1$$

to continue further numerator should be "0" at $x=0$
which is true

$$A = \lim_{x \rightarrow 0} \left(\left(\frac{-2a}{6} \right) \left(\frac{\sin x}{x} \right) + \frac{b}{6} \left(\frac{\sin x}{x} \right) - \frac{a}{6} \cos x \right) = 1$$

$$\Rightarrow A = \frac{-2a}{6} + \frac{b}{6} - \frac{a}{6} = 1 \quad \left(\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right)$$

$$\Rightarrow \boxed{-3a+b=6} \quad \text{--- (2)}$$

From ①, ② we get

$$\begin{aligned} b - a &= 1 \\ -3a + b &= 6 \end{aligned}$$

$$\Rightarrow a = -5/2 ; b = -3/2$$

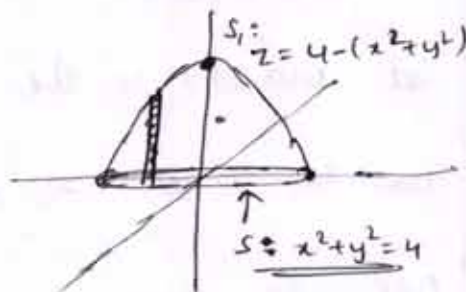
Hence the result

08-

1. (d) Find the volume of the region lying below the paraboloid with equation $z = 4 - x^2 - y^2$ and above the xy -plane. [10]

Given paraboloid $z = 4 - x^2 - y^2$

It makes surface: $x^2 + y^2 = 4$ on xy plane ($z=0$)



To calculate volume over surface S and below Paraboloid

Taking Polar coordinates (for surface S)

08-

$$x = r \cos \theta ; y = r \sin \theta ; 0 \leq r \leq 2 \quad 0 \leq \theta \leq 2\pi$$

$$\text{and } z = 4 - (x^2 + y^2) = 4 - r^2$$

$$\begin{aligned}
 \therefore \text{volume} &= \iint z \, dx \, dy = \iint (4-r^2) \, r \, dr \, d\theta \\
 &= \int_{\theta=0}^{2\pi} \int_{r=0}^2 (4-r^2) r \, dr \, d\theta = \int_{\theta=0}^{2\pi} \int_{r=0}^2 (4r-r^3) \, dr \, d\theta \\
 &= \left[2r^2 - \frac{r^4}{4} \right]_0^2 \cdot \left[\theta \right]_0^{2\pi} = (8-4)(2\pi) \\
 &= \underline{\underline{8\pi}}
 \end{aligned}$$

\therefore volume Enclosed by Paraboloid above xy plane is 8π

Hence the result

1. (c) Find the volume of a tetrahedron in terms of the lengths of the three edges which meet in point and of the angles which these edges make with each other in pairs. [10]

let $O(0,0,0)$ be the vertex

OA - be x -axis $\Rightarrow A(a,0,0)$

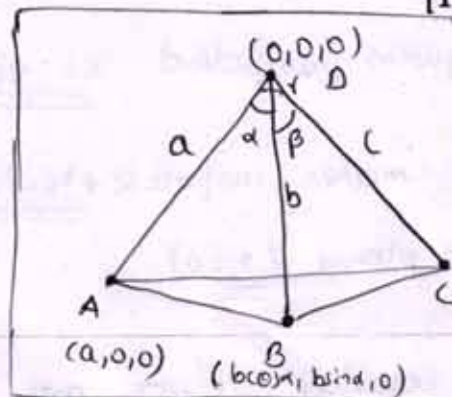
OAB - xy plane

$\Rightarrow DB$ makes angle " α " with DA ; length $DB = b$

\Rightarrow coordinates of B $(b \cos \alpha, b \sin \alpha, 0)$

NOW vertex C

let D.C's of line DC be l, m, n



length $\underline{DC} = c$; makes angles β with \underline{DB} ; γ with \underline{DA}

$$\Rightarrow \cos \beta = \frac{l \cdot b \cos \alpha + m \cdot b \sin \alpha + n \cdot 0}{b} \quad \text{--- (1)}$$

$$\Rightarrow \boxed{\cos \beta = l \cos \alpha + m \sin \alpha} \quad \text{--- (1)}$$

and

$$\cos \gamma = \frac{l \cdot a + m \cdot 0 + n \cdot 0}{a}$$

$$\Rightarrow \boxed{l = \cos \gamma} \quad \text{--- (2)}$$

from (1), we get

$$m = \frac{\cos \beta - \cos \gamma \cos \alpha}{\sin \alpha} = \cos \beta \operatorname{cosec} \alpha - \cos \gamma$$

NOW

$$n = \sqrt{1 - l^2 - m^2} = \sqrt{1 - \cos^2 \gamma - \cos^2 \beta \operatorname{cosec}^2 \alpha}$$

$$= \sqrt{1 - \cos^2 \gamma - \cos^2 \beta \operatorname{cosec}^2 \alpha - \cos^2 \gamma \cot^2 \alpha + 2 \cos \beta \cos \gamma \operatorname{cosec} \alpha \cot \alpha} \quad \text{--- (3)}$$

$$\Rightarrow \underline{C(l, m, n)}$$

NOW

volume of the tetrahedron =

$$= \frac{1}{6} \begin{vmatrix} 0 & 0 & 0 & 1 \\ a & 0 & 0 & 1 \\ b \cos \alpha & b \sin \alpha & 0 & 0 \\ l c & m c & n c & 0 \end{vmatrix} = \frac{1}{6} \cdot abc \cdot \sin \alpha \cdot (n)$$

Hence the result

(put value of (n) from (3))

2. (a) Investigate for what values of λ, μ the simultaneous equations.
 $x + y + z = 6, x + 2y + 3z = 10, x + 2y + \lambda z = \mu$
have (i) no solution, (ii) a unique solution, (iii) an infinite number of solutions.

[10]

3. (a) Let $W = \{[x_1 \ x_2 \ x_3 \ x_4]^t \in \mathbb{R}^4 \mid 2x_1 + 3x_2 = 4x_3 + x_4\}$. Show that W is a subspace of \mathbb{R}^4 . Find a basis of W and extend it to form a basis of \mathbb{R}^4 . Do the same if $W = \{[x_1 \ x_2 \ x_3 \ x_4]^t \in \mathbb{R}^4 \mid x_1 + x_2 = 0, x_3 - x_4 = 0\}$. [15]

Given $W = \{ (x_1 \ x_2 \ x_3 \ x_4)^t \mid 2x_1 + 3x_2 = 4x_3 + x_4 \}$

*One condition \Rightarrow ~~only~~ 3 free variables

let x_1, x_2, x_3 be free variables

$$\Rightarrow x_4 = 2x_1 + 3x_2 - 4x_3 \quad \text{--- (1)}$$

$$\Rightarrow W = \{ x_1, x_2, x_3, 2x_1 + 3x_2 - 4x_3 \}$$

$$= x_1(1, 0, 0, 2) + x_2(0, 1, 0, 3) + x_3(0, 0, 1, -4)$$

$$\therefore \text{Basis of } W = (0, 1, 0, 3)^t; (1, 0, 0, 2)^t; (0, 0, 1, -4)^t$$

It is subspace ~~because~~ because

linear span of 3 vectors is the shortest subspace containing 3 vectors

Proof let $a \in W$; $a = x_1(1, 0, 0, 2) + x_2(0, 1, 0, 3) + x_3(0, 0, 1, -4)$

$b \in W$; $b = y_1(1, 0, 0, 2) + y_2(0, 1, 0, 3) + y_3(0, 0, 1, -4)$

$$\therefore \underline{a-b} = \underline{(x_1 - y_1)(1, 0, 0, 2)} \in W \quad \because \underline{(x_1 - y_1)} \in \mathbb{R}$$

$\therefore W$ is a subspace

Extending it to basis of \mathbb{R}^4 - consider $(1, 1, 1, 2)^t \notin W$

$$\therefore S = \{ (0, 1, 0, 3)^t; (1, 0, 0, 2)^t; (0, 0, 1, -4)^t; (1, 1, 1, 2)^t \}$$

forms basis for \mathbb{R}^4

Now $W = \{ (x_1, x_2, x_3, x_4)^T \mid x_1 + x_2 = 0; x_3 - x_4 = 0 \}$
 2 conditions \Rightarrow 2 free variables

$\Rightarrow W = (x_1, -x_1, x_3, x_3) = x_1(1, -1, 0, 0)^T + x_3(0, 0, 1, 1)^T$ let x_1, x_3

Basis of W is $\{ (1, -1, 0, 0)^T; (0, 0, 1, 1)^T \}$

subspace - linear span of the vectors is the smallest subspace
 \therefore is a subspace

Extension of basis of R^4 = consider $(1, 0, 0, 0)^T$ $(0, 0, 0, 1)^T$
 both doesn't belong to W ; both linearly independent

\therefore Extended Basis $S = \{ (1, -1, 0, 0)^T; (0, 0, 1, 1)^T; (1, 0, 0, 0)^T; (0, 0, 0, 1)^T \}$

Hence the result

3. (b) (i) The temperature at a point (x, y) on a metal plate is $T(x, y) = 4x^2 - 4xy + y^2$. An ant on the plate walks around the circle of radius 5 centered at the origin. What are the highest and lowest temperatures encountered by the ant?

(ii) Evaluate the integral $\int_0^x \int_0^x x e^{-x^2/y} dx dy$ by changing the order of integration.

[12+08=20]

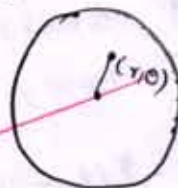
(i)

Equation of plate

$$x^2 + y^2 \leq 25$$

let at any point (r, θ)

$$\Rightarrow x^2 + y^2 = r^2$$



Temperature equation on plate

$$T(x, y) = 4x^2 - 4xy + y^2 = (2x - y)^2$$

(a) for any point inside boundary (for stationary point)

$$T_x = 8x - 4y = 0 \quad (\text{differentiating w.r.t } x)$$

$$T_y = -4x + 2y = 0 \quad (\text{differentiating w.r.t } y)$$

$$\Rightarrow y = 2x$$

$$\underline{T_{xy} = -4} \quad ; \quad T_{xx} = 8; \quad T_{yy} = 2$$

$$\Rightarrow D = T_{xx} T_{yy} - T_{xy}^2 = \underline{0}$$

$$\underline{T_{xx} = 8}$$

value of (x, y) ~~(x, y)~~ $(x, 2x)$

$$\text{value of } T(x, y) = \underline{8x^2 - 8x^2 = 0}$$

$$\therefore T(x, y) = (2x - y)^2$$

$T=0$ is the lowest value of temperature

(b) consider rim of the plate

$$\Rightarrow y^2 = 25 - x^2$$

$$\Rightarrow T(x, y) = 4x^2 - 4x\sqrt{25-x^2} + 25 - x^2$$

$$= 3x^2 + 25 - 4x\sqrt{25-x^2}$$

$$\text{Now } T_x = 0 = 6x - 4\sqrt{25-x^2} = \frac{4x \cdot (-x)}{\sqrt{25-x^2}}$$

$$= 6x - \frac{4x^2}{\sqrt{25-x^2}}$$

$$\Rightarrow 100 = 6x\sqrt{25-x^2}$$

$$\Rightarrow \left(\frac{50}{3}\right)^2 = x^2(25-x^2)$$

$$\text{we get } x = \underline{\sqrt{20}}$$

$$T(x, y) = (2x - y)^2$$

Take $g(x) = 2x - y$
maximize this

$$y = \sqrt{25-x^2}$$

$$g(x) = 2x + \sqrt{25-x^2}$$

$$g'(x) = 2 - \frac{x}{\sqrt{25-x^2}}$$

$$\Rightarrow xy = \frac{100}{6} = \underline{\frac{50}{3}}$$

$$\text{we get } x = \underline{\sqrt{20}}$$

$$y = \underline{-\sqrt{5}}$$

the temperature is maximum for value $(x, y) = (\underline{\sqrt{20}}, \underline{-\sqrt{5}})$

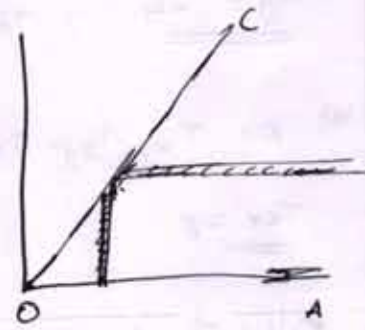
$$\underline{T} = 80 + 5 + 4 \cdot 10 = \underline{125}$$

$\underline{T=125}$ is the maximum temperature that will come across

(ii) Integral $I = \int_0^{\infty} \int_0^x x e^{-x^2/y} dx dy$

On changing order of integration, we get

$$I = \int_{y=0}^{\infty} \int_{x=y}^{\infty} x \cdot e^{-x^2/y} dy dx$$



region of integration

OAC

let $-x^2/y = t$

$\Rightarrow x dx = -y \cdot dt/2$

$$I = \int_{y=0}^{\infty} \left[\int_{-y}^{-\infty} e^t (-y) \cdot \frac{dt}{2} \right] dy$$

To change order of integration

$x: y \text{ to } \infty$

$y: 0 \text{ to } \infty$

$$= \int_0^{\infty} dy \left(\frac{-y}{2} \right) \left[e^t \right]_{-y}^{-\infty} = \int_0^{\infty} (dy) \left(\frac{y}{2} \right) (e^{-y})$$

$\int_0^{\infty} \frac{e^{-y} \cdot y \cdot dy}{2}$ — Γ -function

$\Gamma(2)$

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx$$

$= \frac{\Gamma(2)}{2} = \boxed{\frac{1}{2}}$

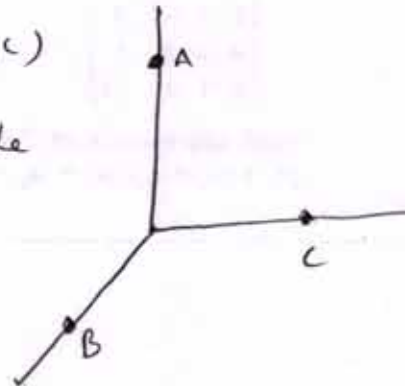
\therefore value of integral $I = \int_0^{\infty} \int_0^x x e^{-x^2/y} dx dy = \underline{\underline{\frac{1}{2}}}$

Hence the result

3. (c) A sphere of constant radius $2k$ passes through the origin and meets the axes in A, B, C. Find the locus of the centroid of the tetrahedron OABC. [15]

Let $A(a, 0, 0)$ $B(0, b, 0)$ $C(0, 0, c)$

where sphere of radius $2k$ cuts the
Axes.



We know that Equation of

sphere OABC is $x^2 + y^2 + z^2 - ax - by - cz = 0$ - (1)

The sphere has radius $2k$

$$\Rightarrow OE = 2k$$

$$\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2 + \left(\frac{c}{2}\right)^2 = 4k^2$$

$$u = -a/2$$

$$v = -b/2$$

$$w = -c/2$$

(\Rightarrow Centre $(-a/2, -b/2, -c/2)$)

$$a^2 + b^2 + c^2 = 16k^2$$
 - (2)

Now centroid of the tetrahedron OABC $\left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4}\right)$

$$x_1 = a/4; y_1 = b/4; z_1 = c/4$$
 - (3)

\Rightarrow substituting (3) in (2), we get

$$x_1^2 + y_1^2 + z_1^2 = k^2$$

\Rightarrow locus is sphere of radius k ; centre at origin

$$x^2 + y^2 + z^2 = k^2$$

Hence the required locus

4. (a) Let V be a 4-dimensional vector space over \mathbb{R} and let $T \in L(V)$ whose matrix with respect to an ordered basis $\{u_1, u_2, u_3, u_4\}$ is

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

Find the matrix of T with respect to the basis $\{u_1 + u_2 + u_3, u_1 + u_2 - u_4, u_3 - u_4, u_1 - u_2\}$.

[20]

SECTION - B

5. (a) Find the orthogonal trajectories of
- $r = a(1 + \cos n\theta)$
- .

[10]

Given equation of curve $r = a(1 + \cos n\theta)$ — (1)

\Rightarrow applying logarithmic

$$\log r = \log a + \log(1 + \cos n\theta)$$

now differentiating, we get

$$\frac{dr}{r} = \frac{-n \sin(n\theta)}{1 + \cos n\theta} d\theta$$

$$\Rightarrow \boxed{\frac{dr}{d\theta} = \frac{r \cdot (-n) \sin(n\theta)}{1 + \cos(n\theta)}}$$

— (2) gives differential equation of (1)

To get orthogonal trajectories replace

$$\frac{dr}{d\theta} \text{ by } -r^2 \frac{d\theta}{dr}$$

$$\Rightarrow -r^2 \frac{d\theta}{dr} = r \frac{(-n) \sin(n\theta)}{1 + \cos(n\theta)} \Rightarrow n \frac{dr}{r} = d\theta \frac{1 + \cos(n\theta)}{\sin(n\theta)}$$

$$\Rightarrow n^2 \frac{dr}{r} = \frac{2 \cdot \cos(n\theta/2)}{\sin(n\theta/2)} \cdot d(n\theta/2)$$

on integration we get

$$\left(\begin{aligned} \because 1 + \cos n\theta &= 2 \cos^2(n\theta/2) \\ \sin n\theta &= 2 \sin(n\theta/2) \cos(n\theta/2) \end{aligned} \right)$$

$$\Rightarrow n^2 \log r = 2 \log(\sin(n\theta/2)) + \log c$$

$$\Rightarrow \text{or } r = \left(c (\sin(n\theta/2))^2 \right)^{1/n^2} = \left(c^2 (1 - \cos(n\theta)) \right)^{1/n^2}$$

(3)

$$\boxed{r^2 = (c^2 (1 - \cos(n\theta)))^2}$$

— (3)

is the required orthogonal trajectory

5. (b) Use the variation of parameters method to show that the solution of equation $d^2 y/dx^2 + k^2 y = \phi(x)$ satisfying the initial conditions $y(0) = 0$, $y'(0) = 0$ is

$$y(x) = \frac{1}{k} \int_0^x \phi(t) \sin k(x-t) dt.$$

[10]

Given differential equation $(D^2 + k^2)y = \phi(x)$

where $R = \phi(x)$

complementary functions of $(D^2 + k^2)y = 0$ is

$$u = \cos(kx) ; v = \sin(kx) \quad \text{i.e.}$$

$$C.F. = c_1 \cos(kx) + c_2 \sin(kx)$$

Now Particular Integral $y_p = Au + Bv$

where

$$A = \int \frac{-vR}{W}$$

$$B = \int \frac{uR}{W}$$

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos(kx) & \sin(kx) \\ -k \sin(kx) & k \cos(kx) \end{vmatrix} = k$$

$$\Rightarrow A = \int \frac{-\sin(kx) \phi(x)}{k} ; B = \int \frac{\cos(kx) \phi(x)}{k}$$

$$\therefore y(x) = c_1 \cos(kx) + c_2 \sin(kx) + \frac{\cos(kx)}{k} \int -\sin(kx) \phi(x) + \frac{\sin(kx)}{k} \int \cos(kx) \phi(x)$$

Initial conditions : $y(0) = 0$

$$\Rightarrow y(0) = c_1 + \frac{1}{k} \left(\int -\sin kx \phi(x) \right)_{x=0} = 0$$

$$\text{And } y'(0) = 0 = kc_2 + \left(\int \cos(kx) \phi(x) \right)_{x=0} = 0$$

We get $c_1 = c_2 = 0$

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NOW

$$\begin{aligned}
 y(x) &= -\frac{\cos(kx)}{k} \int \sin(kt) \phi(t) dt + \frac{\sin(kx)}{k} \int \cos(kt) \phi(t) dt \\
 &= \int \frac{(\phi(t))}{k} (\sin kx \cos(kt) - \sin(kt) \cos(kx)) dt \\
 &= \frac{1}{k} \int \phi(t) \sin k(x-t) dt
 \end{aligned}$$

 \Rightarrow we get

$$y(x) = \frac{1}{k} \int_0^x \phi(t) \sin k(x-t) dt$$

Hence the result

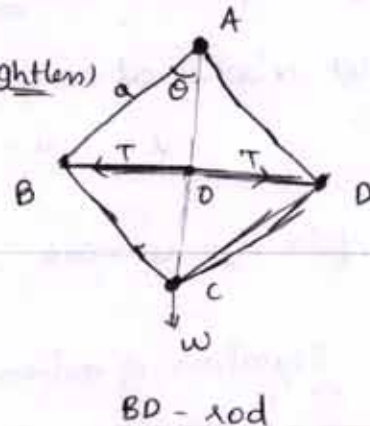
5. (c) A frame work ABCD consists of four equal, light rods smoothly jointed together to form a square, it is suspended from a peg at A, and a weight W is attached to C, the framework being kept in shape by a light rod connecting B and D. Determine the thrust in this rod. [10]

Given ABCD - four equal, light rods (weightless)
smoothly jointed - form square

• weight w at C let AB = a

• light rod connecting BD

Thrust be T in the rod



Let for a small gentle disturbance, we have
virtual work equations (for static equilibrium)

$$\underline{T \delta(BD) + w \delta(AC) = 0} \quad \text{--- (1)}$$

from figure $AC = 2AD = 2a \cos \theta$

$$BD = 2BO = 2a \sin \theta$$

$$\Rightarrow T(2a \sin \theta) + w(2a \cos \theta) = 0$$

$$\Rightarrow 2aT \cos \theta - 2aw \sin \theta = 0$$

$$\Rightarrow T = w \tan \theta \quad \therefore \text{for square } \theta = \pi/4$$

$$\Rightarrow \boxed{T = w}$$

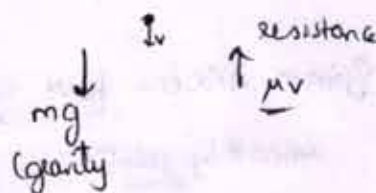
Hence the thrust in the rod = weight of the Particle

5. (d) A particle of mass m , is falling under the influence of gravity through a medium whose resistance equals μ times the velocity. If the particle were released from rest, determine the distance fallen through in time t . [10]

Given particle of mass m

let velocity at any point be

$$v = \dot{x} = \frac{dx}{dt}$$



$$\Rightarrow \text{force of resistance} = \mu \dot{x}$$

\therefore Equation of motion is

$$m \cdot \ddot{x} = mg - \mu \dot{x}$$

$$\underline{\underline{0 = d/dt}}$$

$$\Rightarrow m \left(\frac{d^2}{dt^2} + \left(\frac{\mu}{m} \right) \frac{d}{dt} \right) x = g$$

- is the required differential equation

\Rightarrow Auxiliary equation

$$D(D + (\mu/m)) = 0 \Rightarrow C.F. = C_1 + C_2 \cdot e^{-\mu/m t}$$

Particular integral = $\frac{1}{D(0^2 + 1/m)} \cdot g \cdot e^{0t} = \frac{1}{D(1/m)} (g) = \left(\frac{m \cdot g}{\mu}\right)$

$\therefore x = c_1 + c_2 e^{-(1/m)t} + \frac{mgt}{\mu}$ ← at $t=0$ $x=0$
 put in this (Initial condition)

$\Rightarrow c_1 + c_2 = 0$; at $t=0$; velocity $\dot{x} = 0$
 (from $t=0$ $x=0$)

$\Rightarrow \dot{x} = -(1/m)c_2 \cdot e^{-(1/m)t} + \frac{mg}{\mu} = 0$
 \Rightarrow at $t=0$; $c_2 = \frac{m^2 g}{\mu^2}$

$\Rightarrow x = \frac{m^2 g}{\mu^2} (e^{-(1/m)t} - 1) + \frac{mgt}{\mu}$

Hence the distance the particle has fallen through in time t

5. (e) Represent the vector $A = z\mathbf{i} - 2x\mathbf{j} + y\mathbf{k}$ in cylindrical coordinates. Thus determine A_ρ , A_ϕ and A_z . [10]

Given vector $A = z\mathbf{i} - 2x\mathbf{j} + y\mathbf{k}$ ①

- to convert to cylindrical coordinates (ρ, ϕ, z)

we have

$r = \overset{(x)}{\rho \cos \phi} \hat{i} + \overset{(y)}{\rho \sin \phi} \hat{j} + \overset{(z)}{z} \hat{k}$

- in cylindrical coordinates ②

$\hat{\rho} = \frac{\partial r}{\partial \rho} = \cos \phi \hat{i} + \sin \phi \hat{j}$

$\hat{\phi} = \frac{1}{\rho} \cdot \frac{\partial r}{\partial \phi} = -\sin \phi \hat{i} + \cos \phi \hat{j}$

$\hat{z} = \hat{k}$

$\Rightarrow \begin{bmatrix} \hat{\rho} \\ \hat{\phi} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} \Rightarrow \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\rho} \\ \hat{\phi} \\ \hat{z} \end{bmatrix}$

$\Rightarrow \hat{i} = \cos \phi \hat{\rho} - \sin \phi \hat{\phi} ; \hat{j} = \sin \phi \hat{\rho} + \cos \phi \hat{\phi} ; \hat{k} = \hat{z}$ ③

Put ②, ③ in ①, we get

$$\begin{aligned}
 A &= z(\cos\phi \hat{r} - \sin\phi \hat{\phi}) - 2(f\cos\phi)(\sin\phi \hat{r} + \cos\phi \hat{\phi}) + (f\sin\phi) \hat{z} \\
 &= (z\cos\phi - f\sin(2\phi)) \hat{r} + (-2\sin\phi - 2f\cos^2\phi) \hat{\phi} \\
 &\quad + f\sin\phi \hat{z}
 \end{aligned}$$

is the required vector in cylindrical coordinates

$$A_r = z\cos\phi - f\sin(2\phi)$$

$$A_\phi = -2\sin\phi - 2f\cos^2(\phi)$$

$$A_z = f\sin\phi$$

Hence the result

6. (a) Solve $(D^2 - 1)y = \cosh x \cos x + a^x$.

[13]

Auxiliary Equation is $D^2 - 1 = 0$

complementary function is

$$C.F. = \underline{c_1 e^x + c_2 e^{-x}} \quad \text{--- ①}$$

now Particular Integral

$$P.I. = \frac{1}{(D^2 - 1)} (\cosh x \cos x + a^x)$$

$$= \frac{e^x \cos x + e^{-x} \cos x}{2(D^2 - 1)} + \frac{a^x}{(D^2 - 1)}$$

now take

$$\begin{aligned}
 \frac{e^x \cos x}{2(D^2-1)} &= \frac{\cancel{e^x \cos x}}{(D-1)(D+1)^2} \cdot \frac{e^x}{(D+1)^2-1} \quad \left(\because \frac{e^{ax} v}{f(D)} = e^{ax} \left[\frac{v}{f(D+a)} \right] \right) \\
 &= \frac{e^x}{2} \left(\frac{\cos x}{D^2+2D} \right) = \frac{e^x}{2} \left(\frac{\cos x}{-1+2D} \right) \\
 &= \frac{e^x}{2} \left[\frac{(1+2D)\cos x}{4D^2-1} \right] = \frac{e^x}{2} \left[\frac{\cos x - 2\sin x}{(-5)} \right] \\
 &= \frac{e^x}{10} (2\sin x - \cos x) \quad - (2)
 \end{aligned}$$

Now take

$$\begin{aligned}
 \frac{e^{-x} \cos x}{2(D^2-1)} &= \frac{e^{-x}}{2} \left[\frac{\cos x}{(D-1)^2-1} \right] = \frac{e^{-x}}{2} \left[\frac{\cos x}{D^2-2D} \right] \\
 &= \frac{e^{-x}}{2} \left[\frac{\cos x}{-1-2D} \right] = -\frac{e^{-x}}{2} \left[\frac{\cos x}{(1+2D)} \right] \\
 &= -\frac{e^{-x}}{2} \left[\frac{(1-2D)\cos x}{1-4D^2} \right] = -\frac{e^{-x}}{2} \left(\frac{\cos x + 2\sin x}{5} \right) \\
 &= -\frac{e^{-x}}{10} (\cos x + 2\sin x) \quad - (3)
 \end{aligned}$$

Now take

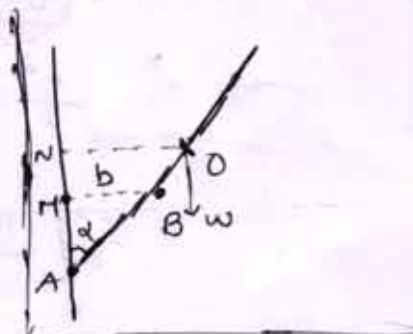
$$\begin{aligned}
 \frac{a^x}{D^2-1} &= \frac{1}{2} \left(\frac{a^x}{D-1} + \frac{-a^x}{D+1} \right) = \frac{1}{2} \left(e^x \int a^x \cdot e^x dx - e^x \int a^x e^x dx \right) \\
 &= \frac{1}{2} \left(e^x \cdot \frac{(a/e)^x}{\log(a/e)} - e^x \cdot \frac{(ae)^x}{\log(ae)} \right) = \frac{1}{2} \left(\frac{a^x}{\log a - 1} - \frac{a^x}{\log a + 1} \right)
 \end{aligned}$$

$$\begin{aligned}
 \therefore y &= c_1 e^x + c_2 e^{-x} + \frac{e^x}{10} (2\sin x - \cos x) + \frac{-e^{-x}}{10} (\cos x + 2\sin x) \\
 &\quad + \frac{1}{2} \left(\frac{a^x}{\log a - 1} - \frac{a^x}{\log a + 1} \right)
 \end{aligned}$$

Hence the result

6. (b) A uniform beam of length $5a$, rests in equilibrium against a smooth vertical wall and upon a smooth peg at a distance b from the wall. Show that in the position of equilibrium the beam is inclined to the wall at an angle $\sin^{-1}(b/a)^{1/3}$. [10]

Let length of beam be $2a$;
equilibrium against smooth vertical wall



- B - be peg - at distance b from wall $\Rightarrow BM = b$
O - be centre of the rod $\Rightarrow OA = a$

NOW $NM = NA - MA$
 $= a \cos \alpha - b \cot \alpha$ - (1)

NOW, virtual work due to weight of the rod

is $w \delta(NM) = 0$ (for equilibrium)

$\Rightarrow \delta(NM) = 0$

$\Rightarrow \delta(a \cos \alpha - b \cot \alpha) = 0$

$\Rightarrow -a \sin \alpha + b \operatorname{cosec}^2 \alpha = 0$

$\Rightarrow \sin^3 \alpha = b/a$

$\Rightarrow \alpha = \sin^{-1}(\sqrt[3]{b/a})$

Hence the result

6. (c) The end links of a uniform chain slide along a fixed rough horizontal rod. Prove that the ratio of the maximum span to the length of the chain is

$$\mu \log \left\{ \frac{1 + \sqrt{1 + \mu^2}}{\mu} \right\}.$$

[10]

Let catenary hangs from the points

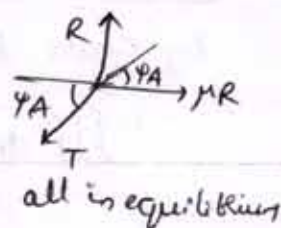
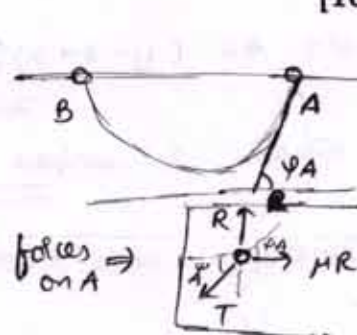
A, B

— given rod is rough ; let
Coefficient of friction be μ

→ at maximum span \Rightarrow Friction is also maximum = μR

from free body diagram of forces on A

we get $\tan \psi_A = R / \mu R = \underline{\underline{1/\mu}}$



$\Rightarrow \therefore s = c \tan \psi_A = c/\mu$ (half length of the chain)

$\Rightarrow x = c \log (\sec \psi_A + \tan \psi_A) = c \log \left(\frac{1 + \sqrt{1 + \mu^2}}{\mu} \right)$
(half span of the chain)

\therefore asked to find $\frac{2x}{2s} = \frac{\log \left(\frac{1 + \sqrt{1 + \mu^2}}{\mu} \right)}{(1/\mu)} = \mu \log \left(\frac{1 + \sqrt{1 + \mu^2}}{\mu} \right)$

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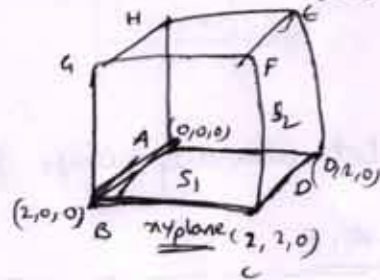
$$= \boxed{\mu \log \left(\frac{1 + \sqrt{1 + \mu^2}}{\mu} \right)}$$

Hence the result

6. (d) Verify Stokes theorem for $\mathbf{A} = (y - z + 2)\mathbf{i} + (yz + 4)\mathbf{j} - xz\mathbf{k}$, where S is the surface of the cube $x = 0, y = 0, z = 0, x = 2, y = 2, z = 2$ above the xy plane. [17]

Given $\mathbf{A} = (y - z + 2)\mathbf{i} + (yz + 4)\mathbf{j} - xz\mathbf{k}$

where S - surface of the cube



To verify Stokes theorem

$$\iint_S (\text{curl } \mathbf{A}) \cdot d\mathbf{S} = \oint_{\text{boundary}} \mathbf{A} \cdot d\mathbf{r}$$

(Surface Integral) (Line Integral) boundary is $ABCD$

(i) Surface Integral

By Gauss divergence theorem - Over close surface $S = S_1 + S_2$

$$\iiint_S \text{div}(\text{curl } \mathbf{A}) \, dV = \iint_S (\text{curl } \mathbf{A}) \cdot d\mathbf{S} = 0$$

(0) $(\because \text{div}(\text{curl } \mathbf{A}) = 0)$

$$\Rightarrow \iint_{S_1 + S_2} \text{curl } \mathbf{A} \cdot d\mathbf{S} = 0 \Rightarrow \iint_{S_2} \text{curl } \mathbf{A} \cdot d\mathbf{S} = - \iint_{S_1} \text{curl } \mathbf{A} \cdot d\mathbf{S}$$

for $S_1 \rightarrow \hat{n} = -\hat{k}$
(normal)

$$\text{curl } \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y - z + 2 & yz + 4 & -xz \end{vmatrix} = -y\mathbf{i} + (-1 + z)\mathbf{j} + -\mathbf{k}$$

$$\Rightarrow \text{curl } \mathbf{A} \cdot \hat{n} = (-1)$$

$$\therefore \iint_{S_2} \text{curl } \mathbf{A} \cdot d\mathbf{S} = - \iint_{S_1} \text{curl } \mathbf{A} \cdot \hat{n} \, dS = - \iint_{S_1} dS$$

$$\therefore \iint_{S_2} \text{curl } \mathbf{A} \cdot d\mathbf{S} = -4$$

(Area of the square $ABCD$)

(ii) Line Integral $\int \vec{A} \cdot d\vec{r} = \int (A_x dx + A_y dy + A_z dz)$

along AB : $x=0 \text{ to } 2; y=0; z=0; dy=dz=0$

$$\Rightarrow \int \vec{A} \cdot d\vec{r} = \int_0^2 2 dx = 4$$

along BC : $x=2; y=0 \text{ to } 2; z=0; dx=dz=0$

$$\Rightarrow \int_0^2 4 dy = \underline{8}$$

along CD : $x=2 \text{ to } 0; y=2; z=0$

$$\Rightarrow \int_2^0 4 dx = -8$$

along DA : $x=0; y=2 \text{ to } 0; z=0$

$$\Rightarrow \int_2^0 4 dy = -8$$

$$\Rightarrow \oint \vec{A} \cdot d\vec{r} = -4 \quad \text{--- (2)}$$

From (1), (2) we proved Stokes Theorem. Hence the result

7. (a) Find the general and singular solution of $y^2 (y - xp) = x^4 p^2$. [12]

8. (a) By using Laplace transform method solve the $(D^3 - 2D^2 + 5D)y = 0$ if $y(0) = 0$, $y'(0) = 1$, $y(\pi/8) = 1$ [15]

Gives differential equation $(D^3 - 2D^2 + 5D)y = 0$

Applying Laplace on both sides

$$y(0) = 0$$

$$y'(0) = 1$$

$$L(D^3 y) + L(-2D^2 y) + L(5Dy) = 0$$

$$\Rightarrow p^3 L(y) - p^2 y(0) - p y'(0) - y''(0)$$

$$+ (-2) (p^2 L(y) - p y(0) - y'(0))$$

$$+ 5 (p L(y) - y(0)) = 0$$

$$\Rightarrow L(y) (p^3 - 2p^2 + 5p) - p + 2 = y''(0)$$

$$\Rightarrow L(y) = \frac{y''(0)}{p(p^2 - 2p + 5)} + \frac{(p-2)}{p(p^2 - 2p + 5)}$$

$$\Rightarrow y = L^{-1} \left\{ \frac{(y''(0) - 2)}{p((p-1)^2 + 4)} + \frac{1}{(p-1)^2 + 4} \right\}$$

$$\text{we know } L^{-1} \left(\frac{1}{(p-1)^2 + 4} \right) = \frac{1}{2} e^t \sin(2t)$$

$$\text{now } L \left(\int_0^t \frac{1}{2} e^t \sin(2t) dt \right) = \frac{1}{p} \cdot \frac{1}{((p-1)^2 + 4)}$$

$$\frac{1}{2} \int_0^t e^t \sin(2t) dt = \left| e^t \left(\frac{\sin 2t - 2 \cos 2t}{2 \cdot 5} \right) \right|_0^t$$

$$= \frac{e^t (\sin 2t - 2 \cos 2t)}{10} + \frac{1}{5}$$

$$\therefore y = (y''(0) - 2) \left\{ \frac{e^t (\sin 2t - 2 \cos 2t)}{10} + \frac{1}{5} \right\} + \frac{1}{2} e^t \sin(2t) \quad (3)$$

satisfies initial conditions $y(0) = 0; y'(0) = 1$

now put $y(\pi/8) = 1 \rightarrow$ we get value of $y''(0)$

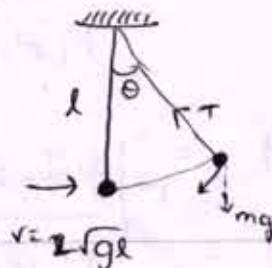
$$1 = (y''(0) - 2) \left[\frac{e^{\pi/8} (-\frac{1}{\sqrt{2}}) + \frac{1}{5}}{10} \right] + \frac{1}{2} \cdot \frac{1}{\sqrt{2}} e^{\pi/8}$$

Above equation gives $(y''(0) - 2)$ value substitute in (3)

Hence the result

8. (b) A heavy particle hanging vertically from a fixed point by a light inextensible cord of length l is struck by a horizontal blow which imparts it a velocity $2\sqrt{gl}$, prove that the cord becomes slack when the particle has risen to a height $\frac{2}{3}l$ above the fixed point. [15]

Given Heavy particle hanging vertically from a cord of length l
equation of motion



$$m \frac{d^2 r}{dt^2} = -mg \sin \theta$$

$$\text{where } r = l \theta$$

$$\Rightarrow l \ddot{\theta} = -g \sin \theta \quad (1)$$

now multiply $2 \cdot \dot{\theta}$ on both sides and integrate

$$\Rightarrow \int 2 \dot{\theta} (2 \ddot{\theta}) dt = -2 \int g \sin \theta \cdot \dot{\theta} dt$$

$$\Rightarrow A + l\ddot{\theta}^2 = 2g|\cos\theta|_0^{\theta} = 2g\cos\theta - 2g$$

$$\Rightarrow \because \text{at } t=0; \dot{\theta}^2 = \frac{2gl}{l^2} = 4g/l; \theta=0$$

$$\Rightarrow A + 4g = 0 \Rightarrow \underline{A = -4g}$$

$$\Rightarrow \boxed{l\ddot{\theta}^2 = 2g + 2g\cos\theta} \quad \text{--- (2) Equation for Angular velocity}$$

Tension in the string

$$T = mg\cos\theta + ml\ddot{\theta}^2$$

$$= m(2g + 3g\cos\theta)$$

string becomes slack when tension becomes zero

$$2) T=0 = m(2g + 3g\cos\theta)$$

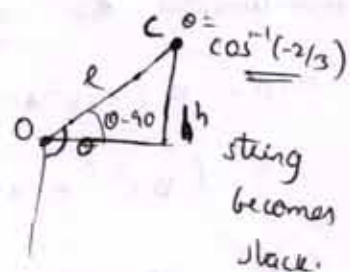
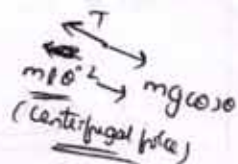
$$2) \boxed{\cos\theta = -2/3}$$

Height OC when it becomes slack

$$h = l\sin(\theta-90) = -l\cos\theta$$

$$\therefore \text{at height } \boxed{h = \frac{2}{3}l} \text{ string becomes slack}$$

Hence the result



8. (c) Show that $\mathbf{A} = (2x^2 + 8xy^2z)\mathbf{i} + (3x^3y - 3xy)\mathbf{j} - (4y^2z^2 + 2x^3z)\mathbf{k}$ is not solenoidal but $\mathbf{B} = xyz^2\mathbf{A}$ is solenoidal. [08]

solenoidal vector \mathbf{A} if $\nabla \cdot \mathbf{A} = 0$ ($\text{div } \mathbf{A} = 0$)

Given $\mathbf{A} = (2x^2 + 8xy^2z)\mathbf{i} + (3x^3y - 3xy)\mathbf{j} - (4y^2z^2 + 2x^3z)\mathbf{k}$

$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$

$(A_x) \qquad (A_y) \qquad (A_z)$

Now $\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

$$= (4x + 8y^2z) + (\cancel{3x^3} - 3) - (8y^2z + 2x^3)$$

$$\nabla \cdot \mathbf{A} = \underline{\underline{x + x^3}} \neq 0$$

\therefore not solenoidal

Now consider $\mathbf{B} = \underline{\underline{xyz^2\mathbf{A}}}$

$$(\because \nabla(\phi \mathbf{A}) = \nabla\phi \cdot \mathbf{A} + \phi \text{div}(\mathbf{A}))$$

$$\nabla \cdot \mathbf{B} = \nabla \cdot (xyz^2\mathbf{A}) = \nabla(xyz^2) \cdot \mathbf{A} + xyz^2(\text{div } \mathbf{A})$$

$$= (yz^2\mathbf{i} + xz^2\mathbf{j} + 2xyz\mathbf{k}) \cdot \mathbf{A} + xyz^2(x + x^3)$$

$$= yz^2(2x^2 + 8xy^2z) + xz^2(3x^3y - 3xy) + 2xyz(-4y^2z^2 - 2x^3z) + xyz^2(x + x^3)$$

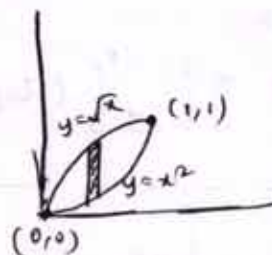
$$= \underline{\underline{0}}$$

hence the result

8. (d) Verify Green's theorem in the plane for $\oint_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$, where C is the boundary of the region defined by: $y = \sqrt{x}$, $y = x^2$ [12]

Given function $\oint_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$
 $(M) \quad (N)$

By Green's theorem $\oint_C Mdx + Ndy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$



calculating $I = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -6y + 4y = -2y$

$I = \int_{x=0}^1 \int_{y=x^2}^{y=\sqrt{x}} -2y \cdot dx dy = \int_0^1 (-2y^2) dy = -\frac{2}{3} y^3 \Big|_0^1 = -\frac{2}{3}$

$= -\frac{2}{3} (1 - 0) = -\frac{2}{3}$

$= -\frac{2}{3} (1 - 0) = -\frac{2}{3}$

$\therefore I = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = -\frac{2}{3}$

calculating Line Integral $\int Mdx + Ndy$

Along $C_1: y = x^2 \Rightarrow (t, t^2) \quad dy = 2x dx \quad ; y = x^2$

$\int_{C_1} (3x^2 - 8y^2) dx + (4y - 6xy) dy = \int_0^1 (3x^2 - 8x^4) dx + \int_0^1 (4x^2 - 6x^3) 2x dx$
 $= \int_0^1 (3x^2 - 8x^4) dx + \int_0^1 (8x^3 - 12x^4) dx = 1 - \frac{8}{5} + \frac{8}{4} - \frac{12}{5}$
 $= -1$

Along curve $c_2: y = \sqrt{x} \Rightarrow \underline{y^2 = x} \Rightarrow \underline{dx = 2y dy}$

$$\int_{y=1}^0 (3y^4 - 8y^2)(2y dy) + \int_{y=1}^0 (4y - 6y^3) dy$$

y from 1 to 0

$$= - \left[\int_{y=0}^1 (6y^5 - 22y^3 + 4y) dy \right] = - \left[1 - \frac{22}{4} + 2 \right]$$

$$= \underline{\underline{5/2}}$$

$$\therefore \oint M dx + dy = \oint_{c_1} + \oint_{c_2} = -1 + 5/2 = \underline{\underline{3/2}}$$

$$\boxed{I_1 = \oint M dx + ndy = 3/2}$$

$$\Rightarrow \underline{\underline{① = ②}}$$

Thus $\boxed{I = I_1}$ Green Theorem Verified

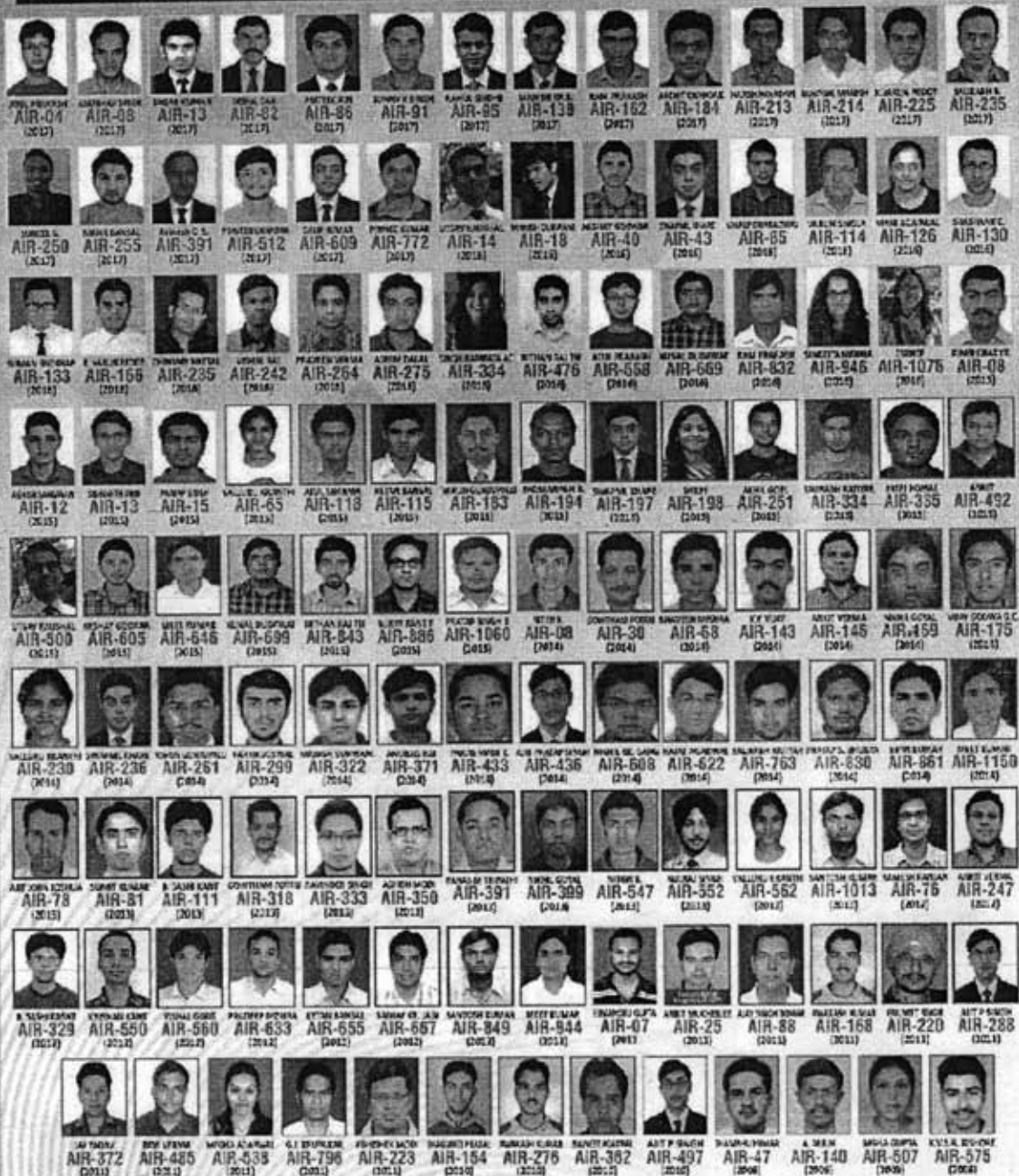
Hence the result

END OF THE EXAMINATION

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