INSTRUCTIONS: Books, notes, and electronic devices are **not** permitted. Write (1) **your name**, (2) **1350/Test 2**, (3) **lecture number/instructor name** and (4) **SUMMER 2015** on the front of your bluebook. Also make a **grading table** with room for 5 problems and a total score. **Start each problem on a new page.** $\boxed{\textbf{Box}}$ **your answers.** A correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit. **SHOW ALL WORK! JUSTIFY ALL YOUR ANSWERS!**

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1. Consider the equation $f(x) = \sqrt{x}$

   (a) (6 pts) Find dy/dx and d^2y/dx^2 at the point $(4, 2)$

   (b) (6 pts) Find the linearization of the equation at the point $(4, 2)$. Use the linearization to estimate the y-value when $x = 4.1$

   (c) (4 pts) Does the approximation in part (b) overestimate or underestimate the actual value of y at $x = 4.1$? Explain.

   (d) (4 pts) What is dy/dx at $x = 0$?

Solution:
(a)

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} \implies \frac{dy}{dx}|_{(4,2)} = \frac{1}{4}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{4}x^{-3/2} \implies \frac{d^2y}{dx^2}|_{(4,2)} = -\frac{1}{32}$$

(b)

$$L(x) = f(4) + f'(4)(x-4) = 2 + \frac{1}{4}(x-4) = 2 + \frac{1}{4}x - 1 = \frac{1}{4}x + 1$$

So then $\sqrt{4.1} \approx L(4.1) = \frac{1}{4} \cdot 4.1 + 1 = 2.025$

(c) Since the function is concave down at $(4, 2)$, the tangent line lies above the graph of $f(x) = \sqrt{x}$. Therefore this linear approximation must be an overestimate.

(d) $\frac{dy}{dx}|_{(0,0)}$ is undefined. The slope has "infinite steepness" here and so is undefined.

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2. (25 pts) A rocket is launched vertically and is tracked by a radar station located on the ground 4 miles from the launch site. What is the vertical speed of the rocket at the instant when its distance from the radar station is 5 miles and this distance is increasing at the rate of 3600 mi/hr ?

Solution:
We may relate the distance between the radar station and the rocket launcher, the rocket launcher position and the vertically moving rocket, and the distance between the rocket and the radar station with a right triangle. Let $x = 4$ miles be the leg of the right triangle between the radar station and the rocket. Let y denote the vertical height of the rocket (also the other leg of the right triangle) and let z denote the distance between the rocket and the radar station (the hypotenuse of our right triangle). Then we can set up the following relationship based on the pythagorean theorem:

$$4^2 + y^2 = z^2$$

We are given that when $z = 5$, $\frac{dz}{dt} = 3600$ miles/hr. Next, taking the necessary derivatives we arrive at:

$$2y\frac{dy}{dt} = 2z\frac{dz}{dt} \implies y\frac{dy}{dt} = z\frac{dz}{dt}$$

Note that $\frac{dy}{dt}$ is the quantity we are trying to solve for. When $z = 5, 25 = y^2 + 16 \implies y = 3$. Then we have:

$$3 \cdot \frac{dy}{dt} = 5 \cdot 3600$$

Therefore

$$\frac{dy}{dt} = \frac{5}{3} \cdot 3600 = 6000$$

So the vertical speed of the rocket is 6000 miles/hour.

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3. Calculate dy/dx for each of the following: (After finding dy/dx, do not simplify. In the case of implicit differentiation, solve for dy/dx but you do not need to simplify further.)

(a) (5 pts) $y = \frac{\tan x}{1+\cos x}$

(b) (5 pts) $y = (x + \frac{1}{x^2})^{\sqrt{7}}$

(c) (5 pts) $\sin(xy) = x^2 - y$

(d) (5 pts) $y = \tan^2(\sin\theta)$

Solution:

(a)

$$\frac{dy}{dx} = \frac{(1+\cos x)(\sec^2 x) - \tan x(-\sin x)}{(1+\cos x)^2} = \frac{(1+\cos x)(\sec^2 x) + \tan x(\sin x)}{(1+\cos x)^2}$$

(b)

$$\frac{dy}{dx} = \sqrt{7}\left(x + \frac{1}{x^2}\right)^{\sqrt{7}-1}\left(1 - \frac{2}{x^3}\right)$$

(c)

$$\cos(xy)\left(x\frac{dy}{dx} + y\right) = 2x - \frac{dy}{dx}$$

$$x\cos(xy)\frac{dy}{dx} + y\cos(xy) = 2x - \frac{dy}{dx}$$

$$x\cos(xy)\frac{dy}{dx} + \frac{dy}{dx} = 2x - y\cos(xy)$$

$$\implies \frac{dy}{dx} = \frac{2x - y\cos(xy)}{x\cos(xy) + 1}$$

(d)

$$\frac{dy}{dx} = 2\tan(\sin\theta) \cdot \sec^2(\sin\theta) \cdot \cos\theta$$

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4. Consider the function

$$f(x) = \frac{2 + x - x^2}{(x-1)^2}, \quad f'(x) = \frac{x-5}{(x-1)^3}, \quad f''(x) = \frac{2(7-x)}{(x-1)^4}$$

(a) (3 pts) Find any vertical, horizontal, or slant asymptotes of f. Use appropriate limits to justify your answer.

(b) (3 pts) On what intervals is f increasing? decreasing?

(c) (4 pts) Find all local maximum and minimum values of f.

(d) (4 pts) On what intervals is f concave up? concave down?

(e) (3 pts) Find all inflection points of f.

(f) (3 pts) Using the information from (a)-(e), sketch a graph of f. Clearly label any points of interest, including any asymptotes, local extrema, and inflection points.

Solution:

(a) There is a vertical asymptote at $x = 1$. To prove this:

$$\lim_{x\to 1^+} \frac{2+x-x^2}{(x-1)^2} = \lim_{x\to 1^-} \frac{2+x-x^2}{(x-1)^2} = \lim_{x\to 1^\pm} \frac{1}{(x-1)^2} = +\infty$$

There is a horizontal asymptote at $y = -1$ since $\displaystyle\lim_{x\to\pm\infty} \frac{2+x-x^2}{(x-1)^2} = \lim_{x\to\pm\infty} \frac{2/x^2 + x/x^2 - x^2/x^2}{x^2/x^2 - 2x/x^2 + 1/x^2} = -1$

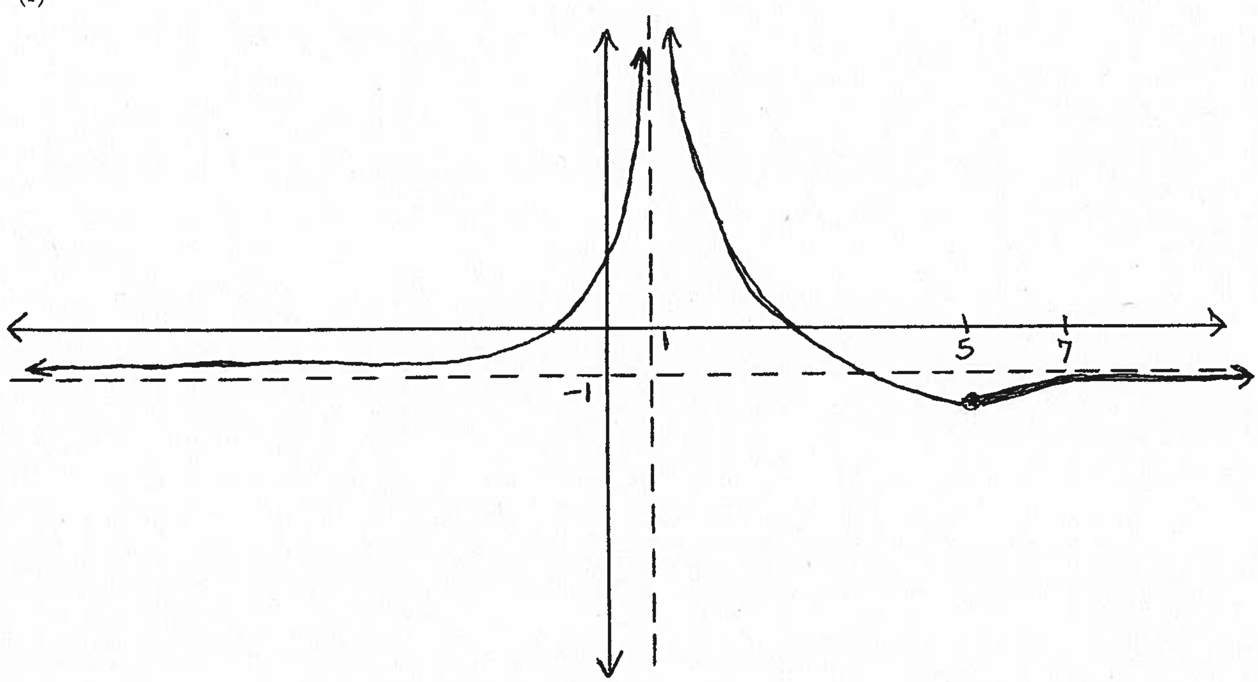
(b) The critical point for $f'(x)$ is at $x = 5$. Note that $f'(x)$ is undefined at $x = 1$. So we must check the intervals $(-\infty, 1), (1, 5), (5, \infty)$. By choosing test values in each interval, one may see that $f'(x)$ is increasing on $(-\infty, 1) \cup (5, \infty)$ and decreasing on $(1, 5)$.

(c) Using the first derivative test and the results from part (b), there must be a local minimum at $x = 5$. $f(5) = \frac{2+5-25}{4^2} = -\frac{9}{8}$. Therefore there is a local minimum at $(5, -\frac{9}{8})$. There are no local maximums.

(d) The only place that an inflection point may occur is at $x = 7$ since this makes the given second derivative 0. We check the sign of $f''(x)$ on the intervals $(-\infty, 1), (1, 7), (7, \infty)$ since the second derivative is also undefined at $x = 1$. By choosing test values in each interval, we find that $f(x)$ is concave up on $(-\infty, 1) \cup (1, 7)$ and concave down on $(7, \infty)$.

(e) Using the result from part (d), the only inflection point is at $x = 7$.

(f)



5. (a) (7 pts) State the Mean Value Theorem

(b) (8 pts) Suppose that f and g are continuous on $[a, b]$ and differentiable on (a, b). Suppose also that $f(a) = g(a)$ and $f'(x) < g'(x)$ for $a < x < b$. Prove that $f(b) < g(b)$. [Hint: Apply the Mean Value Theorem to the function $h = f - g$.]

Solution:

(a) The Mean Value Theorem states: Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there is a number c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

(b) Let $h = f - g$. Then since f and g are continuous on $[a, b]$ and differentiable on (a, b), so is h. Thus h satisfies the assumptions of the Mean Value Theorem. Therefore, there is a number c with $a < c < b$ such that

$$h(b) - h(a) = h'(c)(b - a)$$

Now since $(b - a) > 0$ and $h'(c) < 0$ (since $c \in (a, b), f'(c) < g'(c)$ by our given), then we can say that

$$h'(c)(b - a) < 0$$

This implies that $h(b) - h(a) = h(b) < 0$ where we get the first equality because $h(a) = f(a) - g(a) = 0 \implies h(b) - h(a) = h(b)$. Therefore,

$$h(b) < 0$$

$$\implies f(b) - g(b) < 0 \implies f(b) < g(b)$$

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