Formal Languages and Automata (CS452) - Homework Assignment #6 Formal Languages and Automata (CS452) - Homework Assignment #6

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Problem 4.20, Claim: The problem of determining whether a DFA and a regular expression are equivalent is decidable.

Proof. Let M be a DFA and R be a regular expression. Define the language $C = \{\langle M, R\rangle \mid M \text{ is a DFA and } R \text{ a regular expression with} L(M) = L(R)\}$.
Sipser defines a Turing machine F that decides C. Thus, the claim holds. □

Problem 4.26: Answer the following for the given functions.

(a) **Is f one-to-one?**

No, each of 6 and 7 in Y are mapped to multiple values in X.

(b) **Is f onto?**

No, each of 8, 9, and 10 in Y are not mapped to any input in X.

(c) **Is f a correspondence?**

Yes; each input maps to a single output, thus it is well-defined.

(d) **Is g one-to-one?**

Yes; g satisfies $g(a) = g(b)$ iff $a = b$, for all $a, b \in X$.

(e) **Is g onto?**

Yes; each $y \in Y$ has $x \in X$ s.t. $g(x) = y$.

(f) **Is g a correspondence?**

Yes; since g is a bijection, it must be well-defined.

Problem 5.21, Claim: The language of ambiguous CFGs is undecidable.

Proof. Let $L = \text{AMBIG}_{\text{CFG}}$ for brevity, and define finite sequences $(a_N), (b_N)$ of words in L. We define the following CFG G:

$$\begin{aligned} S &\to A \mid B \\ A &\to a_1 A\sigma_1 \mid a_2 A\sigma_2 \mid \cdots \mid a_n A\sigma_n \mid a_1\sigma_1 \mid \cdots \mid a_n\sigma_n \\ B &\to b_1 B\sigma_1 \mid b_2 B\sigma_2 \mid \cdots \mid b_n B\sigma_n \mid b_1\sigma 1 \mid \cdots \mid b_n\sigma_n \end{aligned}$$

The σ_i are new characters to the alphabet, i.e. $\sigma_i = i$.

Similarly, if there is no ambiguity, then the PCP cannot be solved for w. Now, we carry out the reduction from PCP to show that L is undecidable, as required. □

Problem 5.24, Claim: Neither the given language nor its complement are Turing-recognizable.

Proof. Let A be the language defined. Trivially, M is not recognizable. We reduce A to J with the reduction function f defined by $w \mapsto 1w$. In particular, $w \in A$ iff $f(w) \in J$; obviously, f is compputable, so the reduction holds. This suffices to show that J is not recognizable.

Similarly, we reduce $A_{\text{TM}}^{\complement}$ to $J^{\complement}$ with the reduction function induced by $w \mapsto 0w$. Again, since this reduction holds, $J^{\complement}$ cannot be recognizable. □

Problem 5.25: Provide an example of an undecidable language that can be reduced to its complement.

Let L be recursively enumerable and undecidable. The language $0L \cup 1L^{\complement}$ is not recursively enumerable, thus it is not decidable.

Problem 5.30(c): Prove the undecidability of the given language.

Proof. With Rice's Theorem, it suffices to show that L is nontrivial and that $L(M_1) = L(M_2)$ implies that $M_1 \in P$ iff $M_2 \in P$.

We can easily show that $L = \mathrm{ALL}_{\mathrm{TM}}$ is nontrivial, i.e. L and its complement both contain well-defined Turing machines. Now, suppose M_1 and M_2 are Turing machines. If $L(M_1) = L(M_2)$, then either $L(M_1) = L(M_2) = \sum^\star$ or $L(M_1) = L(M_2) \neq \sum^\star$. Equivalently, $M_1 \in L$ iff $M_2 \in L$, as required. □

Problem 6.13, Claim: For each ring defined by $\mathbb{Z}_n$, the theory $\mathrm{Th}(\mathbb{Z}_n)$ is decidable.

Proof. Let $\phi = Q_1 x_1 Q_2 x_2 \dots Q_n x_n \psi(x_1, \dots, x_n)$ be a formula, where the Q_i are quantifiers and ψ has no quantifiers. We define operators $I_k(x_1, \dots, x_k)$ as follows:

If $k = n$, then $I_k = \psi$. Otherwise, if $Q_k = \exists$, then put $I_{k-1}(x_1, \dots, x_{k-1}) = \vee_{i=0}^{n-1} I_k(x_1, \dots, x_{k-1}, i)$. Finally, if $Q_k = \forall$, put $I_{k-1}(x_1, \dots, x_{k-1}) = \wedge_{i=0}^{n-1} I_k(x_1, \dots, x_{k-1}, i)$.

By this definition, I_0 will have no inputs, and simply return a Boolean value; return I_0. To demonstrate that this machine is well-defined, see by induction that $\phi \iff Q_1 Q_2 \dots Q_k I_k$.

Thus, the theories of rings of the integers modulo n are decidable. □