Lecture-20
Thursday
30/3/17.

Vector Calculus

30/3/17 → Line integrals.
Green's theorem

31/3/17 → surface integrals
Gauss divergence theorem.

$$\iint_S \cdots ds \longleftrightarrow \iiint_V \cdots dV$$

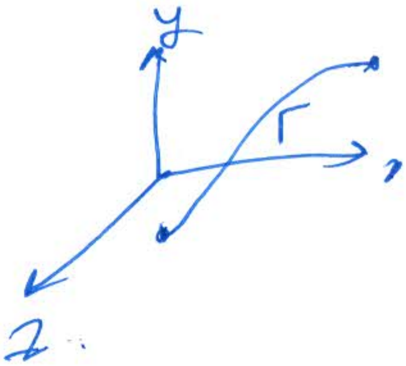
Line integrals

$\int_a^b f(x)\,dx \rightarrow$ integration along x-axis (a st. line).

In line integrals, integration is taken along any curved line either in 2D or in 3D.

2D

$$\int_C V(x,y)\,dx + W(x,y)\,dy.$$



$$\int_\Gamma V_1(x,y,z)\,dx + V_2(x,y,z)\,dy + V_3(x,y,z)\,dz$$

Computation of line integral.

To compute

$$I = \int_C V_1(x,y)\,dx + V_2(x,y)\,dy.$$

C is the arc of the circle

$x^2+y^2=4$, from $(\sqrt{2},\sqrt{2})$, to $(1,\sqrt{3})$:

Parametric Representation of a curve.

$x = 2\cos\theta$, $y = 2\sin\theta$. $\frac{\pi}{4} \le \theta \le \frac{\pi}{3}$.

(In general a curve in 3D can be represented as $x = f_1(t)$, $y = f_2(t)$, $z = f_3(t)$; $t_1 \le t \le t_2$)

$V_j(x,y) = V_j(2\cos\theta, 2\sin\theta)$; $j = 1, 2$.

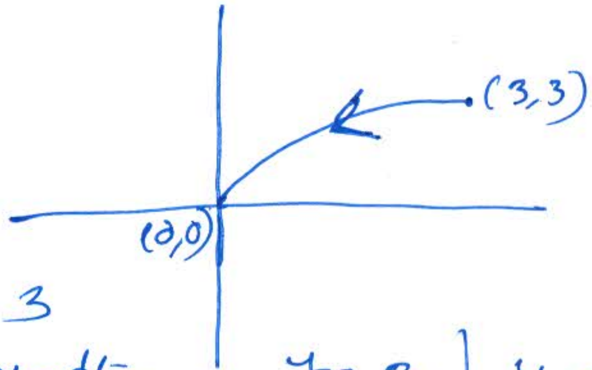
$dx = -2\sin\theta\,d\theta$, $dy = 2\cos\theta\,d\theta$.

$$I = \int_{\theta=\frac{\pi}{4}}^{\pi/3} \Big[V_1(2\cos\theta, \sin\theta) \times -2\sin\theta\,d\theta + V_2(2\cos\theta, \sin\theta)\,2\cos\theta\,d\theta \Big]$$

Ex-1 Compute

$$I = \int_C -y\,dx + x\,dy$$

along $y^2 = 3x$ from the point $(3,3)$ to the pt. $(0,0)$.



$y^2 = 3x$.

$x = \frac{t^2}{3}$, $y = t$, $0 \le t \le 3$

$dx = \frac{2t}{3}\,dt$, $dy = dt$.

$y = 3$, $t = 3$ | $y = 0$, $t = 0$.

$$I = \int_{t=3}^{0} -t \times \frac{2t}{3}\,dt + \frac{t^2}{3}\,dt$$

$$= \int_{t=3}^{0} \left(-\frac{2}{3}t^2 + \frac{1}{3}t^2\right) dt = \int_{t=0}^{3} \frac{t^2}{3}\,dt = \frac{3^3}{3^2} = 3$$

Ex-2. Find the work that is done by a force $\vec{F} = (x+y)\hat{i} + xy\hat{j} - z^2\hat{k} = (x+y, xy, -z^2)$ acting on a particle that moves along the line segment from $(0,0,0)$ to $(1,3,1)$ & then along $(1,3,1)$ to $(2,-1,4)$.

$W = \int_{P_1}^{P_2} \vec{F}\cdot d\vec{r}$ = work done by the force $\vec{F}$ in moving a particle from P_1 to P_2 along some curve

$\vec{r}$ = position vector. $d\vec{r} \to$ directed line segment along that curve.

$= x\hat{i} + y\hat{j} + z\hat{k}$

$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$, $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$.

$$\vec{F}\cdot d\vec{r} = (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

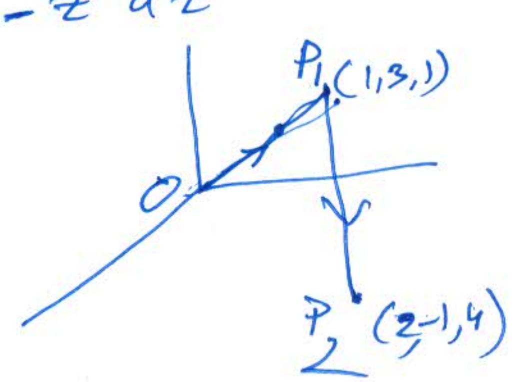
$$= F_1\,dx + F_2\,dy + F_3\,dz.$$

$$W = \int_{P_1}^{P_2} F_1\,dx + F_2\,dy + F_3\,dz.$$

In the present problem,

$$W = \int (x+y)\,dx + xy\,dy - z^2\,dz$$

$$= \int_0^{P_1} (\quad) + \int_{P_1}^{P_2} (\quad)$$



$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0} = t.$$

→ Parametric representation of a line

of OP_1

" "

$\underline{OP_1}$: $(x_0, y_0, z_0) = (0,0,0)$, $(x_1, y_1, z_1) = (1,3,1)$

$$\frac{x-0}{1-0} = \frac{y-0}{3-0} = \frac{z-0}{1-0} = t \;; \quad x = t,\; y = 3t,\; z = t \quad \Big|\; dx = dt,\; dy = 3dt,\; dz = dt$$

$$I_1 = \int_0^{P_1} (x+y)\,dx + xy\,dy - z^2\,dz$$

$$= \int_{t=0}^{t=1} (t + 3t)\,dt + 3t^2 \times 3\,dt - t^2\,dt$$

$$= \int_0^1 4t\,dt + 9t^2\,dt - t^2\,dt$$

$$= \frac{14}{3}.$$

$$I_2 = \int_{P_1}^{P_2} (x+y)\,dx + xy\,dy - z^2\,dz.$$

P_1P_2: $(x_0, y_0, z_0) = (1, 3, 1)$, $(x_1, y_1, z_1) = (2, -1, 4)$.

$$\frac{x-1}{2-1} = \frac{y-3}{-1-3} = \frac{z-1}{4-1} = u.$$

$$x = 1+u,\quad y = 3-4u,\quad z = 1+3u.$$

$$dx = du,\quad dy = -4du,\quad dz = 3du.$$

$$I_2 = \int_{u=0}^{u=1} (1+u+3-4u)\,du + (1+u)(3-4u)\times -4du - (1+3u)^2 3\,du.$$

$$= \int_0^1 \left[(4-3u) + 4(4u-3)(u+1) - 3(1+9u^2+6u) \right] du$$

$$= -\frac{139}{6}.$$

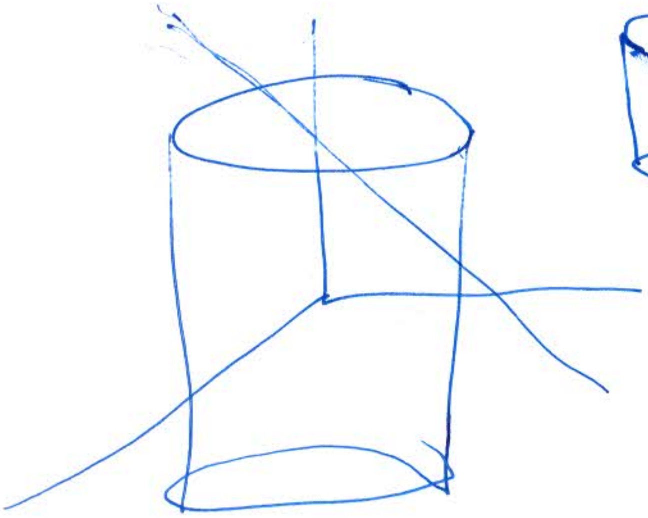
$$I_1 + I_2 = \int_{0}^{P_1} \cdots + \int_{P_1}^{P_2} \cdots = \frac{14}{3} - \frac{139}{6}$$

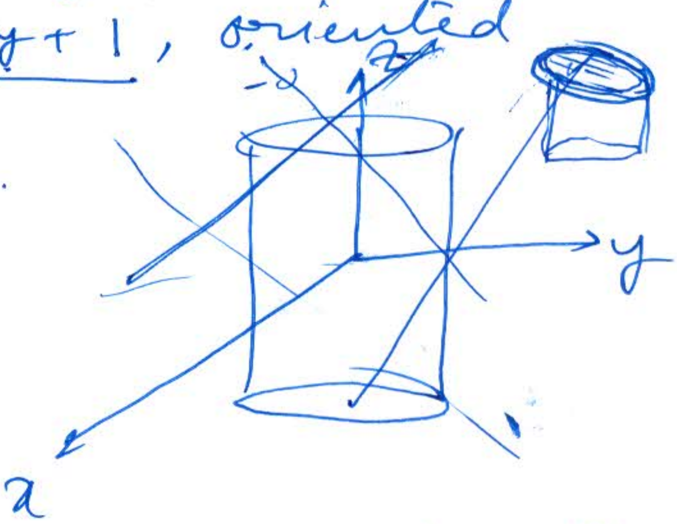
$$= -\frac{37}{2} \;//$$

Ex. Evaluate.

$$\int_C y\,dx + z\,dy + x\,dz.$$

C is the ellipse formed by the intersection of the cylinder $x^2+y^2=1$ and the plane $z = 2y+1$, oriented counterclockwise.





Parametric representation of the curve

$$x^2+y^2=1,\quad z=2y+1.$$

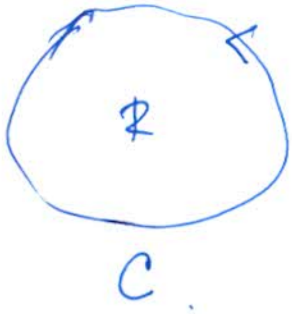
$$x=\cos\theta,\quad y=\sin\theta,\quad z=2\sin\theta+1,$$

$$0\le\theta\le 2\pi.$$

$$\int_{\theta=0}^{2\pi}\Big[\sin\theta \times -\sin\theta\, d\theta + (2\sin\theta+1)\cos\theta\, d\theta + \cos\theta \times 2\cos\theta\, d\theta\Big]$$

$$=\pi.$$

Green's theorem in plane.



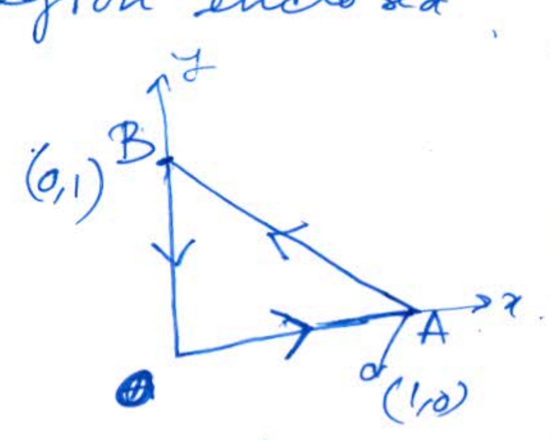
Suppose R is a simply connected region bounded by a curve C. (taken in anticlockwise direction)

then $$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Ex1. Verify Green's thm in the plane for

$$\oint \left[(3x^2-8y^2)\,dx + (4y-6xy)\,dy\right]$$

$C \to$ boundary of the region enclosed by $x=0,\ y=0,\ x+y=1$



$$\oint_C = \int_O^A + \int_A^B + \int_B^O$$

$$\int_O^A (3x^2-8y^2)\,dx + (4y-6xy)\,\underbrace{dy}_{=0}$$

$$= \int_{x=0}^{1} (3x^2-8.(0)^2)\,dx = 1.$$

Along OA, $y=0$, $0 \le x \le 1$ $dy=0$.

$$\int_B^O (3x^2-8y^2)\,dx + (4y-6xy)\,dy$$

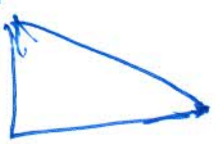
Along BO,
$x=0$
$dx=0.$
$0 \le y \le 1$

$$= \int_{y=1}^{0} \cancel{3\cdot 0^2}\ (4y - 6\times 0\times y)\,dy$$

$$= -2.$$

Along AB.
$x+y=1$
$y=t$
$x=1-t$
$0 \le t \le 1.$

$$\int_A^B (3x^2-8y^2)\,dx + (4y-6xy)\,dy$$



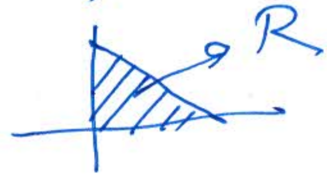
$$= \int_{t=0}^{1} \left\{3(1-t)^2 - 8t^2\right\}\times -dt + \left\{4t - 6(1-t)t\right\}dt.$$

$$= \int_0^1 \left[\left\{8t^2 - 3(1+t^2-2t)\right\} + \left\{4t - 6t + 6t^2\right\}\right] dt$$

$$= \frac{8}{3}$$

$$\int_O^A + \int_A^B + \int_B^O = 1 + \frac{8}{3} - 2 = \frac{8}{3} - 1 = \frac{5}{3}.$$

R.H.S $= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$

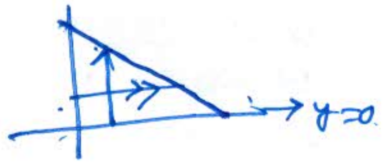


$Q = 4y - 6xy$; $\frac{\partial Q}{\partial x} = -6y$

$P = 3x^2 - 8y^2$; $\frac{\partial P}{\partial y} = -16y.$

$$R.H.S = \iint (-6y + 16y)\, dx\, dy.$$

$$= \int_{x=0}^{1} \int_{y=0}^{y=1-x} 10y\, dy\, dx.$$



$$= \int_0^1 \left[5y^2\right]_0^{1-x} dx.$$

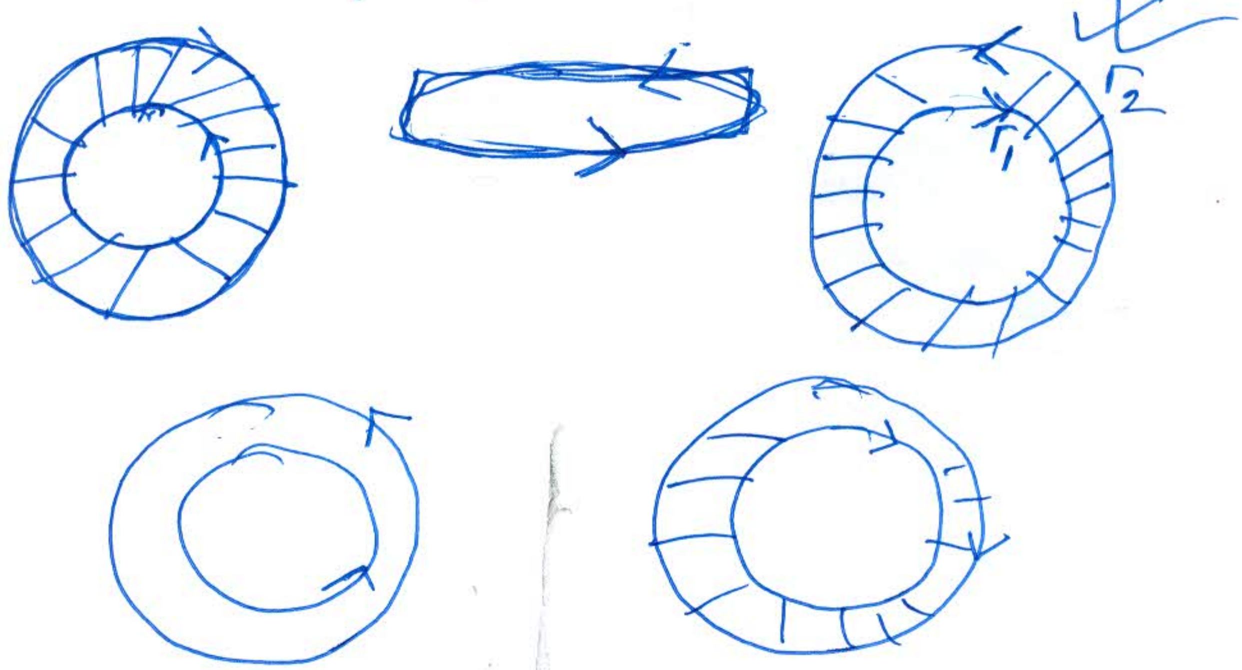
$$= 5\int_0^1 (1-x)^{2-0}\, dx = \frac{5}{3}(1-x)^3\Big]_1^0$$

$$= \frac{5}{3}$$

(2). Verify the Green's theorem for

$$\oint (xy^2\, dy - x^2 y)\, dx$$

C = boundary of the annulus $1 \leq x^2 + y^2 \leq 4$.



L.H.S

$$\oint_C \left(xy^2\,dy - x^2 y\,dx\right)$$

$$= \oint_{\Gamma_1 : x^2+y^2=1} \cdots + \int_{\Gamma_2 : x^2+y^2=4}$$

$x=\cos\theta,\ y=\sin\theta$ $\qquad$ $x = 2\cos\theta,\ y = 2\sin\theta.$

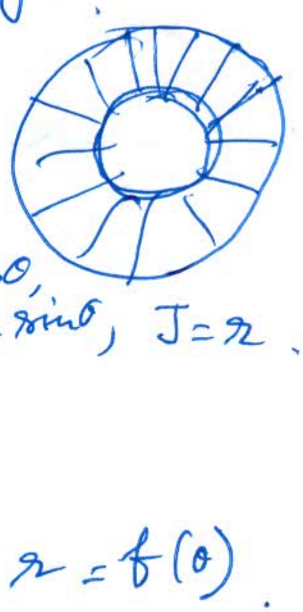
$$= \int_{\theta=2\pi}^{0} \left(\cos\theta \times \sin^2\theta \times \cos\theta + \cos^2\theta \sin\theta \sin\theta\right) d\theta + 2^4 \int_{\theta=0}^{2\pi} \left(\cos^2\theta \sin^2\theta + \cos^2\theta \sin^2\theta\right) d\theta.$$

$$= \left(2^4 - 1\right) \int_0^{2\pi} 2\cos^2\theta \sin^2\theta\, d\theta.$$

$$= \frac{15\pi}{2}.$$

R.H.S $$= \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

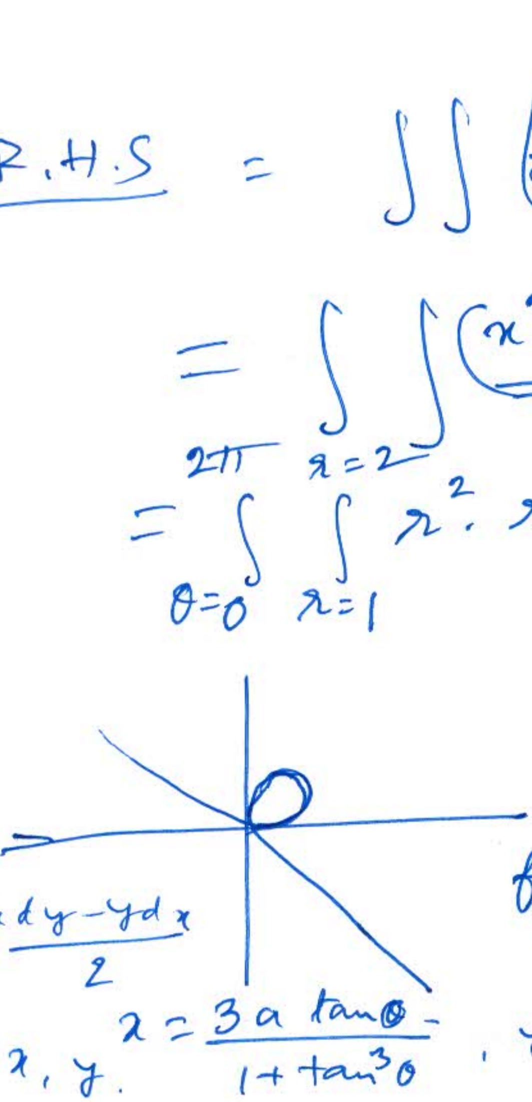
$$= \iint \underline{\underline{\left(x^2+y^2\right)}}\, dx\,dy.$$



$$= \int_{\theta=0}^{2\pi} \int_{r=1}^{r=2} r^2 \cdot r\, dr\, d\theta.$$

$x = r\cos\theta,\ y = r\sin\theta,\ J = r.$

$$= \frac{15}{2}\pi.$$



$$\int \frac{x\,dy - y\,dx}{2}$$

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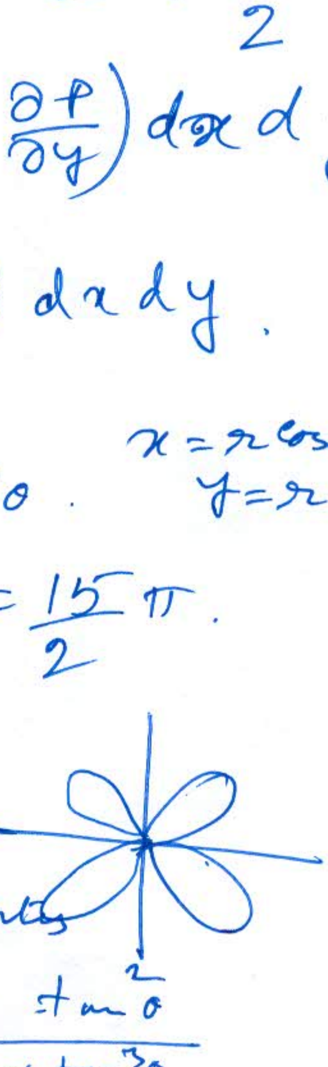
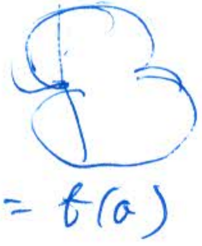


$r = f(\theta).$



$x, y.$ $\quad x = \dfrac{3a\tan\theta}{1+\tan^3\theta}, \quad y = \dfrac{3\ldots\tan^2\theta}{1+\tan^3\theta}$

$r = f(\theta).$

Ex 1 Compute

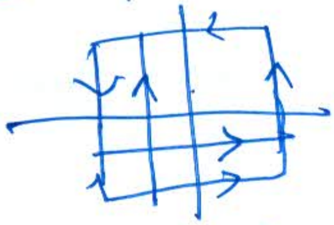
$$\oint_C x e^{-y^2} dx + \left\{ -x^2 y e^{-y^2} + \frac{1}{x^2+y^2} \right\} dy.$$

using suitable theorem of Vector Calculus.

C is bounded by $|x| \le a$, $|y| \le a$.

By Green's theorem

Given integral



$$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy.$$

$$Q = -x^2 y e^{-y^2} + \frac{1}{x^2+y^2}$$

$$\frac{\partial Q}{\partial x} = -2xy e^{-y^2} - \frac{2x}{(x^2+y^2)^2}$$

$$P = x e^{-y^2}$$

$$\frac{\partial P}{\partial y} = -2yx e^{-y^2}$$

$$= \iint_R \left[\cancel{-2xy e^{-y^2}} - \frac{2x}{(x^2+y^2)^2} + \cancel{2xy e^{-y^2}} \right] dx dy$$

$$= \int_{y=-a}^{a} \int_{x=-a}^{a} -\frac{2x}{(x^2+y^2)^2} dx dy.$$

$$= \int_{y=-a}^{a} \left[\frac{1}{x^2+y^2} \right]_{-a}^{a} dy = 0$$

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Ex 2 Compute using double integrals.

$$\oint_{C} \cancel{e^{x}(\sin y\, dx + \cos y\, dy)}.$$

$C \rightarrow$ ~~boundary~~

$$\oint (xy+y^2)\,dx + x^2\,dy$$

C = closed boundary of the region enclosed by $y = x$ & $y = x^2$.

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Evaluate

$$= -\frac{1}{20}.$$

Finding area using Green's thm.

We know, $\iint_R dx\,dy$ = area of R.

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy. \qquad Q = \frac{x}{2} \quad \frac{\partial Q}{\partial x} = \frac{1}{2}, \qquad P = -\frac{y}{2} \quad \frac{\partial Q}{\partial y} = -\frac{1}{2}$$

$$= \iint_R \left(\frac{1}{2}+\frac{1}{2}\right) dx\,dy = \text{area of } R.$$

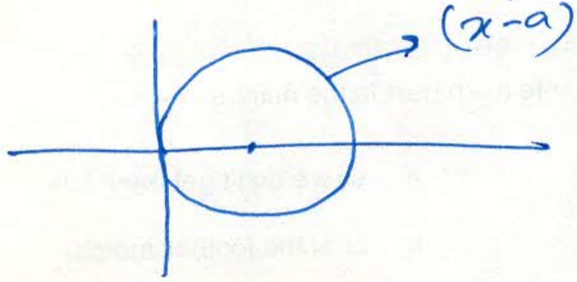
$$= \oint_C P\,dx + Q\,dy = \oint \left(\frac{x}{2}\,dy - \frac{y}{2}\,dx\right)$$

$$= \frac{1}{2}\oint_C x\,dy - y\,dx.$$

area of $R = \frac{1}{2} \oint_C (x\,dy - y\,dx)$

$C \rightarrow$ boundary of the region R.

area of polar curves.



$(x-a)^2 + y^2 = a^2$

$r = 2a\cos\theta$.

Find the area of the circle $r = 2a\cos\theta$ using line integral.

In general

$x = r\cos\theta, \quad y = r\sin\theta.$

$dx = dr\cos\theta - r\sin\theta\,d\theta.$

$dy = dr\sin\theta + r\cos\theta\,d\theta.$

$$\frac{1}{2} \oint_C (x\,dy - y\,dx)$$

$$= \frac{1}{2} \int \left(r\cos\theta \times r\cos\theta\,d\theta - r\sin\theta \times -r\sin\theta\,d\theta\right)$$

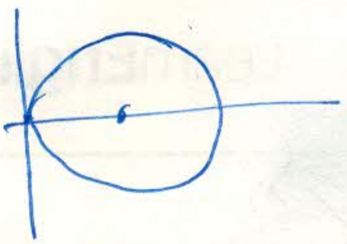
$$= \frac{1}{2} \int r^2\,d\theta = \frac{1}{2} \int f^2(\theta)\,d\theta.$$

$$x\,dy - y\,dx = r\cos\theta\,(dr\sin\theta + r\cos\theta\,d\theta) - r\sin\theta\,(dr\cos\theta - r\sin\theta\,d\theta).$$

$$= \cancel{r\sin\theta\cos\theta\,dr} + r^2\cos^2\theta\,d\theta \cancel{- r\sin\theta\cos\theta\,dr} + r^2\sin^2\theta\,d\theta$$

$$= r^2\,d\theta$$

$$= \frac{1}{2} \int_{\theta=\theta_1}^{\theta_2} f^2(\theta)\,d\theta. \qquad r = f(\theta)$$



$r = 2a \cos\theta$.

Area of this circle $= \int r^2 \, d\theta$

$$= \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} 4a^2 \cos^2\theta \, d\theta \quad = \pi a^2$$

, check by computation //.

Ex. Using line integral, compute the area of the region bounded by $y = x^2$ & $x = y^2$

$$\oint \frac{x\,dy - y\,dx}{2},$$

$$= \int_0^A \frac{x\,dy - y\,dx}{2} + \int_A^0 \frac{x\,dy - y\,dx}{2},$$

$y = x^2$: $x = t,\ y = t^2$. $x = y^2$: $x = u^2,\ y = u$

$$= \int_{t=0}^{1} + \int_{u=1}^{0} = \frac{1}{3}.$$