1. (15 points) Evaluate the following:

(a) $\displaystyle\int \frac{a+bx^2}{\sqrt{3ax+bx^3}}\,dx$ (b) $\displaystyle\int \frac{\cos\left(\frac{\pi}{x}\right)}{x^2}\,dx$ (c) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin(x)}{\cos^3(x)}dx$

Solution:

(a) $u = 3ax + bx^3$
$du = (3a + 3bx^2)dx$
$\frac{1}{3}du = (a + bx^2)dx$
$$\int \frac{a+bx^2}{\sqrt{3ax+bx^3}}dx = \frac{1}{3}\int u^{-\frac{1}{2}}du = \frac{1}{3}[2u^{\frac{1}{2}} + C_1] = \boxed{\tfrac{2}{3}\sqrt{3ax+bx^3} + C}$$

(b) $u = \frac{\pi}{x}$
$du = -\frac{\pi}{x^2}dx$
$-\frac{1}{\pi}du = \frac{1}{x^2}dx$
$$\int \frac{\cos\left(\frac{\pi}{x}\right)}{x^2}dx = -\frac{1}{\pi}\int \cos u\,du = -\frac{1}{\pi}\sin u + C = \boxed{-\tfrac{1}{\pi}\sin\left(\tfrac{\pi}{x}\right) + C}$$

(c) $u = \cos x$
$du = -\sin x dx$
$x = 0 \Rightarrow u = 1$
$x = \frac{\pi}{4} \Rightarrow u = \frac{\sqrt{2}}{2}$
$$\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos^3 x}dx = -\int_1^{\frac{\sqrt{2}}{2}} u^{-3}du = \frac{1}{2}u^{-2}\Big|_1^{\frac{\sqrt{2}}{2}} = \frac{1}{2}\left(\left(\frac{2}{\sqrt{2}}\right)^2 - \frac{1}{1}\right) = \frac{1}{2}\left(\frac{4}{2} - 1\right) = \boxed{\tfrac{1}{2}}$$

OR

$$\int_0^{\frac{\sin x}{\cos^3 x}} dx = \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x}\frac{1}{\cos^2 x}dx = \int_0^{\frac{\pi}{4}} \tan x \sec^2 x dx$$
$u = \tan x$
$du = \sec^2 x dx$
$x = 0 \Rightarrow u = 0$
$x = \frac{\pi}{4} \Rightarrow u = 1$
$$\int_0^1 u du = \frac{1}{2}u^2\Big|_0^1 = \boxed{\tfrac{1}{2}}$$

2. (15 points) The expression $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{4}{n}\sqrt{\frac{4}{n}i}$ describes the area of a region bounded by some function $f(x)$ on $1 \le x \le 5$ using subintervals of equal width and right endpoints.

(a) What is the function $f(x)$?

(b) Set up a definite integral to compute the area of the region.

(c) Find the area of the region.

Solution:

(a) on $[1,5]$, $\Delta x = \frac{5-1}{n} = \frac{4}{n}$ and right end-points $x_i = 1 + i\Delta x = 1 + \frac{4}{n}i$.
$\frac{4}{n}\sqrt{\frac{4}{n}i} = \frac{4}{n}\sqrt{\frac{4}{n}i + 1 - 1} = \Delta x\sqrt{x_i - 1}$
Therefore $\boxed{f(x) = \sqrt{x-1}}$

(b) Area of region $= \displaystyle\int_1^5 \sqrt{x-1}dx$

(c) $\int_1^5 \sqrt{x-1}dx = \frac{2}{3}\,(x-1)^{\frac{3}{2}}\Big|_1^5 = \frac{2}{3}[8-0] = \boxed{\frac{16}{3}}$

3. (12 points) Suppose that at any time t (seconds) the current i (amp) in an alternating current circuit is $i = 2\cos t + 2\sin t$. What is the peak (largest positive magnitude) current for this circuit?

Solution:

(a) $\dfrac{di}{dt} = -2\sin t + 2\cos t$

$$\frac{di}{dt} = 0 \quad \Rightarrow \quad \sin t = \cos t \quad \Rightarrow \quad \tan t = 1 \quad \Rightarrow \quad t = \frac{(4n-3)}{\pi} = \left\{\frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \dots\right\}$$

Determine if these optimums are minimums or maximums.

$\frac{d^2 i}{dt^2} = -2\cos t - 2\sin t \quad$ or $\quad i''(t) = -2(\cos t + \sin t)$

$\Rightarrow \quad i''(\frac{\pi}{4}) < 0 \quad \Rightarrow \quad i$ is CCD at $t = \frac{\pi}{4}$.

$\Rightarrow \quad i''(\frac{5\pi}{4}) > 0 \quad \Rightarrow \quad i$ is CCU at $t = \frac{5\pi}{4}$.

$\Rightarrow \quad i''(\frac{9\pi}{4}) < 0 \quad \Rightarrow \quad i$ is CCD at $t = \frac{9\pi}{4}$.

$\Rightarrow \quad i''(\frac{13\pi}{4}) > 0 \quad \Rightarrow \quad i$ is CCU at $t = \frac{13\pi}{4}$. etc.

The periodicity of $\tan(x)$ implies that $i(t)$ obtains a maximum at $t = \frac{(8n-7)\pi}{4}$.

$i\left(\frac{\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}}\right) + 2\left(\frac{1}{\sqrt{2}}\right) = \boxed{2\sqrt{2}}$.

4. The following questions are not related:

(a) (12 points) The temperature T (degrees) inside a furnace is described by the function $T(t) = 1000 + 100\sin(\frac{\pi}{12}t + \frac{\pi}{6})$ where t is the time in hours, $t = 0$ corresponding to when the furnace is first fired up. Find the average temperature in the furnace during its first two hours of operation.

(b) (12 points) Recalling that a function is constant on an interval if and only if its derivative is zero on that interval, show that the following function is constant on $(0, \infty)$.

$$f(x) = \int_0^{\frac{2}{x}} \frac{1}{t^2+1}dt + \int_0^x \frac{2}{t^2+4}dt.$$

Solution:

(a) $T_{ave} = \dfrac{1}{2-0}\displaystyle\int_0^2 \left(1000 + 100\sin\left(\frac{\pi}{12}t + \frac{\pi}{6}\right)\right) dt$

$= \dfrac{1}{2}\left[1000\displaystyle\int_0^2 dt + 100\int_0^2 \sin\left(\frac{\pi}{12}t + \frac{\pi}{6}\right)dt\right]$

$u = \dfrac{\pi}{12}t + 6$

$du = \dfrac{\pi}{12}dt$

$t = 0 \Rightarrow u = \dfrac{\pi}{6}$

$t = 2 \Rightarrow u = \dfrac{\pi}{3}$

$T_{ave} = 500\ t|_0^2 + 50\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\pi}{12}\sin u du$

$1000 + \dfrac{600}{\pi}\cos u\Big|_{\frac{\pi}{3}}^{\frac{\pi}{6}}$

$1000 + \dfrac{600}{\pi}\left(\cos\left(\dfrac{\pi}{6}\right) - \cos\left(\dfrac{\pi}{3}\right)\right) = 1000 + \dfrac{600}{\pi}\left(\dfrac{\sqrt{3}}{2} - \dfrac{1}{2}\right)$

$= \boxed{1000 + \frac{300}{\pi}(\sqrt{3}-1)}$ degrees.

(b) $f(x) = \displaystyle\int_0^{\frac{2}{x}} \frac{1}{t^2+1}dt + \int_0^x \frac{2}{t^2+4}dt$

$$f'(x) = \frac{1}{\left(\frac{2}{x}\right)^2+1}\frac{d}{dx}\left(\frac{2}{x}\right) + \frac{2}{x^2+4} = \frac{1}{\frac{4}{x^2}+1}\left(\frac{-2}{x^2}\right) + \frac{2}{x^2+4} = \frac{-2}{4+x^2} + \frac{2}{x^2+4} = 0$$

Since $f'(x) = 0$ for $x > 0$, $f(x)$ is constant on $(0, \infty)$.

5. (14 points) A cyclist pedals along a straight road with velocity $v(t) = 2t^2 - 8t + 6$ miles per hour for three hours.

(a) Find the displacement of the cyclist (in miles) on the time interval $[0, 3]$.

(b) Find the distance traveled over the interval $[0, 3]$.

Solution:

(a) Displacement $= \int_0^3 (2t^2 - 8t + 6)dt$

$= \frac{2}{3}t^3 - 4t^2 + 6t\Big|_0^3$

$= 18 - 36 + 18 = \boxed{0}$

(b) $v(t) > 0$ on $0 < t < 1$ and $v(t) < 0$ on $1 < t < 3$.

$$\text{Total distance} = \left|\int_0^1 v(t)dt\right| + \left|\int_1^3 v(t)dt\right|$$

$$\left|\int_0^1 v(t)dt\right| = \left|\frac{2}{3}t^3 - 4t^2 + 6t\Big]_0^1\right| = \left|\frac{8}{3} - 0\right| = \frac{8}{3}.$$

$$\int_1^3 v(t)dt = \left|\frac{2}{3}t^3 - 4t^2 + 6t\Big]_1^3\right| = \left|0 - \frac{8}{3}\right| = \frac{8}{3}$$

Therefore, the total distance is $\frac{8}{3} + \frac{8}{3} = \boxed{\frac{16}{3}}$ miles.

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6. (20 points) Produce an answer with a short, succinct explanation. Box only your answer (not the explanation).

   (a) Consider using Newton's method to find the root of a function, $f(x)$. Suppose that for your initial guess, x_1, you discover that $f(x_1) = 0$. Assuming that $f'(x_1) \neq 0$ (and $f'(x_1)$ is defined), what is $f(x_3)$?

   (b) Which of the following statements (i, ii, iii, or iv) is NOT asking for the same information?

      i. Find the x-coordinates of the points where the curve $y = x^3 - 3x$ crosses the horizontal line $y = -1$.

      ii. Find the roots of $f(x) = x^3 - 3x - 1$.

      iii. Find the x-coordinates of the intersections of the curve $y = x^3$ with the line $y = 3x + 1$.

      iv. Find the values of x where the derivative of $g(x) = (\frac{1}{4})x^4 - (\frac{3}{2})x^2 - x + 5$ equals zero.

   (c) If $\displaystyle\int_0^{\pi} \cos(\sin x)dx = 2.4$, then $\displaystyle\int_{-\pi}^{\pi} \cos(\sin x)dx =?$

   (d) For some function $h(x)$, it is known that $h'(x) = 2$ for all x in the interval $[0, 6]$ and $h(0) = -4$. Find $\displaystyle\int_0^6 h(x)dx$.

   (e) Is it true or false that there exists a c in $[1, 4]$ such that the rectangle with length 3 and height $\frac{c}{\sqrt{1+2c}}$ has an area of $\displaystyle\int_1^4 \frac{x}{\sqrt{1+2x}}dx$.

Solution:

(a) $x_1 = x_3 \quad \Rightarrow \quad f(x_1) = f(x_3) = \boxed{0}$

(b) $\boxed{\text{i}}$

(c) $\cos(\sin(x))$ is an even function $\quad \Rightarrow \quad \displaystyle\int_{-\pi}^{\pi} \cos(\sin x)dx = \boxed{4.8}$

(d) Geometrically this is the area of two triangles whose sum is $\boxed{12}$

(e) True

$\boxed{\text{END of Exam}}$