1. (21 pts) Evaluate the following integrals. You do not need to use the limit definition of an integral for the definite integral.

   (a) $\displaystyle\int \frac{x^4+x+3}{\sqrt[3]{x}}\,dx$ (b) $\displaystyle\int_{-\pi/3}^{\pi/2} (\sin x - 2\cos x)\,dx$ (c) $\displaystyle\int_0^2 \left|x^2-1\right|\,dx$
   (*Hint: Sketch the graph*)

   Solution:

   (a) $$\int \frac{x^4+x+3}{\sqrt[3]{x}}\,dx = \int \left(x^{11/3}+x^{2/3}+3x^{-1/3}\right)\,dx = \frac{x^{14/3}}{14/3}+\frac{x^{5/3}}{5/3}+3\frac{x^{2/3}}{2/3}+C$$
   $$= \frac{3x^{14/3}}{14}+\frac{3x^{5/3}}{5}+\frac{9x^{2/3}}{2}+C$$

   (b) $$\int_{-\pi/3}^{\pi/2} (\sin x - 2\cos x)\,dx = [-\cos x - 2\sin x]_{-\pi/3}^{\pi/2} = -\cos\frac{\pi}{2} - 2\sin\frac{\pi}{2} - \left(-\cos\left(-\frac{\pi}{3}\right) - 2\sin\left(-\frac{\pi}{3}\right)\right)$$
   $$= 0 - 2 + \frac{1}{2} - 2\cdot\frac{\sqrt{3}}{2} = -\frac{3}{2} - \sqrt{3}$$

   (c) Notice that x^2-1 is negative on $(-1,1)$ and positive on $(-\infty,-1)\cup(1,\infty)$. In particular, x^2-1 is negative on $(0,1)$ and positive on $(1,2)$. We have
   $$\begin{aligned}\int_0^2 \left|x^2-1\right|\,dx &= -\int_0^1 (x^2-1)\,dx + \int_1^2 (x^2-1)\,dx\\ &= -\left[\frac{x^3}{3}-x\right]_0^1 + \left[\frac{x^3}{3}-x\right]_1^2\\ &= -\left[\frac{1}{3}-1\right] + \left[\frac{8}{3}-2-\frac{1}{3}+1\right]\\ &= 2.\end{aligned}$$

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2. (24 points)

   (a) State the two parts of the Fundamental Theorem of Calculus, including any hypotheses.

   (b) Suppose $h(x) = \displaystyle\int_0^{2x+2} \sin(t-2)\,dt$ and $f(s) = \displaystyle\int_{\pi/2}^{3s} h(x)\,dx$.

      i. Show that $f''(s) = 9h'(3s)$.

      ii. Find $f''(\pi/12)$.

   Solution:

   (a) The Fundamental Theorem of Calculus states that if f is continuous on $[a,b]$ then

   1. $\displaystyle\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

   2. $\displaystyle\int_a^b f(x)\,dx = F(b)-F(a)$, where F is any antiderivative of f (that is, $F' = f$).

(b) i. First, we find $f'(s)$. By the Fundamental Theorem of Calculus, $f'(s) = h(3s) \cdot \dfrac{d}{ds}(3s) = 3\,h(3s)$. Now we find $f''(s)$:

$$\begin{aligned} f''(s) &= 3 \cdot \frac{d}{ds}\left[h\left(3s\right)\right] \\ &= 3 \cdot \frac{d}{ds}\left(3s\right) h'\left(3s\right) \\ &= 9h'\left(3s\right). \end{aligned}$$

ii. We need $h'(x)$. Again, by the Fundamental Theorem of Calculus,

$$h'(x) = \sin\left[(2x+2) - 2\right] \cdot \frac{d}{dx}(2x+2) = 2\sin 2x.$$

Substituting into f'', we have

$$f''(s) = 9 \cdot 2\sin\left(2 \cdot 3s\right) = 18\sin 6s.$$

Finally, with $s = \frac{\pi}{12}$, we obtain $f''(\pi/12) = 18\sin\left(\pi/2\right) = 18$.

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3. (30 pts) Consider $\displaystyle\int_0^3 \left(x^2 - 6x + 10\right)\,dx$.

(a) Write down a Riemann sum approximation to this integral by dividing $[0,3]$ into four equal subintervals and using the **left endpoint** of each subinterval for evaluation. You do not need to evaluate the sum.

(b) Is your sum an overestimate or an underestimate? Briefly explain your answer.

(c) Write down a Riemann sum to approximate the integral by dividing $[0,3]$ into n equal subintervals and using the **right endpoint** of each subinterval for evaluation.

(d) Evaluate the limit as $n \to \infty$ of the sum in (c). You must explicitly show your work. Check your answer by evaluating the integral.

Note: The following formulas may be relevant:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Solution:

(a) Let $f(x) = x^2 - 6x + 10$, and let $I = \displaystyle\int_0^3 f(x)\,dx$. Then

$$I \approx \sum_{i=1}^{4} f(x_i)\Delta x,$$

where $\Delta x = \dfrac{3-0}{4} = \dfrac{3}{4}$ and $x_i = 0 + (i-1)\Delta x = \dfrac{3(i-1)}{4}$ for $i = 1,2,3,4$ (since we're using left endpoints). That is,

$$I \approx \left[\frac{3}{4}\,f(0) + \frac{3}{4}\,f\left(\tfrac{3}{4}\right) + \frac{3}{4}\,f\left(\tfrac{3}{2}\right) + \frac{3}{4}\,f\left(\tfrac{9}{4}\right)\right]$$

(b) Notice that $f'(x) = 2x - 6$, so $f' < 0$ on $[0, 3]$. That is, f is a decreasing function on $[0, 3]$, so the left-endpoint sum gives an overestimate.

(c) Now we have $\Delta x = \dfrac{3-0}{n} = \dfrac{3}{n}$ and $x_i = 0 + i\Delta x = \dfrac{3i}{n}$.

$$\begin{aligned} I &\approx \sum_{i=1}^{n} f(x_i)\Delta x \\ &= \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^2 - 6\left(\frac{3i}{n}\right) + 10 \right] \cdot \frac{3}{n} \end{aligned}$$

(d) Let $S_n = \displaystyle\sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^2 - 6\left(\frac{3i}{n}\right) + 10 \right] \cdot \frac{3}{n}$. Simplifying, we have

$$\begin{aligned} S_n &= \frac{3}{n} \sum_{i=1}^{n} \left[\frac{9i^2}{n^2} - \frac{18i}{n} + 10 \right] \\ &= \frac{27}{n^3} \sum_{i=1}^{n} i^2 - \frac{54}{n^2} \sum_{i=1}^{n} i + \frac{30}{n} \sum_{i=1}^{n} 1 \\ &= \frac{27}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{54}{n^2} \cdot \frac{n(n+1)}{2} + \frac{30}{n} \cdot n \\ &= \frac{9}{2} \cdot \frac{(n+1)(2n+1)}{n^2} - 27 \cdot \frac{n+1}{n} + 30. \end{aligned}$$

Taking the limit as $n \to \infty$, we have

$$\begin{aligned} \lim_{n\to\infty} S_n &= \lim_{n\to\infty} \left[\frac{9}{2} \cdot \frac{(n+1)(2n+1)}{n^2} - 27 \cdot \frac{n+1}{n} + 30 \right] \\ &= \frac{9}{2} \cdot 2 - 27 \cdot 1 + 30 \\ &= 12. \end{aligned}$$

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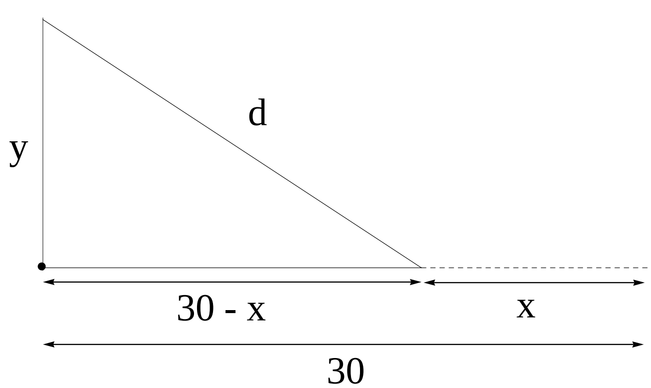
4. (25 points) A car leaves an intersection at noon and drives due north at 40 miles per hour. Another car is 30 miles due east of the intersection at noon. It heads due west at 30 miles per hour and arrives at the intersection at 1 pm. When are the cars closest? In the diagram below, $y(t)$ is the distance the first car has driven north since noon, and $x(t)$ is the distance the second car has driven west since noon.



Solution:

Let $d(t)$ be the distance between the cars, where t is the number of hours since noon. We want to minimize $d(t)$, or equivalently, $d^2(t)$. Notice that x, y, and d are related via

$$d^2 = y^2 + (30 - x)^2.$$

Additionally, since the first car is traveling at 40 miles per hour, it follows that

$$y(t) = 40\,\frac{\text{miles}}{\text{hour}} \times t\,\text{hours} = 40t.$$

Similarly, the second car is traveling at 30 miles per hour, so $x(t) = 30t$. Then we have

$$d^2 = (40t)^2 + (30 - 30t)^2.$$

Let $D(t) = d^2(t)$. We want to minimize

$$D(t) = (40t)^2 + (30 - 30t)^2 = 1600t^2 + 900(1 - t)^2.$$

We have $D'(t) = 3200t - 1800(1 - t) = 5000t - 1800$. Thus $D'(t) = 0$ if $t = \frac{1800}{5000} = \frac{9}{25}$.
Furthermore, $D''(t) = 5000$, which is always positive, so we must have a global minimum. Thus, the two cars are closest to each other when $t = \frac{9}{25} = 0.36$ hours past noon (or approximately 12:22 pm).
