MATH 1300: HW #7

Due on March 2, 2017 at 10:00am

Professor Braden Balentine Section 005

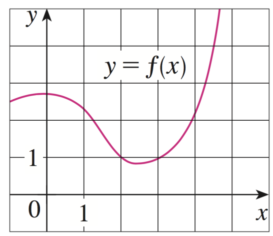
John Keller

Section 3.4

56. If f is the function whose graph is shown, let $h(x) = f(f(x))$ and $g(x) = f(x^2)$. Use the graph of f to estimate the value of each derivative.

    (a) $h'(2) = f'(1) \cdot -1 = -1 \cdot -1 = 1$

    (b) $g'(2) = f'(2^2) \cdot 2(2) = 2 \cdot 4 = 8$



74. Under certain circimstances a rumor spreads according to the equation

$$p(t) = \frac{1}{1 + ae^{-kt}}$$

where $p(t)$ is the proportion of the population that knows the rumor at time t and a and k are positive constants. [In section 7.5 we will see that this is a reasonable equation for $p(t)$.]

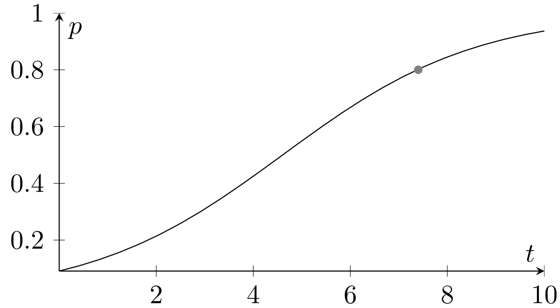
(a) Find $\lim_{t\to\infty} p(t)$.

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{1 + ae^{-kt}} &= \lim_{t\to\infty} \frac{1}{1 + \frac{a}{e^{kt}}} \\ &= \frac{1}{1 + 0} \\ &= 1 \end{aligned}$$

(b) Find the rate of spread of the rumor.

$$\begin{aligned} p'(t) &= \frac{(1 + ae^{-kt}) \cdot 0 - 1 \cdot (0 + ae^{-kt}(-k)}{(1 + ae^{-kt})^2} \\ &= \frac{-(-ake^{-kt}}{(1 + ae^{-kt})^2} \\ &= \frac{ake^{-kt}}{(1 + ae^{-kt})^2} \end{aligned}$$

(c) Graph p for the case $a = 10$, $k = 0.5$ with t measured in hours. Use the graph to estimate how long it will take for 80% of the population to hear the rumor.



Using the graph, it appears the rumor will take 7.4 hours to spread to 80% of the population.

Section 3.5

35. If $xy + e^y = e$, find the value of y'' at the point where $x = 0$.

$$\begin{aligned}
xy + e^y &= e \\
0y + e^y &= e \\
e^y &= e \\
y + xy' + y'e^y &= 0 \\
1 + 0y' + y'e^1 &= 0 \\
1 + y'e &= 0 \\
y' &= \frac{-1}{e} \\
2y' + (x + e^y)y'' + (y')e^y &= 0 \\
2(\frac{-1}{e}) + (0 + e^1)y'' + (-1/e)^2 e^1 &= 0 \\
-\frac{1}{e} + ey'' &= 0 \\
y'' &= \frac{1}{e^2}
\end{aligned}$$

44. The two curves are **orthogonal** if their tangent lines are perpendicular at each point of intersection. Show that the given families of curves are orthogonal trajectories of each other, that is, every curve in one family is orthogonal to every curve in the other family. Sketch both families of curves on the same axis.

$$y = ax^3, \qquad x^2 + 3y^2 = b$$

$$\begin{aligned}
\frac{dy}{dx} &= 3(ax)^2 \\
2x + 6y\frac{dy}{dx} &= b\frac{dy}{dx} \\
6y\frac{dy}{dx} - b\frac{dy}{dx} &= -2x \\
\frac{dy}{dx}(6y - b) &= -2x \\
\frac{dy}{dx} &= 2x \\
3(ax)^2 &= 2x
\end{aligned}$$

Section 4.1

10. A particle moves along the curve $y = \sqrt{1+x^3}$. As it reaches the point $(2, 3)$, the y-coordinate is increasing at a rate of 4 cm/s. How fast is the x-coordinate of the point changing at that instant?

$$\begin{aligned}
\frac{dy}{dx} &= 4 \\
y &= \frac{1}{2}(1+x^3)^{-\frac{1}{2}}(3x^2) \\
3\frac{dx}{dx} &= \frac{1}{2}(1+2^3\frac{dy}{dx})^{-\frac{1}{2}}(3(2)^2\frac{dy}{dx}) \\
3 &= \frac{1}{2}(1+2^3\frac{dy}{dx})^{-\frac{1}{2}}(3(2)^2\frac{dy}{dx}) \\
3 &= \frac{1}{2}(1+2^3\frac{dy}{dx})^{-\frac{1}{2}}(3(2)^2\frac{dy}{dx})
\end{aligned}$$

$$\frac{dy}{dx} = 1 + \frac{\sqrt{5}}{2} \approx \boxed{2.1}$$

14. A street light is mounted on top of a 15-ft-tall pole. A man 6 ft tall walks away from the pole with a speed of 5 ft/s along a straight path. How fast is the top of his shadow moving when he is 40 ft from the pole?

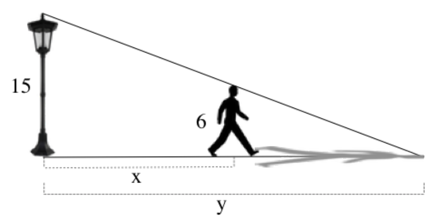
(a) What quantities are given in the problem?

- $\frac{dx}{dt} = 5$ ft/s
- $h = 15$ ft
- $h_2 = 6$ ft
- $b = 40$ ft

(b) What is the unknown?

- y

(c) Draw a picture of the situation for any time t.



(d) Write an equation that relates the quantities

$$\frac{15}{y} = \frac{6}{y-x} \qquad \text{or} \qquad 9y = 15x$$

(e) Finish solving the problem.

$$\begin{aligned}
9\frac{dy}{dt} &= 15\frac{dx}{dt} \\
9\frac{dy}{dt} &= 15(5) \\
\frac{dy}{dt} &= \frac{25}{3} \text{ ft/s}
\end{aligned}$$

26. Water is leaking out of an inverted conical tank at a rate of 10,000 cm^3/min at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 m and the diameter at the top is 4 m. If the water level is rising at a rate of 20 cm/min when the height of the water is 2 m, dins the rate at which water is being pumped into the tank.

   - $\frac{dV}{dt} = -10,000 \text{ cm}^3/\text{min}$
   - $\frac{dh}{ht} = 200 \text{ cm}$
   - $r = 400$ cm
   - $h = 600$ cm

$$\begin{aligned}
V &= \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h \\
V &= \frac{1}{3}\pi\frac{1}{4}h^2 h \\
\frac{dV}{dt} &= \frac{1}{3}\pi\frac{3}{4}\cdot 200^2 \cdot 20 \\
\frac{dV}{dt} &= \frac{3}{12}\pi 40,000(20) \\
\frac{dV}{dt} &= \frac{1}{4}\pi 800,000 \\
\frac{dV}{dt} &= \boxed{200,000\pi - 10,000}
\end{aligned}$$

44. The minute hand on a watch is 8 mm long and the hour hand is 4 mm long. How fast is the distance between the hands changing at one o'clock?

   - Law of Cosines: $c^2 = a^2 + b^2 - 2ab\cos(C)$

$$\begin{aligned}
c^2 &= 8^2 + 4^2 - 2(4)(8)\cos(C) \\
c^2 &= 80 - 64\cos(C) \\
c^2 &= 80 - 64\cos\left(\frac{\pi}{6}\right) \\
c^2 &= 80 - 64\cdot\frac{\sqrt{3}}{2} \\
c^2 &= 80 - 32\sqrt{3} \\
c &= \sqrt{80 - 32\sqrt{3}} \\
2c\frac{dc}{dt} &= 64\sin(C)\frac{dC}{dt} \\
2(\sqrt{80 - 32\sqrt{3}})\frac{dc}{dt} &= 64\sin(\frac{\pi}{6})(-\frac{11\pi}{6}) \\
2(\sqrt{80 - 32\sqrt{3}})\frac{dc}{dt} &= 64\frac{1}{2}(-\frac{11\pi}{6}) \\
\sqrt{80 - 32\sqrt{3}}\frac{dc}{dt} &= -\frac{88\pi}{3} \\
\frac{dc}{dt} &= -\frac{88\pi}{3\sqrt{80 - 32\sqrt{3}}} \approx 18 \text{ mm/hr}
\end{aligned}$$