Iterative methods for solving system (linear) of equations:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

Direct methods: Gauss, Gauss Jordan

Iterative methods: Gauss Jacobi, Gauss-seidel method

Assume: 1) There are n linear equations in n unknowns.

2) unique solution exists.

Gauss - Jacobi method:

$x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)} \rightarrow$ initial guess $\rightarrow$ prescribed.

$$x_1^{(1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)} - \cdots - a_{1n}x_n^{(0)}\right]$$

$$x_2^{(1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(0)} - a_{23}x_3^{(0)} - \cdots - a_{2n}x_n^{(0)}\right]$$

$$x_3^{(1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(0)} - a_{32}x_2^{(0)} - \cdots - a_{3n}x_n^{(0)}\right]$$

$$\vdots$$

$$x_n^{(1)} = \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1^{(0)} - a_{n2}x_2^{(0)} - \cdots - a_{n\,n-1}x_{n-1}^{(0)}\right]$$

$$x_1^{(2)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(1)} - a_{13}x_3^{(1)} - \cdots - a_{1n}x_n^{(1)}\right]$$

$$x_2^{(2)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(1)} - \cdots - a_{2n}x_n^{(1)}\right]$$

$$x_3^{(2)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(1)} - a_{32}x_2^{(1)} - \cdots - a_{3n}x_n^{(1)}\right]$$

$$\vdots$$

$$x_n^{(2)} = \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1^{(1)} - a_{n2}x_2^{(1)} - \cdots - a_{n\,n-1}x_{n-1}^{(1)}\right]$$

$$x_1^{(r+1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(r)} - a_{13}x_3^{(r)} - \cdots - a_{1n}x_n^{(r)}\right]$$

$$x_2^{(r+1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(r)} - a_{23}x_3^{(r)} - \cdots - a_{2n}x_n^{(r)}\right]$$

$$\vdots$$

$$x_n^{(r+1)} = \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1^{(r)} - a_{n2}x_2^{(r)} - \cdots - a_{nn-1}x_n^{(r)}\right]$$

$r = 0, 1, 2, \ldots$

$\boxed{\text{Solve}}$

Ex.

$$6x_1 - 2x_2 + x_3 = 11$$
$$-2x_1 + 7x_2 + 2x_3 = 5$$
$$x_1 + 2x_2 - 5x_3 = -1$$

by gauss- Jacobi method correct upto 2-decimal places.

Given $x_1^{(0)} = 0 = x_2^{(0)} = x_3^{(0)}$

Solⁿ

$$x_1^{(k+1)} = \frac{1}{6}\left[11 + 2x_2^{(k)} - x_3^{(k)}\right] \quad ; \quad k = 0, 1, 3, \ldots$$

$$x_2^{(k+1)} = \frac{1}{7}\left[5 + 2x_1^{(k)} - 2x_3^{(k)}\right]$$

$$x_3^{(k+1)} = -\frac{1}{5}\left[-1 - x_1^{(k)} - 2x_2^{(k)}\right] = \frac{1}{5}\left[1 + x_1^{(k)} + 2x_2^{(k)}\right]$$

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	0	0	0 → (given)
1	1.833	0.7143	0.2
2	2.0381	1.1809	0.85234
3	2.08491	1.0530	1.07996.
4	2.0043	1.0012	1.0381
5	1.99405	0.9903	1.00134
6	1.99654	0.9979	0.99493
7	2.000145	1.00046	0.998468
8	2.0004 ≈ 2.00	1.00047 ≈ 1.00	1.000213 ≈ 1.00
9	2.0001 ≈ 2.00	1.00005 ≈ 1.00	1.000268 ≈ 1.00

∴ $x_1 = 2.00$, $x_2 = 1.00$, $x_3 = 1.00$

Gauss - Seidel method :

$x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}$ are prescribed.

$$x_1^{(1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)} - \cdots - a_{1n}x_n^{(0)}\right].$$

$$x_2^{(1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(0)} - \cdots - a_{2n}x_n^{(0)}\right].$$

$$x_3^{(1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(1)} - a_{32}x_2^{(1)} - \cdots - a_{2n}x_n^{(0)}\right]$$

$$\vdots$$

$$x_n^{(1)} = \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1^{(1)} - a_{n2}x_2^{(1)} - \cdots - a_{nn-1}x_{n-1}^{(1)}\right]$$

$$x_1^{(k+1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} - \cdots - a_{1n}x_n^{(k)}\right]$$

$$x_2^{(k+1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)} - \cdots - a_{2n}x_n^{(k)}\right]$$

$$x_3^{(k+1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(k+1)} - a_{32}x_2^{(k+1)} - a_{34}x_4^{(k)} \cdots - a_{3n}x_n^{(k)}\right]$$

$$\vdots$$

$$x_n^{(k+1)} = \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1^{(k+1)} - a_{n2}x_2^{(k+1)} - \cdots - a_{nn-1}x_{n-1}^{(k+1)}\right]$$

To find 3 iterations $\Rightarrow$ find $x_i^{(1)}, x_i^{(2)}, x_i^{(3)}$

Initial guess $x_i^{(0)}$ will be prescribed

Ex. Solve

$$6x_1 - 2x_2 + x_3 = 11$$
$$-2x_1 + 7x_2 + 2x_3 = 5$$
$$x_1 + 2x_2 - 5x_3 = -1.$$

by Gauss-seidel method $x_1^{(0)} = 0 = x_2^{(0)} = x_3^{(0)}$.

Given $x_1^{(0)} = 0 = x_2^{(0)} = x_3^{(0)}$.

Ans:

$$x_1^{(k+1)} = \frac{1}{6}\left[11 + 2x_2^{(k)} - x_3^{(k)}\right]$$

$$x_2^{(k+1)} = \frac{1}{7}\left[5 + 2x_1^{(k+1)} - 2x_3^{(k)}\right] \qquad k = 0, 1, 2, \ldots$$

$$x_3^{(k+1)} = \frac{1}{5}\left[1 + x_1^{(k+1)} + 2x_2^{(k+1)}\right]$$

K	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	0	0	0
1	1.833	1.2381	1.0619
2	2.06905	1.60204	1.0814
3	1.9982 ≈ 2.00	0.9953 ≈ 1.00	0.99776 ≈ 1.00
4	1.9988 ≈ 2.00	1.0001 ≈ 1.00	0.99776 ≈ 1.00

$x_1 = 2.00$, $x_2 = 1.00$, $x_3 = 1.00$

Dont write $x_1 = 2$, $x_2 = 1$, $x_3 = 1$

Note in the given system of equations

$$|6| > |-2| + |1|$$

$$|7| > |-2| + |2|$$

$$|-5| > |1| + |2|$$

Such system is called strictly diagonally dominant.

A square matrix is said to be strictly diagonally dominant if

$$|a_{ii}| > |a_{i1}| + |a_{i2}| + \cdots + |a_{i\,i-1}| + |a_{i\,i+1}| + \cdots + |a_{in}| \quad i = 1, 2, \ldots, n.$$

or $$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$$

Also $AX = B$ is strictly diagonally dominant system of equations, if A is strictly diagonally dominant.

Theorem: A sufficient condition for convergence of the Gauss-Jacobi / Gauss-seidel is that the coefficient matrix (system of equations) is strictly diagonally dominant (by row).

Note: If the system is d.d then the Gauss-Jacobi or Gauss-seidel scheme converges for any choice of initial guess.

Ex:

$$\left.\begin{aligned} x_1 - 2x_2 + 14x_3 - 7x_4 &= 3 \\ -21x_1 + x_2 - 4x_3 + 5x_4 &= -6 \\ 3x_1 - 50x_2 + 16x_3 - 3x_4 &= 45 \\ 5x_1 - 9x_2 + 3x_3 - 22x_4 &= 16 \end{aligned}\right\} \quad (1)$$

Now the d.d form of the above system of eq^{n}s (1) is

$$\begin{aligned} -21x_1 + x_2 - 4x_3 + 5x_4 &= -6 \\ 3x_1 - 50x_2 + 16x_3 - 3x_4 &= 45 \\ x_1 - 2x_2 + 14x_3 - 7x_4 &= 3 \\ 5x_1 - 9x_2 + 3x_3 - 22x_4 &= 16 \end{aligned}$$