Solutions

Math 1300-005 - Spring 2017

The Intermediate Value Theorem - 1/31/17

Guidelines: Please work in groups of two or three. Please show all work and clearly denote your answer.

1. Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval. Remember, in solving each problem, you must verify that each of the hypotheses of the IVT are satisfied.

   (a) $x^4 + x - 3 = 0, \quad (1, 2)$

   Let $f(x) = x^4 + x - 3$, which is continuous on $[1,2]$ b/c it is a polynomial.
   Since $f(1) = 1+1-3 < 0$, $f(2) = 16+2-3 > 0$, 0 is between $f(1)$ & $f(2)$, so by IVT
   there exists c in $(1,2)$ such that $f(c) = 0$, ie, $c^4 + c - 3 = 0$.

   (b) $\sqrt[3]{x} = 1 - x, \quad (0, 1)$

   Rearrange first as $\sqrt[3]{x} + x - 1 = 0$. Let $f(x) = \sqrt[3]{x} + x - 1$, which is cont. on $[0,1]$ b/c it is a root function plus a polynomial. Since
   $f(0) = -1 < 0$, $f(1) = 1 > 0$, 0 is between $f(0)$ and $f(1)$. By the IVT
   there exists c in $(0,1)$ such that $f(c) = 0$, ie, $\sqrt[3]{c} + c - 1 = 0$.

   (c) $e^x = 3 - 2x, \quad (0, 1)$

   Rearrange first as $e^x + 2x - 3 = 0$. Let $f(x) = e^x + 2x - 3$, which is cont. on $[0,1]$ b/c it is an exponential function plus a polynomial. Since
   $f(0) = e^0 + 2(0) - 3 = 1 - 3 < 0$
   $f(1) = e + 2 - 3 = e - 1 > 0$ (since $e \approx 2.7$), 0 is between $f(0)$ and $f(1)$. By the IVT, there exists c in $(0,1)$ such that $f(c) = 0$, ie, $e^c + 2c - 3 = 0$.

   (d) $\sin(x) = x^2 - x, \quad (1, 2)$

   Rearrange first as $\sin(x) + x - x^2 = 0$. Let $f(x) = \sin(x) + x - x^2$, which is continuous on $[1,2]$ b/c it is a trig function plus a polynomial. Since
   $f(1) = \sin(1) + 1 - (1)^2 = \sin(1) > 0$
   $f(2) = \sin(2) + 2 - (2)^2 = \sin(2) - 2 < 0$, 0 is between $f(1)$ and $f(2)$. By the IVT, there exists c in $(1,2)$ such that $f(c) = 0$, ie, $\sin(c) + c - c^2 = 0$.

   To get that $\sin(1) > 0$, note 1 is between 0 and $\frac{\pi}{2}$ and $\sin(x) > 0$ on $(0, \frac{\pi}{2})$ [draw a graph].

   To get $\sin(2) - 2 < 0$, note $-1 \le \sin(2) \le 1$, so at most $\sin(2) = 1$, and $1 < 2$, so $\sin(2) - 2 < 0$

The following problems are review of the material we covered Monday 1/30 over the definition of continuity.

2. State the interval(s) where the following function is continuous.

$$f(x) = \begin{cases} \cos(x) & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 - x^2 & \text{if } x > 0 \end{cases}$$

We need only check continuity at $a=0$.

① $f(0)$ is defined and $f(0)=0$.

② $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \cos(x) = \cos(0) = 1$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 1-x^2 = 1-(0)^2 = 1$ $\rightarrow$ so $\lim_{x\to 0} f(x) = 1.$

③ Since $\lim_{x\to 0} f(x) \neq f(0)$, f is not continuous at 0. Hence

$\boxed{f \text{ is continuous on } (-\infty, 0) \cup (0, \infty)}$

3. For what value of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} x+2 & \text{if } x < 0 \\ e^{2x+c} & \text{if } x \geq 0 \end{cases}$$

We need only check continuity at $a=0$.

① $f(0)$ is defined and $f(0) = e^{2(0)+c} = e^c$

② $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x+2) = 2$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{2x+c} = e^c$

We need ~~RHL=LHL~~ RHL = LHL = $f(0)$, so we need $2 = e^c$ or $\boxed{c = \ln(2)}$