Lecture - 14
Friday
3/3/17

Type III & IV integrals

Type III $\int_{a^+}^{b} f(x)\,dx$. $f(x) \in C(a,b]$, $\lim_{x\to a^+} f(x)$ does not exist

Type IV $\int_{a}^{b^-} f(x)\,dx$, $f(x) \in C[a,b)$; $\lim_{x\to b^-} f(x)$ doesn't exist

Ex 1. $\int_0^1 \frac{dx}{(1+x)\sqrt{x}}$.

$f(x) = \frac{1}{(1+x)\sqrt{x}}$, $g(x) = \frac{1}{\sqrt{x}}$.

$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{1}{1+x} = 1$. $\int_0^1 f\,dx$ & $\int_0^1 g\,dx$ converge/diverge together.

$\int_0^1 \frac{dx}{\sqrt{x}}$. Here $\mu = \frac{1}{2}$.

$\int_0^1 \frac{dx}{x^\mu}$ converges when $\mu < 1$.

So, $\int_0^1 g(x)\,dx$ converges.

Hence $\int_0^1 \frac{dx}{(1+x)\sqrt{x}}$ converges.

Ex-2. $\int_0^1 \frac{\log x}{\sqrt{x}}\,dx$. $f(x) = \frac{\log x}{\sqrt{x}}$ has an infinite discont. at $x=0$.

μ test: $\lim_{x\to 0} x^\mu f(x) = l$.

$\lim_{x\to 0} x^\mu \frac{\log x}{\sqrt{x}} = \lim_{x\to 0} \frac{\log x}{x^{\frac{1}{2}-\mu}}$ $\left(\frac{\infty}{\infty}\right)$ $\frac{1}{2}-\mu < 0$.

We need $\frac{1}{2}-\mu < 0$ so that $\frac{\log x}{x^{\frac{1}{2}-\mu}} \sim \frac{\infty}{\infty}$.

$\mu > \frac{1}{2} = \frac{2}{4}$, $\mu = \frac{3}{4}, \frac{5}{4}$.

$\mu = \frac{3}{4}$.

$$\lim_{x\to 0} x^{3/4} \frac{\log x}{\sqrt{x}} = \lim_{x\to 0} \frac{\log x}{x^{-\frac{1}{4}}} \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{4} x^{-\frac{5}{4}}} = -4 \lim_{x\to 0} \frac{x^{\frac{5}{4}}}{x}$$

$$= 0.$$

$\mu = \frac{5}{4}$.

$$\lim_{x\to 0} x^{5/4} \frac{\log x}{\sqrt{x}}$$

$$= 0$$

Don't show this in exam.

$\therefore$ if $\mu = \frac{3}{4}$, $\lim_{x\to 0} x^{\mu} f(x) = 0$.

$\therefore$ by μ-test, $\because \mu < 1$ & $l = 0$, $\int_0^1 \frac{\log x}{\sqrt{x}}\, dx$ converges.

Ex 3. $\int_1^2 \frac{\sqrt{x}}{\log x}\, dx$. $f = \frac{\sqrt{x}}{\log x}$ has an ∞ discontinuity at $x = 1$.

μ-test. $\lim_{x\to 1} (x-1)^{\mu} \frac{\sqrt{x}}{\log x}$. $\mu > 0$ so that we get $\left(\frac{0}{0}\right)$.

$$= \lim_{x\to 1} \frac{\mu (x-1)^{\mu-1}\sqrt{x} + (x-1)^{\mu} \frac{1}{2\sqrt{x}}}{\frac{1}{x}}.$$

$$= \lim_{x\to 1} \mu (x-1)^{\mu-1} x^{3/2} + \frac{1}{2}(x-1)^{\mu}\sqrt{x}.$$

Case 1 $\mu < 1$, $l = \infty$.

$\boxed{\text{Case 2} \quad \mu = 1 \quad l = 1}$ $\Longrightarrow$ Conclude $\int_1 f\, dx$ diverges

Case 3 $\mu > 1$, 0.

Ex-4. Show that $\displaystyle\int_0^{\pi/2} \frac{dx}{(\cos x)^{1/n}}$ converges for $n>1$.

$$\lim_{x\to\frac{\pi}{2}} \left(\frac{\pi}{2}-x\right)^{\frac{1}{n}} \times \frac{1}{(\cos x)^{1/n}} = \lim_{x\to\frac{\pi}{2}} \left(\frac{\frac{\pi}{2}-x}{\cos x}\right)^{1/n} = 1$$

$\therefore \displaystyle\int_0^{\pi/2} f\,dx$ conveges if $\frac{1}{n} < 1 \Rightarrow n>1$.

Absolute convergence

If $f(x)$ changes sign over $[a,b]$, one cannot apply the tests.

Thm. If $\displaystyle\int_a^b |f(x)|\,dx$ converges, then $\displaystyle\int_a^b f(x)\,dx$ converges.

Ex.1 $\displaystyle\int_0^1 \frac{|\sin\frac{1}{x}|}{\sqrt{x}}\,dx$ converges.

$$\Rightarrow \int_0^1 \frac{\sin\frac{1}{x}}{\sqrt{x}}\,dx \text{ converges.}$$

Ex 2. $\displaystyle\int_0^1 \frac{\sin\frac{1}{x}}{x^{3/2}}\,dx$ converges, whereas,

$$\int_0^1 \frac{|\sin\frac{1}{x}|}{x^{3/2}}\,dx \text{ diverges.}$$

In Ex 1, $\displaystyle\int_0^1 \frac{\sin\frac{1}{x}}{\sqrt{x}}\,dx$ is said to be absolutily convergent.

En Ex 2, $\displaystyle\int_0^1 \frac{\sin\frac{1}{x}}{x^{3/2}}\,dx$ is said to be conditionally convergent.

Leibnitz rule.

I. Simple Leibnitz rule.

Consider $I = \int_c^d f(x,y)\,dy$.

Let 1) $f(x,y)$ be continuous in the rectangle

$R : \{(x,y) ; a \le x \le b,\ c \le y \le d\}$.

2) $f_x(x,y)$ exists in R and is continuous in R.

If $g(x) = \int_c^d f(x,y)\,dy$, then

$$g'(x) = \frac{d}{dx} g(x) = \int_c^d \frac{\partial}{\partial x} f(x,y)\,dy .$$

II Generalized Leibnitz rule.

Consider $I = \int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy$.

Let assumptions 1) & 2) above hold.

3) $\alpha(x), \beta(x)$ are differentiable in $[a,b]$.

If $g(x) = \int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy$, then.

$$g'(x) = \frac{dg(x)}{dx} = \int_{\alpha(x)}^{\beta(x)} \frac{\partial}{\partial x} f(x,y)\,dy + \beta'(x) f(x, \beta(x)) - \alpha'(x) f(x, \alpha(x)).$$

Ex 1 Evaluate $I(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\ln x} dx \quad (\alpha > 0)$. $\longrightarrow (1)$

$$\frac{dI}{d\alpha} = \int_0^1 \frac{\partial}{\partial \alpha}\left(\frac{x^\alpha - 1}{\ln x}\right) dx.$$

$$\frac{d a^x}{dx} = a^x \ln a.$$

$$= \int_0^1 \frac{\partial}{\partial \alpha} \frac{x^\alpha}{\ln x} dx = \int_0^1 \frac{x^\alpha \ln x}{\ln x} dx.$$

$$= \int_0^1 x^\alpha dx = \left.\frac{x^{\alpha+1}}{\alpha+1}\right|_0^1 = \frac{1}{\alpha+1}.$$

$$\frac{dI}{d\alpha} = \frac{1}{\alpha+1}$$

Integrating w.r.to α,

$$I(\alpha) = \int \frac{d\alpha}{\alpha+1} + c = \ln|\alpha+1| + c \longrightarrow (2).$$

From (1), $I(0) = 0$.

From (2), $0 = \ln|0+1| + c \Rightarrow c = 0$.

$$\therefore I(\alpha) = \ln|\alpha+1|.$$

$$\frac{d}{dx}\int_{-1}^{1} \frac{dx}{x} \quad \times$$

Ex-2. $I(\alpha) = \int_0^{\alpha^2} \tan^{-1}\left(\frac{x}{\alpha}\right) dx$.

$$g(x) = \int_{\alpha(x)}^{\beta(x)} f(x,y)\, dy.$$

$$g'(x) = \int_{\alpha(x)}^{\beta(x)} f_x(x,y)\, dy + \beta'(x) f(x,\beta(x)) - \alpha'(x) f(x,\alpha(x)).$$

$$\beta(\cancel{x}) = \cancel{\alpha}\, x^2 \quad \alpha(x) = 0$$

$$\frac{dI}{d\alpha} = \int_0^{\alpha^2} \frac{1}{1+\frac{x^2}{\alpha^2}} \times -\frac{x}{\alpha^2}\, dx + 2\alpha \tan^{-1}\left(\frac{\alpha^2}{\alpha}\right).$$

$$= -\int_0^{\alpha^2} \frac{x\, dx}{x^2+\alpha^2} + 2\alpha \tan^{-1}\alpha$$

$$= -\frac{1}{2} \ln(x^2+\alpha^2) \Big]_0^{\alpha^2} + 2\alpha \tan^{-1}\alpha.$$

$$= -\frac{1}{2}\left[\ln(\alpha^4+\alpha^2) - \ln(\alpha^2)\right] + 2\alpha \tan^{-1}\alpha$$

$$= -\frac{1}{2} \ln(\alpha^2+1) + 2\alpha \tan^{-1}\alpha.$$

$$I(\alpha) = -\frac{1}{2}\int \ln(\alpha^2+1)\, d\alpha + \int \underline{2\alpha} \tan^{-1}\alpha\, d\alpha + C.$$

$$= -\frac{1}{2}\left[\alpha \ln(\alpha^2+1) - \cancel{\int \frac{2\alpha^2}{\alpha^2+1} d\alpha}\right]$$

$$+ \alpha^2 \tan^{-1}\alpha - \cancel{\int \frac{\alpha^2}{1+\alpha^2} d\alpha} + C$$

$$= -\frac{1}{2}\alpha \ln(\alpha^2+1) + \alpha^2 \tan^{-1}\alpha + C.$$

$I(0) = 0 \quad \therefore I(\alpha) = \int_0^{\alpha^2} \tan^{-1}\left(\frac{x}{\alpha}\right) dx.$

$I(0) = -\frac{1}{2} 0 . \ln(0^2+1) + 0^2 \tan^{-1} 0 + c$

$\Rightarrow c = 0.$

$I(\alpha) = \alpha^2 \tan^{-1}\alpha - \frac{1}{2}\alpha \ln(1+\alpha^2).$

1. 9th Thurs → Copy showing 5-7 p.m. Copies will be given 5:10 - 6:30 p.m.

2. Extra classes on 17, 24, 31 (Friday). 3-5 p.m.

3. marks ≤ 15. They <u>must</u> attend extra slots on 21/3, 28/3, 4/4, 11/4, 18/4
   5:15 - 6:15 → sec 1. (Tuesday)
   6:30 - 7:30 → sec 2