TARUN KAUSHIK

MATHEMATICS FOR UPSC CSE MAINS

Topic: CALCULAS PART 1

RECAP

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PAPER 1

1. LINEAR ALGEBRA*
2. CALCULAS*

* Some part is still remaining

PAPER 2

1. COMPLEX ANALYSIS
2. LPP
3. NM

VECTOR / ODE / PDE (Upcoming)

$\sin 2x = 2\sin x\cos x = \frac{2\tan x}{\ldots}$

$\cos 2x = \cos^2 x - \sin^2 x = \ldots - 2\sin^2 x = \frac{1-\tan^2 x}{1+\tan^2 x}$

$\sin 3x = 3\sin x - 4\sin^3 x$, $\cos 3x = \ldots\cos^3 x - 3\cos x$

$\tan 3x = \frac{3\tan x - \tan^3 x}{\ldots\tan^2 x}$

$\sin x\cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$

$\cos x\sin y = \frac{1}{2}[\sin(x+y) - \sin(x-y)]$

$\cos x\cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$

$\sin x\sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$

$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$

$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$

$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$

$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2} = 2\sin\frac{x+y}{2}\sin\frac{y-x}{2}$

$a\sin x + b\cos x = r\sin(x+\theta)$, $a\cos x + b\sin x = r\cos(x-\theta)$, where $a = r\cos$

so that $r = \sqrt{a^2+b^2}$, $\theta = \tan^{-1}\left(\frac{b}{a}\right)$.

In any ΔABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$ (sine formula)

Imp. Basic Formula To used in Diff. & Int.

$\cos A = \frac{b^2+c^2-a^2}{2bc}$ (cosine formula)

$a = b\cos C + c\cos B$ (projection formula)

Area of $\Delta ABC = \frac{1}{2}bc\sin A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$, where R is the circum-radius of Δ ABC.

$R = \frac{abc}{4\Delta}$

$r = \frac{\Delta}{s}$, where r is the radius of inscribed circle of Δ ABC.

Demoivre's Theorem: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Euler's Theorem: $\cos\theta + i\sin\theta = e^{i\theta}$.

Hyperbolic Functions

$\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}$, $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$\text{cosech}\, x = \frac{1}{\sinh x}$, $\text{sech}\, x = \frac{1}{\cosh x}$, $\coth x = \frac{1}{\tanh x}$

$\tanh x = \frac{\sinh x}{\cosh x}$, $\coth x = \frac{\cosh x}{\sinh x}$, $\cosh^2 x - \sinh^2 x = 1$

$\sin ix = i\sinh x$, $\cos ix = \cosh x$, $\tan ix = i\tanh x$

$\sinh^{-1} x = \log\left(x + \sqrt{x^2+1}\right)$, $\cosh^{-1} x = \log\left(x + \sqrt{x^2-1}\right)$,

$\tanh^{-1} x = \frac{1}{2}\log\frac{1+x}{1-x}$.

Series

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \infty$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \infty$, $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \infty$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \infty$, $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \infty$

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \infty$, $\log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \infty\right)$

$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \infty$ (Gregory series)

$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots \infty$

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FUNCTIONS OF SINGLE VARIABLE

LIMITS.

By limit we are able to find out behaviour of a function $f(x)$ near a point c, when distance between x and c is very small i.e. $|x-c|$ is small.

Specifically, we would like to know whether there is some real number k such that $f(x)$ approaches k as x approaches c.

We write as : $\lim_{x\to c} f(x) = k$

Right hand and Left hand limits

Left-hand limit: $\lim_{x\to c^-} f(x) = k$

Right-hand limit: $\lim_{x\to c^+} f(x) = k$

If $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)$, then limit of $f(x)$ as $x \to c$ exists and is written as $\lim_{x\to c} f(x)$.

The limit of a function f(x) may not exist in any of the following situations :

(i) $\lim_{x\to c^-} f(x)$ does not exist

(ii) $\lim_{x\to c^+} f(x)$ does not exist

(iii) Both $\lim_{x\to c^-} f(x)$, and $\lim_{x\to c^+} f(x)$ exist but are unequal

Theorems on Limits

If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist, then

(1) $\lim_{x\to c} (f(x) \pm g(x)) = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$

(2) $\lim_{x\to c} (f(x).g(x)) = \lim_{x\to c} f(x).\lim_{x\to c} g(x)$

(3) $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ where $\left(\lim_{x\to c} g(x) \neq 0\right)$

Some useful Limits

(1) $\lim_{x\to 0} \dfrac{\sin x}{x} = 1 = \lim_{x\to 0} \cos x = \lim_{x\to 0} \dfrac{\tan x}{x}$

(2) $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e = \lim_{x\to\infty} \left(1+\dfrac{1}{x}\right)^x$

(3) $\lim_{x\to 0} \dfrac{\log_a(1+x)}{x} = \log_a e \quad (a>0, a\neq 0)$

(4) $\lim_{x\to 0} \dfrac{\log(1+x)}{x} = 1$

(5) $\lim_{x\to 0} \dfrac{a^x - 1}{x} = \log a \quad (a>0)$

(6) $\lim_{x\to 0} \dfrac{(1+x)^m - 1}{x} = m$

(7) $\lim_{x\to\infty} \dfrac{\log x}{x^m} = 0 \quad (m>0)$

(8) $\lim_{x\to a} \dfrac{x^n - a^n}{x-a} = na^{n-1}$

L'HOSPITAL'S RULE.

It states that for functions f and g if

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$$

or $\lim_{x\to c} g(x) = \pm\infty$

(condition that $\lim_{x\to c} f(x) = \pm\infty$ is not necessary)

and if $\lim_{x\to c} \dfrac{f'(x)}{g'(x)}$ exists then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

Note : If f' and g' again satisfy above conditions of f and g, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f''(x)}{g''(x)} \text{ and so on}$$

e.g. $\lim_{x\to 0} \dfrac{\sin x}{x} = \dfrac{0}{0}$

By L' Hospital Rule

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{x} = \frac{1}{1} = 1$$

e.g. $\lim_{x\to 0} \dfrac{x^2 + x - \sin x}{x^2} = \dfrac{0}{0}$

By L' Hospital Rule

$$\Rightarrow \lim_{x\to 0} \frac{x^2 + x - \sin x}{x^2} = \lim_{x\to 0} \frac{2x + 1 - \cos x}{2x} = \frac{0}{0}$$

$$\Rightarrow \lim_{x\to 0} \frac{x^2 + x - \sin x}{x^2} = \lim_{x\to 0} \frac{2 + \sin x}{2} = 1$$

14. Calculus

(a) Standard Limits

(i) $\operatorname{Lim}_{x\to a} \dfrac{x^n - a^n}{x-a} = na^{n-1}, n \in Q$

(ii) $\operatorname{Lim}_{x\to 0} \dfrac{\sin x}{x} = 1$

(iii) $\operatorname{Lim}_{x\to 0} \dfrac{\sin^{-1} x}{x} = 1$

(iv) $\operatorname{Lim}_{x\to 0} \dfrac{\tan x}{x} = 1$

(v) $\operatorname{Lim}_{x\to 0} \dfrac{\tan^{-1} x}{x} = 1$

(vi) $\operatorname{Lim}_{x\to 0} (1+x)^{\frac{1}{x}} = e$

(vii) $\operatorname{Lim}_{x\to 0} \dfrac{a^x - 1}{x} = \log_e a$

(viii) $\operatorname{Lim}_{x\to\infty} x^{1/x} = 1$

(ix) If $f(a) = g(a) = 0$, then

$$\operatorname{Lim}_{x\to a} \frac{f(x)}{g(x)} = \operatorname{Lim}_{x\to a} \frac{f'(x)}{g'(x)} \qquad \text{(L' Hospital's Rule)}$$

(Differentiate the numerator and denominator separately)

(b) Differentiation

(i) $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$ (Product Rule)

(ii) $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$ (Quotient Rule)

(iii) $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ (Chain Rule)

(iv) If $x = f(t)$, $y = g(t)$, then $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$

(v) $\dfrac{d}{dx}(c) = 0$

(vi) $\dfrac{d}{dx}(x^n) = nx^{n-1}$

(vii) $\dfrac{d}{dx}(e^x) = e^x$

(viii) $\dfrac{d}{dx}(a^x) = a^x \log_e a$

(ix) $\dfrac{d}{dx}(\log_e x) = \dfrac{1}{x}$

(x) $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x \log a}$

(xi) $\dfrac{d}{dx}(\sin x) = \cos x$

(xii) $\dfrac{d}{dx}(\cos x) = -\sin x$

(xiii) $\dfrac{d}{dx}(\tan x) = \sec^2 x$

(xiv) $\dfrac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(xv) $\dfrac{d}{dx}(\sec x) = \sec x \tan x$

(xvi) $\dfrac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

(xvii) $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$

(xviii) $\dfrac{d}{dx}(\cos^{-1} x) = \dfrac{-1}{\sqrt{1-x^2}}$

Imp. Basic Formula To used in Diff. & Int.

(xix) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

(xx) $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

(xxi) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

(xxii) $\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$

(xxiii) $\frac{d}{dx}(\sinh x) = \cosh x$

(xxiv) $\frac{d}{dx}(\cosh x) = \sinh x.$

(c) **Integration**

(i) $\int x^n\, dx = \frac{x^{n+1}}{n+1}\ (n \neq -1)$

(ii) $\int \frac{1}{x}\, dx = \log_e x$

(iii) $\int e^x\, dx = e^x$

(iv) $\int a^x\, dx = \frac{a^x}{\log_e a}$

(v) $\int \sin x\, dx = -\cos x$

(vi) $\int \cos x\, dx = \sin x$

(vii) $\int \tan x\, dx = -\log \cos x$

(viii) $\int \cot x\, dx = \log \sin x$

(ix) $\int \sec x \tan x\, dx = \sec x$

(x) $\int \text{cosec}\, x \cot x\, dx = -\text{cosec}\, x$

(xi) $\int \sec^2 x\, dx = \tan x$

(xii) $\int \text{cosec}^2 x\, dx = -\cot x$

(xiii) $\int \sec x\, dx = \log(\sec x + \tan x)$

(xiv) $\int \text{cosec}\, x\, dx = \log(\text{cosec}\, x - \cot x) = \log \tan \frac{x}{2}$

(xv) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$

(xvi) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$

(xvii) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\frac{a+x}{a-x}$

(xviii) $\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\frac{x}{a}$

(xix) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\frac{x-a}{x+a}$

(xx) $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a}$

(xxi) $\int \sqrt{a^2-x^2}\, dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$

(xxii) $\int \sqrt{a^2+x^2}\, dx = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a}$

(xxiii) $\int \sqrt{x^2-a^2}\, dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}$

(xxiv) $\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) = \frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left(bx - \tan^{-1}\frac{b}{a}\right)$

(xxv) $\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) = \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left(bx - \tan^{-1}\frac{b}{a}\right)$

(xxvi) $\int \sinh x\, dx = \cosh x$

(xxvii) $\int \cosh x\, dx = \sinh x$

(xxviii) $\int \tanh x\, dx = \log \cosh x$

(xxix) $\int \coth x\, dx = \log \sinh x$

(xxx) $\int \text{sech}^2 x\, dx = \tanh x$

(xxxi) $\int \text{cosech}^2 x\, dx = -\coth x$

(xxxii) $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

(xxxiii) $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx,$ a<

(xxxiv) $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$

(xxxv) $\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$

(xxxvi) $\int_{-a}^a f(x)\, dx = \begin{cases} 0, & \text{if } f \text{ is an odd function} \\ 2\int_0^a f(x)\, dx, & \text{if } f \text{ is an even function} \end{cases}$

(xxxvii) $\int_0^{2a} f(x)\, dx = \begin{cases} 0, & \text{if } f(2a-x) = -f(x) \\ 2\int_0^a f(x)\, dx, & \text{if } f(2a-x) = f(x) \end{cases}$

(xxxviii) $\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx \quad (n > 1)$

$$= \frac{(n-1)(n-3)(n-5)\ldots}{n(n-2)(n-4)\ldots} \times \left(\frac{\pi}{2}, \text{only if } n \text{ is even}\right)$$

(xxxix) $\int_0^{\pi/2} \sin^m x \cos^n x\, dx$

$$= \frac{(m-1)(m-3)\ldots \times (n-1)(n-3)\ldots}{(m+n)(m+n-2)(m+n-4)\ldots} \times \left(\frac{\pi}{2}, \text{only if both } m \text{ and } n \text{ are even}\right)$$

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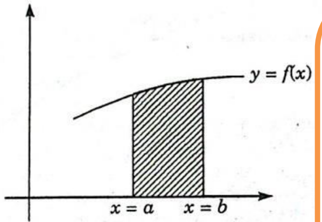
Imp. Basic Formula To used in Diff. & Int.



2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

where c is a point lies between the interval $[a, b]$. and $f(x)$ integrable over $[a, c]$ and $[c, b]$.

4. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

5. $\int_{-a}^a f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(-x)\,dx$

If function is even, i.e. $f(-x) = f(x)$, then

$$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$$

If function is odd, i.e. $f(-x) = -f(x)$, then

$$\int_{-a}^a f(x)\,dx = 0$$

6. $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

If $f(x) = f(2a-x)$, then

$$\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

and if $f(x) = -f(2a-x)$, then

$$\int_0^{2a} f(x)\,dx = 0$$

Imp. Basic Formula To used in Diff. & Int.

"Calculas"

Dated: 26/08/2016

(6)

"Limits" :- A no. "l" is stb the limit of function $f(x)$ as $x \to 0$ if $\forall\, \epsilon > 0$ (however small) $\exists$ a $\delta > 0$

i.e. $|f(x) - l| < \epsilon \quad \forall \quad |x-a| < \delta$

$\Rightarrow \boxed{\lim_{x \to a} f(x) = l}$

$l - \epsilon < f(x) < l + \epsilon \quad \forall \quad a - \delta < x < a + \delta$

left limit :- $\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h), \quad a - \delta < x < a$

Right Limit :- $\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h), \quad a < x < a + \delta$

Continuity :-

$\boxed{LHL = RHL = f(x)\big|_{x = \text{given value}}}$

Differentiability :- A function f is said to be differentiable if both LHD & RHD are equal.

i.e.

$$\underbrace{\lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}}_{LHD} = \underbrace{\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}_{RHD}$$

Mean Value Theorem :-

let $f(x)$ be a function defined such that

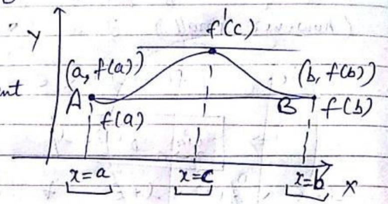
i) $f(x) \leftarrow$ Continuous [closed interval]

ii) " $\leftarrow$ differentiable (open interval)

iii) $f(a) = f(b)$

Rolle's M.V.T

then $\exists$ at least one value $c \in (a,b)$ such that $f'(c) = 0$



$f'(c) =$ slope of tangent at $x=c$

= slope of line AB

$= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{f(b) - f(a)}{b-a} = \dfrac{0}{b-a} = 0$

Eg. find c of the $f(x) = \dfrac{\sin x}{e^x}$ in $[0, \pi]$

$f'(x) = \dfrac{e^x \cos x - e^x \sin x}{e^{2x}} = \dfrac{\cos x - \sin x}{e^x} = 0$

$x = \pi/4 \Rightarrow c = \pi/4$

Que:- $f(x) = x^3 - 4x$; $x \in [-2, 2]$

Sol^n $f(-2) = -8+8 = 0$; $f(+2) = 8-8 = 0$

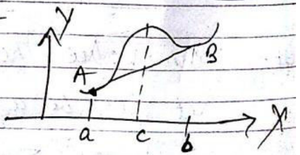
$\because f(a) = f(b)$

$f'(x) = 3x^2 - 4 = 0 \Rightarrow x = 4/\sqrt{3} \Rightarrow c = 4/\sqrt{3}$

Langrange's M.V.T :-

if $f(a) \neq f(b)$

then $f'(c) = \dfrac{f(b) - f(a)}{b-a}$



Cauchy's M.V.T :-

Let $f(x)$ and $g(x)$ be two functions defined such that

i) f & g $\leftarrow$ Conti. $[a,b]$

ii) diff.^{r} (a,b)

iii) $g'(x) \neq 0$ $\forall$ x (a,b) then

$\exists$ at-least one value $c \in (a, b)$

such that $\dfrac{f'(c)}{f(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$

$\rightarrow$ It is like L.M.V.T but with two diff.^{t} funct.

Indeterminate form :-

1. $0/0$ or ∞/∞ : Use L'Hopital
2. $0 \times \infty$: "
3. 0^0 or 1^∞ ∞^0 : "

Maxima & Minima :-

~~f(1) = 4~~

$\rightarrow$ $f(x,y) = 4x^2 + 6y^2 - 8x - 4y + 8$ (Two Variable)

$\rightarrow$ $f(x) = 2x^3 - 9x^2 + 12x - 5$ in $[0, 3]$

Sol^n

$\Rightarrow f'(x) = 6x^2 - 18x + 12 = 0$

$6(x^2 - 3x + 2) = 0 \Rightarrow (x-2)(x-1) = 0$ $\boxed{x = 1, 2}$

$f''(x) = 12x - 18 \Rightarrow f''(1) = -6 < 0$ MAXIMA

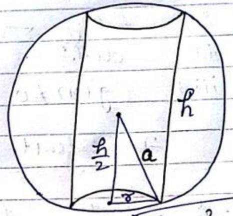
$f''(2) = +ve > 0$ MINIMA

$\boxed{f(1) = 0}$, $\boxed{-1 = f(2)}$ Ans

15 Marks

Q(2014) find the height of cylinder of maximum volume that can be inscribed in a sphere of radius a.

Sol.^{n}



$$V = \pi r^2 h$$

$$V = \pi\left(a^2 - \frac{h^2}{4}\right)h$$

$$V = \pi\left(a^2 h - \frac{h^3}{4}\right)$$

$$\boxed{a^2 = \frac{h^2}{4} + r^2}$$

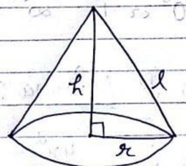
$$\frac{dV}{dh} = \pi\left(a^2 - \frac{3h^2}{4}\right) = 0$$

$$\Rightarrow a^2 = \frac{3}{4}h^2 \Rightarrow h^2 = \frac{4}{3}a^2$$

$$\Rightarrow \boxed{h = \frac{2}{\sqrt{3}}a} \quad \text{Ans.}$$

Q(2015) 13 Marks :- A conical tent is of given capacity. For the least amount of canvas required for it. find the Ratio of its height to the radius of its base.

Sol.^{n}



Given $V = \frac{1}{3}\pi r^2 h$

$$S = \pi r l$$

$$= \pi r\left(\sqrt{r^2 + h^2}\right)$$

$$\boxed{l = \sqrt{h^2 + r^2}}$$

$$= \pi\sqrt{r^4 + h^2 r^2}$$

$$\because V = \frac{1}{3}\pi r^2 h \Rightarrow \left(h = \frac{3V}{\pi r^2}\right)$$

$$= \pi\sqrt{r^4 + r^2 \times \frac{9V^2}{\pi^2 r^4}} = \pi\sqrt{r^4 + \frac{9V^2}{\pi^2 r^2}}$$

$$\frac{dS}{dr} = \frac{\pi\left(4r^3 - \frac{18V^2}{\pi^2 r^3}\right)}{2\sqrt{r^4 + 9V^2/\pi^2 r^2}} = 0$$

$$\Rightarrow 4r^3 = \frac{18V^2}{\pi^2 r^3} \Rightarrow r^6 = \frac{9V^2}{2\pi^2}$$

$$\boxed{r = \left(\frac{3V}{\sqrt{2}\pi}\right)^{1/3}} \Leftarrow \boxed{r^3 = \frac{3V}{\sqrt{2}\pi}}$$

$$\because V = \frac{1}{3}\pi r^2 h \Rightarrow h = \frac{3V}{\pi r^2} = \frac{3V}{\pi} \times \left(\frac{\sqrt{2}\pi}{3V}\right)^{2/3}$$

$$= \frac{3V}{(3V)^{2/3}} \times \frac{\pi^{2/3}}{\pi} \times (\sqrt{2})^{2/3}$$

$$\boxed{h = \frac{(3V)^{1/3}}{\pi^{1/3}} \times (2)^{1/3}}$$

Now → Ratio of $h/r = \frac{(3V)^{1/3}\, 2^{1/3}}{\pi^{1/3}} \times \frac{(\sqrt{2})^{1/3}\, \pi^{1/3}}{(3V)^{1/3}}$

$$= (\sqrt{2})^{2/3}(\sqrt{2})^{1/3} = \sqrt{2}$$

$$\Rightarrow \frac{h}{r} = \frac{\sqrt{2}}{1} \quad \text{i.e.} \quad \boxed{h : r = \sqrt{2} : 1} \quad \text{Ans.}$$

Topic: Limits:

Que. (Mains 2015); 10 Marks

Evaluate $\lim_{x \to a} \left(2 - \frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$

Sol: Let $A = \left(2 - \frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$

Multiplying both side by log.

$$\Rightarrow \log A = \log \left(2 - \frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$$

$$\Rightarrow \log A = \tan\left(\frac{\pi x}{2a}\right) \cdot \log\left(2 - \frac{x}{a}\right)$$

$$\Rightarrow \log A = \frac{\log\left(2 - \frac{x}{a}\right)}{\frac{1}{\tan\left(\frac{\pi x}{2a}\right)}}$$

Using L'Hopital Rule; do the derivate of Numerator & denominator seperately.

$$\frac{d}{dx}(\log A) = \frac{\frac{1}{(2 - x/a)} \times \frac{-1}{a}}{\frac{-1}{\tan^2\left(\frac{\pi x}{2a}\right)} \times \sec^2\left(\frac{\pi x}{2a}\right) \times \frac{\pi}{2a}}$$

$$= \lim_{x \to a} \frac{\frac{+1}{a}}{+\frac{\cos^2(\pi x/2a)}{\sin^2(x\pi/2a)} \times \sec^2\left(\frac{\pi x}{2a}\right) \times \frac{\pi}{2a}}$$

$$= \lim_{x \to a} \frac{2a}{a\pi} \times \frac{\sin^2\left(\frac{x\pi}{2a}\right)}{(\;)^2} = \frac{2}{\pi} \times (1)^2 = \frac{2}{\pi}$$

Ans

Q (2011): 8 Marks

$$\lim_{x \to 2} f(x), \text{ where } f(x) = \begin{cases} \frac{x^2-4}{x-2}, & x \neq 2 \\ \pi, & x = 2 \end{cases}$$

Sol: L.H.L $\quad x \to 2^- \Rightarrow x = 2 - h$; where $h \to 0$

$$f(x) = \frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{(x-2)} = (x+2)$$

$$f(2-h) = (2-h+2) = (4-h)$$

$$\lim_{h \to 0} f(2-h) = \lim_{h \to 0} (4-h) = 4$$

R.H.L: $\quad x \to 2^+ \Rightarrow x = 2 + h$; where $h \to 0$

$$f(x) = \frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{(x-2)} = (x+2)$$

$$\lim_{h \to 0} f(x) = \lim_{h \to 0} 2+h+2 = \lim_{h \to 0} 4+h = 4$$

Now $f(2) = \pi$ (Given)

$$\Rightarrow \boxed{L.H.L = R.H.L \neq f(2)} \Rightarrow$$ function is discontinuous.

Topic: Integration

Q (2011); ← Main; 12 Marks

$$\int_0^1 \log x \, dx$$

(Solve using "ILATE"
Inverse
Logrithm
Algebra
Trigonometry
Exponential

Sol: $\displaystyle\int_0^1 \underbrace{\log x}_{I} \cdot \underbrace{1}_{II} \, dx$

ILATE $\Leftarrow$ Product of two different functions

$$\boxed{\int I\cdot II = I\cdot\int II - \int\left[dI\cdot\int II\right]}$$

$$= \log x\cdot x - \int \frac{1}{x}\cdot x\,dx\Big|_0^1$$

$$= \log x\cdot x - x\Big|_0^1$$

$$= x(\log x - 1)\Big|_0^1 = (\log 1 - 1) - 0$$

$$= 0 - 1 = -1 \text{ Ans}$$

Que:- (2014) 10 : Marks

$$\int_0^1 \frac{\log_e(1+x)}{1+x^2}\,dx.$$

Ref^r: 7.11 (8) NCERT (12)

Sol. Put $x = \tan\theta \Rightarrow 0 \le \theta \le \pi/4 \Rightarrow dx = \sec^2\theta\cdot d\theta$

$$\Rightarrow \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{1+\tan^2\theta}\cdot\sec^2\theta\,d\theta$$

$$I = \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{\cancel{\sec^2\theta}}\times\cancel{\sec^2\theta}\,d\theta = \int_0^{\pi/4}\log(1+\tan\theta)\,d\theta$$

$$\boxed{\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx}$$ Using this property

$$I = \int_0^{\pi/4}\log\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right)d\theta$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B}$$

$$\because \left[\tan\left(\frac{\pi}{4}-\theta\right) = \frac{\tan\frac{\pi}{4} - \tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right]$$

$$\Rightarrow I = \int_0^{\pi/4}\log\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)d\theta$$

$$I = \int_0^{\pi/4}\log\left(\frac{1+\cancel{\tan\theta}+1-\cancel{\tan\theta}}{1+\tan\theta}\right)d\theta$$

$$I = \int_0^{\pi/4}\log\left(\frac{2}{1+\tan\theta}\right)d\theta = \int_0^{\pi/4}\log 2\,d\theta - \int_0^{\pi/4}\log(1+\tan\theta)\,d\theta$$

I

$$I = \int_0^{\pi/4}\log 2\,d\theta - I$$

$$\Rightarrow 2I = \int_0^{\pi/4}\log 2\,d\theta = \log 2\times\theta\Big|_0^{\pi/4}$$

$$\Rightarrow 2I = \log 2\left(\frac{\pi}{4}-0\right) = \frac{\pi}{4}\log 2$$

$$\Rightarrow \boxed{I = \frac{\pi}{8}\log 2} \text{ Ans}$$

Q. (2015) $\displaystyle\int_{\pi/6}^{\pi/3}\frac{\sqrt[3]{\sin x}}{\sqrt[3]{\sin x}+\sqrt[3]{\cos x}}\,dx$ (10 Marks)

Solution

$$\int_b^a f(x)\,dx = \int_b^a f(a+b-x)\,dx$$

$$I = \int_{\pi/6}^{\pi/3}\frac{\sqrt[3]{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt[3]{\sin x}+\sqrt[3]{\cos x}}\,dx = \int_{\pi/6}^{\pi/3}\frac{\sqrt[3]{\cos x}}{\sqrt[3]{\sin x}+\sqrt[3]{\cos x}}\,dx$$

$$I + I = \int_{\pi/6}^{\pi/3}\cancel{\frac{\sqrt[3]{\sin x}+\sqrt[3]{\cos x}}{\sqrt[3]{\sin x}+\sqrt[3]{\cos x}}}\,dx = \int_{\pi/6}^{\pi/3}dx$$

$$2I = x\Big|_{\pi/6}^{\pi/3} = \left(\frac{\pi}{3}-\frac{\pi}{6}\right) = \frac{2\pi-\pi}{6} = \frac{\pi}{6}$$

$$\boxed{I = \frac{\pi}{12}} \text{ Ans}$$

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Topic : Limits { Continuity & differentiability for two variable.

Que: (2015); 12 Marks

$$f(x,y) = \begin{cases} \frac{x^2 - x\sqrt{y}}{x^2+y}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Solution :- Put $\sqrt{y} = mx$

$$\Rightarrow f(x,y) = \frac{x^2 - x(mx)}{x^2 + m^2x^2} = \frac{\cancel{x^2}(1-m)}{\cancel{x^2}(1+m^2)}$$

$$\underset{x\to 0, y\to 0}{Lt} f(x,y) = \frac{(1-m)}{1+m^2}$$

where $m \in N$ $\Rightarrow$

$f(x,y)$ does not have a unique value for Limit $x \to 0$, $y \to 0$.

Hence $f(x,y)$ = Discontinuous at $(0,0)$.

Ans

Que: (2012) 12 Marks

$$f(x,y) = \begin{cases} \frac{x^3 \cos\frac{1}{y} + y^3 \cos\frac{1}{x}}{x^2+y^2}, & x,y \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

Sol Put $y = mx$

$$\Rightarrow f(x,y) = \frac{x^3 \cos\frac{1}{mx} + m^3x^3 \cos\frac{1}{x}}{x^2 + m^2x^2}$$

$$= \frac{x^{\cancel{3}}\left(\cos\frac{1}{mx} + m^3 \cos\frac{1}{x}\right)}{\cancel{x^2}(1+m^2)}$$

$$= \underset{x\to 0}{Lt}\; x\left\{\cos\frac{1}{mx} + m^3 \cos\frac{1}{x}\right\}$$

$$= 0$$

$\Rightarrow$ Limit exist at $x \to 0$, $y \to 0$ and hence it is Continuous function.

Que: (2011); 10 Marks

Find $\lim\limits_{(x,y)\to 0} \frac{x^2y}{x^3+y^3}$ if it exist.

Solution Put $y = mx$ for y along a straight line.

$$f(x,y) = \frac{x^2(mx)}{x^3+(mx)^3} = \frac{mx^3}{x^3(1+m^3)} = \frac{m}{1+m^3}$$

~~He~~ Since the value of $f(x,y)$ is not unique at $(0,0)$. Hence the limit does not exist.

Q(2009) 20 Marks

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Sol. Put $y = mx$

$$\Rightarrow \quad f(x,y) = \frac{x(mx)}{\sqrt{x^2+(mx)^2}} = \frac{mx^2}{x\sqrt{1+m^2}} = \frac{mx}{\sqrt{1+m^2}}$$

$$\lim_{(x,y)\to 0} f(x,y) = \lim_{(x,y)\to 0} \frac{mx}{\sqrt{1+m^2}} = 0$$

$\Rightarrow$ limit exist and hence the function is continuous.

ASYMPTOTES

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i.e., ... and $x + y = 0$

Question

(*ii*) The equation of curve is $(x-y)^2(x+2y-1) = 3x + y - 7$

or $$(x-y)^2(x+2y) - (x-y)^2 - 3x - y + 7 = 0 \quad \ldots(1)$$

Since the co-efficients of x^3 and y^3, the highest degree terms in x and y, are constants, there are no asymptotes parallel to the x-axis or y-axis.

To find oblique asymptotes, putting $x = 1$, $y = m$ in the third, second and first degree terms in (1)

$$\phi_3(m) = (1-m)^2(1+2m) = (1-2m+m^2)(1+2m) = 2m^3 - 3m^2 + 1$$

$$\phi_2(m) = -(1-m)^2 = -m^2 + 2m - 1, \quad \phi_1(m) = -3 - m$$

The slopes of asymptotes are the roots of $\phi_3(m) = 0$

$$\phi_3(m) = 0 \Rightarrow (1-m^2)(1+2m) = 0 \quad \therefore \quad m = 1, 1, -\frac{1}{2}$$

Also $\quad \phi_3'(m) = 6m^2 - 6m, \quad \therefore \quad \phi_3''(m) = 12m - 6, \quad \phi_2'(m) = -2m + 2$

For the non-repeated value $m = -\frac{1}{2}$, c is given by

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} \quad \text{or} \quad c = -\frac{-m^2 + 2m - 1}{6m^2 - 6m} = \frac{m^2 - 2m + 1}{6m^2 - 6m}$$

When $\quad m = -\frac{1}{2}, c = \dfrac{\frac{1}{4} + 1 + 1}{\frac{3}{2} + 3} = \frac{9}{4} \times \frac{2}{9} = \frac{1}{2}$

Corresponding asymptote is $\quad y = mx + c$ *i.e.,* $y = -\frac{1}{2}x + \frac{1}{2}$ or $x + 2y - 1 = 0$

For the twice repeated value $m = 1, 1$, c is given by

$$\frac{c^2}{2!}\phi_3''(m) + c\phi_2'(m) + \phi_1(m) = 0$$

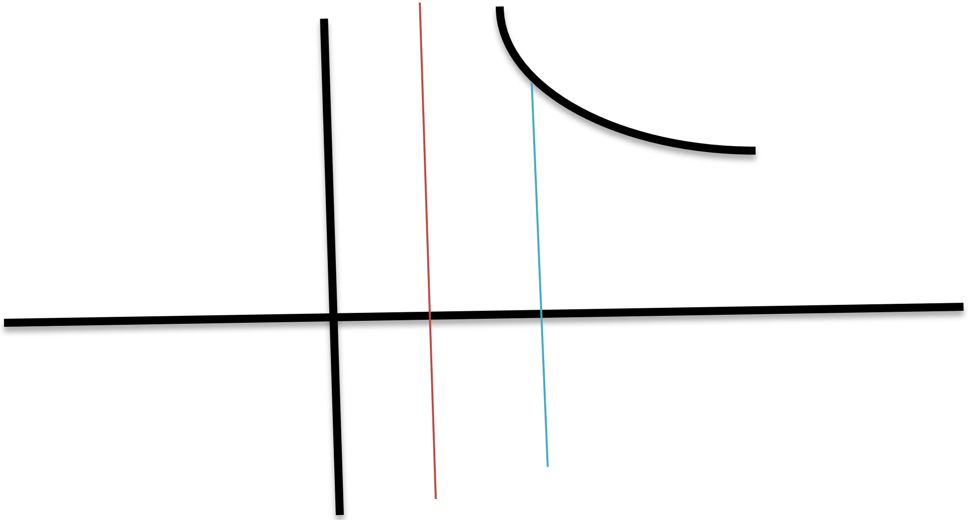
or $$\frac{c^2}{2}(12m - 6) + c(-2m + 2) + (-3 - m) = 0$$

or $$c^2(6m - 3) + c(-2m + 2) + (-3 - m) = 0$$

When $m = 1$, we get $\quad 3c^2 - 4 = 0 \quad$ or $\quad c = \pm\frac{2}{\sqrt{3}}$

Corresponding asymptotes are $\quad y = mx + c \quad$ *i.e.,* $\quad y = x \pm \frac{2}{\sqrt{3}}$

Hence the asymptotes are $\quad x + 2y - 1 = 0 \quad$ and $\quad y = x \pm \frac{2}{\sqrt{3}}$.



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THANKS

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