1. (24 pts) Evaluate the following limits:

(a) $\displaystyle\lim_{x\to 3} \frac{x^2+x-12}{x-3}$ (b) $\displaystyle\lim_{x\to a^-} \left(1+\frac{x-a}{|x-a|}\right)$ (c) $\displaystyle\lim_{x\to -\infty} \sqrt{9x^2+1}-3x$

For (a) and (b) only, locate and classify any discontinuities as either *jump*, *removable*, or *infinite*.

Solution:

(a) $\displaystyle\lim_{x\to 3} \frac{x^2+x-12}{x-3} = \lim_{x\to 3} \frac{(x+4)(x-3)}{x-3} = \lim_{x\to 3} (x+4) = 7.$

There is a removable discontinuity at $x=3$.

(b) Recall that

$$|x-a| = \begin{cases} x-a, & x-a \geq 0 \\ -(x-a), & x-a < 0 \end{cases} = \begin{cases} x-a, & x \geq a \\ -(x-a), & x < a. \end{cases}$$

Since we consider the limit as x approaches a from the left ($\lim_{x\to a^-}$), it follows that $x<a$, so that $|x-a| = -(x-a)$. Thus we have

$$\lim_{x\to a^-} \left(1+\frac{x-a}{|x-a|}\right) = \lim_{x\to a^-} \left(1+\frac{x-a}{-(x-a)}\right) = 1-1=0.$$

There is a jump discontinuity at $x=a$.

(c) Notice that the term under the square root is large and positive as $x\to-\infty$, as is the term $-3x$ (since we consider large negative values of x). Thus we expect that $\displaystyle\lim_{x\to-\infty}\sqrt{9x^2+1}-3x = +\infty$. We can show this as follows

$$\begin{aligned}
\lim_{x\to-\infty}\sqrt{9x^2+1}-3x &= \lim_{x\to-\infty}\sqrt{x^2\left(9+\frac{1}{x^2}\right)}-3x \\
&= \lim_{x\to-\infty}|x|\sqrt{9+\frac{1}{x^2}}-3x \\
&= \lim_{x\to-\infty}-x\sqrt{9+\frac{1}{x^2}}-3x \\
&= \lim_{x\to-\infty}-x\left(\sqrt{9+\frac{1}{x^2}}+3\right) \\
&= \left[\lim_{x\to-\infty}(-x)\right]\cdot\left[\lim_{x\to-\infty}\left(\sqrt{9+\frac{1}{x^2}}+3\right)\right] \\
&= \left[\lim_{x\to-\infty}(-x)\right]\cdot\left(\sqrt{9+0}+3\right) \\
&= 6\cdot\lim_{x\to-\infty}(-x) \\
&= +\infty.
\end{aligned}$$

2. (16 pts) Let $f(x) = \cos x$

   (a) State the limit definition of the derivative of a function.

   (b) Use your definition from (a) to show that $\frac{d}{dx}(\cos x) = -\sin x$.

   Hint: $\cos(a+b) = \cos a \cos b - \sin a \sin b$, and $\lim\limits_{\theta \to 0} \dfrac{\cos\theta - 1}{\theta} = 0$.

Solution:

(a) The derivative of a function $f(x)$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

(b)

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h} \\
&= \lim_{h \to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\
&= \lim_{h \to 0} \left[\frac{\cos x \cos h - \cos x}{h} - \frac{\sin x \sin h}{h} \right] \\
&= \lim_{h \to 0} \left[\cos x \cdot \frac{\cos h - 1}{h} - \sin x \cdot \frac{\sin h}{h} \right] \\
&= \left(\lim_{h \to 0} \cos x \right) \cdot \left(\lim_{h \to 0} \frac{\cos h - 1}{h} \right) - \left(\lim_{h \to 0} \sin x \right) \cdot \left(\lim_{h \to 0} \frac{\sin h}{h} \right) \\
&= \cos x \cdot 0 - \sin x \cdot 1 \\
&= -\sin x.
\end{aligned}$$

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3. (24 pts) Consider the function

$$g(x) = \begin{cases} x^2 + cx + 2, & x \le 0 \\ 7x + d, & x > 0, \end{cases}$$

where c and d are real constants.

   (a) What is the domain of $g(x)$?

   (b) What does it mean for a function to be continuous at $x = a$? Your definition must include limits.

   (c) Find the value of d that makes $g(x)$ continuous at $x = 0$, **using your definition from (b)**. Are there any restrictions on the value of c?

Solution:

(a) The domain of $g(x)$ is all real numbers, or $(-\infty, \infty)$.

(b) A function $f(x)$ is continuous at $x = a$ if $\lim\limits_{x \to a} f(x) = f(a)$. Alternatively, we can write this in three parts:

   (i) $f(a)$ exists.

   (ii) $\lim\limits_{x \to a} f(x)$ exists.

   (iii) $\lim\limits_{x \to a} f(x) = f(a)$.

(c) (i) First, notice that $g(0) = 2$ from the first line of the function (so $g(0)$ exists).

(ii) Next, we compute $\lim_{x\to 0} g(x)$ by examining the left and right-hand limits.

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} \left(x^2 + cx + 2\right) = 2.$$
$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} (7x + d) = d.$$

For $\lim_{x\to 0} g(x)$ to exist, the left and right-hand limits must be equal. This is the case if $d = 2$.

(iii) Lastly, we check that $g(0) = \lim_{x\to 0} g(x)$. Since $g(0) = 2$ and $\lim_{x\to 0} g(x) = 2$, it follows that $g(x)$ is continuous when $d = 2$.

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4. (16 pts) Let $f(x) = -3x^5 + 2\sin x + 3$.

(a) Find the equation of the tangent line to $f(x)$ at $x = 0$. You may use rules of differentiation.

(b) Show that there is at least one solution to $f(x) = 4$ on $(-\pi, \pi)$.

Solution:

(a) The equation of the tangent line to $y = f(x)$ at $x = 0$ is

$$y - f(0) = f'(0)(x - 0) = f'(0)x.$$

First compute $f'(x)$ using rules of differentiation: $f'(x) = -15x^4 + 2\cos x$.
Next, substitute $x = 0$ into $f(x)$ and $f'(x)$ to obtain $f(0) = 3$ and $f'(0) = 2$.
Lastly, substitute into the equation for the tangent line to get $y - 3 = 2x$ or $y = 2x + 3$.

(b) First, notice that $f(x)$ is a continuous function for all x since polynomials are continuous and $\sin x$ is continuous. In particular, $f(x)$ is continuous on $[-\pi, \pi]$. Next, notice that

$$f(-\pi) = -3(-\pi)^5 + 2\sin(-\pi) + 3 = 3\pi^5 + 3 \approx 3^6 + 3 >> 4.$$

since $\sin(-\pi) = 0$ and $\pi \approx 3$. Additionally, we have

$$f(\pi) = -3(\pi)^5 + 2\sin\pi + 3 = -3\pi^5 + 3 \approx -3^6 + 3 << 4.$$

Since $f(-\pi) > 4$ and $f(\pi) < 4$, it follows from the Intermediate Value Theorem that there is some number c in $(-\pi, \pi)$ such that $f(c) = 4$.

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5. (14 pts) Let the displacement of an object traveling along a straight line be given by

$$s(t) = t^3 - 12t^2 + 45t + 2.$$

(a) Find the velocity and acceleration functions of the object. You may use rules of differentiation.

(b) When is the object moving in the positive direction? When is the object at rest?

Solution:

(a) $v(t) = s'(t) = 3t^2 - 24t + 45$ and $a(t) = v'(t) = s''(t) = 6t - 24$.

(b) Notice that $v(t) = 3(t^2 - 8t + 15) = 3(t-5)(t-3)$. The object is at rest when $v(t) = 0$, which occurs at $t = 3$ and $t = 5$.
Now check the sign of $v(t)$ in the following intervals:

$$\begin{aligned} t < 3: &\quad v(t) > 0 \\ 3 < t < 5: &\quad v(t) < 0 \\ t > 5: &\quad v(t) > 0. \end{aligned}$$

The object is moving in the positive direction when $v(t) > 0$. That is, the object is moving in the positive direction when t is in $(-\infty, 3) \cup (5, \infty)$.

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6. (6pts) True or False: (Write the word **True** or **False**, do not write T/F.)

(a) If $f(x)$ is odd, then $f(x+1)$ is odd.

(b) If a function is continuous at $x = a$, then it is differentiable at $x = a$.

(c) $\lim_{x\to 2} \left[(x-2)^2 \sin\left(\frac{2}{x-2}\right)\right] = 0.$

Solution:

(a) False. Consider $f(x) = x$, which is an even function. Now define $g(x) = f(x+1) = x+1$. We have $g(-x) = -x + 1 \neq -g(x)$. So $f(x+1)$ is not odd.

(b) False. The function $f(x) = |x|$ is continuous everywhere, but is not differentiable at $x = 0$.

(c True. We can use the Squeeze Theorem to verify:
First, notice that $-1 \leq \cos\theta \leq 1$ for all θ. Consequently, $-1 \leq \sin\left(\frac{2}{x-2}\right) \leq 1$. Next, multiply the inequality by $(x-2)^2$, which gives

$$-(x-2)^2 \leq (x-2)^2 \sin\left(\frac{2}{x-2}\right) \leq (x-2)^2.$$

Then since $\lim_{x\to 2}\left[-(x-2)^2\right] = \lim_{x\to 2}(x-2)^2 = 0$, and since $-(x-2)^2 \leq (x-2)^2 \sin\left(\frac{2}{x-2}\right) \leq (x-2)^2$, by the Squeeze Theorem we know that $\lim_{x\to 2}\left[(x-2)^2 \sin\left(\frac{2}{x-2}\right)\right] = 0.$
