PROBLEM SET - 5

1) Given:

$$\left(\frac{1-\phi}{\phi}\right)\frac{c_t}{l_t} = (1-\alpha)\frac{y_t}{n_t}$$

$$E_{t-1}\left[\beta\frac{u(c_t,l_t)}{c_t}\left(\alpha\frac{y_t}{k_t}+(1-\delta)\right)\right] = \frac{u(c_{t-1},l_{t-1})}{c_{t-1}}$$ ①

$$y_t = z_t \cdot k_t^{\alpha} n_t^{1-\alpha}$$

$$1 = n_t + l_t$$

$$y_t = c_t + i_t$$

$$k_t = i_{t-1} + (1-\delta)k_{t-1}$$

$$u(c,l) = \frac{(c^{\phi} l^{1-\phi})^{1-\rho}}{1-\rho}$$

$$\ln(z_t) = (1-\rho)\ln\bar{z} + \rho\ln z_{t-1} + e_t$$

Performing steady state calculations:

$$① \rightarrow \left(\frac{1-\phi}{\phi}\right)\frac{\bar{c}}{\bar{l}} = (1-\alpha)\frac{\bar{y}}{\bar{n}}$$

$$\therefore \left(\frac{1-\phi}{\phi}\right)\frac{\bar{c}}{\bar{n}} = \frac{\bar{l}}{\bar{n}}(1-\alpha)\frac{\bar{y}}{\bar{n}}$$

$$② \rightarrow \beta\bar{u}\bar{c}^{-1}\left(\alpha\frac{\bar{y}}{\bar{k}}+1-\delta\right) = \bar{u}\bar{c}^{-1}$$

$$\frac{\bar{y}}{\bar{n}} : \frac{\bar{k}}{\bar{n}} = \frac{\left(\frac{1}{\beta}-1+\delta\right)}{\alpha}$$

③ $\Rightarrow$ $\frac{\bar{y}}{\bar{n}} = \left(\frac{\bar{K}}{\bar{n}}\right)^{\alpha}$

$\therefore \frac{\bar{K}}{\bar{n}} = \left[\frac{\frac{1}{\beta} - 1 + \delta}{\alpha}\right]^{1/\alpha-1} = \omega$

④ $\Rightarrow$ $1 = \bar{n} + \bar{l}$

⑤ $\Rightarrow$ $\bar{y} = \bar{c} + \bar{l} = \bar{c} + \bar{i}$

$\frac{\bar{y}}{\bar{n}} = \frac{\bar{c}}{\bar{n}} + \frac{i}{n}$

$\therefore \omega^{\alpha} = \frac{\bar{c}}{n} + \frac{i}{n}$

⑤ $\Rightarrow$ $\cancel{\bar{y} = \bar{c} + \bar{l} = \bar{c} + \bar{i}}$

⑥ $\Rightarrow$ $\bar{K} = i + (1-\delta)\bar{K}$

$\therefore \bar{K} = \delta \frac{\bar{K}}{\bar{n}} = \frac{\bar{i}}{n}$

Solving the above 6 equations,

$\left(\frac{\bar{K}}{\bar{i}}\right) = \left[\frac{1-(1-\delta)}{\alpha\beta}\beta\right]^{1/\alpha-1}$

$\frac{\bar{K}}{\bar{n}} = \left[\frac{1/\beta - 1 + \delta}{\alpha}\right]^{1/\alpha-1} = \omega$ — I

$\frac{\bar{y}}{\bar{n}} = \left(\frac{\bar{K}}{\bar{n}}\right)^{\alpha} = \omega^{\alpha}$ — II

$$\frac{\bar{c}}{\bar{n}} = w^{\alpha} - \delta w \quad \text{— III}$$

$$\frac{\bar{i}}{\bar{n}} = \delta w \quad \text{— IV}$$

$$\frac{\bar{I}}{\bar{n}} = \frac{(1-\phi)}{\phi} \frac{w^{\alpha} - \delta w}{(1-\alpha) w^{\alpha}} = \bar{c}\beta \quad \text{— V}$$

$$\frac{\bar{k}}{\bar{n}} = w \quad \text{— } \cancel{\text{VI}} \text{ I}$$

① $\rightarrow$ $\bar{c} + \bar{k} = \bar{k}^{\alpha} \bar{n}^{(1-\alpha)} + (1-\delta)\bar{k}$

$$1/\beta = (\alpha \bar{k}^{\alpha-1} \bar{n}^{1-\alpha} + 1 - \delta) \quad \text{— ⓐ}$$

<u>Log linearization :</u>

Now,

$y_t = z_t k_t^{\alpha} n_t^{1-\alpha}$

Applying log : $\ln y_t = \ln z_t + \alpha \ln k_t + (1-\alpha) \ln n_t$

Taylor Series : $\ln \bar{y} + \left(\frac{y_t - \bar{y}}{\bar{y}}\right)$

$$= \ln \bar{z} + \left(\frac{z_t + \bar{z}}{\bar{z}}\right) + \alpha \ln \bar{k} + \frac{\alpha}{\bar{k}}(k_t - \bar{k})$$

$$+ (1-\alpha) \ln \bar{n} + \left(\frac{1-\alpha}{\bar{n}}\right)(n_t - \bar{n})$$

In steady state :

$$\ln \bar{y} = \ln \bar{z} + \alpha \ln \bar{k} + (1-\alpha) \ln \bar{n}$$

Cancelling these terms in above eqⁿ :

$$\frac{y_t - \bar{y}}{\bar{y}} = \left(\frac{z_t - \bar{z}}{\bar{z}}\right) + \alpha\left(\frac{k_t - \bar{k}}{\bar{k}}\right) + (1-\alpha)\left(\frac{n_t - n}{n}\right)$$

$$\tilde{y}_t = \tilde{z} + \alpha \tilde{k}_t + (1-\alpha)\tilde{\eta}_t \quad — \quad T3$$

where $\tilde{y}_t = \frac{y_t - \bar{y}}{\bar{y}}$ = Percentage deviation from steady state

Applying same steps as above,

$$y_t = c_t + i_t$$

log: $\ln y_t = \ln(c_t + i_t)$

Applying Taylor $\Rightarrow \frac{y_t - \bar{y}}{\bar{y}} = \left(\frac{\bar{c}}{\bar{c}+\bar{i}}\right)\left(\frac{c_t - \bar{c}}{\bar{c}}\right) + \left(\frac{\bar{i}}{\bar{c}+\bar{i}}\right)\left(\frac{i_t - i}{i}\right)$

steady state $\bar{y} = \bar{c} + \bar{i}$

$$\tilde{y}_t = \frac{\bar{c}}{\bar{y}}\tilde{c}_t + \frac{\bar{i}}{\bar{y}}\tilde{i}_t \quad — \quad T5$$

$\Rightarrow k_t = i_{t-1} + (1-\delta)k_{t-1}$

log: $\ln k_t = \ln(i_{t-1} + (1-\delta)k_{t-1})$

Taylor: $\ln \bar{k}_t + \left(\frac{k_t - \bar{k}}{k}\right) = \ln(i + (1-\delta)\bar{k}) + \frac{1}{(i+(1-\delta)\bar{k})}(i_{t-1} - i)$

$$+ \frac{(1-\delta)}{(i+(1-\delta)\bar{k})}(k_{t-1} - \bar{k})$$

simplifying,

$$\tilde{k}_t = \frac{i}{k}\tilde{i}_{t-1}(1-\delta)\tilde{k}_{t-1} \quad — \quad T6$$

$\Rightarrow \quad 1 = n_t + l_t$

log : $\ln(1) = \ln(n_t + l_t)$

$\ln(n_t + l_t) = 0$

Taylor : $\ln(\bar{n} + \bar{l}) + \frac{1}{(\bar{n}+\bar{l})}(n_t - \bar{n})$

At steady $\bar{n} + \bar{l} = 1 \Rightarrow \ln(\bar{n} + \bar{l}) - \ln(1) = 0$

$\therefore \left(\frac{n_t - \bar{n}}{\bar{n}}\right) \cdot \frac{\bar{n}}{(\bar{n}+\bar{l})} = \left(\frac{l_t - \bar{l}}{\bar{l}}\right) \times \frac{\bar{l}}{\bar{n}+\bar{l}} = 0$

$\therefore \bar{n}\,\tilde{n}_t + \bar{l}\,\tilde{l}_t = 0 \longrightarrow T4$

$\Rightarrow \quad \frac{(1-\phi)}{\phi} \frac{c_t}{l_t} = (1-\alpha)\frac{y_t}{n_t}$

By following the same approach,

$\frac{c_t - \bar{c}}{\bar{c}} = \frac{(l_t - \bar{l})}{\bar{l}} = \left(\frac{y_t - \bar{y}}{\bar{y}}\right) - \left(\frac{n_t - \bar{n}}{\bar{n}}\right)$

$\therefore \tilde{c}_t + \tilde{n}_t = \tilde{y}_t + \tilde{l}_t \quad — T1$

$\Rightarrow \quad \ln(z_t) = (1-\rho)\ln\bar{z} + \rho \ln z_{t-1} + \epsilon_t$

Removing steady states from both sides

$\tilde{z}_t = \rho\,\tilde{z}_{t-1} + \epsilon_t \longrightarrow T8$

Now, from the above equations

$E_{t-1}\left[\beta \cdot c_t^{\phi(1-\rho)-1}\, l_t^{(1-\phi)(1-\rho)}\left(\alpha z_t k_t^{\alpha-1} n_t^{1-\alpha} + 1 - \delta\right)\right]$

$= c_t^{\phi(1-\rho)-1}\, l_{t-1}^{(1-\phi)(1-\rho)}$

Assume: $\phi(1-\rho)-1 = \gamma$ & $(1-\phi)(1-\rho) = \lambda$

log: $\gamma \ln c_{t-1} + \lambda \ln l_{t-1} = \ln\beta + E_{t-1}\Big[\gamma \ln c_t + \lambda \ln l_t + \ln(\alpha z_t K_t^{\alpha-1} n_t^{1-\alpha} + 1-\delta)\Big]$

Applying Taylor Series,

$$\ln(\alpha z_t k_t^{\alpha-1} n_t^{1-\alpha} + 1-\delta) \simeq \ln(\alpha \bar{K}^{\alpha-1}\bar{n}^{1-\alpha} + 1-\delta) + \frac{\alpha \bar{K}^{\alpha-1}\bar{n}^{1-\alpha}(z_t-\bar{z})}{\alpha \bar{K}^{\alpha-1}\bar{n}^{1-\alpha}+1-\delta} + \frac{\alpha(\alpha-1)\bar{K}^{\alpha-1}\bar{n}^{1-\alpha}}{\alpha \bar{K}^{\alpha-1}\bar{n}^{1-\alpha}+1-\delta}(n_t-\bar{n}) + \frac{\alpha(\alpha-1)\bar{K}^{\alpha-2}\bar{n}^{1-\alpha}}{\alpha \bar{K}^{\alpha-1}\bar{n}^{1-\alpha}+1-\delta}(K_t-\bar{K})$$

$$\gamma \ln\bar{c} + \gamma\frac{(c_{t-1}-\bar{c})}{\bar{c}} + \lambda \ln\bar{l} + \lambda\frac{(l_{t-1}-\bar{l})}{\bar{l}} = \ln\beta + E_{t-1}\Big[\gamma\ln\bar{c} + \gamma\frac{(c_t-c)}{c} + \lambda\ln\bar{l} + \lambda\frac{(l_t-\bar{l})}{\bar{l}}$$

Cancelling similar terms,

$$\gamma\tilde{c}_{t-1} + \lambda\tilde{l}_{t-1} = E_{t-1}\Big[\gamma\tilde{c}_t + \lambda\tilde{l}_t + (1-(1-\delta)\beta)\tilde{z}_t + (\alpha-1)(1-(1-\delta)\beta)\tilde{K}_t + (1-\alpha)(1-(1-\delta)\beta)\tilde{n}_t \qquad \text{T2}$$

T1 to T8 are the log linearized system of eq^n using 1st order Taylor Series Expansion