⇒ LPP :

1) $\boxed{\text{Convex Sets}}$

A set S is convex if for any 2 points X_1, X_2 in the set, the line segment joining these points is also in the set. ,ie, $X_1, X_2 \in S \Rightarrow \lambda X_2 + (1-\lambda) S_1 \in S$ for $\lambda \in [0,1]$

2) $\boxed{\text{Extreme Point (or vertex)}}$ of a convex set is a point of the set which doesn't lie on any segment joining ~~#~~ 2 other points of the set.

x is vertex $\Longleftrightarrow$ there do not exit $x_1, x_2\ (x_1 \neq x_2) \in S$

st $x = \lambda x_1 + (1-\lambda) x_2$ $\quad \lambda \in (0,1)$

not its open interval

Every pt on a circle is an extreme pt.

3) Hyperplane and Half spaces:

$$S = \{ x \in E^n \mid c_1 x_1 + c_2 x_2 + \cdots + c_n x_n = d \}$$

ie $S = \{ x \in E^n \mid CX = d \}$ $\quad C = [c_1\ c_2 \cdots c_n]$

$X = [x_1\ x_2 \cdots x_n]^T$

S divides E^n into 3 mutually exclusive & exhaustive regions.

$S_1 = \{ x : Cx < d \}$

$S_2 = \{ x : Cx = d \}$

$S_3 = \{ x : Cx > d \}$

S_1 & S_3 are open half spaces

$\{x : Cx \le d\}$ & $\{x : Cx \ge d\}$ are closed half spaces

Hyperplane is a convex set

4.) Convex Combination :

$X_1, X_2, \ldots X_n$ are finite no of pts in E^n

A ~~conex~~ convex combination is a pt

$$X = \sum_{i=1}^{n} \mu_i X_i \qquad \mu_i \geq 0 \quad \text{where } \sum_{i=1}^{n} \mu_i = 1$$

5.) Convex Hull:

If A is not convex, then the smallest convex set containing A is the convex hull of A.

∴ convex hull of A = intersection of all convex sets containing A.

a) CH of finite no of pts is the set of all convex combinations $= \{ X \in E^n \mid X = \sum_{i=1}^{m} \mu_i X_i \text{ when } \sum_{i=1}^{m} \mu_i = 1 \text{ \& } \mu_i \geq 0 \}$

b) This convex hull of finite no of points is called convex Polyhedron.

6.) Structure of LPP problems :

We have to optimize a linear function (objective function) subject to some constraints which can be a set of equations or inequalities or both.

The variables are called the Decision Variables. We have to identify the values for these DVs which optimise the ObFn.

Additionally, non-negativity constraint is also imposed on DVs.

7) Solution : values of DVs which satisfy the given constraints of LPP

Feasible solution: any solution which also satisfies the non(-ve) constraint

Optimal Feasible solution: FS which optimizes the objective fn of LPP

Feasible Region: common region determined by all the constraints & non- -ve restriction. For an LPP it's always a convex region.

⇒ Graphical Method :

- Generally used when 2 variables are given

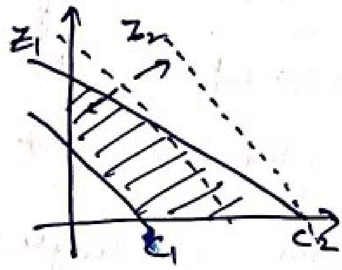
1> CORNER POINT METHOD :

a> Draw the feasible region

b> Find the coordinates of the vertices of the region.

c> Evaluate Z at each vertex obtained in (b)

2> ISO-PROFIT OR ISO-COST METHOD :



a> Draw the region

b> Draw a line by putting $Z = Z_1$

c> Move the line drawn in (b) depending on whether to minimise or maximise

d> The pt in the region this line touches last is the optimal point.

If it overlaps with a boundary of the region, we have ∞ no of optimal solutions ⇒ We have Alternative Optimal Solution

3> **Unbounded Problem** :

We have to maximise Z but the feasible region is unbounded, then optimal $Z \to \infty$ & hence optimal solⁿ can't be found.

4> **Unfeasible Problem** :

Constraints are such that no feasible region is possible.

5> **Redundant constraint** :

Removing it doesn't alter the feasible region.

NOTE: This method uses the fact that if optimal solution exists, it ~~occurs~~ occurs at an extreme pt of the feasible region.

⇒ Important insights into SIMPLEX METHOD :

a) Whenever feasible solutions existed, the region of feasible solutions was convex

b) If the Z's optimal value was finite, it occured at some vertex of the convex region.

No corner point was optimal in case of unbounded solⁿ.

c) In case of n-dimensional space also, optimal solution corresponds to a particular vertex of the feasible convex region. But there is no way to reach the vertex in a single step.

d) In simplex method, we move step by step from one vertex to another adjacent vertex till we reach an optimal vertex.

e) We move to next vertex only if it gives improved value to the objective function.

f) Since the no of vertices is finite, this method terminates.

⇒ Important definitions & forms :

a) Canonical Form : $\text{Max } Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$

st $a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \leq b_i$

$i = 1, 2, \dots, m$

$x_1, x_2, \dots, x_n \geq 0$

(i) Z is of max. type
(ii) inequalities are ≤ type
(iii) All DVs are ≥ 0

No restriction on sign of bis.

It is useful in Duality theory.

b) Standard Form : Max|Min $z = c_1x_1 + c_2x_2 + \dots + c_nx_n$

st $a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i$

$i = 1, 2, \dots, m$

$x_1, x_2, \dots, x_m \geq 0$

(i) constraints are = type
(ii) $b_i \geq 0 \; \forall \; i = 1, 2, \dots, m$
(iii) All DVs are ≥ 0
(iv) z is Max or Min type.

c) Slack variable : added to a $\leq$ constraint to make it =

d) Surplus variable : subtracted from a $\geq$ constraint to make it =

eg. $a_1x_1 + a_2x_2 \leq b \quad (x_1, x_2 \geq 0)$
$\hookrightarrow a_1x_1 + a_2x_2 + \underset{\text{slack}}{S_1} = b \quad (x_1, x_2, S_1 \geq 0)$

$a_1x_1 + a_2x_2 \geq b \quad (x_1, x_2 \geq 0)$
$\hookrightarrow a_1x_2 + a_2x_2 - \underset{\text{surplus}}{S_1} = b \quad (x_1, x_2, S \geq 0)$

To get IFBS (initial feasible basic solution), Artificial Variable is added when a surplus variable is subtracted. It is also added in case of equality constraints.

eg. $a_1x_1 + a_2x_2 \geq b \quad (x_1, x_2 \geq 0)$
$\hookrightarrow a_1x_1 + a_2x_2 - S_1 + A_1 = b \quad (x_1, x_2, S_1, A \geq 0)$ $\Rightarrow$ $x_1 = x_2 = S_1 = 0$, $A_1 = b$ is an IFBS

eg $a_1x_1 + a_2x_2 = b \quad (x_1, x_2 \geq 0)$
$\hookrightarrow a_1x_1 + a_2x_2 + A_1 = b \quad (x_1, x_2, A_1 \geq 0)$ $\Rightarrow$ $x_1 = x_2 = 0$, $A_1 = b$ is an IFBS

e) Unrestricted variables : If any variable x_i doesn't have sign constraints (i.e, ≥ 0) then it can be replaced by $x_i = x_i' - x_i''$ where $x_i', x_i'' \geq 0$.

⇒ Generating an Initial Feasible Solution :

Put slack & artificial variables as RHS in all the equations in which they occur & assign 0 to all other variables in the equation including surplus variables.

Penalty costs : slack & surplus variables do not change the nature of the constraints but artificial variables do. So in the objective function, slack & surplus are given 0 coefficient & AVs are given very large +ve (+M for minimization) or −ve (−M for max) prob.

∴ eg LPP : Max $Z = 80x_1 + 60x_2$

st $x_1 + 3x_2 \le 25$

$x_1 + x_2 = 10$

LPP : Max : $Z = 80x_1 + 60x_2 + 0S_1 - MA_1$

st
$$\left.\begin{aligned} x_1 + 3x_2 + S_1 &= 25 \\ x_1 + x_2 + A_1 &= 10 \end{aligned}\right\} \quad \text{IFBS} = x_1 = x_2 = 0, \; S_1 = 25 \quad A_1 = 10$$

⇒ Matrix form of LPP :

optimize (max | min) $Z = C^T X$

st $AX = B$

$X \ge 0$

X : column vector of unknowns

C^T : row vector of corresponding costs

A : coefficient matrix of constraint equations

B : column vector of RHS of constraints

NOTE : If a variable appears only once in the set of constraints, it can be used as the artificial variable by making its coefficient as 1. But it should have had a +ve coefficient before making it 1.

⇒ Different type of solutions :

① BASIC SOLUTION :

Given a system of m simultaneous linear equations in n unknowns $(m < n)$ $AX = b$ $X^T \in \mathbb{R}^n$

where A is an $m \times n$ matrix of rank 'm'.

We can form a submatrix $B_{m \times m}$ having m LI column vectors of A.

Rest of the $(n-m)$ variables, called non-basic variables can be set as 0 & value of the rest of the variables can be obtained by solving the resulting system thus obtained using matrix B. These 'm' variables are called basic variables and the solution a Basic Solution.

Matrix B is called the Basis Matrix & the columns are the Basis Vectors.

obviously these basis vectors $\boxed{\text{have to be LI}}$ for it to be a basic solution.

eg is $[1, 0, 1, 0, 0, 0]^T$ a basic solution of

$$x_1 + 2x_2 + 2x_3 - x_4 + x_5 = 3$$
$$2x_1 + 3x_2 + 4x_3 + x_6 = 6 \quad ?$$

Basis vectors are $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ which are LI, so NO.

If $X_B = B^{-1}b$ then $\boxed{[X_B^T, 0] \text{ is a basic solution}}$

Total no of Basic solutions $= {}^nC_m$

② BASIC FEASIBLE SOLUTION:

Basic solution which satisfies non-−ve restrictions

③ If one of or more basic variables are 0, the BFS is called Degenerate

④ If all the basic variables are >0, BFS is Non-degenerate

⑤ The BFS which optimizes the given LPP function is called optimal Basic Feasible Solution

NOTE: A BFS corresponds to an extreme pt of the convex set K of all FSs & conversely every extreme pt of K corresponds to a BFS to an LPP

Example Types :

1) Find all BSs of $2x_1 + x_2 + 4x_3 = 11$; $3x_1 + x_2 + 5x_3 = 14$ identifying basic & non-basic variables in each case. Also check if the solutions is feasible, degenerate or optimal?

⇒ Go for table approach.

3 variables, 2 eq^n ⇒ $^3C_2 = 3$ BSs

No.	BV	NBV	Values of BVs	Is it feasible?	Is it degenerate?
1	x_1, x_2	$x_3 = 0$	$2x_1 + x_2 = 11$ $3x_1 + x_2 = 14$ $\therefore x_1 = 3 \quad x_2 = 5$	Yes	No

Proceed like this.

2) Consider the system of eq^n: $2x_1 + 3x_2 - x_3 = 4$
$-5x_1 + 6x_2 + x_3 = 2$

A feasible sol^n $(x_1 = 1, x_2 = 1, x_3 = 1)$ is given. Reduce it to a basic feasible solution

$$\Rightarrow AX = B \quad \Rightarrow A = \begin{bmatrix} 2 & 3 & -1 \\ -5 & 6 & 1 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad B = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\Rightarrow A \quad x_1 \underbrace{\begin{bmatrix} 2 \\ -5 \end{bmatrix}}_{A_1} + x_2 \underbrace{\begin{bmatrix} 3 \\ 6 \end{bmatrix}}_{A_2} + x_3 \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{A_3} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$\rho(A) = 2$ & A_1, A_2 & A_3 are LD vectors.

$\therefore \exists\ \lambda_1, \lambda_2, \lambda_3$ not all zero st $A_1\lambda_1 + A_2\lambda_2 + A_3\lambda_3 = 0$ — (1)

Since $\rho(A) = 2$, we need to make one of the variables $= 0$ in the BFS & other variables have to be ≥ 0 [very imp^t to ensure this]

Find a set of $\lambda_1, \lambda_2, \lambda_3$ ~~at~~ which satisfy (1).

Pick 'r' st $\frac{x_r}{\lambda_r} = \min_{i=1,2,3} \left\{ \frac{x_i}{\lambda_i} \;\middle|\; \lambda_i > 0 \right\}$ where x_1, x_2, x_3 are the FS given

In the above system, we get $\frac{x_r}{\lambda_r} = \left\{ \frac{1}{2}, \frac{1}{2/3}, \frac{1}{6} \right\} = \frac{1}{6}$ for $r = 3$

so we make $x_3 = 0$

& obtain $x_1' = x_1 - x_3 \frac{\lambda_1}{\lambda_3} = \frac{2}{3}$ & $x_2' = x_2 - x_3 \frac{\lambda_2}{\lambda_3} = \frac{8}{9}$

& $x_3' = x_3 - x_3 \frac{\lambda_3}{\lambda_3} = 0$ $\quad$ A_1 & A_2 are LI.

Hence (x_1', x_2', x_3') is a BFS.

⇒ Simplex Method :

Steps :

1) Reduce / Express equations in standard form.
st all b_is are +ve, & appropriate slack / surplus / artificial variables are added.
Also, objective function Z should be made maximisation type.

2) Find an IBFS ; n variables & m equations means n−m variables can be put to 0.

3) Put in standard format table.

coeff of BVS — coeff. in objective function. — key element

θ denotes how much more freedom we have w/i the given constraints ⇒ limit of +ve θ ⇒ most. So choose min +ve θ.

	c_j →	3	4	0	0	−M		
C_B ↓	Basic Variable ↓	x_1	x_2	s_1	s_2	A_1	b	θ
0	S_1	Coefficients of the decision variables in constraint matrix.					b	$\theta = \frac{b}{a_j}$
−M	A_1							
	$z_j = \sum a_{ij} C_B$							
	$C_j = c_j - z_j$							

select min +ve θ

a_j is the key column

C_j denotes per unit increase if the variable is brought in the solution

maximum +ve C_j is the incoming variable [if multiple C_js are maximum, select arbitrarily]

If all C_js ≤ 0 optimality achieved

If all θs are negative, & some C_j are +ve, we have unbounded solution

4) Repeat the iterations (update matrix for new incoming variable) till optimality is reached.

NOTE : If a non-basic variable has $C_j = 0$, it means it can also be made basic w/o changing Z. So it means an alternate optimal solution exists.

Big - M Method :

1) Equations involving $\geq$ or $=$ constraints have artificial variables introduced to which act as basis variables in IBFS.

A huge penalty $-Mx$ is added in the objective function relative to these artificial variables.

The normal simplex procedure is then followed.

Following cases may arise :

a) optimality is reached & no AVs in BVs $\Rightarrow$ OBFS

b) optimality reached but AVs in Basis at 0 level (L-value)) $\Rightarrow$ degenerate OBFS

c) Optimality reached but atleast one AV in the basis at non-zero level $\Rightarrow$ Pseudo optimal solution as it satisfies the constraints but doesn't optimize Z.

d) NOTE : Once an AV leaves the basis, its column can be removed from the table for next iteration.

Two - Phase Method :

PHASE - I :

Try to force all AVs to be zero by maximising $Z^* = -A_1 - A_2 \cdots - A_m$

Cases

i) Max $Z^* < 0$ & atleast one AV in Basis at +ve level $\Rightarrow$ no OBFS

ii) Max $Z^* = 0$ ~~bld~~ & no AV in basis $\Rightarrow$ proceed to phase II with the same simplex table but original Z

iii) Max $Z^* = 0$ but one AV exists in Basis at 0 level

There is no need for phase II, this is the OBFS

Just prolong phase I to push out all AVs out of basis.

eg Use Two-Phase method to min $Z = 7.5x_1 - 3x_2$

st $3x_1 - x_2 - x_3 \geq 3$

$x_1 - x_2 + x_3 \geq 2$ $\quad x_1, x_2, x_3 \geq 0$

We have to min Z = max$(W = -Z) = -7.5x_1 + 3x_2 + 0S_1 + 0S_2 - MA_1 - MA_2$

st $3x_1 - x_2 - x_3 - S_1 + A_1 = 3$

$x_1 - x_2 + x_3 - S_2 + A_2 = 2$

Phase-I: Max $Z^* = -A_1 - A_2$

		0	0	0	0	0	-1	-1		
C_B	B	x_1	x_2	x_3	S_1	S_2	A_1	A_2	b	θ
-1	A_1	(3)	-1	-1	-1	0	1	0	3	$3/3 = 1$ →
-1	A_2	1	-1	1	0	-1	0	1	2	$2/1 = 2$
	Z_j	-4	2	0	1	1	-1	-1		
	C_j	4 ↑	-2	0	-1	-1	0	0		

⇓

Can't just remove columns for A_i in Phase-I

	C	0	0	0	0	0	-1	-1		
C_B	B	x_1	x_2	x_3	S_1	S_2	A_1	A_2		
0	x_1	1	-1/3	-1/3	-1/3	0	1/3	0	1	-3
-1	A_2	0	-2/3	(4/3)	1/3	-1	-1/3	1	1	3/4 →
	Z_j	0	2/3	-4/3	-1/3	1	1/3	-1		
	C_j	0	-2/3	4/3 ↑	1/3	-1	-1/3	0		

		0	0	0	0	0		-1	-1	
	B	x_1	x_2	x_3	S_1	S_2	~~b~~	A_1	A_2	b
0	x_1	1	-1/2	0	-1/4	-1/4	~~5/4~~	1/4	1/4	5/4
0	x_3	0	-1/2	1	1/4	-3/4	~~3/4~~	-1/4	3/4	3/4
	Z_j	0	0	0	0	0		0	0	
		0	0	0	0	0		-1	-1	

Phase-II $\quad x_1 = 5/4 \quad x_3 = 3/4$ is IFS

		-7.5	3	0	0	0	
		x_1	x_2	x_3	S_1	S_2	
-7.5	x_1	1	-1/2	0	-1/4	-1/4	5/4
0	x_3	0	-1/2	1	1/4	-3/4	3/4
	Z_j	-7.5	15/4	0	15/8	15/8	
		0	–	0	–	–	

It satisfies optimal cond^n

$\therefore x_1 = \frac{5}{4}, x_3 = \frac{3}{4}$ is the OBFS

⇒ Tie - Breakers

a) Multiple incoming variables possible (having same +ve c_j)
Pick any one arbitrarily

b) Tie for outgoing variables, ie, θ is same
  i) If $\theta > 0$, then pick any one arbitrarily
  ii) If $\theta = 0$, simplex method fails

c) Degeneracy - any one of the BV vanishes.
Removing θ arbitrarily may lead to cycles
So follow these steps:
  i) Divide each elt in the row by the +ve coeff of key elt
  ii) Compare the ratios from L → R first in the unit matrix, then the body matrix
  iii) Outgoing variable lies in the rows which first contains the smallest algebraic ratio

eg Table is

		2	1	0	0	0		
		x_1	x_2	s_1	s_2	s_3	b	0
0	s_1	4	3	1	0	0	12	
0	s_2	4	1	0	1	0	8	2
0	s_3	4	-1	0	0	1	8	2
		0	0	0	0	0		

→ These 2 have tie, so use degeneracy tie breaker.
~~Getting~~ Column wise compare 1st in unit matrix

col. s_1 ⇒ $\left[- \ \frac{0}{4} \ \frac{0}{4}\right]$ can't resolve

col. s_2 ⇒ $\left[- \ \frac{1}{4} \ \frac{0}{4}\right]$ minimum to s_3 row goes out.

⇒ Transportation Problem :

1> 'm' sources – $S_1, S_2, \ldots, S_m$ with capacities $a_1, a_2, a_3 \ldots, a_m$
'n' destinations – $D_1, D_2, \ldots, D_n$ with demands $b_1, b_2, \ldots, b_n$

c_{ij} = cost from source S_i to D_j
x_{ij} = amount transferred from S_i to D_j

Problem is to minimise $\sum\sum c_{ij} x_{ij}$

Balanced TP if $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

2> We create a $m \times n$ table with sources as rows & destinations as columns.

3> (m, n) TP has $m+n$ constraints, mn variables

In LPP $AX = B$ if matrix A is $p \times q$ & $\rho(A) = p$ then BFS has atmost p +ve variables.

Also if $\rho(A) = r < p$, then $p - r$ constraints are reduntant.

'A' matrix for (m,n) TP is $(m+n) \times mn$ & its rank is $(m+n)-1$

∴ BFS of (m,n) TP consists of $(m+n)-1$ variables which are +ve at the max.

∴ We need to transport on at max $(m+n)-1$ routes.

4> Determining an Initial Basic Feasible Solution:

a> North West Corner Method :

Starting from NW corner, assign as much value as possible & move down or right depending on left over demand or capacity. With each move cross the ~~d~~ row or column cells which can't be further assigned any value

b) Matrix - Minima or Least cost Method :

Select the cheapest route and assign as much value as possible. Select any of the routes in case of a tie.

c) Vogel Approximation Method :

Find difference b/w min & 2nd min costs in each row & column. Allot max possible at min. cost cell in the row/column having maximum difference. In case of ties, choose least cost cell. Recalculate differences again in ~~least~~ reduced table.

5) Checking for 'basic'ness of the IFS obtained, ie, cells are in LI positions.

Closed Chain : If a closed chain can be formed using the cells or any subset of the cells of IFS, then they are LD.

Open chain : Given solution is basic if no closed chain can be formed.

6) Solving Degeneracy :

If the no of BVs in IBFS < (m+n)-1, then the solution is degenerate.

We add additional cells with zero allocation ensuring that no closed chain is formed & the assigned values remain non-negative.

7> U-V Method for Balanced TPs :

a) Find a non-degenerate IBFS.

b) Compute dual variables for each row & column
Assign 0 value to the row/col with max no of Basic Variables
For others use $u_i + v_j = c_{ij}$

c> Find $\Delta_{ij} = u_i + v_j - c_{ij}$ for all other cells.

d> If all $\Delta_{ij} \leq 0$, then optimality is reached

e> Else, select the cell with max +ve Δ_{ij} which becomes the entering cell. Let it be (r, s).

f> Assign θ value to (r,s) & form a closed chain containing this cell. subtract & add θ alternatively at the corners of the chain. The cell with $-\theta$ & smallest allocation goes out.

g> Continue ~~eep~~ from (b) till optimality is reached.

h> Calculate the final cost. DONT FORGET!!

* In case of tie of Δ_{ij}, choose the one with lower cost.

* For unbalanced problems, add new source or destination as needed and assign 0 cost to these new cells & continue with above procedure.

eg. Solve:

9	12	9	6	9	10	5
7	3	7	7	5	5	6
6	5	9	11	3	11	2
6	8	11	2	2	10	9
4	4	6	2	4	2	

IBFS:

9 x	12 x	9 (5)	6 x	9 x	10 x	5	(3) (3) (0) (0) (0)
7 x	3 (4)	7 x	7 x	5 (ε) x	5 (2)	~~6~~ 4	(2) (2) (2) (4)
6 (1)	5 x	9 (1)	11 x	3 x	11 x	2	(2) (2) (2) (2) (1) (3) (3)
6 (3)	8 x	11 x	2 (2)	2 (4)	10 x	~~9~~ 7	(0) (0) (4) (2) (5)
4	4	6	~~2~~ 0	~~4~~ 0	~~2~~ 0		
(0) (0) (3)	(2)	(2) (0) (0)	(4)	(1)	(5)		

U-V

9 −	12 −	9 (5)	6 −	9 −	10 −	5	(−3)
7 2	3 (4)	7 (θ) 5	7 −	5 (ε) −θ	5 (2)	6	(0)
6 (1) +θ	5 −	9 (1) −θ	11 −	3 −	11 −	2	(−3)
6 (3) −θ	8 −	11 −	2 (2)	2 (4) +θ	10 −	9	(−3)
4	4	6	2	4	2		
(9)	(3)	(12)	(5)	(5)	(5)		

⇓

9 −	12 −	9 (5)	6 −	9 −	10 −	5	(2)
7 −	3 (4)	7 (ε)	7 −	5 −	5 (2)	6	(0)
6 (1)	5 0	9 (1)	11 −	3 −	11 −	2	(2)
6 (3)	8 −	11 −	2 (2)	2 (4)	10 −	9	(2)
4	4	6	2	4	2		
(4)	(3)	(7)	(0)	(0)	(5)		

↳ optimality

cost = 45 + 12 + 10 + 6 + 9 + 18 + 4 + 8
= 112

⇒ Assignment Problem :

* Generally done for minimization problems
No of opⁿ to be assigned to diff ~~each~~ operators so that the cost incurred is minimum

* optimal solⁿ doesn't change if constant is added or subtracted from any row or column in the cost matrix

1) Hungarian Method :

a) Subtract min elt of each row from all the elts of the row.
Repeat for all the columns.
Each row & column should have atleast 1 zero.

b) Cover all zeroes ~~rows~~ in reduced matrix by min no. of horizontal & vertical lines, say, 'r' lines

If $r = n$ (order of matrix), then table is optimal
If $r < n$, pick min elt from all the uncovered cells & subtract it from uncovered cells. Add it to the intersection pts of the earlier drawn lines

c) Repeat 'b' till $r = n$

d) Iterate row wise & select those ~~els~~ which ~~are~~ have only 1 zero. Encircle this zero & cross out other 0's in its column
Repeat this for all the columns also.

e) If now all rows & column have atleast 1 encircled 0. we have optimal assignment.
Else we can select any zero arbitrarily for the remaining rows & column & try to cover all.

* If matrix is not square, make it one by adding 0 row / column of accordingly.

⇒ Variations :

1) For maximization problems, convert costs to -ve & proceed normally.

2) Batting Averages given, determine order of Batsmen for max average. Either select the highest average in all cells, subtract each from each cell & convert to minⁿ problem.
It can also be done with rows wise max average. But better go with full steps.

3) Impossible Assignment – give very large cost in case of minimisation problem.

⇒ Duality in LPP :

1> If primal contains large no of constraints and a smaller no of variables, it can be solved more easily in its dual form.

2> Formulation :

a> Max^n becomes min^n & vice versa
b> $\leq$ constraints become $\geq$ type & vice versa
c> $c_1, \ldots, c_n$ become $b_1, b_2, \ldots, b_m$ & vice versa
d> n variables & m constraints become m variables & n constraints.
e> Transpose of the body matrix gives the new body matrix.
f> Variables in both are non-negative.

3> Important changes to be made :

a> In min^m primal all constraints should be $\geq$. So multiply with -1. $\boxed{\text{No sign constraint on } b_i}$

Similarly all constraints in max^m primal should be $\leq$

b> If primal has $=$ constraint, make it into 2 $\geq$ & $\leq$ constraints. ,ie, $a_{11}x_1 + a_{12}x_2 = b_1 \Rightarrow \begin{array}{l} a_{11}x_1 + b_{12}x_2 \leq b_1 \\ a_{11}x_1 + b_{12}x_2 \geq b_1 \end{array}$

Multiply $\leq$ or $\geq$ by -1 as per nature of Z function.

In this case the new variable in dual is unrestricted.

c> If primal has unrestricted variable x_i, replace it with $x_i = x_i' - x_i''$ & form the dual.

In this case we get $=$ constraint in dual.

⇒ Duality Principle :

1) Optimal value of primal & dual are same
$$\text{Max } Z = \text{Min } W$$

2) If primal has an unbounded solution, the dual problem will not have a feasible solution & vice versa.

3) Suppose optimal value / solution to dual is found :

a) if primal variable corresponds to a slack starting variable in dual, its value = coeff of slack variable with changed sign in C_j row of dual simplex table & vice versa.

b) if it corresponds to an artificial variable in dual, its value = coeff of AV with changed sign in C_j row after deleting constant M & vice versa.

eg. Primal : Max $Z = x_1 + 2x_2$ st $3x_1 + 4x_2 \le 5$ & $6x_1 + x_2 \le 8$

Dual : Min $W = 5y_1 + 8y_2$ st $3y_1 + 6y_2 \ge 1$ & $4y_1 + y_2 \ge 2$

std form : Max $W^* = -5y_1 - 8y_2$

st
$$3y_1 + 6y_2 - S_1 + A_1 = 1$$
$$4y_1 + y_2 - S_2 + A_2 = 2$$

~~We~~ So x_1 corresponds to A_1 & x_2 to A_2 :

Don't remove AV column while solving dual !!

⇒ Dual Simplex Method :

1) w/o caring for sign of bi's convert LPP to maximization form.

2) Convert ≥ constraints to ≤ by multiplying with −1 and add appropriate slack variables (we do not have surplus & artificial variables in this method).

3) Find IBFS & form the table.

4) Now essentially we reverse the role of $b_i (\theta_i)$ & C_j from normal simplex method :

a) If all $C_j \le 0$ and all $b_i \ge 0$ ⇒ OBFS reached

b) If ~~all~~ any $C_j > 0$ → method fails

c) If all $c_j \le 0$ & at least one $b_i < 0$, then go to (5)

5) First find outgoing variable → one with most negative b_i

6) Check for key row elts :

a) All are +ve, ie, C_j / a_i ratios are −ve ⇒ no feasible sol^n

b) If atleast one key row elt is −ve, calculate the θ ratio & select the one with least +ve ratio as incoming.

7) Keep on iterating till OBFS or no sol^n is reached