Date : ....

NO.1 INSTITUTE FOR IAS/IFOS EXAMINATIONS

IMS™
(INSTITUTE OF MATHEMATICAL SCIENCES)

MATHEMATICS CLASSROOM TEST

2021-22

Under the guidance *of* K. Venkanna

MATHEMATICS

COMPLEX ANALYSIS CLASS TEST

Date: 3 Jan., 2021

Time: 03:00 Hours **Maximum Marks: 250**

INSTRUCTIONS

1. Write your Name & Name of the Test Centre in the appropriate space provided on the right side.
2. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
3. Candidates should attempt All Question.
4. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
5. Symbols/notations carry their usual meanings, unless otherwise indicated.
6. All questions carry equal marks.
7. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
8. All rough work should be done in the space provided and scored out finally.
9. The candidate should respect the instructions given by the invigilator.
10. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any page from this booklet.

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name:

Mobile No.

Test Centre

Email.:

I have read all the instructions and shall abide by them

Signature of the Candidate

_ _

I have verified the information filled by the candidate above

Signature of the invigilator

_ _

P.T.O.

INDEX TABLE

Question	Page No.	Max. Marks	Marks Obtained
1.		10	
2.		10	
3.		10	
4.		12	
5.		12	
6.		16	
7.		10	
8.		15	
9.		10	
10.		10	
11.		10	
12.		16	
13.		16	
14.		15	
15.		15	
16.		15	
17.		20	
18.		10	
19.		18	

Total Marks

1. Prove that the function f defined by

$$f(z)=\begin{cases}\dfrac{z^5}{|z|^4}, z\neq 0\\ 0,\ z=0\end{cases}$$ is not differentiable at z = 0 **[10]**

2. Expand the function $f(z) = \dfrac{2z^2 + 11z}{(z+1)(z+4)}$ in a Laurent's series valid for 2 < z < 3.

[10]

3. If f(z) = u + iv is analytic function and $u - v = e^x (\cos y - \sin y)$, find f(z) in terms of z. **[10]**

4. Using Cauchy's theorem and / or Cauchy's integral formula, calculate the following integrals :

(i) $\int_C \frac{\cosh(\pi z)\,dz}{z(z^2+1)}$, where C is circle $|z| = 2$

(ii) $\int_C \frac{e^{az}\,dz}{(z-\pi i)}$, where C is the ellipse $|z-2| + |z+2| = 6$.

(iii) $\int_C \frac{(\sin z)^2\,dz}{\left(z-\frac{\pi}{6}\right)^3}$, where C is circle $|z| = 1$. **[12]**

5. (A) Show that the function e^{-1/z^2} has no singularities.

(B) Find residue of $f(z) = e^z \operatorname{cosec}^2 z$ at all poles in the finite plane. **[12]**

6. Evaluate $\int_0^{2\pi} \frac{d\theta}{\left(a+b\cos^2\theta\right)^2}$, where a > b > 0. **[16]**

7. Prove that the function f(z) = u + iv, where

$$f(z)=\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}(z\neq 0), f(0)=0$$

is continuous and that Cauchy-Reimann equations are satisfied at the origin, yet f″(z) does not exist there. **[10]**

8. Show that an isolated singular point z_0 of a function f(z) is a pole of order m if and only if f(z) can be written in the form $f(z) = \frac{\phi(z)}{(z - z_0)^m}$ where $\phi(z)$ is analytic and non-zero at z_0.

Moreover $\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$ if $m \geq 1$. **[15]**

9. Obtain the first three terms of Laurent series expansion of the function $f(z)=\dfrac{1}{(e^z-1)}$ about the point z = 0 valid int he region $0 < |z| < 2\pi$. **[10]**

10. Prove that if $b\, e^{a+1} < 1$ where a and b are positive and real, then the function $z^n e^{-a} - b\, e^z$ has n zeroes in the unit circle. **[10]**

HEAD OFFICE:25/8, Old Rajinder Nagar Market, Delhi-60. Ph.9999197625, 011-45629987.
BRANCH OFFICE: 105-106, Top Floor,Mukherjee Tower, Mukherjee Nagar, Delhi-9.
REGIONAL OFFICE : 1-10-237, IInd Floor, Room No. 202 R.K'S Kancham's Blue Sapphire Ashok Nagar Hyderabad-20. Mobile No : 09652351152, 09652661152
www.ims4maths.com || Email: ims4maths@gmail.com

11. Show that the function defined by

$$f(z)=\begin{cases}\dfrac{x^3y^5(x+iy)}{x^6+y^{10}}, & z\neq 0\\ 0, & z=0\end{cases}$$

is not analytic at the origin though it satisfies Cauchy-Riemann equations at the origin. **[10]**

12. If α, β, γ are real numbers such that $\alpha^2 > \beta^2 + \gamma^2$ then show that:

$$\int_0^{2\pi} \frac{d\theta}{\alpha + \beta\cos\theta + \gamma\sin\theta} = \frac{2\pi}{\sqrt{\alpha^2 - \beta^2 - \gamma^2}}.$$ **[16]**

13. Expand $f(z) = \dfrac{1}{(z+1)(z+3)}$ in a Laurent's series valid for the regions

(i) $|z| < 1$ (ii) $1 < |z| < 3$

(iii) $|z| > 3$ (iv) $0 < |z+1| < 2$. **[16]**

14. Using contour integral method, prove that $\int_0^\infty \frac{x \sin mx}{a^2 + x^2} dx = \frac{\pi}{2} e^{-ma}$ **[15]**

15. For a function f : $\mathbb{C} \to \mathbb{C}$ and n ≥ 1, let f(n) denote the nth derivative of f and f(0) = f. Let f be an entire function such that for some $n \geq 1,\ f^{(n)}\left(\frac{1}{k}\right) = 0$ for all k = 1, 2, 3,.......

Show that f is a polynomial. **[15]**

16. Let f = u + iv be an analytic function on the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$. Show that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0=\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}$$ at all points of D. **[15]**

17. (A) Prove that the function $u = e^{-x}(x \cos y + y \sin y)$ is harmonic and find the corresponding analytic function.

(B) Show that the function $v(x, y) = ln\,(x^2 + y^2) + x + y$ is harmonic. Find its conjugate harmonic function $u(x, y)$. Also find the corresponding analytic function $f(z) = u + iv$ in terms of z. **[10+10=20]**

18. (A) Use Cauchy's theorem and/or Cauchy integral formula to evaluate the following integrals.

(i) $\int_{|z|=1} \frac{z+3}{z^4+az^3}dz; (|a|>1)$ (ii) $\int_{|z|=4} \frac{z^4}{(z-i)^3}dz$

(B) If $f(z)=\frac{x^3y(y-ix)}{x^6+y^2}, z \neq 0$ and f(0) = 0, show that $\frac{f(z)-f(0)}{z} \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in any manner. **[10+10=20]**

19. (A) If a function f(z) is analytic for all finite values of z and as $|z| \to \infty$, $|f(z)| = A\left(|z|^k\right)$, then show that f(z) is a polynomial of degree ≤ k.

(B) Determine all entire functions f(z) such that 0 is a removable singularity of $f\left(\frac{1}{z}\right)$. **[18]**

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