1. The following problems are not related.

(a)(7 pts) Suppose $a(x) = \sqrt{x} + x^2$ and $b(x) = 2x - x^2$ then what is the domain of the function $y = a(x)/b(x)$? Give your answer in interval notation.

(b)(7 pts) Suppose $n(x) = x^2 + 4x + 4$ and $m(x) = \sqrt{x}$, find $(m \circ n)(x)$ and sketch the graph of the composition.

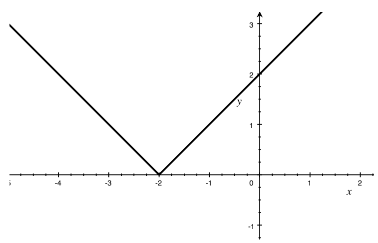
(c)(7 pts) Suppose the function $y = g(x)$ has horizontal asymptote $y = 3$ and vertical asymptote $x = -1$, find all horizontal and vertical asymptotes of the function $h(t) = -g(t-2)/3$. Justify your answer.

(d)(7 pts) Suppose $f(x)$ is an odd function such that $\lim_{x \to 5^-} f(x) = c$, where c is some nonzero constant. Which of the following five limits given below is/are equal to $-c$? [Clearly write down your answer(s) in your bluebook, no justification necessary.]

$$(i) \lim_{x \to 5^+} f(x) \quad (ii) \lim_{x \to -5^+} f(x) \quad (iii) \lim_{x \to -5^-} f(x) \quad (iv) \lim_{x \to +5} f(x) \quad (v) \lim_{x \to -5} f(x)$$

Solution: (a)(7 pts) For the numerator we have $a(x) = \sqrt{x} + x^2$ and so we need $x \geq 0$ and for the denominator, note that $b(x) = 2x - x^2 = x(2-x)$, so we need $x \neq 0$ and $x \neq 2$ and so the domain is $\boxed{\text{all real numbers } x \text{ in either } (0,2) \text{ or } (2,\infty)}$ or, alternately, $(0,2) \cup (2,\infty)$.

(b)(7 pts) Note that $(m \circ n)(x) = \sqrt{x^2+4x+4} = \sqrt{(x+2)^2} = |x+2|$ and so the graph looks like



(c)(7 pts) Note that we can assume that $\lim_{x\to\infty} g(x) = 3$ and $\lim_{x \to -\infty} g(x) = 3$ and so

$$\lim_{t\to\infty} h(t) = \lim_{t\to\infty} -g(t-2)/3 = -\frac{1}{3}\lim_{t\to\infty} g(t-2) = -\frac{1}{3}\cdot 3 = -1$$

and, similarly, $\lim_{t\to-\infty} h(t) = -1$ and so $y = -1$ is a horizontal asymptote of $h(t)$. Now note that the graph of $h(t) = -g(t-2)/3$ is the graph of $g(t)$ reflected about the t-axis, re-scaled and shifted to the right 2 units. Thus $h(t)$ has a vertical asymptote at $t = -1 + 2 = 1$. So $h(t)$ has a horizontal asymptote at $y = -1$ and a vertical asymptote at $t = 1$.

(d)(7 pts) The limit of choice (ii) is $-c$. Consider the limit $\lim_{x\to 5^-} f(x) = c$, now if we let $x = -t$ then $t = -x$ and so if $x < 5$ this implies $t = -x > -5$ and $x \to 5^-$ implies $t = -x \to -5^+$ and now, since $f(x)$ is odd, we have

$$c = \lim_{x\to5^-} f(x) \underbrace{=}_{\text{let } x=-t} \lim_{t\to-5^+} f(-t) = \lim_{x\to-5^+} -f(t) \text{ and so } \lim_{x\to-5^+} f(t) = -c.$$

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2. Evaluate the following limits, (please do not use l'Hospital's Rule) remember to show all work.

(a)(7 pts) $\displaystyle\lim_{x\to\infty} \frac{x^{-1}+x^{-4}}{x^{-2}-x^{-3}}$ (b)(7 pts) $\displaystyle\lim_{\theta\to0} \frac{\cos(\theta)-1}{\sin(\theta)}$ (c)(7 pts) $\displaystyle\lim_{x\to2} \frac{x^2+x-6}{|x-2|}$ (d)(7 pts) $\displaystyle\lim_{x\to4} \frac{1}{-4-x}$

Solution: (a)(7 pts) Here we have

$$\lim_{x\to\infty} \frac{x^{-1}+x^{-4}}{x^{-2}-x^{-3}} \overset{\text{“}0/0\text{”}}{=} \lim_{x\to\infty} \frac{x^{-1}(1+x^{-3})}{x^{-2}(1-x^{-1})} = \lim_{x\to\infty} \frac{x(1+1/x^3)}{(1-1/x)} = \infty \cdot 1 = \infty$$

(b)(7 pts) Note here

$$\lim_{\theta\to0} \frac{\cos(\theta)-1}{\sin(\theta)} = \lim_{\theta\to0} \frac{\cos(\theta)-1}{\theta} \cdot \frac{\theta}{\sin\theta} = 0 \cdot \frac{1}{1} = 0.$$

Alternately, we could also proceed as follows

$$\lim_{\theta\to0} \frac{\cos(\theta)-1}{\sin(\theta)} \cdot \frac{\cos(\theta)+1}{\cos(\theta)+1} = \lim_{\theta\to0} \frac{\cos^2(\theta)-1}{\sin(\theta)(\cos(\theta)+1)} = \lim_{\theta\to0} \frac{-\sin^2(\theta)}{\sin(\theta)(\cos(\theta)+1)} = \lim_{\theta\to0} \frac{-\sin(\theta)}{\cos(\theta)+1} = 0.$$

(c)(7 pts) We check the one-sided limits, thus

$$\lim_{x\to2^-} \frac{x^2+x-6}{|x-2|} = \lim_{x\to2^-} \frac{\cancel{(x-2)}(x+3)}{-\cancel{(x-2)}} = -5 \text{ and } \lim_{x\to2^+} \frac{x^2+x-6}{|x-2|} = \lim_{x\to2^+} \frac{\cancel{(x-2)}(x+3)}{\cancel{(x-2)}} = 5$$

and thus $\displaystyle\lim_{x\to2} \frac{x^2+x-6}{|x-2|}$ does not exist.

(d)(7 pts) Note that $\displaystyle\lim_{x\to4} \frac{1}{-4-x} = -\frac{1}{8}$.

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3. The following problems are not related, remember justify your answers and cite any theorems you use.

(a)(8 pts) Let $q(t) = \begin{cases} kt^2+2, & \text{if } t \leq 3 \\ \dfrac{t^2-9}{t-3}, & \text{if } t > 3 \end{cases}$. Find the value of k that makes $q(t)$ *continuous* on $(-\infty,\infty)$. Justify.

(b)(7 pts) Does the equation $2\sin(x) = 3 - 2x$ have a solution? Why or why not? Justify your answer.

(c)(7 pts) Is the function $f(x) = \begin{cases} \sqrt{-x}\left[1+\cos^2(1/x)\right], & \text{if } x < 0 \\ 0, & \text{if } x = 0 \end{cases}$ *left continuous* at $x = 0$? Why or why not?

Solution: (a)(8 pts) We check the one-sided limits, thus

$$q(3) = \lim_{t\to3^-} q(t) = \lim_{t\to3^-} kt^2+2 = 9k+2 \text{ and } \lim_{t\to3^+} q(t) = \lim_{t\to3^+} \frac{t^2-9}{t-3} \overset{\text{“}0/0\text{”}}{=} \lim_{t\to3^+} \frac{\cancel{(t-3)}(t+3)}{\cancel{(t-3)}} = 6$$

and letting $9k+2 = 6$ we see that $k = 4/9$. Now note that for all $t \neq 3$ we know (by a theorem in the book) that polynomial are continuous and rational functions are continuous on their domains and so we see that $\boxed{q(t) \text{ will be continuous on for all values of } t \text{ if } k = 4/9.}$

(b)(7 pts) Let $f(x) = 2\sin(x) + 2x - 3$ and note that

$$f(0) = -3 < 0 \text{ and } f(\pi/2) = 2\sin(\pi/2) + 2(\pi/2) - 3 = 2 + \pi - 3 = \pi - 1 > 0$$

where the last equality follows since $\pi > 1$, thus we see that $f(0) < 0 < f(\pi/2)$. Finally, note that $f(x)$ is continuous since it is a sum and difference of continuous functions and so, by the Intermediate Value Theorem, there exists at least one number c in $(0,\pi/2)$ such that $f(c) = 0$, that is, the equation $2\sin(x) = 3 - 2x$ has at least one solution.

(c)(7 pts) We need to check that $\lim_{x\to0^-} f(x) = f(0)$. Now note that for all $x < 0$ we have

$$-1 \leq \cos(1/x) \leq 1 \Longrightarrow 0 \leq \cos^2(1/x) \leq 1 \Longrightarrow 1 \leq 1 + \cos^2(1/x) \leq 2 \Longrightarrow \sqrt{-x} \leq \sqrt{-x}\left[1+\cos^2(1/x)\right] \leq 2\sqrt{-x}$$

finally observe that $\lim_{x\to 0^-}\sqrt{-x} = \lim_{x\to 0^-} 2\sqrt{-x} = 0$ and so by Squeeze Theorem we have $\lim_{x\to 0^-}\sqrt{-x}\left[1+\cos^2(1/x)\right] = 0$ and so

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\sqrt{-x}\left[1+\cos^2(1/x)\right] = 0 = f(0)$$

and thus we see that $f(x)$ is left continuous at $x = 0$ since $\lim_{x\to 0^-} f(x) = f(0)$.

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4. The following problems are not related, remember to show all work and justify your answers.

(a)(8 pts) If $y = \sqrt{x+5}$, find dy/dx using the limit definition of the derivative. Simplify your answer.

(b)(7 pts) Suppose $f(x) = \begin{cases} x^2 + x, & \text{if } x \le 0 \\ \sin(x), & \text{if } x > 0 \end{cases}$, is $f(x)$ continuous for all x? Why or why not? Is $f(x)$ *differentiable* at the point $x = 0$? (Use the limit definition of the derivative for this problem). Justify your answer.

(c)(7 pts) Explorers on a small airless planet used a spring gun to launch a ball bearing vertically upward from the surface at a launch velocity of 15 m/sec. The acceleration of gravity at the planet's surface is assumed to be k m/sec^2 and the explorers expect the ball bearing to reach a height of $s(t) = 15t - (1/2)kt^2$ meters t seconds after the launch. The explorers determined that the ball bearing was at rest 20 seconds after being launched. What is the acceleration of gravity, k, at the planet surface?

Solution: (a)(8 pts) Using the limit definition with $f(x) = \sqrt{x+5}$, we have

$$\begin{aligned} f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{\sqrt{(x+h)+5}-\sqrt{x+5}}{h} \\ &\overset{\text{“}0/0\text{”}}{=} \lim_{h\to 0}\frac{\sqrt{(x+h)+5}-\sqrt{x+5}}{h}\cdot\frac{\sqrt{(x+h)+5}+\sqrt{x+5}}{\sqrt{(x+h)+5}+\sqrt{x+5}} \\ &= \lim_{h\to 0}\frac{[(x+h)+5]-(x+5)}{h(\sqrt{(x+h)+5}+\sqrt{x+5})} = \lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}(\sqrt{(x+h)+5}+\sqrt{x+5})} = \boxed{\frac{1}{2\sqrt{x+5}}} \end{aligned}$$

(b)(7 pts) Yes, $f(x)$ is continuous for all real x since for $x < 0$ we have $f(x) = \sin(x)$ which is a continuous function (by a theorem in the book) and for $x > 0$ we have $f(x) = x^2 + x$, and recall that polynomial are continuous. Now, for $x = 0$ we have to check that $\lim_{x\to 0} f(x) = f(0)$, note that

$$f(0) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2 + x = 0 \text{ and } \lim_{x\to 0^+} f(x) = \sin(0) = 0$$

and so $f(x)$ is continuous for all real x. Now we can show $f(x)$ is differentiable, note that

$$\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-}\frac{(h^2+h)-0}{h} = \lim_{h\to 0^-}\frac{\cancel{h}(1+h)}{\cancel{h}} = 1$$

and

$$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+}\frac{\sin(h)-0}{h} = \lim_{h\to 0^+}\frac{\sin(h)}{h} = 1$$

and so, by definition, we see that $f'(0) = 1$ (*i.e.* $f'(0)$ exists) and so $f(x)$ is differentiable at $x = 0$.

(c)(7 pts) We are given that the velocity is zero at $t = 20$. Note that

$$s(t) = 15t - (1/2)kt^2 \Longrightarrow v(t) = s'(t) = 15 - kt \text{ and } v(20) = 15 - k(20) = 0 \Longrightarrow k = 15/20 = 3/4$$

thus the acceleration due to gravity is $\boxed{k = 3/4 \text{ m/sec}^2.}$
