Math 1300-005 - Spring 2017

Indeterminate Forms and l'Hospital's Rule - 4/4/17

Guidelines: Please work in groups of two or three. This will not be handed in, but is a study resource for Midterm 3.

The purpose of this activity is to explore the various indeterminate form limits and how to use l'Hospital's Rule to solve them.

A. Recall ***l'Hospital's Rule***, which says that if

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

is *indeterminate* of form $0/0$ or ∞/∞, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right hand side exists. Note the limit on the right hand side has nothing to do with the quotient rule; we simply take the ratio of derivatives and then compute the limit. This only works for the $0/0$ or ∞/∞ indeterminate forms.

Let us practice using l'Hospital's rule by computing the following limits:

1. $\displaystyle\lim_{x\to\pi/2} \frac{\cos(x)}{1-\sin(x)}$

2. $\displaystyle\lim_{x\to\infty} \frac{\ln(x)}{x}$

Sometimes after applying l'Hospital's rule, the limit is still indeterminate of form $0/0$ or ∞/∞. In this case apply l'Hospital's rule again. This can be repeated indefinitely so long as the resulting limit is still indeterminate. Give this a try below:

3. $\displaystyle\lim_{t\to 0} \frac{e^t-1-t}{t^2}$

There are several more indeterminate forms aside from $0/0$ and ∞/∞. l'Hospital's rule is not directly applicable to these other indeterminate forms, but they can be manipulated using various tricks so that a $0/0$ or ∞/∞ type limit appears.

B. **The Indeterminate Form** $0 \cdot \infty$

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$, then it is not clear what happens with

$$\lim_{x\to a} f(x)g(x).$$

There is a struggle as the function f wants trys to pull the limit to 0 whereas the function g trys to pull the limit to infinity.

l'Hospital's Rule does not apply to this indeterminate form directly, but if we rewrite the product $f(x)g(x)$ as either

$$fg = \frac{f}{1/g} \quad \text{or} \quad fg = \frac{g}{1/f},$$

then we notice that $\lim_{x\to a} f(x)g(x)$ has been converted into a limit of the form $0/0$ or ∞/∞, respectively.

Let's see how this works with an example. Consider

$$\lim_{x\to\infty} e^{-x}\ln(x),$$

which is indeterminate of form $0 \cdot \infty$. Then

$$\lim_{x\to\infty} e^{-x}\ln(x) = \lim_{x\to\infty} \frac{\ln(x)}{1/e^{-x}} = \lim_{x\to\infty} \frac{\ln(x)}{e^{x}} \overset{\textbf{l'H}}{=} \lim_{x\to\infty} \frac{1/x}{e^{x}} = \lim_{x\to\infty} \frac{1}{xe^{x}} = 0,$$

where l'Hospital's rule was performed at the step with the "**l'H**" over the equal sign. Give this a try below:

1. $\lim_{x\to 0^+} x^3 \ln(x)$

2. $\lim_{x\to -\infty} x^2 e^x$

C. **Indeterminate Powers**

If we consider

$$\lim_{x\to a}[f(x)]^{g(x)}$$

then several indeterminate forms can arise:

- If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ form 0^0.
  Here f is trying to make the limit 0, whereas g is trying to make the limit 1, since any nonzero number raised to the power 0 is equal to 1.

- If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$ form ∞^0.
  Here f is trying to make the limit infinite, whereas g is trying to make the limit 1.

- If $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \infty$ form 1^∞.
  Here f is trying to make the limit equal to 1, whereas g is trying to make the limit infinite.

In each case, we solve the limit using a method quite similar to what we used in taking the derivative of functions of the form $[f(x)]^{g(x)}$, and that is to use the inverse relationship of e^x and $\ln(x)$ as well as properties of logarithms to write

$$[f(x)]^{g(x)} = e^{g(x)\ln(f(x))}.$$

Thus we are led to the product

$$\lim_{x\to a} g(x)\ln(f(x)),$$

which is indeterminate of form $0\cdot\infty$. Briefly think about and discuss why each of the three cases above leads to this limit being indeterminate of form $0\cdot\infty$.

To see how this works, consider

$$\lim_{x\to 0^+} x^{(x^2)}$$

which is indeterminate of form 0^0. Applying the trick discussed above,

$$x^{(x^2)} = e^{x^2\ln(x)}.$$

Since $\lim_{x\to 0^+} x^2\ln(x)$ is of form $0\cdot\infty$, we have

$$\lim_{x\to 0^+} x^2\ln(x) = \lim_{x\to 0^+}\frac{\ln(x)}{1/x^2} = \lim_{x\to 0^+}\frac{\ln(x)}{x^{-2}} \overset{\text{l'H}}{=} \lim_{x\to 0^+}\frac{1/x}{-2/x^3} = \lim_{x\to 0^+}\frac{x^2}{-2} = 0.$$

Therefore,

$$\lim_{x\to 0^+} x^{(x^2)} = \lim_{x\to 0^+} e^{x^2\ln(x)} = e^0 = 1.$$

Try this for yourself on this next page.

1. $\lim_{x \to 0^+} (2x)^{(x^3)}$

2. $\lim_{x \to \infty} x^{(1/x)}$

3. $\lim_{x \to 0^+} (4x + 1)^{(\cot(x))}$