

# 21.5 Axioms of Probability, Properties and Examples-

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✓ Set-Theory ✓

expt  $S$ : sample space  $U$

$E$ : event : set of outcomes

$E \subseteq S$

(e.g) Throwing 2 coins (distinct)

✓  $E = \{(T,T), (T,H)\}$

✓  $E \cap F = \{(H,H)\}$

✓  $E \cup F = \{(H,H), (T,T), (H,T)\}$

$S = \{(H,H), (H,T), (T,H), (T,T)\}$

$E = \{ \text{all outcomes where 1st coin is H} \}$

$F = \{ \text{all outcomes with both coins of same value} \}$

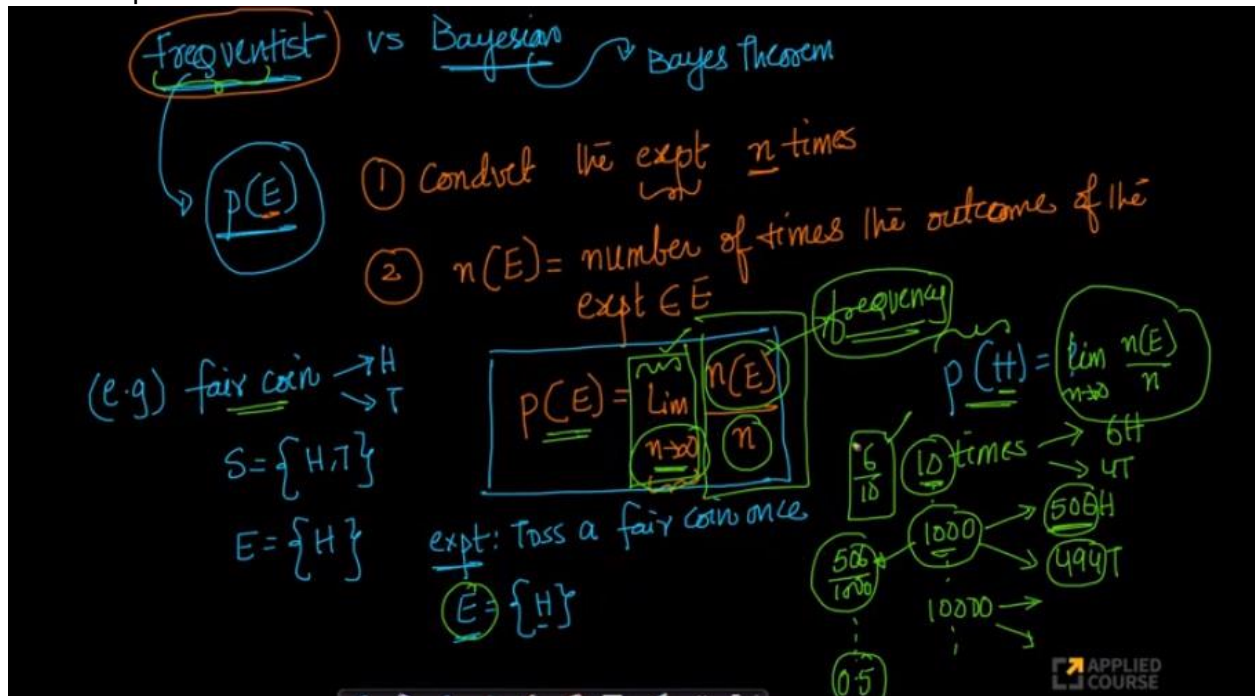
$\{(H,H), (T,T)\} = F$

APPLIED

Let's study probability in set theoretic perspective

1. If we consider an experiment, we define a sample space( $S$ ) which is a set of all outcomes. An event  $E$  which is a set of outcomes (subset of  $S$ ).
2. Let's say we are flipping two coins and we have our sample space as  $s = \{HH, HT, TH, HH\}$
3. Consider an event  $E$  of getting outcomes where first coin is  $H$  ( $E = \{HH, HT\}$  )
4. Also consider another event  $F$  of getting outcomes where both coins are of same value
5. Now we can think of these events  $E$  and  $F$  as sets and can perform all operations which we can perform on sets. i.e.) set union, intersection, complement etc as shown above.

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Let's understand the Frequentist and Bayesian approach of probability

From a frequentist approach probability of an event  $p(E)$  can be defined as

1. let's say we are conducting an experiment  $n$  times (assume the experiment of tossing two fair coins)
2.  $n(E)$  be the number of times the outcome of the experiment belongs to event  $E$

Then  $p(E) = \lim_{n \rightarrow \infty} n(E)/n$

$n(E)$  is the frequency of occurrence of the event and  $n$  is total number of times we are conducting the experiment as  $n$  tends to infinity (we conduct the experiment infinite times)

Sct Thy Axiomatic (Self-evident)

Geometry: parallel lines don't meet

Physics  
3 Newton's laws  
↳ Mechanics

①  $0 \leq p(E) \leq 1$   
 ②  $p(S) = 1$   
 ③ if  $E_1, E_2, E_3 \dots$  are mutually exclusive  
 then  $E_i \cap E_j = \phi$   
 $p(E_1 \cup E_2 \cup E_3 \cup E_4 \dots)$   
 $= \sum_i p(E_i)$   
 $\checkmark \{ |E_1 \cup E_2 \cup E_3 \cup \dots| = |E_1| + |E_2| + |E_3| + \dots$

Axiomatic (self-evident) approach of probability says

1. The probability of any event lies between 0 and 1 i.e.  $0 \leq p(E) \leq 1$
2. The probability of sample space is  $p(S) = 1$
3. If there are events  $E_1, E_2, E_3, \dots$  etc which are mutually exclusive ( $E_1 \cap E_2 = \phi$ ) then  
 $p(E_1 \cup E_2 \cup E_3 \cup E_4 \dots) = \sum_{i=1} p(E_i)$

Using the above axioms and properties in set theory we can actually prove all theorems in probability

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Dice

$S = \{1, 2, 3, 4, 5, 6\}$

mutually exclusive

$E_1 = \{2\} = \frac{1}{6}$   
 $E_2 = \{4\} = \frac{1}{6}$   
 $E_3 = \{6\} = \frac{1}{6}$

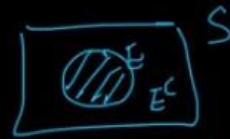
$p(E_1 \cup E_2 \cup E_3)$   
 $= p(\{2, 4, 6\}) = p(E_1) + p(E_2) + p(E_3)$   
 $= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$

the sum of the events.

Above is an example where we have three mutually exclusive events and their union is equal to

Results:

$$(1) P(E^c) = 1 - P(E)$$



$$E \cap E^c = \emptyset$$

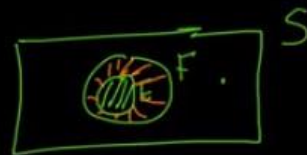
$$(2) \checkmark P(S) = 1$$

$$(1) \underbrace{P(E \cup E^c)} = P(S) = 1$$

$$P(E) + P(E^c) = 1$$

$$P(E^c) = 1 - P(E)$$

$$(2) \text{ if } E \subseteq F \quad P(E) \leq P(F)$$



$$F = \underbrace{E} \cup \underbrace{(F \cap E^c)}$$

$$\checkmark P(F) = P(E \cup (F \cap E^c))$$

$$P(F) = P(E) + \underbrace{P(F \cap E^c)}_{\geq 0}$$

$$P(F) \geq P(E)$$

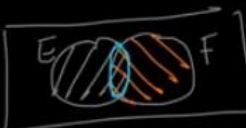
③  $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

principle of Inclusion-Exclusion

$|E \cup F| = |E| + |F| - |E \cap F|$

Set Thy

$P(E_1 \cup E_2 \cup E_3 \cup \dots \cup E_n) = P(\bar{E}_1) + P(\bar{E}_2) + \dots + P(\bar{E}_n)$   
 $- (P(E_1 \cap E_2) + P(E_1 \cap E_3) + \dots \text{2way})$   
 $+ (P(E_1 \cap E_2 \cap E_3) + \dots \text{3way})$   
 $- (P(E_1 \cap E_2 \cap E_3 \cap E_4) + \dots \text{4way})$



- Above are few examples of proof that we can solve all theorems in probability using these axioms and properties in set theory

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Sample Space with equally likely outcomes

die  $S = \{1, 2, 3, 4, 5, 6\}$

$P(E) = \frac{|E|}{|S|}$

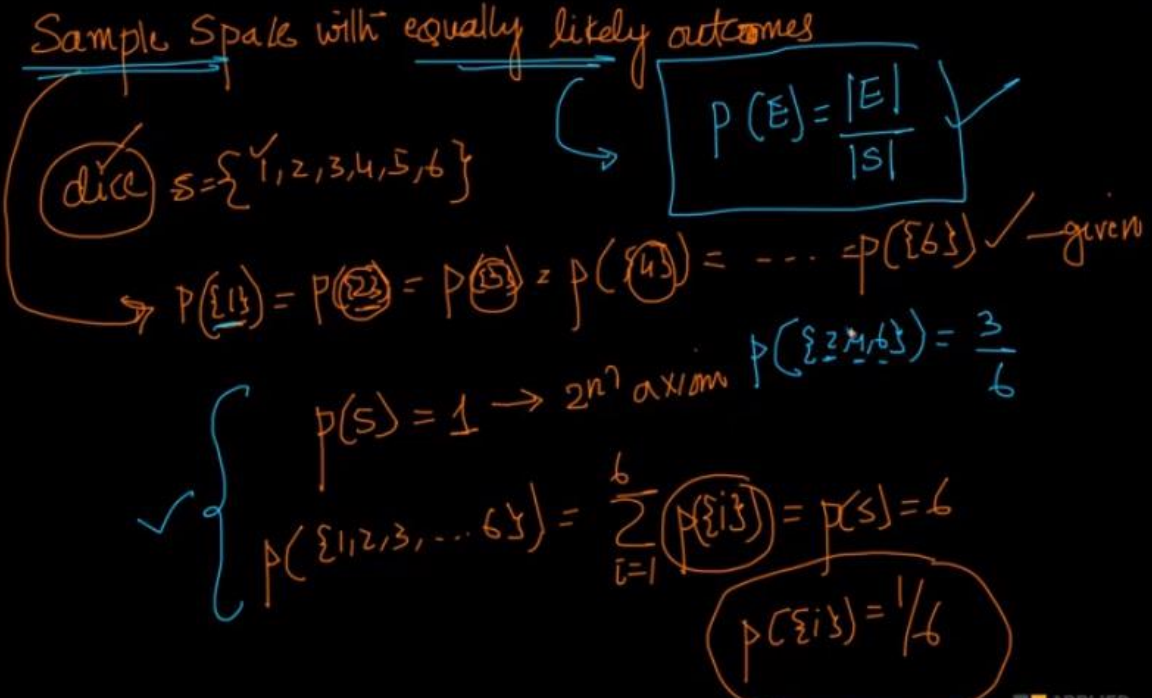
$P(\{1\}) = P(\{2\}) = P(\{3\}) = P(\{4\}) = \dots = P(\{6\})$  ✓ — given

$P(\{2, 4, 6\}) = \frac{3}{6}$

$P(S) = 1 \rightarrow 2^{nd} \text{ axiom}$

$P(\{1, 2, 3, \dots, 6\}) = \sum_{i=1}^6 P(\{i\}) = P(S) = 1$

$P(\{i\}) = \frac{1}{6}$



- In an experiment of rolling a dice, we have a sample space with equally likely outcomes as shown above.

$$S = \{1, 2, 3, 4, 5, 6\}$$

$p(\{1\}) = p(\{2\}) = p(\{3\}) = p(\{4\}) = p(\{5\}) = p(\{6\}) = \frac{1}{6}$  and We know that probability of sample space is  $p(S) = 1$

- Hence,  $p(E) = |E|/|S|$  this is true for sample space with equally likely events

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(e.g.)

6W, 5B  
 1W, 2B  
 3 Balls

"randomly" picked up

$$\frac{{}^6C_1 \cdot {}^5C_2}{{}^{11}C_3} = \frac{4}{11}$$

$|S| = {}^{11}C_3$   
 $|E| = {}^6C_1 \cdot {}^5C_2$

$p(E) = \frac{|E|}{|S|}$

- In the above example we have to find probability that we have to choose 3 balls out of 11 balls such that we need to choose exactly 1 white and 2 black. Out of 11 balls we have 6 are white and 5 are black.
- As shown above using combinatorics we can find the probability is 4/11



(e.g)  $n$  balls  
 one special  
 expt:  $k$  balls drawn one at a time randomly  
 $E$ : special ball is picked

$$|S| = nC_k$$

$$|E| = {}^1C_1 \cdot {}^{n-1}C_{k-1}$$

$$= \frac{{}^{n-1}C_{k-1}}{{}^nC_k} = \frac{k}{n} \checkmark$$

$$\frac{|E|}{|S|} = P(E)$$

1. Above example we have  $n$  balls out of which one ball is special. let's say in an experiment of drawing  $k$  balls out of  $n$  balls randomly.
2. Let say event  $E$  where special ball is picked
3.  $p(E)$  can be calculated using combinatorics as shown above

Matching Problem:

$N$  men —  
 ✓ Throw  
 picks a hat randomly

$p(\text{no one has picked their hat}) = ?$

Counting

HINT: Inclusion-Excl

- In a party where  $N$  men throw their hats into the bag and then each of them picks a hat randomly. We have to find probability that No one picks their own hat. It can be solved using principle of inclusion and exclusion as shown below

$E_i$  = i<sup>th</sup> person has picked the correct hat

$P(E_1 \cup E_2 \cup \dots \cup E_n)$  = prob. of at least one person picking their hat

$1 - P(E_1 \cup E_2 \dots \cup E_n)$  = prob. of no person picking their hat

$$\begin{aligned}
 & \rightarrow \frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n} = \frac{n}{n} = 1/1! \\
 & \checkmark P(E_1) + P(E_2) + \dots + P(E_n) \rightarrow \left(\frac{1}{n} \cdot \frac{1}{n-1}\right)^n C_2 = \frac{n \times n-1}{2!} \times \frac{1}{n} \cdot \frac{1}{n-1} \\
 & - (P(E_1 \cap E_2) + P(E_1 \cap E_3) + \dots) \rightarrow \frac{1}{2!} \\
 & + (P(E_1 \cap E_2 \cap E_3) + \dots) \rightarrow \frac{1}{n} \cdot \frac{1}{n-1} \cdot \frac{1}{n-2} \times \frac{n \times n-1 \times n-2}{3!} = \frac{1}{3!}
 \end{aligned}$$

P3

$$P\left(\bigcup_{i=1}^n E_i\right) = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \dots$$

$$1 - P\left(\bigcup_{i=1}^n E_i\right) = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \dots + \frac{1}{n!}$$

$\infty$ -series  $\rightarrow$

$\lim_{n \rightarrow \infty} e^{-1} = \underline{\underline{0.36788}}$