

## A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET



28

## TEST SERIES (MAIN)-2016

Test Code: PAPER-I: IAS (M)/16-10-16

# MATHEMATICS

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FULL LENGTH TEST

Test - 09

192  
250

Time: Three Hours

Maximum Marks: 250

### INSTRUCTIONS

1. This question paper-cum-answer booklet has 48 pages and has 34 PART/SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated."
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any THREE of the remaining questions selecting at least ONE question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any page from this booklet.

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name

ATUL PRAKASH

Roll No.

Test Centre ORN

Medium ENGLISH

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the invigilator

### IMPORTANT NOTE:

Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

P.T.O.

**DO NOT WRITE ON  
THIS SPACE**

## INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED
1	(a)			7
	(b)			8
	(c)			8
	(d)			8
	(e)			6
2	(a)			6
	(b)			8
	(c)			14
	(d)			13
3	(a)			
	(b)			
	(c)			
	(d)			
4	(a)			12
	(b)			13
	(c)			8
	(d)			6
5	(a)			8
	(b)			8
	(c)			9
	(d)			7
	(e)			8
6	(a)			12
	(b)			14
	(c)			13
	(d)			
7	(a)			
	(b)			
	(c)			
	(d)			
8	(a)			
	(b)			
	(c)			
	(d)			
<b>Total Marks</b>				

**DO NOT WRITE ON  
THIS SPACE**

## SECTION - A

1. (a) (i) Show that the diagonal elements of the square of an anti-Hermitian matrix are either zero or negative.  
(ii) Prove that the eigen values of a hermitian matrix are always real.  
(iii) Use the above result to show that  $\det(H - 3iI)$  can not be zero, if  $H$  is a hermitian matrix and  $I$  is the unit matrix. (10)

(i)  $A$  be skew-Hermitian  $\Rightarrow a_{ij} = -\bar{a}_{ji}$   
 $(n \times n)$

$$A^2 = A \cdot A \Rightarrow (i, i)^{\text{th}} \text{ element} = \sum_{k=1}^n a_{ik} a_{ki} = a_{ii}^2$$

$$a_{ii}^2 = a_{ii} a_{ii} + a_{iz} a_{zi} + a_{iz} a_{zi} + \dots$$

$$= -|a_{1i}|^2 - |a_{2i}|^2 - |a_{3i}|^2 - \dots$$

$$= -\sum_{i=1}^n (|a_{ii}|^2) \quad \begin{array}{l} \text{as } |a_{ii}|^2 \text{ is +ve so,} \\ \text{either } -\sum |a_{ii}|^2 \text{ is } \boxed{\text{+ve or } 0} \end{array}$$

(ii).  $A$  be Hermitian  $\Rightarrow A = A^0$ :

$$AX = \lambda X$$

$$\textcircled{1} - X^0 AX = \lambda X^0 X \Rightarrow (X^0 AX)^0 = (\lambda X^0 X)^0$$

$$\Rightarrow X^0 A^0 X = \bar{\lambda} X^0 X$$

$$\Rightarrow X^0 AX = \bar{\lambda} X^0 X \quad \textcircled{2} \Rightarrow \textcircled{1} = \textcircled{2}$$

$$\cancel{\lambda} X^0 X = \bar{\lambda} X^0 X \Rightarrow X^0 X (\lambda - \bar{\lambda}) = 0$$

$$\lambda - \bar{\lambda} = 0 \Rightarrow \lambda \text{ is real} \quad \boxed{\text{Proved}}$$

(iii)  $\det(H - 3iI) = 0$  iff  $3i$  is eigenvalue of  $H$ .

But as seen, if  $H$  is hermitian  $\Rightarrow H$  has only real eigenvalues  $\Rightarrow$  So,  $\boxed{\det(H - 3iI) \neq 0}$

1. (b) (i) If the vectors  $(0, 1, a), (1, a, 1), (a, 1, 0)$  in  $\mathbb{R}^3(\mathbb{R})$  are linearly dependent, find the value of  $a$ .  
(ii) Show that the vector  $(1+i, 2i), (1, 1+i)$  are linearly dependent in  $\mathbb{C}^2(\mathbb{C})$  and linearly independent in  $\mathbb{C}^2(\mathbb{R})$ .

(i)  $\begin{bmatrix} 0 & 1 & a \\ 1 & a & 1 \\ a & 1 & 0 \end{bmatrix} = A$  *If given vectors are linear independent then  $\det(A) \neq 0$ .*

$$\det(A) = 1(a) + a(1-a^2) = 2a - a^3 = a(2-a^2).$$

So, vectors are linearly dependent iff  $\det(A) = 0$ .

$a=0, a=\pm\sqrt{2}$  vectors are dependent.

(ii) When field is  $\mathbb{C}$  then

$\text{Q8'} \quad (1+i, 2i) = \boxed{(1+i)} \boxed{(1, 1+i)} = \boxed{(1+i, (1+i)^2)} = \boxed{(1+i, 2i)}$

So, linearly dependent as one is multiple of other.

Let  $a, b \in \mathbb{R}$  s.t

$$a(1+i, 2i) + b(1, 1+i) = 0$$

$$(a+ai+b, 2ai+bi+b) = (0, 0)$$

$$a+b+ai=0 \quad b+(2a+b)i=0$$

$$a=0, b=0$$

So linearly independent on  $\mathbb{R}$

1. (c) Show that  $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx = \frac{1}{\sqrt{2}} \log(\sqrt{2} + 1)$

(10)

$$I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx = f(x)$$

by property of definite integrals,

$$f(a+b-x) = f(x)$$

$a, b$  are lower and  
upper limits

$$f(\frac{\pi}{2}-x) = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\sin x + \cos x} dx = I$$

$$2I = \int_0^{\frac{\pi}{2}} \frac{1}{\sin x + \cos x} dx = \int_0^{\frac{\pi}{2}} \frac{1/\sqrt{2}}{\frac{\sin x}{\sqrt{2}} + \frac{\cos x}{\sqrt{2}}} dx$$

$$2I = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{2}} \csc(\frac{\pi}{4} + x) dx = \frac{1}{\sqrt{2}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \csc t dt$$

$$t = \frac{\pi}{4} + x$$

$$\begin{aligned}
 & \text{2I} = \frac{1}{\sqrt{2}} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \log(\tan t) dt \\
 & \Rightarrow \frac{1}{\sqrt{2}} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec t dt = \sqrt{2} \int_0^{\frac{\pi}{4}} \sec t dt = 2I \\
 & I = \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{4}} \sec t dt = \boxed{\frac{1}{\sqrt{2}} \log(\sqrt{2} + 1)} \quad \text{Ans}
 \end{aligned}$$

1. (d) Evaluate the integral  $\int_0^1 \int_x^1 e^{xy} dx dy$ , by changing the order of integration. (10)

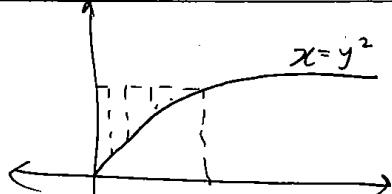
Changing order of integration  $\Rightarrow$

$$\begin{aligned}
 & x \rightarrow 0 \text{ to } y^2 \quad x \rightarrow y^2 \text{ to } 1 \\
 & y \rightarrow 0 \text{ to } 1 \quad y \rightarrow 0 \text{ to } 1 \\
 & \int_0^1 \int_{x=y^2}^{y^2} e^{xy} dx dy = \int_0^1 e^{xy} \Big|_{x=y^2}^y dy \\
 & = \int_0^1 e^y - e^{y^3} dy \\
 & \int_0^1 \int_{y^2}^y e^{xy} dy dx = \int_0^1 \left[ \frac{e^{xy}}{y} \right]_{y^2}^y dx = \int_0^1 \frac{e^y - e^{y^3}}{y} dx
 \end{aligned}$$

Changing order of integration

$$x \equiv 0 \text{ to } y^2$$

$$y \equiv 0 \text{ to } 1.$$



$$\int_0^1 \int_0^{y^2} e^{xy} dx dy = \int_0^1 y e^{xy} \Big|_0^{y^2} dy.$$

$$= \int_0^1 y e^{y^3} - y^2 dy$$

$$= \left[ y e^y - e^y - \frac{y^2}{2} \right]_0^1$$

$$= e - e^{-\frac{1}{2}} - (0 - 1 - 0)$$

$$= \boxed{\frac{1}{2}} \text{ Ans}$$

$\therefore 8'$

1. (e) Spheres are described to contain the circle  $z = 0$ .  $x^2 + y^2 = a^2$ . Prove that the locus of the extremities of their diameters which are parallel to the  $x$ -axis is the rectangular hyperbola  $x^2 - z^2 = a^2$ ,  $y = 0$ . (10)

Diameters  $\parallel$  to  $x$ -axis, extremities at  $(\alpha, \beta, \gamma)$

$$\Rightarrow \frac{x-\alpha}{1} = \frac{y-\beta}{0} = \frac{z-\gamma}{0}.$$

~~Q8~~ Sphere through circle  $\Rightarrow x^2 + y^2 - k^2 = a^2$ .

Centre at  $(0, 0, k)$ .

~~Q8~~ For dia to pass through centre  $\Rightarrow$

$$\cancel{\alpha} \quad \cancel{\beta} \quad (\gamma + \alpha, \beta, \gamma) = (0, 0, k) \text{ for some } \delta.$$

$$\textcircled{2} - \boxed{R=2\gamma}, \boxed{\gamma = -\alpha}$$

$$\boxed{\beta = 0} - \textcircled{1}$$

Now  $k = 2\gamma -$  from ②.

$(\alpha, \beta, \gamma)$  lies on sphere  $\Rightarrow$

$$\alpha^2 + \beta^2 - \gamma^2 = a^2$$

$$\alpha^2 + \beta^2 - \gamma^2 = a^2$$

$$\alpha^2 - \gamma^2 = a^2 - \beta^2 \quad (B=0) \quad \text{from ①}$$

$$\text{So, } \alpha^2 - \gamma^2 = a^2 \Rightarrow \boxed{\alpha^2 - z^2 = c^2, y=0} \text{ Ans}$$

2. (a) If  $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$ , obtain  $A^2$ . Find scalars  $a$  and  $b$  such that  $I + aA + bA^2 = \mathbf{O}$ , where  $I$  is the unit matrix and  $\mathbf{O}$  is the null matrix both of order two. (08)

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix}$$

$$I + aA + bA^2 \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a & 2a \\ -2a & a \end{bmatrix} + \begin{bmatrix} -3b & 4b \\ -4b & -3b \end{bmatrix} = \mathbf{O}$$

$$\Rightarrow \begin{bmatrix} a-3b+1 & 2a+4b \\ -2a-4b & a-3b+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Solving

$$a - 3b + 1 = 0.$$

$$a + 2b = 0.$$

$$5b = 1 \Rightarrow b = \frac{1}{5}, a = -\frac{2}{5}.$$

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2. (b) Let  $T$  be the linear operator on  $\mathbb{R}^3$  which is represented in the standard ordered basis by the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}. \text{ Find the minimal polynomial for } T. \quad (10)$$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 1-\lambda & -1 \\ 0 & 2 & 4-\lambda \end{vmatrix}$$

$$= (2-\lambda)[(1-\lambda)(4-\lambda) + 2]$$

$$= (2-\lambda)[\lambda^2 - 5\lambda + 6]$$

$$= (2-\lambda)(\lambda-3)(\lambda-2).$$

*Characteristic polynomial.*

As characteristic polynomial =  $(\lambda-2)^2(\lambda-3)$

Minimal polynomial candidates are.

$$(\lambda-2)(\lambda-3)$$

$$(\lambda-2)^2(\lambda-3)$$

$$A-2I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{bmatrix} \neq 0$$

$$A-3I = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{bmatrix} \neq 0$$

$$(A-2I)(A-3I) = \begin{bmatrix} 0 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq 0. \quad -(1)$$

So, as characteristic polynomial is satisfied by a matrix (Cayley Hamilton Theorem)

and  $(1) \Rightarrow$  minimal polynomial is

$$\boxed{(\lambda-2)^2(\lambda-3)}$$

2. (c) (i) Find  $\lim_{x \rightarrow 0} \left( \frac{\tan x}{x} \right)^{1/x}$ .

(ii) If  $v = At^{1/2} e^{-x^2/4a^2t}$ , prove that  $\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}$ . (16)

(i) Let limit reqd be  $L$ .

$$\log L = \frac{1}{x} \log \left( \frac{\tan x}{x} \right)$$

$$= \lim_{x \rightarrow 0} \frac{\log \left( \frac{\tan x}{x} \right)}{x}$$

$$\begin{aligned} (\cancel{\frac{d}{dx}}) &= \lim_{x \rightarrow 0} \frac{x}{\tan x} \left[ \frac{\sec^2 x}{x} - \frac{\tan x}{x^2} \right] \\ &= \lim_{x \rightarrow 0} \left[ \frac{\sec^2 x}{\tan x} - \frac{1}{x} \right] = \lim_{x \rightarrow 0} \frac{x \sec^2 x - \tan x}{x \tan x} \end{aligned}$$

$$\begin{aligned} (\cancel{\frac{d}{dx}}) &= \frac{\sec^2 x + 2x \sec^2 x \tan x - \sec^2 x}{\tan x + x \sec^2 x} \\ &= \lim_{x \rightarrow 0} \frac{2 \sec^2 x \tan x + 2x \sec^4 x + 4x \tan^2 x \sec^2 x}{\sec^2 x + \sec^2 x + 2x \sec^2 x \tan x} \end{aligned}$$

$$= \frac{0}{2} = 0.$$

$$\log L = 0 \Rightarrow L = e^0 = \boxed{1} \text{ is the required limit.}$$

$$(ii) V = A t^{-\frac{1}{2}} e^{-\frac{x^2}{4a^2 t}}$$

$$\frac{\partial V}{\partial x} = A t^{-\frac{1}{2}} e^{-\frac{x^2}{4a^2 t}} \cdot \frac{(-1) \cdot 2x}{4a^2 t}$$

$$\begin{aligned} \frac{\partial^2 V}{\partial x^2} &= \frac{2A t^{-\frac{3}{2}}}{4a^2} \cdot \frac{\partial}{\partial x} \left( x e^{-\frac{x^2}{4a^2 t}} \right) = -At^{-\frac{3}{2}} \left[ e^{-\frac{x^2}{4a^2 t}} + x e^{-\frac{x^2}{4a^2 t}} \frac{(-2x)}{4a^2 t} \right] \\ &= \frac{At^{-\frac{5}{2}}}{4a^4} x^2 e^{-\frac{x^2}{4a^2 t}} - \frac{At^{-\frac{3}{2}}}{2a^2} e^{-\frac{x^2}{4a^2 t}} \end{aligned} \quad \text{--- (1)}$$

$$\begin{aligned} \frac{\partial V}{\partial t} &= -\frac{1}{2} At^{-\frac{3}{2}} e^{-\frac{x^2}{4a^2 t}} + At^{-\frac{1}{2}} e^{-\frac{x^2}{4a^2 t}} \frac{(-x^2)(-1)}{4a^2 \cdot t^2} \\ &= \frac{At^{-\frac{5}{2}}}{4a^2} x^2 e^{-\frac{x^2}{4a^2 t}} - \frac{At^{-\frac{3}{2}}}{2} e^{-\frac{x^2}{4a^2 t}} \end{aligned} \quad \text{--- (2)}$$

$$a^2 \frac{\partial^2 V}{\partial x^2} = \frac{At^{-\frac{5}{2}}}{4a^2} x^2 e^{-\frac{x^2}{4a^2 t}} - \frac{At^{-\frac{3}{2}}}{2} e^{-\frac{x^2}{4a^2 t}}$$

$$= \frac{\partial V}{\partial t} \quad (\text{from (1) and (2)})$$

Hence proved

2. (d) The section of a cone with vertex at P and guiding curve  $(x^2/a^2) + (y^2/b^2) = 1, z=0$  by the plane  $x=0$  is a rectangular hyperbola. Show that the locus of P is  $(x^2/a^2) + \{(y^2+z^2)/b^2\} = 1$ . (16)

Let P be  $(f, g, h)$ .

$$\frac{x-f}{l} = \frac{y-g}{m} = \frac{z-h}{n} \quad \text{be generator of cone. - (1)}$$

$$z=0 \Rightarrow \frac{x-f}{l} = \frac{y-g}{m} = -\frac{h}{n} = (\sigma).$$

$x, y$  satisfies  $x^2/a^2 + y^2/b^2 = 1$ .

$$\Rightarrow \frac{(f-\frac{lh}{n})^2}{a^2} + \frac{(g-\frac{mh}{n})^2}{b^2} = 1.$$

$$\text{from (1)} \Rightarrow \frac{l}{n} = \frac{x-f}{z-h}$$

$$\frac{m}{n} = \frac{y-g}{z-h}.$$

$$b^2 \left[ f - \frac{h(x-f)}{z-h} \right]^2 + a^2 \left[ g - \frac{h(y-g)}{z-h} \right]^2 = a^2 b^2.$$

$$b^2 \left[ f(z-h) - h(x-f) \right]^2 + a^2 \left[ g(z-h) - h(y-g) \right]^2 = a^2 b^2 (z-h)^2 \quad (2)$$

If  $z=0$  leads to rectangular hyperbola  $\Rightarrow$  Coeff of  $y^2$  = - Coeff of  $z^2$  in (2)

$$a^2 h^2 = - (b^2 f^2 + a^2 g^2 - a^2 b^2).$$

$$\Rightarrow b^2 f^2 + a^2 (g^2 + h^2) = a^2 b^2.$$

$$\Rightarrow \frac{f^2}{a^2} + \frac{(g^2+h^2)}{b^2} = 1 \Rightarrow \boxed{\frac{x^2}{a^2} + \frac{y^2+z^2}{b^2} = 1}$$

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3. (a) (i) Evaluate  $A^{50}$  for the matrix  $A = \begin{bmatrix} 4/3 & \sqrt{2}/3 \\ \sqrt{2}/3 & 5/3 \end{bmatrix}$
- (ii) Prove that it is impossible to find a matrix  $P$  such that  $P^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} P = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$  for any  $\alpha, \beta \in \mathbb{R}$ . (18)



3. (b) (i) By using the transformation  $x = u(1+v)$ ,  $y = v(1+u)$ , prove that

$$\int_0^2 \int_0^1 [(x+y+1)^2 - 4xy]^{-1/2} dx dy = 2 \log 2 - \frac{1}{2}.$$

- (ii) Show that the function  $f$  defined by setting

$$f(x, y) = \frac{x^3 y}{x^6 + y^2}, \text{ when } (x, y) \neq (0, 0) \text{ and } f(0, 0) = 0, \text{ is not continuous at the origin.} \quad (16)$$



3. (c) A variable generator meets two generators of the same system through the extremities B and B' of the minor axis of the principal elliptic section of the hyperboloid in P and P', prove that  $BP \cdot B'P' = b^2 + c^2$ . (16)

4. (a) Let  $T$  be the linear transformation from  $\mathbb{R}^3 \rightarrow \mathbb{R}^2$  defined by  
 $T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1)$ .

If  $\beta = \{(1, 0, -1), (1, 1, 1), (1, 0, 0)\}$ ,  $\beta' = \{(0, 1), (1, 0)\}$ , be ordered bases of  $\mathbb{R}^3, \mathbb{R}^2$ , respectively, then find the matrix of  $T$  relative to  $\beta, \beta'$ . Also find rank ( $T$ ) and nullity ( $T$ ). (15)

$$T(1, 0, -1) = (1, -3) = -3(0, 1) + 1(1, 0)$$

$$T(1, 1, 1) = (2, 1) = 1(0, 1) + 2(1, 0)$$

$$T(1, 0, 0) = (1, -1) = -1(0, 1) + 1(1, 0)$$

$T$  relative to  $\beta, \beta'$  is.

$$\begin{bmatrix} -3 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix} \equiv [T]_{\beta' \beta}$$

• Null space of  $T \Rightarrow$

$$x_1 + x_2 = 0 \Rightarrow x_1 = -x_2$$

$$2x_3 = x_1$$

$(c, -c, c_2) \in \text{Null space of } T$  spans

$(1, -1, 1/2)$  spans  
null space of  $T$ .

So, nullity  $T = 1$ .

for  $T: V \rightarrow W$

Using rank-nullity theorem

$$\text{rank } T + \text{nullity } T = \dim V$$

$$\boxed{\dim V = 3}$$

$$\text{So, rank } T = 3 - 1 = \boxed{2} \text{ by}$$

4. (b) Find the points on the sphere  $x^2 + y^2 + z^2 = 25$  where  $f(x, y, z) = x + 2y + 3z$  has its maximum and minimum values. (15)

Construct the Lagrangian  $\Rightarrow$

$$L = x + 2y + 3z + \lambda (x^2 + y^2 + z^2 - 25)$$

$$\frac{\partial L}{\partial x} = 1 + 2\lambda x = 0 \Rightarrow x = -\frac{1}{2}\lambda.$$

$$\frac{\partial L}{\partial y} = 2 + 2\lambda y = 0 \Rightarrow y = -\frac{1}{\lambda}.$$

$$\frac{\partial L}{\partial z} = 3 + 2\lambda z = 0 \Rightarrow z = -\frac{3}{2}\lambda.$$

Putting  $x, y, z$  in  $\frac{\partial L}{\partial \lambda} = 0$ .

$$\frac{1}{4\lambda^2} + \frac{1}{\lambda^2} + \frac{9}{4\lambda^2} = 25.$$

$$\frac{14}{4\lambda^2} = 25 \Rightarrow \lambda^2 = \frac{14}{100}$$

$$\lambda^2 \left[ \lambda = \pm \frac{\sqrt{14}}{10} \right]$$

$$x, y, z = \begin{cases} \left( \frac{10}{2\sqrt{14}}, \frac{10}{\sqrt{14}}, \frac{30}{2\sqrt{14}} \right) & \text{--- (1)} \\ \left( -\frac{10}{2\sqrt{14}}, -\frac{10}{\sqrt{14}}, -\frac{30}{2\sqrt{14}} \right) & \text{--- (2)} \end{cases}$$

① and ② are extrema values

$$x+2y+3z \begin{cases} +ve \text{ for } ① \\ -ve \text{ for } ② \end{cases}$$

Hence, minima point =  $\left( -\frac{10}{2\sqrt{14}}, -\frac{10}{\sqrt{14}}, -\frac{30}{2\sqrt{14}} \right)$

maxima point =  $\left( \frac{10}{2\sqrt{14}}, \frac{10}{\sqrt{14}}, \frac{30}{2\sqrt{14}} \right) \approx$

4. (c) Prove that the lines  $\frac{x-a+d}{\alpha-\delta} = \frac{y-a}{\alpha} = \frac{z-a-d}{\alpha+\delta}$  and

$$\frac{x-b+c}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-b-c}{\beta+\gamma}$$
 are coplanar and find the equation to the plane in which they lie. (12)

Lines are coplanar if  $\Rightarrow$

$$[-a+d+b-c, b-a, -a-d+b+c] \cdot ((\alpha-\delta, \alpha, \alpha+\delta) \times (\beta-\gamma, \beta, \beta+\gamma)) = 0$$

$$\Rightarrow [-a+b-c+d, b-a, -a+b+c-d] \cdot (x\gamma-\beta\delta, 2\beta\gamma-2x\gamma, \alpha\gamma-\beta\delta) = 0$$

$$\Rightarrow \alpha\gamma(-a+b-c+d - 2b + 2a - d + \beta + c - \delta) + \beta\delta(0) = 0$$

$\Rightarrow 0$  Hence lines are coplanar.

i	j	k
$\alpha-\delta$	$\alpha$	$\alpha\beta - \alpha\gamma + \delta\beta - \delta\gamma$
$\beta-\gamma$	$\beta$	$\alpha\beta - \alpha\gamma + \delta\beta + \delta\gamma$
$-\delta\beta + \gamma$	$\alpha\beta + \alpha\gamma - \alpha\beta - \beta\delta$	

Normal to plane  $\equiv (\alpha\gamma - \beta\delta, 2\delta\beta - 2\alpha\gamma, \alpha\gamma - \beta\delta)$

So plane is  $\Rightarrow$

$$\begin{aligned} & (\alpha\gamma - \beta\delta) [x - a + d] + (2\delta\beta - 2\alpha\gamma) [y - a] \\ & \quad + (\alpha\gamma - \beta\delta) (z - a - d) = 0 \end{aligned}$$

Ans

$\checkmark$  Q8

4. (d) Show that the plane  $8x - 6y - z = 5$  touches the paraboloid  $(x^2/2) - (y^2/3) = z$ , and find the point of contact. (08)

For the paraboloid, tangent at  $(\alpha, \beta, \gamma)$

$$\frac{\alpha x}{2} - \frac{\beta y}{3} = \left(\frac{z+1}{2}\right) \Rightarrow \frac{\alpha x}{2} - \frac{\beta y}{3} - \frac{z}{2} = \frac{1}{2}.$$

Comparing given plane,

$$\frac{8}{\alpha/2} = \frac{-6}{-\beta/3} = \frac{-1}{-1/2} = \frac{5}{\gamma/2} \quad (2),$$

Given plane is tangent if (2) holds simultaneously

$$\checkmark Q6 \quad \frac{16}{\alpha} = \frac{18}{\beta} = 2 = \frac{10}{\gamma}$$

$$\Rightarrow \alpha = 8, \beta = 9, \gamma = 5.$$

Hence,

The plane is a tangent to paraboloid  
at  $(8, 9, 5)$  ~~An~~

## SECTION - B

5. (a) Solve  $(x^3 D^3 + 2xD - 2)y = x^2 \ln x + 3x.$

(10)

Put  $x = e^z$ .

$$(D_1(D_1-1)(D_1-2) + 2D_1 - 2)y = z e^{2z} + 3e^z$$

$$(D_1^3 - 3D_1^2 + 4D_1 - 2)y = z e^{2z} + 3e^z$$

C. F.  $\Rightarrow$  roots are  $1, 1+i, 1-i$ .

$$\text{Homogeneous solution} = C_1 e^z + e^z (C_2 \cos z + C_3 \sin z)$$

$$(D_i - 1)(D_i^2 - 2D_i + 2)$$

$$\text{PI} \Rightarrow e^z \Rightarrow \frac{1}{(1^2 - 2z + 2)(D_i - 1)} e^z \Rightarrow \frac{e^z}{1} \frac{1}{(D_i + z - 1)} + \\ = ze^z.$$

$$\Rightarrow e^{2z} z \Rightarrow \frac{e^z}{(D_i + 2 - 1)(D_i^2 + 2D_i + 4 - 2D_i - z + 2)} z^2$$

$$\text{Q8} \quad = \frac{e^{2z}}{(D_i + 1)(D_i^2 + 2D_i + 2)} z = \frac{e^{2z}}{2(D_i + 1)} \left(1 - \frac{D_i^2 - 2D_i}{2}\right) z.$$

$$\text{PI} \equiv 3ze^z + (z-2) \frac{e^{2z}}{2} = \cancel{ze^z} \\ = 3\log x \cdot x + \frac{1}{2} (\log x - 2) x^2$$

$$= \frac{e^{2z}}{2} \frac{1}{(D_i + 1)} (z-1).$$

$$= \frac{e^{2z}}{2} (1 - D_i \cdot) (z-1) \\ = \frac{e^{2z}}{2} ((z-1) - 1).$$

General solution  $\Rightarrow$

$$C_1 x + x \left( C_2 \cos(\log x) + C_3 \sin(\log x) \right) + 3x \log x \\ + \frac{1}{2} (\log x - 2) x^2$$

5. (b) Find the orthogonal trajectories of cardioids  $r = a(1 - \cos \theta)$ ,  $a$  being parameter. (10)

$$\frac{dr}{d\theta} = a \sin \theta \Rightarrow a = \frac{dr}{d\theta} \frac{1}{\sin \theta}$$

$$\Rightarrow r = \frac{dr}{d\theta} \frac{(1 - \cos \theta)}{\sin \theta}$$

$$\text{Replace } \frac{dr}{d\theta} \text{ by } -r^2 \frac{d\theta}{dr}.$$

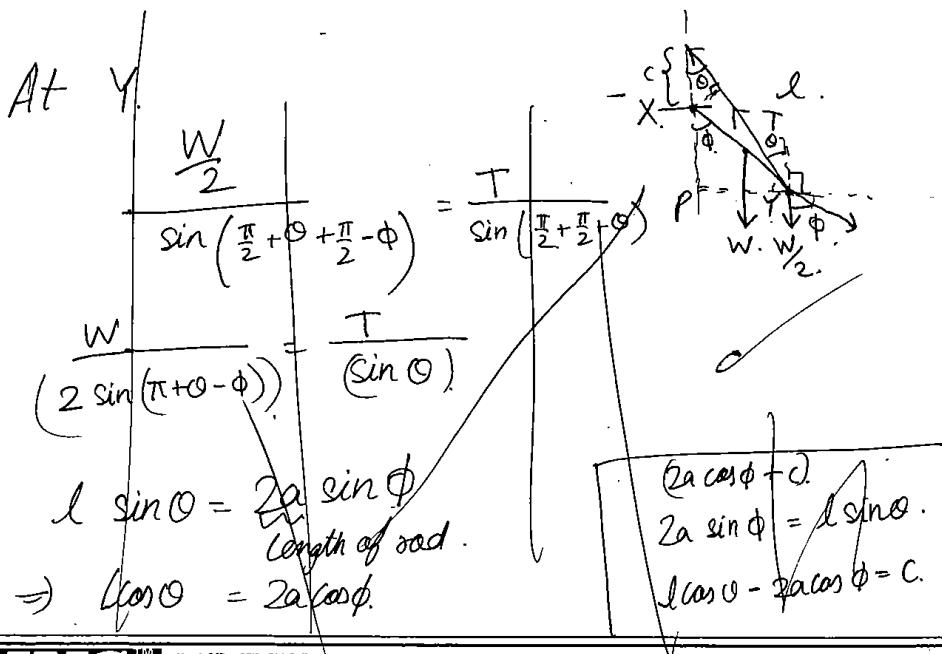
$$r = -r^2 \frac{d\theta}{dr} \frac{(1 - \cos \theta)}{\sin \theta} \Rightarrow \frac{dr}{r} = (\cot \theta - \operatorname{cosec} \theta) d\theta.$$

$$\text{Q8} \log r = \log \sin \theta - \log (\tan \frac{\theta}{2}) + C.$$

$$\frac{r \tan \frac{\theta}{2}}{\sin \theta} = C \Rightarrow r = 2C \cos^2 \frac{\theta}{2}.$$

$$r = \frac{2C}{2} (1 + \cos \theta) \quad \underline{\underline{\text{AM}}}$$

5. (c) A rod is movable in a vertical plane about a smooth hinge at one end, and at the other end is fastened a weight  $W/2$ , the weight of the rod being  $W$ . This end is fastened by a string of length  $l$  to a point at a height  $c$  vertically over the hinge. Show that the tension of the string is  $Wl/c$ . (10)



$$l \cos\theta - 2a \cos\phi = C \Rightarrow l \cos\theta - C = 2a \cos\phi$$

At X  $\Rightarrow$  Balancing Torque.

$$T_c \sin\theta = W \cdot a \sin\phi + \frac{W}{2} \cdot 2a \sin\phi$$

$$T_c \sin\theta = 2Wa \sin\phi$$

$$PY = 2a \sin\phi = l \sin\theta$$

$$T_c \sin\theta = Wl \sin\theta \Rightarrow T = \frac{Wl}{C}$$

$Q4^r$

5. (d) A particle whose mass is  $m$  is acted upon by a force  $m\mu \left[ x + \frac{a^4}{x^3} \right]$  towards origin, if it starts from rest at a distance  $a$  show that it will arrive at origin in time  $\pi/(4\sqrt{\mu})$ . (10)

$$\text{acceleration} = \frac{d^2x}{dt^2} = -\mu \left[ x + \frac{a^4}{x^3} \right].$$

$$2 \frac{dx}{dt} \cdot \frac{d^2x}{dt^2} = -2\mu \left[ x + \frac{a^4}{x^3} \right] \frac{dx}{dt}$$

$$\Rightarrow \left( \frac{dx}{dt} \right)^2 = -2\mu \left( \frac{x^2}{2} - \frac{a^4}{2x^2} \right) + C$$

$$\text{At } x=a, \frac{dx}{dt}=0 \Rightarrow C = 2\mu \left[ \frac{a^2}{2} - \frac{a^2}{2} \right] = 0$$

$$\left( \frac{dx}{dt} \right)^2 = \frac{a^4 \mu}{x^2} - x^2 \mu. \Rightarrow \frac{dx}{dt} = \frac{-\sqrt{\mu} \sqrt{a^4 - x^4}}{x}$$

$$\frac{dx}{\sqrt{a^2-x^2}} = -\sqrt{\mu} dt$$

$$x^2 = t$$

$$2x dx = dt$$

$$\frac{1}{2} \frac{(-1) dt}{\sqrt{a^2-t^2}} = \sqrt{\mu} dt \Rightarrow t \Big|_0^{t_f} = \frac{1}{2\sqrt{\mu}} \cos^{-1}\left(\frac{t}{a}\right)$$

$$t_f = \frac{1}{2\sqrt{\mu}} \cos^{-1}\left(\frac{x^2}{a}\right) \Big|_a^0$$

$$t_f = \frac{1}{2\sqrt{\mu}} \left(\frac{\pi}{2}\right)$$

$$\boxed{t_f = \frac{\pi}{4\sqrt{\mu}}} \quad \text{Ans}$$

5. (e) Using Green's theorem, evaluate  $\int_C (x^2 y dx + x^2 dy)$  where C is boundary described counter-clock wise of the triangle with vertices (0, 0), (1, 0), (1, 1). (10)

$$\int_C M dx + N dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

Green's Theorem

$$\int_C x^2 y dx + x^2 dy = \iint_R 2x - x^2 dxdy.$$

Area =  $\Delta ACX$   $\Rightarrow y < 0$   
 $\Delta ABC$   $\Rightarrow$   
 $\Rightarrow$   $y$  is 0 to  $x$   
 $x$  is 0 to 1.

$$\int_0^1 \int_0^x (2x-x^2) dy dx$$

$$\int_0^1 2x^2 - x^3 dx$$

$$\left[ \frac{2}{3}x^3 - \frac{x^4}{4} \right]_0^1 = \frac{2}{3} - \frac{1}{4}$$

$$= \frac{8-3}{12} = \boxed{\frac{5}{12}}$$

6. (a) Solve the equation  $xy'' - 2(x+1)y' + (x+2)y = (x-2)e^x$ , ( $x > 0$ ) by changing into normal form. (16)

Equation is:  $y'' - \frac{2(x+1)}{x}y' + \frac{(x+2)}{x^2}y = \frac{(x-2)}{x}e^x$

Let  $u = e^{-\frac{1}{2}\int P dx} = e^{-\frac{1}{2}\int \frac{2(x+1)}{x} dx} = e^{\int \frac{x+1}{x} dx} = e^{x + \log x} = xe^x$

The corresponding normal form is:

$$\frac{d^2V}{dx^2} + \left[ \frac{0^2}{4} - \frac{R^2}{4} - \frac{1}{2} \frac{dP}{dx} \right] V = \frac{R}{u}$$

$$\frac{d^2V}{dx^2} + \left[ \frac{(x+2)^2}{x^2} - \frac{(x+1)^2}{x^2} - \frac{1}{x^2} \right] V = \frac{(x-2)e^x}{x \times xe^x}$$

$$\frac{d^2V}{dx^2} + \left[ \frac{2x+2}{x^2} \right] V = \frac{x-2}{x^2}$$

~~① is in exact form.~~

So,  $\text{IF} = e^{\int \frac{2x+2}{x^2} dx} = e^{\int \left(\frac{2}{x} + \frac{2}{x^2}\right) dx} = e^{2\log x - 2/x}$

 $= x^2 e^{-2/x}$

$C^{mx}$  gives  $m=1$  as a solution.  $\Rightarrow e^x$  is a solution

Normal form  $\Rightarrow$  Consider  $e^x v$  to be solution.

~~$$\frac{d^2v}{dx^2} + \left(p + \frac{2}{x} \frac{dv}{dx}\right) \frac{dv}{dx} \neq \frac{R}{u}$$~~

$$\frac{d^2v}{dx^2} + \left(-\frac{2(x+1)}{x} + \frac{2}{e^x} ex\right) \frac{dv}{dx} = \frac{(x-2)}{x}$$

$$\frac{dv}{dx} = q \Rightarrow \frac{dq}{dx} + -\frac{2}{x}q = \frac{x-2}{x}$$

$$\text{IF} = e^{\int -\frac{2}{x} dx} = \boxed{\frac{1}{x^2}}$$

$$q \times \frac{1}{x^2} = \int \frac{x-2}{x^3} dx = -\frac{1}{2x} + \frac{1}{x^2} + C$$

$$\frac{dv}{dx} = -x+1+x^2 \Rightarrow \boxed{v = \frac{x^3}{3} - \frac{x^2}{2} + x + d}$$

Solution is  $\Rightarrow \boxed{\left(\frac{x^3}{3} - \frac{x^2}{2} + x + d\right) e^x}$

END OF THE EXAMINATION

6. (b) Show that the length of an endless chain which will hang over a circular pulley of radius  $a$  so as to be in contact with two-thirds of the circumference of the pulley is

$$a \left\{ \frac{3}{\log(2+\sqrt{3})} + \frac{4\pi}{3} \right\} \quad (18)$$

Let the *chain* be placed over the pulley as shown.

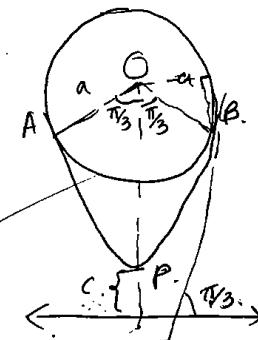
- So, as chain is in contact with  $\frac{2}{3}$  of pulley  $\angle AOB = \frac{2\pi}{3}$ .

$$\text{Angle at end } B = \frac{\pi}{3} = (\theta)$$

$$\text{We know that } x_B = C \log(\sec \theta + \tan \theta)$$

$$x_B = a \cos 30^\circ = \frac{\sqrt{3}a}{2}$$

$$\frac{\sqrt{3}a}{2} = C \log(2 + \sqrt{3}) \Rightarrow C = \frac{\sqrt{3}a}{2 \log(2 + \sqrt{3})}$$



$$S_1 = C \tan \theta \quad (\text{length of chain } PB)$$

$$S_1 = \frac{\sqrt{3}a}{2 \log(2 + \sqrt{3})} \times \sqrt{3} = \boxed{\frac{3a}{2 \log(2 + \sqrt{3})}}$$

$$\text{Total hanging chain} = 2S_1$$

$$= \boxed{\frac{3a}{\log(2 + \sqrt{3})}} \quad \text{--- (1)}$$

$$\text{Chain in contact with sphere} = \frac{4\pi}{3}a. -\textcircled{2}$$

So, Total length of chain

$$\textcircled{1} + \textcircled{2}$$

$$a \left\{ \frac{3}{\log(1+\beta)} + \frac{4\pi}{3} \right\} \text{dm}$$

6. (c) (i) Show that  $r^n \vec{r}$  is an irrotational vector for any value of n, but is solenoidal only if  $n = -3$  ( $\vec{r}$  is position vector of a point).

- (ii) Find the value of a, b and c such that

$$\mathbf{F} = (3x - 4y + az)\hat{i} + (cx + 5y - 2z)\hat{j} + (x - by + 7z)\hat{k}$$

is irrotational.

(16)

$$(\text{i}) \quad \nabla \times r^n \vec{r} = \sum i \times \frac{\partial}{\partial x} (r^n \vec{r})$$

$$= \sum i \times (n r^{n-1} \vec{r})$$

$$(\text{i}) \quad \text{curl}(\phi A) = \text{grad} \phi \times A + \phi (\text{curl} A).$$

$$\text{So, } \nabla \times r^n \vec{r} = (\text{grad } r^n) \times \vec{r} + r^n (\text{curl } \vec{r}).$$

$$= \left( n r^{n-1} \frac{x \hat{i}}{r} + \frac{y \hat{j}}{r} + \frac{z \hat{k}}{r} \right) \times \vec{r} \\ + r^n \text{curl}(\vec{r})$$

$$\begin{aligned}
 &= (n \bar{\gamma}^{n-1} \bar{\gamma}) \times \bar{\gamma} + \bar{\gamma}^n \cdot 0. \\
 &= (n \bar{\gamma}^n) (\bar{\gamma} \times \bar{\gamma}) + 0 \\
 &= 0 \quad \text{Hence proved.}
 \end{aligned}$$

and  $\bar{\gamma} = \begin{matrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{matrix}$

i) Solenoidal  $\equiv \nabla \cdot A = 0$ .

$$\nabla \cdot (\phi A) = (\nabla \phi) \cdot A + \cancel{\phi} (\nabla \cdot A) \phi.$$

$$\nabla \cdot (\bar{\gamma}^n \bar{\gamma}) = (\nabla \phi) \cdot \bar{\gamma} + \bar{\gamma}^n (3).$$

$$= n \bar{\gamma}^{n-2} (\bar{\gamma} \cdot \bar{\gamma}) + \cancel{3 \bar{\gamma}^n}$$

$$= n \bar{\gamma}^n + 3 \bar{\gamma}^n$$

$$(n+3) \bar{\gamma}^n = 0 \text{ iff } \boxed{n = -3} \quad \text{Hence solenoidal for } n = -3 \text{ only.}$$

(ii)

For irrotational,  $\nabla \times F = 0$ .

$$\begin{aligned}
 \nabla \times F &= i (-b+2) + j (a-1) \\
 &\quad + k (c+4)
 \end{aligned}$$

$$\begin{matrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x-4y+az & (a+5)y-2z & x-by+7z \end{matrix}$$

$$\nabla \times F = 0 \Rightarrow \boxed{b=2, a=1, c=-4}$$

for F to be irrotational

7. (a) By using Laplace transform method, solve  
 $(D^2 + m^2)x = a \cos nt, t > 0$  if  $x = Dx = 0$  when  $t = 0$  (16)



7. (b) A heavy hemispherical shell of radius  $r$  has a particle attached to a point on the rim, and rests with the curved surface in contact with a rough sphere of radius  $R$  at the highest point. Prove that if  $R/r > \sqrt{5}-1$ , the equilibrium is stable, whatever be the weight of the particle. (18)

7. (c) (i) Find the value of  $a$  if  $A = a\hat{i} + \hat{j} + \sqrt{5}\hat{k}$  subtends an angle of  $60^\circ$  with  $4\hat{i} - 5\hat{j} + \sqrt{5}\hat{k}$ .  
(ii) Find the directional derivative of the scalar function  $\phi = 4e^{(2x-y+z)}$  at the point  $(1, 1, -1)$  in a direction towards the point  $(-3, 5, 6)$ . (16)



8. (a) Reduce the equation  $xyp^2 - (x^2 + y^2 - 1)p + xy = 0$  to Clairaut's form. Hence show that the equation represents a family of conics touching the four sides of a square. (12)

8. (b) Solve  $(2+2x^2y^{1/2})y dx + (x^2y^{1/2}+2)x dy = 0$ . (07)

8. - (c) A particle moves with a central acceleration  $\mu(r + a^4/r^3)$  being projected from an apse at a distance 'a' with a velocity  $2a\sqrt{\mu}$ . Prove that it describes the curve  $r^2(2 + \cos\sqrt{3}\theta) = 3a^2$ . (15)

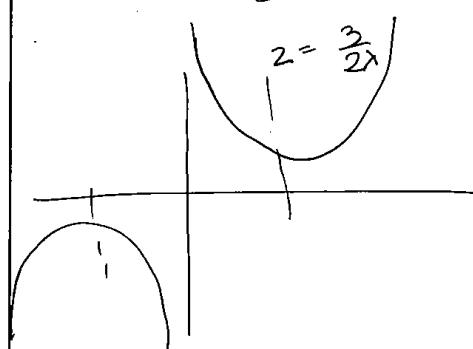
### ROUGH SPACE

$$1 + 2x\lambda = 0 \quad x = -\frac{1}{2\lambda}$$

$$2 + 2y\lambda = 0$$

$$3 + 2z\lambda = 0$$

$$z = \frac{3}{2\lambda}$$



$$\frac{1}{4\lambda^2} + \frac{1}{\lambda^2} + \frac{9}{4\lambda^2}$$

$$\frac{4+1+9}{4\lambda^2} = 25$$

$$\frac{14}{100} = \lambda^2$$

$$\boxed{\pm \sqrt{\frac{14}{10}} = \lambda}$$

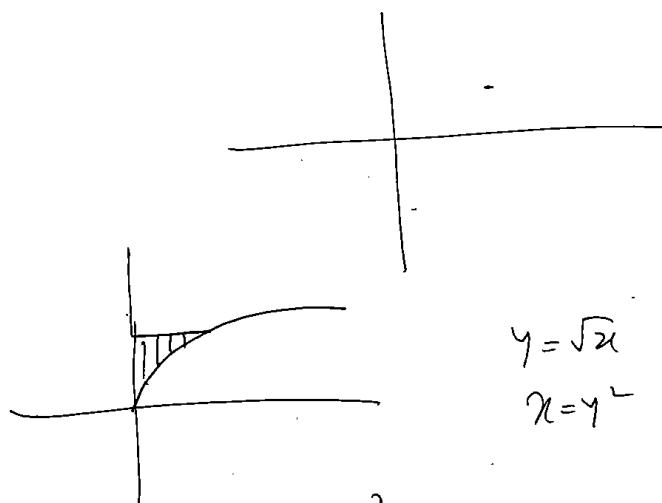
$$x^2 + y^2 - kz = a^2.$$

$$\frac{x-\alpha}{1} \rightarrow y = \beta, z = \gamma.$$

$$(x+\alpha, \beta, \gamma)$$

$$(x+\alpha)^2$$

$$x^2 + \beta^2 - kz = a^2$$



$$\int_0^{y^2} e^{xy} dz dy \cdot e^{xy}.$$

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