CS 215 : Assignment 3

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Question 4

Data : Assume given data is $x_1, x_2, x_3, ..., x_n$ is from U(0,θ)
Prior : Given prior is

$$P(\theta) \propto \begin{cases} \left(\frac{\theta_m}{\theta}\right)^\alpha & \theta \geq \theta_m \\ 0 & \text{otherwise} \end{cases}$$

Likelihood : Likelihood for uniform distribution is

$$Likelihood = \left(\frac{1}{\theta^n}\right)$$

For $\hat{\theta}^{\text{ML}}$, Likelihood has to be maximized then θ to be minimized

$$x_1, x_2, .., x_n \in (0, \theta)$$

$$\implies \theta = max(x_i) \ where \ i \in [1, n]$$

Posterior :

$$Posterior \propto Likelihood * Prior$$

$$P(\theta|x_1, x_2, ..., x_n) \propto \begin{cases} \frac{\theta_m^\alpha}{\theta^{n+\alpha}} & \theta \geq max\{\theta_m, max\{x_i\}\} \\ 0 & \text{otherwise} \end{cases}$$

For $\hat{\theta}^{\text{MAP}}$, Posterior has to be maximized then θ to be minimized

$$\theta = max\{\theta_m, max\{x_i\}\}$$

Since other part is zero
Now, if $\theta_m > \theta$, $\hat{\theta}^{\text{MAP}}$ is always θ_m, which doesn't depend on sample size. So, $\hat{\theta}^{\text{MAP}}$ need not tend to $\hat{\theta}^{\text{ML}}$ as n tending to infinity.
As MAP estimator always doesn't tend to ML estimator, MAP estimator is not desirable.

Posterior Mean :

$$
\begin{aligned}
\hat{\theta}^{\text{PM}} &= E_{Posterior}[\theta] \\
&= \frac{\int_P^\infty \frac{\theta_m^\alpha d\theta}{\theta^{n+\alpha-1}}}{\int_P^\infty \frac{\theta_m^\alpha d\theta}{\theta^{n+\alpha}}} \\
where\ P &= max\{\theta_m, max\{x_i\}\} \\
&= \frac{\left(\frac{P^{n+\alpha-2}}{n+\alpha-2}\right)}{\left(\frac{P^{n+\alpha-1}}{n+\alpha-1}\right)} \\
&= P\left(\frac{n+\alpha-1}{n+\alpha-2}\right)
\end{aligned}
$$

$$
\implies \hat{\theta}^{\text{PM}} = \hat{\theta}^{\text{MAP}}\left(\frac{n+\alpha-1}{n+\alpha-2}\right)
$$

Now, for n tending to infinity, $\hat{\theta}^{\text{PM}}$ tends to $\hat{\theta}^{\text{MAP}}$ which doesn't tends $\hat{\theta}^{\text{ML}}$ always. As Posterior Mean estimator always doesn't tend to ML estimator, Posterior Mean estimator is not desirable.