5.3 ONE DIMENSIONAL HEAT EQUATION BY EXPLICIT AND IMPLICIT METHODS

5.3 (a) BENDER-SCHMIDT'S DIFFERENCE EQUATION

One dimensional Heat equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \text{ where } c^2 = \frac{k}{s\rho} \qquad \dots (1)$$

Put $c^2 = \frac{1}{a}$ then

$$(1) \Rightarrow \frac{\partial^2 u}{\partial x^2} - a\frac{\partial u}{\partial t} = 0,\ k = \frac{s\rho}{a} \qquad \dots (2)$$

(a) **Explicit method :**

In a rectangular mesh in the xt plane with spacing h along x direction and spacing k along t direction

Denote, $(x, t) = (ih, jk)$

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{k}\ ; \quad \frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

$$(2) \Rightarrow u_{i,j+1} = \lambda u_{i+1,j} + (1-2\lambda) u_{i,j} + \lambda u_{i-1,j} \qquad \dots (3)$$

$$\text{where, } \lambda = \frac{k}{ah^2}$$

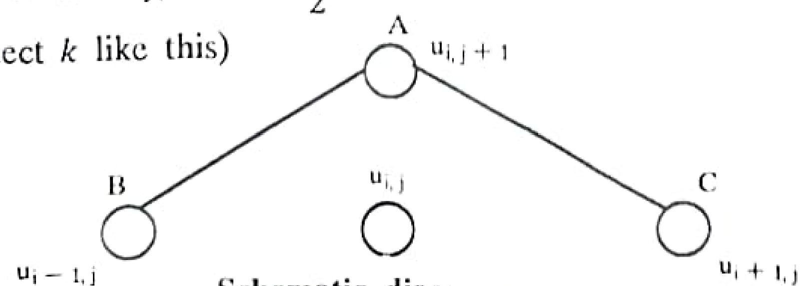
Hence, (3) is called the Schmidt explicit formula which is valid only for $0 < \lambda \le \frac{1}{2}$

Note : If $\lambda = \frac{1}{2}$ then (3) reduces to

$$u_{i,j+1} = \frac{1}{2}[u_{i-1,j} + u_{i+1,j}] \text{ where } k = \frac{ah^2}{2}$$

This is valid only, if $k = \frac{a}{2}h^2$

(so, select k like this)



Schematic diagram

Value of u at A $= \frac{1}{2}$ [Value of u at B + Value of u at C]

1. Solve $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial t} = 0$ given $u(0, t) = 0$, $u(4, t) = 0$, $u(x, 0) = x(4 - x)$. Assume $h = 1$. Find the values of u upto $t = 5$. [A.U CBT M/J 2010, Trichy M/J 2010, AU M/J 2012]

Solution :

$$\text{Given : } \quad \frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial t} = 0$$

$$\text{i.e., } \quad u_{xx} - 2u_t = 0$$

We know that, one dimensional heat equation is

$$u_{xx} - a\,u_t = 0$$

Here, $a = 2$, $k = \frac{ah^2}{2} \Rightarrow k = h^2$

Given : $h = 1 \Rightarrow k = 1$, To find : u upto $t = 5$

Given :

$u(0, t) = 0$	$u(4, t) = 0$	$u(x, 0) = x(4 - x)$
$u(0, 0) = 0$	$u(4, 0) = 0$	$u(1, 0) = 1(4 - 1) = 3$
$u(0, 1) = 0$	$u(4, 1) = 0$	$u(2, 0) = 2(4 - 2) = 4$
$u(0, 2) = 0$	$u(4, 2) = 0$	$u(3, 0) = 3(4 - 3) = 3$
$u(0, 3) = 0$	$u(4, 3) = 0$	
$u(0, 4) = 0$	$u(4, 4) = 0$	
$u(0, 5) = 0$	$u(4, 5) = 0$	

We have Bender-Schmidt relation

$$u_{i,j+1} = \frac{1}{2}[u_{i-1,j} + u_{i+1,j}]$$

x-direction, $h = 1$

$t \backslash x$	0	1	2	3	4
0	0	3	4	3	0
1	0	$\frac{0+4}{2} = 2$	$\frac{3+3}{2} = 3$	$\frac{4+0}{2} = 2$	0
2	0	$\frac{0+3}{2} = 1.5$	$\frac{2+2}{2} = 2$	$\frac{3+0}{2} = 1.5$	0
3	0	$\frac{0+2}{2} = 1$	$\frac{1.5+1.5}{2} = 1.5$	$\frac{2+0}{2} = 1$	0
4	0	$\frac{0+1.5}{2} = 0.75$	$\frac{1+1}{2} = 1$	$\frac{1.5+0}{2} = 0.75$	0
5	0	$\frac{0+1}{2} = 0.5$	$\frac{0.75+0.75}{2} = 0.75$	$\frac{1+0}{2} = 0.5$	0

b

c

$= \frac{a+b}{2}$

Given $\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial t} = 0$, $f(0, t) = f(5, t) = 0$, $f(x, 0) = x^2(25 - x^2)$, find f in the range taking $h = 1$ and upto 5 seconds.

Solution : **[A.U A/M 2011][A.U A/M 2015 (R8-10)]**

Given : $\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial t} = 0$

Here, $a = 1,\ k = \frac{a h^2}{2} = \frac{h^2}{2}$

Given : $h = 1 \Rightarrow k = \frac{1}{2} = 0.5$

To find f upto $t = 5$

Given:

$f(0,t) = 0$	$f(5,t) = 0$	$f(x,0) = x^2(25 - x^2)$
$f(0,0) = 0$	$f(5,0) = 0$	$f(1,0) = 1(25-1) = 24$
$f(0,0.5) = 0$	$f(5,0.5) = 0$	$f(2,0) = 4(25-4) = 84$
$f(0,1) = 0$	$f(5,1) = 0$	$f(3,0) = 9(25-9) = 144$
$f(0,1.5) = 0$	$f(5,1.5) = 0$	$f(4,0) = 16(25-16) = 144$
$f(0,2) = 0$	$f(5,2) = 0$	
$f(0,2.5) = 0$	$f(5,2.5) = 0$	other values find by using the following diagram
$f(0,3) = 0$	$f(5,3) = 0$	$a \searrow c \swarrow b$
$f(0,3.5) = 0$	$f(5,3.5) = 0$	
$f(0,4) = 0$	$f(5,4) = 0$	
$f(0,4.5) = 0$	$f(5,4.5) = 0$	$c = \dfrac{a+b}{2}$
$f(0,5) = 0$	$f(5,5) = 0$	

We have, Bender-Schmidt relation

$$u_{i,j+1} = \frac{1}{2}[u_{i-1,j} + u_{i+1,j}]$$

$u(0,t) = 0$ $\boxed{\rightarrow x \text{ direction } [h = 1]}$ $u(5,t)=0$

$\downarrow$ t-direction $\left[k=\frac{1}{2}\right]$

t \ x	0	1	2	3	4	5
0	0	24	84	144	144	0
0.5	0	42	84	114	72	0
1	0	42	78	78	57	0
1.5	0	39	60	67.5	39	0
2	0	30	53.25	49.5	33.75	0
2.5	0	26.625	39.75	43.5	24.75	0
3	0	19.875	35.0625	32.25	21.75	0
3.5	0	17.5312	26.0625	28.4062	16.125	0
4	0	13.0312	22.9687	21.0938	14.2031	0
4.5	0	11.4843	17.0625	18.5859	10.5469	0
5	0	8.5312	15.0351	13.8047	9.2929	0

3. Solve $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$, $0 \le x \le 1$, $t \ge 0$ with $u(x, 0) = x(1 - x)$, $0 < x < 1$ and $u(0, t) = u(1, t) = 0$, $\forall\, t > 0$, using explicit method with $\Delta x = 0.2$ for 3 time steps. **[A.U. A/M 2004]**

Solution : Given : $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$

Here, $a = 1$, $k = \dfrac{ah^2}{2} \Rightarrow k = \dfrac{h^2}{2}$

Given : $\Delta x = 0.2$ i.e., $h = 0.2 \Rightarrow k = \dfrac{(0.2)^2}{2} = 0.02$

To find u upto $t = 3$ time steps

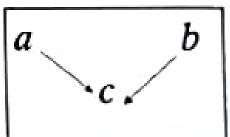
$\Rightarrow u$ for $t = 0,\ 0.02,\ 0.04,\ 0.06$

Given :

$u(0,t) = 0$	$u(1,t) = 0$	$u(x,0) = x(1-x)$
$u(0,0) = 0$	$u(1,0) = 0$	$u(0.2,0) = (0.2)(1-0.2) = 0.16$
$u(0,0.02) = 0$	$u(1,0.02) = 0$	$u(0.4,0) = (0.4)(1-0.4) = 0.24$
$u(0,0.04) = 0$	$u(1,0.04) = 0$	$u(0.6,0) = (0.6)(1-0.6) = 0.24$
$u(0,0.06) = 0$	$u(1,0.06) = 0$	$u(0.8,0) = (0.8)(1-0.8) = 0.16$

We have Bender-Schmidt relation

$$u_{i,j+1} = \frac{1}{2}[u_{i-1,j} + u_{i+1,j}]$$



$c = \dfrac{a+b}{2}$

x-direction, $h = 0.2$

t \ x	0	0.2	0.4	0.6	0.8	1.0
0	0	0.16	0.24	0.24	0.16	0
0.02	0	$\frac{0+0.24}{2}=0.12$	$\frac{0.16+0.24}{2}=0.2$	$\frac{0.24+0.16}{2}=0.2$	$\frac{0.24+0}{2}=0.12$	0
0.04	0	$\frac{0+0.2}{2}=0.1$	$\frac{0.12+0.2}{2}=0.16$	$\frac{0.2+0.12}{2}=0.16$	$\frac{0.2+0}{2}=0.1$	0
0.06	0	$\frac{0+0.16}{2}=0.08$	$\frac{0.1+0.16}{2}=0.13$	$\frac{0.16+0.1}{2}=0.13$	$\frac{0.16+0}{2}=0.08$	0

4. **Solve $u_{xx} = 32\, u_t$, taking $h = 0.25$ for $t > 0$, $0 < x < 1$ and $u(x, 0) = 0$, $u(0, t) = 0$, $u(1, t) = t$.** **[A.U. May, 2000]**

Solution : Given : $u_{xx} - 32u_t = 0$ **[A.U Tvli M/J 2010]**

We know that, one dimension heat equation is **[A.U M/J 2016 R8-10]**

$$x_{xx} - au_t = 0$$

Here, $a = 32$, $k = \dfrac{ah^2}{2} = 16\,h^2$

Given : $h = 0.25 = \dfrac{1}{4} \Rightarrow k = (16)\left(\dfrac{1}{16}\right) = 1$

Given :

$u(0, t) = 0$	$u(1, t) = t$	$u(x, 0) = 0$
$u(0, 0) = 0$	$u(1, 0) = 0$	$u(0.25, 0) = 0$
$u(0, 1) = 0$	$u(1, 1) = 1$	$u(0.5, 0) = 0$
$u(0, 2) = 0$	$u(1, 2) = 2$	$u(0.75, 0) = 0$
$u(0, 3) = 0$	$u(1, 3) = 3$	other values find by using
$u(0, 4) = 0$	$u(1, 4) = 4$	a b c
$u(0, 5) = 0$	$u(1, 5) = 5$	
⋮	⋮	$c = \dfrac{a+b}{2}$

We have Bender-Schmidt relation $u_{i,j+1} = \dfrac{1}{2}[u_{i-1,j} + u_{i+1,j}]$

t \ x	0	0.25	0.5	0.75	1
0	0	0	0	0	0
1	0	$\frac{0+0}{2} = 0$	$\frac{0+0}{2} = 0$	$\frac{0+0}{2} = 0$	1
2	0	$\frac{0+0}{2} = 0$	$\frac{0+0}{2} = 0$	$\frac{0+1}{2} = 0.5$	2
3	0	$\frac{0+0}{2} = 0$	$\frac{0+0.5}{2} = 0.25$	$\frac{0+2}{2} = 1$	3
4	0	$\frac{0+0.25}{2} = 0.125$	$\frac{0+1}{2} = 0.5$	$\frac{0.25+3}{2} = 1.625$	4
5	0	$\frac{0+0.5}{2} = 0.25$	$\frac{0.125+1.625}{2} = 0.875$	$\frac{0.5+4}{2} = 2.25$	5
⋮	⋮	⋮	⋮	⋮	⋮

5. **Using Bender-Schmidt's method, solve : $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ given $u(0, t) = 0$, $u(1, t) = 0$, $u(x, 0) = \sin \pi x$, $0 < x < 1$ and $h = 0.2$. Find the value of u upto $t = 0.1$**

[A.U. Nov. 1996, AU M/J 2012, M/J 2014][A.U A/M 2015 (R8)]
[A.U N/D 2016 (R13)]

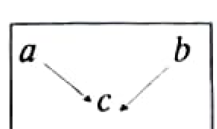
Solution : Given : $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$

Here, $a = 1$, $k = \frac{ah^2}{2} \Rightarrow k = \frac{h^2}{2}$

Given : $h = 0.2$, $\Rightarrow k = \frac{(0.2)^2}{2} = 0.02$

To find : u upto $t = 0.1$

Given :

$u(0, t) = 0$	$u(1, t) = 0$	$u(x, 0) = \sin \pi x$
$u(0, 0) = 0$	$u(1, 0) = 0$	$u(0.2, 0) = 0.5878$
$u(0, 0.02) = 0$	$u(1, 0.02) = 0$	$u(0.4, 0) = 0.9511$
$u(0, 0.04) = 0$	$u(1, 0.04) = 0$	$u(0.6, 0) = 0.9511$
$u(0, 0.06) = 0$	$u(1, 0.06) = 0$	$u(0.8, 0) = 0.5878$
$u(0, 0.08) = 0$	$u(1, 0.08) = 0$	Other values are found by,
$u(0, 0.1) = 0$	$u(1, 0.1) = 0$	 where $c = \frac{a+b}{2}$

We have Bender-Schmidt relation

$$u_{i,j+1} = \frac{1}{2}[u_{i-1,j} + u_{i+1,j}]$$

$x \rightarrow$ direction $h = 0.2$

	$t \backslash x$	0	0.2	0.4	0.6	0.8	1.0
t-direction	0	0	0.5878	0.9511	0.9511	0.5878	0
$k = 0.02$	0.02	0	0.4756	0.7695	0.7695	0.4756	0
$\downarrow$	0.04	0	0.3848	0.6225	0.6225	0.3848	0
	0.06	0	0.3113	0.5036	0.5036	0.3113	0
	0.08	0	0.2518	0.4074	0.4074	0.2518	0
	0.1	0	0.2037	0.3296	0.3296	0.2037	0

6. **Find the values of $u(x, t)$ satisfying the parabolic equation $\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$ and the boundary conditions $u(0, t) = 0 = u(8, t)$ and $u(x, 0) = 4x - \frac{x^2}{2}$ at the points $x = i$, i = 0, 1, 2, ..., 7 and $t = \frac{1}{8}j$, j = 0, 1, 2, ..., 5**

Solution : Given : $\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$ Here, $a = \frac{1}{4}$, $k = \frac{ah^2}{2} \Rightarrow k = \frac{h^2}{8}$

Given : $h = 1$, $\Rightarrow k = \frac{1}{8}$

Given :	$u(0, t) = 0$	$u(8, t) = 0$	$u(x, 0) = 4x - \frac{x^2}{2}$
	$u(0, 0) = 0$	$u(8, 0) = 0$	$u(1, 0) = 3.5$
	$u(0, 1) = 0$	$u(8, 1) = 0$	$u(2, 0) = 6$
	$u(0, 2) = 0$	$u(8, 2) = 0$	$u(3, 0) = 7.5$
	$u(0, 3) = 0$	$u(8, 3) = 0$	$u(4, 0) = 8$
	$u(0, 4) = 0$	$u(8, 4) = 0$	$u(5, 0) = 7.5$
	$u(0, 5) = 0$	$u(8, 5) = 0$	$u(6, 0) = 6$
			$u(7, 0) = 3.5$

We have Bender-Schmidt relation

$$u_{i,j+1} = \frac{1}{2}[u_{i-1,j} + u_{i+1,j}]$$

$t \backslash x$	0	1	2	3	4	5	6	7	8
0	0	3.5	6	7.5	8	7.5	6	3.5	0
1	0	3	5.5	7	7.5	7	5.5	3	0
2	0	2.75	5	6.5	7	6.5	5	2.75	0
3	0	2.5	4.625	6	6.5	6	4.625	2.5	0
4	0	2.3125	4.25	5.5625	6	5.5625	4.25	2.3125	0
5	0	2.125	3.9375	5.125	5.5625	5.125	3.9375	2.125	0

5.3.(b) CRANK-NICHOLSON'S DIFFERENCE EQUATION, CORRESPONDING TO THE PARABOLIC EQUATION.

$\left[\frac{\partial \mathbf{u}}{\partial \mathbf{t}} = \alpha^2 \frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}\right]$ **[Implicit method]**

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \qquad \dots (1)$$

We replace, $\left(\frac{\partial^2 u}{\partial x^2}\right)_{(x,y)}$ by

$$\frac{1}{2}\left[\left(\frac{\partial^2 u}{\partial x^2}\right)_{(x_i, y_i)} + \left(\frac{\partial^2 u}{\partial x^2}\right)_{(x_i, y_{j+1})}\right]$$

and $\frac{\partial u}{\partial t}$ by $\left(\frac{u_{i,j+1} - u_{i,j}}{k}\right)$ in (1).

Thus, the difference equation corresponding to (1), which is true at the point (x_i, y_j) is

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{\alpha^2}{2}\left[\left\{\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}\right\} + \left\{\frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}}{h^2}\right\}\right]$$

i.e., $u_{i,j+1} - u_{i,j} = \dfrac{k\alpha^2}{2h^2}[u_{i-1,j} - 2u_{i,j} + u_{i+1,j} + u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}]$ ... (2)

Taking $\dfrac{k\alpha^2}{h^2} = \lambda$ in (2), it becomes

$$2u_{i,j+1} - 2u_{i,j} = \lambda\,[u_{i-1,j} - 2u_{i,j} + u_{i+1,j} + u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}]$$

i.e., $-\lambda\, u_{i-1,j+1} + (2\lambda + 2)\, u_{i,j+1} - \lambda\, u_{i+1,j+1}$

$$= \lambda\, u_{i-1,j} + (2 - 2\lambda)\, u_{i,j} + \lambda\, u_{i+1,j} \quad \text{... (3)}$$

Equation (3) is known as Crank-Nicholson's difference equation in the general form. This equation is also known as the implicit formula, as it does not give the value of u at $t = t_{j+1}$, directly in terms of the values of u at $t = t_j$.

Though λ can take any value, we take $\lambda = 1$ in order to simplify the numerical work involved.

When $\lambda = 1$, the Crank-Nicholson's difference equation takes the simplest form, namely,

$$-u_{i-1,j+1} + 4u_{i,j+1} - u_{i+1,j+1} = u_{i-1,j} + u_{i+1,j} \quad \text{... (4)}$$

As far as possible, we should try to make use of Equation (4), by proper choice of h and/or k, so that $\lambda = \dfrac{k\alpha^2}{h^2} = 1$.

1. Solve $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ in $0 < x < 5$, $t \geq 0$ given that $u(x, 0) = 20$, $u(0, t) = 0$, $u(5, t) = 100$. Compute u for the time-step with $h = 1$ by Crank-Nicholson's method. [A.U A/M 2005, M/J 2006]

Solution : Given : $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ Here, $a = 1$, $k = \lambda\, ah^2$

A convenient choice of λ makes the Crank-Nicholson difference scheme simple.

Let $\lambda = 1$, Given : $h = 1 \Rightarrow k = 1$

Given :

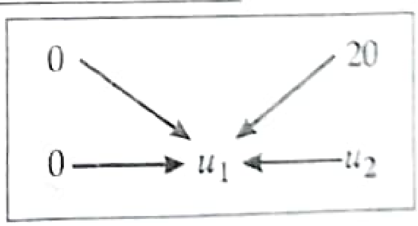
$u(0, t) = 0$	$u(5, t) = 100$	$0 < x < 5$
		$u(x, 0) = 20$
$u(0, 0) = 0$ $u(0, 1) = 0$	$u(5, 0) = 100$ $u(5, 1) = 100$	$u(1, 0) = 20$ $u(2, 0) = 20$ $u(3, 0) = 20$ $u(4, 0) = 20$

$t \backslash x$	0	1	2	3	4	5
0	0	20	20	20	20	100
1	0	u_1	u_2	u_3	u_4	100

The Crank-Nicholson's formula becomes

$$u_{i,j+1} = \frac{1}{4}\left[u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}\right]$$

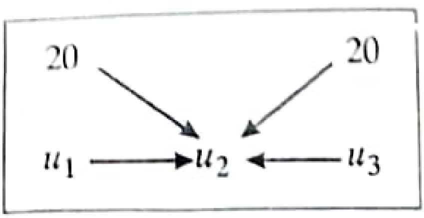
$$\boxed{u_1 = \frac{1}{4}[0 + 0 + 20 + u_2]}$$



$$\Rightarrow \quad 4u_1 = 20 + u_2$$

$$\Rightarrow \quad 4u_1 - u_2 = 20 \qquad \text{... (1)}$$

$$\boxed{u_2 = \frac{1}{4}[u_1 + 20 + 20 + u_3]}$$



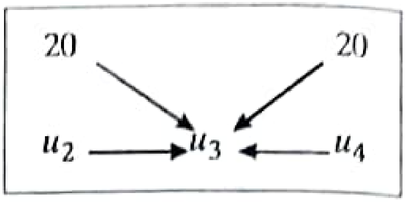
$$4u_2 = u_1 + u_3 + 40$$

$$-u_1 + 4u_2 - u_3 = 40 \quad \text{... (2)}$$

$$\boxed{u_3 = \frac{1}{4}[u_2 + 20 + 20 + u_4]}$$

$$4u_3 = u_2 + u_4 + 40$$

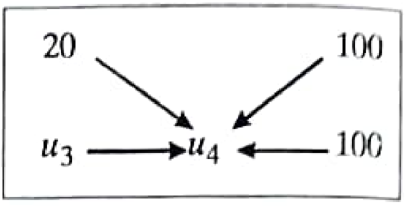
$$-u_2 + 4u_3 - u_4 = 40 \quad \dots (3)$$



$$\boxed{u_4 = \frac{1}{4}[u_3 + 20 + 100 + 100]}$$

$$4u_4 = u_3 + 220$$

$$-u_3 + 4u_4 = 220 \quad \dots (4)$$



$(3) \times 4 + (4) \Rightarrow -4u_2 + 15u_3 = 380 \quad \dots (5)$

$(2) \times 15 + (5) \Rightarrow -15u_1 + 56u_2 = 980 \quad \dots (6)$

$(1) \times 15 + 4 \times (6) \Rightarrow 209\,u_2 = 4220$

$$\boxed{u_2 = 20.19}$$

$(1) \Rightarrow \quad 4u_1 = 20 + u_2$

$4u_1 = 20 + 20.19$

$4u_1 = 40.19$

$$\boxed{u_1 = 10.05}$$

$(2) \Rightarrow u_3 = -u_1 + 4u_2 - 40$

$= -10.05 + 4\,(20.19) - 40$

$$\boxed{u_3 = 30.71}$$

$(4) \Rightarrow \quad 4u_4 = 220 + u_3$

$4u_4 = 220 + 30.71$

$4u_4 = 250.71$

$$\boxed{u_4 = 62.68}$$

Hence, the values are $u_1 = 10.05$, $u_2 = 20.19$, $u_3 = 30.71$, $u_4 = 62.68$

2. Solve by Crank-Nicholson's method the equation $u_{xx} = u_t$ subject to $u(x, 0) = 0$, $u(0, t) = 0$ and $u(1, t) = t$, for two time steps.
[A.U. N/D 2003, AU M/J 2012] [A.U N/D 2016 (R8-10)]

Solution : Given : $u_{xx} = u_t$ Here, $a = 1$, $k = \lambda a h^2$

Let $\lambda = 1$, $h = \frac{1}{4}$ [h value not given]

$$\therefore\ k = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$$

Given :

$u(0,t) = 0$	$u(1,t) = t$	$u(x,0) = 0$
$u(0,0) = 0$ $u\left(0, \frac{1}{16}\right) = 0$ $u\left(0, \frac{2}{16}\right) = 0$	$u(1,0) = 0$ $u\left(1, \frac{1}{16}\right) = \frac{1}{16}$ $u\left(1, \frac{2}{16}\right) = \frac{2}{16}$	$u(0.25, 0) = 0$ $u(0.5, 0) = 0$ $u(0.75, 0) = 0$

$t \backslash x$	0	0.25	0.5	0.75	1
0	0	0	0	0	0
$\frac{1}{16}$	0	u_1	u_2	u_3	$\frac{1}{16}$
$\frac{2}{16}$	0	u_4	u_5	u_6	$\frac{2}{16}$

The Crank-Nicholson's formula becomes,

$$\boxed{u_{i,j+1} = \frac{1}{4}\left[u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}\right]}$$

I set :

$$\boxed{u_1 = \frac{1}{4}[0 + 0 + 0 + u_2]}$$

$$\begin{tikzcd} 0 \arrow[dr] & & 0 \arrow[dl] \\ 0 \arrow[r] & u_1 & u_2 \arrow[l] \end{tikzcd}$$

$$4u_1 = u_2 \qquad \ldots (1)$$

$$\boxed{u_2 = \frac{1}{4}[u_1 + 0 + 0 + u_3]}$$

$$\begin{tikzcd} 0 \arrow[dr] & & 0 \arrow[dl] \\ u_1 \arrow[r] & u_2 & u_3 \arrow[l] \end{tikzcd}$$

$$4u_2 = u_1 + u_3 \qquad \ldots (2)$$

$$16u_1 = u_1 + u_3 \text{ by } (1)$$

$$15u_1 = u_3 \qquad \ldots (3)$$

$$\begin{tikzcd} 0 \arrow[dr] & & 0 \arrow[dl] \\ u_2 \arrow[r] & u_3 & 1/16 \arrow[l] \end{tikzcd}$$

$$\boxed{u_3 = \frac{1}{4}\left[u_2 + 0 + 0 + \frac{1}{16}\right]}$$

$$4u_3 = u_2 + \frac{1}{16}$$

$$60u_1 = 4u_1 + \frac{1}{16} \text{ by (1) \& (3)}$$

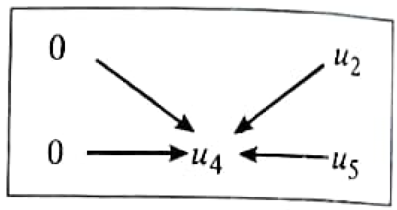
$$56u_1 = \frac{1}{16}$$

$$u_1 = \frac{1}{896} = 0.0011$$

i.e., $\boxed{u_1 = 0.0011}$

(1) $\Rightarrow$ $u_2 = 4u_1 = 4(0.0011)$

$$\boxed{u_2 = 0.0044}$$

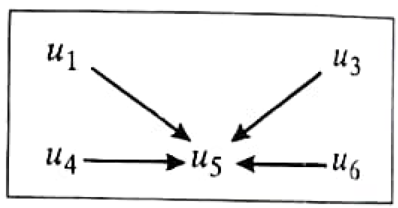


(3) $\Rightarrow$ $u_3 = 15u_1 = 15(0.0011)$

$$\boxed{u_3 = 0.0165}$$

II set : $\boxed{u_4 = \frac{1}{4}[0 + 0 + u_2 + u_5]}$

$$4u_4 = u_2 + u_5$$



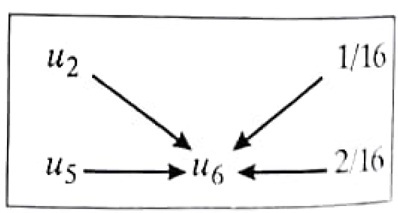
$$4u_4 - u_5 = 0.0044 \quad \text{... (4)}$$

$$\boxed{u_5 = \frac{1}{4}[u_4 + u_1 + u_3 + u_6]}$$

$$4u_5 = u_4 + 0.0011 + 0.0165 + u_6$$

$$-u_4 + 4u_5 - u_6 = 0.0176 \quad \text{... (5)}$$

$$\boxed{u_6 = \frac{1}{4}\left[u_5 + u_2 + \frac{1}{16} + \frac{2}{16}\right]}$$



$$4u_6 = u_5 + 0.0044 + \frac{3}{16}$$

$$-u_5 + 4u_6 = 0.1919 \quad \text{... (6)}$$

(5) × 4 + (6) × 1 $\Rightarrow$ $-4u_4 + 15u_5 = 0.2623$ ... (7)

(4) + (7) $\Rightarrow$ $14u_5 = 0.2667$

$$\boxed{u_5 = 0.0191}$$

(4) $\Rightarrow$
$$4u_4 = 0.0044 + u_5$$
$$4u_4 = 0.0044 + 0.0191$$
$$4u_4 = 0.0235$$
$$\boxed{u_4 = 0.0059}$$

(6) $\Rightarrow$
$$4u_6 = 0.1919 + u_5$$
$$4u_6 = 0.1919 + 0.0191$$
$$4u_6 = 0.211$$
$$\boxed{u_6 = 0.0528}$$

Hence, the values are

$u_1 = 0.0011$, $u_2 = 0.0044$, $u_3 = 0.0165$

$u_4 = 0.0059$, $u_5 = 0.0191$, $u_6 = 0.0528$

3. **Solve $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$, $0 < x < 2$, $t > 0$, $u(0, t) = u(2, t) = 0$, $t > 0$ and $u(x, 0) = \sin\dfrac{\pi x}{2}$, $0 \le x \le 2$ using $\Delta x = 0.5$, and $\Delta t = 0.25$ for two times steps by Crank-Nicholson's implicit finite difference method.** **[A.U. A/M 2003]**

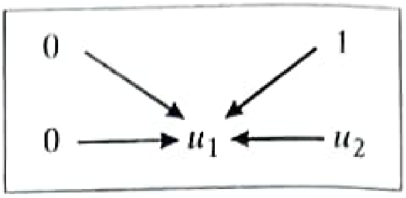
Solution : Here, $a = 1$, $h = \Delta x = 0.5$, $k = \Delta t = 0.25$

The Crank-Nicholson's formula

$$u_{i,j+1} = \frac{1}{4}[u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}]$$

j \ i	0	0.5	1	1.5	2
0	0	0.707	1	0.707	0
0.25	0	u_1	u_2	u_3	0
0.5	0	u_4	u_5	u_6	0

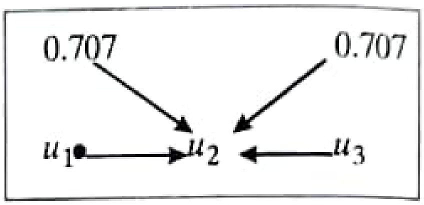
I set : $\boxed{u_1 = \frac{1}{4}[0 + 0 + 1 + u_2]}$



$$4u_1 = 1 + u_2 \quad \text{... (1)}$$

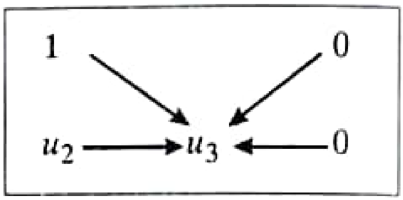
$$u_2 = \frac{1}{4}[u_1 + 0.707 + 0.707 + u_3]$$

$$4u_2 = u_1 + 1.414 + u_3 \quad \text{... (2)}$$



$$u_3 = \frac{1}{4}[u_2 + 1 + 0 + 0]$$

$$4u_3 = 1 + u_2 \quad \text{... (3)}$$



From (1) & (3), we get

R.H.S. (1) = R.H.S (3)

$$4u_1 = 4u_3$$

$$\Rightarrow \boxed{u_1 = u_3} \quad \text{... (4)}$$

$$(2) \Rightarrow 4u_2 = 2u_1 + 1.414 \quad \text{... (5)}$$

$$(5) \times 2 \Rightarrow 8u_2 = 4u_1 + 2.828$$

$$8u_2 = 1 + u_2 + 2.828 \text{ by (1)}$$

$$7u_2 = 3.828$$

$$\boxed{u_2 = 0.5469}$$

$$(1) \Rightarrow 4u_1 = 1 + 0.5469$$

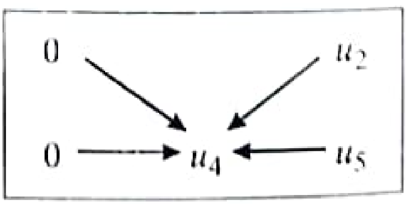
$$\boxed{u_1 = 0.3867}$$

$$(4) \Rightarrow \boxed{u_3 = 0.3867}$$

II set : $\boxed{u_4 = \frac{1}{4}[0 + 0 + u_2 + u_5]}$

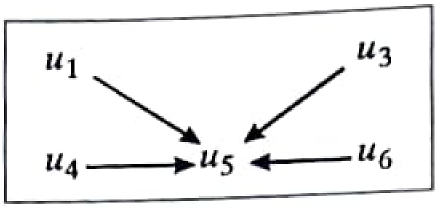
$$4u_4 = u_2 + u_5$$

$$4u_4 = 0.5469 + u_5 \quad \text{... (A)}$$



$$4u_4 - u_5 = 0.5469 \quad \text{... (6)}$$

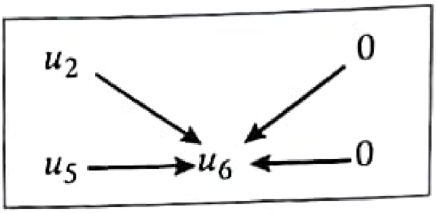
$$\boxed{u_5 = \frac{1}{4}[u_4 + u_1 + u_3 + u_6]}$$



$$4u_5 = u_4 + 0.3867 + 0.3867 + u_6$$

$$-u_4 + 4u_5 - u_6 = 0.7734 \quad \dots (7)$$

$$\boxed{u_6 = \frac{1}{4}[u_5 + u_2 + 0 + 0]}$$



$$4u_6 = u_5 + 0.5469 \quad \dots (B)$$

$$-u_5 + 4u_6 = 0.5469 \quad \dots (8)$$

(A) & (B), we get

$$4u_4 = 4u_6 \Rightarrow u_4 = u_6 \quad \dots (9)$$

$$(7) \Rightarrow \quad -2u_4 + 4u_5 = 0.7734 \dots (10) \text{ by } (9)$$

$$(6) \times 4 + (10) \Rightarrow \quad 14u_4 = 2.961$$

$$\boxed{u_4 = 0.2115}$$

$$(9) \Rightarrow \quad \boxed{u_6 = 0.2115}$$

$$(8) \Rightarrow \quad u_5 = 4u_6 - 0.5469$$

$$u_5 = 4(0.2115) - 0.5469$$

$$\boxed{u_5 = 0.2991}$$

Hence, the values are

$$u_1 = u_3 = 0.3867, \; u_2 = 0.5469$$

$$u_4 = u_6 = 0.2115, \; u_5 = 0.2991$$

$a^2 = 16$

$a = 4$

4. Use Crank-Nicholson's scheme to solve $\frac{\partial^2 u}{\partial x^2} = 16 \frac{\partial u}{\partial t}$, $0 < x < 1$ and $t > 0$ given $u(x, 0) = 0$, $u(0, t) = 0$ and $u(1, t) = 100t$. Compute $u(x, t)$ for one time step, taking $\Delta x = 1/4$. **[A.U M/J 2007, Tvli N/D 2010] [A.U CBT N/D 2011] [A.U A/M 2017 R-13]**

Solution : Here, $a = 16,\ h = \frac{1}{4}$

$$\therefore\ k = ah^2 = 16\left(\frac{1}{16}\right) = 1\ ;$$

$$u_{i,j+1} = \frac{1}{4}\left[u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j} + u_{i-1,j}\right]$$

$\rightarrow x$ increasing

$\downarrow t$

$j \backslash i$	0	0.25	0.5	0.75	1
0	0	0	0	0	0
1	0	u_1	u_2	u_3	100

$$u_1 = \frac{1}{4}(0 + 0 + 0 + u_2) \qquad \therefore\ u_1 = \frac{1}{4}u_2 \qquad \text{... (2)}$$

$$u_2 = \frac{1}{4}(0 + 0 + u_1 + u_3) \qquad \therefore\ u_2 = \frac{1}{4}(u_1 + u_3) \qquad \text{... (3)}$$

$$u_3 = \frac{1}{4}(0 + 0 + u_2 + 100) \qquad \therefore\ u_3 = \frac{1}{4}(u_2 + 100) \qquad \text{... (4)}$$

Substitute u_1, u_3 values in (3),

$$u_2 = \frac{1}{4}\left[\frac{1}{4}(2u_2 + 100)\right] = \frac{1}{8}u_2 + \frac{25}{4}$$

$$\therefore\quad u_2 = \frac{50}{7} = 7.1429$$

$$u_1 = 1.7857\ ;\ u_3 = 26.7857$$

The values are $u_1 = 1.7857$, $u_2 = 7.1429$, $u_3 = 26.7857$.

SHORT QUESTIONS AND ANSWERS

1. **Classify the following equation :** **[A.U A/M 2015]**

$$\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0.$$ **[A.U N/D 2016 (R-13)]**

Solution : Here $A = 1,\ B = 4,\ C = 4,\ D = -1,\ E = 2$

$$\begin{aligned} B^2 - 4AC &= 4^2 - 4(1)(4) \\ &= 16 - 16 \\ &= 0 \end{aligned}$$

$\therefore$ The given equation is parabolic.

2. **Classify the equation :** **[A.U N/D 2016 (R8-10)]**

$u_{xx} + 2u_{xy} + 4u_{yy} = 0.$

Solution : Here $A = 1,\ B = 2,\ C = 4$

$$\begin{aligned} B^2 - 4AC &= 4 - 4(1)(4) \\ &= -12 < 0 \end{aligned}$$

$\therefore$ The given equation is elliptic.

3. **Classify the partial differential equation :**

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Solution : Here $A = 1,\ B = -2,\ C = 1$

$$\begin{aligned} B^2 - 4AC &= 4 - 4 \\ &= 0 \end{aligned}$$

$\therefore$ The given equation is said to be parabolic.

4. **What is the classification of $f_x - f_{yy} = 0$?**

[M.U. Oct. 97, Apr 9

Solution : Here, $A = 0,\ B = 0,\ C = -1$

$$B^2 - 4AC = 0 - 4 \times 0 \times -1 = 0$$

So, the equation is parabolic.

5. Give an example of a parabolic equation.

Solution : The one dimensional heat equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ is parabolic.

6. State Schmidt's explict formula for solving heat flow equation.
[A.U CBT M/J 2010, Tvli M/J 2010, CBT N/D 2010]
[A.U CBT A/M 2011, AU M/J 2012]

Solution : $u_{i,j+1} = \lambda u_{i+1,j} + (1 - 2\lambda) u_{i,j} + \lambda u_{i-1,j}$

$$\text{If } \lambda = \frac{1}{2}, \quad u_{i,j+1} = \frac{1}{2}[u_{i+1,j} + u_{i-1,j}]$$

7. Fill in the blank.

Bender-Schmidt recurrence scheme is useful to solve .... equation.

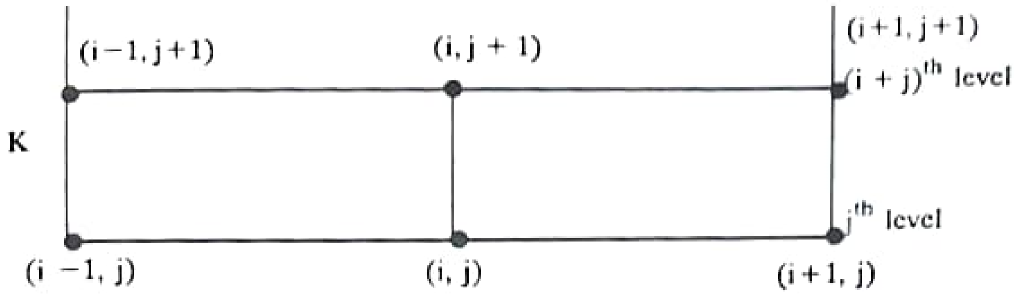
Solution : One dimensional heat

8. Write an explicit formula to solve numerically the heat equation (parabolic equation) [A.U CBT N/D 2011][A.U A/M 2017 R-13]
$u_{xx} - au_t = 0$

Solution : $u_{i,j+1} = \lambda u_{i+1,j} + (1 - 2\lambda) u_{i,j} + \lambda u_{i-1,j}$

where, $\lambda = \frac{k}{h^2 a}$ (h is the space for the variable x and k is the space in the time direction).

The above formula is a relation between the function values at the two levels $j + 1$ and j and is called a two level formula. The solution value at any point $(i, j + 1)$ on the $(j + 1)$th level is expressed in terms of the solution values at the points $(i - 1, j)$, (i, j) and $(i + 1, j)$, on the jth level. Such a method is called explicit formula. The formula is geometrically represented below.



9. **What is the value of k to solve $\dfrac{\partial u}{\partial t} = \dfrac{1}{2} u_{xx}$ by Bender-Schmidt method with $h = 1$ if h and k are the increments of x and t respectively ?** **[A.U CBT A/M 2011]**

Solution : Given : $u_{xx} = 2\dfrac{\partial u}{\partial t}$ Here $\alpha^2 = 2,\ h = 1$

$$\lambda = \frac{k\alpha^2}{h^2} = \frac{k\,(2)}{1} = 2k$$

$$\lambda = 2k = \frac{1}{2} \qquad [\because 0 < \lambda \leq \tfrac{1}{2}]$$

$$k = \frac{1}{4}$$

10. **What is the classification of one dimensional heat flow equation.**

Solution : One dimensional heat flow equation is $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{\alpha^2}\dfrac{\partial u}{\partial t}$

Here, A = 1, B = 0, C = 0

$$\therefore B^2 - 4AC = 0$$

Hence, the one dimensional heat flow equation is parabolic.

11. **Write down the Crank-Nicholson's formula to solve $u_t = u_{xx}$.** **[A.U. A/M, 2004]**

Solution : $\frac{1}{2}\lambda\, u_{i+1,j+1} + \frac{1}{2}\lambda\, u_{i-1,j+1} - (\lambda + 1)\, u_{i,j+1}$

$$= -\frac{1}{2}\lambda\, u_{i+1,j} - \frac{1}{2}\lambda\, u_{i-1,j} + (\lambda - 1)\, u_{i,j}$$

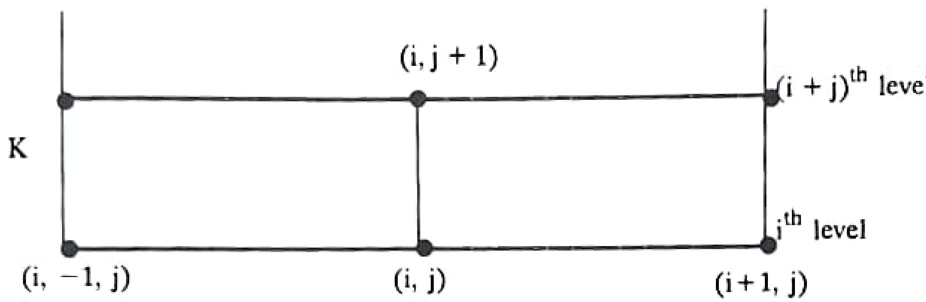
$$\text{(or)}\ \lambda\,(u_{i+1,j+1} + u_{i-1,j+1}) - 2(\lambda + 1)\, u_{i,j+1} = 2(\lambda - 1)\, u_{i,j} - \lambda\,(u_{i+1,j} + u_{i-1,j})$$

12. Write down the implicit formula to solve one dimensional heat flow equation. $u_{xx} = \frac{1}{c^2} u_t$ **[A.U Nov/Dec. 2004]** **[A.U CBT M/J 2010]**

Solution : Same as (11)

13. Why is Crank-Nicholson's scheme called an implicit scheme ? **[A.U, April, 1996] [A.U A/M 2015 (R8)]**

Solution : The schematic representation of Crank Nicholson's method is shown below.



The solution value at any point $(i, j+1)$ on the $(j+1)^{th}$ level is dependent on the solution values at the neighbouring points on the same level and on three values on the j^{th} level. Hence it is an implicit method.

14. Fill up the blanks.

In the parabolic equation. $u_t = \alpha^2 u_{xx}$ if $\lambda = \frac{k\alpha^2}{h^2}$

where $k = \Delta t$ and $h = \Delta x$, then

(a) explicit method is stable only if $\lambda =$ .....

(b) implicit method is convergent when $\lambda =$ **[M.U. April 96]**

Solution : (a) explicit method is stable only if $\lambda < \frac{1}{2}$

(b) Implicit method is convergent when $\lambda = \frac{1}{2}$

15. What type of equations can be solved by using Crank-Nicholson's difference formula ? **[A.U. April 1998]**

Solution : Crank-Nicholson's difference formula is used solve parabolic equations of the form.

$$u_{xx} = a u_t$$

16. Write the Crank-Nicholson's difference scheme to solve $u_{xx} = a\,u_t$ with $u(0, t) = T_0$, $u(l, t) = T_l$ and the initial condition as $u(x, 0) = f(x)$

Solution : The scheme is **[A.U CBT A/M 2011]**

$$\frac{1}{2}\lambda\, u_{i-1,j+1} - (\lambda + 1)\, u_{i,j+1} + \frac{1}{2}\lambda\, u_{i+1,j+1}$$

$$= -\frac{1}{2}\lambda\, u_{i-1,j} + (\lambda - 1)\, u_{i,j} - \frac{1}{2}\lambda\, u_{i+1,j}$$

17. For what purpose Bender-Schmidt recurrence relation is used ?

Solution : To solve one dimensional heat equation.

18. Mention any two single step methods for solving an ordinary differential equation, subject to initial condition.

[AU CBT. N/D 2010]

Solution : 1. Bender-Schmidt ; 2. Crank-Nicholson

19. What is the condition of stability for the Schmidt method?

Solution : λ satisfies the condition $\lambda \le \frac{1}{2}$

20. What is the order of the Crank-Nicholson's method for solving the heat conduction equation?

Solution : $O(k^2 + h^2)$

21. What is the condition of stability for the Crank-Nicholson's method?

Solution : The Crank-Nicholson's method is stable for the values of the mesh ratio parameter λ. The method is also called an unconditionally stable method.

22. What type of system of equations do we get, when we apply the Crank-Nicholson's method to solve the one dimensional heat conduction equation?

Solution : We obtain a linear tridiagonal system of algebraic equations.