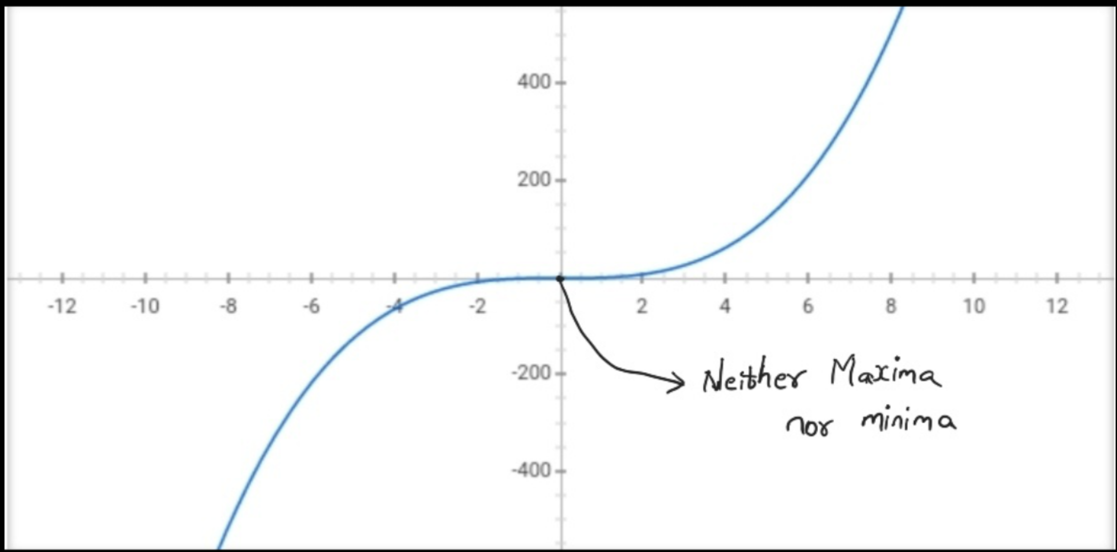
Saddle point

Now lets see an interesting function $f(x) = x^3$, which is a continuously increasing function as shown below.



Now lets try to compute its optima

$$f(x) = x^3$$

$$f'(x) = 3x^2 \Rightarrow \text{Optima lies at } x = 0$$

$\Downarrow$

But by looking at plot we can say at $x=0$, its neither minima nor maxima and its an increasing function.

$\downarrow$

Such points are called Saddle points

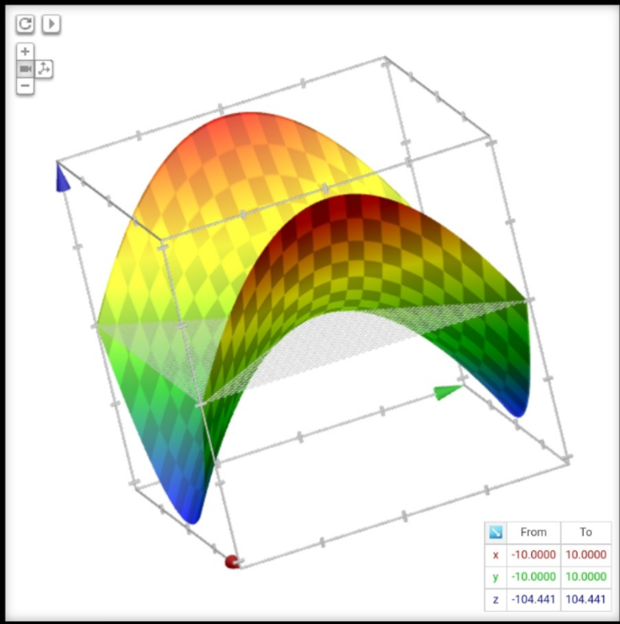
lets try to compute second order derivative at $x=0$ for $f(x)$

$$f''(x) = 6x$$

$$f''(x)\Big|_{x=0} = \underline{\underline{0}} \longrightarrow \text{Neither +ve nor -ve}$$

lets look at one more example where

$z = f(x,y) = x^2 - y^2$ which is as shown below



$$\nabla z = \begin{bmatrix} \partial z/\partial x \\ \partial z/\partial y \end{bmatrix} = \begin{bmatrix} 2x \\ -2y \end{bmatrix}$$

Optima exists at $\nabla z = 0$ i.e $x=0; y=0$
But by observing the plot $(x=0, y=0)$ is neither minima nor maxima. which is one more example for saddle point.

lets see at $f''(x,y)$ @ $x=0$ & $y=0$

$$\nabla(\nabla z) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$ $\Longrightarrow$ one is +ve while other is -ve

this is one interesting observation at saddle point

Note: Be it saddle point or not, first order derivative satisfies the necessary condition for the presence of optima. Whether it is a maximum or a minimum or a saddle point can be told only by the second order derivative. If the second order derivative is positive then you have a minimum, if the second order derivative is -ve then you have a maximum, if the second order derivative $= 0$ then it is a saddle point.

Ily in multi-variant functions can be done by det (Hessian matrix)

$$\text{Hessian matrix} = Hf(x_0, y_0) = \begin{bmatrix} f_{xx}(x_0,y_0) & f_{yx}(x_0,y_0) \\ f_{xy}(x_0,y_0) & f_{yy}(x_0,y_0) \end{bmatrix}$$

$$\det(Hf(x,y)) = f_{xx}(x_0,y_0)\cdot f_{yy}(x_0,y_0) - f_{xy}(x_0,y_0)^2$$

if $\det(H) < 0$, then (x_0,y_0) is a saddle point.