

Online

DATE: _____

A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET



MAINS TEST SERIES-18

JUNE-2018 TO SEPT.-2018

Under the guidance of K. Venkanna

MATHEMATICS

PAPER - 1 : FULL SYLLABUS

TEST CODE: TEST-05: IAS(M)/08-JULY-2018

167
250

Time: Three Hours

Maximum Marks: 250

INSTRUCTIONS

1. This question paper-cum-answer booklet has 52 pages and has 32 PART/SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated.
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any THREE of the remaining questions selecting at least ONE question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any page from this booklet.

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name Karishak Kataria

Roll No. 1133664

Test Centre

Jaipur

Medium

English

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the invigilator

IMPORTANT NOTE:

Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

P.T.O.

(continued)

100

Age Group	Total	Female	Male	Unknown
18-24	100	85	15	0
25-34	100	75	25	0
35-44	100	85	15	0
45-54	100	75	25	0
55-64	100	85	15	0
65+	100	75	25	0

1-34

10. $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

3-83

24

1. What is the purpose of the study?

6-46

32

10. *Chlorophyll a* (Chl a) is the primary photosynthetic pigment in most plants and algae. It is a green pigment that absorbs light energy in the blue and red regions of the visible spectrum. Chl a is located in the thylakoid membranes of chloroplasts. It plays a central role in the light-dependent reactions of photosynthesis, where it transfers energy to other pigments and eventually to the reaction center, where water is split and oxygen is produced.

12

164

Handwritten note: *Handwritten note: ...*

[illegible]

100

SECTION - A

1. (a) Let $W_1 = \langle (2, 0, 3, 1, 1)', (1, 0, 2, 1, 1)', (2, 0, 3, 1, 3)' \rangle$

and $W_2 = \langle (2, 1, 1, 0, 1)', (3, 2, 3, 2, 3)', (1, 1, 1, 1, 1)' \rangle$

be subspaces of \mathbb{R}^5 . Find a basis for $W_1 + W_2$ and a basis for $W_1 \cap W_2$. [10]

$$W_1 = \{ (2, 0, 3, 1, 1)', (1, 0, 2, 1, 1)', (2, 0, 3, 1, 3)' \}$$

$$W_2 = \{ (2, 1, 1, 0, 1)', (3, 2, 3, 2, 3)', (1, 1, 1, 1, 1)' \}$$

To find basis for $W_1 + W_2$ consider $A = \begin{bmatrix} 2 & 0 & 3 & 1 & 1 \\ 1 & 0 & 2 & 1 & 1 \\ 2 & 0 & 3 & 1 & 3 \\ 2 & 1 & 1 & 0 & 1 \\ 3 & 2 & 3 & 2 & 3 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

Apply transformations to reduce it to row echelon form:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 & 1 \\ 2 & 0 & 3 & 1 & 3 \\ 2 & 1 & 1 & 0 & 1 \\ 3 & 2 & 3 & 2 & 3 \\ 2 & 0 & 3 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -2 & 1 & -1 & 1 \\ 0 & -1 & -1 & -2 & -1 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & -2 & 1 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & -2 & -2 & -1 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 & -1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{This is the row echelon form}$$

$$\therefore \text{Basis for } W_1 + W_2 = \{ (1, 1, 1, 1, 1)^T, (0, 1, -1, 0, 0)^T, (0, 0, 1, 1, -1)^T, (0, 0, 0, 0, 1)^T \}$$

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

$$\therefore \dim(W_1 \cap W_2) = 3 + 3 - 4 = 2$$

1. (b) Find the characteristic polynomial of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & i & 0 & 0 \\ 2 & \frac{1}{2} & -i & 0 \\ \frac{1}{3} & -i & \pi & -1 \end{bmatrix}$$

Diagonalise this matrix, if possible.

[10]

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & i & 0 & 0 \\ 2 & \frac{1}{2} & -i & 0 \\ \frac{1}{3} & -i & \pi & -1 \end{bmatrix} \Rightarrow A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & 0 & 0 \\ -1 & i-\lambda & 0 & 0 \\ 2 & \frac{1}{2} & -i-\lambda & 0 \\ \frac{1}{3} & -i & \pi & -1-\lambda \end{bmatrix}$$

$$\begin{aligned} \therefore |A - \lambda I| &= (1-\lambda)(i-\lambda)(-i-\lambda)(-1-\lambda) \\ &= (1-\lambda)(1+\lambda)(i-\lambda)(i+\lambda) \\ &= (1-\lambda^2)(-1-\lambda^2) = -(1-\lambda^2)(1+\lambda^2) \end{aligned}$$

$$\text{characteristic polynomial} = \lambda^4 - 1$$

Eigenvalues $\lambda = \{1, -1, i, -i\}$ Let $X = (x_1, x_2, x_3, x_4)$

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P.T.O.

for $\lambda = 1$ $A - \lambda I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & i-1 & 0 & 0 \\ 2 & 1-i & i-1 & 0 \\ 1/3 & -i & \pi & -2 \end{bmatrix}$

$(A - I)x = 0$ Given all the eigenvalues all unique & distinct we can say that A is diagonalizable to the A is similar to the diagonal matrix

$$\begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}$$

$\therefore A$ can be diagonalised to $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}$

with matrix $P = [x_1^T \ x_2^T \ x_3^T \ x_4^T]$

where x_1, x_2, x_3, x_4 are eigenvectors corresponding to $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = i, \lambda_4 = -i$

1. (c) Find the values of a and b in order that

$$\lim_{x \rightarrow 0} \frac{x(1+a \cos x) - b \sin x}{x^3} \text{ may be equal to } 1.$$

[10]

$$\lim_{x \rightarrow 0} \frac{x(1+a \cos x) - b \sin x}{x^3} = L$$

To make the function of indeterminate form $\frac{0}{0}$

Numerator has to be 0 as $x \rightarrow 0$

$$\therefore \lim_{x \rightarrow 0} x(1+a \cos x) - b \sin x = 0 \text{ which is true}$$

Applying L'Hospital Rule $L = \lim_{x \rightarrow 0} \frac{(1+a \cos x) + x(-a \sin x) - b \cos x}{3x^2}$

$$L = \lim_{x \rightarrow 0} \frac{1 + (a-b) \cos x - ax \sin x}{3x^2}$$

Again to make indeterminate $\lim_{x \rightarrow 0} \frac{1 + (a-b) \cos x - ax \sin x}{3x^2} = \frac{0}{0}$
 $\Rightarrow \boxed{1 + a - b = 0} \quad \text{--- (1)}$

Applying L'Hospital Rule: $L = \lim_{x \rightarrow 0} \frac{-(a-b) \sin x - a \sin x - ax \cos x}{6x}$

$$L = \lim_{x \rightarrow 0} \frac{(b-2a) \sin x - ax \cos x}{6x}$$

which is again indeterminate of form $\frac{0}{0}$

\therefore Applying L'Hospital rule again $L = \lim_{x \rightarrow 0} \frac{(b-2a) \cos x - a \cos x + ax \sin x}{6}$

$$\Rightarrow L = \frac{(b-2a) - a}{6} = 1 \text{ (given)}$$

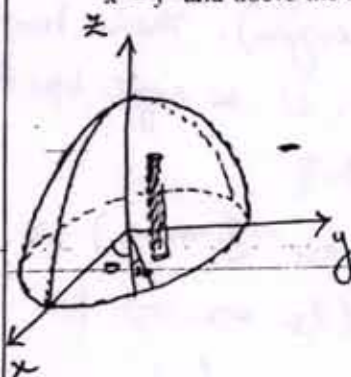
$$\therefore b - 3a = 6 \quad \text{--- (11)}$$

using (1) & (11) \Rightarrow

$$\begin{aligned} b - a &= 1 \\ b - 3a &= 6 \end{aligned} \Rightarrow \begin{aligned} a &= -5/2 \\ b &= -3/2 \end{aligned}$$

$$a = -5/2 \quad b = -3/2$$

1. (d) Find the volume of the region lying below the paraboloid with equation $z = 4 - x^2 - y^2$ and above the xy -plane. [10]



paraboloid: $z = 4 - x^2 - y^2$

$$\text{Volume} = \iint_R z \, dx \, dy$$

where R is the circle

$$x^2 + y^2 = 4 \text{ in } xy \text{ plane}$$

$$\therefore V = \iint_R (4 - x^2 - y^2) \, dx \, dy \Rightarrow \text{converting to polar coordinates}$$

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned}$$

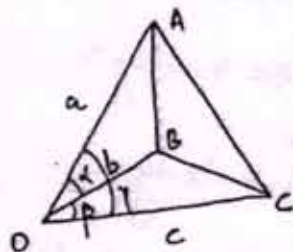
r varies from 0 to 2 & θ from 0 to 2π

$$\therefore V = \int_{\theta=0}^{2\pi} \int_{r=0}^2 (4 - r^2 \cos^2 \theta - r^2 \sin^2 \theta) r \, dr \, d\theta$$

$$\begin{aligned}
 V &= \int_0^{2\pi} \int_0^2 (4-r^2)r dr d\theta = \int_0^{2\pi} d\theta \int_0^2 (4r-r^3) dr \\
 &= \left[\theta \right]_0^{2\pi} \left[2r^2 - \frac{r^4}{4} \right]_0^2 = (2\pi-0) \left(2 \times 4 - \frac{16}{4} \right) \\
 &= (2\pi)(8-4) \\
 &= 8\pi
 \end{aligned}$$

$$\therefore \text{volume} = 8\pi \text{ units}^3$$

1. (c) Find the volume of a tetrahedron in terms of the lengths of the three edges which meet in point and of the angles which these edges make with each other in pairs. [10]



Let the 3 edges meet in the point O (origin). Their lengths are a, b, c & angle b/w them are α, β, γ

Let their direction cosines be (l_1, m_1, n_1) for OA, (l_2, m_2, n_2) for OB and (l_3, m_3, n_3) for OC.

As their lengths are given $A = (al_1, am_1, an_1)$
 $B = (bl_2, bm_2, bn_2)$
 $C = (cl_3, cm_3, cn_3)$

and

$$\begin{aligned}
 l_1 l_2 + m_1 m_2 + n_1 n_2 &= \cos \alpha \\
 l_1 l_3 + m_1 m_3 + n_1 n_3 &= \cos \gamma \\
 l_2 l_3 + m_2 m_3 + n_2 n_3 &= \cos \beta
 \end{aligned}$$

3. (a) Let $W = \{[x_1, x_2, x_3, x_4]^T \in \mathbb{R}^4 \mid 2x_1 + 3x_2 = 4x_3 + x_4\}$. Show that W is a subspace of \mathbb{R}^4 . Find a basis of W and extend it to form a basis of \mathbb{R}^4 . Do the same if $W = \{[x_1, x_2, x_3, x_4]^T \in \mathbb{R}^4 \mid x_1 + x_2 = 0, x_3 - x_4 = 0\}$. [15]

$$W = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid 2x_1 + 3x_2 = 4x_3 + x_4\}$$

To prove W is a subspace of \mathbb{R}^4 , we need to prove that

for $\alpha, \beta \in W$ and $a, b \in \mathbb{R}$

$$a\alpha + b\beta \in W$$

$$\text{let } \alpha = (x_1, x_2, x_3, x_4) \in W$$

$$\beta = (y_1, y_2, y_3, y_4) \in W$$

$$a\alpha + b\beta = a(x_1, x_2, x_3, x_4) + b(y_1, y_2, y_3, y_4)$$

$$= (ax_1, ax_2, ax_3, ax_4) + (by_1, by_2, by_3, by_4)$$

$$= (ax_1 + by_1, ax_2 + by_2, ax_3 + by_3, ax_4 + by_4)$$

$$\text{Now } \alpha, \beta \in W$$

$$2x_1 + 3x_2 = 4x_3 + x_4$$

$$2y_1 + 3y_2 = 4y_3 + y_4$$

$$\text{Take } 2(ax_1 + by_1) + 3(ax_2 + by_2)$$

$$= a(2x_1 + 3x_2) + b(2y_1 + 3y_2)$$

$$= a(4x_3 + x_4) + b(4y_3 + y_4)$$

$$= 4(ax_3 + by_3) + (ax_4 + by_4)$$

$$\Rightarrow a\alpha + b\beta \in W \Rightarrow \therefore W \text{ is subspace of } \mathbb{R}^4$$

$$\text{If } (x_1, x_2, x_3, x_4) \in W$$

$$\Rightarrow x_4 = 2x_1 + 3x_2 - 4x_3$$

$$\alpha = (x_1, x_2, x_3, 2x_1 + 3x_2 - 4x_3)$$

$$= x_1(1, 0, 0, 2) + x_2(0, 1, 0, 3)$$

$$+ x_3(0, 0, 1, -4)$$

\therefore Any element in W can be expressed as a linear combination of $(1, 0, 0, 2), (0, 1, 0, 3), (0, 0, 1, -4)$

$$\therefore \text{Basis of } W = \{(1, 0, 0, 2), (0, 1, 0, 3), (0, 0, 1, -4)\}$$

Now we add an element 'a' to the basis which doesn't make the set present in linear span of the basis

$$\text{Take } \alpha = (1, 1, 1, 0)$$

$$S = \{(1, 0, 0, 2), (0, 1, 0, 3), (0, 0, 1, -4), (1, 1, 1, 0)\}$$

$$a(1, 0, 0, 2) + b(0, 1, 0, 3) + c(0, 0, 1, -4) + d(1, 1, 1, 0) = (a+d, b+d, c+d, 2a+3b-4c) = 0$$

$$\Rightarrow a = b = c = d = 0$$

$\therefore S$ is linearly independent with dimension 4. $\therefore S$ is a basis of \mathbb{R}^4

$$\text{Volume of tetrahedron} = \frac{1}{6} \begin{vmatrix} x_0 & y_0 & z_0 & 1 \\ x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \end{vmatrix}$$

$$= \frac{1}{6} \begin{vmatrix} 0 & 0 & 0 & 1 \\ a_1 & a_m & a_n & 1 \\ b_1 & b_m & b_n & 1 \\ c_1 & c_m & c_n & 1 \end{vmatrix} \Rightarrow \frac{1}{6} \begin{vmatrix} a_1 & a_m & a_n \\ b_1 & b_m & b_n \\ c_1 & c_m & c_n \end{vmatrix}$$

$$= \frac{abc}{6} \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

$$\text{Consider } D = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

$$\text{Also } D = \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}$$

$$\therefore D^2 = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} = \begin{vmatrix} 1 & \cos \alpha & \cos \gamma \\ \cos \alpha & 1 & \cos \beta \\ \cos \gamma & \cos \beta & 1 \end{vmatrix}$$

[\therefore transpose doesn't alter determinant]

$$\therefore D^2 = \begin{vmatrix} 1 & \cos \alpha & \cos \gamma \\ \cos \alpha & 1 & \cos \beta \\ \cos \gamma & \cos \beta & 1 \end{vmatrix} \therefore D = \sqrt{\begin{vmatrix} 1 & \cos \alpha & \cos \gamma \\ \cos \alpha & 1 & \cos \beta \\ \cos \gamma & \cos \beta & 1 \end{vmatrix}}$$

$$= \sqrt{1 - \cos^2 \alpha - \cos^2 \beta - \cos^2 \gamma - 2 \cos \alpha \cos \beta \cos \gamma}$$

$$\therefore \text{Volume of tetrahedron} = \frac{abc}{6} \sqrt{1 - \cos^2 \alpha - \cos^2 \beta - \cos^2 \gamma - 2 \cos \alpha \cos \beta \cos \gamma}$$

If $W = \{(x_1, x_2, x_3, x_4) \mid x_1 + x_2 = 0, x_3 - x_4 = 0\}$

$$v = (x_1, x_2, x_3, x_4) = (x_1, -x_1, x_3, x_3) \\ = x_1(1, -1, 0, 0) + x_3(0, 0, 1, 1)$$

\therefore Basis of $W = \{(1, -1, 0, 0), (0, 0, 1, 1)\}$

$\therefore (1, -1, 0, 0), (0, 0, 1, 1)$ are linearly independent & they can be used to represent any element of W by linear combination

To extend the basis, we add $(0, 1, 1, 0) \wedge (1, 0, 0, 1)$ to the basis to form

$$S' = \{(1, -1, 0, 0), (0, 0, 1, 1), (0, 1, 1, 0), (1, 0, 0, 1)\}$$

$$\dim(S') = 4$$

$$\text{Also } a(1, -1, 0, 0) + b(0, 0, 1, 1) + c(0, 1, 1, 0) + d(1, 0, 0, 1)$$

$$= (a+d, -a+c, b+c, b+d) = (0, 0, 0, 0)$$

$$\Rightarrow a = b = c = d = 0$$

\therefore they are linearly independent & hence form basis of \mathbb{R}^4

3. (b) (i) The temperature at a point (x, y) on a metal plate is $T(x, y) = 4x^2 - 4xy + y^2$. An ant on the plate walks around the circle of radius 5 centered at the origin. What are the highest and lowest temperatures encountered by the Ant?

- (ii) Evaluate the integral $\int_0^{\pi} \int_0^{\pi} e^{-x^2-y^2} dx dy$ by changing the order of integration.

[12+08=20]

(i) $T(x, y) = 4x^2 - 4xy + y^2$ Ant walks on $x^2 + y^2 = 25$.

$$\therefore F = (4x^2 - 4xy + y^2) + \lambda(x^2 + y^2 - 25)$$

$$F_x = \frac{\partial F}{\partial x} = 8x + 2\lambda x - 4y = 0 \quad F_y = (-4x + 2y + 2\lambda y)$$

At stationary points $F_x = F_y = 0$

$$\Rightarrow (8+2\lambda)x - 4y = 0 \Rightarrow \frac{2y}{x} = \frac{(8+2\lambda)}{4}$$

$$4x - (2+2\lambda)y = 0 \Rightarrow \frac{y}{x} = \frac{4}{2+2\lambda}$$

$$\Rightarrow \frac{8+2\lambda}{4} = \frac{4}{2+2\lambda} \Rightarrow 16 + 4\lambda + 6\lambda + 4\lambda^2 = 16 \\ \Rightarrow 4\lambda^2 + 20\lambda = 0$$

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$$\Rightarrow 4\lambda(\lambda+5)=0 \Rightarrow \lambda=0, -5$$

for $\lambda=0 \Rightarrow y=2x$ & $x^2+y^2=25 \Rightarrow 5x^2=25$
 $x=\pm\sqrt{5}$
 $y=\pm 2\sqrt{5}$

for $\lambda=-5 \Rightarrow x=-2y$ & $x^2+y^2=25 \Rightarrow 5y^2=25$
 $y=\pm\sqrt{5}$
 $x=\mp 2\sqrt{5}$

$$F_{xx} = (6+2\lambda) \quad F_{yy} = (2+2\lambda)$$

$$F_{xy} = -4 \quad (F_{xx})(F_{yy}) - (F_{xy})^2 = (6+2\lambda)(2+2\lambda) - (-4)^2 > 0$$

at $\lambda=0 \quad F_{xx} > 0 \Rightarrow$ Temperature is minimum

at $\lambda=-5 \quad F_{xx} < 0 \Rightarrow$ Temperature is maximum

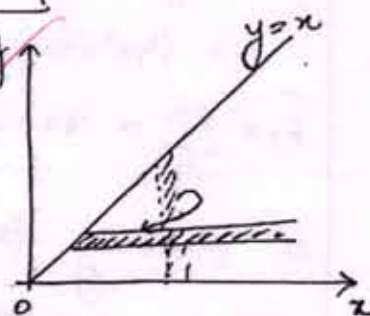
$$\therefore T_{\min} = 4(\sqrt{5})^2 - 4(\sqrt{5})(2\sqrt{5}) + (2\sqrt{5})^2 = 20 - 40 + 20 = 0$$

$$T_{\max} = 4(2\sqrt{5})^2 - 4(2\sqrt{5})(-\sqrt{5}) + (\sqrt{5})^2 = 80 + 40 + 5 = 125$$

$$\therefore T_{\max} = 125 \quad \& \quad T_{\min} = 0$$

$$(ii) \int_0^{\infty} \int_0^x x e^{-x^2/y} dx dy$$

changing order of integration



$$I = \int_{y=0}^{\infty} \int_{x=y}^{\infty} x e^{-x^2/y} dx dy$$

Let $\frac{x^2}{y} = t$
 $\Rightarrow \frac{2x}{y} dx = dt \Rightarrow \frac{2x}{y} dx = dt$

$$I = \int_{y=0}^{\infty} \int_{t=y}^{\infty} \frac{y}{2} dt \cdot e^{-t} dy = \frac{1}{2} \int_0^{\infty} \int_y^{\infty} y dy e^{-t} dt$$

$$= \frac{1}{2} \int_0^{\infty} y dy [-e^{-t}]_0^{\infty} = \frac{1}{2} \int_0^{\infty} y dy [-\lim_{t \rightarrow \infty} (He^{-t} - e^{-y})]$$

$$= \frac{1}{2} \int_0^{\infty} y e^{-y} dy$$

$$= \frac{1}{2} \left[\frac{y e^{-y}}{-1} \Big|_0^{\infty} - \int_0^{\infty} 1 \cdot \frac{e^{-y}}{-1} dy \right]$$

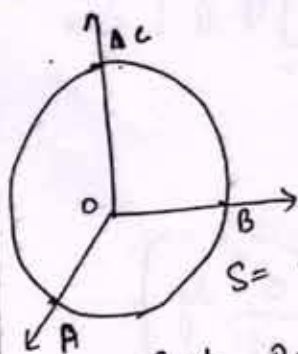
$$= \frac{1}{2} \left[\lim_{y \rightarrow \infty} \frac{-y}{e^y} + 0 \right] + \frac{1}{2} \int_0^{\infty} e^{-y} dy$$

$$= 0 + \frac{1}{2} [-e^{-y}]_0^{\infty} \quad \left[\because \lim_{y \rightarrow \infty} \frac{-y}{e^y} = 0 \right]$$

$$= \frac{1}{2} [0 + 1] = \frac{1}{2} \quad \left[\because \lim_{y \rightarrow \infty} e^{-y} = 0 \right]$$

$$\therefore \boxed{I = \frac{1}{2}}$$

3. (c) A sphere of constant radius $2k$ passes through the origin and meets the axes in A, B, C. Find the locus of the centroid of the tetrahedron OABC. [15]



Let the points A, B, C be $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$.

Let the equation of sphere be
 $S = x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

Points O, A, B, C lie on S

$$\therefore 0 + 0 + 0 + 0 + 0 + 0 + d = 0 \Rightarrow d = 0$$

$$a^2 + 0 + 0 + 2ua + 0 + 0 = 0 \Rightarrow u = -a/2$$

$$\text{Similarly } v = -b/2 \quad w = -c/2$$

$$\therefore S = x^2 + y^2 + z^2 - ax - by - cz = 0$$

Centroid of $\Delta ABC = \left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right) = (X, Y, Z)$

Also S has constant radius $= 2k = \sqrt{u^2 + v^2 + w^2 - d}$

$$\Rightarrow 4k^2 = \frac{a^2}{4} + \frac{b^2}{4} + \frac{c^2}{4}$$

Putting $a = 3X$, $b = 3Y$, $c = 3Z$ to get the locus of centroid

$$4k^2 = \frac{9X^2}{4} + \frac{9Y^2}{4} + \frac{9Z^2}{4}$$

$$\Rightarrow \boxed{\frac{16k^2}{9} = X^2 + Y^2 + Z^2} \text{ is the required locus}$$

SECTION - B

5. (a) Find the orthogonal trajectories of
- $r = a(1 + \cos n\theta)$
- .

[10]

$$r = a(1 + \cos n\theta) \Rightarrow \frac{r}{dr} = \frac{1 + \cos n\theta}{-n \sin n\theta}$$

$$\frac{dr}{d\theta} = -an \sin n\theta$$

Replacing $\frac{dr}{d\theta}$ by $-\frac{r^2}{dr}$

$$\Rightarrow \frac{r dr}{-r^2 d\theta} = \frac{1 + \cos n\theta}{-n \sin n\theta}$$

$$\Rightarrow \frac{n dr}{r} = \frac{1 + \cos n\theta}{\sin n\theta} d\theta = \frac{1 + \cos n\theta}{1 - \cos^2 n\theta} \sin n\theta d\theta$$

Put $\cos n\theta = t \Rightarrow -n \sin n\theta d\theta = dt$

$$\Rightarrow \frac{n dr}{r} = \frac{-dt(1+t)}{n(1-t)(1+t)} \Rightarrow \int \frac{dr}{r} = \int \frac{-dt}{1-t}$$

$$\Rightarrow n^2 \log r = \log(1 - \cos n\theta) + \log b$$

$$\Rightarrow r^{n^2} = b(1 - \cos n\theta)$$

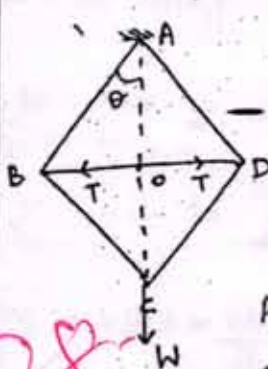
Q8

5. (b) Use the variation of parameters method to show that the solution of equation $d^2 y/dx^2 + k^2 y = \phi(x)$ satisfying the initial conditions $y(0) = 0$, $y'(0) = 0$ is

$$y(x) = \frac{1}{k} \int_0^x \phi(t) \sin k(x-t) dt.$$

[10]

5. (c) A frame work ABCD consists of four equal, light rods smoothly jointed together to form a square, it is suspended from a peg at A, and a weight W is attached to C, the framework being kept in shape by a light rod connecting B and D. Determine the thrust in this rod. [10]



Height of the 4 rods & the middle rod is negligible.

Weight W is acting at C and thrust in rod BD.

Replacing rod by the thrusts T for calculation, we get :-

Let $\angle ABC$ be θ ($=\angle DAC$) & the system given a displacement $\delta\theta$. As a result there is change in length BD & AC.

By Principle of Virtual Work $\delta W = W(\delta AC) + T(\delta BD) = 0$
let lengths of the 4 rods be 'a'

$$AC = AD + DC = 2 \times AD = 2a \cos \theta \Rightarrow \delta(AC) = -2a \sin \theta \delta \theta$$

$$BD = 2 BO = 2a \sin \theta \Rightarrow \delta(BD) = 2a \cos \theta \delta \theta$$

$$\therefore W(-2a \sin \theta \delta \theta) + T(2a \cos \theta \delta \theta) = 0$$

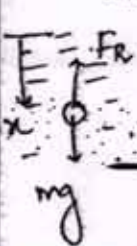
$$\Rightarrow W \sin \theta = T \cos \theta$$

$$\Rightarrow T = W \tan \theta$$

At equilibrium, $\theta = 45^\circ$ as ABCD is a square

$$\therefore T = W \tan 45^\circ = W$$

5. (d) A particle of mass m , is falling under the influence of gravity through a medium whose resistance equals μ times the velocity. If the particle were released from rest, determine the distance fallen through in time t . [10]



$$F_R = -\mu \frac{dx}{dt} \quad [\text{acting opposite to velocity}]$$

$$\text{Net force on particle} = mg - \mu \frac{dx}{dt}$$

$$\text{Let acceleration be } \frac{d^2x}{dt^2} \text{ at time } t$$

$$\therefore mg - \mu \frac{dx}{dt} = m \frac{d^2x}{dt^2} \Rightarrow m \frac{dv}{dt} - \mu v = mg$$

$$\Rightarrow \frac{dv}{dt} + \frac{\mu}{m} v = g \quad \text{Integrating factor} \Rightarrow e^{\int \frac{\mu}{m} dt} = e^{\frac{\mu t}{m}}$$

$$\therefore v e^{\frac{\mu t}{m}} = \int g e^{\frac{\mu t}{m}} dt + c \Rightarrow v e^{\frac{\mu t}{m}} = \frac{g e^{\frac{\mu t}{m}}}{\frac{\mu}{m}} + c$$

$$\text{at } t=0, v=0 \text{ as released from rest} \Rightarrow c = -\frac{mg}{\mu}$$

$$\Rightarrow v = \frac{mg}{\gamma} (1 - e^{-\frac{\gamma t}{m}}) = \frac{dx}{dt}$$

$$\Rightarrow \int_0^x dx = \int_0^t \frac{mg}{\gamma} (1 - e^{-\frac{\gamma t}{m}}) dt$$

$$\Rightarrow x = \frac{mg}{\gamma} \left[t - \frac{m}{-\gamma} e^{-\frac{\gamma t}{m}} \right]_0^t = \frac{mg}{\gamma} \left[t + \frac{m}{\gamma} e^{-\frac{\gamma t}{m}} - \frac{m}{\gamma} \right]$$

$$x = \frac{mg}{\gamma} t - \frac{m^2 g}{\gamma^2} \left[1 - e^{-\frac{\gamma t}{m}} \right]$$

5. (c) Represent the vector $A = z\mathbf{i} - 2x\mathbf{j} + y\mathbf{k}$ in cylindrical coordinates. Thus determine A_r , A_θ and A_z . [10]

6. (a) Solve $(D^2 - 1)y = \cosh x \cos x + a^2$.

[13]

$$(D^2 - 1)y = \cosh x \cos x + a^2$$

Complementary function

$$y_c = (D^2 - 1)y_c = 0$$

$$(D-1)(D+1)y_c = 0$$

auxiliary equation: $m^2 - 1 = 0$
 $m = \pm 1$

$$y_c = c_1 e^x + c_2 e^{-x} \quad \text{--- (1)}$$

Particular Integral:

$$y_p = \frac{1}{(D^2 - 1)} (\cosh x \cos x + a^2)$$

$$= \frac{1}{2(D-1)} - \frac{1}{2(D+1)} (\cosh x \cos x + a^2) \quad \text{--- (2)}$$

$$\frac{1}{(D-a)} f(x) = e^{+ax} \int e^{-ax} f(x) dx$$

$$\cosh x \cos x = \frac{(e^x + e^{-x})}{2} \cos x$$

$$I_1 = \frac{1}{D-1} \cosh x \cos x$$

$$= e^x \int e^{-x} \frac{(e^x + e^{-x})}{2} \cos x dx$$

$$= e^x \int \frac{(1 + e^{-2x})}{2} \cos x dx$$

$$= \frac{e^x}{2} \int (\cos x + \cos x \cdot e^{-2x}) dx$$

$$= \frac{e^x}{2} \left[\sin x + \frac{e^{-2x} (\sin x - 2 \cos x)}{5} \right] \quad \text{--- (11)}$$

\therefore using Integration by parts
on $\int \cos x \cdot e^{-2x} \cdot dx$

$$= \frac{e^{-2x}(\sin x - 2\cos x)}{5}$$

$$I_2 = \frac{1}{D-1} a^x = e^x \int e^{-x} a^x dx$$

$$= e^x \times \left(\frac{-a^x e^{-x}}{1-\ln a} \right)$$

$$= \frac{a^x}{1-\ln a} \quad \text{--- (iii)}$$

$$I_3 = \frac{1}{D+1} \left[\cosh x \cos x \right]$$

$$= e^{-x} \int e^x \frac{(e^x + e^{-x})}{2} \cos x dx$$

$$= \frac{e^{-x}}{2} \left[\int \cos x + e^{2x} \cos x dx \right]$$

$$= \frac{e^{-x}}{2} \left[\sin x + e^{2x} \frac{(\sin x + 2\cos x)}{5} \right] \quad \text{--- (iv)}$$

$$I_4 = \frac{1}{D+1} a^x \frac{1}{2} e^{-x} \int e^x a^x dx$$

$$= e^{-x} \times \frac{a^x e^x}{1+\ln a}$$

$$= \frac{a^x}{1+\ln a} \quad \text{--- (v)}$$

Putting (iii), (iv), (v) in (i)

$$I = \frac{1}{2} \left[\frac{e^x \sin x}{2} + \frac{e^{-x}(\sin x - 2\cos x)}{10} \right. \\ \left. + \frac{a^x}{1-\ln a} - \frac{e^{-x} \sin x}{2} \right. \\ \left. - \frac{e^x(\sin x + 2\cos x)}{10} - \frac{a^x}{1+\ln a} \right]$$

$$= \frac{1}{2} \left(\frac{e^x - e^{-x}}{2} \right) \sin x$$

$$+ \frac{1}{10} \left[\frac{e^{-x}}{2} \right]$$

$$= \frac{a^x \ln a}{1-(\ln a)^2} + \frac{\sinh x \cosh x}{2}$$

$$+ \frac{(\sin x - 2\cos x)e^{-x}}{20} - \frac{(\sin x + 2\cos x)e^x}{20}$$

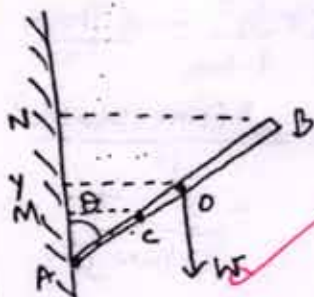
\therefore Integral = $y_c + y_p$

$$= C_1 e^x + C_2 e^{-x} + \frac{a^x \ln a}{1-(\ln a)^2}$$

$$+ \frac{\sinh x \cosh x}{2} + \frac{(\sin x - 2\cos x)e^{-x}}{20}$$

$$- \frac{(\sin x + 2\cos x)e^x}{20}$$

6. (b) A uniform beam of length $5a$, rests in equilibrium against a smooth vertical wall and upon a smooth peg at a distance b from the wall. Show that in the position of equilibrium the beam is inclined to the wall at an angle $\sin^{-1} (b/5a)^{1/3}$ [10]



Let the peg be at 'c' such that $CM = b$

Length $AB = 5a$.

Point C is fixed, upon a small displacement on point O moves.

\therefore By principle of virtual work: $\delta W = 0 = W \delta(MY)$

$$MY = AY - AM = OA \cos \theta - CM \cot \theta$$

$$OA = \frac{AB}{2} = \frac{5a}{2} \quad CM = b \text{ given}$$

$$\therefore \delta(MY) = \delta \left(\frac{5a}{2} \cos \theta - b \cot \theta \right) = -\frac{5a}{2} \sin \theta \delta \theta + b \operatorname{cosec}^2 \theta \delta \theta$$

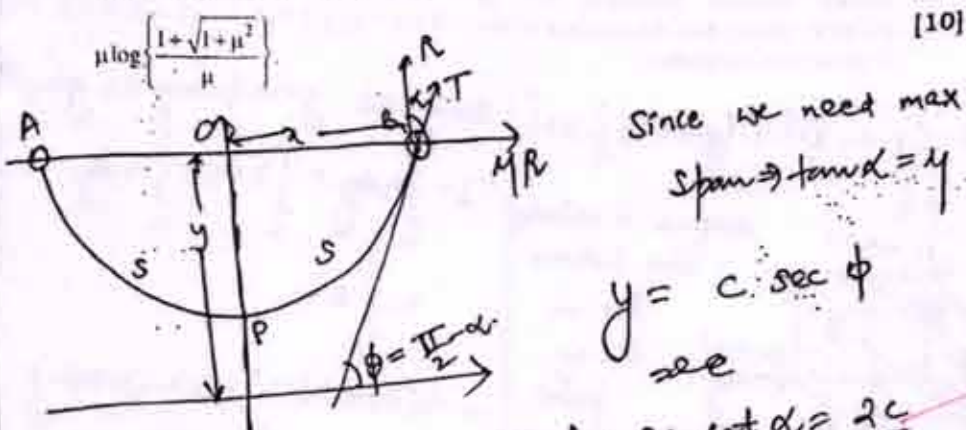
$$\therefore W \left[-\frac{5a}{2} \sin \theta + b \operatorname{cosec}^2 \theta \right] \delta \theta = 0$$

$$\Rightarrow \frac{5a}{2} \sin \theta = \frac{b}{\sin^2 \theta} \Rightarrow \sin^3 \theta = \frac{2b}{5a}$$

$$\therefore \sin \theta = \left(\frac{2b}{5a} \right)^{1/3}$$

$$\therefore \theta = \sin^{-1} \left(\frac{2b}{5a} \right)^{1/3}$$

6. (c) The end links of a uniform chain slide along a fixed rough horizontal rod. Prove that the ratio of the maximum span to the length of the chain is



length = $2s = 2c \tan \phi = 2c \cot \alpha = \frac{2c}{\mu}$

span = $2x = 2c \log(\sec \phi + \tan \phi)$

$= 2c \log(\csc \alpha + \cot \alpha)$

$= 2c \log\left(\frac{1 + \sqrt{1 + \mu^2}}{\mu}\right)$

$\therefore \frac{\text{span}}{\text{length}} = \frac{2x}{2s} = \frac{2c \log\left(\frac{1 + \sqrt{1 + \mu^2}}{\mu}\right)}{\frac{2c}{\mu}}$

$= \mu \log\left(\frac{1 + \sqrt{1 + \mu^2}}{\mu}\right)$

$I_3 = \iint_{A-DH} (-\hat{o} + (z-1)\hat{j} - \hat{k}) \cdot \hat{j} \, dx \, dz$
 $= \iint_{A-DH} -(z-1) \, dx \, dz$
 $= -2\left(\frac{z^2}{2} - z\right) = 0$

$I_4 = \iint_{A-BEH} (-y\hat{i} + (z-1)\hat{k} - \hat{k}) \cdot \hat{i} \, dy \, dz$
 $= \iint_{A-BEH} -y \, dy \, dz = -4$

$I_5 = \iint_{EHGF} (-y\hat{i} + \hat{j} - \hat{k}) \cdot (\hat{k}) \, dx \, dy$
 $= \iint_{EHGF} -1 \, dx \, dy = -4$

Putting these values in (1)

$\iint_S (\vec{\nabla} \times \vec{A}) \cdot \hat{n} \, ds = 0 + 4 + 0 - 4$
 $= -4 \text{ sq units}$
 — (III)

From (I) & (III)

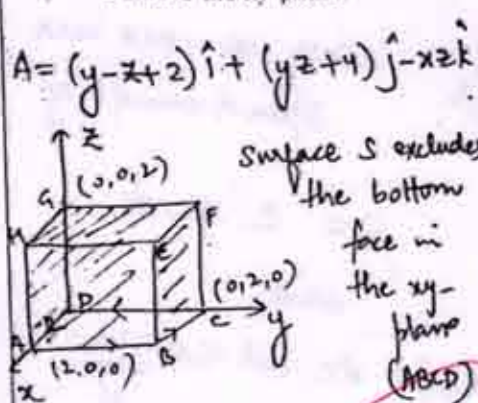
$\oint_C \vec{A} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{A}) \cdot \hat{n} \, ds$

Hence verified.

7. (a) Find the general and singular solution of $y^2(y - xp) = x^4p^2$.

[12]

6. (d) Verify Stokes theorem for $\vec{A} = (y - z + 2)\hat{i} + (yz + 4)\hat{j} - xz\hat{k}$, where S is the surface of the cube $x = 0, y = 0, z = 0, x = 2, y = 2, z = 2$ above the xy plane. [17]



Stokes's theorem:

$$\oint_C \vec{A} \cdot d\vec{r} = \iint_S (\nabla \times \vec{A}) \cdot \hat{n} ds$$

LINE INTEGRAL

C is the path ABCD in anticlockwise direction

$$\vec{A} = (y - 0 + 2)\hat{i} + (y \cdot 0 + 4)\hat{j} - x \cdot 0\hat{k} \\ = (y + 2)\hat{i} + 4\hat{j} \quad [z = 0]$$

$$\therefore \oint_C \vec{A} \cdot d\vec{r} = \int_{AB} \vec{A} \cdot d\vec{r} + \int_{BC} \vec{A} \cdot d\vec{r} + \int_{CD} \vec{A} \cdot d\vec{r} + \int_{DA} \vec{A} \cdot d\vec{r}$$

Along AB & CD ($dx = 0$) & 2

Along AD & BC ($dy = 0$)

$$\therefore \oint_C \vec{A} \cdot d\vec{r} = \int_{AB} 4 dy + \int_{BC} (y + 2) dx + \int_{CD} 4 dy + \int_{DA} (y + 2) dx$$

$$\text{Along BC } y = 2 \text{ along DA } y = 0 \\ \therefore I = \int_0^2 4 dy + \int_2^0 4 dx + \int_2^0 4 dy + \int_0^2 2 dx \\ = 4(2) + 4(-2) + 4(-2) + 2(2) \\ = -4 \quad \text{--- (1)}$$

SURFACE INTEGRAL

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz + 2 & yz + 4 - xz & \end{vmatrix}$$

$$= \hat{i}[0 - y] - \hat{j}[-z + 1] + \hat{k}[0 - 1] \\ = -y\hat{i} + (z - 1)\hat{j} - \hat{k}$$

$$\iint_S (\nabla \times \vec{A}) \cdot \hat{n} ds = I_1 + I_2 + I_3 + I_4 + I_5 \quad \text{--- (2)}$$

$$I_1 = \iint_{EBCF} (-y\hat{i} + (z - 1)\hat{j} - \hat{k}) \cdot \hat{j} dy dz$$

$$= \iint_{EBCF} (z - 1) dy dz \\ = (2) \left(\frac{2^2}{2} - 2 \right) = 0$$

$$I_2 = \iint_{DCFG} (-y\hat{i} + (z - 1)\hat{j} - \hat{k}) \cdot (-\hat{i}) dy dz \\ = \iint_{DCFG} y dy dz = \frac{2^2}{2} \cdot 2 = 4$$

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8. (a) By using Laplace transform method solve the $(D^3 - 2D^2 + 5D)y = 0$ if $y(0) = 0, y'(0) = 1, y(2/8) = 1$ [15]

$$(D^3 - 2D^2 + 5D)y = 0 \Rightarrow y''' - 2y'' + 5y' = 0$$

$$\mathcal{L}\{F''(t)y\} = p^n \mathcal{L}\{F(t)y\} - p^{n-1}F(0) - p^{n-2}F'(0) - \dots - F^{(n-1)}(0)$$

$$\mathcal{L}\{y''' - 2y'' + 5y'\} = 0$$

$$\Rightarrow p^3 \mathcal{L}(y) - p^2 y(0) - p y'(0) - y''(0) - [p^2 \mathcal{L}(y) - p y(0) - y'(0)] + 5[p \mathcal{L}(y) - y(0)] = 0$$

$$\Rightarrow (p^3 - 2p^2 + 5p) \mathcal{L}(y) - [p^2 + p - 5](0) - [p - 2](1) - y''(0) = 0$$

$$\mathcal{L}(y) = \frac{y''(0) + (p - 2)}{p(p^2 - 2p + 5)}$$

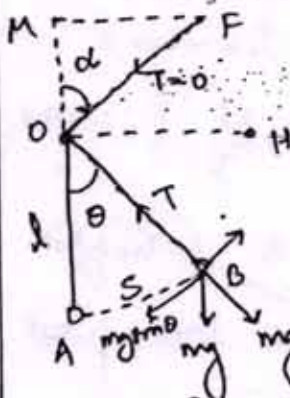
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8. (b) A heavy particle hanging vertically from a fixed point by a light inextensible cord of length l is struck by a horizontal blow which imparts it a velocity $2\sqrt{gl}$, prove that the cord becomes slack when the particle has risen to a height $\frac{2}{3}l$ above the fixed point. [15]



Initial velocity at $t=0 = 2\sqrt{gl}$
 Let the particle be at point B,
 at time t , subtending angle θ with
 the vertical and having
 tension T in the string.

Now writing the net eqns of motion on the
 particle :

$$m \frac{d^2 s}{dt^2} = -mg \sin \theta \quad \text{--- (I)}$$

$$T - mg \cos \theta = \frac{mv^2}{l} \quad \text{--- (II)}$$

$$\text{Also } s = l\theta$$

$$s = l\theta \Rightarrow \frac{ds}{dt} = l \frac{d\theta}{dt}$$

Put in (1) $\Rightarrow \frac{d^2\theta}{dt^2} = -\frac{g}{l} \sin\theta$

Multiply both sides with $2 \frac{d\theta}{dt}$ and integrate.

$$\int 2 \frac{d\theta}{dt} \frac{d^2\theta}{dt^2} = \int \frac{2g \sin\theta}{l} \frac{d\theta}{dt}$$

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{2g \cos\theta}{l} + C_1 \quad \text{--- (iii)}$$

At $t=0$, $\theta=0$ and $v = l \frac{d\theta}{dt}$

$$\therefore \left.\frac{d\theta}{dt}\right|_{t=0} = 2\sqrt{\frac{g}{l}} = 2\sqrt{gl}$$

\Rightarrow Putting in (iii)

$$\Rightarrow \frac{4g}{l} = \frac{2g}{l} + C_1$$

$$\Rightarrow C_1 = 2g/l$$

$$\therefore \frac{d\theta}{dt} = \sqrt{\frac{2g}{l}(1+\cos\theta)}$$

$$\begin{aligned} \therefore T - mg \cos\theta &= \frac{m}{l} \left(l \frac{d\theta}{dt} \right)^2 \\ &= \frac{m}{l} \left[l^2 \times \frac{2g}{l} (1+\cos\theta) \right] \end{aligned}$$

$$T - mg \cos\theta = 2mg(1+\cos\theta)$$

$$\Rightarrow T = 2mg + 3mg \cos\theta$$

Cord becomes slack when

$$T=0 \Rightarrow 2mg + 3mg \cos\theta = 0$$

$$\Rightarrow \cos\theta = -\frac{2}{3}$$

which means that particle crosses the horizontal point H.

New height of particle

above O when $T=0 = OM$
 $= OF \cos \alpha$

But $\alpha = \pi - \theta$ at $t=0$

$$\therefore OM = OF \cos(\pi - \theta)$$

$$= l \times -\cos\theta$$

$$= -\frac{2l}{3} \quad [\because \cos\theta = -\frac{2}{3}]$$

\therefore Particles is at height

$$\frac{2l}{3} \text{ above fixed point}$$

O when cord slacks

8. (c) Show that $A = (2x^2 + 8xy^2z) \hat{i} + (3x^3y - 3xy) \hat{j} - (4y^2z^2 + 2x^3z) \hat{k}$ is not solenoidal but $B = xyz^2 A$ is solenoidal. [08]

Vector is solenoidal if

$$\nabla \cdot \vec{A} = 0$$

$$A = (2x^2 + 8xy^2z) \hat{i} + (3x^3y - 3xy) \hat{j} - (4y^2z^2 + 2x^3z) \hat{k}$$

$$\therefore \nabla \cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$= 4x + 8y^2z + 3x^3 - 3x - 8y^2z - 2x^3$$

$$= x + x^3 \neq 0 \quad \text{--- (1)}$$

when $x \neq 0$.

$\therefore \vec{A}$ is not solenoidal.

Now $B = xyz^2 A$

$$\nabla \cdot (\phi A) = (\nabla \phi) \cdot A + \phi (\nabla \cdot A)$$

$$\therefore \nabla \cdot (xyz^2 A) = \nabla (xyz^2) \cdot A + xyz^2 (\nabla \cdot A)$$

$$\nabla (xyz^2) = \frac{\partial (xyz^2)}{\partial x} \hat{i} + \frac{\partial (xyz^2)}{\partial y} \hat{j} + \frac{\partial (xyz^2)}{\partial z} \hat{k}$$

$$\Rightarrow yz^2 \hat{i} + xz^2 \hat{j} + 2xy z \hat{k}$$

$$\therefore \nabla (xyz^2) \cdot A = (2x^2 + 8xy^2z) yz^2 + (3x^3y - 3xy) xz^2 - (4y^2z^2 + 2x^3z) (2xyz)$$

$$= 2x^2 y z^2 + 8x^2 y^3 z^2 + 3x^4 y z^2 - 3x^2 y z^2 - 8x y^3 z^2 - 4x^4 y z^2$$

$$= -x^2 y z^2 - 4x^4 y z^2$$

$$= -x^2 y z^2 (1 + x^2)$$

Also $\phi (\nabla \cdot A) = xyz^2 (x + x^3)$ [using (1)]

$$= x^2 y z^2 (1 + x^2)$$

$$\therefore \nabla \cdot B = \nabla (xyz^2 A) \cdot A$$

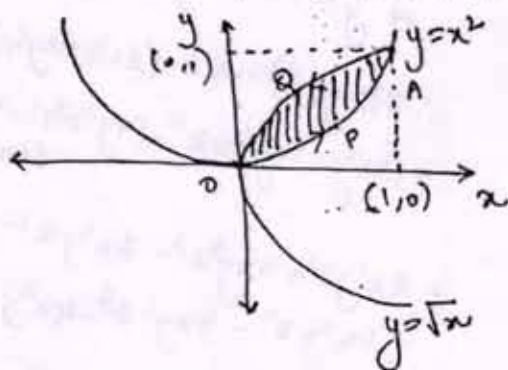
$$= -x^2 y z^2 (1 + x^2)$$

$$+ x^2 y z^2 (1 + x^2)$$

$$= 0$$

$\therefore \vec{B}$ is solenoidal vector

8. (d) Verify Green's theorem in the plane for $\oint_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$, where C is the boundary of the region defined by: $y = \sqrt{x}, y = x^2$ [12]



C is the boundary OPAQO for the shaded region

Green's theorem states:

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

where P < bounds R & travels in positive sense.

LINE INTEGRAL:

Path OPA = $t\hat{i} + t^2\hat{j}$
t varies from 0 to 1

Path AQO = $t^2\hat{i} + t\hat{j}$
t varies from 1 to 0

$$\begin{aligned} \therefore \oint_C (3x^2 - 8y^2)dx + (4y - 6xy)dy \\ = \int_{OPA} (3t^2 - 8t^4)(dt) + (4t^2 - 6t^3)(2t dt) \\ + \int_{AQO} (3t^2 - 8t^2)(2t dt) + (4t - 2t^3)dt \end{aligned}$$

$$\begin{aligned} &= \int_0^1 (3t^2 + 6t^3 - 20t^4) dt \\ &+ \int_1^0 (4t - 22t^3 + 6t^5) dt \\ &= [t^3 + 2t^4 - 4t^5]_0^1 - [2t^2 - \frac{22t^4}{4} + t^6]_0^1 \\ &= (1 + 2 - 4) - (2 - \frac{22}{4} + 1) \\ &= \frac{22}{4} - 4 = \frac{6}{4} = \frac{3}{2} \end{aligned}$$

SURFACE INTEGRAL

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

$$= \iint_R (-6y - 6x) dx dy$$

$$= -6 \int_0^1 \int_{x^2}^{\sqrt{x}} (x+y) dx dy$$

$$= -6 \int_0^1 \left(xy + \frac{y^2}{2} \right) \Big|_{x=x^2}^{\sqrt{x}} dx$$

$$= -6 \int_0^1 \left(x^{\frac{3}{2}} + \frac{x}{2} - x^3 - \frac{x^4}{2} \right) dx$$

$$= -6 \left[\frac{2x^{\frac{5}{2}}}{5} + \frac{x^2}{4} - \frac{x^4}{4} - \frac{x^5}{10} \right]_0^1 = \frac{3}{2} \text{ Hence proved}$$