Lecture - 23
Friday
7/4/17

Conservative vector field -

$\vec{F} = \vec{\nabla}\phi$.

Thm. If $\vec{F} = \vec{\nabla}\phi \iff \text{Curl}\,\vec{F} = 0$,

Line integrals -

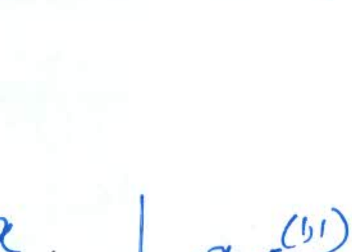
$$\int_{P_1}^{P_2} \vec{F}\cdot d\vec{r}$$

In 2D, $\displaystyle\int_{P_1}^{P_2} V_1(x,y)\,dx + V_2(x,y)\,dy$

3D, $\displaystyle\int_{P_1}^{P_2} V_1(x,y,z)\,dx + V_2(x,y,z)\,dy + V_3(x,y,z)\,dz$.

Ex1 (2D). $\vec{F} = y^2\hat{i} + x^2\hat{j}$.

Compute $\displaystyle\int_{(0,0)}^{(1,1)} \vec{F}\cdot d\vec{r}$ along $y = x$.



$$= \int_{(0,0)}^{(1,1)} (y^2\,dx + x^2\,dy)$$

$$= \int_{t=0}^{1} t^2\,dt + t^2\,dt. \qquad y = x = t. \quad dy = dx = dt$$

$$= \int_{t=0}^{1} 2t^2\,dt = \frac{2}{3}$$

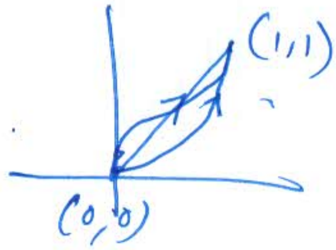
$\boxed{\text{Line integrals generally depend on paths.}}$

Compute $\displaystyle\int_{(0,0)}^{(1,1)} \vec{F}\cdot d\vec{r}$ along $y = x^2$ $\qquad x = t,\ dx = dt,\ y = t^2,\ dy = 2t\,dt$

$$= \int_{t=0}^{1} (y^2\,dx + x^2\,dy) = \frac{7}{10}$$

Ex 2. (2D) $\vec{F} = y\hat{i} + x\hat{j}$.

compute $\int_{(0,0)}^{(1,1)} \vec{F}\cdot d\vec{r}$ along $y = x$.



$$= \int_{(0,0)}^{(1,1)} y\,dx + x\,dy \qquad x = y = t.$$

$$= \int_{t=0}^{1} t\,dt + t\,dt = 1$$

Compute $\int_{(0,0)}^{(1,1)} \vec{F}\cdot d\vec{r}$ along $y = x^2$

$$= \int_{(0,0)}^{(1,1)} y\,dx + x\,dy \qquad \begin{array}{l} x = t;\ dx = dt \\ y = t^2,\ dy = 2t\,dt. \end{array}$$

$$= 1$$

Note, $\int_{(0,0)}^{(1,1)} y\,dx + x\,dy = \int_{(0,0)}^{(1,1)} d(xy)$

$$= [xy]_{(0,0)}^{(1,1)} = 1.$$

$$\int_{P_1}^{P_2} V_1\,dx + V_2\,dy + V_3\,dz.$$

If $\vec{F}$ is conservative $(\Leftrightarrow \text{Curl}\,\vec{F} = 0)$ $(\vec{F} = \vec{\nabla}\phi)$

then $\int_{P_1}^{P_2} F_1\,dx + F_2\,dy + F_3\,dz$ is independent of path.

$$\vec{F} = (F_1, F_2, F_3) = \vec{\nabla}\phi := \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right).$$

$$\therefore F_1 = \frac{\partial \phi}{\partial x}, \quad F_2 = \frac{\partial \phi}{\partial y}, \quad F_3 = \frac{\partial \phi}{\partial z}.$$

$$\therefore \int_{P_1}^{P_2} F_1\,dx + F_2\,dy + F_3\,dz.$$

$$= \int_{P_1}^{P_2} \left(\frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy + \frac{\partial \phi}{\partial z}\,dz\right)$$

$$= \int_{P_1}^{P_2} d\phi = \left[\phi\right]_{P_1}^{P_2} = \phi(P_2) - \phi(P_1).$$

To ~~E~~examine whether $\text{Curl}\,\vec{F} = 0$. If so, find ~~grad~~ ϕ. (such that $\vec{F} = \vec{\nabla}\phi$)

Hence find $\int_{P_1}^{P_2} \vec{F}\cdot d\vec{r}$ (along, say, $y = x^2$, $x = t$, $y = t^2$, $z = t^3$).

$$= \left[\phi\right]_{P_1}^{P_2}$$

You show,

$$= \int_{P_1}^{P_2} \left(\frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy + \frac{\partial \phi}{\partial z}\,dz\right)$$

$$= \int_{P_1}^{P_2} d\phi = \left[\phi\right]_{P_1}^{P_2}$$

<u>Ex.</u> Compute $\int_C \vec{F}\cdot d\vec{r}$.

where $\vec{F} = \sin x\,\hat{i} + \cos y\,\hat{j} + zx\,\hat{k}$.

for two paths. 1) $C: \vec{r}(t): t^3\hat{i} + t^2\hat{j} + t\hat{k}$, $0 \le t \le 1$

2) C: st line joining $(0,0,0)$ & $(1,1,1)$

Are the values of the line integral same along path 1 & path 2? If not, state the reason

1) $\int_C \vec{F} \cdot d\vec{r}$

$= \int (\sin x \, dx + \cos y \, dy + zx \, dz)$

$x = t^3, \quad y = t^2, \quad z = t$

$$I_1 = \int_{t=0}^{1} \left[\sin(t^3) \times 3t^2 dt + \cos(t^2) \cdot 2t \, dt + t \cdot t^3 dt \right]$$

$$= \sin 1 + \frac{6}{5} - \cos 1.$$

$$\frac{x-0}{1-0} = \frac{y-0}{1-0} = \frac{z-0}{1-0} = t.$$

2) $$\int_{t=0}^{1} \left[\sin t \, dt + \cos t \, dt + t^2 dt \right]$$

$$= [\cos t]_1^0 + [\sin t]_0^1 + \left[\frac{t^3}{3}\right]_0^1$$

$$= 1 - \cos 1 + \sin 1 + \frac{1}{3}$$

The values of line integral is different.

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin x & \cos y & zx \end{vmatrix} = -z\hat{j} \neq \vec{0}.$$

Q. Given $\vec{F} = e^{y}\hat{i} + xe^{y}\hat{j} + (z+1)e^{z}\hat{k}$

1). Show that $\vec{F}$ is conservative.

$(\text{Curl } \vec{F} = \vec{0})$.

2) Find ϕ : $\vec{F} = \vec{\nabla}\phi$

$\phi = xe^{y} + ze^{z} + C$.

(3) Hence compute $\displaystyle\int_{(0,0,0)}^{(1,1,1)} \left[e^{y}\hat{i} + xe^{y}\hat{j} + (z+1)e^{z}\hat{k}\right] d\vec{r}$.

along the st. line joining $(0,0,0)$ to $(1,1,1)$.

$$= \left[xe^{y} + ze^{z}\right]_{(0,0,0)}^{(1,1,1)} = e + e = 2e.$$

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

Green's thm in plane.

1) verification of Green's thm
compute L.H.S & R.H.S & match the value.

2) Compute line integral using Green's thm.
$\Rightarrow$ Compute double integral on r.h.s.

3). Find the area of R using Green's thm.

$$\oint_C \frac{x\,dy - y\,dx}{2}$$