Math 351 - Homework Assignment #1

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Problem 0.7: Supply a relation that satisfies each constraint.

(a) **Reflexive and symmetric, but not transitive.**

The relation between two people x, y given iff x is blood-related to y is reflexive and symmetric, but not transitive.

(b) **Reflexive and transitive, but not symmetric.**

$R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a \geq b\}$ is reflexive and transitive, but not symmetric.

(c) **Symmetric and transitive, but not reflexive.**

$R = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ in $X = \{0, 1, 2\}$. This is symmetric and transitive, but not reflexive since $(2, 2) \notin R$.

Problem 0.9: Formally describe the given graph.

The given graph is the complete bipartite graph $K_{3,3}$ with formal description $(V = \{1, 2, 3, 4, 5, 6\}, E = \{(1, 4), (2, 4), (3, 4), (1, 5), (2, 5), (3, 5), (1, 6), (2, 6), (3, 6)\})$.

Problem 0.10: Find the error in the given proof.

The error in the given proof is that the writer defines $a = b = 1$. Thus, when he transforms the equality $(a + b)(a - b) = b(a + b)$ to $a + b = b$, he divides by $a - b = 0$. Thus, the proof is incorrect.

Problem 0.12: Find the error in the given proof.

Let $P(h)$ be the predicate that all horses in a set of h horses are the same color. As per the base case, $P(1)$ is true. However, $P(2)$ is not implied by the base case $P(1)$.

Take two horses x, y of different color. By the inductive hypothesis, all the horses in $\{x\}$ and those in $\{y\}$ are the same color. However, the horses in $\{x, y\}$ are not the same color by definition. Thus, the claim of the author that $P(k)$ implies $P(k + 1)$ is untrue.

Problem 0.13, Claim: It is not true that every graph of two or more vertices contain two vertices of equal vertices when self-loops are allowed.

Proof. The graph G with two vertices x, y where x has a self-loop and y has no edge is a counterexample. $\square$

Problem 0.14, Claim: Ramsey's Theorem.

Proof. Let G be a graph of size n. Now, assume a 2-coloring on the edges of the complete graph K_n of size n s.t. red edges are in G and blue edges are not in G.

If there are $k \geq \frac{1}{2}\lg n$ red edges, then we are done. So, assume that this is not the case, i.e. $k \leq \frac{1}{2}\lg n$. Therefore, there are $n^2 - n - k$ blue edges in our coloring of K_n.

Finally, observe that $k + \frac{1}{2}\lg n \leq \lg n \leq n(n-1) = n^2 - n$. Thus, $n^2 - n - k \geq \frac{1}{2}\lg n$; so, it follows that there are more than $\frac{1}{2}\lg n$ blue edges.

We know that the blue edges of K_n are not connected by an edge in G. Thus, the set of vertices connected with blue edges in K_n forms an independent set in G, as required. $\square$

Extra Problem, Claim: The relation S defined by $(1, 1) \in S$ and $(a + 1, b + 2a + 1) \in S$ for all $(a, b) \in S$ is equivalent to the function $f : \mathbb{N} \to \mathbb{N}$ induced by $n \mapsto n^2$.

Proof. Let $P(n)$ be the predicate that $(n, n^2) inS$ for a particular n. We prove the claim with mathematical induction.

<u>Base Case:</u> It is given that $(1, 1) = (1, 1^2) \in S$. Thus, $P(1)$ holds.

<u>Inductive Step:</u> Suppose that $P(k)$ holds for all k s.t. $1 \leq k \leq m$, where $m \geq 1$. It follows from the inductive hypothesis that $m + 1 \mapsto m^2 + 2m + 1 = (m + 1)^2$. Thus, $P(m + 1)$ holds.

Hence, by the Principle of Mathematical Induction, the claim holds. $\square$