1. (16 points) Let $f(x) = \dfrac{2+e^x}{3-e^x}$.

   (a) Does f have any horizontal or vertical asymptotes? Justify your answer using appropriate limits.

   (b) Find the instantaneous rate of change of f with respect to x. Simplify your answer.

   (c) Find the linearization of f centered at $x = 0$.

   (d) Find the inverse function $f^{-1}(x)$. You may assume that f is one-to-one.

Solution:

(a) Horizontal asymptotes:

$$\lim_{x\to\infty} \frac{2+e^x}{3-e^x} = \lim_{x\to\infty} \frac{2e^{-x}+1}{3e^{-x}-1} = \frac{0+1}{0-1} = -1$$
$$\lim_{x\to-\infty} \frac{2+e^x}{3-e^x} = \frac{2+0}{3-0} = \frac{2}{3}$$

There are horizontal asymptotes at $\boxed{y=-1}$ and $\boxed{y=2/3}$.
Vertical asymptotes: Check $x = \ln 3$ where the denominator approaches 0.

$$\lim_{x\to(\ln 3)^+} \frac{2+e^x}{3-e^x} = \frac{2+e^{\ln 3}}{0^-} = -\infty$$

There is a vertical asymptote at $\boxed{x = \ln 3}$.

(b) $f'(x) = \dfrac{(3-e^x)e^x - (2+e^x)(-e^x)}{(3-e^x)^2} = \dfrac{3e^x - e^{2x} + 2e^x + e^{2x}}{(3-e^x)^2} = \boxed{\dfrac{5e^x}{(3-e^x)^2}}$

(c) $f(0) = \dfrac{2+1}{3-1} = \dfrac{3}{2}$ and $f'(0) = \dfrac{5}{4}$. The linearization of f is

$$L(x) = f(0) + f'(0)(x-0) = \boxed{\frac{3}{2} + \frac{5}{4}x}$$

(d) First solve for x, then swap x and y.

$$\begin{aligned} y &= \frac{2+e^x}{3-e^x} \\ 2+e^x &= 3y - e^x y \\ e^x(1+y) &= 3y-2 \\ e^x &= \frac{3y-2}{1+y} \\ x &= \ln\left(\frac{3y-2}{1+y}\right) \\ f^{-1}(x) &= \boxed{\ln\left(\frac{3x-2}{1+x}\right)} \end{aligned}$$

2. (36 points) The following problems are not related.

(a) Evaluate $\lim_{x\to\infty}(1+2^x)^{1/x}$.

(b) Use the Intermediate Value Theorem to show that $1+u+\arctan u=0$ has at least one real solution.

(c) Find $\frac{dy}{dx}$ if $y=(\ln x)^x$.

(d) Evaluate $\lim_{h\to 0}\dfrac{\arccos(1/2+h)-\pi/3}{h}$.

(e) Evaluate $\displaystyle\int_0^{1/6}\frac{\sin^{-1}(3t)}{\sqrt{1-9t^2}}\,dt$.

(f) A monster eel is growing exponentially, increasing in length by 30 percent every 2 months. If its length at birth was b inches, how long will it take for the eel to reach 8 times its initial length?

Solution:

(a) Let

$$\begin{aligned}
L &= \lim_{x\to\infty}(1+2^x)^{1/x}.\\
\ln L &= \lim_{x\to\infty}\ln(1+2^x)^{1/x}\\
&= \lim_{x\to\infty}\frac{\ln(1+2^x)}{x}\\
&\overset{L'H}{=} \lim_{x\to\infty}\frac{\frac{1}{1+2^x}\cdot 2^x(\ln 2)}{1} = \lim_{x\to\infty}\frac{2^x(\ln 2)}{1+2^x}\\
&\overset{L'H}{=} \lim_{x\to\infty}\frac{2^x(\ln 2)^2}{2^x(\ln 2)} = \ln 2\\
L &= e^{\ln 2} = \boxed{2}
\end{aligned}$$

(b) Let $f(u)=1+u+\arctan u$. Then $f(0)=1+0+0=1>0$ and $f(-1)=1-1-\frac{\pi}{4}<0$. By the Intermediate Value Theorem, since f is a continuous function, there is a number c in $(-1,0)$ such that $f(u)=0$.

(c) Use logarithmic differentiation.

$$\begin{aligned}
y &= (\ln x)^x\\
\ln y &= \ln(\ln x)^x = x\ln(\ln x)\\
\frac{1}{y}\frac{dy}{dx} &= x\cdot\frac{1}{\ln x}\cdot\frac{d}{dx}(\ln x)+\ln(\ln x)\\
&= \frac{x}{\ln x}\cdot\frac{1}{x}+\ln(\ln x)\\
&= \frac{1}{\ln x}+\ln(\ln x)\\
\frac{dy}{dx} &= \boxed{(\ln x)^x\left(\frac{1}{\ln x}+\ln(\ln x)\right)}
\end{aligned}$$

(d) This is the definition of derivative for $\arccos(\frac{1}{2})$.

$\frac{d}{dx}[\arccos(x)] = -\frac{1}{\sqrt{1-x^2}} \Rightarrow \frac{d}{dx}[\arccos(x)]\,|_{x=\frac{1}{2}} = \frac{-1}{\sqrt{1-(\frac{1}{2})^2}} = \frac{-1}{\sqrt{\frac{3}{4}}} = \boxed{-\frac{2}{\sqrt{3}}}$.

(e) Let $u=\sin^{-1}(3t)$, $du=3\,dt/\sqrt{1-9t^2}$. The new u limits are 0 to $\pi/6$.

$$\int_0^{1/6}\frac{\sin^{-1}(3t)}{\sqrt{1-9t^2}}\,dt = \frac{1}{3}\int_0^{\pi/6}u\,du = \left[\frac{u^2}{6}\right]_0^{\pi/6} = \boxed{\frac{\pi^2}{216}}$$

(f) Let $y(t)$ represent the length of the eel at t months of age. We are given that $y(0) = b$ and $y(2) = 1.3b$. First find the rate constant k.

$$\begin{aligned} y(t) &= y(0)e^{kt} \\ y(2) &= be^{2k} = 1.3b \\ e^{2k} &= 1.3 \\ 2k &= \ln 1.3 \\ k &= \frac{\ln 1.3}{2} \end{aligned}$$

Now find t when $y(t) = 8b$.

$$\begin{aligned} y(t) &= be^{kt} = 8b \\ e^{kt} &= 8 \\ kt &= \ln 8 \\ t &= \frac{\ln 8}{k} = \boxed{\frac{2\ln 8}{\ln 1.3} \text{ months}} \approx 15.85 \text{ months} \end{aligned}$$

3. (12 points) Let $f(x) = \int_0^x \left(e^{t^3-9t^2+24t+1}\right)\, dt$, $x > 0$.

(a) Find the intervals where the graph of $f(x)$ is concave upwards.

(b) Find the intervals where the graph of $f(x)$ is concave downwards.

(c) Find the values of x where the graph of $f(x)$ has inflection points.

Solution:

$f'(x) = e^{x^3-9x^2+24x+1}$

$f'' = e^{x^3-9x^2+24x+1}(3x^2 - 18x + 24)$

$f'' = 0$ when $3x^2 - 18x + 24 = 3(x-2)(x-4) = 0$

(a) Concave upward on $(0,2) \cup (4,\infty)$

(b) Concave downward on $(2,4)$.

(c) Inflection points occur at $x = 2$ and $x = 4$.

4. (25 points) The following questions are not related:

(a) Evaluate $\int \left(\frac{\sin\theta - 1}{\cos^2\theta} \right) \, d\theta$

(b) Evaluate $\int \left(\frac{1}{x\ln(x^3)} \right) \, dx$

(c) Solve the following differential equation: $\frac{dy}{dx} = 2e^{-x}\cosh x$ where $y(0) = 1$.

(d) Name all points where the tangent line to the function $y = \frac{1}{x}$ is parallel to $y = 2x - 3$.

(e) Name all points where the tangent line to the function $y = 3x - \frac{4}{3}x^3$ is perpendicular to $x + 2y = 2$.

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Solution:

(a) $\displaystyle\int \left(\frac{\sin\theta - 1}{\cos^2\theta}\right)\, d\theta = \int \left(\frac{\sin\theta}{\cos^2\theta} - \frac{1}{\cos^2\theta}\right) d\theta = \int \left(\frac{\sin\theta}{\cos\theta}\frac{1}{\cos\theta} - \sec^2\theta\right) d\theta$

$\displaystyle= \int \left(\tan\theta\sec\theta - \sec^2\theta\right) d\theta = \boxed{\sec\theta - \tan\theta + C}.$

(b) $\displaystyle\int \left(\frac{1}{x\ln(x^3)}\right)\, dx = \int \frac{1}{3x\ln x}dx = \frac{1}{3}\int \frac{1}{x\ln x}dx$

$u = \ln x$ so $du = \frac{1}{x}dx$

$\displaystyle\frac{1}{3}\int \frac{1}{u}du = \frac{1}{3}\ln|u| + C = \boxed{\tfrac{1}{3}\ln|\ln x| + C}$

(c) $\displaystyle\frac{dy}{dx} = 2e^{-x}\cosh x \Rightarrow \int dy = \int 2e^{-x}\cosh x dx = 2\int \frac{\cosh x}{e^x}dx$

$\displaystyle y = \int \frac{e^x + e^{-x}}{e^x}dx = \int 1 + e^{-2x}dx$

$y(x) = x - \frac{1}{2}e^{-2x} + C$

$y(0) = 0 - \frac{1}{2}e^0 + C = 1 \Rightarrow C = \frac{3}{2}$

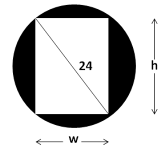
$\boxed{y(x) = x - \tfrac{1}{2}e^{-2x} + \tfrac{3}{2}}$

(d) There are no points parallel to $y = 2x - 3$ since this line has a slope of $m = 2$ and $y = \frac{1}{x}$ has no positive slopes.

(e) $y' = 3 - 4x^2 = 2 \Rightarrow 4x^2 = 1 \Rightarrow x = \pm\frac{1}{2}$

points are then $(\frac{1}{2}, \frac{4}{3})$ and $(-\frac{1}{2}, -\frac{4}{3})$.

5. (12 points) A wooden beam has a rectangular cross section of height h and width w. The strength of the beam is given by $s = kh^2w$ where k is a positive constant. What are the dimensions of the strongest beam that can be cut from a round log of diameter 24 inches? Justify.



Solution: Maximize $S(h, w) = kh^2w$
$h^2 + w^2 = 24^2 = 576 \Rightarrow h^2 = 576 - w^2$

$S(w) = k(576 - w^2)w = k(576w - w^3)$

$S'(w) = k(576 - 3w^2) = 0 \Rightarrow \frac{576}{3} = w^2 = 8^2(3) = 8\sqrt{3}$

so $h = \sqrt{8^2 3^2 - 8^2 3} = 8\sqrt{6}$.

$S''(w) = -6kw < 0$ at $8\sqrt{3}$, so this is a maximum location.

dimensions are $\boxed{8\sqrt{3} \text{ by } 8\sqrt{6}}$ in.

6. (6 points) A particle is moving along the curve $x^2y^2 = 81$ in the fourth quadrant. When it reaches $x = 1$, its x-coordinate is increasing at a rate of $\frac{1}{2}$ unit per second. At what rate is the y-coordinate changing? Choose the most appropriate answer; you need not justify your answer:

(A) $\dfrac{1}{18}$ units per second. (B) $-\dfrac{9}{2}$ units per second. (C) $\dfrac{9}{2}$ units per second.

(D) $-\dfrac{1}{18}$ units per second. (E) y is constant (not changing). (F) None of the above.

Solution: C

7. (24 points) Answer the following statements as Always True, or False. No justification is necessary.

Consider the function $g(x) = \begin{cases} x^2 & : -1 \leq x < 2 \\ 3 - 3(x-2)^2 & : 2 \leq x \leq 3 \end{cases}$

(a) $g(x)$ is differentiable.

(b) $g(x)$ is integrable.

(c) $g(x)$ has an absolute maximum.

(d) The area under the graph of $g(x)$ is 5.

(e) $\lim_{x \to 2} g(x)$ exists.

(f) Rolle's Theorem guarantees that $g(x)$ has a horizontal tangent for some c in $(-1, 1)$.

Solution:

(a) False

(b) Always True

(c) False

(d) Always True

(e) False

(f) Always True

8. Consider the following short answer questions; no justification is necessary.

(a) (2 points) Which of the following are not approximation methods?
(I) Newton's Method (II) Average rate of change (III) Linearization (IV) Riemann Sum

(b) (5 points) State whether each of the following functions is continuous or has a jump discontinuity or has an infinite discontinuity or has a removable discontinuity on their respective domains.

(I) $f(x) = \dfrac{x-4}{x^2+1}$ (II) $f(x) = \dfrac{|x-4|}{x-4}$ (III) $f(x) = \dfrac{x^2-1}{x-4}$

(IV) $f(x) = \dfrac{x^2-4x}{x-4}$ (V) $f(x) = \begin{cases} \frac{x^2-4x}{x-4} & : x \neq 4 \\ 6 & : x = 4 \end{cases}$

Solution:

(a) II, Average rate of change is NOT an approximation method.

(b) I Continuous
II Jump discontinuity
III infinite discontinuity
IV removable discontinuity
V removable discontinuity

9. (12 points) Answer the following, no justification is necessary.
Suppose g is a continuous function defined on $[0,5]$ that satisfies the following conditions:

$$g(3) = g(5) = 0, \qquad g' > 0 \text{ on } (0,4), \qquad g' < 0 \text{ on } (4,5).$$

(a) To maximize the value of $\displaystyle\int_a^b g(x)\,dx$ on a subinterval of $[0,5]$, what values of a and b should be chosen if $a < b$?

(b) The expression $\displaystyle\sum_{i=1}^{3} g\left(2 + \frac{2i}{3}\right)\left(\frac{2}{3}\right)$ is a left endpoint rectangle approximation for what integral?

(c) Which of the following expressions computes the total area between the curve $y = g(x)$ and the x-axis on $[0,5]$?

$$\text{(I)} \int_0^5 g(x)\,dx \quad \text{(II)} \left|\int_0^5 g(x)\,dx\right| \quad \text{(III)} -\int_0^4 g(x)\,dx + \int_4^5 g(x)\,dx \quad \text{(IV) none of the above}$$

Solution:

(a) Since g is positive on $(3,5)$ and negative otherwise, choose $\boxed{a = 3, b = 5}$.

(b) $L_3 = \sum_{i=1}^{3} g\left(2 + \frac{2i}{3}\right)\left(\frac{2}{3}\right) = g\left(\frac{8}{3}\right)\left(\frac{2}{3}\right) + g\left(\frac{10}{3}\right)\left(\frac{2}{3}\right) + g\left(\frac{12}{3}\right)\left(\frac{2}{3}\right)$ is an approximation of $\boxed{\displaystyle\int_{8/3}^{14/3} g(x)\,dx}$.

(c) The function g is negative on $[0,3)$ and positive on $(3,5)$ so the answer is $\boxed{\text{IV}}$.

$\boxed{\text{END of Exam}}$