Lecture 11
Thursday
23/2/17.

Interpolation

Ex1 Find Newton's forward & backward interpolating polynomials corresponding to the following data. Show that both polynomials are the same.

x	$1=x_0$	$2=x_1$	$3=x_2$	$4=x_3$
$f(x)$	$5=y_0$	$9=y_1$	$14=y_2$	$20=y_3$

Difference table

x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$
1	$5=y_0$			
		$4=\Delta y_0$		
2	9		$1=\Delta^2 y_0$	
		$5=\Delta y_1$		$0=\Delta^3 y_0 = \nabla^3 y_0$
3	14		$1=\Delta^2 y_1 = \nabla^2 y_3$	
		$6=\Delta y_2 = \nabla y_3$		
4	$20=y_3$			

$\boxed{\nabla^2 y_3 = \Delta^2 y_{3-2}}$

$\nabla y_3 = \Delta y_2$, $\nabla^2 y_3 = \Delta^2 y_1$

$\nabla^3 y_3 = \Delta^3 y_0$

$\boxed{h=1}$

$$p_3(x) = y_0 + \frac{\Delta y_0}{h}(x-x_0) + \frac{\Delta^2 y_0}{2!\,h^2}(x-x_0)(x-x_1) + \frac{\overset{=0}{\cancel{\Delta^3 y_0}}}{3!\,h^3}(x-x_0)(x-x_1)(x-x_2)$$

$$p_3(x) = 5 + 4\cdot(x-1) + 1\,\frac{(x-1)(x-2)}{2}$$

$$= \frac{x^2+5x+4}{2} = \frac{1}{2}x^2 + \frac{5}{2}x + 2.$$

$$q_3(x) = y_3 + \frac{\nabla y_3}{h}(x-x_3) + \frac{\nabla^2 y_3}{2!\,h^2}(x-x_3)(x-x_2)$$

$$+ \frac{\cancel{\nabla^3 y_3}}{3!\,h^3}(x-x_3)(x-x_2)(x-x_1) \quad \nabla^3 y_3 = 0$$

$$= 20 + 6(x-4) + \frac{(x-4)(x-3)}{2}$$

Actually we get

$$q_2(x) = \frac{x^2 + 5x + 4}{2} = p_2(x)$$

2. Find Newton's forward & backward polynomials from the following data.

x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$
1	5			
		4		
2	9		1	
		5		1
3	14		2	
		7		
4	21			

$$\frac{x^3 - 3x^2 + 26x + 6}{6}$$

Note. In Ex1 & Ex2 there are 4 sets of data viz. $(x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3)$, ~~$(x_4, y_4)$~~.

So, it is possible to have a polynomial of degree $\boxed{\text{less or equal to } 3}$.

In Ex1, we got a polynomial of degree 2.

In " 2, " " " degree 3.

Lagrange polynomial.

x	1	3	6	7	9
y	20	66	79	95	102

$x_1 - x_0 = 2$

$x_2 - x_1 = 3$.

$$L_n(x_j) = f(x_j)\,; \quad j = 0, 1, 2, \ldots, n. \longrightarrow (1)$$

$L_n(x) \to$ Lagrange polynomial.

$$L_n(x) = \sum_{i=0}^{n} \underline{l_i(x)}\, f(x_i) \longrightarrow (2)$$

$l_i(x)$ are such that

$$l_i(x_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

In (2), put $x = x_j$

$$L_n(x_j) = \sum_{i=0}^{n} l_i(x_j) f(x_i) = f(x_j)$$

$$= \underbrace{l_0(x_j)}_{=0} f(x_0) + \underbrace{l_1(x_j)}_{=0} f(x_1) + \underbrace{l_2(x_j)}_{=0} f(x_2)$$

$$+ \cdots + \underbrace{l_{j-1}(x_j)}_{=0} f(x_{j-1}) + \underbrace{l_j(x_j)}_{=1} f(x_j)$$

$$+ \underbrace{l_{j+1}(x_j)}_{=0} f(x_{j+1}) + \cdots + \underbrace{l_n(x_j)}_{=0} f(x_n).$$

$$l_i(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$$

We need

$$l_i(x) = C_0 (x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)$$

C_0 is determined, from $l_i(x_i) = 1$

$$l_i(x_i) = 1 = C_0(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)$$

$$C_0 = \frac{1}{\prod\limits_{\substack{j=0 \\ j \neq i}}^{n} (x_i - x_j)} .$$

$$l_i(x) = \frac{\prod\limits_{\substack{j=0 \\ j \neq i}}^{n} (x - x_j)}{\prod\limits_{\substack{j=0 \\ j \neq i}}^{n} (x_i - x_j)} .$$

$$L_n(x) = \sum_{i=0}^{n} l_i(x) f(x_i) = \sum_{i=0}^{n} l_i(x) y_i$$

$$= l_0(x) y_0 + l_1(x) y_1 + l_2(x) y_2 + \cdots + l_n(x) y_n$$

$$= \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)\cdots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\cdots(x_2-x_n)} y_2$$

$$+ \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} y_n .$$

Ex-1. Find the Lagrange interpolating polynomial corr. to the data points $(1,2), (3,8), (6,64)$

x	$1 = x_0$	$3 = x_1$	$6 = x_2$
y	$2 = y_0$	$8 = y_1$	$64 = y_2$

$$L_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

$$= \frac{47x^2 - 143x + 126}{15} .$$

Error in Interpolation

$$E_n(x) = f(x) - p_n(x)$$

$$= \frac{\omega(x)\, f^{n+1}(\xi)}{(n+1)!}$$

$$\omega(x) = (x-x_0)(x-x_1) \cdots (x-x_n) = \prod_{i=0}^{n} (x-x_i)$$

1) $x_0 < \xi < x_n$, for interpolation

2) For extrapolation $\quad x \ldots x_0$

either $x < \xi < x_0$. $\quad x_n \; x$.

or, $x_n < \xi < x$.

For interpolation

$$E_n(x) = \frac{\omega(x)\, f^{(n+1)}(\xi)}{(n+1)!}; \quad x_0 < \xi < x_n.$$

$$\omega(x) = (x-x_0)(x-x_1)(x-x_2) \cdots (x-x_n)$$

Newton's forward interpolating pol.

$$E_n(x) \simeq \omega(x) \frac{\Delta^{n+1} f(x_0)}{h^{n+1}}$$

Newton's backward interpolating pol.

$$E_n(x) \simeq \omega(x) \frac{\nabla^{n+1} f(x_n)}{h^{n+1}}.$$

1

2 $\quad E_3$

3

4

5 extrapolate & get y_5 $\quad$ 5.

<u>Ex. 1.</u> Find a third order polynomial and hence find the values of $f(2.5)$, $f(0.7)$, $f(14.2)$, $f(19.1)$ using appropriate Newton's interpolating polynomials.

	x_0	x_1	x_2	x_3	x_4	x_5
x	2	5	8	11	14	17
y	16	21	29	37	49	61

Hint. 1. Use 1st 4 data (x_i, y_i) $(i = 0, 1, 2, 3)$ to find N forward int. polynomial and compute $f(0.7)$, $f(2.5)$

2. Use last 4 data (x_i, y_i) $(i = 2, 3, 4, 5)$ to find N backward int. polynomial & compute $f(14.2)$, $f(19.1)$.

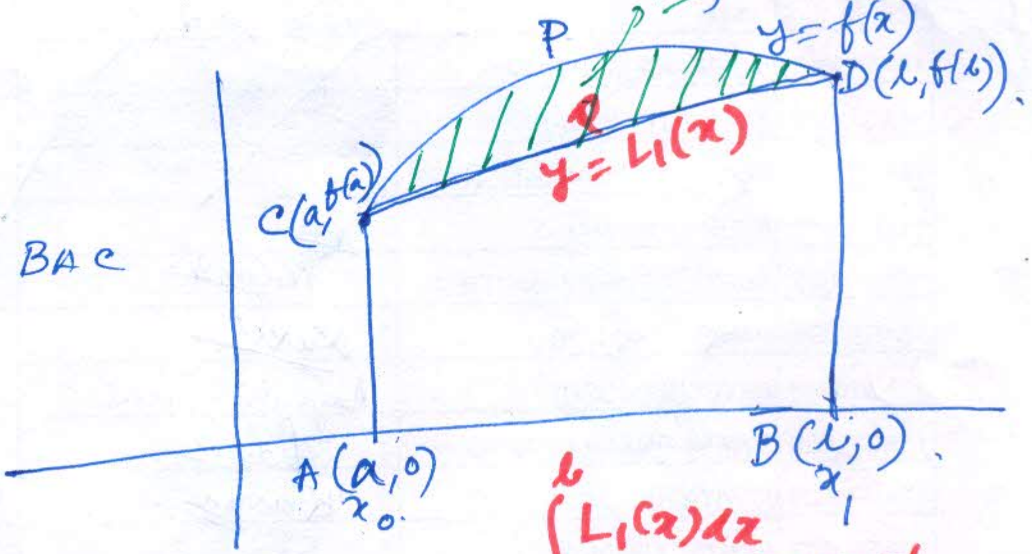
Verify that the two polynomials are different. But if ~~you~~ you'd have used all 6 pairs of data, you'll get a unique pol. of degree ≤ 5.

Numerical Integration

To evaluate

$$I = \int_a^b f(x)\,dx$$

$=$ area of CPDBAC



$\int_a^b L_1(x)\,dx$ = area of CQDBAC

Replace $f(x)$ by $L_n(x)$

$$L_1(x) = \frac{(x-x_1)}{x_0-x_1}y_0 + \frac{(x-x_0)}{x_1-x_0}y_1 .$$

x_0	x_1
y_0	y_1

~~[$a \to x_0$, $b \to x_1$]~~

$$y = \frac{(x-x_1)}{x_0-x_1}y_0 + \frac{(x-x_0)}{x_1-x_0}y_1 .$$

$$y - y_0 = \frac{x-x_1}{x_0-x_1}y_0 - y_0 + \frac{x-x_0}{x_1-x_0}y_1 .$$

$$= \frac{x-\cancel{x_1}-x_0+\cancel{x_1}}{x_0-x_1}y_0 + \frac{x-x_0}{x_1-x_0}y_1$$

$$= -\frac{x-x_0}{x_1-x_0}y_0 + \frac{x-x_0}{x_1-x_0}y_1$$

$$= \frac{(x-x_0)(y_1-y_0)}{x_1-x_0} .$$

$$\frac{y-y_0}{y_1-y_0} = \frac{x-x_0}{x_1-x_0} . \quad a \to x_0, \quad b \to x_1 .$$

$$\int_a^b f(x)\,dx \approx \int_a^b L_n(x)\,dx.$$

↳ Newton-Cote's formula.

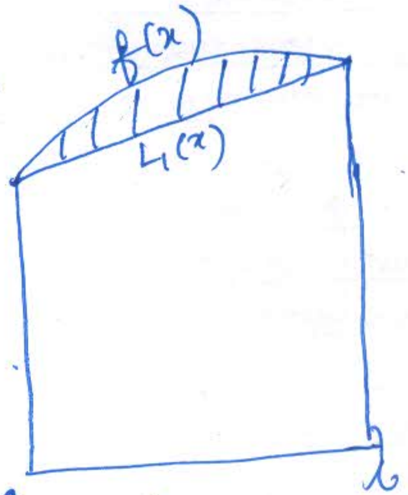
$n=1 \rightarrow$ Trapezoidal rule.

$n=2 \rightarrow$ Simpson's $\frac{1}{3}$rd rule

} Maths II.

$n=3 \rightarrow$ Simpson's $\frac{3}{8}$th rule.

Simple Trapezoidal rule.



$$I \approx \int_a^b L_1(x)\,dx.$$

$$= \int_{x_0}^{x_1}\left[\frac{x-x_1}{x_0-x_1}y_0 + \frac{x-x_0}{x_1-x_0}y_1\right]dx.$$

$$= \frac{y_0}{x_0-x_1}\int_{x_0}^{x_1}(x-x_1)\,dx + \frac{y_1}{x_1-x_0}\int_{x_0}^{x_1}(x-x_0)\,dx.$$

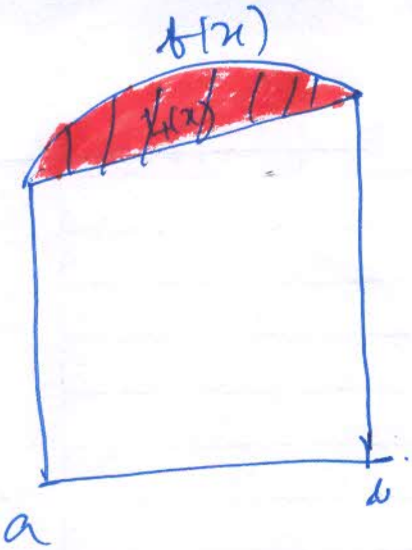
$$= \frac{y_0}{x_0-x_1}\left[\frac{(x-x_1)^2}{2}\Big|_{x_0}^{x_1}\right] + \frac{y_1}{x_1-x_0}\left[\frac{(x-x_0)^2}{2}\Big|_{x_0}^{x_1}\right]$$

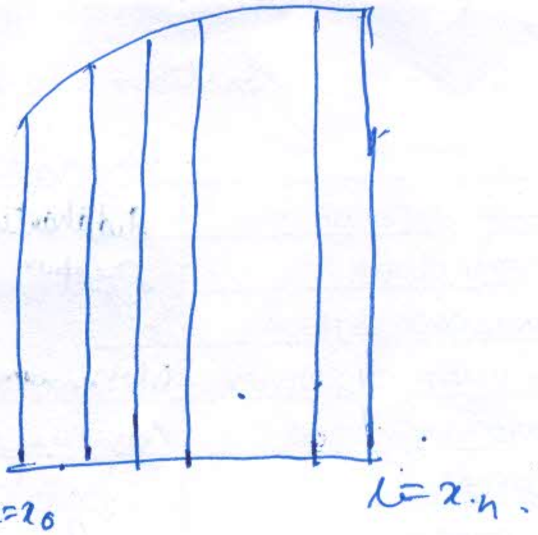
$$= \frac{y_0}{x_0-x_1}\left[-\frac{(x_0-x_1)^2}{2}\right] + \frac{y_1}{\cancel{(x_1-x_0)}}\left[\frac{(x_1-x_0)^{\cancel{2}}}{2}\right]$$

$$= -\frac{(x_0-x_1)}{2}.y_0 + \frac{(x_1-x_0).y_1}{2}.$$

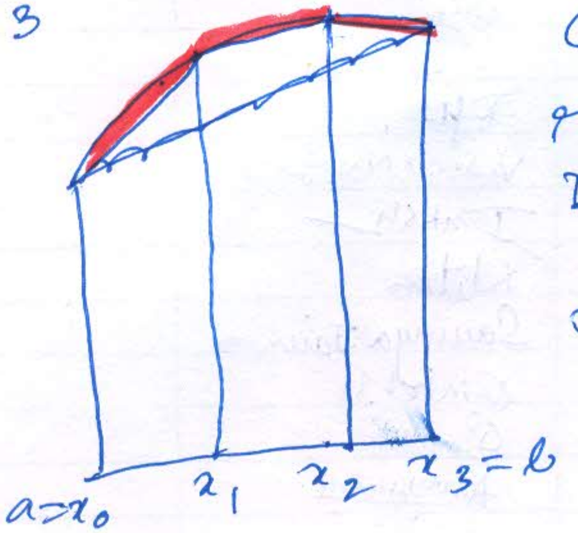
$$= \frac{x_1-x_0}{2}y_0 + \frac{(x_1-x_0)}{2}y_1$$

$$T^{simple} = \frac{(b-a)}{2}\left[f(a)+f(b)\right] \approx \int_a^b f(x)\,dx$$





n = 3



Composite Trapezoidal rule.

Divide $[a, b]$ into. $=[x_0, x_n]$

n equal parts/intervals each of length h.

$$\frac{b-a}{n} = h.$$

& apply Trapezoidal rule to $[x_0, x_1]$, $[x_1, x_2]$ - - , $[x_{n-1}, x_n]$.

Apply Trap. rule to $[x_0, x_1] = \frac{\overbrace{x_1 - x_0}^{=h}}{2}\left[f(x_0) + f(x_1)\right]$

" Trap. " " $[x_1, x_2] = \frac{\overbrace{x_2 - x_1}^{=h}}{2}\left[f(x_1) + f(x_2)\right]$.

⋮

" " " " $[x_{n-1}, x_n] = \frac{\overbrace{x_n - x_{n-1}}^{=h}}{2}\left[f(x_{n-1}) + f(x_n)\right]$

Adding $= \frac{h}{2}\Big[f(x_0) + 2\big(f(x_1) + f(x_2) + \cdots + f(x_{n-1})\big) + f(x_n)\Big]$