APPM 1350 Exam 2 Solutions Spring 2013

1. (15 points)

   (a) Find the linearization of $f(x) = \sqrt[4]{1-x}$ at $x = 0$.

   (b) Use the linearization to approximate the value of $\sqrt[4]{0.92}$.

Solution:

(a) First find the values of $f(0)$ and $f'(0)$.

$$f(x) = (1-x)^{1/4}$$
$$f(0) = 1$$
$$f'(x) = -\frac{1}{4}(1-x)^{-3/4} = \frac{-1}{4(1-x)^{3/4}}$$
$$f'(0) = -\frac{1}{4}$$

Then find $L(x)$.

$$L(x) = f(0) + f'(0)(x) = \boxed{1 - \frac{1}{4}x}$$

(b) Note that $\sqrt[4]{0.92} = f(0.08) \approx L(0.08)$.

$$L(0.08) = 1 - \frac{1}{4}(0.08) = 1 - 0.02 = \boxed{0.98}$$

2. (30 points) Consider the function $f(x) = \dfrac{-2x}{x^2-3}$, $f'(x) = \dfrac{2(x^2+3)}{(x^2-3)^2}$, $f''(x) = \dfrac{-4x(x^2+9)}{(x^2-3)^3}$.

   (a) Find any vertical, horizontal, or slant asymptotes of f. Use appropriate limits to justify your answer.

   (b) On what intervals is f increasing? decreasing?

   (c) Find all local maximum and minimum values of f.

   (d) On what intervals is f concave up? concave down?

   (e) Find all inflection points of f.

   (f) Using the information from (a) to (e), sketch a graph of f. Clearly label any asymptotes, local extrema, and inflection points.

Solution:

(a) f is not defined at $x = \pm\sqrt{3}$ so we check at these x-values for asymptotes.

$$\lim_{x\to-\sqrt{3}^-} \frac{-2x}{x^2-3} = \frac{2\sqrt{3}}{0^+} = +\infty,$$
$$\lim_{x\to-\sqrt{3}^+} \frac{-2x}{x^2-3} = \frac{2\sqrt{3}}{0^-} = -\infty,$$
$$\lim_{x\to\sqrt{3}^-} \frac{-2x}{x^2-3} = \frac{-2\sqrt{3}}{0^-} = +\infty,$$
$$\lim_{x\to\sqrt{3}^+} \frac{-2x}{x^2-3} = \frac{-2\sqrt{3}}{0^+} = -\infty.$$

So there are vertical asymptotes at $\boxed{x = \pm\sqrt{3}}$.

$$\lim_{x\to-\infty} \frac{-2x}{x^2-3} = \lim_{x\to-\infty} \frac{x^2}{x^2}\cdot\frac{-2/x}{(1-3/x^2)} = \lim_{x\to-\infty} \frac{-2/x}{1-3/x^2} = \frac{0}{1-0} = 0,$$
$$\lim_{x\to\infty} \frac{-2x}{x^2-3} = \lim_{x\to\infty} \frac{x^2}{x^2}\cdot\frac{-2/x}{(1-3/x^2)} = \lim_{x\to\infty} \frac{-2/x}{1-3/x^2} = \frac{0}{1-0} = 0.$$

So there is a horizontal asymptote at $\boxed{y = 0}$.

(b) $f'(x) = \dfrac{2(x^2+3)}{(x^2-3)^2} > 0$ so f is always increasing where defined on

$$\boxed{\left(-\infty, -\sqrt{3}\right) \cup \left(-\sqrt{3}, \sqrt{3}\right) \cup \left(\sqrt{3}, \infty\right)}.$$

(c) Since f is always increasing there are $\boxed{\text{no local extrema}}$.

(d) $f''(x) = \dfrac{-4x(x^2+9)}{(x^2-3)^3}$ so $f'' = 0$ at $x = 0$ and f'' is undefined at $x = \pm\sqrt{3}$. Testing values on the four subintervals, we find that f is concave up on $\boxed{\left(-\infty, -\sqrt{3}\right), \left(0, \sqrt{3}\right)}$ and concave down on $\boxed{\left(-\sqrt{3}, 0\right), \left(\sqrt{3}, \infty\right)}$.

(e) f switches from concave up to concave down at $x = 0$ and is continuous at $x = 0$, so $(0, f(0)) = \boxed{(0,0)}$ is the only inflection point.

(f)

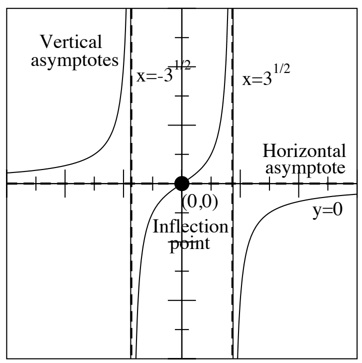
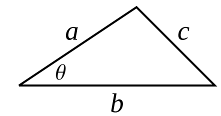


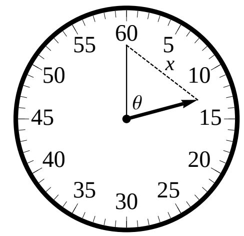
Figure 1: Graph of $y = \dfrac{-2x}{x^2-3}$

3. (15 points) The second hand on a stopwatch, 5 centimeters in length, makes a full revolution every minute. Let x represent the distance between the tip of the hand and its starting position at the 60-second mark. At what rate is x increasing when the hand reaches the 15-second mark? Express your answer in centimeters per second.

Hint: Use the Law of Cosines:

$c^2 = a^2 + b^2 - 2ab\cos\theta.$





Solution:

We wish to find dx/dt when $\theta = \pi/2$. The second hand makes a full revolution each minute, or 2π radians every 60 seconds, so $d\theta/dt = 2\pi/60 = \pi/30$ rad/sec. Use the Law of Cosines.

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos\theta \\ x^2 &= 5^2 + 5^2 - 2(5)(5)\cos\theta \\ x^2 &= 50 - 50\cos\theta \end{aligned}$$

Differentiate with respect to time.

$$\begin{aligned} 2x\frac{dx}{dt} &= 50\sin\theta\,\frac{d\theta}{dt} \\ x\frac{dx}{dt} &= 25\sin\theta\,\frac{d\theta}{dt} \end{aligned}$$

Note that when $\theta = \pi/2$, then $x = \sqrt{50} = 5\sqrt{2}$.

$$\begin{aligned} 5\sqrt{2}\,\frac{dx}{dt} &= 25\left(\sin\frac{\pi}{2}\right)\left(\frac{\pi}{30}\right) = 25(1)\left(\frac{\pi}{30}\right) \\ 5\sqrt{2}\,\frac{dx}{dt} &= \frac{5\pi}{6} \end{aligned}$$

$$\frac{dx}{dt} = \boxed{\frac{\pi}{6\sqrt{2}} \text{ cm/sec}}$$

4. (12 points) Let $f(x) = 1/x$, where $0 < a < b$.

   (a) Verify that f satisfies the hypotheses of the Mean Value Theorem.

   (b) Find the value(s) of c that satisfy the conclusion of the Mean Value Theorem. Express your answer in terms of a and b.

Solution:

(a) The derivative of f is $f'(x) = -1/x^2$. Both f and f' are undefined at $x = 0$ but we are given that a and b are positive so f is continuous on $[a, b]$ and differentiable on (a, b), satisfying the hypotheses of the Mean Value Theorem.

(b) The Mean Value Theorem states there there is a c in (a, b) such that $f'(c) = (f(b) - f(a))/(b - a)$.

$$\begin{aligned} f'(c) &= \frac{\frac{1}{b} - \frac{1}{a}}{b - a} \\ -\frac{1}{c^2} &= \frac{1}{b-a} \cdot \frac{a-b}{ab} = -\frac{1}{ab} \\ c^2 &= ab \\ c &= \boxed{\sqrt{ab}} \end{aligned}$$

5. (12 points) For the following statements, answer TRUE if the statement is always true and justify your answer. Otherwise provide a sketch of a COUNTEREXAMPLE to show that the statement may be false.

   (a) If f is differentiable for all x, then f has an absolute minimum value on $[-5, 5]$.

   (b) If g is decreasing for $x < -2$ and increasing for $x > -2$, then g has a local minimum value at $x = -2$.

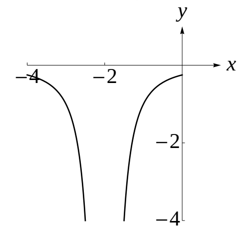
   (c) If h is continuous and $h(-3) = h(7)$, then there is a number c in $(-3, 7)$ such that $h'(c) = 0$.

Solution:

(a) TRUE. If f is differentiable for all x, then it is continuous for all x. By the Extreme Value Theorem, a continuous function must have absolute extrema on a closed interval.

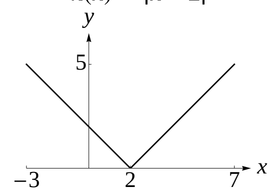
(b) FALSE. A counterexample is $g(x) = -1/(x + 2)^2$.

$$g(x) = -\frac{1}{(x+2)^2}$$

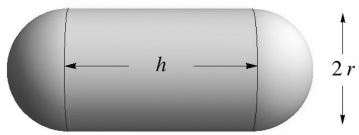


(c) FALSE. A counterexample is $h(x) = |x - 2|$.

$$h(x) = |x - 2|$$



6. (16 points) Hank Hill is designing a propane tank with a volume of 64π cubic meters. The tank is cylindrical with spherical endcaps. The spherical endcaps cost $8/3$ as much per square meter as the cylindrical body. What dimensions will minimize the cost of materials for the tank?



Solution:

We start by writing down the volume and surface area,

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 + \pi r^2 h = 64\pi, \\ S &= 4\pi r^2 + 2\pi r h. \end{aligned}$$

Since the spherical endcaps cost eight-thirds as much the sidewalls we write down the cost function to be minimized,

$$C(r,h) = \frac{32}{3}\pi r^2 + 2\pi r h.$$

To get cost as a function of a single variable we look to the volume equation,

$$\begin{aligned} V = \frac{4}{3}\pi r^3 + \pi r^2 h &= 64\pi \\ \pi r^2 h &= 64\pi - \frac{4}{3}\pi r^3 \\ h &= \frac{64}{r^2} - \frac{4}{3}r, \end{aligned}$$

and substitute to get

$$\begin{aligned} C(r) &= \frac{32}{3}\pi r^2 + 2\pi r\left(\frac{64}{r^2} - \frac{4}{3}r\right) \\ &= \frac{32}{3}\pi r^2 + \frac{128\pi}{r} - \frac{8}{3}\pi r^2 \\ &= 8\pi r^2 + \frac{128\pi}{r}. \end{aligned}$$

We check for critical numbers. The domain of $C(r)$ is $(0,\infty)$ and $C(r)$ is defined for all $r > 0$ so the only critical points we have are those when $C'(r) = 0$.

$$\begin{aligned} C'(r) = 16\pi r - \frac{128\pi}{r^2} &= 0 \\ 16\pi r^3 &= 128\pi \\ r^3 &= 8 \\ r &= 2. \end{aligned}$$

So $r = 2$ is the only critical number of $C(r)$.

Now,

$$\begin{aligned} C'(r) = 16\pi r - \frac{128\pi}{r^2} &> 0 \\ 16\pi r^3 &> 128\pi \\ r^3 &> 8 \\ r &> 2, \end{aligned}$$

and

$$\begin{aligned} C'(r) = 16\pi r - \frac{128\pi}{r^2} &< 0 \\ 16\pi r^3 &< 128\pi \\ r^3 &< 8 \\ r &< 2. \end{aligned}$$

So $r = 2$ is a critical number of $C(r)$ and $C(r)$ is increasing when $r > 2$ and $C(r)$ is decreasing when $r < 2$. Thus by the First Derivative Test for Absolute Extrema the cost is minimized when $r = \boxed{\text{2 meters}}$ and

$h = \dfrac{64}{2^2} - \dfrac{4}{3}(2) = 16 - \dfrac{8}{3} = \dfrac{48-8}{3} = \boxed{\dfrac{40}{3}\text{ meters}}$.