Numerical Analysis :

⇒ ERRORS :

1) Significant Digits : any non-zero digit in its decimal repr" or any 0 lying b/w significant digits or used as place holder to indicate a retained place.

eg 0.0005010 : 1st 4 0's are non - SD.
other 2 0's are SD.

0's to right of decimal point & at the same time right of a non-zero digit. 0's at end of non-decimal pt no are non-SD.

In scientific notation $M \times 10^n$, SD's are all the digits explicitly in M.

SD's are counted from L ⟶ R starting with left most non-zero digit.

eg

No.	SD	No of SDS
37.99	3, 7, 9, 9	4
0.00082	8, 2	2
0.000620	6, 2, 0	3
3.506×10^4	3, 5, 0, 6	4
12300 → non SD as no • pt	1, 2, 3	3

2) Rounding Off Numbers

We prefer to keep even digit at end so 0.125 ⟶ 0.12
0.135 ⟶ 0.14

if last digit is < 5 → keep as it is
if last digit is > 5 ⟶ increment by 1.

Round-off to 4 SD's.

No.	
9.6782	9.678
29.1568	29.16
8.24159	8.242
30.0567	30.06

3) ERROR :

Error = True value − approximate value

$|\text{error}| = |TV - AV|$

$|\text{Relative error}| = \left|\frac{TV - AV}{TV}\right| = \left|\frac{\text{error}}{\text{True value}}\right|$

% error = 100 × relative error

Inherent error : already present in statement before solⁿ

Round off error : bcoz of rounding off

Truncation error : we just discard some of the terms in an infinite sum eg Taylor expansion

4) Floating Point Representation :

$x = \pm(\cdot d_1 d_2 \dots d_n)10^m$ $d_1, d_2 \dots \in \mathbb{N}$ between 0 to 9.

If $d_1 \neq 0$, it is Normalised representation.

Given the precision 'n', we want the representation

		Chopped FP	Rounded FP
fl(537)	n=1	0.5×10^3	0.5×10^3
	n=2	0.53×10^3	0.54×10^3
	n=3	0.537×10^3	0.537×10^3

Round-off error reduces when n ↑es.

* Let x be a real number & x^* be approximation to x.

Then x^* is accurate to k decimal places if

$$\frac{1}{2}\times 10^{-(k+1)} \leq |x - x^*| \leq \frac{1}{2}\times 10^{-k}$$

eg π & $\frac{22}{7}$ $\pi = 3.14159265$ $\frac{22}{7} = 3.14285714$

$\therefore |\pi - \frac{22}{7}| = 0.00126499$ ⇒ $0.0005 < 0.00126 < 0.005$

$\frac{1}{2}\times 10^{-3} <$ error $< \frac{1}{2}\times 10^{-2}$

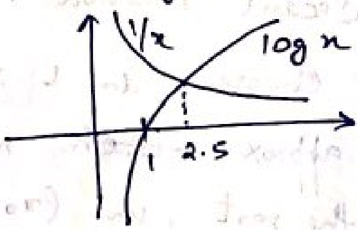
∴ it's accurate upto 2 decimal places.

⇒ Solution of Algebraic & Transcendental Equations :

1) pure polynomial equation ⟶ Algebraic
having other functions such as trig, log, exp ⟶ Transcendental

2) Two methods to solve equations :
Direct : No initial approx needed
Iterative : Start with initial approx & iterate.
↓
use graphical method to get this.

If it's hard to draw $f(x)$, split it as $\phi(x) = \psi(x)$ which are easier to plot, their pt of intersection gives initial solⁿ. eg $x \log x = 1 \Rightarrow \log x = \frac{1}{x}$



3) BISECTION METHOD :

i) Find $x_1 = a, x_2 = b$ st $f(x_1) \cdot f(x_2) < 0$.
ii) Approximate root as $x_0 = \frac{a+b}{2}$.
iii) Check $f(x_0)$ & based on sign choose next interval

stop when iteration doesn't change the reqd no. of SDS

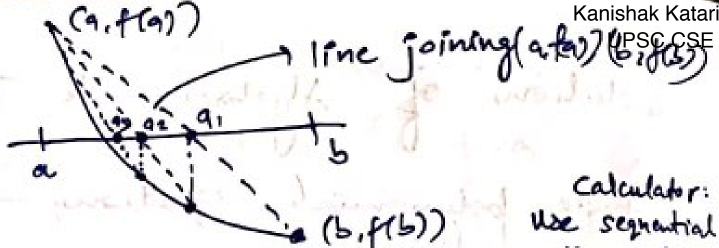
Example : Solve $x^3 - 9x + 1$ for the root lying b/w 2 & 3. correct to 3 significant digits.

$f(x) = x^3 - 9x + 1$ $\quad f(2) = -9$ $\quad f(3) = 1$ $\quad \therefore f(2) \cdot f(3) < 0$

n	a_n (-ve)	b_n (+ve)	$x_{n+1} = \frac{a_n + b_n}{2}$	$f(x_{n+1})$
0	2	3	2.5	−5.875 (<0)
1	2.5	3	2.75	−2.95313 (<0)
2	2.75	3	2.875	−1.1113 (<0)
3	2.875	3	2.9375	−0.09003 (<0)
4	2.9375	3	2.96875	0.44626 (>0)
5	2.9375	2.96875	2.953	0.17552 (>0)
6	2.9375	2.953	2.9455	0.04557 (>0)
7	2.938	2.9455	2.9418	−0.01816 (<0)
8	2.9418	2.9455	2.9436	0.014 (>0)

we stop as all 3 a_n, b_n, x_{n+1} are equal upto 2 SDs.

4) Regula Falsi Method :-



(i) Let x_0, x_1 be nos such that $f(x_0) \cdot f(x_1) < 0$

then $x_2 = \dfrac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$

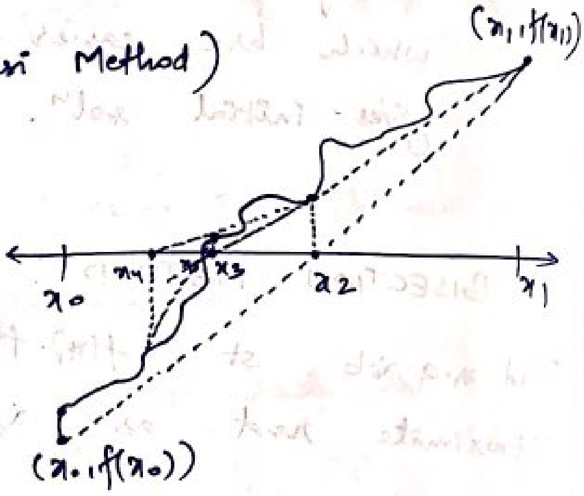
Calculator: Use sequential equations using $C = f(A) : D = f(B) : X = \dfrac{AD-BC}{D-C}$ & press CALC. $f(x)$

(ii) If $f(x_0) f(x_2) < 0$; choose $[x_0\ x_2]$ as next interval ~~else~~ else choose $[x_2, x_1]$

(iii) Stop iterations if interval is of sufficiently small length or difference b/w 2 approximations is negligible.

NOTE: Use calculator to get the ~~interva~~ solution first & then decide the interval.

5) Secant Method : (Modified Regula-Falsi Method)



(i) Choose x_0 & x_1 as any 2 approx ~~which~~ need not contain the root in (x_0, x_1).

(ii) Find $x_{n+1} = \dfrac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$

In answer write as :

x_i	$f(x_i)$	$x_{i+1} = \dfrac{x_{i-1}(f(x_i)) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$

TIP : Calculator : $C = f(A) : D = f(B) : X = \dfrac{AD - BC}{D - C} : A = B : B = X$

Press CALC & input initial A & B.

Then keep noting X & press C=' sequentially.

(iii) Example : Compare Regula-Falsi & Secant method for $x^3 + x^2 - 3x - 3 = 0$ using $x_0 = 1$ & $x_1 = 2$ rounded off to 3 decimal places

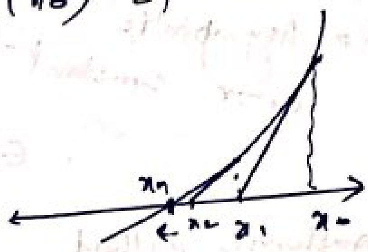
For regula-falsi $x_5 = 1.73140$ $x_6 = 1.73194$ we say 1.731 is soln since upto 3 sig digits

For Secant method $x_5 = 1.73199$ ∴ $x_6 = 1.73205$ we say $\underline{1.732}$ is solution

errors in secant = $|x_6 - x_5| = 0.00006$ < error in regula-falsi = 0.00011

5) Newton - Raphson Method :

Taylor expansion : $f(x_0+h) = f(x_0) + h f'(x_0)$ [for small h]



If $f(x_0+h) = 0$ then $h = -\frac{f(x_0)}{f'(x_0)}$

$\therefore$ $x_1 = x_0 + h \simeq x_0 - \frac{f(x_0)}{f'(x_0)}$

CALCULATOR : solve using calc. & then choose : Input x_0 & press ~~ANS~~ =,

Put $ANS - \frac{f(ANS)}{f'(ANS)}$ form

we it to find $\sqrt{N}$ by considering $x^2 - N = 0$

$$x_{n+1} = x_n - \frac{x_n^2 - N}{2x_n} = \frac{x_n}{2} + \frac{N}{2x_n} = \frac{1}{2}\left[x_n + \frac{N}{x_n}\right]$$

to find $\frac{1}{N}$. consider $f(x) = \frac{1}{x} - N \Rightarrow x_{n+1} = x_n(2 - x_n N)$

to find $\frac{1}{\sqrt{N}}$ consider $f(x) = x^2 - \frac{1}{N} = 0 \Rightarrow x_{n+1} = \frac{1}{2}\left[x_n + \frac{1}{N x_n}\right]$

to find $\sqrt[K]{N}$ consider $f(x) = x^K - N = 0 \Rightarrow x_{n+1} = \frac{1}{K}\left[(K-1)x_n + \frac{N}{x_n^{K-1}}\right]$

This method fails if $f'(x_i) = 0$

7) Convergence Criterion :

let actual root be 'α' & approximations $x_0, x_1, \ldots, x_K, \ldots$

$|\epsilon_{n+1}| = |x_{n+1} - \alpha|$ $|\epsilon_n| = |x_n - \alpha|$

$\boxed{\text{Order of convergence} = p \text{ if } |\epsilon_{n+1}| \leq \lambda |\epsilon_n|^p}$ for some $\lambda > 0$

Larger $p \Rightarrow$ iterations converge rapidly

for $p = 1$ we require $\lambda < 1$ $\longrightarrow$ linear convergence

Bisection Method : we go from $[a_0, b_0] \longrightarrow [a_1, b_1] \longrightarrow [a_2, b_2] \ldots \rightarrow [a_n, b_n]$

$\therefore$ $b_1 - a_1 = (b_0 - a_0)/2$ $b_2 - a_2 = (b_0 - a_0)/2^2$ ... $(b_n - a_n) = (b_0 - a_0)/2^N$

We want no of iterations to have error $< \epsilon$

We want N st $\frac{b_0 - a_0}{2^N} < \epsilon$ $\therefore N = \text{int}\left[\frac{\log(b_0 - a_0) - 10^{-M} \text{ or } (\epsilon)}{\log 2}\right]$

So for 10^{-5} error we need 17 iterations in $[0,1]$!!

So bisection is slow method

Secant Method : order of convergence $\boxed{p = \frac{1+\sqrt{5}}{2}} = 1.618$

λ = Asymptotic error constant $\quad \lambda = \left[\frac{f''(\alpha)}{2f'(\alpha)}\right]^{p/1+p} = \left[\frac{f''(\alpha)}{2f'(\alpha)}\right]^{0.618}$

$$\epsilon_{n+1} = \lambda \epsilon_n^p$$

Newton Raphson Method : order of convergence

$$\epsilon_{i+1} = \frac{f''(\alpha)}{2f'(\alpha)} \epsilon_i^2$$

But if α is ~~linear~~ double root it becomes linear

eg. $f(x) = (x-2)^4 \quad x_0 = 2.1$, compute x_1, x_2, x_3, x_4 & check order of convergence :

$x_0 = 2.1 \quad \epsilon_0 = 0.1$ $\quad \times 3/4$
$x_1 = 2.075 \quad \epsilon_1 = 0.075$ $\quad \times 3/4$
$x_2 = 2.05625 \quad \epsilon_2 = 0.05625$ $\quad \times 3/4$
$x_3 = 2.0422 \quad \epsilon_3 = 0.0422$ $\quad \times 3/4$
$x_4 = 2.0316 \quad \epsilon_4 = 0.0316$

$\lambda = 3/4$ & linear convergence

Derivation : $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$ $\quad x_{i+1} - \alpha = \epsilon_{i+1} \Rightarrow x_{i+1} = \epsilon_{i+1} + \alpha$ $\quad x_i - \alpha = \epsilon_i \quad x_i = \epsilon_i + \alpha$

$$\Rightarrow \epsilon_{i+1} = \epsilon_i - \frac{f(\epsilon_i + \alpha)}{f'(\epsilon_i + \alpha)} = \frac{\epsilon_i f'(\epsilon_i + \alpha) - f(\epsilon_i + \alpha)}{f'(\epsilon_i + \alpha)}$$

Expand $f(\epsilon_i + \alpha) = f(\alpha) + f'(\alpha)\epsilon_i + f''(\alpha)\frac{\epsilon_i^2}{2} + f'''(\alpha)\frac{\epsilon_i^3}{3!}$ $\quad (f(\alpha) = 0)$

$$\Rightarrow \epsilon_{i+1} = \frac{\epsilon_i\left[f'(\alpha) + \epsilon_i f''(\alpha) + \frac{\epsilon_i^2}{2} f'''(\alpha) + \ldots\right] - \left[f(\alpha) + \epsilon_i f'(\alpha) + \ldots\right]}{f'(\alpha) + \epsilon_i f''(\alpha) + \ldots}$$

$f(\alpha) = 0$ & $f'(\alpha) \neq 0$

$$\therefore \epsilon_{i+1} = \frac{\epsilon_i^2}{2}\left[f''(\alpha) + \ldots\right] \Big/ f'(\alpha)\left[1 + \epsilon_i \frac{f''(\alpha)}{f'(\alpha)} + \ldots\right]$$

$$= \frac{\epsilon_i^2\left[f''(\alpha) + \ldots\right]}{2f'(\alpha)}\left[1 - \epsilon_i \frac{f''(\alpha)}{f'(\alpha)} + \ldots\right]$$ Neglect ϵ_i^3 & above.

we get $\epsilon_{i+1} = \frac{f''(\alpha)}{2f'(\alpha)} \epsilon_i^2$

Similarly approach for secant method.

There assume $\epsilon_{i+1} = \lambda \epsilon_i^p$ & put in the relations you get & on RHS & LHS compare powers of ϵ_i you'll get $p = 1 + \frac{1}{p}$.

⇒ System of linear equations :

1) $AX = B$ $\quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$ $\quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$ $\quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$

Restrict $A \to n\times n$
& non-homogenous system

2) DIRECT METHODS :

(i) A is $\text{diag}(a_{11}, a_{22}, \ldots, a_{nn})$ then $x_i = \dfrac{b_i}{a_{ii}}$ $\quad i = 1, 2, 3 \ldots n$

(ii) Forward substitution $\quad A = L \quad (a_{ij} = 0,\ j > i)$

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{n1} & a_{n2} & \cdots & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \qquad x_1 = \frac{b_1}{a_{11}} \qquad x_2 = \frac{b_2 - a_{21}x_1}{a_{22}}$$

$$x_i = \frac{\left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j\right)}{a_{ii}}$$

(iii) Backward substitution $\quad A = U \quad (a_{ij} = 0,\ j < i)$

$$x_i = \frac{b_i - \sum_{j=i+1}^{n} a_{ij} x_j}{a_{ii}} \qquad i = 1, 2, 3 \ldots n$$

(iv) Gaussian Elimination Method :

Reduce to upper Δ form & apply back substitution.

No need to make pivot = 1 if it is cumbersome.

3 cases (i) $r = n$ unique solⁿ

(ii) $r < n$, $b_{r+1}^{(r-1)} \ldots b_n^{(r-1)}$ all zero ⇒ ∞ solⁿ

(iii) $r < n$ & Not all zero ⇒ no solution

r = stage at which it becomes upper Δ. It is $\rho(A)$

Write in augmented $[A \,\vdots\, B]$ form & do.

Gaussian elimination with pivoting :

If any of the pivot elements vanishes or becomes too small as compared to other elements in its row, then we apply pivoting

(i) Partial Pivoting : at each stage bring the row having highest elt in the first column to the top.

(ii) Complete Pivoting : Search for highest at any posⁿ & make it the first pivot even if change of variable is needed.
It is generally not used.

Example :
$$x_1 + x_2 + x_3 = 6$$
$$3x_1 + 3x_2 + 4x_3 = 20$$
$$2x_1 + x_2 + 3x_3 = 13$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 3 & 3 & 4 & 20 \\ 2 & 1 & 3 & 13 \end{bmatrix} \sim \begin{bmatrix} 3 & 3 & 4 & 20 \\ 1 & 1 & 1 & 6 \\ 2 & 1 & 3 & 13 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 3 & 4 & 20 \\ 0 & 0 & -1/3 & -2/3 \\ 0 & -1 & 1/3 & -1/3 \end{bmatrix} \sim \begin{bmatrix} 3 & 3 & 4 & 20 \\ 0 & -1 & 1/3 & -1/3 \\ 0 & 0 & -1/3 & -2/3 \end{bmatrix} \sim \begin{bmatrix} 3 & 3 & 4 & 20 \\ 0 & +1 & -1/3 & +1/3 \\ 0 & 0 & 1/3 & 2/3 \end{bmatrix}$$

$$\therefore x_3 = 2 \quad x_2 = 1 \quad x_1 = 3.$$

It's handy when coeff. are very small. Since multiplication with large no in case of very small pivot has big chances of errors.

eg
$$0.0003x_1 + 1.566x_2 = 1.569$$
$$0.3454x_1 - 0.436x_2 = 3.018$$
$$\sim \left[\begin{array}{cc|c} 0.0003 & 1.566 & 1.569 \\ 0.3454 & -0.436 & 3.018 \end{array}\right]$$

w/o pivot :
$$\begin{bmatrix} 0.0003 & 1.566 & 1.569 \\ 0 & -1802.0 & -1803.0 \end{bmatrix}$$
$$x_2 = 1.001$$
$$x_1 = 3.333$$

With pivot
$$\left[\begin{array}{cc|c} 0.3454 & -0.436 & 3.018 \\ 0.0003 & 1.566 & 1.569 \end{array}\right]$$
$$\sim \begin{bmatrix} 0.3454 & -0.436 & 3.018 \\ 0 & 1.566 & 1.566 \end{bmatrix}$$
$$\Rightarrow x_2 = 1 \;\&\; x_1 = 10$$

vi) Gauss - Jordan Elimination Method :

Using partial pivoting, convert A to a diagonal matrix we get $[A|B] \longrightarrow [I|D]$ so $x_i = d_i \quad i = 1, 2, 3 \ldots$
It takes more time than Gauss-elimination
It is also used to find A^{-1} $\quad [A|I] \xrightarrow[\text{Jordan}]{\text{Gauss}} [I|A^{-1}]$

(VII) Gauss ~~Jordan~~ Elimination method to find A^{-1}:

Augmented matrix $[A|I] \longrightarrow$ convert A to upper Δ

$[I|A^{-1}] \longleftarrow$ row operations to convert to I.

Gauss Jordan: No stage I, directly apply row operations to convert to I.

3) INDIRECT METHODS:

(i) GAUSS - SIEDEL ITERATION METHOD: (too many 0s)

generally used when matrix A is sparse or of very large order

In Normal iteration: $x_1 = \frac{-1}{a_{11}}(a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n) + \frac{b_1}{a_{11}}$

$\vdots$

$x_i = \frac{-1}{a_{ii}}(a_{i2}x_2 + a_{i3}x_3 + \cdots + a_{in}x_n) + \frac{b_i}{a_{ii}}$

$\therefore X = HX + C$ where $H = \begin{bmatrix} 0 & -a_{12}/a_{11} & -a_{13}/a_{11} & \cdots & -a_{1n}/a_{11} \\ -a_{21}/a_{22} & 0 & -a_{23}/a_{22} & \cdots & -a_{2n}/a_{22} \\ \vdots & & & & \\ -a_{n1}/a_{nn} & -a_{n2}/a_{nn} & \cdots & & 0 \end{bmatrix}$

& $C = \begin{bmatrix} b_1/a_{11} \\ b_2/a_{22} \\ \vdots \\ b_n/a_{nn} \end{bmatrix}$

$\therefore X^{(k+1)} = HX^{(k)} + C.$ $k = 0, 1, 2, \ldots$

In Gauss-siedel, we make use of an improved values as soon as it is available.

TIP: calculator: $A = \frac{(b_1 - B - C)}{a_{11}}$: $B = \frac{(b_2 - A - C)}{a_{22}}$: $C = \frac{(\ )}{(\ )}$
Press CALC & input B^0 & C^0.

Example: $-8x_1 + x_2 + x_3 = 1$
$x_1 - 5x_2 + x_3 = 16$
$x_1 + x_2 - 4x_3 = 7$

$\rightarrow x_1^{(k+1)} = \frac{-1}{8}(1 - x_2^{(k)} - x_3^{(k)})$

$x_2^{(k+1)} = \frac{-1}{5}(16 - x_1^{(k+1)} - x_3^{(k)})$

$x_3^{(k+1)} = -\frac{1}{4}(7 - x_1^{(k+1)} - x_2^{(k+1)})$

Take $X^{(0)} = [0, 0, 0]$

$X^{(1)} = [-0.1250, -3.2250, -2.5875]$

$X^{(2)} = [-0.8516, -3.8878, -2.9348]$

$X^{(3)} = [-0.9778, -3.9825, -2.9801]$

$X^{(4)} = [-0.9966, -3.9973, -2.9985]$

Very important: The iterations will not converge in some cases if diagonal elements are not heavy. So better to rearrange the ~~di~~ rows to make diagonals heavy & then start Gauss-siedel:

eg. $4x - y + 8z = 26$
$5x + 2y - z = 6$
$x - 10y + 2z = -13$

Rearrange as
$5x + 2y - z = 6$
$x - 10y + 2z = -13$
$4x - y + 8z = 26$

$$x^{k+1} = \frac{1}{5}\left[6 - 2y^k + z^k\right]$$
$$y^{k+1} = \frac{-1}{10}\left[-13 - 2z^k - x^{k+1}\right]$$
$$z^{k+1} = \frac{1}{8}\left[26 - 4x^{k+1} + y^{k+1}\right]$$

This will converge quickly o/w in given state it wont.

$X^0 = (0, 0, 0)$
$X^1 = (1.2, 1.42, 2.8275)$
$X^2 = (1.19, 1.98, 2.8994)$
$X^3 = (0.98, 1.97, 3.004)$
$X^4 = (1.009, 2.001, 2.995)$

so $x = 1$ $y = 2$ $z = 3$

(ITERATION METHOD)

eg Find root for $e^{-x} = 10x$ using method of iteration

$f(x) = e^{-x} - 10x$ $\quad f(0) = 1$ $\quad f(-1) = -9.86321$

$f(0) < f(1) \Rightarrow$ root is near 0

Rewrite $x = \frac{1}{10}e^{-x} = \phi(x)$ $\quad \phi'(x) = -\frac{e^{-x}}{10}$ $\quad |\phi'(x)| = \frac{1}{10e^x} < 1$

Take $x_0 = 0 \Rightarrow$ $x_1 = \phi(x_0) = 0.1$ $\quad f(x_1) = -0.09516$

$x_2 = \phi(x_1) = 0.09048$ $\quad f(x_2) = 0.00869$

$x_3 = \phi(x_2) = --$

we get $x_4 = 0.091274$ $\quad f(x_4) = 2.75787 \times 10^{-5}$

$\Downarrow$
root.

⇒ Interpolation :

1) $\phi(x)$ is interpolating polynomial for $f(x)$ st that
$$\phi(x_j) = f_j \quad j = 0, 1, 2, \ldots, n$$

2) Forward Difference : $\Delta y_{n-1} = y_n - y_{n-1}$
$$\Delta^2 y_{n-1} = \Delta y_n - \Delta y_{n-1}$$
$$\Delta^r y_{n-1} = \Delta^{r-1} y_n - \Delta^{r-1} y_{n-1}$$

x	y	Δy	Δ²y	Δ³y	Δ⁴y	Δ⁵y
0.1	0.003					
		0.064				
0.3	0.067		0.017			
		0.081		0.002		
0.5	0.148		0.019		0.001	
		0.100		0.003		0.000
0.7	0.248		0.022		0.001	
		0.122		0.004		
0.9	0.370		0.026			
		0.148				
1.1	0.518					

NOTE : 2 keep x_is equally spaced

3) Backward Difference : $\nabla^k y_n = \nabla^{k-1} y_n - \nabla^{k-1} y_{n-1}$
$$\nabla y_n = y_n - y_{n-1}$$
$$\nabla^2 y_n = \nabla y_n - \nabla y_{n-1}$$

4) Central Difference : $\delta y_n = y_{n+\frac{1}{2}} - y_{n-\frac{1}{2}}$
$$\delta^k y_n = \delta^{k-1} y_{n-\frac{1}{2}} - \delta^{k-1} y_{n+\frac{1}{2}}$$

Also $\Delta y_x = y_{x+h} - y_x = f(x+h) - f(x)$ $\quad \nabla y_x = y_x - y_{x-h} = f(x) - f(x-h)$

Also $y_1 - y_0 = \Delta y_0 = \nabla y_1 = \delta y_{1/2}$ ⇒ same nos appear in the tables

5) Shift Operator (E) : $E f(x) = f(x+h)$ $\quad E^n f(x) = f(x+nh)$
$$E^{-1} f(x) = f(x-h)$$

Average operator (μ) : $\mu f(x) = \frac{1}{2}\left[f(x+\frac{h}{2}) + f(x-\frac{h}{2})\right] = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right]$

Differential operator (D) : $D f(x) = \frac{d}{dx} f(x) = f'(x)$

Now $\Delta y_x = y_{x+h} - y_x = E y_x - y_x$ ⇒ $\boxed{\Delta = E - 1 \quad \nabla = 1 - E^{-1} \quad \delta = E^{1/2} - E^{-1/2} \quad hD = \log E}$

6) Differences of a polynomial:

n^{th} differences of a polynomial of the n^{th} degree are constant. Converse is not true.

7) Effect of an error on a difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$
x_1	y_1	Δy_1	$\Delta^2 y_1$	$\Delta^3 y_1$	$\Delta^4 y_1 + \epsilon$
x_2	y_2	Δy_2	$\Delta^2 y_2$	$\Delta^3 y_2 + \epsilon$	$\Delta^4 y_2 - 4\epsilon$
x_3	y_3	Δy_3	$\Delta^2 y_3 + \epsilon$	$\Delta^3 y_3 - 3\epsilon$	$\Delta^4 y_3 + 6\epsilon$
x_4	y_4	$\Delta y_4 + \epsilon$	$\Delta^2 y_4 - 2\epsilon$	$\Delta^3 y_4 + 3\epsilon$	$\Delta^4 y_4 - 4\epsilon$
x_5	$y_5 + \epsilon$	$\Delta y_5 - \epsilon$	$\Delta^2 y_5 + \epsilon$	$\Delta^3 y_5 - \epsilon$	$\Delta^4 y_5 + \epsilon$
x_6	y_6	Δy_6	$\Delta^2 y_6$	$\Delta^3 y_6$	
x_7	y_7	Δy_7	$\Delta^2 y_7$		
x_8	y_8	Δy_8			
x_9	y_9				

(i) error ↑es with order of difference

(ii) coeff of ϵ follow binomial expansion coeff of $(1-\epsilon)^n$

(iii) Σ ~~sum~~ errors $= 0$ in any column

(iv) Max error in each column is opposite to the entry containing initial error.

eg. Following is a table of values of a polynomial of degree 5. It is given that $f(3)$ has an error. Correct it

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0		1	30	$160+\epsilon$	$200-4\epsilon$	$220+10\epsilon$
1	2	31	$190+\epsilon$	$360-3\epsilon$	$420+6\epsilon$	$20-10\epsilon$
2	33	$221+\epsilon$	$550-2\epsilon$	$780+3\epsilon$	$440-4\epsilon$	
3	$254+\epsilon$	$771-\epsilon$	$1330+\epsilon$	$1220-\epsilon$		
4	1025	2101	2550			
5	3126	4651				
6	7777					

Now $\Delta^5 y$ is constant as it is polynomial of degree 5

$\therefore 220 + 10\epsilon = 20 - 10\epsilon$

$\Rightarrow \epsilon = -10$

$\therefore$ correct $f(3)$ is $254 - 10$
$= 244$

Similarly end points can be extended if the polynomial degree is given

8) Newton's Binomial Expansion Formula :-

Let $y_0, y_1, y_2, \ldots y_n$ be values corresponding to $x_0, x_0+h, \ldots x_0+nh$ & let one of y's is missing. Since n values are known.

$$\Delta^n y_0 = 0 \Rightarrow (E-1)^n y_0 = 0$$

$$\Rightarrow \left[E^n - {}^nC_1 E^{n-1} + \ldots + (-1)^n\right] y_0 = 0$$

$$\Rightarrow E^n y_0 - nE^{n-1} y_0 + \ldots + (-1)^n y_0 = 0$$

$$y_n - n y_{n-1} + \frac{n(n-1)}{2} y_{n-2} + \ldots + (-1)^n y_0 = 0$$

OR go by table directly

eg. If we know values for m equally spaced x_i, then we can fit polynomial of degree $(m-1)$ & hence $\Delta^m y = 0$

Eg Find missing term :

x	0	1	2	3	4	5
$y=f(x)$	0	—	8	15	—	35

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	0	x	$8-2x$	$3x-9$	$y-4x-12$
1	x	$8-x$	$x-1$	$y-x-21$	$93-4y+x$
2	8	7	$y-22$	$72-3y$	
3	15	$y-15$	$50-2y$		
4	y	$35-y$			
5	35				

4 values are given so let $f(x) = \deg(3)$

$\therefore \Delta^4 y = 0$

$y - 4x - 12 = 0$

$x - 4y + 93 = 0$

$x = +3 \quad y = 24$

9) Newton's Forward Interpolation Formula :-

Let $y_0, y_1, y_2, \ldots, y_n$ be values for $x_0, x_1, \ldots, x_n$ equally spaced by 'h'. Let $\phi(x)$ be the n^{th} degree polynomial st $f(x_r) = \phi(x_r)$ $r=0$ to n & for other $f(x) = \phi(x) + \underbrace{R(x)}_{\text{error}}$

$$\therefore f(x) \approx \phi(x) \approx a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \ldots + a_n(x-x_0)(x-x_1) \ldots (x-x_n) \quad (1)$$

we get $a_0 = y_0 \quad a_1 = \frac{\Delta y_0}{h} \quad a_2 = \frac{\Delta^2 y_0}{2!h^2} \ldots \quad a_n = \frac{\Delta^n y_0}{n!h^n}$

Now to get ~~ge~~ $f(x)$ put $u = (x-x_0)/h$ in (1) then

$$\phi(x) = y_0 + u \cdot \frac{\Delta y_0}{1!} + \frac{u(u-1)}{2!}\Delta^2 y_0 + \ldots + \frac{u(u-1)\ldots(u-n+1)}{n!}\Delta^n y_0$$

Note: (a) It's useful for beginning of interval, ie, close to x_0.

(b) If we have $f(x)$ for $x = 1, 2, 3, 4, 5, 6$ & want $f(x)$ for 3.1, better ignore $x = 1$ & 2 & start using $y_0 = f(3)$ ...

TIP: Put $u = A$ in calci & then press CALC & input value for u saves time

Example: Interval wise marks distribution is given like [10 - 20], [20 - 30], [30 - 40] ...
Find students in [15 - 20]

In such questions create the cumulative table & ~~then~~ such that $f(x)$ = no of students below 'x'.
Then for [15 - 20] do $f(20) - f(15)$ → use ~~lagrang~~ newton's method
directly from table

10) Newton - Gregory Backward interpolation formula:

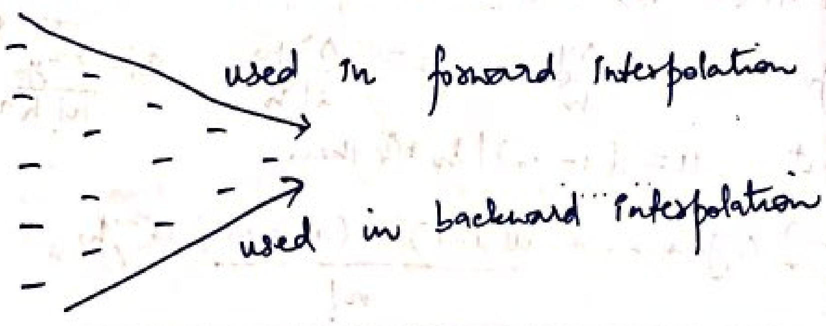
Used to find values near the end of the table.

Here $f(x) \simeq \phi(x) \approx a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + \dots + a_n(x - x_n)(x - x_{n-1}) \dots (x - x_1)$

$\therefore \quad a_0 = y_n \qquad a_1 = \dfrac{\nabla y_n}{1!\, h} \qquad a_2 = \dfrac{\nabla^2 y_n}{2!\, h^2} \quad \dots \quad a_n = \dfrac{\nabla^n y_n}{n!\, h^n}$

take $u = \dfrac{x - x_n}{h}$

$$\therefore f(x) = y_n + \frac{u \nabla y_n}{1!\cancel{h}} + \frac{u(u+1)\nabla^2 y_n}{2!} + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \dots + \frac{u(u+1)(u+2)\dots(u+n-1)\nabla^n y_n}{n!}$$



Calci = Input formula

$y_n + \dfrac{\nabla y_n A}{1!} + \dfrac{\nabla^2 y_n A(A+1)}{2!} + \dfrac{\nabla^3 y_n A(A+1)(A+2)}{3!}$

Press CALC & input $A = u$

Remember this formula

11) Error in interpolation formula :

Forward Interpolation $R(x) = \frac{\Delta^{n+1} y_0}{(n+1)!} u(u-1)(u-2)\ldots(u-n)$ $u = \frac{x - x_0}{h}$

Backward interpolation $R(x) = \frac{\nabla^{n+1} y_n}{(n+1)!} u(u+1)(u+2)\ldots(u+n)$ $u = \frac{x - x_n}{h}$.

Just take the next term we could have written for $f(x)$ expansion

12) Lagrange's Interpolation formula :

Don't need equally spaced x_i's. Unequal intervals also work

Let $x_0, x_1, \ldots, x_n$ be $(n+1)$ distinct points on the real line & let $f(x)$ be a real valued function in the interval containing these points. Then $\exists$ exists. EXACTLY ONE polynomial $P_n(x)$ of degree $\leq n$ st $P_n(x_i) = f_i$ .. $i = 0, 1, 2, 3, \ldots n$

Proof: Let $P_n(x)$ & $Q_n(x)$ be 2 such polynomials, then

$H_n(x) = P_n(x) - Q_n(x)$ is of degree $\leq n$ having $(n+1)$ roots

$\Rightarrow H_n(x) \equiv 0 \Rightarrow P_n(x) = Q_n(x)$

we get $f(x)$ as :-

$$f(x) = \sum_{i=0}^{n} l_i(x) f_i \quad \text{where} \quad L_i(x) = \frac{\Pi_{n+1}(x)}{(x - x_i)\Pi'_{n+1}(x_i)}$$

$$= \frac{(x - x_0)(x - x_1)\ldots(x - x_{i-1})(x - x_{i+1})\ldots(x - x_n)}{(x_i - x_0)(x_i - x_1)\ldots(x_i - x_{i-1})(x_i - x_{i+1})\ldots(x_i - x_n)}$$

$l_i(x)$ are Lagrange interpolation coefficients

$\Sigma l_i(x) = 1$ & $l_i(x_j) = \left\{ \begin{matrix} 0 & i \neq j \\ 1 & i = j \end{matrix} \right\}$.

13) Inverse interpolation formula : [given $\bar{y}$, find $\bar{x}$ st $f(\bar{x}) = \bar{y}$]

$$g(y) = f^{-1}(y) = \sum_{i=1}^{n} \frac{\Pi_{n+1}(y)}{(y - y_i)\Pi'_{n+1}(y_i)} x_i$$

14> Partial fractions using Lagrange Interpolation:

eg. Express $\frac{3x^2+x+1}{(x-1)(x-2)(x-3)}$ as Partial fraction

Calculate $f(x) = 3x^2+x+1$ at 1, 2, 3

x	1	2	3
$f(x)$	5	15	31

$$\therefore f(x) = 3x^2+x+1 = \frac{(x-2)(x-3)}{(1-2)(1-3)}5 + \frac{(x-1)(x-3)}{(2-1)(2-3)}15 + \frac{(x-1)(x-2)}{(3-1)(3-2)}\times 31$$

$$= \frac{5}{2}(x-2)(x-3) - 15(x-1)(x-3) + \frac{31}{2}(x-1)(x-2)$$

$$\therefore \frac{3x^2+x+1}{(x-1)(x-2)(x-3)} = \frac{5}{2(x-1)} - \frac{15}{(x-2)} + \frac{31}{2(x-3)}$$

15> Truncation Error Bounds : Remember this for exam!

$$E_1(f:x) = f(x) - P(x)$$

at $x = x_0$, $x = x_1$ $E_1(f:x) = 0$

for $x \in [x_0, x_1]$ Define $g(t) = f(t) - P(t) - [f(x) - P(x)]\frac{(t-x_0)(t-x_1)}{(x-x_0)(x-x_1)}$

$\therefore g(t) = 0$ at $t = x_0, x_1, x$

$$g''(t) = f''(t) - \frac{2(f(x)-P(x))}{(x-x_0)(x-x_1)} \quad \exists \xi \text{ st } g''(\xi) = 0$$

$$\therefore f(x) = P(x) + \frac{1}{2}(x-x_0)(x-x_1)f''(\xi)$$

$$\therefore E_1(f:x) = \frac{1}{2}(x-x_0)(x-x_1)f''(\xi)$$ $\Rightarrow$ truncation error in linear interpolation

If $|f''(x)| < M_2$ $x \in [x_0, x_1]$

then $|E_1(f:x)| \le \frac{1}{2} \text{Max} |(x-x_0)(x-x_1)| M_2$

$$\boxed{|f(x) - P(x)| \le \frac{1}{8}(x_1-x_0)^2 M_2}$$

$\epsilon = |f(x) - P(x)| \le \frac{1}{8}h^2 M_2 \le \epsilon$

h is step size

So we can get max 'h'.

NOTE: For n-degree polynomial:

$$E_n(f:x) = \frac{\pi_{n+1}(x)}{(n+1)!}f^{n+1}(\xi)$$

Example: Determine 'h' so that linear interpolation of $f(x) = \sin x$ in $x \in [1,3]$ is correct to 4 decimal places.

$f(x) = \sin x \qquad f''(x) = -\sin x$

$$\text{Error} = \frac{1}{8} h^2 \text{ Max } (f''(x)) \le \underbrace{\frac{1}{2} \times 10^{-4}}_{\text{for upto 4 decimal correctness}}$$

$$\Rightarrow \frac{1}{8 \cdot 4} \times h^2 \times \underset{x \in [1,3]}{\text{Max}} |-\sin x| \le \frac{1}{2} \times 10^{-4}$$

$$\frac{h^2}{4} \le 10^{-4} \Rightarrow h \le 0.02$$

Example: Following table gives values of $f(x) = e^x$. If we fit a degree 4 polynomial, find magnitude of maximum possible error in computed value of $f(x)$ when $x = 1.25$

x	1.2	1.3	1.4	1.5	1.6
$f(x)$	3.3201	3.6692	4.0552	4.4817	4.9530

$$E_4(f:x) = (x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4) \frac{f^5(\xi)}{5!}$$

$$|E_4(f:x)| \le |(x-1.2)(x-1.3)(x-1.4)(x-1.5)(x-1.6)| \text{ Max } \frac{f^5(\xi)}{5!}$$

Now $\xi \in [1.2, 1.6] \quad \therefore \text{Max } f^5(\xi) = \text{Max } (e^x) = e^{1.6} = 4.9530$

$$\therefore |\text{Max Error}| = |(0.05)(-0.05)(-0.15)(-0.25)(-0.35)| \times \frac{4.9530}{5!}$$

$$= 0.00000135$$

16) Divided Differences:

		1st DD	2nd DD	3rd DD
x_0	y_0	$[x_0, x_1]$	$[x_0, x_1, x_2]$	$[x_0, x_1, x_2, x_3]$
x_1	y_1	$[x_1, x_2]$	$[x_1, x_2, x_3]$	
x_2	y_2	$[x_2, x_3]$		
x_3	y_3			

$$[x_0 x_1] = \frac{y_1 - y_0}{x_1 - x_0}$$

$$[x_0 x_1 x_2] = \frac{[x_1 x_2] - [x_0 x_1]}{x_2 - x_0}$$

(i) DD's are symmetrical, ie, $[x_0, x_1] = [x_1, x_0] = \frac{y_0}{x_0 - x_1} + \frac{y_1}{x_1 - x_0}$

$[x_0 x_1 x_2] = [x_1 x_0 x_2] = [x_2 x_0 x_1] \ldots$

(ii) For n degree polynomial, n^{th} DD are constant

$$[x_0 x_1] = \frac{\Delta y_0}{h} \quad [x_0 x_1 x_2] = \frac{\Delta^2 y_0}{2h^2} \quad \ldots \quad [x_0 x_1 x_2 \ldots x_n] = \frac{1}{n!\, h^n} \Delta^n y_0$$

17> Newton's Divided Difference Formula :

$$y = f(x) = y_0 + (x-x_0)[x_0,x_1] + (x-x_0)(x-x_1)[x_0,x_1,x_2] + \dots + (x-x_0)(x-x_1)\dots(x-x_n)[x, x_0, x_1, \dots, x_n]$$

Unlike Lagrange, here we can directly use new entries without recalculating the coefficients.

Example: Use NDD to evaluate f(8) & f(15) given

x	$f(x)$					
4	48					
		52				
5	100		15			
		97		1		
7	294		21		0	
		202		1		0
10	900		27		0	
		310		1		
11	1210		33			
		409				
13	2028					

$\therefore f(x) = 48 + (x-4)(52) + (x-4)(x-5)(15) + (x-4)(x-5)(x-7)(1)$

$\therefore f(8) = 48 + 208 + 180 + 12 = 448$

$f(15) = 48 + 572 + 1650 + 880 = 3150$.

Example: Use newton's forward interpolation to find $\Sigma n^3 = \left(\frac{n(n+1)}{2}\right)^2$

construct table directly & apply

	$\Delta y_0 = n^3$	$\Delta^2 y_0 = n^3-(n-1)^3$	...			
1	1					
		8				
2	9		19			
		27		18		
3	36		27		6	
		64		24		0
4	100		61		6	
		125		30		
5	225		91			
		216				
6	441					

$S_n = 1 + (n-1)8 + \frac{(n-1)(n-2)}{2!}19 + \dots$

⇒ Numerical Integration

Quadrature process to integrate an approximate interpolation formula instead of $f(x)$.

1) Newton - Cotes formula : (general formula for equidistant ordinates)

consider $I = \int_a^b f(x)\,dx$ where $f(x_0) = y_0$, $f(x_0+h) = y_1$ ... $f(x_0+nh) = y_n$

$$I = \int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_0+nh} \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots \right] dx$$

using N-Forward formula

$p = (x - x_0)/h$.

$x = x_0 + ph \Rightarrow dx = h\,dp$

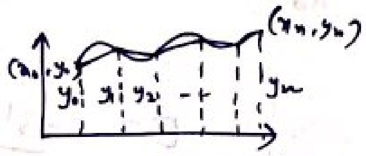
$$= h\int_0^n \left[y_0 + p\Delta y_0 + \frac{p^2 - p}{2!}\Delta^2 y_0 + \cdots \right] dp$$

$$= h\left[p y_0 + \frac{p^2}{2}\Delta y_0 + \frac{\frac{p^3}{3} - \frac{p^2}{2}}{2!}\Delta^2 y_0 + \cdots \right]_0^n$$

$$= h\left[n y_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 y_0 + \cdots \right]$$

Using this basic funda formula's can be obtained for different n

2) TRAPEZOIDAL RULE : $(n = 1)$



$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1}) \right]$$

We have replaced curve b/w (x_0, x_1) by straight line

$$\text{Error} = \frac{-1}{12}h^3\left[y_0'' + y_1'' + \cdots + y_{n-1}'' \right]$$

$$= \frac{-(b-a)}{12} h^2 y''(\bar{x})$$ where $y''(\bar{x})$ is $\max_i y_i''$

∴ Error in trapezoidal : $O(h^2)$

3> Simpson's $1/3^{rd}$ Rule : (use when $n = 2k$)

We take a degree 2 polynomial to approximate curve b/w (x_r & x_{r+2h})

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n) + 2(y_2+y_4+\dots+y_{n-2}) + 4(y_1+y_3+\dots+y_{n-1})\right]$$

$$= \frac{h}{3}\left[(y_0+y_n) + 2(\text{sum of even ordinates}) + 4(\text{sum of remaining odd ordinates})\right]$$

Error : $E = -\frac{nh^5}{90} y^{iv}(\bar{x}) = -\frac{(b-a)h^4}{180} y^{iv}(\bar{x})$ $\quad (2n)h = (b-a)$

$= O(h^4)$

4> Simpson's 3/8 Rule : (use when $n = 3k$)

We fit a cubic curve ($n=3$ in Newton Cotes)

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n) + 3(y_1+y_2+y_4+y_5+\dots+y_{n-2}+y_{n-1}) + 2(y_3+y_6+y_9+\dots+y_{n-3})\right]$$

$$E = -\frac{3h^5}{80} y^{iv}(\bar{x}) = O(h^4)$$

Both have same order but magnitude is less in $1/3^{rd}$ rule

TRICK : $\frac{3h}{8} > \frac{h}{3}$, i.e. coeff of remaining terms in $3/8^{th} < 1/3^{rd}$

TIP : Calculator use functions to tabulate required value

In questions, 5 intervals ≡ 6 points, ie, $x_0, x_1, \dots, x_5$ & $h = \frac{b-a}{5}$

Some applications : if ordinates(y) are areas, then integral is volume; are velocities, then distance.

Example: Solid of revolution is formed by rotating about the x-axis, the area b/w the x-axis, the lines x=0 x=1 & a curve through the pts

x	0	0.25	0.5	0.75	1
y	1	0.9896	0.9589	0.9089	0.8415

Estimate the volume of the solid formed.

Here $h = 0.25$ $\quad V = \int_0^1 \pi y^2 dx = \pi \int_0^1 y^2 dx$ Don't forget such terms in Integrals

x	0	0.25	0.5	0.75	1
y^2	1	0.9793	0.9195	0.8261	0.7081

$$\therefore I = \frac{\pi \times 0.25}{3}\left[(1+0.7081) + 2(0.9195) + 4(0.9793 + 0.8261)\right] = 2.8192$$

Remember just in case: Weddle's Rule: (n = 6) In Newton-Cotes

$$I = \frac{3h}{10}\left[(y_0 + y_6) + 5(y_1 + y_5) + (y_2 + y_4) + 6y_3\right]$$

for bigger size of ordinates: split into 2 weddles:

$$I = \frac{3h}{10}\left[(y_0 + y_6) + 5(y_1 + y_5) + (y_2 + y_4) + 6y_3\right] + \frac{3h}{10}\left[(y_6 + y_{12}) + 5(y_7 + y_{11}) + (y_8 + y_{10}) + 6y_9\right]$$

5) Gaussian Integration:

Allows for unequal spacing of ordinates.

Limits have to be (−1 to +1) $\int_a^b \to \int_{-1}^1$ using $x = \frac{(b-a)u}{2} + \frac{(b+a)}{2}$

$$\int_{-1}^1 f(x)\,dx = w_1 f(x_1) + w_2 f(x_2) + \cdots + w_n f(x_n) = \sum_{i=1}^{n} \underbrace{w_i}_{\text{weights}} \underbrace{f(x_i)}_{\text{abscissae}}$$

One-pt formula (n=1): $\int_{-1}^1 f(x)\,dx = 2f(0)$

Two-pt formula: (n=2) $\quad I = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$ ⇒ Gauss-Legendre Formula

Three-pt formula: $\quad I = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$

In general x_is are roots of Legendre polynomial $P_{n+1}(x)$

where $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$

$P_0(x) = 1 \quad P_1(x) = x$

the weights are given by $w_i = \int_{-1}^{1} \prod_{\substack{j=0 \\ j\neq i}}^{n} \left(\frac{x-x_j}{x_i-x_j}\right) dx$

$P_0(x) = 1$

$P_1(x) = x \qquad \therefore x_i = 0$

$P_2(x) = \frac{1}{2}(3x^2-1) \qquad x_i = \pm\frac{1}{\sqrt{3}}$

$P_3(x) = \frac{1}{2}(5x^3-3x) \qquad x_i = 0, \pm\sqrt{\frac{3}{5}}$

Example : Evaluate $\int_0^1 e^{-x^2} dx$ using 3-pt gaussian quadrature formula [upto 5 decimal points]

$I = \int_0^1 e^{-x^2} dx \qquad$ Put $x = \frac{(1-0)u + (1+0)}{2} = \frac{u+1}{2} \qquad dx = \frac{du}{2}$

$\therefore I = \frac{1}{2}\int_{-1}^{1} e^{-\left(\frac{u+1}{2}\right)^2} du = \frac{1}{2}\left[\frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)\right]$

$f(u) = e^{-\left(\frac{u+1}{2}\right)^2} \qquad \therefore f\left(-\sqrt{\frac{3}{5}}\right) = 0.987379$

$f(0) = 0.778801$

$f\left(+\sqrt{\frac{3}{5}}\right) = 0.455072$

$\therefore I = \frac{1}{2}\left[\frac{5}{9}\times(0.987379 + 0.455072) + \frac{8}{9}(0.778801)\right]$

$= \frac{1.493629}{2} = 0.74681$

Numerical Solution to ODEs :

Taylor's Series Method :-

1) Solution for $\frac{dy}{dx} = f(x,y)$: $y(x_0) = y_0$

If $y(x)$ is the exact solution

$$y(x_0+h) = y_0 + hy_0' + \frac{h^2}{2}y_0'' + \dots \quad \text{where } h = x - x_0$$

$$y' = f \quad y'' = f_x + f_y \cdot f \quad y''' = f_{xx} + 2ff_{xy} + f^2 f_{yy} + f_y(f_x + f_y f)$$
$$= \frac{d}{dx}(f_x + f_y \cdot f)$$

Example : $y' = x - y^2$ & $y(0) = 1$. find $y(0.1)$ [upto 4 decimal]

$$y(x) = 1 + xy_0' + \frac{x^2}{2}y_0'' + \frac{x^3}{6}y_0''' + \dots$$

$$y' = x - y^2 = 0 - 1 = -1$$
$$y'' = 1 - 2yy' = 1 - 2xy + 2y^3 = 1 - 2(1)(-1) = 3$$
$$\cancel{y''' = -2xy' - 2y + 6y^2y' = (6y^2 - 2x)(x - y^2) - 2y}$$
$$y''' = -2yy'' - 2y'^2 = -2(1)(3) - 2(-1)^2 = -8$$
$$y^{iv} = 34$$

$$\therefore y(x) = 1 - x + \frac{3}{2}x^2 - \frac{4}{3}x^3 - \frac{17}{12}x^4 - \frac{31}{20}x^5 + \dots$$

$$y(0.1) = 0.9138$$

If we truncate at x^4, & want to know limits of x so that y is correct upto 4 decimal places then

$$\frac{31}{20}x^5 \le \frac{1}{2}\times 10^{-4} \Rightarrow x \le 0.126.$$

Tip: If $y(0)$ is given & we want $y(0.2)$ with $h = 0.1$
1st calculate $y(0.1)$, $y'(0.1)$, $y''(0.1)$ & then proceed to $y(0.2)$
ie, 2 iterations we have to do.

2) Taylor Series method for simultaneous 1st order ODE :

$$y' = f_1(x,y,z) \quad \& \quad z' = f_2(x,y,z) \quad y(x_0)=y_0 \quad z(x_0) = z_0$$

Here we use 2 taylor expansions .

$$y_1 = y_0 + hy_0' + \frac{h^2 y_0''}{2!} + \frac{h^3 y_0'''}{3!} + \ldots$$

$$z_1 = z_0 + hz_0' + \frac{h^2 z_0''}{2!} + \frac{h^3 z_0'''}{3!} + \ldots$$

eg $y' = 1 + xz \quad z' = -xy \quad x=0,\ y=0,\ z=1 \quad h=0.3$

$\therefore y' = 1+xz \quad y'(0) = 1$

$y'' = z + xz' \quad y''(0) = 1$

$y''' = 2z' + xz'' \quad y'''(0) = 0$

$z' = -xy \quad z'(0) = 0$

$z'' = -y - xy' \quad z''(0) = 0$

$z''' = -2y' - xy'' \quad z'''(0) = -2$

$$\therefore y_1 = 0 + (0.3)(1) + \frac{(0.3)^2 \times 1}{2}$$

$$= 0.3 + 0.045 = 0.345$$

$$z_1 = 1 + (0.3)(0) + (0.3)^2 \times 0 + \frac{(0.3)^3 \times -2}{6}$$

$$= 1 - 0.009$$

$$= 0.991$$

3) Taylor series for 2nd order DE :

$$y'' = f(x,y,y') \quad y(x_0) = y_0 \quad \& \quad y'(x_0) = y_0'$$

No difference same as above . Get values for y', y'', y''' & put in taylor's

eg $y'' = y + xy' \quad y(0)=1 \quad y'(0) = 0$ calculate $y(0.1)$

$y(0) = 1 \quad y'(0) = 0 \quad y'' = y + xy' \quad y''' = 2y' + xy'' \quad y^{iv} = 3y'' + xy'''$

$y''(0) = 1 \quad y'''(0) = 0 \quad y^{iv}(0) = 3$

$$\therefore y(0.1) = 1 + (0.1)\times 0 + (0.1)^2 \times \frac{1}{2!} + \frac{(0.1)^3 \times 0}{3!} + (0.1)^4 \times \frac{3}{4!}$$

$$= 1 + 0.005 + 0.0000125 = 1.0050125$$

47 EULER'S METHOD :

given $\frac{dy}{dx} = f(x,y) \qquad y(x_0) = y_0$ — (1)

$$y_r = y_{r-1} + h\, f(x_{r-1}, y_{r-1})$$

Slow process, Need small h to get accuracy o/w errors gets accrued.

$\Rightarrow \quad y_1 = y(x_1) = y_0 + h f(x_0, y_0)$

$y_2 = y(x_2) = y_1 + h f(x_1, y_1)$

$\vdots$

$y_n = y(x_n) = y_{n-1} + h f(x_{n-1}, y_{n-1})$

Modified Euler method :

Use Trapezoid rule in interval $[x_{r-1}, x_r]$

$\therefore \; y_r^{(0)} = y_{r-1} + h f(x_{r-1}, y_{r-1}) \xrightarrow{\text{becomes}} y_r^{(0)} = y_{r-1} + \frac{h}{2}\left[f(x_{r-1}, y_{r-1}) + f(x_r, y_r)\right]$

$$\therefore \; y_r^{(n)} = y_{r-1} + \frac{h}{2}\left[f(x_{r-1}, y_{r-1}) + f(x_r, y_r^{(n-1)})\right]$$

Example: $y' = x^2 + y \quad , \quad y(0) = 1 \quad h = 0.01 \quad$ find $y(0.02)$

$y_0 = 1 \qquad y_1^{(0)} = y_0 + h f(x_0, y_0) = 1 + 0.01(0^2 + 1) = 1.01$

$\therefore \; y_1^{(0)} = 1.01$

Now $\quad y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{0})\right]$

$= 1 + \frac{0.01}{2}\left[(0^2 + 1^2) + (0.01^2 + 1.01^2)\right] = 1.01005$

$y_1^{(2)} = 1 + \frac{0.01}{2}\left[0^2 + 1^2 + 0.01^2 + 1.01005\right] = 1.01005$

$\therefore \; y_1^{(1)} = y_1^{(2)} = 1.01005$

$y_2^{(0)} = y_1 + \frac{h}{2}\left[f(x_1, y_1)\right] = 1.01005 + 0.01\left[0.01^2 + 1.01005\right]$
$= 1.02015$

$\therefore \; y_2^{(1)} = 1.01005 + \frac{0.01}{2}\left[0.01^2 + 1.01005 + 0.02^2 + 1.02015\right] = 1.020204$

$y_2^{(2)} = 1.01005 + \frac{0.01}{2}\left[0.01^2 + 1.01005 + 0.02^2 + 1.020204\right] = 1.020204$

Remember : Initial value of y is calculated using normal euler, then we iterate upon it

5) Runge-Kutta Method :

$$y' = f(x,y) \qquad y(x_0) = y_0$$

Don't need to calculate higher order derivative

It agrees with Taylor's series solution upto the term in h^r where 'r' depends on the method & is called its order

(a) 1st order R-K : Euler's Method.

$$y_1 = y_0 + hy_0' \qquad \& \quad y_1 = y_0 + hy_0' + \frac{h^2 y_0''}{2!} + \cdots$$

$\therefore$ match upto h

(b) 2nd order R-K : Modified Euler's method

Put $f(x_0+h, y_1(0))$ as $f(x_0+h, y_0+hf_0)$ & expand

using $f(x_0+h, y_0+hf_0) = f(x_0,y_0) + h\left(\frac{\partial f}{\partial x}\right)_0 + hf_0\left(\frac{\partial f}{\partial y}\right)_0 + O(h^2)$

On rearranging you get

$$y_1 = y_0 + hy_0' + \frac{h^2}{2}y_0'' + O(h^3)$$

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

where $k_1 = hf(x_0, y_0)$
$k_2 = hf(x_0+h, y_0+k_1)$

(c) 3rd order R-K :

$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

where $k_1 = hf(x_0, y_0)$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$k_3 = hf(x_0+h, y_0+k')$

& $k' = hf(x_0+h, y_0+k_1)$

(4) 4^{th} order : (MOST USED)

$$k_1 = hf(x_0, y_0)$$
$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$k_4 = hf(x_0 + h, y_0 + k_3)$$

then $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$\Rightarrow \quad y_1 = y_0 + k$

Try to take appropriate 'h' for extra marks questions rather than single step.

TIP: Calculator keep on STO in A, B, C, D to get final 'k' quickly

6) Runge-Kutta Method for a system of equations :

$y'' = f(x, y, y') \qquad y(x_0) = y_0 \quad y'(x_0) = y_0'$

Put $\frac{dy}{dx} = p$

$\Rightarrow \frac{dy}{dx} = p = f(x, y, p) \quad \& \quad \frac{dp}{dx} = f_2(x, y, p)$

with conditions $y(x_0) = y_0$ & $y'(x_0) = y_0' = p_0$.

Apply R-K starting from (x_0, y_0, p_0) with step sizes (h, k, L)

$$k_1 = hf_1(x_0, y_0, p_0) \qquad L_1 = hf_2(x_0, y_0, p_0)$$
$$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, p_0 + \frac{L_1}{2}\right) \qquad L_2 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, p_0 + \frac{L_1}{2}\right)$$
$$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, p_0 + \frac{L_2}{2}\right) \qquad L_3 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, p_0 + \frac{L_2}{2}\right)$$
$$k_4 = hf_1(x_0 + h, y_0 + k_3, p_0 + L_3) \qquad L_4 = hf_2(x_0 + h, y_0 + k_3, p_0 + L_3)$$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad p_1 = p_0 + \frac{1}{6}(L_1 + 2L_2 + 2L_3 + L_4)$$

Pretty symmetric