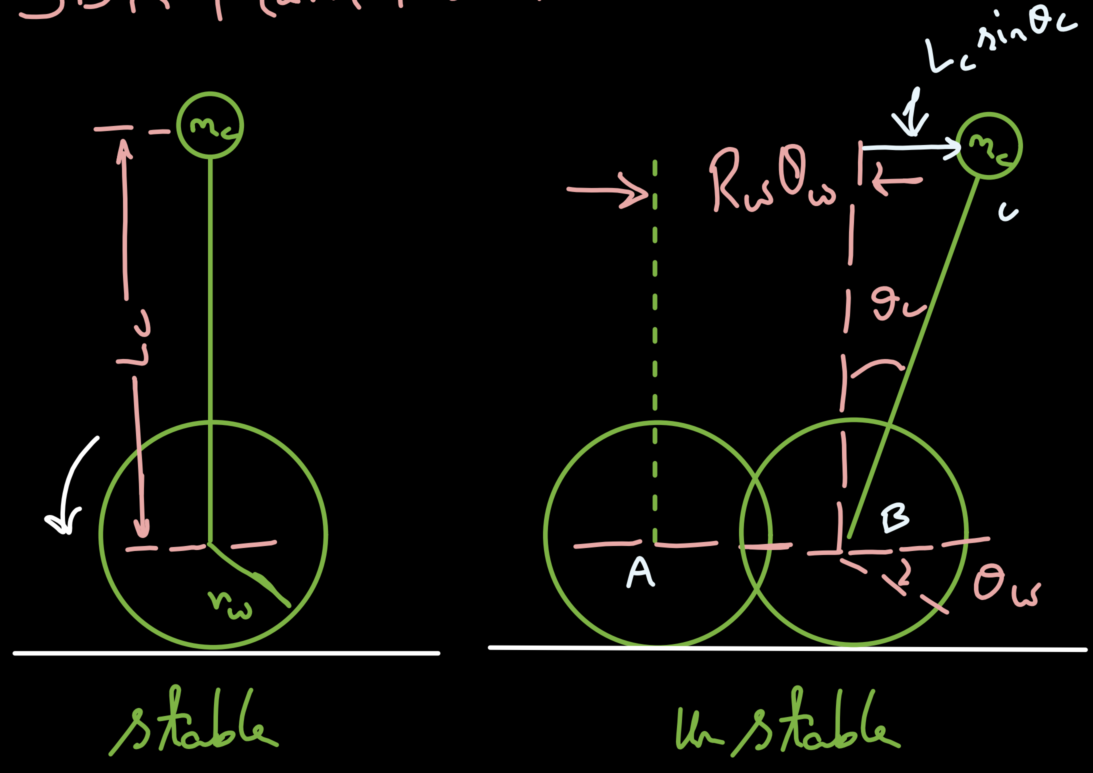
SBR Math model

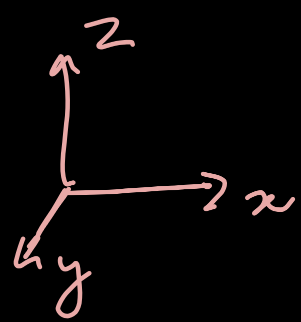


r_w - radius of wheel

m_c - centre of mass COM

θ_c - angular displacement of COM

θ_w - angular displacement of wheel



$$\dot{X} = AX + BU$$

X → state of system

A, B → constant parameter matrix

$\dot{x}$ → rate of change of states

u → input matrix (T_1, T_2)

Following varies based on given system

T_1, T_2 → torque for two wheels,

$X(\dot{\theta}_w, \dot{\theta}_c, \ddot{\theta}_w, \ddot{\theta}_c)$ → can add more state based on the system considered.

$$\begin{bmatrix} \dot{\theta}_w \\ \dot{\theta}_c \\ \ddot{\theta}_w \\ \ddot{\theta}_c \end{bmatrix} = \overset{A}{\begin{bmatrix} \quad & \quad \\ & \\ & \\ & \end{bmatrix}} \begin{bmatrix} \theta_w \\ \theta_c \\ \dot{\theta}_w \\ \dot{\theta}_c \end{bmatrix} + \overset{B}{\begin{bmatrix} \quad & \quad \\ & \\ & \\ & \end{bmatrix}} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$$

→ A

$$A = \begin{bmatrix} \frac{\partial \dot\theta_w}{\partial \theta_w} & \frac{\partial \dot\theta_w}{\partial \theta_c} & \frac{\partial \dot\theta_w}{\partial \dot\theta_w} & \frac{\partial \dot\theta_w}{\partial \dot\theta_c} \\ \frac{\partial \dot\theta_c}{\partial \theta_w} & \frac{\partial \dot\theta_c}{\partial \theta_c} & \frac{\partial \dot\theta_w}{\partial \dot\theta_w} & \frac{\partial \dot\theta_c}{\partial \dot\theta_c} \\ - & - \cdots & - & - \\ - & - & - & \cdot \end{bmatrix}$$

A, B matrices are used for LQR (K) matrix computation

Displacement of Centre of Gravity $[y_c = 0]$

$$\begin{bmatrix} X_c \\ Z_c \end{bmatrix} = \begin{bmatrix} R_w \theta_w + L_c \sin\theta_c \\ L_c \cos\theta_c \end{bmatrix}$$

Length of AB

BC component

$$\begin{bmatrix} \dot X_c \\ \dot Z_c \end{bmatrix} = \begin{bmatrix} R_w \dot\theta_w + L_c \dot\theta_c \cos\theta_c \\ - L_c \dot\theta_c \sin\theta_c \end{bmatrix}$$

Wheel Displacement $[y_w = 0]$

$$\begin{bmatrix} X_w \\ Z_w \end{bmatrix} = \begin{bmatrix} R_w \theta_w \\ 0 \end{bmatrix} \qquad \begin{bmatrix} \dot X_w \\ \dot Z_w \end{bmatrix} = \begin{bmatrix} R_w \dot\theta_w \\ 0 \end{bmatrix}$$

Lagrangian energy Equation [2 wheels, $PE_w = 0$]

$$\boxed{L = K.E_c + 2\,KE_w - PE_c - 2\,PE_w}$$

$$L = \left[\tfrac{1}{2} M_c V_c^2 + \tfrac{1}{2} I_c \theta_c^2\right] + \left[m_\omega V_\omega^2 + I_\omega \theta_\omega^2\right] - \left[M_c g\, z_c\right]$$

$$V_c^2 = \dot{x}_c^2 + \dot{z}_c^2 = R_\omega^2 \theta_\omega^2 + L_c^2 \theta_c^2 + 2R_\omega L_c \dot{\theta}_\omega \dot{\theta}_c \cos\theta_c$$

$$V_\omega^2 = R_\omega^2 \dot{\theta}_\omega^2$$

$$L = \frac{1}{2} m_c \left(R_\omega^2 \theta_\omega^2 + L_c^2 \theta_c^2 + 2R_\omega L_c \dot{\theta}_\omega \dot{\theta}_c \cos\theta_c\right) + \frac{1}{2} I_c \theta_c^2$$

$$+ M_\omega R_\omega^2 \dot{\theta}_\omega^2 + I_\omega \theta_\omega^2 - M_c g L_c \cos\theta_c$$

$$= \left(\frac{1}{2} M_c + M_\omega\right) R_\omega^2 \dot{\theta}_\omega^2 + \left(\frac{1}{2} I_c + \frac{1}{2} m_c L_c^2\right) \dot{\theta}_c^2 + I_\omega \dot{\theta}_\omega^2$$

$$+ m_c L_c \cos\theta_c \left(R_\omega \dot{\theta}_\omega \dot{\theta}_c - g\right)$$

$$L = \left[\left(\frac{m_c}{2} + m_\omega\right) R_\omega^2 + I_\omega\right] \dot{\theta}_\omega^2 + \left(\frac{I_c + m_c L_c^2}{2}\right) \dot{\theta}_c^2 + m_c L_c \cos\theta_c \left(R_\omega \dot{\theta}_\omega \dot{\theta}_c - g\right)$$

Euler - Lagrangian Equation

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = F}$$

$$\frac{\partial L}{\partial \theta_\omega} = 0 \quad \rightarrow ①, \qquad \frac{\partial L}{\partial \theta_c} = -M_c L_c \left(R_\omega \dot{\theta}_\omega \dot{\theta}_c - g\right) \sin\theta_c \quad \rightarrow ②$$

$$\frac{\partial L}{\partial \dot{\theta}_\omega} = 2\dot{\theta}_\omega \left(I_\omega + m_\omega R_\omega^2 + \frac{m_c R_\omega^2}{2}\right) + M_c L_c R_\omega \dot{\theta}_c \cos\theta_c \quad \rightarrow ③$$

$$\frac{\partial L}{\partial \dot{\theta}_c} = \dot{\theta}_c \left(I_c + M_c L_c^2\right) + M_c L_c R_\omega \dot{\theta}_\omega \cos\theta_c \quad \rightarrow ④$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_w}\right) = 2\ddot{\theta}_w\left(I_w + m_w R_w^2 + \frac{m_c R_w^2}{2}\right) + m_c L_c R_w\left(\ddot{\theta}_c \cos\theta_c - \dot{\theta}_c^2 \sin\theta_c\right) \rightarrow (5)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_c}\right) = \ddot{\theta}_c\left(I_c + m_c L_c^2\right) + m_c L_c R_w\left(\ddot{\theta}_w \cos\theta_c - \dot{\theta}_w \dot{\theta}_c \sin\theta_c\right) \rightarrow (6)$$

E.L eqn

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_w}\right) - \frac{\partial L}{\partial \theta_w} = T_1$$

$$2\ddot{\theta}_w\left(I_w + m_w R_w^2 + \frac{m_c R_w^2}{2}\right) + \ddot{\theta}_c\left(m_c L_c R_w \cos\theta_c\right) - \dot{\theta}_c^2\left(m_c L_c R_w \sin\theta_c\right) = T_1 \rightarrow (7)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_c}\right) - \frac{\partial L}{\partial \theta_c} = T_2$$

$$\ddot{\theta}_c\left(I_c + m_c L_c^2\right) + m_c L_c R_w \ddot{\theta}_w \cos\theta_c - m_c L_c g \sin\theta_c = T_2$$

$$\ddot{\theta}_c = \frac{T_2 + m_c L_c g \sin\theta_c - m_c L_c R_w \ddot{\theta}_w \cos\theta_c}{I_c + m_c L_c^2} \rightarrow (8)$$

$\Rightarrow$ (8) in (7)

$$2\ddot{\theta}_w\left(I_w + m_w R_w^2 + \frac{m_c R_w^2}{2}\right) - \dot{\theta}_c^2\left(m_c L_c R_w \sin\theta_c\right) + \frac{T_2 m_c L_c R_w \cos\theta_c}{I_c + m_c L_c^2} + \frac{m_c^2 L_c^2 R_w g \sin\theta_c \cos\theta_c}{I_c + m_c L_c^2} - \left(\frac{m_c^2 L_c^2 R_w^2 \cos^2\theta_c}{I_c + m_c L_c^2}\right)\ddot{\theta}_w = T_1$$

$$\ddot{\theta}_w = \frac{T_1 + \dot{\theta}_c^2 (M_c L_c R_w \sin\theta_c) - T_2\left(\frac{M_c L_c R_w}{I_c + M_c L_c^2}\right)\cos\theta_c - \frac{1}{2}\left(\frac{M_c^2 L_c^2 R_w g}{I_c + M_c L_c^2}\right)\sin 2\theta_c}{2 I_w + 2 M_w R_w^2 + M_c R_w^2 - \frac{M_c^2 L_c^2 R_w^2 \cos^2\theta_c}{I_c + M_c L_c^2}} \longrightarrow (9)$$

from (7)

$$\ddot{\theta}_w = \frac{T_1 + \dot{\theta}_c^2 (M_c L_c R_w \sin\theta_c) - \ddot{\theta}_c (M_c L_c R_w \cos\theta_c)}{(2 I_w + 2 M_w R_w^2 + M_c R_w^2)}$$

$$\ddot{\theta}_c (I_c + M_c L_c^2) + \frac{M_c L_c R_w \cos\theta_c}{2 I_w + 2 M_w R_w^2 + M_c R_w^2} T_1 + \frac{M_c^2 L_c^2 R_w^2 \dot{\theta}_c^2 \sin 2\theta_c}{2(2 I_w + 2 M_w R_w^2 + M_c R_w^2)}$$

$$- \frac{M_c^2 L_c^2 R_w^2 \cos^2\theta_c}{2 I_w + 2 M_w R_w^2 + M_c R_w^2} \ddot{\theta}_c - M_c L_c g \sin\theta_c = T_2$$

$$\ddot{\theta}_c = \frac{T_2 + M_c L_c g \sin\theta_c - \frac{M_c L_c R_w \cos\theta_c}{2 I_w + 2 M_w R_w^2 + M_c R_w^2} T_1 - \frac{M_c^2 L_c^2 R_w^2 \dot{\theta}_c^2 \sin 2\theta_c}{2(2 I_w + 2 M_w R_w^2 + M_c R_w^2)}}{I_c M_c L_c^2 - \frac{M_c^2 L_c^2 R_w^2 \cos^2\theta_c}{2 I_w + 2 M_w R_w^2 + M_c R_w^2}} \longrightarrow (10)$$

Simplifying eqn (9) & (10) [$\sin\theta_c = \theta_c$]

$$\ddot{\theta}_w = \frac{(I_c + M_c L_c^2) T_1 + (I_c + M_c L_c^2)(M_c L_c R_w)(\dot{\theta}_c^2)(\sin\theta_c) - (M_c L_c R_w)\cos\theta_c T_2 - (M_c^2 L_c^2 R_w g)(\sin 2\theta_c)(1/2)}{(I_c + M_c L_c^2)(2 I_w + 2 M_w R_w^2 + M_c R_w^2) - (M_c^2 L_c^2 R_w^2 \cos^2\theta_c)} \longrightarrow (11)$$

$$\ddot{\theta}_c = \frac{(2 I_w + 2 M_w R_w^2 + M_c R_w^2) T_2 + (2 I_w + 2 M_w R_w^2 + M_c R_w^2)(M_c L_c G) \sin\theta_c - (M_c L_c R_w)(T_1)\cos\theta_c - (M_c^2 L_c^2 R_w^2)\dot{\theta}_c^2 \sin 2\theta_c \left(\frac{1}{2}\right)}{(I_c + M_c L_c^2)(2 I_w + 2 M_w R_w^2 + M_c R_w^2) - (M_c^2 L_c^2 R_w^2 \cos^2\theta_c)} \longrightarrow (12)$$

Assume $K_1 = 2I_w + 2M_w R_w^2 + M_c R_w^2$, $K_2 = I_c + M_c L_c^2$

$K_3 = M_c L_c R_w$, $K_4 = M_c L_c G$, $K_5 = K_1 K_2 - K_3^2$

All constants are computable from robot properties.

(11) $\Rightarrow$

$$\ddot{\theta}_w = \frac{K_2 T_1 + K_2 K_3 (\dot{\theta}_c^2)(\sin\theta_c) - K_3 (T_2)\cos\theta_c - K_3 K_4 (0.5)\sin 2\theta_c}{K_1 K_2 - K_3^2(\cos^2\theta_c)} \rightarrow (13)$$

(12) $\Rightarrow$

$$\ddot{\theta}_c = \frac{K_1 T_2 + K_1 K_4 (\sin\theta_c)^2 - K_3 (T_1)(\cos\theta_c) - K_3^2 (0.5)(\dot{\theta}_c^2)\sin 2\theta_c}{K_1 K_2 - K_3^2 (\cos^2\theta_c)} \longrightarrow (14)$$

Equilibrium point $\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ i.e stable state

$\theta_c = \theta_w = \dot{\theta}_c = \dot{\theta}_w = 0$

$$\frac{\partial \ddot{\theta}_w}{\partial \theta_w} = 0 \quad \frac{\partial \ddot{\theta}_c}{\partial \theta_w} = 0 \quad \frac{\partial \ddot{\theta}_w}{\partial \dot{\theta}_w} = 0 \quad \frac{\partial \ddot{\theta}_c}{\partial \dot{\theta}_w} = 0$$

$$\frac{\partial \ddot{\theta}_w}{\partial \theta_c} = \frac{(K_1 K_2 - K_3^2 \cos^2\theta_c)\left[(K_2 K_3)(\dot{\theta}_c^2)(\cos\theta_c) + K_3 T_2 \sin\theta_c - K_3 K_4 \cos 2\theta_c\right] - \left[K_2 T_1 + K_3 K_2 (\dot{\theta}_c^2)(\sin\theta_c) - K_3 T_2 \cos\theta_c - K_3 K_4 (0.5)(\sin 2\theta_c)\right] K_3^2 \sin 2\theta_c}{\left[K_1 K_2 - K_3^2 \cos^2\theta_c\right]^2}$$

at $\theta_c = \theta_w = \dot{\theta}_c = \dot{\theta}_w = 0$

$$\frac{(K_1 K_2 - K_3^2)\left[-K_3 K_4\right]}{\left[K_1 K_2 - K_3^2\right]^2} \Rightarrow \frac{\partial \ddot{\theta}_w}{\partial \theta_c} = \frac{-K_3 K_4}{K_5}$$

$$\frac{\partial \ddot{\theta}_c}{\partial \theta_c} = \frac{\left(K_1K_2 - K_3^2\cos^2\theta_c\right)\left[K_1K_4\cos\theta_c + K_3T_1\sin\theta_c - K_3^2(\dot{\theta}_c^2)\cos 2\theta_c\right] - \left[K_1T_2 + K_1K_4\sin\theta_c - K_3T_1\cos\theta_c - K_3^2(\theta_c^2)\left(\frac{\sin 2\theta_c}{2}\right)\right]K_3^2\sin 2\theta_c}{\left[K_1K_2 - K_3^2(\cos^2\theta_c)\right]^2}$$

at $\theta_c = \theta_w = \dot{\theta}_c = \dot{\theta}_w = 0$

$$\frac{(K_1K_2 - K_3^2)(K_1K_4)}{\left[K_1K_2 - K_3^2\right]^2} \Rightarrow \frac{\partial \ddot{\theta}_c}{\partial \theta_c} = \frac{K_1K_4}{K_5}$$

$$\frac{\partial \ddot{\theta}_w}{\partial \dot{\theta}_c} = \frac{2\dot{\theta}_c(K_2K_3)(\sin\theta_c)}{K_1K_2 - K_3^2(\cos^2\theta_c)} = 0$$

$$\frac{\partial \ddot{\theta}_c}{\partial \dot{\theta}_c} = \frac{-K_3^2(\sin 2\theta_c)(\dot{\theta}_c)}{K_1K_2 - K_3^2(\cos^2\theta_c)} = 0$$

$\Rightarrow$ A

$$\begin{bmatrix} \dot{\theta}_w \\ \dot{\theta}_c \\ \ddot{\theta}_w \\ \ddot{\theta}_c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{K_3K_4}{K_5} & 0 & 0 \\ 0 & \frac{K_1K_4}{K_5} & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_w \\ \theta_c \\ \dot{\theta}_w \\ \dot{\theta}_c \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{K_2}{K_5} & -\frac{K_3}{K_5} \\ -\frac{K_3}{K_5} & \frac{K_1}{K_5} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{K_3K_4}{K_5} & 0 & 0 \\ 0 & \frac{K_1K_4}{K_5} & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{K_2}{K_5} & -\frac{K_3}{K_5} \\ -\frac{K_3}{K_5} & \frac{K_1}{K_5} \end{bmatrix}$$