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MATHEMATICS FOR UPSC CSE MAINS

CALCULUS PART 3

Calculas Part-3
(Integral)

Indefinite Integral :-

→ Basic formula's :
→ Substitution
→ Byparts
→ P, Q (Denominator) form
→ Partial fraction

} 12th NCERT More than sufficient

indefinite + limits = "definite integral"

→ Properties (Already discussed in last videos)

Improper Integral :-

An integral $\int_a^b f(x)\,dx$ is stb an improper integral if

a) $f(x)$ becomes infinite in interval of integer.

b) one or both of the limits are infinite.

eg. $\frac{-1}{2}\int_0^{-\infty} -2x\, e^{-x^2} dx$. Put $-x^2 = t$, $-2x\,dx = dt$

$= \frac{-1}{2}\int_0^{-\infty} e^t\,dt = \frac{1}{2}\int_{-\infty}^{0} e^t\,dt$

$= \frac{1}{2}\, e^t \Big|_{-\infty}^{0} = \frac{1}{2}(e^0 - e^{-\infty})$

$= \frac{1}{2}$

Double Integral [Evaluation technique Only]

$\left[\iint_R f(x,y)\,dx\,dy\right]$ ⇐ Given

1.) Integrate w.r.t. "x" keeping "y" as constant.

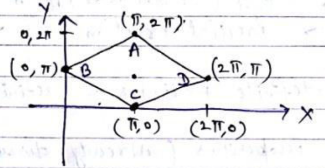
2.) Now " " "y" since "x" is eliminated or Converted to constant value after or in terms of "y".

Que:-(2015): Evaluate the integral $\boxed{12 \text{ MARKS}}$

$$\iint_R (x-y)^2 \cos^2(x+y)\,dx\,dy$$

where R is the Rhombus with successive vertices as $(\pi,0)$, $(2\pi,\pi)$, $(\pi,2\pi)$, $(0,\pi)$.

Sol^n:



→ eq^n of CD :-

$(y-0) = \frac{\pi - 0}{2\pi - \pi}(x-\pi)$

$y = x - \pi \Rightarrow x - y = \pi \Rightarrow \boxed{x - y - \pi = 0}$

→ "Neglecting Constant Part"

let $\boxed{x - y = v}$

→ eq^n of BC :

$y - 0 = \frac{0-\pi}{\pi - 0}(x-\pi) \Rightarrow y = -1(x-\pi)$

$\Rightarrow x + y = \pi \Rightarrow x + y - \pi = 0$

→ Again Neglecting Constant term

$\Rightarrow$ let $\boxed{x + y = u}$

$u = x + y$ and $v = x - y$

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \Rightarrow J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

$J\left(\frac{x,y}{u,v}\right) = \frac{-1}{2}$

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$$\Rightarrow \left[\iint_R f(x,y)\,dx\,dy = \iint_R f(u,v)\,|J|\,du\,dv\right]$$

$$= \iint_R v^2 \cos^2 u \times \frac{1}{2}\, du\cdot dv$$

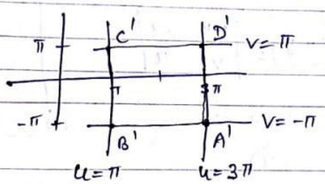
Now limits :- $u = x+y$; $v = x-y$

At point A $(\pi, 2\pi) \Rightarrow (u,v) = (3\pi, -\pi)$

" " B $(0,\pi) \Rightarrow (u,v) = (\pi, -\pi)$

" " C $(\pi,0) \Rightarrow (u,v) = (\pi, \pi)$

" " D $(2\pi,\pi) \Rightarrow (u,v) = (3\pi, \pi)$



$$= \frac{1}{2}\int_{-\pi}^{\pi}\int_{\pi}^{3\pi} v^2\cos^2 u\, du\cdot dv$$

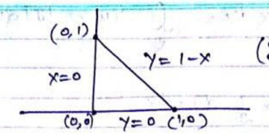
$$= \frac{1}{2}\int_{-\pi}^{\pi} v^2\left[\int_{\pi}^{3\pi} \frac{1+\cos 2u}{2}\, du\right] dv = \frac{1}{4}\int_{-\pi}^{\pi} v^2\cdot\left(u+\frac{\sin 2u}{2}\right)\Big|_{\pi}^{3\pi} dv$$

$$= \frac{1}{4}\int_{-\pi}^{\pi} v^2\cdot(3\pi-\pi)\,dv = \frac{2\pi}{4}\int_{-\pi}^{\pi} v^2\,dv = \frac{\pi}{2}\times 2\int_0^{\pi} v^2\,dv$$

$$= \pi\times\frac{v^3}{3}\Big|_0^{\pi} = \pi\times\frac{\pi^3}{3} = \boxed{\frac{\pi^4}{3}} \text{ Ans.}$$

Que (2014): 15 Marks ; By using X-formation $\boxed{x+y=u}$; $\boxed{y=uv}$, Evaluate the integral $\iint [xy(1-x-y)]^{1/2}\,dx\,dy$ taken over the area enclosed by straight lines $x=0$, $y=0$ & $x+y=1$

Solution:-



$\boxed{y = uv}$
$(y+x) = u$ } Given

$\Rightarrow x = u - y = u - uv$

$\boxed{x = u(1-v)}$

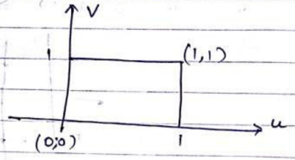
when $(x,y) = (0,0) \Rightarrow u=0$; $v=0$

when $x=0 \Rightarrow u=0;\ v=1$

$y=0 \Rightarrow u=0;\ v=0$

$x+y=1 \Rightarrow u=1$

$$J\left(\frac{x,y}{u,v}\right) = \begin{vmatrix}1-v & -u\\ v & u\end{vmatrix}$$



$$= u - \cancel{uv} + \cancel{uv}$$

$$= u$$

$$= \iint [xy(1-x-y)]^{1/2}\,dx\,dy$$

$$= \int_0^1\int_0^1 \left[u(1-v)\cdot u\cdot v\,\left(1-\underbrace{(x+y)}_{u}\right)\right]^{1/2} u\,du\,dv$$

$$= \int_0^1 u^2\sqrt{1-u}\,du \int_0^1 \sqrt{v-v^2}\,dv$$

Now put $1-u=t \Rightarrow u = 1-t$

$\Rightarrow \boxed{du = -dt}$ $\quad 1<t<0$

$$= \int_0^1 (1-t)^2\sqrt{t}\,dt \int_0^1 \sqrt{\left(\frac{1}{2}\right)^2-\left(v-\frac{1}{2}\right)^2}\,dv$$

$$= \int_0^1 (1+t^2-2t)\sqrt{t}\,dt \times \int_0^1 \sqrt{(1/2)^2-(v-1/2)^2}\,dv$$

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$$= \int_0^1 t^{1/2} + t^{5/2} - 2t^{3/2} \times \left[\cancel{\frac{v\sqrt{v-v^2}}{2}}^{\,0} + \frac{1}{8}\sin^{-1}\frac{v-1/2}{1/2}\right]\Bigg|_0^1$$

$$= \frac{t^{3/2}}{3/2} + \frac{t^{7/2}}{7/2} - \frac{2t^{5/2}}{5/2}\Bigg|_0^1 \times \left[0 + \left(\frac{1}{8}\sin^{-1}1 - \frac{1}{8}\sin^{-1}-1\right)\right]$$

$$= \left(\frac{2}{3} + \frac{2}{7} - \frac{4}{5}\right) \times \frac{1}{8}\left(\sin^{-1}\sin\frac{\pi}{2} - \sin^{-1}\sin\frac{3\pi}{2}\right)$$

$$= \frac{70+30-84}{105} \times \frac{1}{8}\left(\frac{\pi}{2} - \frac{3\pi}{2}\right) = \frac{\cancel{16}^{\,2}}{105} \times \frac{1}{\cancel{8}}(-\pi)$$

$$= -\frac{2\pi}{105} \quad \text{Ans.}$$

Q(2015) : 13 Marks Evaluate $\iint_R \sqrt{|y-x^2|}\,dx\,dy$

where $R = [-1,1\ ;\ 0,2]$

$\underbrace{-1,1}_{x \text{ limits}}$; $\underbrace{0,2}_{y = \text{limits}}$

Sol.^{n}

So put $y - x^2 = 0 \Rightarrow y = x^2$

$$= \int_{-1}^{1}\int_0^2 \sqrt{|y-x^2|}\,dx\,dy$$

$$= \int_{-1}^{1}\left[\int_0^{x^2}\sqrt{-(y-x^2)}\,dy + \int_{x^2}^{2}\sqrt{y-x^2}\,dy\right]dx$$

$$= \int_{-1}^{1}\left[\int_0^{x^2}\sqrt{x^2-y}\,dy + \int_{x^2}^{2}\sqrt{y-x^2}\,dy\right]dx$$

$$= \int_{-1}^{1} -\frac{(x^2-y)^{3/2}}{3/2}\Bigg|_0^{x^2} + \frac{(y-x^2)^{3/2}}{3/2}\Bigg|_{x^2}^{2}\,dx$$

$$= \int_{-1}^{1}\left(-\frac{2}{3}\left[0 - x^3\right] + \frac{2}{3}(2-x^2)^{3/2}\right)dx$$

$$= \int_{-1}^{1}\frac{2}{3}x^3 + \frac{2}{3}(2-x^2)^{3/2}\,dx$$

$$= \frac{2}{3}\left[\underbrace{\cancel{\int_{-1}^{1}x^3\,dx}^{\,0}}_{\text{odd}} + \underbrace{\int_{-1}^{1}(2-x^2)^{3/2}\,dx}_{\text{even.}}\right]$$

$$= \frac{2}{3}\times 2\int_0^1 (2-x^2)^{3/2}\,dx = \frac{4}{3}\int_0^1 (2-x^2)^{3/2}\,dx$$

Put $x = \sqrt{2}\sin t$; $0 \le x \le 1$

$dx = \sqrt{2}\cos t\,dt$; $0 \le t \le \pi/4$

$$= \frac{4}{3}\int_0^{\pi/4}(2 - 2\sin^2 t)^{3/2}\sqrt{2}\cos t\,dt$$

$$= \frac{4\sqrt{2}}{3}\int_0^{\pi/4} 2^{3/2}(1-\sin^2 t)^{3/2}\cos t\,dt$$

$$= \frac{4\times\sqrt{2}\times 2^{3/2}}{3}\int (\cos^2 t)^{3/2}\cos t\,dt$$

$$= \frac{8\times 2}{3}\int_0^{\pi/4}\cos^4 t\,dt = \frac{16}{3}\int_0^{\pi/4}\cos^4 t\,dt$$

Now $\cos^4 t = (\cos^2 t)^2 = \left(\frac{1+\cos 2t}{2}\right)^2$

$= \frac{1}{4}\left(1+\cos^2 2t + 2\cos 2t\right)$

$= \frac{1}{4}\left[1 + 2\cos 2t + \frac{1}{2} + \frac{\cos 4t}{2}\right]$

$\Rightarrow \left[\cos^4 t = \frac{1}{4}\left[\frac{3}{2} + 2\cos 2t + \frac{1}{2}\cos 4t\right]\right]$

Now $\Rightarrow \frac{1}{4}\times\frac{16}{3}\int_0^{\pi/4}\left(\frac{3}{2} + 2\cos 2t + \frac{1}{2}\cos 4t\right)dt$

$= \frac{4}{3}\left[\frac{3}{2}t + 2\frac{\sin 2t}{2} + \frac{1}{2}\frac{\sin 4t}{4}\right]\Bigg|_0^{\pi/4}$

$= \frac{4}{3}\left[\frac{3\pi}{8} + 1 + \frac{1}{8}(0)\right] = \frac{4}{3}\left(\frac{3\pi}{8}+1\right)$

$= \left(\frac{\pi}{2} + \frac{4}{3}\right)$ Ans

Q(2013) : 20 marks : Using Langrange's Multiplier method, find the shortest distance b/w the line $y = 10-2x$ and ellipse $\frac{x^2}{4}+\frac{y^2}{9} = 1$

Sol^n

$F(x,y) = (2x+y-10) + \lambda\left(\frac{x^2}{4}+\frac{y^2}{9}-1\right)$

$F'(x,y) = \left(2+\frac{2x\lambda}{4}\right)dx + \left(1+\frac{2\lambda y}{9}\right)dy = 0$

$2 + \frac{x\lambda}{2} = 0$ --- (1) and $1 + \frac{2\lambda}{9}y = 0$ --- (2)

$\Rightarrow \left[x = \frac{-4}{\lambda}\right]$ and $\left[y = \frac{-9}{2\lambda}\right]$

Now multiply eq^n (1) & eq^n (2) by x & y respect^y and add those.

$\Rightarrow 2x + x^2\frac{\lambda}{2} + y + y^2\frac{\lambda 2}{9} = 0$

$\Rightarrow (2x+y) + 2\lambda\left(\frac{x^2}{4}+\frac{y^2}{9}\right) = 0$

$\Rightarrow 10 + 2\lambda(1) = 0 \Rightarrow \boxed{\lambda = -5}$

Put $\lambda = -5$ & in eq^n (1) & eq^n (2); we get

$\Rightarrow x = 4/5$ and $y = 9/10$

Now Point is $(4/5, 9/10)$ & line is $2x+y = 10$

$d = \frac{|2x+y-10|}{\sqrt{2^2+1^2}} = \frac{\left|2\left(\frac{4}{5}\right)+\frac{9}{10}-10\right|}{\sqrt{4+1}}$

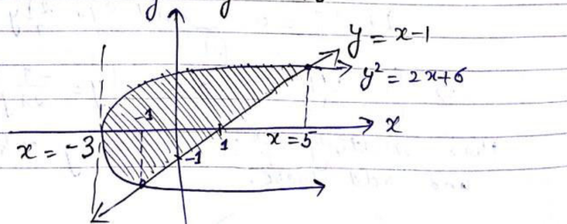
$= \frac{1}{\sqrt{5}}\left[\frac{8}{5}+\frac{9}{10}-10\right] = \left|\frac{1}{\sqrt{5}}\left(\frac{16}{10}+\frac{9}{10}-10\right)\right|$

$= \frac{1}{\sqrt{5}}\left(\frac{16+9-100}{10}\right) = \left|\frac{-75}{10\sqrt{5}}\right| = \frac{7.5}{\sqrt{5}}\times\frac{\sqrt{5}}{\sqrt{5}}$

$= 1.5\sqrt{5}$ or $\frac{3}{2}\sqrt{5}$ Ans

Q(2013) $\iint_D xy\, dA$ D: region bounded by the line b/w $y = x-1$ and the line $y^2 = 2x+6$

Sol^n



Point of intersection b/w line and parabola is:

Line: $y = x-1$ and $y^2 = 2x+6$: Parabola

$$\Rightarrow (x-1)^2 = 2x+6 \Rightarrow x^2+1-2x = 2x+6$$

$$\Rightarrow x^2-4x-5 = 0$$

$$\Rightarrow (x-5)(x+1) = 0$$

$$\Rightarrow [x=5,\ x=-1]$$

So limits of x are from -3 to 5.

Now " " y are from ~~the~~ down to UP

i.e. $(y = x-1)$ line to Parabola $(y = \sqrt{2x+6})$

$$\Rightarrow \int_{-3}^{5}\int_{x-1}^{\sqrt{2x+6}} xy\, dx\, dy = \int_{-3}^{5} x \left.\frac{y^2}{2}\right|_{x-1}^{\sqrt{2x+6}} dx$$

$$= \frac{1}{2}\int_{-3}^{5} x \times \left[(2x+6) - (x-1)^2\right] dx$$

$$= \frac{1}{2}\int_{-3}^{5} x\left[2x+6-x^2-1+2x\right] dx$$

$$= \frac{1}{2}\int_{-3}^{5} (4x^2+5x-x^3)\, dx$$

$$= \frac{1}{2}\left(\frac{4x^3}{3} + \frac{5x^2}{2} - \frac{x^4}{4}\right)\Bigg|_{-3}^{5}$$

$$= \frac{1}{2}\left[\left(\frac{500}{3} + \frac{125}{2} - \frac{625}{4}\right) - \left(-36 + \frac{45}{2} - \frac{81}{4}\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{2000+750-1875}{12}\right) - \left(\frac{-144+90-81}{4}\right)\right]$$

$$= \frac{1}{2}\left(\frac{875}{12} - \frac{-135}{4}\right) = \frac{1}{2}\left(\frac{875+135\times 3}{12}\right)$$

$$= \frac{1}{2} \times \frac{1280}{12} = \frac{640}{12} = \frac{160}{3}$$

$$= \boxed{53\tfrac{1}{3}} \text{ sq. unit}$$

IMPROPER INTEGRAL

Q(2010) : 12 Marks : Evaluate $\displaystyle\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\, dx$

Sol^n Put $x = \cos 2\theta$

$$-1 \le x \le 1 \Rightarrow \frac{\pi}{2} \le \theta \le 0$$

$$\Rightarrow dx = -2\sin 2\theta\, d\theta$$

$$= -2\int_{\pi/2}^{0}\sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}}\, \sin 2\theta\, d\theta = 2\int_{0}^{\pi/2}\frac{\cos^2\theta}{\cancel{\sin\theta}} \times 2\cancel{\sin\theta}\, d\theta$$

$$= 4\int_{0}^{\pi/2}\cos^2\theta\, d\theta = \frac{4}{2} \times \int_{0}^{\pi/2} 1+\cos 2\theta\, d\theta$$

$$= 2 \times \left(\theta + \frac{\sin 2\theta}{2}\Bigg|_{\theta=0}^{\pi/2}\right) = 2\left[\left(\frac{\pi}{2}-0\right) + (0-0)\right]$$

$$= \pi \text{ Ans.}$$

Que:- (2012) 20 Marks

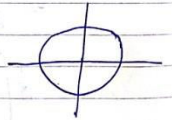
Compute the volume of the solid enclosed between the surface $x^2+y^2 = 9$ and $x^2+z^2 = 9$

Sol^n

$$\text{Vol.} = \iint z\, dy\, dx$$

$$z = \sqrt{9-x^2}$$

$$0 \le y \le \sqrt{9-x^2}$$



$$0 \le x \le a$$

$$= 8\int_{0}^{3}\int_{0}^{\sqrt{9-x^2}}\sqrt{3^2-x^2}\, dy\, dx$$

$$= 8\int_{0}^{a}\sqrt{(9-x^2)}\ y\Big|_{0}^{\sqrt{9-x^2}} dx = 8\int_{0}^{3}(9-x^2)\, dx$$

$$= 8\left(9x - \frac{x^3}{3}\right)\Big|_0^3 = 8[27-9] = 144 \text{ Ans}$$

Que:- (2011) : 20 Marks

Find the volume of the solid that lies under the paraboloid $z = x^2+y^2$ above the xy plane and inside the cylinder $x^2+y^2 = 2x$

Sol.ⁿ Req. volume is in the upper half of xy plane.

i.e. $V = \iint z \, dy\, dx$

$$V = \iint (x^2+y^2)\, dy\, dx$$

Now changing it to polar Coordinate i.e.

$x = r\cos\theta$ and $y = r\sin\theta$

$$0 \le \theta \le \pi/2$$

$$\cancel{\le r \le}$$

$$r^2\cos^2\theta + r^2\sin^2\theta = 2r\cos\theta$$

$r = 2\cos\theta$ i.e. $0 \le r \le 2\cos\theta$

The volume of the circle's ~~too~~ above two parts

$$= 2\int_0^{\pi/2}\int_0^{2\cos\theta} r^2 \cdot r\, dr\, d\theta = 2\int_0^{\pi/2} \frac{r^4}{4}\Big|_0^{2\cos\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 2^4\cos^4\theta\, d\theta = 8\int_0^{\pi/2}\cos^4\theta\, d\theta$$

$$= \cancel{8} \times \left[\frac{3\times 1}{\cancel{4\times 2}}\right] = 3 \text{ Ans}$$

$$\int_0^{\pi/2}\cos^m\theta\, d\theta = \frac{(m-1)(m-3)(m-5)\ldots}{m(m-2)(m-4)\ldots}$$

$\times \frac{\pi}{2}$ if m = even

$\times 1$ m = odd

Q (2010) : 20 Marks

Let D be the ~~inequality~~ region determined by the inequalities $x>0$, $y>0$, $z<8$ and $z > x^2+y^2$. Compute $\iiint_D 2x\, dx\, dy\, dz$

Sol.ⁿ

$$= \iiint_D 2x \, dz\, dy\, dx$$

$$x^2+y^2 < z < 8$$

$$0 < y \le \sqrt{8-x^2}$$

$$0 < x < 2\sqrt{2}$$

$$= \int_{x=}\int_{y=}\int_{z=x^2+y^2}^{8} 2x\, dz\, dy\, dx$$

$$= \int_{x=0}^{2\sqrt{2}}\int_{y=0}^{\sqrt{8-x^2}} 2x\left(8-(x^2+y^2)\right) dy\, dx$$

Now changing to polar Coordinate. i.e.

$x = r\cos\theta$; $y = r\sin\theta$

$0 \le \theta \le \pi/2$ $\qquad x^2+y^2 = 8$

$r^2 = 8 \Rightarrow r = 2\sqrt{2}$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{2\sqrt{2}} 2r\cos\theta\,[8-r^2]\, r\, dr\, d\theta$$

$$= 2\int_0^{\pi/2}\cos\theta\, d\theta \times \int_0^{2\sqrt{2}} (8r^2 - r^4)\, dr$$

$$= 2\left(\sin\theta\Big|_0^{\pi/2}\right) \times \frac{8}{3}r^3 - \frac{r^5}{5}\Big|_0^{\sqrt{8}}$$

$$= 2\times\left(\frac{8}{3}(8)^{3/2} - \frac{8^{5/2}}{5}\right) = 2\left[\frac{8\times8\times2\sqrt{2}}{3} - \frac{8\times8\sqrt{8}}{5}\right]$$

$$= 2\times 8^{5/2}\left(\frac{1}{3}-\frac{1}{5}\right) = 2\times64\times2\sqrt{2}\left(\frac{2}{15}\right)$$

$$= \frac{2^9\sqrt{2}}{15} = \frac{512\sqrt{2}}{15} \text{ Ans.}$$

Q(2009) : 20 Marks

Evaluate $I = \iint_S x\,dy\,dz + dz\,dx + xz^2\,dx\,dy$

where S is the outer side of the part of the sphere $x^2+y^2+z^2=1$ in the first octant.

Sol.

$$I = \iint_S \underbrace{x\,dy\,dz}_{I_1} + \underbrace{dz\,dx}_{I_2} + \underbrace{xz^2\,dx\,dy}_{I_3}$$

$$I_1 = \iint_S x\,dy\,dz \qquad \because x^2+y^2+z^2=1,\quad x=\sqrt{1-y^2-z^2}$$

$$= \iint \sqrt{1-y^2-z^2}\,dy\,dz \qquad \text{In } yz \text{ plane; } x=0 \Rightarrow y^2+z^2=1$$

$$z^2 = 1-y^2,\quad z=\sqrt{1-y^2}$$

$$= \int_0^1\int_0^{\sqrt{1-y^2}} \sqrt{\left(\sqrt{1-y^2}\right)^2 - z^2}\,dy\,dz$$

$$= \int_0^1 \left[\cancelto{0}{\frac{z\sqrt{1-y^2-z^2}}{2}} + \frac{1-y^2}{2}\sin^{-1}\frac{z}{\sqrt{1-y^2}}\right]_0^{\sqrt{1-y^2}} dz$$

$$= \int_0^1 \frac{1-y^2}{2}\times\frac{\pi}{2}\,dy = \frac{\pi}{4}\int_0^1 (1-y^2)\,dy$$

$$= \frac{\pi}{4}\left(y-\frac{y^3}{3}\right)\Big|_0^1 = \frac{\pi}{4}\left(1-\frac{1}{3}\right) = \frac{\pi}{6}$$

Now $I_2 = \iint_S dz\,dx$; Here xz plane is given so $y=0$

$$\Rightarrow x^2+z^2=1,\quad z=\sqrt{1-x^2} \text{ and } x=1$$

$$= \int_0^1\int_0^{\sqrt{1-x^2}} dz\,dx$$

$$= \int_0^1 z\Big|_0^{\sqrt{1-x^2}} dx = \int_0^1 \sqrt{1-x^2}\,dx$$

$$= \cancelto{0}{\frac{x\sqrt{1-x^2}}{2}} + \frac{1}{2}\sin^{-1}\frac{x}{1}\Big|_0^1$$

$$= \frac{1}{2}\times\frac{\pi}{2} = \pi/4$$

Now $I_3 = \iint_S xz^2\,dx\,dy \qquad \because z=\sqrt{1-x^2-y^2}$

$$I_3 = \iint_S x\sqrt{1-x^2-y^2}\,dx\,dy$$

Here; in xy-plane $\Rightarrow z=0$ so $x^2+y^2=1$

Now changing cartesian coordinate into polar coordinate i.e. $x=r\cos\theta$ and $y=r\sin\theta$

$$\Rightarrow r^2=1 \Rightarrow \boxed{r=1}$$

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$$I_3 = \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} r \cos\theta \sqrt{1-r^2} \cdot r \cdot dr \cdot d\theta$$

$$= \int_{0}^{\pi/2} \cos\theta \, d\theta \times \int_{0}^{1} r^2 - r^4 \, dr$$

$$= (1) \times \left. \frac{r^3}{3} - \frac{r^5}{5} \right|_0^1 = \frac{1}{3} - \frac{1}{5} = \frac{2}{15} \text{ Ans.}$$

so $I = I_1 + I_2 + I_3$

$$= \frac{\pi}{6} + \frac{\pi}{4} + \frac{2}{15} = \left(\frac{5\pi}{12} + \frac{2}{15}\right) \text{ Ans.}$$

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THANKS