MATHEMATICS
for
UPSC CSE MAINS

TARUN KAUSHIK

TOPIC: *2017 Solution (Analytical Geometry)*

Analytical Geometry
UPSC - 2017.

Que :- 1(d) [10 Marks] Find the Equation of the Tangent plane at the point $(1,1,1)$ to the conicoid $3x^2 - y^2 = 2z$.

Sol.ⁿ Point passing from $(1,1,1)$ is being given by

$$\frac{x-1}{l} = \frac{y-1}{m} = \frac{z-1}{n} = r$$

Gen. Point $(lr+1, mr+1, nr+1)$

Since this point lies on this conicoid :-

$$\Rightarrow 3(rl+1)^2 - (mr+1)^2 = 2(nr+1)$$

$$\Rightarrow 3(r^2l^2+1+2rl) - (m^2r^2+1+2mr) = 2nr+2$$

$$\Rightarrow r^2(3l^2-m^2) + r(6l-2m-2n) + 3-1 = 2$$

$$\Rightarrow r^2(3l^2-m^2) + r(6l-2m-2n) = 0$$

∴ at Tangent we will have one value of r & it is obtained by putting the coeff. of r is equal to zero. i.e. sum of two roots is zero.

$$6l - 2m - 2n = 0 \Rightarrow 3l - m - n = 0$$

Now put l, m, n as $(x-1, y-1, z-1)$

$$\Rightarrow 3(x-1) - (y-1) - (z-1) = 0$$

$$\Rightarrow 3x - 3 - y + 1 - z + 1 = 0$$

$$\Rightarrow \boxed{3x - y = z + 1} \quad \text{Ans}$$

Que. 1 (e) [10 Marks] Find the shortest distance b/w the skew line :

$$\frac{x-3}{3} = \frac{8-y}{1} = \frac{z-3}{1} \quad \& \quad \frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$$

Sol.ⁿ = $\vec{a_1} = 3\hat{i} + 8\hat{j} + 3\hat{k}$; $\vec{b_1} = 3\hat{i} + (\hat{-j}) + \hat{k}$

$\vec{a_2} = -3\hat{i} - 7\hat{j} + 6\hat{k}$; $\vec{b_2} = -3\hat{i} + 2\hat{j} + 4\hat{k}$

$\vec{a_2} - \vec{a_1} = (-6\hat{i} - 15\hat{j} + 3\hat{k})$; $\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 1 \\ -3 & 2 & 4 \end{vmatrix}$

Now

$S.D = \dfrac{(\vec{a_2} - \vec{a_1}) \cdot \vec{b_1} \times \vec{b_2}}{|\vec{b_1} \times \vec{b_2}|}$

$= \hat{i}(-6) - \hat{j}(15) + \hat{k}(3)$

$\vec{b_1} \times \vec{b_2} = (-6\hat{i} - 15\hat{j} + 3\hat{k})$
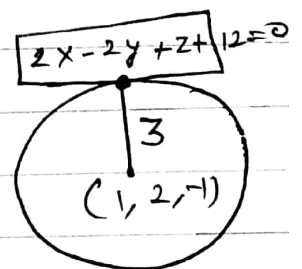
$|\vec{b_1} \times \vec{b_2}| = \sqrt{36 + 225 + 9}$

$= \dfrac{36 + 225 + 9}{\sqrt{36 + 225 + 9}}$ $\sqrt{270} = \sqrt{9 \times 30}$

$= \boxed{3\sqrt{30}}$ units.

2(c)
Que Show that the plane $2x - 2y + z + 12 = 0$ touches
[10 marks] the sphere $x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$ and
find the point of contact.

Sol.ⁿ $x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$

Centre = $(1, 2, -1)$

radius = $\sqrt{1 + 4 + 1 + 3} = \sqrt{9} = \boxed{3}$

Now, If it (plane) touches the sphere then
the distance of plane from the centre should be
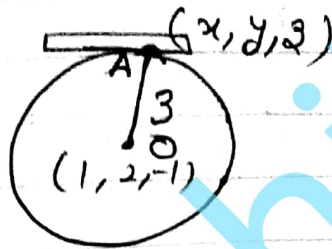equal to radius of the sphere.

So $\perp$ dist. from Centre to Plane is:

$P = \left| \dfrac{2(1) - 2(2) + (-1) + 12}{\sqrt{4 + 4 + 1}} \right| = \left| \dfrac{2 - 4 - 1 + 12}{3} \right|$

$= \dfrac{9}{3} = \boxed{3}$

Hence it touches the sphere.

Now the point of contact; let it be (x, y_3)

So dir. ratio of this line is (radius) $\overset{\circ}{OA}$

$(2, -2, 1)$.

And the eq^n of line is given

$$\frac{x-1}{2} = \frac{y-2}{-2} = \frac{z+1}{1} = \gamma$$

Gen. Point $\left(x = 2\gamma+1, \quad y = -2\gamma+2, \quad z = \gamma-1) \right)$

Since this pt. is on the plane, hence it will satisfied the equation.

$$2(2\gamma+1) - 2(-2\gamma+2) + (\gamma-1) + 12 = 0$$

$$\Rightarrow \quad 4\gamma+2 + 4\gamma-4 + \gamma-1 + 12 = 0$$

$$\Rightarrow \quad \quad 9\gamma + 9 = 0 \quad \Rightarrow \boxed{\gamma = -1}$$

$$\Rightarrow \quad (-1, 4, -2) \quad \text{Point of Contact. Ans}$$

Que: 3(d)
10 Marks.

Find the Locus of the point of intersection of three mutually perpendicular tangent planes to $ax^2 + by^2 + cz^2 = 1$.

Sol^n Given conicoid $ax^2 + by^2 + cz^2 = 1$ - - - - ①

Let one of the 3 mutually $\perp$ Tangent plane is

$$l_1 x + m_1 y + n_1 z = P \quad - - - - - - ②$$

By the condition of tangency $\Rightarrow$

$$\frac{l_1^2}{a} + \frac{m_1^2}{b} + \frac{n_1^2}{c} = P_1^2$$

$$\Rightarrow \quad P_1 = \pm \sqrt{\frac{l_1^2}{a} + \frac{m_1^2}{b} + \frac{n_1^2}{c}}$$

Now eqⁿ (2) becomes :-

$$l_1 x_0 + m_1 y + n_1 z = \sqrt{\frac{l_1^2}{a} + \frac{m_1^2}{b} + \frac{n_1^2}{c}} \quad \boxed{3}$$

similarly

$$l_2 x + m_2 y + n_2 z = \sqrt{\frac{l_2^2}{a} + \frac{m_2^2}{b} + \frac{n_2^2}{c}} \quad \boxed{4}$$

$$l_3 x + m_3 y + n_3 z = \sqrt{\frac{l_3^2}{a} + \frac{m_3^2}{b} + \frac{n_3^2}{c}} \quad \boxed{5}$$

Now to find the locus we need to eliminate the l, m, n ; sqⁿ both side :-

$$\Rightarrow x^2 \underset{1}{\underbrace{\left(l_1^2 + l_2^2 + l_3^2 \right)}} + y^2 \underset{1}{\underbrace{\left(m_1^2 + m_2^2 + m_3^2 \right)}} + z^2 \underset{1}{\underbrace{\left(n_1^2 + n_2^2 + n_3^2 \right)}}$$

$$+ 2xy \underset{0}{\underbrace{\left(l_1 m_1 + l_2 m_2 + l_3 m_3 \right)}} + 2yz \underset{0}{\underbrace{\left(m_1 n_1 + m_2 n_2 + m_3 n_3 \right)}} +$$

$$2zx \underset{0}{\underbrace{\left(n_1 l_1 + n_2 l_2 + n_3 l_3 \right)}} = \frac{1}{a} \underset{1}{\underbrace{\left(l_1^2 + l_2^2 + l_3^2 \right)}} + \frac{1}{b} \underset{}{\underbrace{\left(m_1^2 + m_2^2 + m_3^2 \right)}}$$

$$+ \frac{1}{c} \underset{1}{\underbrace{\left(n_1^2 + m_2^2 + n_3^2 \right)}}$$

$$\Rightarrow \boxed{x^2 + y^2 + z^2 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}} \quad \underline{\underline{Ans}}$$
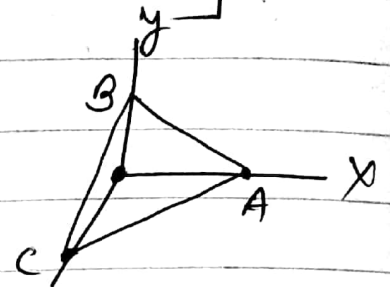
Que2(b) A plane passes through a fixed point

(a, b, c) and cuts the axes at the points A, B, C respectively. Find the locus of the centre of the sphere which passes through the origin O & A, B, C.

Sol.ⁿ Let the eqⁿ of the sphere $OABC$, through $O(0,0,0)$ be

$$\left[x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = 0 \right]$$

Now at pt. $A \Rightarrow y = z = 0$

$$\Rightarrow x^2 + 2ux = 0 \Rightarrow \boxed{x = -2u}$$

Simillarly at pt. B & pt. C we get $\boxed{y = -2v}$ & $\boxed{z = -2w}$

Now the eqⁿ of plane ABC is

$$\Rightarrow \frac{x}{-2u} + \frac{y}{-2v} + \frac{z}{-2w} = 1$$

$$\Rightarrow \frac{x}{-u} + \frac{y}{-v} + \frac{z}{-w} = 2$$

Now this plane passes through (a, b, c)

$$\Rightarrow \frac{a}{-u} + \frac{b}{-v} + \frac{c}{-w} = 2$$

Now Here $(-u, -v, -w)$ are the centre of the sphere, so the locus is given by

$$\Rightarrow \boxed{\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2} \quad \text{Ans}$$

Example 1. *Show that the plane* $2x - 2y + z + 12 = 0$ *touches the sphere* $x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$ *and find the point of contact.*

[K.U.

Solution. The centre of the sphere is $(1, 2, -1)$ and its radius $= \sqrt{1 + 4 + 1 + 3} = 3$

$\therefore$ Perpendicular distance of the centre $(1, 2, -1)$ from the plane

$$= \frac{2(1) - 2(2) + (-1) + 12}{\sqrt{4 + 4 + 1}}$$

$$= \frac{2 - 4 - 1 + 12}{3} = 3, \text{ which is the radius}$$

Since the distance of the centre from the plane = radius of the sphere, hence the pl touches the sphere.

To find the point of contact :

The point of contact is the foot of perpendic

Analytical Geometry Part 5 (Sphere -2)

TARUN KAUSHIK

Q 4(d) 2016
15 Marks
Sol:

Find the Locus of the point of intersection of three mutually perpendicular tangent planes to the conicoid $ax^2 + by^2 + cz^2 = 1$.

Conicoid $\Rightarrow$ $ax^2 + by^2 + cz^2 = 1$ ————(i)

Let one of the mutually $\perp$ tangent plane to eq.(1) is

$$l_1 x + m_1 y + n_1 z = P$$ ————(2)

ACC. to the condition of Tangency:

$$\frac{l_1^2}{a} + \frac{m_1^2}{b} + \frac{n_1^2}{c} = P^2$$

$$l_1^2 + m_1^2 + n_1^2$$

19%

Analytical Geometry Part 7 (UPSC Questions Solutions)