

Multiple integrals

Lecture- 18

Thursday

23/3/17

Change in variables in triple integrals.

$$(x, y, z) \rightarrow (u, v, w)$$

$$\iiint f(x, y, z) dx dy dz$$

$$R_{xyz} = \iiint_{R_{uvw}} f(x, u, w) |J| du dv dw$$

$$(x, y, z) \rightarrow (r, \theta, z)$$

(cylindrical polar coordinates)

$$\textcircled{a}, (x, y, z) \rightarrow (r, \theta, \phi)$$

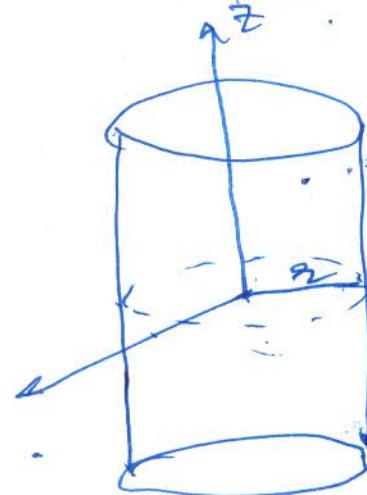
(spherical polar coordinates)

$$\textcircled{b}, (x, y, z) \rightarrow (u, v, w)$$

(general coordinates)

$$\downarrow J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Transformation to cylindrical polar coordinates



$$(x, y, z) \rightarrow (r, \theta, z)$$

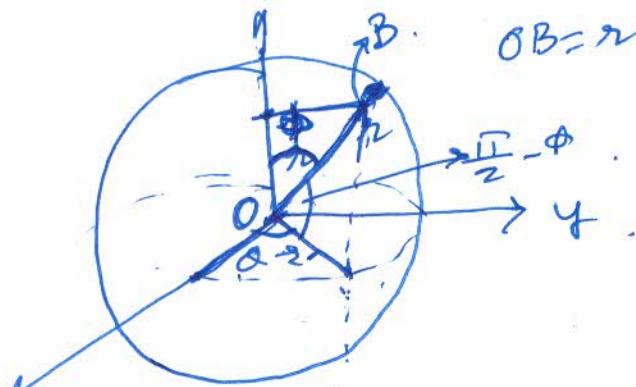
$x = r \cos \theta, y = r \sin \theta, z = z$
 $0 \leq r \leq a$ (a = radius of cylinder),
 $0 \leq \theta \leq 2\pi$,
 $z_1 \leq z \leq z_2$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos \theta & \sin \theta & 0 \\ -r \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

$$J = \begin{vmatrix} \cos \theta & \sin \theta & 0 \\ -r \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

Transformation to spherical polar coordinates

$$(x, y, z) \rightarrow (r, \theta, \phi)$$



$$x = r' \cos \theta = r \sin \theta \cos \phi$$

$$y = r' \sin \theta = r \sin \theta \sin \phi$$

$$z = r' \cos \phi = r \cos \phi$$

$0 \leq r \leq a$; a = radius of sphere
 $0 \leq \theta \leq \pi$ or, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$,
 $0 \leq \phi \leq 2\pi$

$$r' \rightarrow \text{projection of } r \text{ on } xy\text{-plane}$$

$$= r \cos \left(\frac{\pi}{2} - \phi \right) = r \sin \phi$$

~~$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ -r \sin \phi & r \sin \phi & 0 \\ r \cos \phi & r \cos \phi & -r \end{vmatrix} = -r^2 \sin \phi$$~~

$$|J| = (-r^2 \sin \phi) = -1/2 r^2 |2 \sin \phi| = r^2 / \sin \phi$$

In general,

Use cylindrical polar, if

$$f(x, y, z) = f_1(x^2 + y^2) f_2(z)$$

Use spherical polar, if -

$$f(x, y, z) = f_1(x^2 + y^2 + z^2)$$

Ex 1 Evaluate

$$\iiint (10 - x^2 - y^2 - z^2) dx dy dz$$

R: a sphere of radius 3

$$\begin{aligned} x &= r \sin \phi \cos \theta, \quad y = r \sin \phi \sin \theta, \quad z = r \cos \phi \\ \Rightarrow r^2 &= x^2 + y^2 + z^2 = r^2 \\ 0 &\leq r \leq 3, \quad 0 \leq \phi \leq \pi, \quad 0 \leq \theta \leq 2\pi \end{aligned}$$

$$J = -r^2 \sin \phi$$

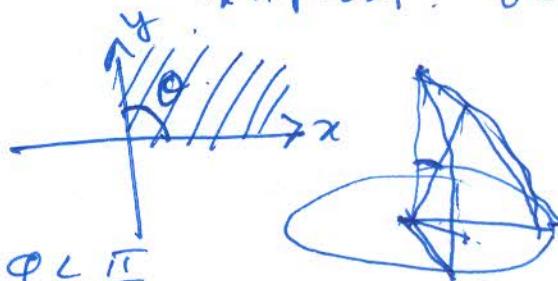
(Determine
in rough)

$$I = \int_{\phi=0}^{\pi} \int_{\theta=0}^{2\pi} \int_{r=0}^3 (10 - r^2) r^2 \sin \phi dr d\theta d\phi$$

$$\begin{aligned} &= \left(\int_{r=0}^3 (10 - r^2) r^2 dr \right) \left(\int_{\phi=0}^{\pi} (\sin \phi) d\phi \right) \\ &= \left[\frac{10r^3}{3} - \frac{r^5}{5} \right]_0^3 \left[\cos \phi \right]_0^{\pi} \times 2\pi \left(\int_{\theta=0}^{2\pi} d\theta \right) = \frac{828\pi}{5} \end{aligned}$$

$$2. \iiint \frac{dx dy dz}{\sqrt{1-x^2-y^2-z^2}}$$

\mathcal{D} = portion of the sphere in the 1st octant.
 $x^2+y^2+z^2=1$.



For 1st octant,

$$0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \varphi \leq \frac{\pi}{2}$$

$$0 \leq r \leq 1$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \int_{\phi=0}^{\theta=0} \frac{r^2 |\sin \varphi| dr d\theta d\varphi}{\sqrt{1-r^2}}$$

$$= \left(\int_0^{\pi/2} \frac{r^2 dr}{\sqrt{1-r^2}} \right) \times \left(\int_{\phi=0}^{\pi/2} \sin \varphi d\varphi \right) \times \left(\int_{\theta=0}^{\pi/2} d\theta \right)$$

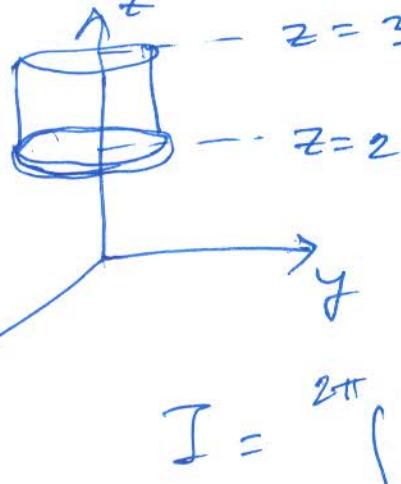
$$= \int_{t=0}^{\pi/2} \frac{\sin^2 t \cdot \cos t dt}{\sqrt{1-\sin^2 t}} \cdot \left(\cos \theta \right)_{\pi/2}^0 \times \frac{\pi}{2}$$

$$= \left(\frac{1}{2} \int_0^{\pi/2} 2 \sin^2 t dt \right) \frac{\pi}{2} = \frac{\pi}{4} \int_0^{\pi/2} (1 - \cos 2t) dt$$

$$= \frac{\pi}{4} \times \frac{\pi}{2} = \frac{\pi^2}{8}$$

3. $\iiint z(x^2 + y^2) dx dy dz$

R = portion of the cylinder $x^2 + y^2 = 1$
 between $z=2$ and $z=3$.



$x = r \cos \theta, y = r \sin \theta, x^2 + y^2 = r^2, z = z$.

$J = r, 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi, 2 \leq z \leq 3$.

$$I = \int_{0}^{2\pi} \int_{r=0}^{z=2} \int_{z=2}^3 z r^2 \times |J| dr d\theta dz$$

$$= \left(\int_{r=0}^{r^3 dr} \right) \left(\int_{z=2}^3 z dz \right) \left(\int_{\theta=0}^{2\pi} d\theta \right)$$

$$= \frac{1}{4} \times \frac{3^2 - 2^2}{2} \times 2\pi = \frac{5\pi}{4}$$

4. Evaluate the integral using cylindrical polar coordinates.

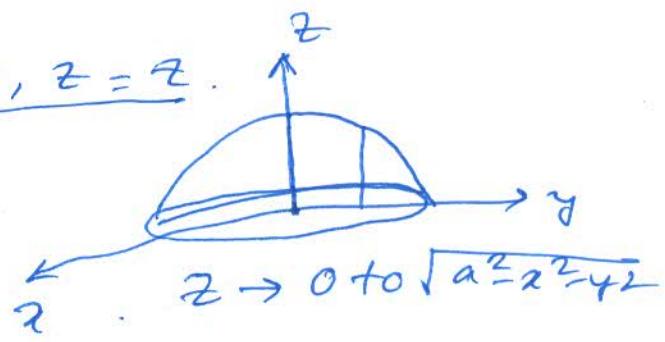
$\iiint z dx dy dz$

R = upper hemisphere $x^2 + y^2 + z^2 = a^2$

Sol. $x = r \cos \theta, y = r \sin \theta, z = z$.

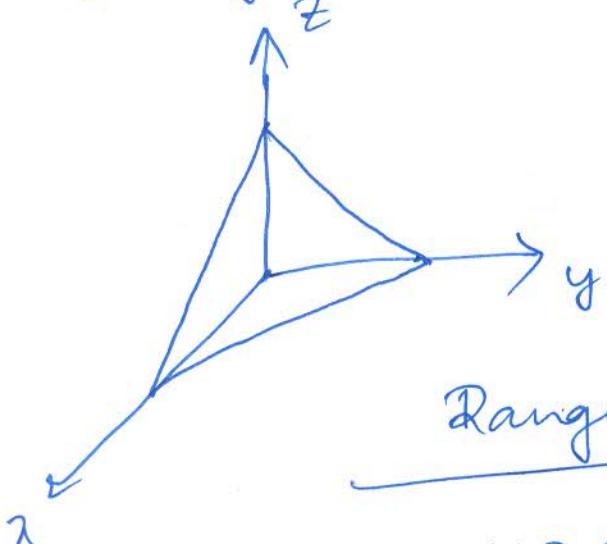
$$I = \int_{\theta=0}^{2\pi} \int_{r=0}^{a\sqrt{a^2 - z^2}} \int_{z=0}^{z} z r dz dr d\theta$$

$$= \frac{\pi a^4}{4}$$



Ex 5 Evaluate $\iiint \frac{[1-x-y-z]}{xyz} dx dy dz$.

3-region bounded by the planes $x \geq 0, y \geq 0, z \geq 0, x+y+z \leq 1$.



$$x+y+z = u \rightarrow (1)$$

$$y+z = uv \rightarrow (2)$$

$$z = uvw \rightarrow (3)$$

Ranges of u, v, w :

$$\because x \geq 0, y \geq 0, z \geq 0 \text{ & } u = x+y+z. \quad \therefore u \geq 0.$$

$$\because x+y+z \leq 1 \quad \therefore u \leq 1 \quad \text{i.e. } 0 \leq u \leq 1$$

From (1) & (2)

$$v = \frac{y+z}{x+y+z} \quad \underline{v \geq 0},$$

~~$x+y+z \leq x+y+z$~~

$$\therefore x \geq 0 \quad \therefore x+y+z \geq y+z \quad \therefore y+z \text{ are also } \geq 0$$

$$v = \frac{y+z}{x+y+z} \leq 1. \quad 0 \leq v \leq 1$$

From (2) & (3),

$$w = \frac{z}{y+z} \quad \therefore y \geq 0, z \geq 0. \quad \therefore w \geq 0$$

$$\therefore z \geq 0, \therefore y+z \geq z$$

$$w = \frac{z}{y+z} \leq 1, \quad 0 \leq w \leq 1$$

Find x, y, z as functions of u, v, w .

$$x = u - uv$$

$$y = uv - uvw$$

$$z = uvw$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1-u & v-vw & vw \\ -u & u-uw & uw \\ 0 & -uv & uv \end{vmatrix} = u^2 v^2.$$

$$\begin{aligned} I &= \iint_{\substack{0 \\ w=0 \\ v=0 \\ u=0}}^1 \frac{(1-u)^{1/2}}{u^{1/2}(1-v)^{1/2}} \frac{u^{1/2}v^{1/2}(1-w)^{1/2}u^{1/2}v^{1/2}w^{1/2}}{dudvdw} \\ &= \iint_{\substack{0 \\ 0 \\ 0}}^1 u^{1/2}(1-u)^{1/2} (1-v)^{1/2} w^{-1/2} (1-w)^{1/2} dudvdw \\ &= \left\{ \int_{u=0}^1 \frac{u^{1/2}(1-u)^{1/2} du}{u^{m-1}(1-u)^{n-1}} \right\} \left\{ \int_{v=0}^1 (1-v)^{1/2} dv \right\} \left\{ \int_{w=0}^1 w^{-1/2} (1-w)^{1/2} dw \right\} \\ &= B\left(\frac{3}{2}, \frac{3}{2}\right) B\left(1, \frac{1}{2}\right) B\left(\frac{1}{2}, \frac{1}{2}\right). \end{aligned}$$

$$= \frac{\Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{3}{2}\right)}{\Gamma(3)} \cdot \frac{\Gamma(1) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} \cdot \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma(1)}$$

$$\begin{aligned} \Gamma\left(\frac{1}{2}+1\right) &= \frac{\frac{1}{2} \Gamma\left(\frac{1}{2}\right)}{2 \Gamma(2)} \cdot \sqrt{\pi} \cdot (\sqrt{\pi})^2 \\ &= \frac{\pi^2}{4}. \end{aligned}$$

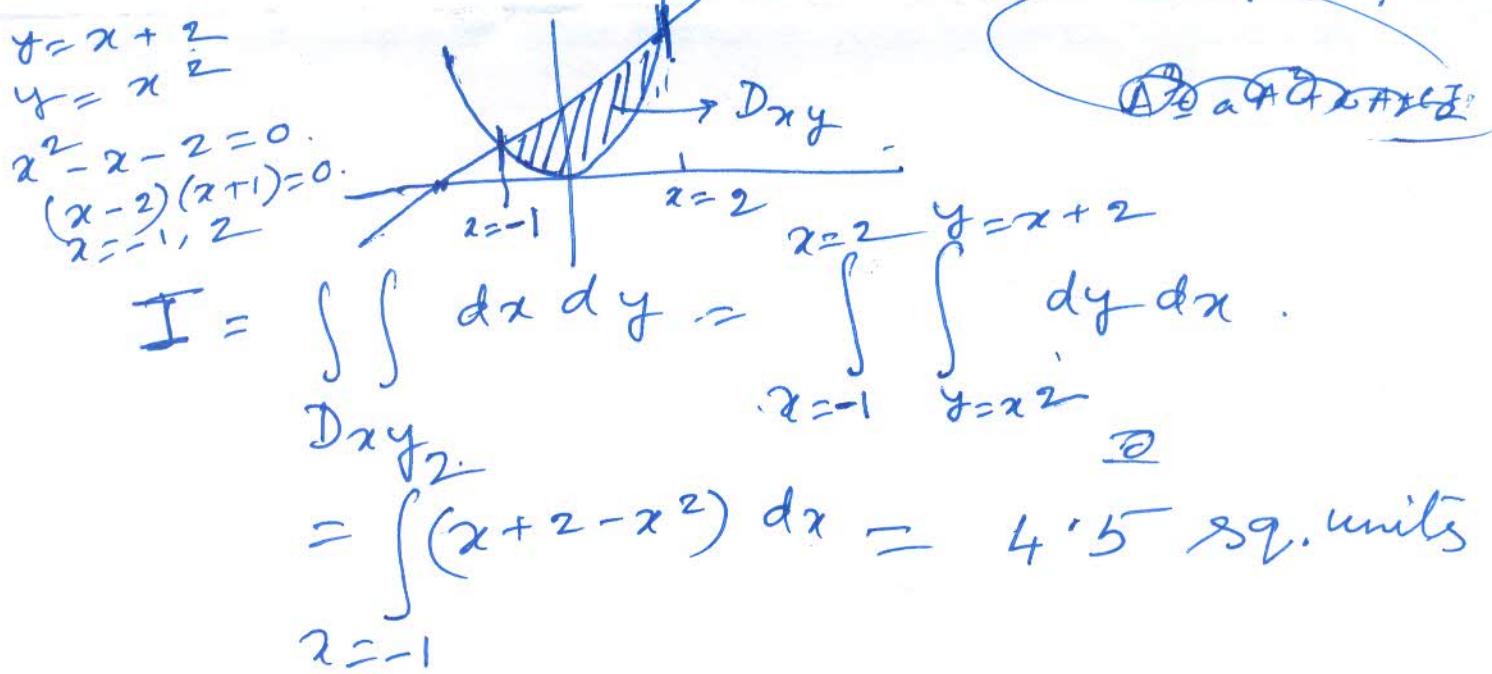
Application of multiple integrals

- ① Evaluation of area . $\iint_{D_{xy}} dx dy = \text{area of } D_{xy}$
- ② Evaluation of volume - $\iiint_R dx dy dz = \text{volume of } R$.
- ③ " " surface area .

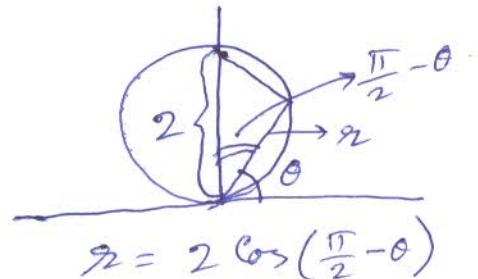
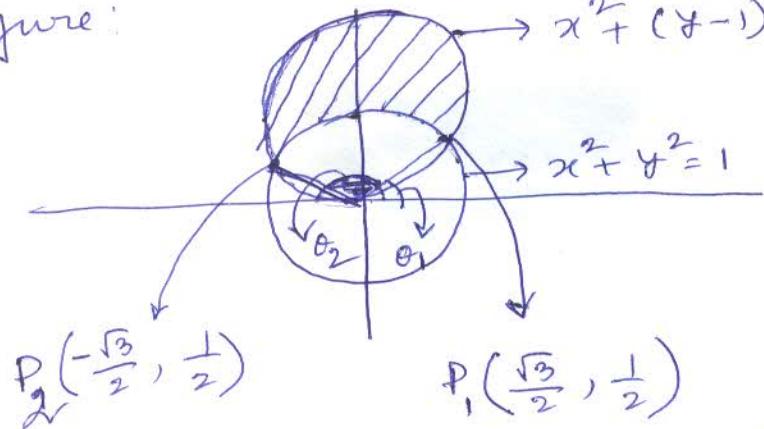
Area -

$$\iint_{D_{xy}} dx dy = \text{area of } D_{xy}$$

- ① (Using double integrals)
 Find the area of the region bounded
 between the line $y = x + 2$ & the parabola



2. Find the area of the portion shown in the figure:



$$\theta_1 \text{ at } P_1 = \tan^{-1} \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{\pi}{6}$$

$$r_2 = 2 \sin \theta$$

$$\theta_2 \text{ at } P_2 = \tan^{-1} \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{5\pi}{6}.$$

$$\iint_{D_{xy}} dx dy = \int_{\theta=0}^{\frac{5\pi}{6}} \int_{r=1}^{2 \sin \theta} r dr d\theta.$$

$$= \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$

Exercise Find the area bounded by
 $y=x$, $y=3x$, $y+x=4$ using the transformation
 $2x=u-v$, $2y=u+v$.

Ans 2 sq. units.