Assignment 2 : CS215

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4. Assume our data is given by $x_1, x_2, \dots x_n$.
   Now, for $\hat{\theta}^{ML}$,
$$\text{Likelihood} = \begin{cases} \dfrac{1}{\theta^n}, & \text{if } \forall i, \theta \geq x_i \\ 0, & \text{otherwise} \end{cases}$$
   Therefore,
$$\text{Likelihood} = \begin{cases} \dfrac{1}{\theta^n}, & \text{if } \theta \geq max(x_i) \\ 0, & \text{otherwise} \end{cases}$$
   Now, since the non-zero part is decreasing, $\hat{\theta}^{ML} = max(x_i)$.
   Now, for $\hat{\theta}^{MAP}$, our prior is,
$$P(\theta) \propto \begin{cases} \left(\dfrac{\theta_m}{\theta}\right)^{\alpha}, & \text{if } \theta \geq \theta_m \\ 0, & \text{otherwise} \end{cases}$$
   Thus, our posterior is,
$$P(\theta | x_1, x_2, \dots x_n) \propto \begin{cases} \dfrac{\theta_m^{\alpha}}{\theta^{n+\alpha}}, & \text{if } \theta \geq max(\theta_m, max(x_i)) \\ 0, & \text{otherwise} \end{cases}$$
   Again, to maximise this, we just take $\hat{\theta}^{MAP} = max(\theta_m, max(x_i))$ as the non-zero part of the expression is decreasing.
   Now, if $\theta_m > \theta_{true}$, $\hat{\theta}^{MAP}$ will always be equal to θ_m, no matter what the sample size. So, it is not necessary that $\hat{\theta}^{MAP}$ will tend to $\hat{\theta}^{ML}$ as n tends to infinity. This is not desirable as we want our estimator to tend to ML estimator.

Now, for $\hat{\theta}^{PM}$,

$$\hat{\theta}^{PM} = E[\theta|Posterior] = \frac{\displaystyle\int_{\theta_l}^{\infty} \frac{\theta_m^{\alpha} d\theta}{\theta^{n+\alpha-1}}}{\displaystyle\int_{\theta_l}^{\infty} \frac{\theta_m^{\alpha} d\theta}{\theta^{n+\alpha}}} \qquad \text{(where } \theta_l = max(\theta_m, max(x_i))\text{)} \tag{1}$$

$$= \frac{\dfrac{\theta_l^{-(n+\alpha-2)}}{n+\alpha-2}}{\dfrac{\theta_l^{-(n+\alpha-1)}}{n+\alpha-1}} \qquad \text{(as } n+\alpha > 2\text{)} \tag{2}$$

$$= \theta_l(\frac{n+\alpha-1}{n+\alpha-2}) \tag{3}$$

Thus, $\hat{\theta}^{PM} = \hat{\theta}^{MAP}(\frac{n+\alpha-1}{n+\alpha-2})$.

Now, as n tends to infinity, $\hat{\theta}^{PM}$ tends to $\hat{\theta}^{MAP}$. So it does not always tend to $\hat{\theta}^{ML}$. Thus, similarly, posterior mean estimator is also not desirable.