


21.7 Multiplication theorem

Timestamp

(e.g) $\underbrace{\text{Select } n}_{\text{balls}}$ $\left\{ \begin{array}{l} \text{sequentially \& randomly chosen} \\ \text{without replacement} \end{array} \right. \checkmark$ 

urn \rightarrow 'r' red, 'b' blue balls

$n \leq r+b$

Given that K out of n balls chosen are blue, } cond. probs
what's the prob of the 1st ball picked is blue

- For understanding conditional probability let's consider the above example
- There are r red balls and b blue balls in an urn, we have to select n balls sequentially and randomly chosen without replacement.
- Given that out of n balls chosen K balls are blue. We need to find the probability of the 1st ball being picked is blue.

B = event that 1st picked ball is blue
 B_k = event that k out of n balls picked are blue

$$P(B|B_k) = \frac{P(B \cap B_k)}{P(B_k)}$$

$$P(B_k) = \frac{{}^b C_k {}^r C_{n-k}}{{}^{b+r} C_n} = \frac{n(B_k)}{n(S)}$$

We define events as shown above and the we need to find.

$$P(B|B_k) = P(B \cap B_k) / P(B_k)$$

We calculate the probability of event B_k (k out of n , balls picked are blue) as shown above.

- We are choosing n balls out of $b+r$ balls, this is our sample space.
- We select k balls out of b blue balls and $n-k$ balls out of r red balls

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B = event that 1st picked ball is blue
 B_k = event that k out of n balls picked are blue

$$P(B|B_k) = \frac{P(B \cap B_k)}{P(B_k)} = \frac{P(B_k|B)P(B)}{P(B_k)}$$

$\frac{b}{r+b}$

Using conditional probability, we can rewrite $P(B \cap B_k)$. Now we find $P(B)$, the probability that first picked ball is blue i.e.) $b/r+b$ as shown above.

$\checkmark B = \text{event that 1st picked ball is blue}$
 $B_k = \text{event that } k \text{ out of } n \text{ balls picked are blue}$

$$P(B|B_k) = \frac{P(B \cap B_k)}{P(B_k)} = \frac{P(B_k|B)P(B)}{P(B_k)}$$

$\checkmark P(B_k) = \frac{\binom{b}{k} \binom{r}{n-k}}{\binom{r+b}{n}}$

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$$P(B_k|B) = \frac{n(k-1 \text{ blue balls} \& n-k \text{ red balls})}{n(n-1 \text{ balls} \& n-1)}$$

$$= \frac{\binom{b-1}{k-1} \binom{r}{n-k}}{\binom{r+b-1}{n-1}}$$

- Given that we have already picked first ball blue, now we have to find the probability that we pick k blue balls out of n balls **i.e.) $P(B_k|B)$**
- Since we have already picked first ball blue we choose k-1 balls out of b-1 blue balls and n-k balls out of r red balls as shown above

Now we have our numerator and denominator, we just substitute the values and we get
 $P(B|B_k) = P(B \cap B_k)/P(B_k) = k/n$

$$P(E_1|E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} \quad \text{if } P(E_2) \neq 0$$

Above is the definition of conditional probability, We can think of multiplication rule as a generalization of conditional probability

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$$\Rightarrow P(E_1 \cap E_2) = P(E_1|E_2) P(E_2)$$

$$P(\underline{E_1 E_2}) = P(E_1|E_2) P(E_2) \quad \checkmark$$

Gen: \checkmark $P(\underbrace{E_1 E_2 E_3 \dots E_n}) = P(E_1) \cdot P(E_2|E_1) \cdot P(E_3|E_2 E_1) \dots$

$$P(E_n|E_{n-1} E_{n-2} \dots E_1)$$

By repeatedly applying conditional probability we obtain the multiplication rule as shown above.

(e.g.) Matching problem: n persons $\rightarrow n$ hats

✓ randomly pick hats back

✓ $P(\text{no one correctly their hat}) = \sum_{i=0}^n (-1)^i / i!$ Ind-Excl

(Q) $P(\text{exactly } k \text{ persons have picked correctly}) = ?$

n_c k $\leftarrow A = \{k \text{ persons who would pick correctly}\}$

$(E) \Rightarrow$ everyone in A has picked correctly

$(G) \Rightarrow$ every one other than people in Set A have picked incor?

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$P(\overbrace{G \cap E}^{\text{AND}}) = P(G \cap E) = P(G|E)P(E)$

$\uparrow \quad \uparrow$

prob that $n-k$ people have not picked the correct hat

$n-k$

$= \sum_{i=0}^{n-k} (-1)^i / i!$

$F_1 =$ event that 1st per in A has picked corr.

$F_2 =$ " 2nd " "

$F_3 =$ " 3rd " "

$P(E) = P(F_1 F_2 F_3 \dots F_k)$

$= P(F_1) P(F_2|F_1) P(F_3|F_2 F_1) \dots P(F_k|F_1 \dots F_{k-1})$

$\frac{(n-k)!}{n!} = \frac{1}{n} \cdot \frac{1}{n-1} \cdot \frac{1}{n-2} \dots \frac{1}{n-k+1}$

- Above is a problem where n persons randomly pick their hats will be solved as shown above