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MATHEMATICS FOR UPSC CSE MAINS

(CALCULUS PART 2)

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Q(2013): 10 Marks Evaluate Dated: 30/08/2016

$$\int_0^1 2x \sin\frac{1}{x} - \cos\frac{1}{x} \, dx$$

Sol.^{n} Put $x = \frac{1}{y} \Rightarrow dx = -\frac{1}{y^2} dy$

and limits $0 \le x \le 1 \Rightarrow \infty \le y \le 1$

$$= -\int_\infty^1 \frac{2}{y} \sin y - \cos y \frac{dy}{y^2} = \int_1^\infty \frac{2}{y^3} \sin y dy - \int_1^\infty \frac{\cos y}{y^2} dy$$

$$= \int_1^\infty \frac{2}{y^3} \sin y \, dy - \left[\frac{1}{y^2} (\sin y) - \int \frac{-2}{y^3} (\sin y) dy \right]$$

$$= \underbrace{\int_1^\infty \frac{2}{y^3} \sin y \, dy} - \frac{\sin y}{y^2}\Big|_1^\infty - \underbrace{\int_1^\infty \frac{2}{y^3} \sin y \, dy}$$

$$= -\left[\frac{\sin\infty}{\infty} - \frac{\sin 1}{1}\right] = -\left[0 - \sin 1\right]$$

$$= \sin 1 = \boxed{0.8414} \text{ Ans}$$

Standard Result :- [n^{th} derivative]

1. $y = x^m$ then $y_n = \frac{m!}{m-n!} x^{m-n}$

2. $y = (ax+b)^m$ then $y_n = \frac{m!}{m-n!} (ax+b)^{m-n} \cdot a^n$

3. $y = \frac{1}{ax+b}$ then $y_n = \frac{(-1)^n n! \, a^n}{(ax+b)^{n+1}}$

4. $y = \log(ax+b)$ then $y_n = \frac{(-1)^{n-1} (n-1)! \, a^n}{(ax+b)^n}$

5. $y = a^{mx}$ then $y_n = a^{mx} \cdot (\log a)^n \cdot m^n$

6. $y = e^{mx}$ then $y_n = e^{mx} \cdot m^n$

7. $y = \sin(ax+b)$ then $y_n = a^n \sin\left(ax+b+\frac{n\pi}{2}\right)$

8. $y = \cos(ax+b)$ then $y_n = a^n \cos\left(ax+b+\frac{n\pi}{2}\right)$

9. $y = e^{ax} \cdot \sin(bx+c)$ then $y_n = (a^2+b^2)^{n/2} \times e^{ax} \times \sin\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$

10. $y = e^{ax}\cos(bx+c)$ then $y_n = (a^2+b^2)^{n/2} \times e^{ax} \times \cos\left(bx+c+n\tan^{-1} b/a\right)$

☆ [What If Product of two different function is given? Solution :- Leibnitz's Theorem]

$y = u \cdot v$ then

$$\boxed{y_n = {}^nc_0 u_n v + {}^nc_1 u_{n-1} v_1 + {}^nc_2 u_{n-2} v_2 + \cdots + {}^nc_n u v_n}$$

eg. $y = x^2 \sin x$; find n^{th} derivative

$$y_n = {}^nc_0 u_n v + {}^nc_1 u_{n-1} v_1 + {}^nc_2 u_{n-2} v_2 + \cdots\cdots$$

Here $y = x^2 \sin x = u \cdot v$ let $u = \sin x$, $v = x^2$

$u_n = \sin\left(x+\frac{n\pi}{2}\right)$ $\quad v_1 = 2x$

$u_{n-1} = \sin\left(x+\frac{(n-1)\pi}{2}\right)$ $\quad v_2 = 2$

$u_{n-2} = \sin\left(x+\frac{n-2}{2}\pi\right)$ $\quad v_3 = 0$

Taylor Series :-

Que:- Express this polynomial $2x^3+7x^2+x-6$ in power of $(x-2)$.

Sol.ⁿ

$$f(x) = 2x^3+7x^2+x-6 \Rightarrow f(2) = 40$$

$$f(2+\underline{x-2}) = 2x^3+7x^2+x-6$$

$$f(2+h) = f(2) + h f'(2) + \frac{h^2}{2!} f''(2) + \dots$$

$$f'(x) = 6x^2+14x+1 \Rightarrow f'(2) = 53$$

$$f''(x) = 12x+14 \Rightarrow f''(2) = 38$$

$$f'''(x) = 12 \Rightarrow f'''(2) = 12$$

$$f(2+h) = f(x) = 40 + (x-2)\,53 + \frac{(x-2)^2}{2!}\times 38 + \frac{(x-2)^3}{3!}\times 12$$

$$= \boxed{40 + 53(x-2) + \frac{38}{2!}(x-2)^2 + \frac{12}{3!}(x-2)^3} \text{ Ans.}$$

Partial Differentiation :-

$$Z = f(x,y)$$

$\frac{\partial z}{\partial x}$ = Considering y as constant

$\frac{\partial z}{\partial y} \rightarrow$ " x " "

* Euler's theorem on homogenous function

(1.) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$ | degree : 1

(2) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$ | for degree 2

Composit function :-

if $u = f(x,y)$ where $x = \phi(t)$ & $y = \psi(t)$

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}}$$

Que:- $u = \sin^{-1}(x-y)$, $x = 3t$, $y = 4t^3$; find $\frac{du}{dt}$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-(x-y)^2}} \;;\; \frac{\partial u}{\partial y} = \frac{-1}{\sqrt{1-(x-y)^2}} \;;\; \frac{dx}{dt} = 3,\; \frac{dy}{dt} = 12t^2$$

$$\frac{du}{dt} = \frac{3}{\sqrt{1-(x-y)^2}} - \frac{12t^2}{\sqrt{1-(x-y)^2}} = \frac{3(1-4t^2)}{\sqrt{1-(x-y)^2}}$$

$$= \frac{3(1-4t^2)}{\sqrt{1-(3t-4t^3)^2}} = \frac{3}{\sqrt{1-t^2}} \text{ Ans.}$$

Jacobians :-

Two functions with two / three independent variable.

i.e. u, v both are function (x,y) variable.

(1) $$\boxed{J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}}$$

(2.) $$J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Que:- $r = \sqrt{x^2+y^2}$; $\theta = \tan^{-1}\frac{y}{x}$; evaluate $\frac{\partial(r,\theta)}{\partial(x,y)}$

Sol.ⁿ

$$J = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix}$$

Taylor Series (Two variable) :-

$$f(x+h, y+k) = f(x,y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f + \frac{\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f}{2!}$$

Que:- Expand $x^2y + 3y - 2$ in power of $(x-1)$ & $(y+2)$ using Taylor Series.

Sol.^{n} $f(x,y) = f(1+\underline{(x-1)}, -2+\underline{(y+2)}) = x^2y+3y-2$

$f(1,-2) = -10$

$f_x = 2xy \Rightarrow f_x(1,-2) = -4$; $f_{xx} = 2y \Rightarrow f_{xx}(1,-2) = -4$

$f_y = x^2+3 \Rightarrow f_y(1,-2) = 4$; $f_{yy} = 0$; $f_{xxx} = 0$

$f_{xy} = 2x \Rightarrow f_{xy}(1,-2) = 2$; $f_{yyy} = 0$; $f_{xxy} = 2$

All higher Order will Vanish.

$$f(x,y) = -10 + [(x-1)(-4) + (y+2)4] + \frac{1}{2}[(x-1)^2(-4) + 2(x-1)(y+2)(2) + (y+2)^2(0)] + \frac{1}{6}[(x-1)^3(0) + 3(x-1)^2(y+2)(2) + 3(x-1)(y+2)^2(0) + (y+2)^3 0]$$

$$= \left[-10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2)\right]$$

Ans

Maxima - Minima [2- variable]

$\boxed{f(x,y)}$: Given

(1) $[f_x , f_y]$ ← Calculate

(2) $[f_{xx} , f_{yy} , f_{xy}$ or $f_{yx}]$ ← Calculate

Here $f_{xx} = r$; $f_{yy} = t$ & $f_{xy} = f_{yx} = s$

Conditions :-

I) $rt - s^2 > 0$ & $r > 0 \Rightarrow$ MAXIMA

II) $rt - s^2 > 0$ & $r < 0 \Rightarrow$ MINIMA

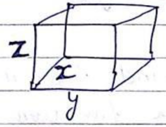
III) $rt - s^2 < 0 \Rightarrow$ No Extreme Value.

(IV) $rt - s^2 = 0 \Rightarrow$ The Case is doubtful & require further investigation.

Que:- A rectangular box, open at the top, is to have a given capacity. Find the dimensions of the box required least material for its construction.

Sol.^{n}

V (Given) $= xyz$



$z = \frac{V}{xy}$

$S = xy + 2yz + 2xz$

$S = xy + 2y\frac{V}{xy} + 2x\frac{V}{xy} = xy + \frac{2V}{x} + \frac{2V}{y}$

$S_x = y - \frac{2V}{x^2} = 0 \Rightarrow y = \frac{2V}{x^2}$ —— (1)

$S_y = x - \frac{2V}{y^2} = 0 \Rightarrow x = \frac{2V}{y^2}$ —— (2)

$S_{xx} = \frac{4V}{x^3}$; $S_{yy} = \frac{4V}{y^3}$; $S_{xy} = S_{yx} = 1$

Put eq.^{n} (2) in eq.^{n} (1) :- we get

$$x = \frac{2V}{y^2} = \frac{2V}{\left(\frac{2V}{x^2}\right)^2} = \frac{2V}{(2V)^2}x^4 = \frac{x^4}{2V}$$

$$\Rightarrow \frac{x^4}{2V} - x = 0 \Rightarrow x\left(\frac{x^3}{2V} - 1\right) = 0 \Rightarrow x \neq 0$$

$$\frac{x^3}{2V} - 1 = 0 \Rightarrow \boxed{x = (2V)^{1/3}}$$

$$y = \frac{2V}{x^2} = \frac{2V}{(2V)^{2/3}} = (2V)^{1/3}$$

$$z = \frac{V}{xy} = \frac{V}{(2V)^{2/3}} = \frac{1}{2}\frac{2V}{(2V)^{2/3}} = \frac{(2V)^{1/3}}{2}$$

$$\Rightarrow \boxed{x = y = 2z = (2V)^{1/3}} \quad \text{Ans.}$$

Lagrange's Method of Undetermined Multiplier :-

→ To find the maximum or minimum value of a function of three or (more) variables, when the variables are not independent but are connected by some given relation.

Q (2014) : 20 Marks:

Find the stationary values of x^2+y^2+z subject to $ax^2+by^2+cz^2=1$ and $lx+my+nz=0$.

Sol.^{n}

$$F(x,y,z) = x^2+y^2+z^2 + \lambda(ax^2+by^2+cz^2-1) + \mu(lx+my+nz)$$

$$dF = 0 \Rightarrow dF \rightarrow (dx + dy + dz)$$

$$dF = (2x + 2a\lambda x + l\mu)\,dx + (2y + 2b\lambda y + m\mu)\,dy + (2z + 2c\lambda z + n\mu)\,dz = 0$$

$$2x + 2a\lambda x + l\mu = 0 \quad \text{----} (1)$$
$$2y + 2b\lambda y + m\mu = 0 \quad \text{----} (2)$$
$$2z + 2c\lambda z + n\mu = 0 \quad \text{----} (3)$$

Multiply Eq.^{n} (1), (2), (3) by x, y, z respectively & add these three equations :-

$$\Rightarrow (2x^2+2y^2+2z^2) + 2\lambda\underbrace{(ax^2+by^2+cz^2)}_{=1} + \mu\underbrace{(lx+my+nz)}_{=0} = 0$$

$$\Rightarrow 2(x^2+y^2+z^2) + 2\lambda = 0$$

Let $(x^2+y^2+z^2) = V$ ↑ given function.

$$\Rightarrow 2V + 2\lambda = 0 \Rightarrow \boxed{V = -\lambda}$$

from eq. (1) :- $2x(1+a\lambda) = -l\mu$

$$x = \frac{-l\mu}{2(1+a\lambda)} = \frac{-l\mu}{2(1-aV)}$$

$$\Rightarrow \boxed{x = \frac{l\mu}{2(aV-1)}}$$

similarly: $\left(y = \frac{m\mu}{2(bV-1)}\right)$, $\left(z = \frac{n\mu}{2(cV-1)}\right)$

$\because\ lx + my + nz = 0$

$$\text{so} \quad \frac{l^2\mu}{2(aV-1)} + \frac{m^2\mu}{2(bV-1)} + \frac{n^2\mu}{2(cV-1)} = 0$$

$$\Rightarrow \frac{\mu}{2}\left[\frac{l^2}{aV-1} + \frac{m^2}{bV-1} + \frac{n^2}{cV-1}\right] = 0$$

$$\Rightarrow \boxed{\frac{l^2}{(aV-1)} + \frac{m^2}{bV-1} + \frac{n^2}{cV-1}} \quad \text{Ans.}$$

(9.)

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Differ^n under Integral Sign :-

Que: (2014) 10 marks $\int_0^{1} \frac{\log(1+x)}{1+x^2}\,dx$

Sol.^{n}

$$F(a) = \int_0^{a} \frac{\log(1+ax)}{1+x^2}\,dx \quad \text{— (1) where } \boxed{a=1}$$

$$F'(a) = \int_0^{a} \frac{d}{da}\left[\frac{\log(1+ax)}{1+x^2}\right]dx \qquad \left(\because \text{ upper limit is a parameter}\right)$$

$$= \int_0^{a} \frac{1}{1+x^2} \times \frac{x\,dx}{1+ax} + \frac{\log(1+a^2)}{1+a^2} \times \frac{d}{da}(a) - \log 1$$

$$= \int_0^{a} \frac{x}{(1+x^2)(1+ax)}\,dx + \frac{\log(1+a^2)}{(1+a^2)}$$

Using Partial fraction

$$= \frac{1}{1+a^2}\int_0^{a}\left(\frac{-a}{1+ax} + \frac{x+a}{1+x^2}\right)dx + \frac{\log(1+a^2)}{1+a^2}$$

$$= \frac{1}{1+a^2}\left(\frac{-1}{2}\log(1+a^2) + a\tan^{-1}a\right) + \frac{\log(1+a^2)}{(1+a^2)}$$

$$F'(a) = \frac{1}{1+a^2}\left[\frac{1}{2}\log(1+a^2) + a\tan^{-1}a\right]$$

Integrating w.r.t. "$\underline{a}$" : USING By Parts Method

$$F(a) = \frac{1}{2}\log(1+a^2)\tan^{-1}a + c$$

Put. $a=0$ in eq.^{n} (1) $F(0)=0 \Rightarrow \boxed{c=0}$

$$F(a) = \frac{1}{2}\log(1+a^2)\tan^{-1}a$$

$$F(1) = \frac{1}{2}(\log 2)\left(\frac{\pi}{4}\right) = \boxed{\frac{\pi}{8}\log 2} \text{ Ans.}$$

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THANKS

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