Assignment 3 : CS215

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2. $X \sim U(0, 1)$
Let $Y = g(X)$
$\implies X = g^{-1}(Y)$

$$y = \frac{-1}{\lambda} log(x) \tag{1}$$

$$\implies x = e^{-\lambda y} \tag{2}$$

$$\left| \frac{d}{dy} g^{-1}(y) \right| = \lambda e^{-\lambda y}$$

Let PDF of Y be denoted by $q(y)$ and the PDF of X by $p(y)$.

$$\implies q(y) = \left| \frac{d}{dy} g^{-1}(y) \right| p(g^{-1}(y)) = \lambda e^{-\lambda y}$$

Therefore, the transformed data Y will have an **exponential distribution** with $mean = \frac{1}{\lambda}$.

Given :

- $\lambda = 5$
- Gamma prior with parameters $\alpha = 5.5$ and $\beta = 1$

MLE for exponential distribution:
The likelihood function is :

$$L(\lambda; x_1, x_2, ..., x_n) = \lambda^n exp(-\lambda \sum_{j=1}^{n} x_j)$$

The log likelihood function is :

$$l(\lambda, x_1, x_2, ..., x_n) = n ln(\lambda) - \lambda \sum_{j=1}^{n} x_j$$

Therefore, the maximum likelihood estimator of λ is the reciprocal of the sample mean ($\frac{\sum_{j=1}^{n} x_j}{n}$)

$$\hat{\lambda} = \frac{n}{\sum_{j=1}^{n} x_j}$$

Bayesian Posterior Mean Estimator:

$$Prior : \ P(\theta) = \ Gamma(\lambda | \alpha, \beta) \propto \lambda^{\alpha - 1} exp(-\beta \lambda)$$

$$P(\theta | x_1, x_2, ..., x_n) \propto P(x_1, x_2, ..., x_n | \theta) \ P(\theta)$$

$$\propto \lambda^n \ exp(-\lambda \sum_{j=1}^{n} y_j) \ \lambda^{\alpha - 1} \ exp(-\beta \lambda)$$

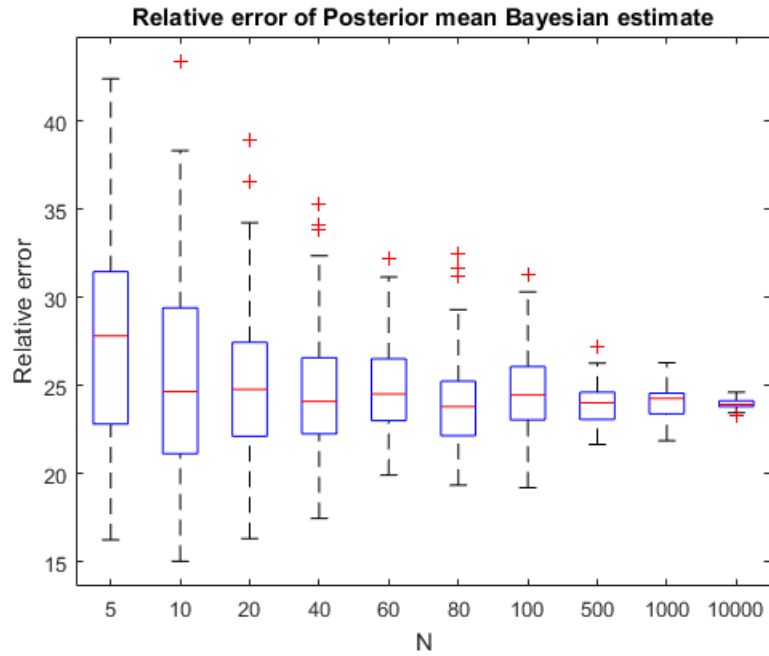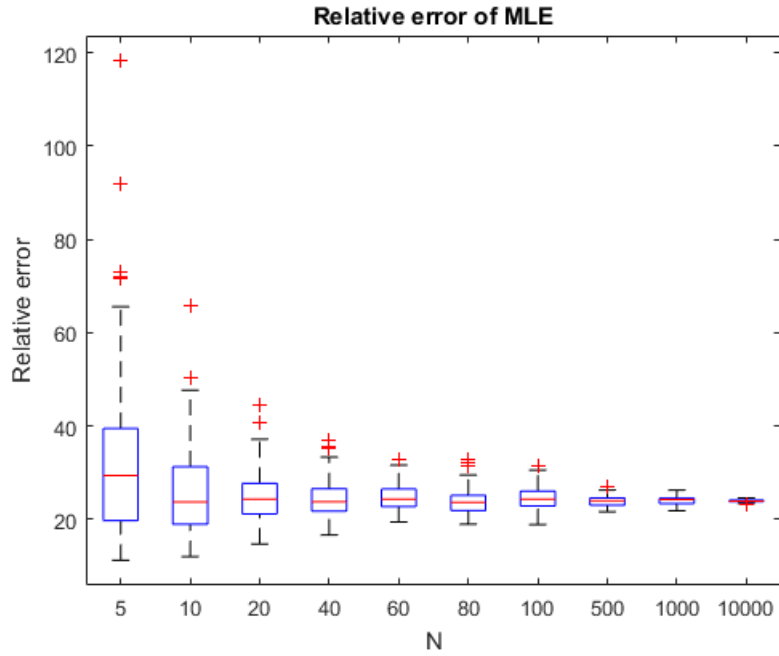$$\propto \lambda^{n + \alpha - 1} \ exp(-\lambda(\beta + \sum_{j=1}^{n} y_j))$$

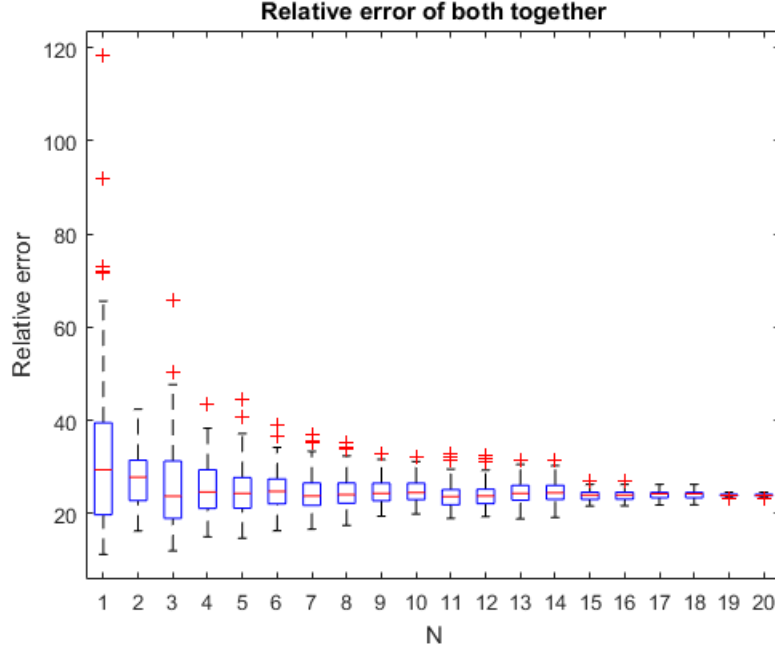$$\propto \ Gamma(\lambda | \alpha + n, \beta + \sum_{j=1}^{n} y_j)$$

We know that, mean of a Gamma distribution with parameters α and β is $\frac{\alpha}{\beta}$.
We also know that Bayesian Posterior mean estimator $= E_{P(\theta|x_1,x_2,...,x_n)}[\theta]$

$$\implies E_{P(\theta|x_1,x_2,...,x_n)}[\theta] = Mean\ of\ Gamma(\lambda|\alpha+n, \beta+\sum_{j=1}^{n} y_j)$$

$$\implies E_{P(\theta|x_1,x_2,...,x_n)}[\theta] = \frac{\alpha+n}{\beta+\sum_{j=1}^{n} y_j}$$

$$\therefore \hat{\lambda}^{PosteriorMean} = \frac{\alpha+n}{\beta+\sum_{j=1}^{n} y_j}$$

Relative error of MLE

Relative error of Posterior mean Bayesian estimate

Relative error of both together

In the above plot, every alternate box starting with the first denotes the relative error plot of MLE and the other set of alternate boxes starting with the second denotes the relative error plot of Bayesian Posterior mean estimate.

$$\hat{\lambda}^{PosteriorMean} = \frac{\alpha + n}{\beta + \sum_{j=1}^{n} y_j}$$

$$\hat{\lambda}^{PosteriorMean} = \left(\frac{\alpha}{\beta}\right)\left(\frac{\beta}{\beta + \sum_{j=1}^{n} y_j}\right) + \left(\frac{n}{\sum_{j=1}^{n} y_j}\right)\left(\frac{\sum_{j=1}^{n} y_j}{\beta + \sum_{j=1}^{n} y_j}\right)$$

$$\hat{\lambda}^{PosteriorMean} = \left(\frac{\alpha}{\beta}\right)w + \left(\frac{n}{\sum_{j=1}^{n} y_j}\right)(1 - w) \quad where \ w = \left(\frac{\beta}{\beta + \sum_{j=1}^{n} y_j}\right)$$

(i) As the value of n increases, the relative error for MLE and Bayesian Posterior Mean Estimate does not follow a strict pattern but may increase or decrease or both.
For very large n, as the Bayesian Posterior Mean Estimate tends to the Maximum Likelihood estimate, the relative error of Bayesian Posterior mean estimate also tends to the relative error of the maximum likelihood estimate.

(ii) Any of the two estimates may be taken