CS 215 Assignment - 2

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1) Given:

Random variables X and Y having pdf's $f_X(x)$ and $f_Y(y)$ resp and joint pdf $f_{XY}(x,y)$.

Now for pdf of random variable $Z = X+Y$. $\longrightarrow$ ①

Let the pdf be $f_Z(z)$ which can be obtained from $F_Z(z)$ where $F_Z(z)$ is the cdf.

$$F_Z(z) = \int_{-\infty}^{z} f_Z(z)\,dz$$

$$\Rightarrow f_Z(z) = \frac{d}{dz}(F_Z(z))$$

From ① for $Z \leq z$, $x+y \leq z$, this can be obtained from $f_{XY}(x,y)$

$$F_Z(z) = \iint_{x+y<z} f_{XY}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{z-x} f_{XY}(x,y)\,dy\right)dx$$

$$f_Z(z) = \frac{d}{dz}(F_Z(z))$$

$$= \frac{d}{dz}\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{z-x} f_{XY}(x,y)\,dy\right)dx\right)$$

$$= \int_{-\infty}^{\infty}\frac{d}{dz}\left(\int_{-\infty}^{z-x} f_{XY}(x,y)\,dy\right)dx$$

On using Leibniz rule

$$= \int_{-\infty}^{\infty}\left(f_{XY}(x,z-x)(1) - \lim_{y\to-\infty} f_{XY}(x,y)(0)\right)dx$$

$$\therefore f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x, z-x)\,dx \longrightarrow ②$$

This can also be implied simply by saying if $x = x$, then $y = z - x$, and x varies $-\infty$ to $+\infty$.

For $P(X \leq Y)$

Now in this case y varies from $-\infty$ to $+\infty$, whereas X varies from $-\infty$ to y (when $Y = y$).

$$\Rightarrow P(X \leq Y) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{y} f_{XY}(x,y)\,dx \right) dy. \longrightarrow ③$$

If 'X' and 'Y' are independent:

$$f_{XY}(x,y) = f_X(x) f_Y(y) \longrightarrow ④$$

Here pdf of $Z = X + Y$ is $\int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx \longrightarrow ⑤$

from ② and ④

$$P(X \leq Y) \text{ is } \int_{-\infty}^{\infty} \left(\int_{-\infty}^{y} f_{XY}(x,y)\,dx \right) dy$$

$$\therefore P(X \leq Y) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{y} f_X(x) \cdot f_Y(y)\,dx \right) dy \longrightarrow ⑥$$

from ③ and ④.

2) Given:

$X_1, X_2, X_3, \ldots, X_n$ be $n > 0$ iid random variables with cdf $F_X(x)$ and pdf $f_X(x) = F_X'(x)$.

$$Y_1 = \max(X_1, X_2, X_3, \ldots, X_n)$$

$$P(Y_1 \leq x) = P\left(\bigcap_{i=1}^{n} (X_i \leq x) \right)$$

$$= P(X_1 \leq x) \cdot P(X_2 \leq x) \cdot P(X_3 \leq x) \cdots P(X_n \leq x)$$

As they are independent.

$= (P(X_1 \le x))^n$ As they are identical

$= (F(x))^n$

$\therefore$ cdf of $Y_1 = (F(x))^n$.

pdf of $Y_1 = \frac{d}{dx}(F(x))^n = n(F(x))^{n-1} \cdot F'(x) = n(F(x))^{n-1} \cdot f(x)$

$Y_2 = \min(X_1, X_2, X_3, \ldots, X_n)$

$P(Y_2 \le x) = 1 - P(Y_2 > x)$

$= 1 - P(\bigcap_{i=1}^{n} (X_i > x))$ using similar arguments as above.

$= 1 - [P(X_1 > x) \cdot P(X_2 > x) \ldots P(X_n > x)]$

$= 1 - (P(X_1 > x))^n$

$= 1 - (1 - P(X_1 \le x))^n$

$= 1 - (1 - F(x))^n$.

$\therefore$ cdf of $Y_2 = 1 - (1 - F(x))^n$
$(F_{Y_2}(x))$

pdf of $Y_2 = \frac{d}{dx}(F_{Y_2}(x))$

$= \frac{d}{dx}\left(1 - (1 - F(x))^n\right)$

$= -n(1 - F(x))^{n-1} \cdot (-1) \cdot F'(x)$

$= n(1 - F(x))^{n-1} \cdot f(x)$.

$\therefore$ pdf of $Y_2 = n(1 - F(x))^{n-1} f(x)$.

$\mathrm{Max}(X_1, X_2, X_3, \ldots, X_n) \le x \Rightarrow \forall i \in [1, n] \quad X_i \le x$

$\mathrm{Min}(X_1, X_2, X_3, \ldots, X_n) > x \Rightarrow \forall i \in [1, n] \quad X_i > x$

3) Markov's inequality:

Let X be a random variable that takes only non-negative values. for any $a>0$,

$$P\{X \geq a\} \leq \frac{E(X)}{a}.$$

To prove:

$$P(X-\mu \geq \tau) \leq \frac{\sigma^2}{\sigma^2+\tau^2} \quad \text{if } \tau>0$$

$$P(X-\mu \geq \tau) \geq 1-\frac{\sigma^2}{\sigma^2+\tau^2} \quad \text{if } \tau<0$$ (Chebyshev's one-sided inequality).

Proof:

Let $Y = X-\mu$

$$E(Y) = E(X-\mu) = E(X) - E(\mu) = \mu - \mu = 0.$$

$$Var(Y) = Var(X-\mu) = Var(X) = \sigma^2$$

if $\lambda>0$, for any $u \geq 0$

$$P(X-\mu \geq \lambda) = P(Y \geq \lambda)$$

$$= P(Y+u \geq \lambda+u)$$

$$\leq P((Y+u)^2 \geq (\lambda+u)^2)$$

$$\leq \frac{E((Y+u)^2)}{(\lambda+u)^2} = \frac{E(Y^2)+E(2uY)+E(u^2)}{(\lambda+u)^2}$$

$$= \frac{\sigma^2+2u(0)+u^2}{(\lambda+u)^2}$$

$$= \frac{\sigma^2+u^2}{(\lambda+u)^2}.$$

if $x,y>0$.
$x<y$
$\Rightarrow x^2<y^2$.
but. if $x^2<y^2$,
$\not\Rightarrow x<y$
so $P(x<y) \leq P(x^2<y^2)$

$$P(X-\mu \geq \lambda) \leq \frac{\sigma^2+u^2}{(\lambda+u)^2} \quad \text{for any } u \geq 0$$

To tighten the boundary choose min. value of $\frac{\sigma^2+u^2}{(\lambda+u)^2}$.

Let $f(u) = \frac{\sigma^2+u^2}{(\lambda+u)^2}.$

$$f'(u) = \frac{(\lambda+u)^2(2u) - (\sigma^2+u^2)\,2(\lambda+u)}{(\lambda+u)^4} = 0$$

$$\Rightarrow \frac{2(\lambda+u)[\cancel{u^2}+\lambda u - \sigma^2 - \cancel{u^2}]}{(\lambda+u)^4} = 0$$

$$\Rightarrow u = \frac{\sigma^2}{\lambda}.$$

for $f'(u^-) < 0$, $f'(u^+) > 0$ $\Rightarrow u = \frac{\sigma^2}{\lambda}$ gives minimum

$$f(u)\Big|_{min} = \frac{\sigma^2 + \frac{\sigma^4}{\lambda^2}}{\frac{(\lambda^2+\sigma^2)^2}{\lambda^2}} = \frac{\frac{\sigma^2}{\cancel{\lambda^2}}\cancel{(\sigma^2+\lambda^2)}}{\frac{(\sigma^2+\lambda^2)^{\cancel{2}}}{\cancel{\lambda^2}}} = \frac{\sigma^2}{\sigma^2+\lambda^2}.$$

$$\therefore\ P(X-\mu \geq \tau) \leq \frac{\sigma^2}{\sigma^2+\tau^2}. \longrightarrow ①$$

If $\tau < 0$ then

$$P(X-\mu < \tau) = P(-(X-\mu) > -\tau)$$

$-\tau > 0.$

$E(-(X-\mu)) = 0.$

$Var(-(X-\mu)) = \sigma^2.$

$P(X-\mu<\tau) \leq \frac{\sigma^2}{\sigma^2+\tau^2}$ from ① and $P(X<x) \leq P(X \leq x)$

$P(X-\mu \geq \tau) \geq 1 - \frac{\sigma^2}{\sigma^2+\tau^2}$ $\because P(X-\mu \geq \tau) = 1 - P(X-\mu<\tau)$

$$\therefore\ P(X-\mu \geq \tau) \geq 1 - \frac{\sigma^2}{\sigma^2+\tau^2}.$$

Hence one-side Chebyshev inequality is proved.

4) To prove:

$$P(X \geq x) \leq e^{-tx} \phi_X(t) \quad \text{for } t>0$$

$$P(X \leq x) \leq e^{-tx} \phi_X(t) \quad \text{for } t<0.$$

Proof:

If $t>0$.

$$P(X \geq x) = \int_x^\infty f_X(x)\, dx$$

$$= \int_x^\infty f_X(y)\, dy$$

$$= \int_x^\infty e^{-ty} f_X(y) e^{ty}\, dy$$

$x \leq y \leq \infty$

$tx \leq ty < \infty$

$-tx \geq -ty > -\infty$

$e^{-tx} \geq e^{-ty} > 0$

$$\leq e^{-tx} \int_x^\infty f_X(y) e^{ty}\, dy$$

(increasing the limits)

$$\leq e^{-tx} \int_{-\infty}^\infty e^{ty} f_X(y)\, dy$$

$$= e^{-tx} \cdot \phi_X(t)$$

(defn of MGF)

If $t<0$

$$P(X \leq x) = \int_{-\infty}^x f_X(y)\, dy$$

~~$x \leq y \leq \infty$~~

$$= \int_{-\infty}^x e^{-ty} f_X(y) e^{ty}\, dy$$

$-\infty \leq y \leq x$

$+\infty > ty \geq tx$

$-\infty < -ty \leq -tx.$

$$\leq e^{-tx} \int_{-\infty}^x f_X(y) e^{ty}\, dy$$

(similar arguments)

$$\leq e^{-tx} \int_{-\infty}^\infty e^{ty} f_X(y)\, dy$$

$$= e^{-tx} \cdot \phi_X(t)$$

Hence proved.

To prove:

$$P(X > (1+\delta)\mu) \le \frac{e^{\mu(e^t-1)}}{e^{(1+\delta)\mu t}} \quad \text{, for any } t \geq 0, \delta > 0 \text{ ,}$$

X is sum of n independent Bernoulli random variables $X_1, X_2, X_3, \ldots, X_n$ where $E(X_i) = P_i$, $\mu = \Sigma P_i$

Proof:

Using the above derived inequality

For $t \geq 0$.

$$P(X > (1+\delta)\mu) \le \phi_X(t) \cdot e^{-(1+\delta)\mu t}$$

As X_i's are independent,

$$\phi_X(t) = \prod_{i=1}^{n} \phi_{X_i}(t)$$

As X_i is Bernoulli random variable.

$$\phi_{X_i}(t) = 1 + P_i(e^t - 1)$$

$$\Rightarrow \phi_X(t) = \left(1+P_1(e^t-1)\right)\left(1+P_2(e^t-1)\right) \ldots \left(1+P_n(e^t-1)\right) \quad \because 1+x \le e^x$$

$$\le e^{P_1(e^t-1)} \cdot e^{P_2(e^t-1)} \cdots e^{P_n(e^t-1)}$$

$$= e^{(P_1+P_2+P_3+\cdots+P_n)(e^t-1)}$$

$$= e^{\mu(e^t-1)}$$

$$\Rightarrow \phi_X(t) \le e^{\mu(e^t-1)}$$

$$\Rightarrow P(X > (1+\delta)\mu) \le \frac{\phi_X(t)}{e^{(1+\delta)\mu t}} \le \frac{e^{\mu(e^t-1)}}{e^{(1+\delta)\mu t}} \quad \text{for any } t \geq 0, \delta > 0$$

To tighten this bound we need to find min. value of $\frac{e^{\mu(e^t-1)}}{e^{(1+\delta)\mu t}}$ as explained in solution of problem 3.

$$\text{Let } f(t) = \frac{e^{\mu(e^t-1)}}{e^{(1+\delta)\mu t}}.$$

$$= e^{\mu(e^t - 1 - t(1+\delta))}$$

$$f'(t) = e^{\mu(e^t - 1 - t(1+\delta))} \cdot \mu(e^t - (1+\delta)) = 0$$

$$\Rightarrow e^t = 1+\delta$$

$$\Rightarrow t = \log(1+\delta)$$

$f'(t^-) < 0$, $f'(t^+) > 0 \Rightarrow t$ gives minimum value.

If we choose $t = \log(1+\delta)$, we can get tight bound compared to our previous bound.

5.) Given: 'k' people

Probability that each person independently of others have a disease $= p$

Probability that exactly 'i' persons have the disease

$$= \underbrace{{}^{k}C_{i}}_{\text{Choose } i \text{ persons from 'k'}} \underbrace{p^{i}(1-p)^{k-i}}_{\text{Probability that 'i' persons have disease and remaining 'k-i' don't.}}$$

$\therefore$ Probability that atleast one have disease

$$= {}^{k}C_{1}\, p^{1}(1-p)^{k-1} + {}^{k}C_{2}\, p^{2}(1-p)^{k-2} + \cdots + {}^{k}C_{k}\, p^{k}$$

$$= \sum_{i=1}^{k} {}^{k}C_{i}\, p^{i}(1-p)^{k-i}$$

$$= \left(\sum_{i=0}^{k} {}^{k}C_{i}\, p^{i}(1-p)^{k-i}\right) - (1-p)^{k}$$

$$= (p+(1-p))^{k} - (1-p)^{k} \quad \text{(By Binomial Theorem)}$$

$$= 1-(1-p)^{k}$$

Expected number of tests in case 1 $= k$ [$\because$ Every time 'k' tests are done i.e., k is a constant]

Expected number of tests in case 2 $= P\left(\begin{smallmatrix}\text{no one}\\ \text{have a}\\ \text{disease}\end{smallmatrix}\right)\cdot 1 + P\left(\begin{smallmatrix}\text{atleast}\\ \text{one have}\\ \text{a disease}\end{smallmatrix}\right)\cdot (k+1)$

$$= (1-p)^{k} + \left(1-(1-p)^{k}\right)(k+1)$$

$$= k+1 - k(1-p)^{k}$$

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Expected number in case 2 $<$ Expected number in case 1

$$K+1-K(1-p)^K < K$$

$$\Rightarrow \quad K(1-p)^K > 1$$

$$\Rightarrow \quad (1-p)^K > \frac{1}{K}$$

$$\Rightarrow \quad (1-p) > \left(\frac{1}{K}\right)^{1/K}$$

$$\Rightarrow \quad 1-p > K^{-1/K}$$

$$\Rightarrow \quad p < 1-K^{-1/K}$$

We need two values of p such that this inequality hols for $K \in [2, 25]$.

We know that $K^{1/K}$ attains maximum value at $K=3$ if K is an integer.

But we need minimum value of $1-K^{-1/K}$, $K \in [2,25]$

$$\because \quad 25^{1/25} < 2^{1/2}$$

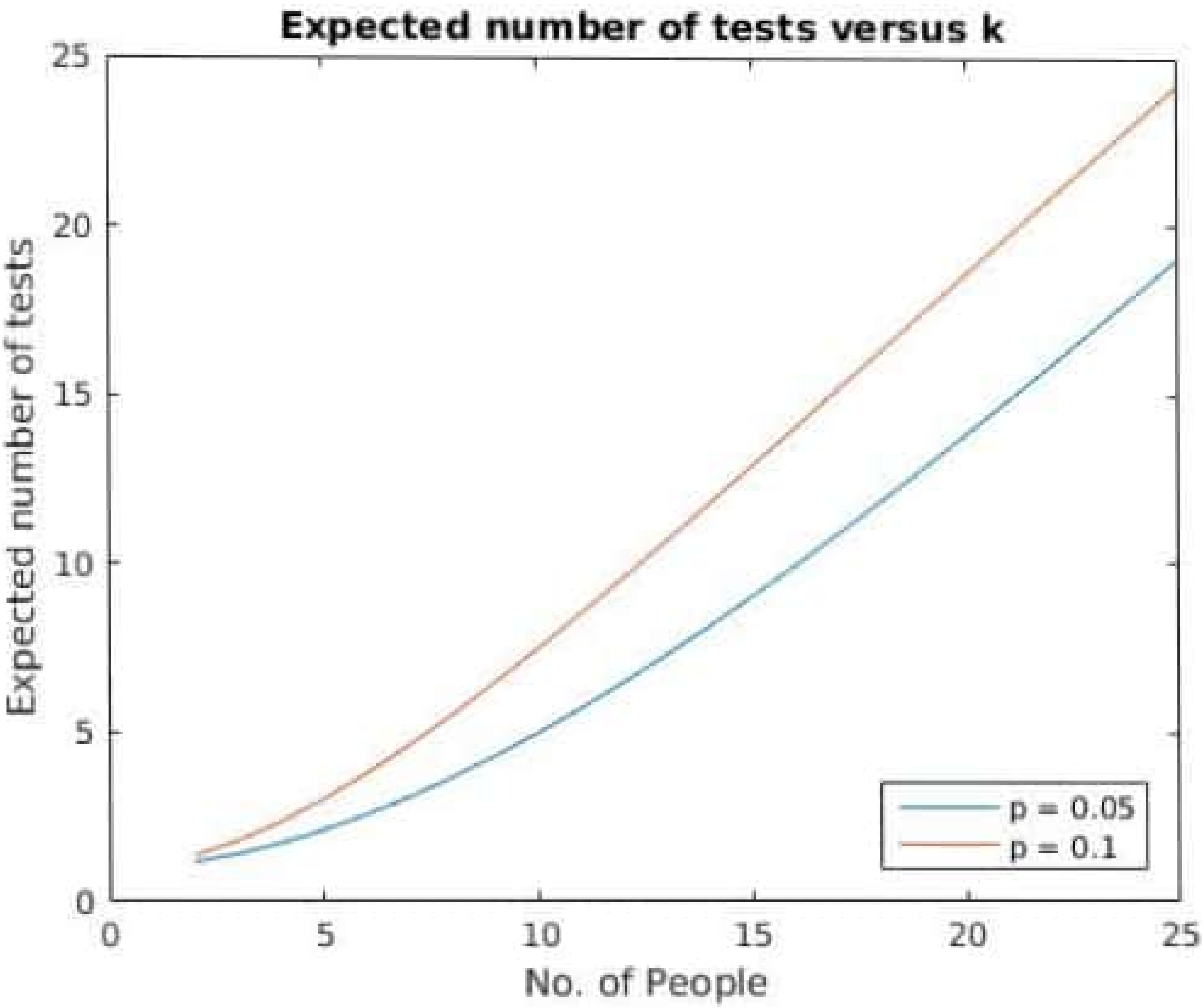
$$2^{-1/2} < 25^{-1/25}$$

$$-25^{-1/25} < -2^{-1/2}$$

$$1-25^{-1/25} < 1-2^{-1/2}$$

$\therefore \quad 1-25^{-1/25} \approx 0.137$. So, $\left(1-25^{-1/25}\right)$ is the minimum value.

So, we take $p=0.05$ and $p=0.1$ and plot the corresponding expected number versus 'k'.


Expected number of tests versus k
Expected number of tests
No. of People
p = 0.05
p = 0.1
0
5
10
15
20
25

Comment on the plot:

Taken $p = 0.05$ and $p = 0.1$.

$k \in [2, 25]$

Expected number of tests in case 2 $= k+1 - k(1-p)^k$

For a constant k,

when p increases, Expected number increases

So, The plot of $p = 0.1$ lies above $p = 0.05$.

The graph is of the form:

$$y = x + 1 - x(1-p)^k$$

$$\frac{dy}{dx} = \underbrace{1 - (1-p)^k}_{>0} - k(1-p)^k \underbrace{\log(1-p)}_{\text{Negative } (\because 1-p<1)} \qquad \left[\because \frac{d}{dx}(a^x) = a^x \log a\right]$$

$\therefore \frac{dy}{dx} > 0$

$\Rightarrow$ Increasing function.

Since, $1-p < 1$,

For very large k, $(1-p)^k$ tends to 0.

then y tends to $1+x$

then it looks like a straight line.

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6.) Comments:

Correlation coefficient of T_1 and T_2:

Here, the correlation coefficient has 2 contradicting factors in the images i.e., the central part of the image is negatively correlated whereas the outer black background of the image is positively correlated which leads to overall positive 'ρ' by the graph. By the graph it has minimum when $t_x = -1$.

When we keep on shifting from here, the positive factor i.e., blackbackground factor (because we add 0's) increases, so 'ρ' increases.

Correlation coefficient of I_1 and $255-I_1$:

At $t_x = 0$, $I_2 = -I_1 + 255$, so they are negatively correlated with 'ρ' $= -1$. When we shift, we increase the black background part i.e., (by adding 0's) which increases the positive factor of 'ρ', so 'ρ' increases.

So, we get a minimum at $t_x = 0$.

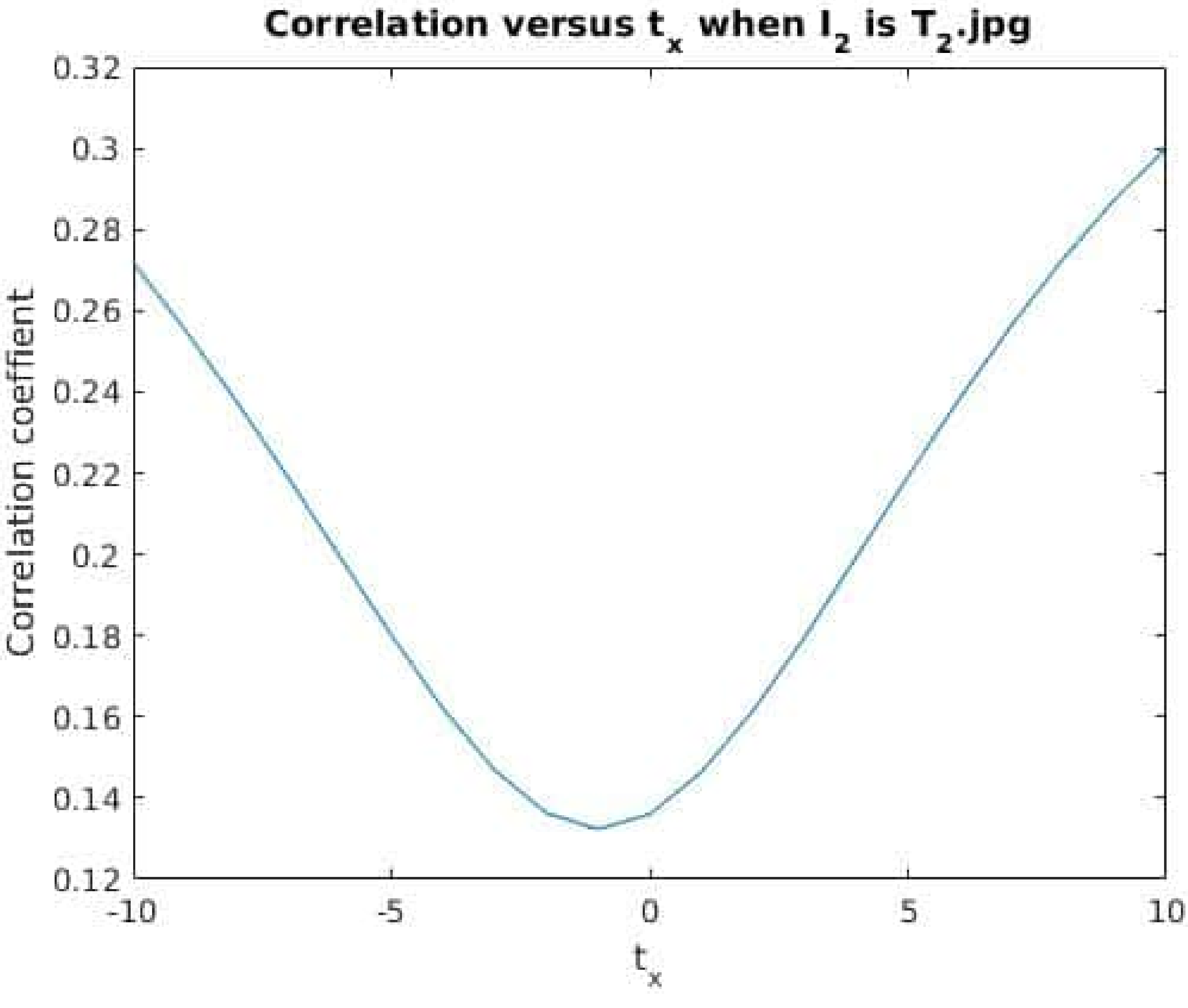
QMI ~~coefficient~~ of I_1 and $255-I_1$:

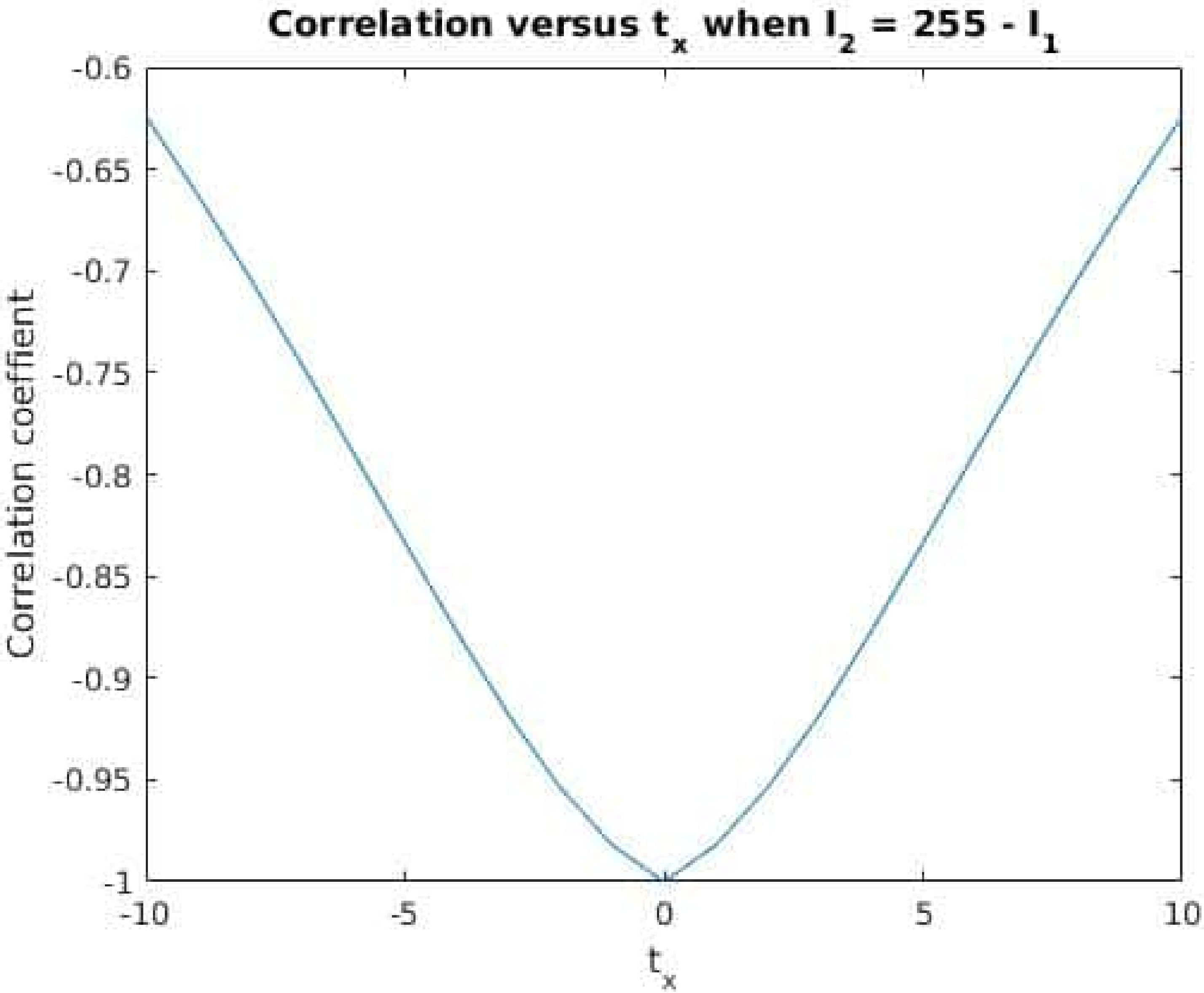
At $t_x = 0$, one image is exactly the inverse of the other, QMI ~~coefficient~~ is maximum. When we shift, the inverse relation is disturbed, so, the QMI ~~coefficient~~ decreases.

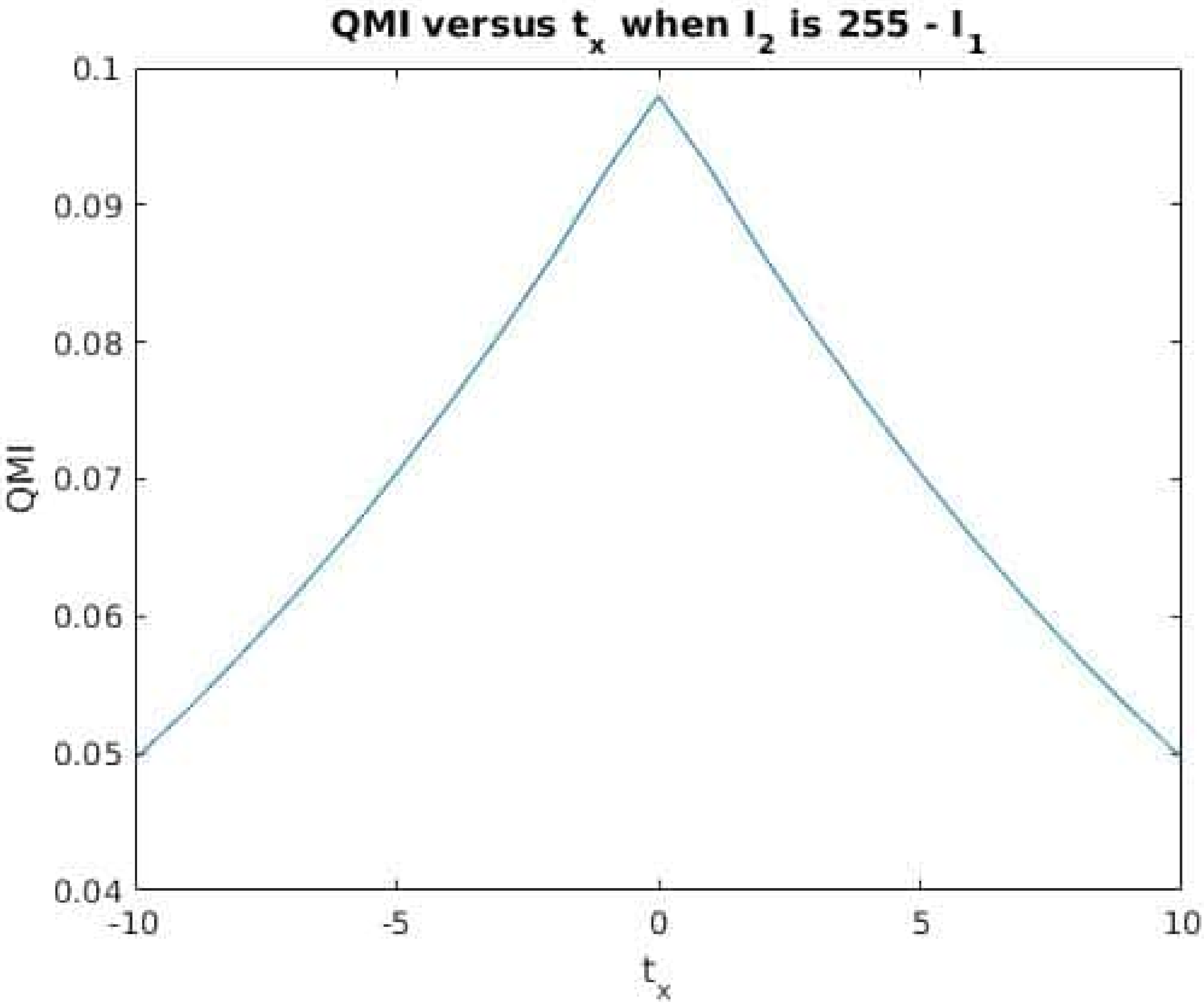
So, we get a maximum at $t_x = 0$.

QMI ~~coefficient~~ of I_1 and I_2:

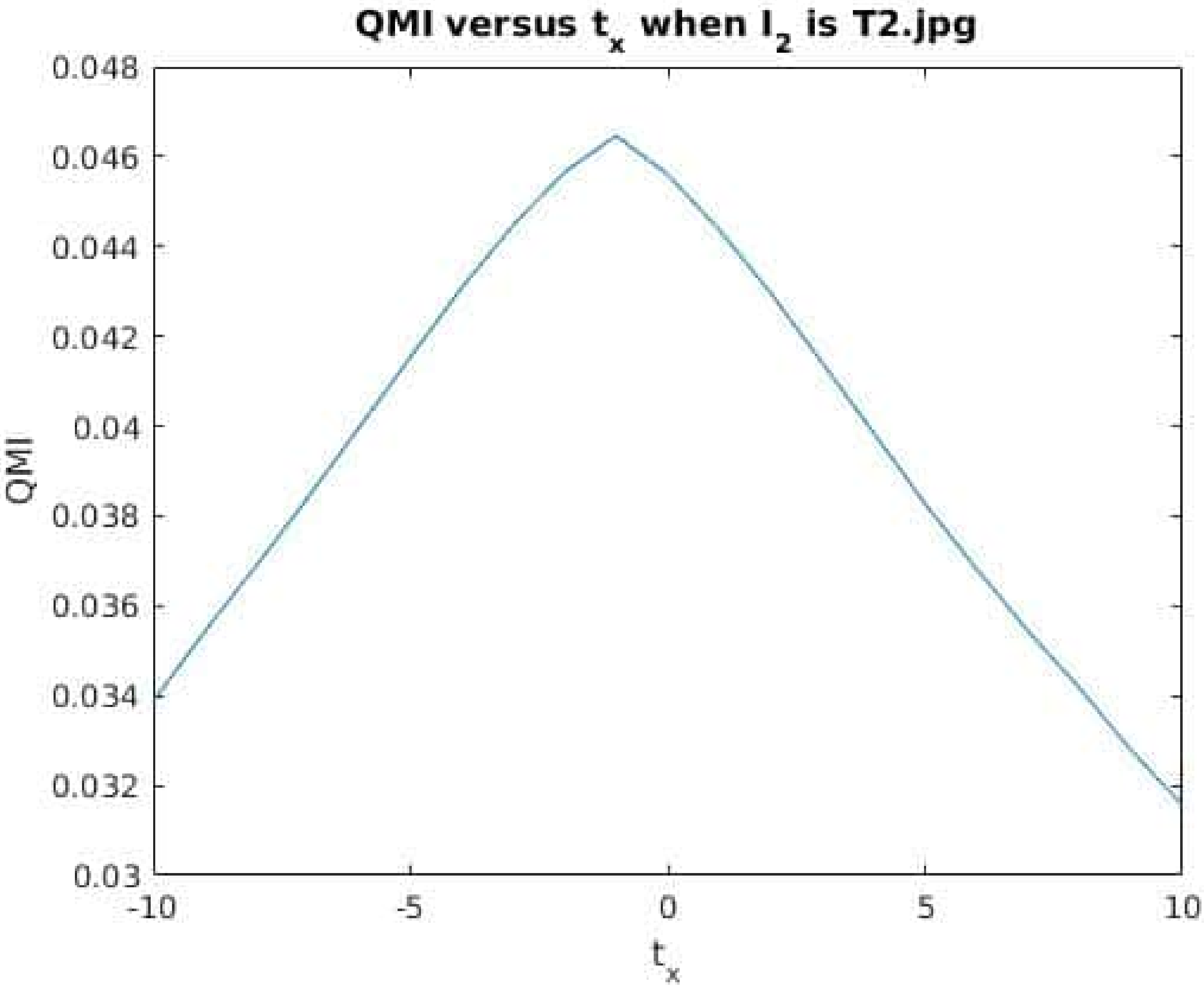
Has a maximum at $t_x = -1$. Since, ~~imo~~ both images have quite difference their QMI is low. When we shift from $t_x = -1$, we can infer from the graph that QMI decreases.


Correlation versus t_x when I_2 is T_2.jpg
Correlation coeffient
0.32
0.3
0.28
0.26
0.24
0.22
0.2
0.18
0.16
0.14
0.12
-10
-5
0
5
10
t_x


Correlation versus t_x when I_2 = 255 - I_1
Correlation coeffient
-0.6
-0.65
-0.7
-0.75
-0.8
-0.85
-0.9
-0.95
-1
-10
-5
0
5
10
t_x



QMI versus t_x when I_2 is 255 - I_1
QMI
t_x
0.1
0.09
0.08
0.07
0.06
0.05
0.04
-10
-5
0
5
10


QMI versus t_x when I_2 is T2.jpg
0.048
0.046
0.044
0.042
0.04
0.038
0.036
0.034
0.032
0.03
QMI
-10
-5
0
5
10
t_x

Conclusion:

In order to examine a shifting, QMI is a good measure whereas correlation coefficient isn't.

When we shift an image, we lose information which can be depicted as QMI is decreasing but correlation coefficient may increase or decrease depending on the new image. So, correlation coefficient is not a good measure.