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Letter to the Editor

What is wrong in Katz's method? Comments on: "A note on fractal dimensions of biomedical waveforms"

On the November issue of this journal, Raghavendra and Dutt [1] showed that the method proposed by Katz [2] for computing the fractal dimension of waveforms, invariably underestimates the true dimension. This was not completely unexpected because also Esteller et al. [3] reported that it largely underestimates the fractal dimension of data generated by Weierstrass cosine functions. However, what appears strikingly surprising in the work of Raghavendra and Dutt is that the Katz's method practically provides the same estimate, an almost constant value close to 1, even when applied to fractional Brownian motions with true dimension between 1.0 and 1.5. Unlike the Weierstrass cosine functions, fractional Brownian motions are realistic models of several physiological processes. Thus, the results of Raghavendra and Dutt imply that the Katz's method is unable to properly describe the fractal properties of most real biomedical signals.

The Katz's method is popular in biological sciences. At the end of 2009 it has been cited by at least 152 papers, according to Google Scholar. The analysis of Raghavendra and Dutt naturally raises a question for the many readers of this journal interested in evaluating the fractal structure of biomedical signals: what is wrong in the Katz's method? This letter answers the question, explaining the poor performances reported in [1], and proposes a correction to a fundamental flaw.

To understand what is wrong in the Katz's method, let us first summarize his approach. As indicated in his original paper [2], Katz was inspired by Mandelbrot [4], who suggested that the fractal dimension of a river can be calculated from the river length, L, and the distance between source and mouth, d, as

$$\mathrm{FD} = \log(L)/\log(d). \tag{1}$$

The formula indicates that the convolutedness of the river increases with its length given the distance as the crow flies between source and mouth. It can be extended to a general fractal curve, with L and d length and extension (i.e., maximum distance between two points) of the curve. Typical fractal curve is the trajectory of a Brownian motion in the X–Y plane. Bi-dimensional trajectories are represented by a series of points (x_k,y_k), ordered according to the sampling index k. The Euclidean distance between two points, i and j, of the curve is

$$l_{i,j} = \sqrt{(y_i - y_j)^2 + (x_i - x_j)^2} \tag{2}$$

If $n+1$ is the number of points, then length L and extension d of the curve are

$$L = \sum_{i=1}^{n} l_{i,i+1} \tag{3}$$

$$d = \max\{l_{i,j}\} \tag{4}$$

Obviously, d and L should be dimensionless numbers to calculate the logarithms in Eq. (1). Thus, they should be normalized if x_k and y_k represent physical quantities. Katz proposed to normalize d and L by the length of the average step, a, defined as L/n. In this way Eq. (1) becomes

$$\mathrm{FD} = \frac{\log(n)}{\log(n) + \log\left(\frac{d}{L}\right)} \tag{5}$$

But unlike the bi-dimensional trajectory of a Brownian motion, a waveform y_k (with k the sampling index) is a mono-dimensional sequence. Therefore Katz generated a bi-dimensional curve from y_k in order to follow the Mandelbrot's approach. The curve was generated in the T–Y space as the sequence of points (t_k,y_k), with t_k time of occurrence of the sample y_k. Katz proposed to estimate the fractal dimension of y_k by calculating Eq. (5) for the (t_k,y_k) sequence, with L and d from Eqs. (2)–(4). Let us call this estimate the Katz's fractal dimension, FD_K.

It should be considered, however, that the variables describing a trajectory in the X–Y plane, x_k and y_k, are homogeneous quantities. For instance, they both correspond to physical distances if they represent the path of a river. What appears critical in the Katz's argument is that t_k and y_k are intrinsically different, being time and the measured quantity. It is true that Eq. (5) implicitly normalizes L and d by a making the estimate independent from the units of measure. However, the normalization is of little use in this case, because the same definition of distance between points is incorrect. In fact, Eq. (2) sums together terms with different units, if x is substituted by t.

A more logical way to apply the Mandelbrot's approach is to calculate Eq. (5) directly in the monodimensional space of y_k (and not on the bi-dimensional sequence generated by associating t_k to y_k, as Katz did). This approach makes the calculation of fractal dimensions computationally fast and simple. In fact, the extension d on the Y-axis is the range of y_k

$$d = \max\{y_k\} - \min\{y_k\} \tag{6}$$

and L is the sum of all the increments, in modulus

$$L = \sum_{k=1}^{n} |y_{k+1} - y_k| \tag{7}$$

To distinguish it from FD_K, let us call Eq. (5) with L and d given by Eqs. (6) and (7) the Mandelbrot's fractal dimension of a waveform, FD_M.

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The use of time in Eq. (2) produces dramatic differences between FD_M and FD_K. In particular, it makes FD_K dependent from the units of measure. To illustrate this point, I calculated FD_K for a real segment of EEG data (128 points from the Oz channel sampled at 256 Hz in a healthy subject with closed eyes) obtaining $FD_K = 1.21$ when the EEG was expressed in μV, and $FD_K = 1.00$ when expressed in mV (by contrast, FD_M was 1.68, independently from the used units). The reason is that only the vertical component of the distance $l_{i,j}$ (Eq. (2)) changes if the unit of measure of the waveform changes. Therefore, the relative distance between points is not preserved, and this affects the fractal dimension. In particular, if the amplitude of the waveform decreases, the relative weight of the horizontal component of $l_{i,j}$ (the "time" component) increases, increasing also the influence of the t_k dynamics on FD_K. Only y_k may be fractal, because t_k increases linearly with k. Thus, a greater influence of the non-fractal t_k dynamics causes FD_K to decrease, tending to 1. This explains why FD_K felt to 1 changing the scale from μV to mV in the EEG example. This also explains why Raghavendra and Dutt found that FD_K fell to 1 decreasing the amplitude of the simulated waveforms (see their figure 4).

For the same reason, in the Katz's method the horizontal component of d and L increases with n faster than the vertical component. In fact, the horizontal component (i.e., the time variable) does not follow a "convoluted" fractal dynamics, but increases linearly with each new sample. This makes FD_K to depend mainly on the linear t_k dynamics when n is large, and thus to converge to 1 when n tends to infinity, as Sevcik observed [5].

Fig. 1 (left panels) illustrates the dependency of the Katz's method on n. It shows FD_K vs. the true fractal dimension for simulated "deterministic" and stocastic fractal processes of different length. The deterministic fractal was a Weierstrass cosine function, WCF, as in [1]. This is a continuous but nowhere differentiable function defined as sum of infinite cosines

$$\mathrm{WCF}(t) = \sum_{k=0}^{\infty} \gamma^{-k(2-FD)} \cos(2\pi\gamma^k t) \tag{8}$$

with FD the fractal dimension. Data were simulated with $\gamma = 4.5$ and by truncating the series to $k = 100$. Three sets of data were synthesized

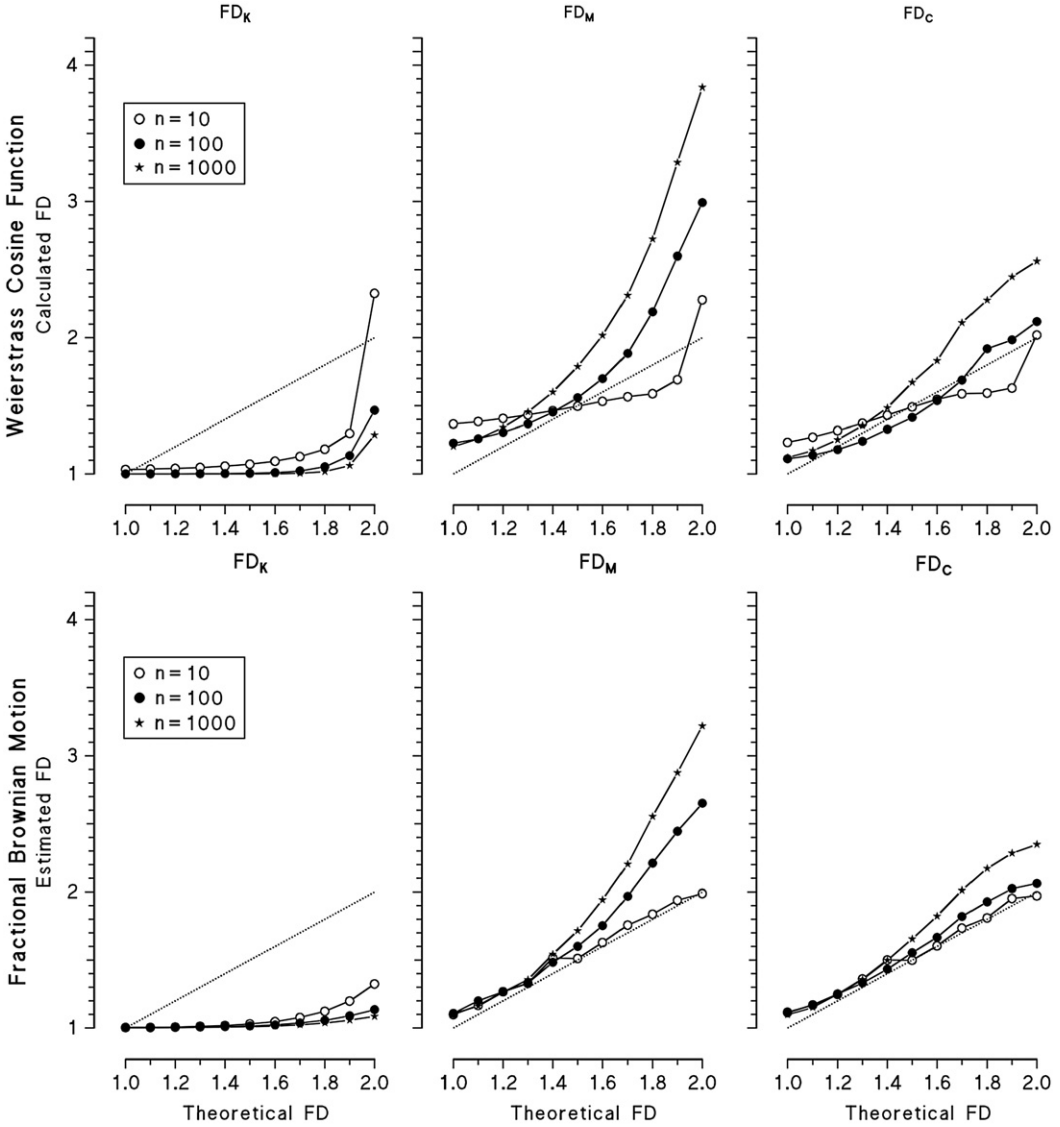

Fig. 1. Fractal dimension estimated by the Katz's method, FD_K (left); by directly applying the Mandelbrot's proposal (Eq. (1)) in one dimension, FD_M (centre); and by correcting FD_M for repetitive recursions on the same points, FD_C (right). Upper panels: values calculated over datasets of sizes n between 10 and 1000, obtained by sampling evenly Weierstrass cosine functions with true fractal dimensions FD between 1 and 2. Lower panels: average of the estimates over 100 simulated fractional Brownian motions with sizes n between 10 and 1000 and FD between 1 and 2.

for each FD. First, t was sampled between 0 and 1 with 2^4 equidistant points, selecting the central segment of $n=10$ consecutive samples; then the range $0 \le t \le 1$ was sampled again with 2^7 and 2^{10} equidistant points, selecting the central segments of $n=100$ and $n=1000$ consecutive samples, respectively. The stochastic fractals were fractional Brownian motions. Series of $n=10$, 100 and 1000 samples were generated. The same Matlab function *wfbm(H,n)* employed in [1], with H the Hurst exponent, was used to synthesize 100 time series for each fractal dimension $FD=2-H$ and for each sample size n. Time series of Weierstrass cosine functions and of fractional Brownian motions were synthesized for 11 fractal dimensions between 1 and 2.

Simulations confirm that FD_K tends to 1 when n increases (Fig. 1, left panels). With $n=100$ and $n=1000$ the trend is substantially the same described by Raghavendra and Dutt [1] in their figures 3(a) and (d): FD_K calculated for WCF or estimated for fractional Brownian motions are practically constant and equal to 1 for true fractal dimension between 1 and 1.5.

By contrast, FD_M (Fig. 1, central panels) shows a different behavior. When it is applied to Weierstrass cosine functions, it increases monotonically with the true FD. In particular, when the sample size n is equal or greater than 100, FD_M is very close to the theoretical value for theoretical FD between 1.0 and 1.5. However, when FD is greater than 1.5, FD_M deviates exponentially from the expected value. This error increases with the sample size n. When FD_M is applied to fractional Brownian motion, it provides values close to the true FD even for n as small as 10 samples. However, also in this case FD_M may deviate from the expected value when FD is greater than 1.5, the error depending on the sample size n. In fact, while FD_M is always close to the true value when $n=10$, it overestimates fractal dimensions greater than 1.5 when $n=100$, the error increasing exponentially with FD also in this case. The overestimation is amplified when $n=1000$.

The same overestimation error seems to also affect FD_K, but, for FD_K, the error appears compensated by the opposite underestimation which increases with n. This error was nevertheless noted by Raghavendra and Dutt who wrote that FD_K "has not provided linear variation but shown an exponential variation with increase of theoretical fractal dimension" [1].

The origin of this overestimation error can be understood considering that it appears only for true dimensions greater than 1.5, i.e., for Hurst exponents lower than 0.5. When $H<0.5$, the Gaussian noise generating the Brownian motion shows negative correlation [6]; thus positive steps of Brownian motion are more likely followed by negative steps, and vice-versa. Therefore when $H<0.5$, the fractional Brownian motion tends to "retrace its steps", and d increases very slowly with n.

This phenomenon is incompatible with FD defined in Eq. (1). In fact, let us consider an example of curve that returns on its steps: a periodic trajectory that repeats itself every P samples, so that $(x_0,y_0)=(x_P,y_P)$. This curve could be an evenly-sampled circle with radius R. For the circle, $d=2R$ and $L=(2\pi R/P)n$ after n steps, with $n>P$. Therefore d is constant, but L increases indefinitely with n. This means that FD increases with the logarithm of n, because the curve goes through the same points many times.

To avoid that the measured fractal dimension increases logarithmically with the length of the data segment, one might calculate L and d for a subset of the available n points, those for which the extension d is half the value measured for the whole dataset. This avoids considering the same points of the trajectory repeatedly. In the example of the circle of radius R and period P, the subset will contain the first $P/4$ points, which have $d=R$ and $L=\pi R/2$. One might apply this procedure to deterministic signals and random processes following these steps: first, the extension d is calculated from the whole dataset of n points as in Eq. (6); second, the dataset is scanned to identify the size n_W of sequences of points with extension at least equal to $d/2$ (in any case n_W should not be lower than 8 samples for a minimal statistical consistency); third, the dataset is split into consecutive, overlapped windows of n_W points, and the fractal dimension evaluated separately in each window by Eqs. (5)–(7); and finally, the corrected FD_M is obtained by averaging the fractal dimensions estimated in each window. Let us call FD_C this corrected estimate.

The right panels of Fig. 1 show FD_C for synthesized deterministic and stochastic series. When applied to the Weierstrass cosine functions, the proposed correction improves the measure of fractal dimension at all the sample sizes n. This was possible because the size n_W of the running window where Eqs. (5)–(7) are calculated decreases with the convolutedness of the curve. For instance, with $n=100$, n_W is equal to 21 samples for $FD=1.2$ and to 11 samples for $FD=1.8$. Compared with FD_M, FD_C calculated for the smallest size, $n=10$, is closer to the true FD and increases with FD more linearly; at $n=100$, FD_C is very close to the true value over the whole range of theoretical FD; and at $n=1000$ the large exponential overestimation, which affects FD_M is almost completely removed.

Similar performances characterize FD_C also when applied to fractional Brownian motions. The estimates are very close to the true values over the whole range of theoretical fractal dimensions, because FD_C removes largely estimation bias and dependency on n. Since fractional Brownian motions may approximate most physiological signals with self-similar structure, FD_C might become particularly useful for those biomedical applications where only small sets of consecutive data (between 10 and 100 samples) are available.

In conclusion, a flaw in the Katz's method is responsible for the low performances reported in previous validation studies [1,3,5]. Because of this flaw, FD_K is strongly influenced by amplitude, duration and units of measure of the waveform, resulting practically useless for any real biomedical application. The correction proposed in this letter provides a computationally simple alternative to the Katz's method.

Conflict of interest statement

The author has no conflicts of interest.

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