Application of multiple integrals

Lecture-19
Friday
24/3/17

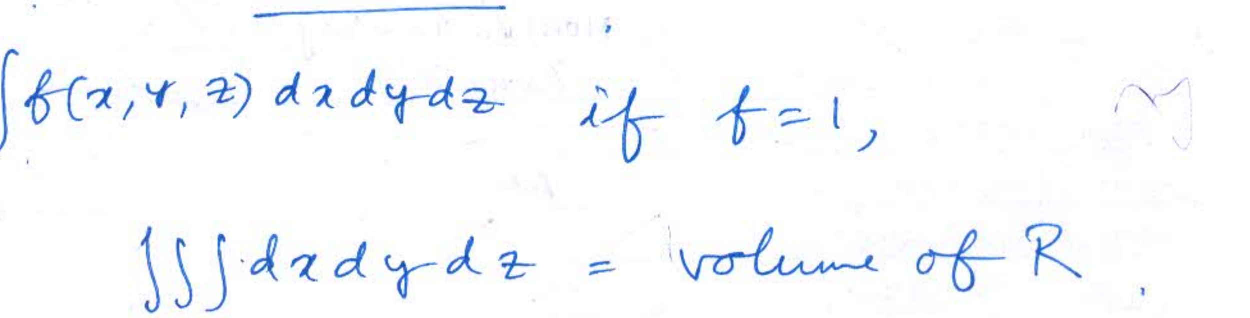
- area $\longrightarrow$ double integral, $\iint\limits_{D_{xy}} dx\,dy = \text{area of } D_{xy}$
- volume $\rightarrow$ triple integral. $\iiint\limits_{R} dx\,dy\,dz = \text{volume of } R$.
- surface area.

Volume

In $\iiint\limits_{R} f(x,y,z)\,dx\,dy\,dz$ if $f=1$,

$$\iiint\limits_{R} dx\,dy\,dz = \text{volume of } R.$$

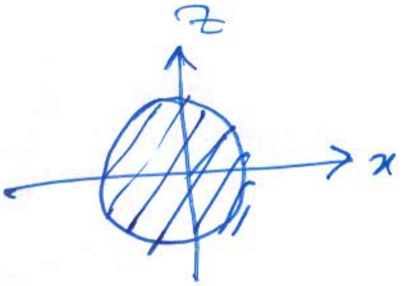
① Find the volume enclosed by the cylinder $x^2+z^2=4$ and the planes $y=-1$, $y+z=4$.



Required volume $= \displaystyle\iint\limits_{D_{zx}} \left(\int_{y=-1}^{y=4-z} 1.\, dy \right) dz\,dx.$

$D_{zx}: x^2+z^2 \le 4.$

$$= \iint\limits_{x^2+z^2 \le 4} (4 - z + 1)\, dz\, dx \; = \iint\limits_{x^2+z^2 \le 4} (5 - z)\, dz\, dx$$



$(x, z) \to (r, \theta)$

$x = r\cos\theta, \; z = r\sin\theta$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r$$

$$\text{Volume} = \int_{\theta=0}^{2\pi}\int_{r=0}^{2} (5 - r\sin\theta)\, \overbrace{r}^{=|J|}\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[\left(\frac{5r^2}{2}\right)_0^2 - \left(\frac{r^3}{3}\right)_0^2 \sin\theta \right] d\theta$$

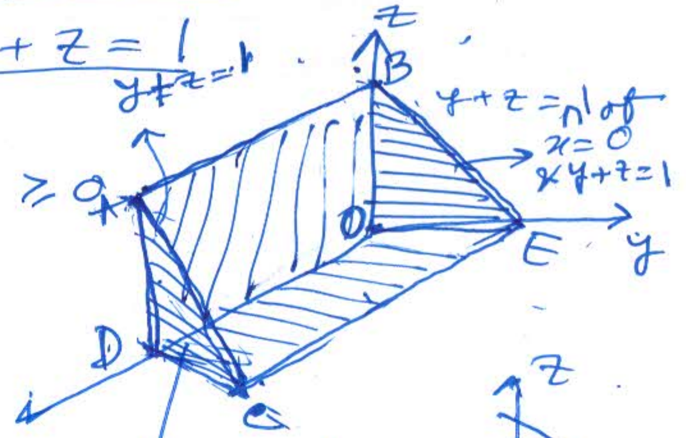
$$= \int_0^{2\pi} \left[10 - \frac{8}{3}\sin\theta \right] d\theta$$

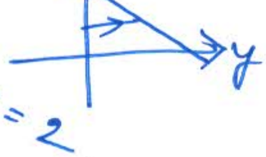
$$\int_0^{2\pi} \sin\theta\, d\theta = \left[\cos\theta\right]_0^{2\pi} = 0$$

$= 20\pi$ cubic units

2. Find the volume of the portion of the solid in the 1st octant bounded by the planes $x = 2$ & $y + z = 1$.

$x \geq 0,\ y \geq 0,\ z \geq 0$





x = 2

Volume $= \iint\limits_{D_{yz}} \int_{x=0}^{x=2} dx\,dy\,dz$

D_{yz}: ΔACD or ΔOBE.

$= \int_{z=0}^{1}\int_{y=0}^{1-z}\int_{x=0}^{2} dx\,dy\,dz \longrightarrow (1)$

$\cap$ between $x = 2$ & $y+z=1$.

Volume $= \iint \left(\int_{y=0}^{1-z} dy \right) dz\,dx$.

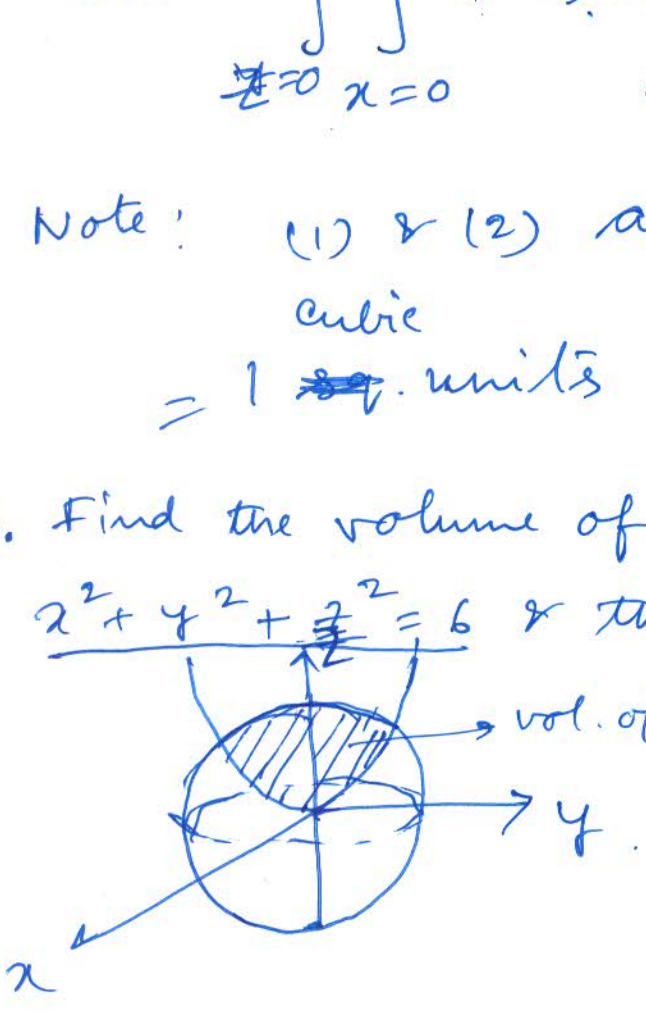
D_{zx} = rectangle ABOD.

Volume $= \int_{z=0}^{1}\int_{x=0}^{2} (1-z)\,dz\,dx \longrightarrow (2)$.

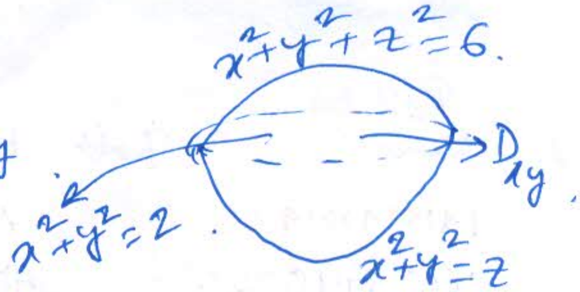
Note: (1) & (2) are the same multiple integrals.

= 1 ~~sq~~ cubic units.

3. Find the volume of solid bounded by $x^2+y^2+z^2=6$ & the paraboloid $x^2+y^2=z$.



$$\text{Volume} = \iint_{D_{xy}} \int_{z=x^2+y^2}^{\sqrt{6-x^2-y^2}} dz \, dx\, dy$$



D_{xy} = intersection of $x^2+y^2+z^2=6 \rightarrow (1)$

& $x^2+y^2=z \rightarrow (2)$.

eliminating x^2+y^2 from (1) & (2)

$$z^2+z-6=0 \implies (z+3)(z-2)=0.$$

$$z = \cancel{-3}, 2. \quad z=2 \implies x^2+y^2=2$$

$$\text{volume} = \iint_{x^2+y^2 \le 2} \Big(\underbrace{\sqrt{6-x^2-y^2} - x^2+y^2}_{f(x^2+y^2)}\Big)\, dx\, dy$$

$x = r\cos\theta$.
$y = r\sin\theta$.

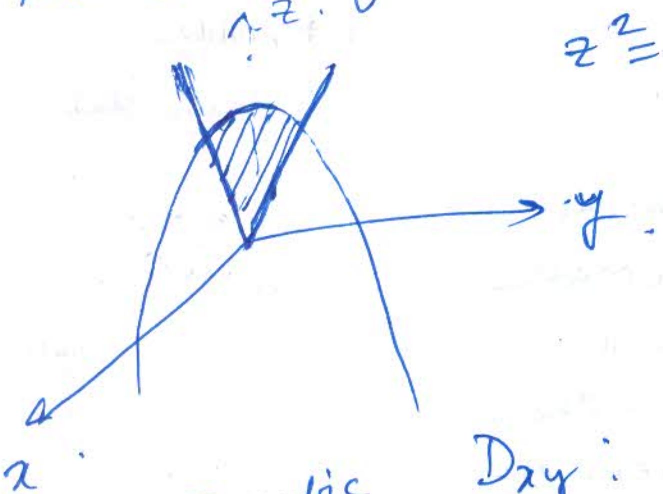
$0 \le r \le \sqrt{2}$

$0 \le \theta \le 2\pi$.

$$\text{volume} = \int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{2}} \left(\sqrt{6-r^2} - r^2\right) r \, dr\, d\theta.$$

$$= \frac{6\sqrt{6}-11}{3} \times 2\pi \text{ cubic units.}$$

Ex Find the volume of the solid bounded by the paraboloid $z = 2 - x^2 - y^2$ & the conic surface $z = \sqrt{x^2+y^2}$.



$z^2 = x^2 + y^2$.

$$\text{volume} = \iint_{D_{xy}} \int_{z=\sqrt{x^2+y^2}}^{2-x^2-y^2} dz\,dx\,dy$$

Ans $\frac{5\pi}{6}$ cubic units.

D_{xy} : $\cap$ of cone & paraboloid.

$z = 2 - z^2$ or, $z^2 + z - 2 = 0$.

$z = -2, 1$ ✓

$x^2 + y^2 = 1$.

$\to x^2 + y^2 \leq 1$.

Surface area.

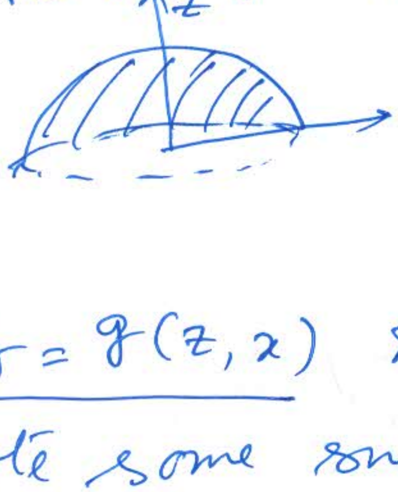
area of curved surface.

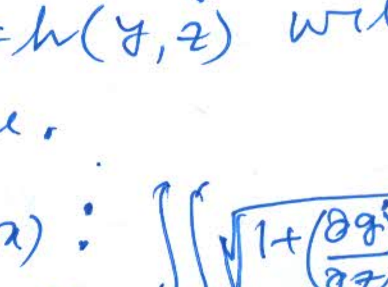
Surface area of the surface. $z = f(x,y)$ $= \iint_{D_{xy}} \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\, dx\,dy$.

D_{xy} = projection of the surface on xy-plane.

$x^2+y^2+z^2=6$; $z = \pm\sqrt{6-x^2-y^2}$.

$z = \sqrt{6-x^2-y^2} = f_1(x,y)$ $z = -\sqrt{6-x^2-y^2} = f_2(x,y)$





$y = g(z,x)$ & $x = h(y,z)$ will also denote some surface.

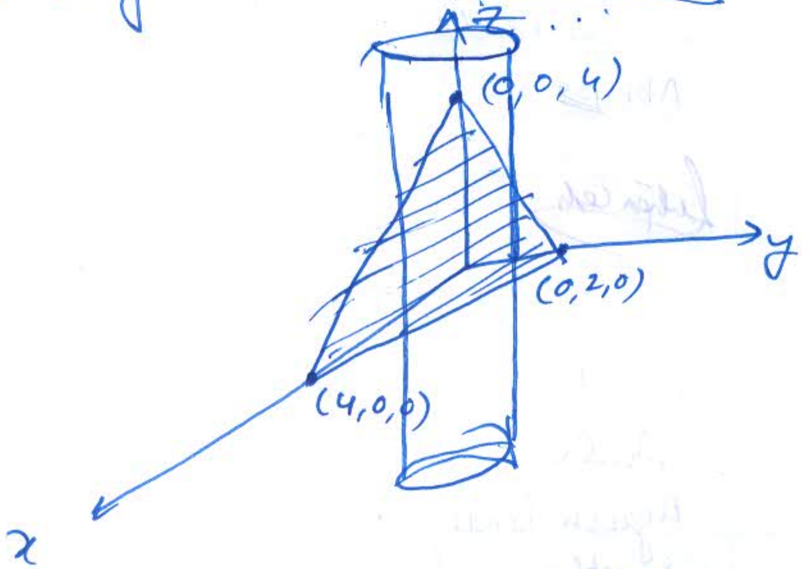
§ surface area of $y = g(z,x)$: $\iint_{D_{zx}} \sqrt{1+\left(\frac{\partial g}{\partial z}\right)^2+\left(\frac{\partial g}{\partial x}\right)^2}\, dz\,dx$.

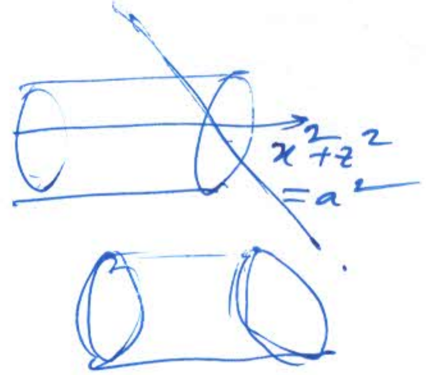
$D_{zx} \to$ projection of $y = g(z,x)$ on (x,z) plane.

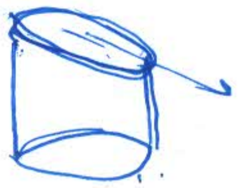
§ surface area of $x = h(y,z)$: $\iint_{D_{yz}} \sqrt{1+\left(\frac{\partial h}{\partial y}\right)^2+\left(\frac{\partial h}{\partial z}\right)^2}\, dy\,dz$

$D_{yz} \to$ projection of $x = h(y,z)$ on (y,z)-plane.

Ex 1. Find the area of the part of the plane $x + 2y + z = 4$ which lies inside the cylinder $x^2 + y^2 = 1$.







$\cap$ of cylinder & plane.

surface are $= \iint \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx\, dy$.

$D_{xy}: x^2 + y^2 \leq 1$.

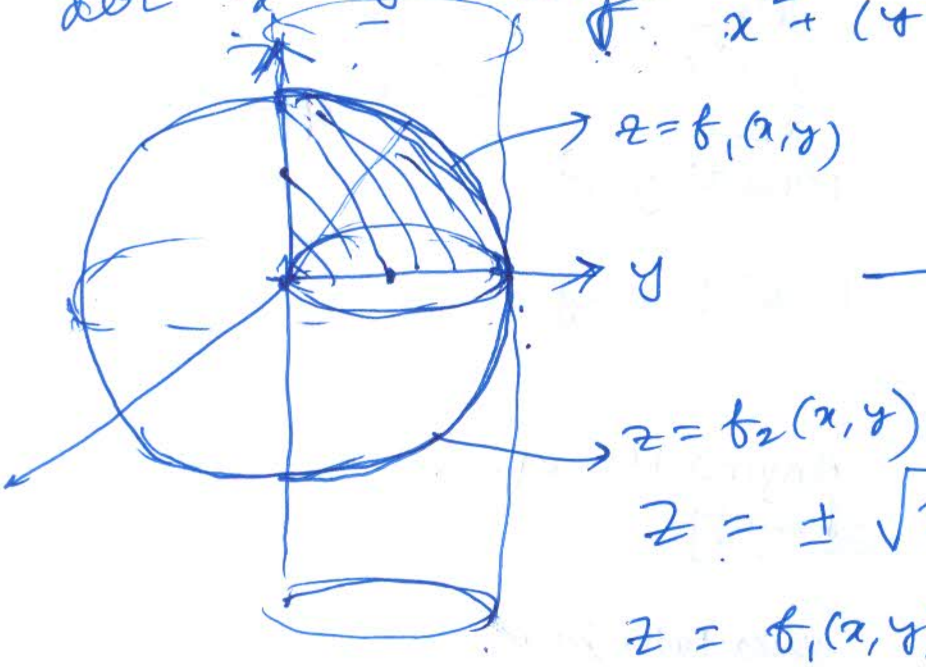
$z = \underbrace{4 - x - 2y}_{f(x,y)}$.

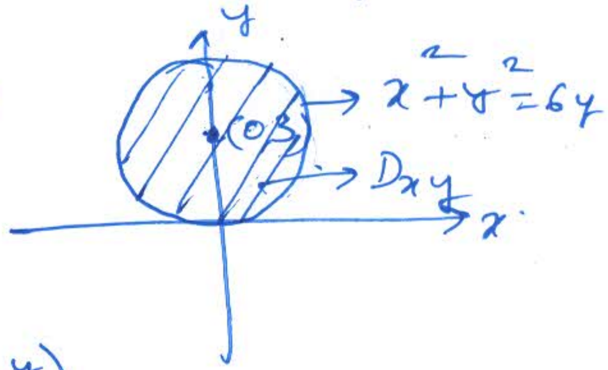
$\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + 1 + 4} = \sqrt{6}$. $\frac{\partial f}{\partial x} = -1, \frac{\partial f}{\partial y} = -2$.

$$= \iint_{D_{xy}: x^2 + y^2 \leq 1} \sqrt{6}\, dx\, dy = \sqrt{6} \iint_{x^2 + y^2 \leq 1} dx\, dy$$

$$= \sqrt{6} \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r\, dr\, d\theta = \sqrt{6}\,\pi \text{ sq. units}.$$

Ex-2 Find the surface area of the sphere $x^2+y^2+z^2=36$ inside the cylinder $x^2+y^2=6y$ $\therefore$ $x^2+(y-3)^2=9$





$$z=\pm\sqrt{36-x^2-y^2}$$

$$z=f_1(x,y)=\sqrt{36-x^2-y^2}$$

$$z=f_2(x,y)=-\sqrt{36-x^2-y^2}$$

surface area of upper ~~portion~~ portion of sphere inside the cylinder.

$$A_1=\iint\limits_{D_{xy}}\sqrt{1+\left(\frac{\partial f_1}{\partial x}\right)^2+\left(\frac{\partial f_1}{\partial y}\right)^2}\,dx\,dy$$

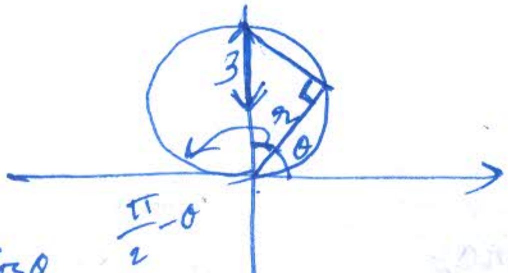
$$f_1=\sqrt{36-x^2-y^2} \quad \therefore\ f_{1x}=\frac{-x}{\sqrt{36-x^2-y^2}}$$

$$f_{1y}=-\frac{y}{\sqrt{36-x^2-y^2}}$$

$$1+f_{1x}^2+f_{1y}^2=1+\frac{x^2+y^2}{36-x^2-y^2}=\frac{36}{36-x^2-y^2}$$

$$A_1=\iint\frac{6\,dx\,dy}{\sqrt{36-x^2-y^2}}$$

D_{xy}: $x^2+(y-3)^2\le 9$



$0 \le r \le 6\sin\theta$

$0 \le \theta \le \pi$

$x = r\cos\theta$
$y = r\sin\theta$

$r = 2\times 3\cos\left(\frac{\pi}{2}-\theta\right) = 6\sin\theta.$

$$A_1 = \int_{\theta=0}^{\pi}\int_{r=0}^{6\sin\theta} \frac{6\,r\,dr\,d\theta}{\sqrt{36-r^2}}$$

$x^2+y^2=6y$
$r^2 = 6r\sin\theta.$
$r^2 - 6r\sin\theta = 0$
$r(r-6\sin\theta)=0$
$r = 0,\ 6\sin\theta.$

$$= \int_{\theta=0}^{\pi} 6\left[\sqrt{36-r^2}\right]_{6\sin\theta}^{0} d\theta.$$

$$= 6\int_0^{\pi}\left\{\sqrt{36} - \sqrt{36-36\sin^2\theta}\right\} d\theta.$$

$$= 6\int_0^{\pi}\left\{6 - 6\,|\cos\theta|\right\} d\theta.$$

$$= 6\int_0^{\pi} 6\,d\theta - 36\left[\int_0^{\pi/2}\cos\theta\,d\theta + \int_{\frac{\pi}{2}}^{\pi}\cos\theta\,d\theta\right].$$

$$= 36\pi - 36\left[1 - 0 \mp (0-1)\right].$$

$= (36\pi - 72)$ sq units

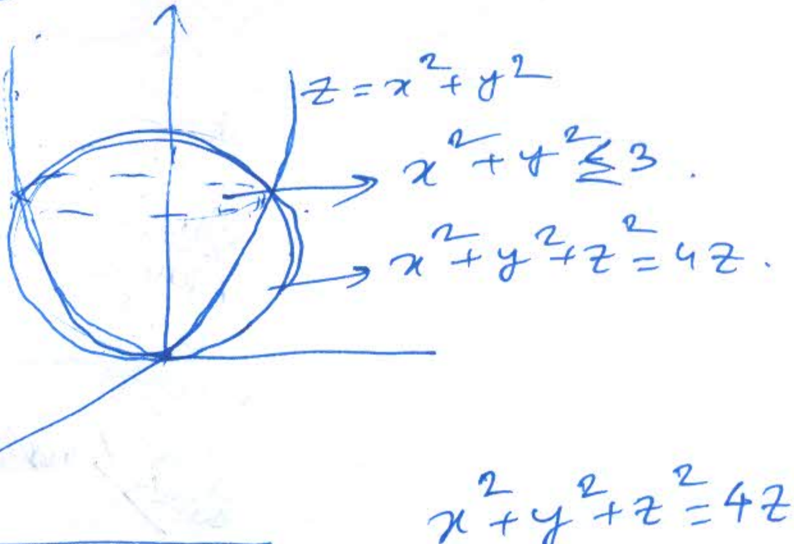
Check.

$A_2 = (36\pi - 72)$ sq. units
(by symmetry)

$\therefore$ required surface area
$= (72\pi - 144)$ sq. units

3. Find surface area of the sphere $x^2+y^2+z^2=4z$ inside the paraboloid

$z = x^2+y^2$



$z = f(x,y)$

$$x^2+y^2+z^2=4z$$
$$x^2+y^2+(z-2)^2=4$$
$$(z-2)^2=4-x^2-y^2$$
$$z=2\pm\sqrt{4-x^2-y^2}$$

$$z = 2+\sqrt{4-x^2-y^2}$$

$$f(x,y) = 2+\sqrt{4-x^2-y^2}$$

$$S.A. = \iint_{x^2+y^2\leq 3} \sqrt{1+f_x^2+f_y^2}\, dx\, dy$$

$$= 4\pi \text{ sq. units.}$$