Numerical Analysis

Lecture-8
Thursday
2/2/17

- **Books:**

① Kreyszig (8th Ed) (Text)

② K. Atkinson (Ref.)

③ Scarborough (Ref.) ④ Gerald and Wheatley

Topics:

A. Solution of a single non-linear equation.

B. Solution of a system of equations (iterative Methods)

C. Interpolation

D. Numerical Integration.

- **Iterative solution of a single non-linear Equation:**

$x = \cosh x$

You find initial guess x_0

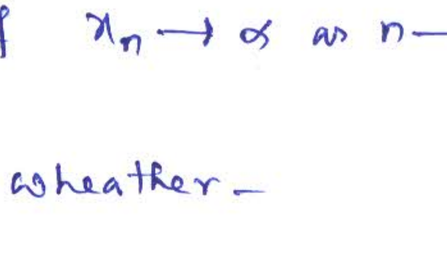
$\alpha = \cosh \alpha$. Considering α as a root of $f(x) = 0$.

Where $f(x) = 0 \Rightarrow x - \cosh x = 0$

that means $f(\alpha) = 0$

Now, initial guess x_0 is given —

How to generate $x_1, x_2, \ldots$



The method is said to converge, if $x_n \to \alpha$ as $n \to \infty$.

$|x_n - \alpha| < \epsilon \quad \forall\, n > N$.

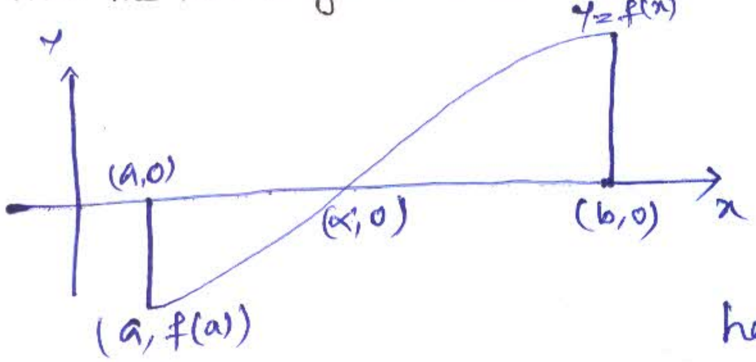
For practical purpose, one checks wheather —

$|x_{n+1} - x_n| < \epsilon \quad \forall\, n > N$.

We say that a method has order of convergence p, if $|\alpha - x_{n+1}| < |\alpha - x_n|^p$

or, $\displaystyle\lim_{n\to\infty} \frac{|\alpha - x_{n+1}|}{|\alpha - x_n|^p} = c \rightarrow$ Asymptotic error constant.

- How the initial guess is determined?



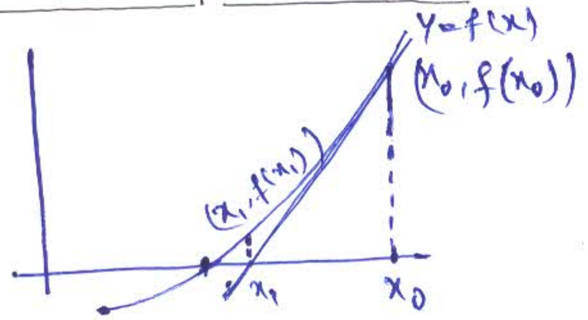
Let $f(x)$ be continuous in $[A,B]$ & $f(a)f(b)<0$ $(A<a<b<B)$ then $f(x)$ has <u>atleast</u> one root in $[a,b]$.

Further if $f(x)$ is strictly monotonic in $[A,B]$ & $f(a)$ & $f(b)$ are of different sign i.e. $f(a)f(b)<0$, then $\exists$ exactly one root of $f(x)$ in $[a,b]$.

For, $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$ (monotonic increasing)

$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ (monotonic decreasing)

- <u>Newton-Raphson method</u>



Equation of tangent at $(x_n, f(x_n))$

$$\frac{f_n - 0}{x_n - x_{n+1}} = f'(x_n)$$

$$\Rightarrow x_n - x_{n+1} = \frac{f(x_n)}{f'(x_n)} \Rightarrow \boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

$n = 0, 1, 2, 3 \ldots$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$ and so on.

Through NR method we actually linearize some non-linear function i.e. given the curve $y = f(x)$. We ~~to~~ are approximating this by a straight line (tangent line) in NR method. This is also done in Secant and Regula-Falsi Method.

Exercise -

Compute the root of $f(x) = 10^x + x - 4$ (by N-R method) correct to 4 decimal places; given $x_0 = 0.5$, ~~unit~~

[Approximation upto 3 decimal places -

$0.345689 \approx 0.346$.
$\quad\quad\downarrow$
$\quad\quad >5$

$10.901256 \approx 10.901$
$\quad\quad\downarrow$
$\quad\quad <5$

0·95[redacted]8562 ≈ 0.958

$0.959562 \approx 0.960$

$0 \cdot d_1 d_2 \cdots d_n d_{n+1} \cdots d_N$

to round of till n decimal place look at d_{n+1}

$d_{n+1} > 5$
approx. no.
$0 \cdot d_1 d_2 \cdots (d_n + 1)$

$d_{n+1} < 5$
approx. no.
$0 . d_1 d_2 \cdots d_n$.

$d_{n+1} = 5$

d_n is even
approx. no. $0 \cdot d_1 d_2 \cdots d_n$

d_n is odd
$0 \cdot d_1 d_2 \cdots (d_n + 1)$]

Ans - $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, Now, $f(x) = 10^x + x - 4$

$\therefore f'(x) = 10^x \log 10 + 1$

$x_0 = 0.5$ and $f'(x_0) = 8.281$

n	x_n	$f(x_n)$
0	·5	-0.3377
1	·54	0.0074
2	·5391	-0.0007
3	·5392	$0.00006 \approx 0.0001$
4	·5392	$-0.0000468 \approx 0$
5	·5392	

The result correct to 4 significant figure is obtained at the 4^{th} step.

Advantage

Order of convergence of NR method is 2.

Disadvantages

1. This method is not guranteed to converge.
2. It may be difficult to compute $f'(x)$.
3. Even if $f'(x)$ exists $f'(x_n)$ may be zero, and there the method will fail.

- When α is a root of multiplicity k -

Case-1 Multiplicity k is known.

N-R formula:

$$x_{n+1} = x_n - k\frac{f(x_n)}{f'(x_n)}.$$

Case-2 Multiplicity k not known.

N-R formula:

$$x_{n+1} = x_n - \left(\frac{ff'}{f'^2 - ff''}\right)_{x=x_n}$$

NR Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

let $x - \dfrac{f(x)}{f'(x)} = g(x)$.

then NR method can be expressed as—

$$x_{n+1} = g(x_n).$$

Fixed-point iteration scheme-

Take $g(x) = x^2$. Find fixed points of $g(x)$.

Def. Fixed points are those for which -

$g(x) = x$.

$x^2 = x$ or $x(x-1) = 0$

$\Rightarrow x = 0, 1$.

$[-2, -1] \rightarrow$ no fixed pt. of $g(x) = x^2$.

$[-1, \frac{1}{2}] \rightarrow$ 1 " " of $g(x) = x^2$.

$[-1, 2] \rightarrow$ 2 " " of $g(x) = x^2$.

let $f(x) = x - g(x)$

α is a root of $f(x) = 0 \Leftrightarrow \alpha$ is a fixed pt. of $g(x)$.

$\therefore$ Root finding problem is equivalent to finding fixed pt. of some function.

To find root of $f(x) = x^2 - 3 = 0$:-

$x^2 = 3$, $x = \frac{3}{x} = g_1(x)$

$x = \frac{1}{2}\left(x + \frac{3}{x}\right) = g_2(x)$

$x = x + c(x^2 - 3)$ let $g_3(x) = x - \frac{1}{6}(x^2 - 3)$

which $g(x)$ will you take?

Answer- That $g(x)$ for which

$|g'(x)| < 1 \ \forall x$ is some prescribed interval

or, $|g'(x_0)| < 1$, for working purpose, where x_0 is the initial guess.

P.T.O

(*) let $x_0 = 1.5$.

Now, In above case -

$$|g_1'(x_0)| = \frac{4}{3} \times, \quad |g_2'(x_0)| = \frac{1}{6} \checkmark$$

$$|g_3'(x_0)| = \frac{1}{2}. \checkmark$$

So, we will take $g_2(x)$, since $\frac{1}{6} < \frac{1}{2}$, so in this case method will converge faster.

<u>Thm</u> If $|g'(x)| < 1$ in some interval $[a, b]$ then the fixed point iteration scheme $x_{n+1} = g(x_n)$ will converge.

<u>Note</u> - If $|g_1'(x)| < A$ & $|g_2'(x)| < B$.

& if $A < 1$, $B < 1$. we choose B if $B < A$.

Q. Find a numerical approximation to the intersection between the line $y = x$ & the curve $y = \frac{1.984}{\ln(x)}$. starting from $x_0 = 2.4$ at $x = 0$, by performing a converget fixed point iteration scheme. Show x_n for $n = 1, 2, 3 \ldots$

Sol- To solve $x = \frac{1.984}{\ln(x)}$ by fixed point iteration.

If $g(x) = \frac{1.984}{\ln(x)}$ find $|g'(x)|_{x=2.4} = 1.078 > 1$

So we can't take $g(x)$ as $\frac{1.984}{\ln(x)}$.

$\ln(x) = \frac{1.984}{x} \Rightarrow x = e^{\frac{1.984}{x}} = g_1(x)$

Now,

$$|g_1'(2.4)| = 0.787 < 1.$$

So,
$$x_{n+1} = g_1(x_n)$$
$$\Rightarrow x_{n+1} = e^{\frac{1.984}{x_n}} \quad ; \; n = 0, 1, 2,$$

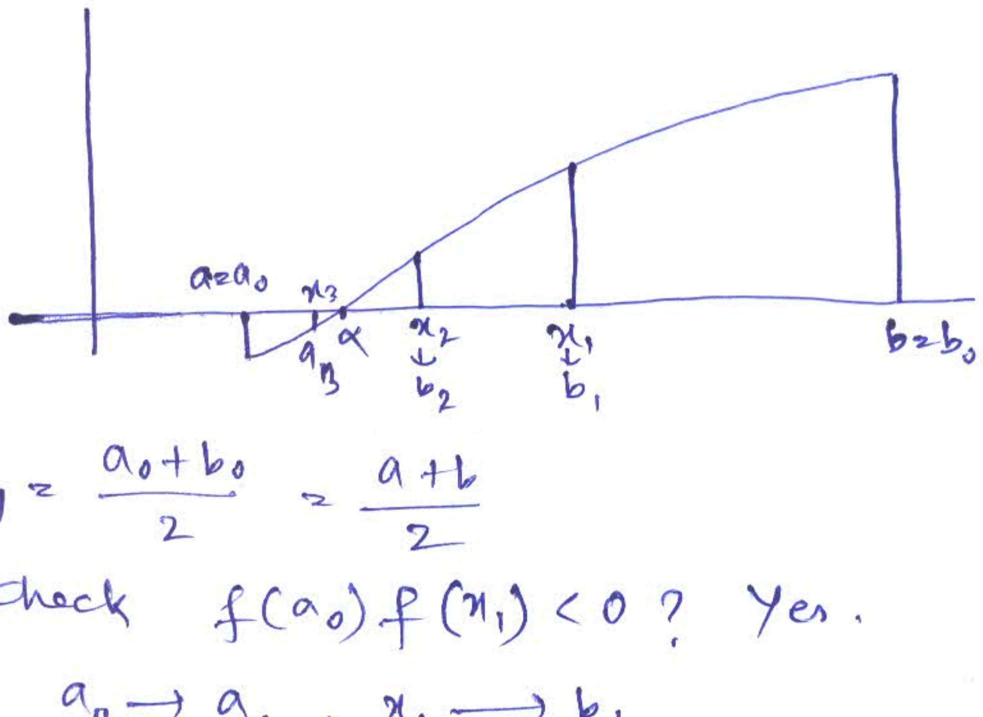
$x_1 = 2.286, \quad x_2 = 2.382, \quad x_3 = 2.3 \ldots$

Q. $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ (N-R method)

What is the sufficient condition that N-R method converges?

- Bisection Method

To find α : $f(\alpha) = 0$



$x_1 = \dfrac{a_0 + b_0}{2} = \dfrac{a + b}{2}$

check $f(a_0) f(x_1) < 0$? Yes.

$a_0 \to a_1, \quad x_1 \longrightarrow b_1$

take $x_2 = \dfrac{a_1 + b_1}{2}$

Now, check $f(a_1) f(x_2) < 0$? If Yes

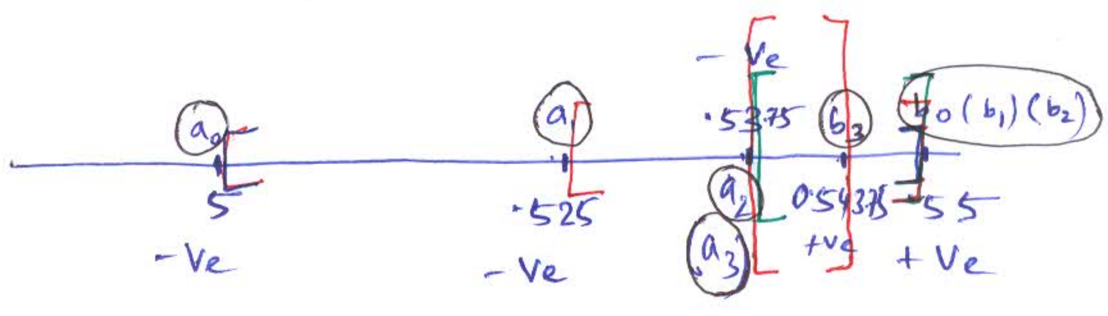
$a_1 \to a_2, \quad x_2 \longrightarrow b_2$ and $x_3 = \dfrac{a_2 + b_2}{2}$

$f(a_2) f(x_3) < 0$? and so on...

Q: Find the root of $f(x) = 10^x + x - 4 = 0$ correct to three decimal places ~~b~~ lying in the interval $[\cdot 5, \cdot 55]$ $\quad [\, f(\cdot 5)\, f(\cdot 55) < 0 \,]$

n	a_n	b_n	$x_{n+1} = \frac{a_n + b_n}{2}$	$f(x_n)$ (sign)
0	·5	·55	0·525	− Ve
1	0·525	·55	0·5375	− Ve
2	0·5375	·55	0·5438	+ Ve
3	0·5375	·5438	0·54065	+ Ve
4	0·5375	·54065	0·53906	− Ve
5	0·53904	·54065	0·53984	+ Ve
6	0·53904	·53984	0·5394	+ Ve
7	0·53904	·5394	0·539	·

Since x_7 & x_8 are matching till 3rd decimal place, therefore the root is 0·539, (correct to 3 decimal places)



- Bisection method is linearly convergent.

i.e. its order of convergence is 1

$$|\alpha - x_{n+1}| \le c|\alpha - x_n|^p$$

$$\Rightarrow |\alpha - x_{n+1}| \le C|\alpha - x_n|$$

$$|\alpha - x_1| \le c|\alpha - x_0|$$

$$|\alpha - x_2| \le c|\alpha - x_1| \le C^2|\alpha - x_0|$$

$$|\alpha - x_n| \le c^n|\alpha - x_0|$$

$\therefore$ As $n \to \infty$, $x_n \to \alpha$ if $0 < c < 1$.

Order of convergence of N-R method = 2.

Order of convergence of F.P method = 1.

" " " " bisection " = 1.

Note: In case of bisection method

$$\frac{|b-a|}{2^n} \le \epsilon$$

$$\therefore n \ge \frac{\log_e\left(\frac{|b-a|}{\epsilon}\right)}{\log_e 2}$$

If $|b-a| = 1$, $\epsilon = 0.001$, $n \ge 10$.