1. The following parts are not related:

(a)(10 pts) Approximate the area of the region bounded by the curve $y = x^2 + 4$ from $x = -4$ to $x = 4$ and the x-axis using a Riemann Sum with 4 subintervals of equal length and taking the sample points to be midpoints.

(b) The following limit of Riemann Sums, $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{16i}{n^2}\sqrt{16 - \frac{16i^2}{n^2}}$, describes the area of the region bounded by some function $f(x)$ for $0 \le x \le 4$ and the x-axis using subintervals of equal length and $x_i^* = x_i$.
(i)(6 pts) What is the function $f(x)$?
(ii)(9 pts) What is the area of the region described by the limit? (Hint: Interpret the limit as a definite integral.)

Solution:

(a)(10 pts) Note that here $\Delta x = \frac{4-(-4)}{4} = 8/4 = 2$ and so the subintervals are $[-4,-2]$, $[-2,0]$, $[0,2]$, and $[2,4]$, so clearly the midpoints are $x_1^* = -3$, $x_2^* = -1$, $x_3^* = 1$, $x_4^* = 3$, thus we have

$$\int_{-4}^{4} (x^2+4)\,dx \approx 2\left[f(-3)+f(-1)+f(1)+f(3)\right] = 2[13+5+5+13] = 2(36) = 72$$

(b)(i)(6 pts) Here we have $\Delta x = \frac{4-0}{n} = \frac{4}{n}$ and $x_i^* = x_i = 0 + i\Delta x = \frac{4i}{n}$, so

$$\begin{aligned}\lim_{n\to\infty} \sum_{i=0}^{n} \frac{16i}{n^2}\sqrt{16-\frac{16i^2}{n^2}} &= \lim_{n\to\infty} \sum_{i=0}^{n} \frac{4i}{n}\sqrt{16-\left(\frac{4i}{n}\right)^2}\cdot\frac{4}{n} \\ &= \lim_{n\to\infty} \sum_{i=0}^{n} x_i^*\sqrt{16-(x_i^*)^2}\Delta x = \lim_{n\to\infty} \sum_{i=0}^{n} f(x_i^*)\Delta x \text{ so, } \boxed{f(x) = x\sqrt{16-x^2}.}\end{aligned}$$

(b)(ii)(9 pts) We determine the limit by evaluating the definite integral using the substitution $u = 16 - x^2$, then $du = -2xdx$ and,

$$\int_0^4 x\sqrt{16-x^2}\,dx = -\frac{1}{2}\int_{16}^{0} u^{1/2}\,du = \frac{1}{2}\int_0^{16} u^{1/2}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_0^{16} = \frac{1}{3}(16)^{3/2} = \frac{64}{3}$$

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2. The following problems are not related.

(a)(10 pts) Use Newton's method to find x_2, the second approximation of the intersection point of the functions $y = \sin(x)$ and $y = \cos(x)$, if the initial approximation is $x_1 = \frac{\pi}{2}$.

(b) A square swimming pool with base width x meters and fixed depth of y meters is being constructed. The inside walls and floor of the pool are to be painted with a special water-proof paint. There is enough paint to cover exactly 300 m^2 of surface and the builder plans to use it all up for the painting of this pool:
(i)(12 pts) What is the largest possible volume of such a pool?
(ii)(3 pts) How do you know your answer is a maximum? (Justify your answer based on the theories of this class.)

Solution:

(a)(10 pts) We wish to approximate a solution to $\sin(x) = \cos(x)$ which implies $\sin(x) - \cos(x) = 0$, so, in Newton's Method, if we let $f(x) = \sin(x) - \cos(x)$ then

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{\pi}{2} - \frac{\sin(\pi/2)-\cos(\pi/2)}{\cos(\pi/2)+\sin(\pi/2)} = \frac{\pi}{2} - \frac{1-0}{1+0} = \frac{\pi}{2} - 1.$$

(b)(i)(12 pts) We wish to maximize the volume $V = x^2y$ subject to the constraint on the surface area $x^2+4xy = 300$ where $x > 0$ and $y > 0$ and so

$$x^2 + 4xy = 300 \Longrightarrow y = \frac{300 - x^2}{4x}, \text{ thus, } V = x^2y = x^2\left(\frac{300 - x^2}{4x}\right) = \frac{300x - x^3}{4} = 75x - \frac{x^3}{4}$$

so we wish to maximize $V(x) = 75x - \dfrac{x^3}{4}$. We now find the critical points, differentiating both sides yields,

$$V'(x) = 75 - \frac{3x^2}{4} \text{ and } V'(x) = 0 \text{ implies } 75 - \frac{3x^2}{4} = 0 \Rightarrow x^2 = \frac{300}{3} = 100 \Rightarrow x = \pm 10$$

and $x > 0$ implies $x = 10$m (we will verify that this does, indeed, yield a maximum in part (ii) below) and so

$$x = 10 \Rightarrow y = \frac{300 - 100}{40} = \frac{200}{40} = 5, \text{ thus largest possible volume is } V = (10)^2 5 = 500\text{m}^3.$$

(b)(ii)(3 pts) Using the Second Derivative Test we note that

$$V''(x) = -\frac{6x}{4} \text{ and } V''(10) = -\frac{6 \cdot 10}{4} < 0 \text{ and so we have a (local) maximum at } x = 10.$$

Note that we could have also use the First Derivative Test to classify the critical point $x = 10$ as a local max.

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3. The following problems are not related:

(a)(10 pts) Given that $g(x)$ is an odd function, $\displaystyle\int_2^7 g(x)\,dx = 13$ and $\displaystyle\int_5^7 g(x)\,dx = 4$, find $\displaystyle\int_{-2}^5 3g(x)\,dx$.

(b) Given that $F(x) = \displaystyle\int_{-2}^{2x} \sqrt{5 + t^2}\,dt$, answer the following questions *without attempting to evaluate any integrals:*
(i)(3 pts) Is $F(-2)$ positive, negative or neither?
(ii)(6 pts) On what interval(s) is the function $F(x)$ increasing? decreasing?
(iii)(6 pts) Find the linearization of $F(x)$ at $x = -1$.

Solution:

(a)(10 pts) Note that since g is odd, $\displaystyle\int_{-2}^2 g(x)\,dx = 0$ and so

$$\int_{-2}^5 3g(x)dx = 3\left[\int_{-2}^2 g(x)dx + \int_2^5 g(x)dx\right] = 3\int_2^5 g(x)dx$$

and now note $\displaystyle\int_2^7 g(x)dx = \int_2^5 g(x)dx + \int_5^7 g(x)dx$, thus

$$\int_2^5 g(x)dx = \int_2^7 g(x)dx - \int_5^7 g(x)dx = 13 - 4 = 9$$

and so, finally, $\displaystyle\int_{-2}^5 3g(x)dx = 3\int_2^5 g(x)dx = 3 \cdot 9 = 27$.

(b)(i)(3 pts) Note that

$$F(-2) = \int_{-2}^{-4} \sqrt{5 + t^2}\,dt = -\int_{-4}^{-2} \sqrt{5 + t^2}\,dt$$

and, since $\sqrt{5 + t^2} > 0$ for $-4 < t < -2$, we have, $\displaystyle\int_{-4}^{-2} \sqrt{5 + t^2}\,dt > 0$ and thus $F(-2) = -\displaystyle\int_{-4}^{-2} \sqrt{5 + t^2}\,dt < 0$ so $F(-2)$ is negative.

(b)(ii)(6 pts) We first find $F'(x)$ by applying the Fundamental Theorem of Calculus,

$$F'(x) = \frac{d}{dx}\left[\int_{-2}^{2x} \sqrt{5 + t^2}\,dt\right] = \sqrt{5 + (2x)^2} \cdot 2 = 2\sqrt{5 + 4x^2}$$

and so $F'(x) > 0$ for all x. So, $F(x)$ is increasing on the interval $(-\infty, \infty)$ and is decreasing for no values of x.

(b)(iii)(6 pts) Note that $F(-1) = \displaystyle\int_{-2}^{-2} \sqrt{5+t^2}\, dt = 0$ and $F'(-1) = 2\sqrt{5+4} = 6$ (from part (ii)) and so the linearization of $F(x)$ is

$$L(x) = L(x) = F(-1) + F'(-1)(x+1) = 6(x+1) = 6x+6.$$

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4. The following problems are not related.

(a)(15 pts) Evaluate these integrals: (i) $\displaystyle\int \sin(x)\cot(x)\, dx$ (ii) $\displaystyle\int_1^{\sqrt{2}} 2x^3\sqrt{x^2-1}\, dx$ (iii) $\displaystyle\int_{-2}^{2} \sqrt{16-4x^2}\, dx$

(b)(10 pts) Show that $\displaystyle\int_0^1 x^{10}(1-x)^6\, dx = \int_0^1 x^6(1-x)^{10}\, dx$. Justify your answer.

Solution:

(a)(i)(5 pts) Here we have,

$$\int \sin(x)\cot(x)\, dx = \int \sin(x)\cdot\frac{\cos(x)}{\sin(x)}\, dx = \int \cos(x)\, dx = \sin(x) + C$$

(a)(ii)(5 pts) Using the substitution $u = x^2 - 1$ we have $du = 2xdx$ and $x^2 = u+1$, thus

$$\int_1^{\sqrt{2}} 2x^3\sqrt{x^2-1}\, dx = \int_1^{\sqrt{2}} x^2\sqrt{x^2-1}\; 2xdx = \int_0^1 (u+1)\sqrt{u}\, du = \int_0^1 (u^{3/2} + u^{1/2})\, du$$

and so

$$\int_0^1 (u^{3/2} + u^{1/2})\, du = \left(\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right)\Bigg|_0^1 = \frac{2}{5} + \frac{2}{3} = \frac{6}{15} + \frac{10}{15} = \frac{16}{15}$$

(a)(iii)(5 pts) First note that

$$\int_{-2}^{2} \sqrt{16-4x^2}\, dx = \int_{-2}^{2} \sqrt{4(4-x^2)}\, dx = 2\int_{-2}^{2} \sqrt{4-x^2}\, dx$$

and the graph of $y = \sqrt{4-x^2}$ is the top half of the circle centered at the origin with radius $r = 2$, so

$$2\int_{-2}^{2} \sqrt{4-x^2}\, dx = 2\cdot\frac{\pi r^2}{2}\Bigg|_{r=2} = 4\pi$$

(b)(10 pts) If we use the substitution $u = 1-x$, then $du = -dx$ and $x = 1-u$, and so

$$\int_0^1 x^{10}(1-x)^6\, dx = -\int_1^0 (1-u)^{10}u^6\, du = \int_0^1 (1-u)^{10}u^6\, du$$

and since the variable "u" is a "dummy variable" in the definite integral we can replace it with "x" and so we have

$$\int_0^1 x^{10}(1-x)^6\, dx = \int_0^1 (1-u)^{10}u^6\, du = \int_0^1 u^6(1-u)^{10}\, du = \int_0^1 x^6(1-x)^{10}\, dx$$
