University of Maryland

ENPM667 – Control of Robotic Systems

Final Project

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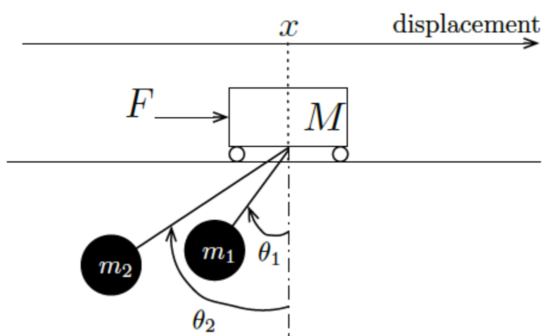
Abstract

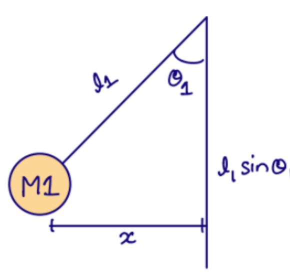
The focus of this project is to understand and implement the core concepts of controls, including State Space Representation, Nonlinear System Design, Linear Quadratic Regulator (LQR) Controller, Linear Quadratic Tracker (LQT) Controller and Luenberger Observer for Double Pendulum on a cart.

Consider a crane that moves along an one-dimensional track. It behaves as a frictionless cart with mass M actuated by an external force F that constitutes the input of the system. There are two loads suspended from cables attached to the crane. The loads have mass m1 and m2, and the lengths of the cables are l1 and l2, respectively. The following figure depicts the crane and associated variables used throughout this project.



Part A – Equations of Motion

We use Eurler-Lagrange to obtain the equations of motion for the system and the corresponding nonlinear state-space representation. To compute the Eurler-Lagrange equation, we calculate the kinetic and potential energy of the system.



We start by defining position of each pendulum as a function of c:

$$x_{m1} = \big(x - l_1 sin(\theta_1)\big)\hat{\imath} + (-l_1 cos(\theta_1))\hat{\jmath}$$

$$x_{m2} = \big(x - l_2 sin(\theta_2)\big)\hat{\imath} + (-l_2 cos(\theta_2))\hat{\jmath}$$

Next, we define velocity of each pendulum as a function of C:

$$v_{m1} = \left(\dot{x} - l_1 cos(\theta_1)\dot{\theta_1}\right)\hat{\imath} + (l_1 cos(\theta_1)\dot{\theta_1})\hat{\jmath}$$

$$v_{m2} = \left(\dot{x} - l_2 cos(\theta_2)\dot{\theta_1}\right)\hat{\imath} + \left(l_2 \cos(\theta_2)\,\dot{\theta_2}\right)\hat{\jmath}$$

Kinetic Energy and Potential Energy are calculated next. We know that total kinetic energy of a system is given as follows:

$$KE = \frac{1}{2} M \left(\frac{dx}{dt}\right)^2$$

$$KE = KE_C + KE_1 + KE_2$$

$$KE = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m_1(\dot{x} - l_2 cos(\theta_2)\dot{\theta_1})^2 + \frac{1}{2}m_1(l_1 cos(\theta_1)\dot{\theta_1})^2 + \frac{1}{2}m_2(\dot{x} - l_2 cos(\theta_2)\dot{\theta_1})^2 + \frac{1}{2}m_2(l_2\cos(\theta_2)\,\dot{\theta_2})^2$$

Similarly, the potential energy of the system is the sum of potential energy the crane and pendulums. However, we consider the case where the crane is positioned at the origin. So, it has no potential energy.

$$PE = PE_1 + PE_2$$

$$PE = -m_1 g l_1 cos(\theta_1) - m_2 g l_2 cos(\theta_2)$$

Creating Langrangian

$$L = KE - PE$$

$$L = \left[\frac{1}{2}M\dot{x}^2 + \frac{1}{2}m_1(\dot{x} - l_2 cos(\theta_2)\dot{\theta_1})^2 + \frac{1}{2}m_1(l_1 cos(\theta_1)\dot{\theta_1})^2 + \frac{1}{2}m_2(\dot{x} - l_2 cos(\theta_2)\dot{\theta_1})^2 + \frac{1}{2}m_2(l_2\cos(\theta_2)\,\dot{\theta_2})^2\right] - [-m_1 g l_1 cos(\theta_1) - m_2 g l_2 cos(\theta_2)]$$

Which simplifies to

$$L = \frac{1}{2}\dot{x}^2(M + m_1 + m_2) - \dot{x} m_1 l_1 cos(\theta_2)\dot{\theta_1} + \dot{x} m_2 l_2 cos(\theta_2)\dot{\theta_2} + \frac{1}{2}\left(m_1\left(l_1\dot{\theta_1}\right)^2 + m_2\left(l_2\dot{\theta_2}\right)^2\right) - m_1 g l_1(1 - cos(\theta_1)) - m_2 g l_2(1 - cos(\theta_2))$$

From $L = KE - PE$, we get

$$\left(\frac{\partial L}{\partial \dot{x}}\right) = \left(\frac{\partial K}{\partial \dot{x}}\right) - \left(\frac{\partial P}{\partial \dot{x}}\right)$$

The Lyapunov equations are

$$\frac{d}{dx}\left(\frac{\partial L}{\partial \dot{q}}\right) - \left(\frac{\partial L}{\partial q}\right) = F$$

$$\frac{d}{dx}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \left(\frac{\partial L}{\partial \theta_1}\right) = 0$$

$$\frac{d}{dx}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \left(\frac{\partial L}{\partial \theta_2}\right) = 0$$

These relations can be computed further. Solving the first Langrange Equation

$$\frac{d}{dx}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = F$$

$$\left(\frac{\partial L}{\partial \dot{x}}\right) = (M + m_1 + m_2)\dot{x} - m_1 l_1 \dot{\theta}_1 cos(\theta_1) - m_2 l_2 \dot{\theta}_2 cos(\theta_2)$$

Differentiate to get

$$\frac{d}{dx}\left(\frac{\partial}{\partial \dot{x}}\right) = (M + m_1 + m_2)\ddot{x} - \left[m_1 l_1 \dot{\theta}_1{}^2 sin(\theta_1) - m_1 l_1 \ddot{\theta}_1 cos(\theta_1)\right] + \left[m_2 l_2 \dot{\theta}_2{}^2 sin(\theta_2) - m_2 l_2 \ddot{\theta}_2 cos(\theta_2)\right]$$

since

$$\frac{\partial L}{\partial x} = 0$$

F is simplified to

$$\frac{d}{dx}\left(\frac{\partial L}{\partial \dot{x}}\right) - 0 = F$$

$$F = (M + m_1 + m_2)\ddot{x} - \left[m_1 l_1 \dot{\theta}_1{}^2 sin(\theta_1) - m_1 l_1 \ddot{\theta}_1 cos(\theta_1)\right] + \left[m_2 l_2 \dot{\theta}_2{}^2 sin(\theta_2) - m_2 l_2 \ddot{\theta}_2 cos(\theta_2)\right]$$

Solving the Second Langrange Equation

$$\frac{d}{dx}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \left(\frac{\partial L}{\partial \theta_1}\right) = 0$$

$$\frac{\partial L}{\partial \theta_1} = m_1 l_1 sin(\theta_2)\left[\dot{x}\dot{\theta}_1 - g\right]$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 - \dot{x} m_1 l_1 cos(\theta_1)$$

$$\frac{d}{dx}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = m_1 l_1^2 \ddot{\theta}_1{}^2 - \left[\ddot{x} m_1 l_1 cos(\theta_1) - \dot{x}\dot{\theta}_1 m_1 l_1 sin(\theta_1)\right]$$

$$\frac{d}{dx}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \left(\frac{\partial L}{\partial \theta_1}\right) = \left[m_1 l_1^2 \ddot{\theta}_1{}^2 - \left[\ddot{x} m_1 l_1 cos(\theta_1) - \dot{x}\dot{\theta}_1 m_1 l_1 sin(\theta_1)\right]\right] - \left[m_1 l_1 sin(\theta_2)\left[\dot{x}\dot{\theta}_1 - g\right]\right] = 0$$

$$m_1 l_1^2 \ddot{\theta}_1 - \ddot{x} m_1 l_1 cos(\theta_1) + m_1 l_1 g sin(\theta_1) = 0$$

Solving the Third Langrange Equation

$$\frac{d}{dx}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \left(\frac{\partial L}{\partial \theta_2}\right) = 0$$

$$\frac{\partial L}{\partial \theta_2} = m_2 l_2 sin(\theta_2)\left[\dot{x}\dot{\theta}_2 - g\right]$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 - \dot{x} m_2 l_2 cos(\theta_2)$$

$$\frac{d}{dx}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = m_2 l_2^2 \ddot{\theta}_2{}^2 - \left[\ddot{x} m_2 l_2 cos(\theta_2) - \dot{x}\dot{\theta}_2 m_2 l_2 sin(\theta_2)\right]$$

$$\frac{d}{dx}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \left(\frac{\partial L}{\partial \theta_2}\right) = \left[m_2 l_2^2 \ddot{\theta}_2{}^2 - \left[\ddot{x} m_2 l_2 cos(\theta_2) - \dot{x}\dot{\theta}_2 m_2 l_2 sin(\theta_2)\right]\right] - \left[m_2 l_2 sin(\theta_2)\left[\dot{x}\dot{\theta}_2 - g\right]\right] = 0$$

$$m_2 l_2^2 \ddot{\theta}_2 - \ddot{x} m_2 l_2 cos(\theta_2) + m_2 l_2 g sin(\theta_2) = 0$$

Using the above derivations, we can build the non-linear state space equation

$$\begin{bmatrix} \ddot{x}(t) \\ \ddot{\theta}_1(t) \\ \ddot{\theta}_2(t) \end{bmatrix} = \begin{bmatrix} \dfrac{F - m_1\left(gS_1C_1 + l_1S_1\dot{\theta}_1{}^2\right) - m_2\left(gS_2C_2 + l_2S_2\dot{\theta}_2{}^2\right)}{\left(M + m_1(S_1{}^2) + m_2(S_2{}^2)\right)} \\ \dfrac{C_1}{l_1}\left[\dfrac{F - m_1\left(gS_1C_1 + l_1S_1\dot{\theta}_1{}^2\right) - m_2\left(gS_2C_2 + l_2S_2\dot{\theta}_2{}^2\right)}{\left(M + m_1(S_1{}^2) + m_2(S_2{}^2)\right)}\right] - g\dfrac{S_1}{l_1} \\ \dfrac{C_2}{l_2}\left[\dfrac{F - m_1\left(gS_1C_1 + l_1S_1\dot{\theta}_1{}^2\right) - m_2\left(gS_2C_2 + l_2S_2\dot{\theta}_2{}^2\right)}{\left(M + m_1(S_1{}^2) + m_2(S_2{}^2)\right)}\right] - g\dfrac{S_2}{l_2} \end{bmatrix}$$

Part B– Linearized System

System will be linearized about the equilibrium point specified at x=0 and $\theta_1 = 0$ and $\theta_2 = 0$. We assume that θ values are very small

$$\sin(\theta) \approx 0$$

$$\sin^2(\theta) \approx 0$$

$$\cos(\theta) \approx 1$$

$$\cos^2(\theta) \approx 1$$

Let Mass M_T = m1 + m2 + M

$$\ddot{x} = \frac{F + m_1 l_1 \ddot{\theta_1} + m_2 l_2 \ddot{\theta_2}}{M_T}$$

$$\ddot{\theta}_1 = -\frac{g}{l_1}\sin(\theta_1) + \frac{\ddot{x}}{l_1}cos(\theta_1)$$

$$\ddot{\theta}_2 = -\frac{g}{l_2}\sin(\theta_2) + \frac{\ddot{x}}{l_2}cos(\theta_2)$$

The state variables are defined as such

$$X = [x, \dot{x}, \theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2]$$

$$\dot{X} = AX + BU$$

Using Lyapunov's Indirect Method,

$$J = \begin{bmatrix} \frac{\partial \ddot{x}}{\partial x} & \frac{\partial \ddot{x}}{\partial \dot{x}} & \frac{\partial \ddot{x}}{\partial \theta_1} & \frac{\partial \ddot{x}}{\partial \dot{\theta_1}} & \frac{\partial \ddot{x}}{\partial \theta_2} & \frac{\partial \ddot{x}}{\partial \dot{\theta_2}} \\ \frac{\partial \ddot{\theta_1}}{\partial x} & \frac{\partial \ddot{\theta_1}}{\partial \dot{x}} & \frac{\partial \ddot{\theta_1}}{\partial \theta_1} & \frac{\partial \ddot{\theta_1}}{\partial \dot{\theta_1}} & \frac{\partial \ddot{\theta_1}}{\partial \theta_2} & \frac{\partial \ddot{\theta_1}}{\partial \dot{\theta_2}} \\ \frac{\partial \ddot{\theta_2}}{\partial x} & \frac{\partial \ddot{\theta_2}}{\partial \dot{x}} & \frac{\partial \ddot{\theta_2}}{\partial \theta_1} & \frac{\partial \ddot{\theta_2}}{\partial \dot{\theta_1}} & \frac{\partial \ddot{\theta_2}}{\partial \theta_2} & \frac{\partial \ddot{\theta_2}}{\partial \dot{\theta_2}} \end{bmatrix}$$

Linearized system can be expressed as

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta_1} \\ \ddot{\theta_1} \\ \dot{\theta_2} \\ \ddot{\theta_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-gm_1}{M} & 0 & \frac{-gm_2}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-g(m_1+M)}{Ml_1} & 0 & \frac{-gm_2}{Ml_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-gm_1}{Ml_2} & 0 & \frac{-g(m_2+M)}{Ml_2} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta_1 \\ \dot{\theta_1} \\ \theta_2 \\ \dot{\theta_2} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{Ml_1} \\ 0 \\ \frac{1}{Ml_2} \end{bmatrix} F$$

Part C – Conditions for Controllability

We calculate the rank using this equation:

$$\text{matrix\_rank} = \text{rank}[\text{B} \quad \text{AB} \quad \text{A}^2\text{B} \quad \text{A}^3\text{B} \quad \text{A}^4\text{B} \quad \text{A}^5\text{B}]$$

We get determinant of the above matrix from Matlab as,

$$|\text{C}| = \frac{-\text{g}^6(\text{l}_1^2 - 2\text{l}_1\text{l}_2 + \text{l}_2^2)}{\text{M}^6\text{l}_1^6\text{l}_2^6}$$

```
rank of matrix:6

ans =

-(g^6*l1^2 - 2*g^6*l1*l2 + g^6*l2^2)/(M^6*l1^6*l2^6)
```

To find conditions for uncontrollability, we set $|\mathrm{C}| = 0$

$$|\mathrm{C}| = \frac{-\mathrm{g}^6(\mathrm{l}_1^2 - 2\mathrm{l}_1\mathrm{l}_2 + \mathrm{l}_2^2)}{\mathrm{M}^6\mathrm{l}_1^6\mathrm{l}_2^6} = 0$$

$$\mathrm{l}_1^2 - 2\mathrm{l}_1\mathrm{l}_2 + \mathrm{l}_2^2 = 0$$

$$(\mathrm{l}_1 + \mathrm{l}_2)^2 = 0$$

We can conclude that the system is uncontrollable when $l_1 = l_2$

Part D – System Controllability and LQR

LQR Cost function

$$J = \int_0^\infty (X^T QX + U^T RU)dt$$

Q and R are solved using the Riccati Equations

$$A^T + PA - PBR^{-1}B^T P = -Q$$

$$K = -R^{-1}B^T P$$

The controller gain K is eventually used for

$$u = -Kx$$

The matrix K is calculated MATLAB using LQR solver functions. The initial conditions that we assume for our state variables are

$$X = [5,0,30,0,45,]$$

We only change the θ angle for the pendulums and the initial displacement of the crane. The symmetric, positive definite Q matrix is initialized as below

$$Q = \begin{bmatrix} 7000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 700000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7000000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 70000000 \end{bmatrix}$$

The R was varied to compare results. Smaller R value helped the system reach steady state much quicker. Ultimately,

$$R = 0.25$$

For the initial conditions we set, the results can be seen below.

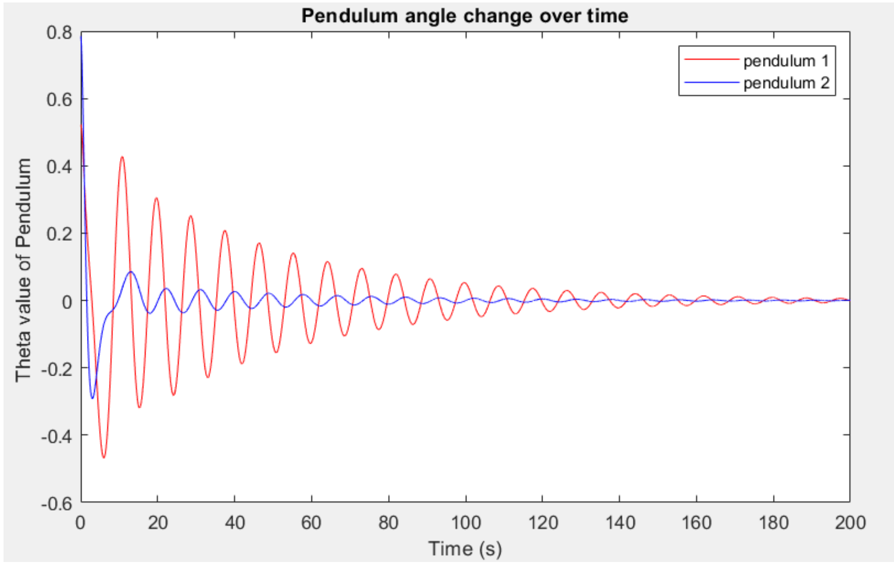


Figure 1: Pendulum Response with LQR controller (Linear System)

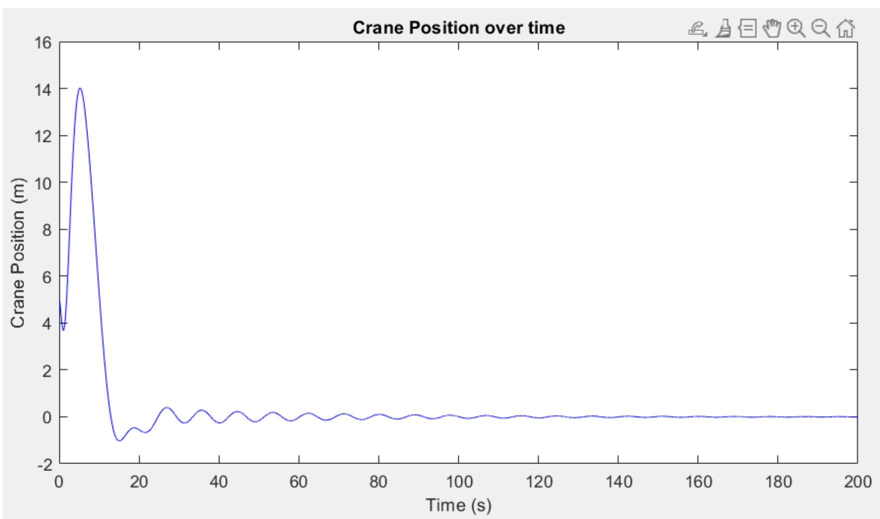


Figure 2: Crane Response with LQR controller (Linear System)

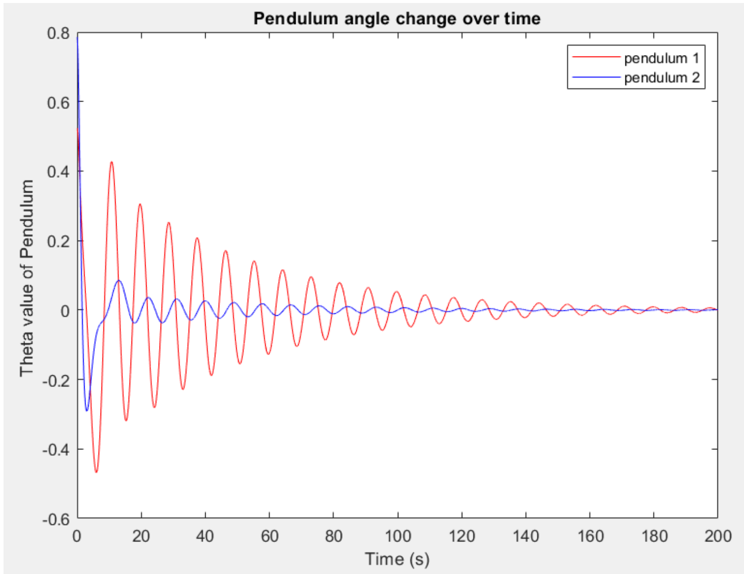


Figure 2: Pendulum Response with LQR controller (Non-Linear System)

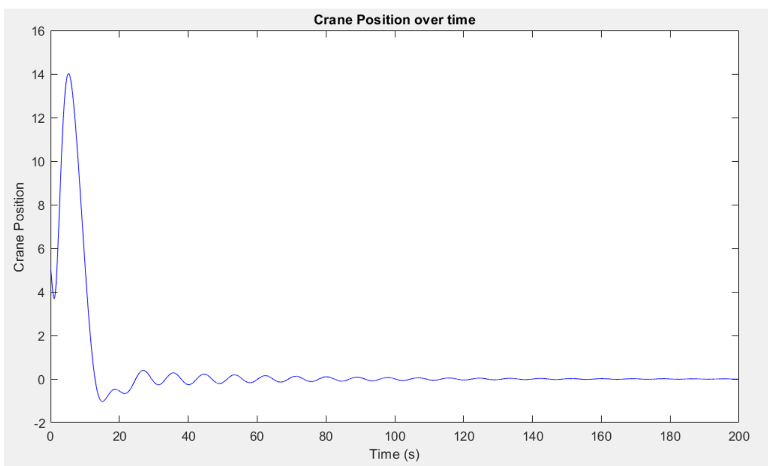


Figure 4: Crane Response with LQR controller (Non-Linear System)

Part E – Observability

Observability is calculated and compared for 4 different cases.

1) $x(t)$,
2) $[\theta_1(t), \theta_2(t)]$,
3) $[x(t), \theta_2(t)]$
4) $[x(t), \theta_1(t), \theta_2(t)]$

This is done by computing the rank of each of the systems and comparing them to the rank of matrix A. It is found that the rank of case 1, case 3 and case 4 is **6**. They are full rank matrices, and hence, observable.

For $[\theta_1(t), \theta_2(t)]$ case, the output rank is 4, i.e. less than 6. Hence the system is not observable in that case.

```
Rank_1 =

     6

Case 1 -->
This System is Observable

Rank_2 =

     4

Case 2 -->
This System is not Observable

Rank_3 =

     6

Case 3 -->
This System is Observable

Rank_4 =

     6

Case 4 -->
This System is Observable
```

Figure 5: MATLAB output for rank and observability for each case

Part F – Implementing Luenberger observer

The Luenberger Observer is expressed using the equation below

$$\dot{x} = Ax + Bu + LC(x - \hat{x})$$

$$\dot{x_e} = Ax + Bu + LC\dot{x}_e$$

We only implement this study on the vectors that give observable systems, hence on

1) $x(t)$,
2) $[x(t), \theta_2(t)]$
3) $[x(t), \theta_1(t), \theta_2(t)]$

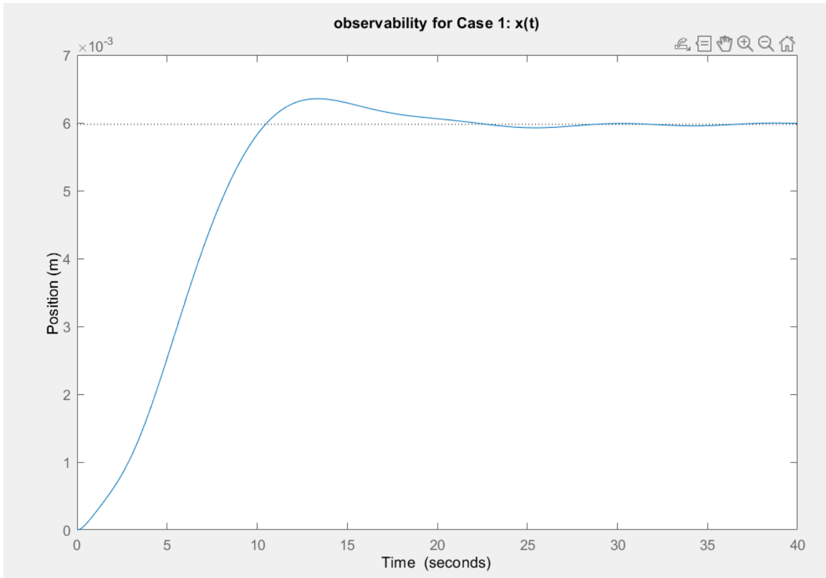


Figure 6: Simulation result with Luenberger Observer, Case 1

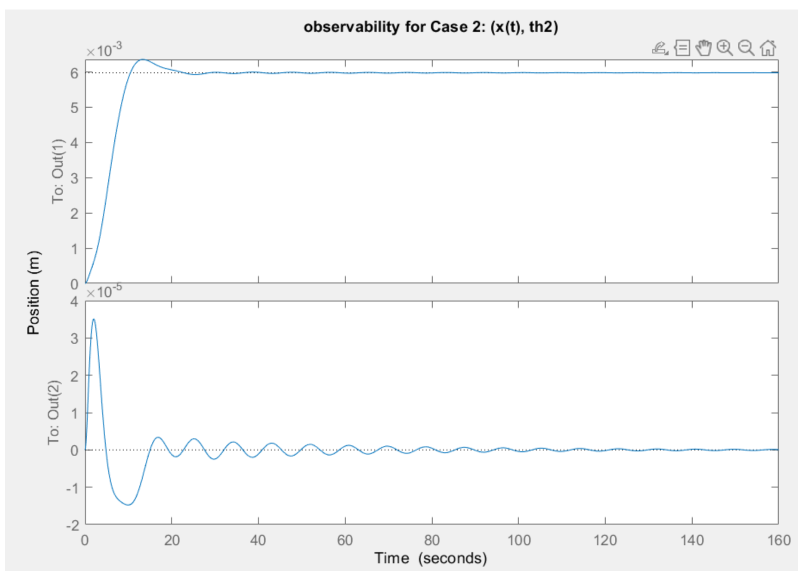


Figure 7: Simulation result with Luenberger Observer, Case 2

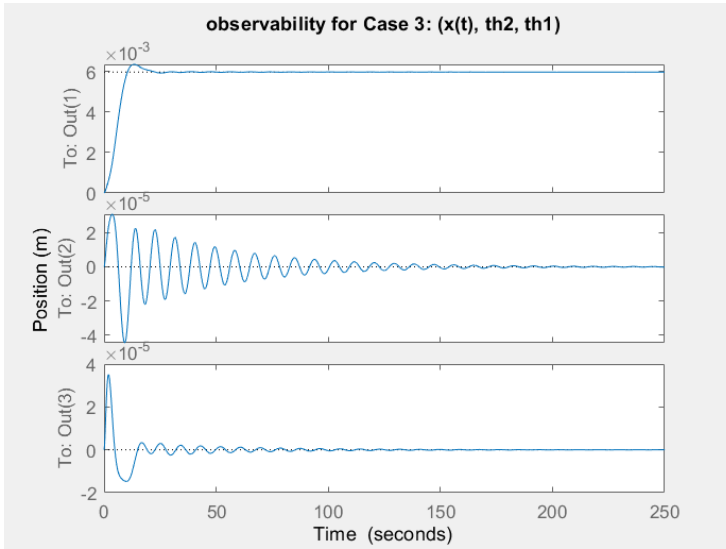


Figure 8: Simulation result with Luenberger Observer, Case 3

```
State FeedBack:
   1.0e+04 *

    0.0167    0.0862   -0.0918    0.2768    1.0913    1.2858

Eigenvalues of (A-B*K) are as follows
  -0.8914 + 0.7031i
  -0.8914 - 0.7031i
  -0.2296 + 0.2685i
  -0.2296 - 0.2685i
  -0.0220 + 0.7071i
  -0.0220 - 0.7071i
```

Figure 9: MATLAB script output

Figure 9 can also be used to show that all systems are Stable. This is because all eigenvalues of the system are on the left hand plane. They are less than zero

Part G – Output Feedback Controller

```
State FeedBack:
   1.0e+04 *

    0.0167    0.0862   -0.0918    0.2768    1.0913    1.2858
```

For the final simulation, the initial state values were chosen as follows

$$X = [10,0,90,0,45,]$$

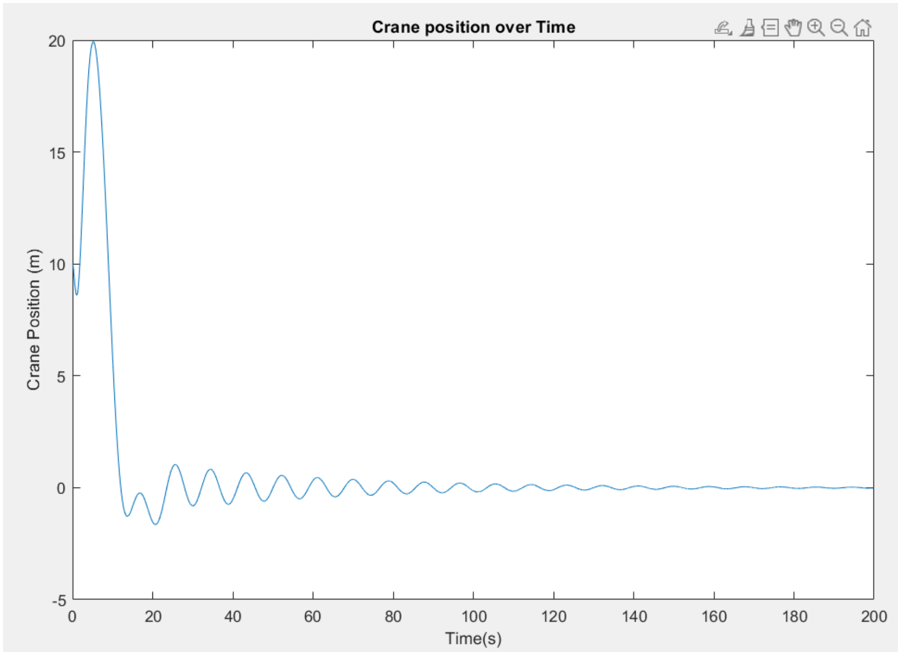


Figure 10: Crane response with the designed LQG controller