MATH 1300: HW #9

Due on March 16, 2017 at 10:00am

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Section 3.7

29. (a) On what interval is $f(x) = x \ln x$ decreasing?

$$\begin{aligned} f'(x) &= \ln(x)\Big(\frac{d}{dx}(x)\Big) + x\Big(\frac{d}{dx}\big(\ln(x)\big)\Big) \\ &= x\Big(\frac{d}{dx}\big(\ln(x)\big)\Big) + \ln(x) \\ &= \ln(x) + \frac{1}{x}x \\ &= 1 + \ln(x) \end{aligned}$$

(b) On what interval is f concave upward?

$$\begin{aligned} f''(x) &= \frac{d}{dx}\big(1 + \ln(x)\big) \\ &= \frac{d}{dx}\big(\ln(x)\big) \\ &= \frac{1}{x} \end{aligned}$$

32. Let $f(x) = \log_a(3x^2 - 2)$. For what value of a is $f'(1) = 3$?

$$\begin{aligned} f'(x) &= \frac{d}{dx}\big(\log_a(3x^2 - 2)\big) \\ &= \frac{d}{dx}\Big(\frac{\log(3x^2 - 2)}{\log(a)}\Big) \\ &= \frac{\frac{d}{dx}(3x^2 - 2)}{(3x^2 - 2)\log(a)} \\ &= \frac{3(\frac{d}{dx}(x^2))}{(3x^2 - 2)\log(a)} \\ &= 2x\frac{3}{(3x^2 - 2)\log(a)} \\ 3 &= \frac{6x}{(3x^2 - 2)\log(a)} \\ 3(3x^2 - 2)\log(a) &= 6x \\ 9x^2 - 6 \cdot 3\log(a) &= 6x \\ 3\log(a) &= 6x - 9x^2 + 6 \\ \log(a) &= 2(1) - 3(1)^2 + 2 \\ \log(a) &= 1 \\ 10^a &= 1 \\ a &= e \end{aligned}$$

34. Use logarithmic differentiation to find the derivative of the function $y = \sqrt{x}e^{x2}(x^2+1)^{10}$.

$$\begin{aligned}
\frac{d}{dx}(y) &= \frac{d}{dx}(\sqrt{x}e^{x2}(x^2+1)^{10}) \\
&= \sqrt{x}(1+x^2)^{10}(\frac{d}{dx}(e^{x^2}) + e^{x^2}(\frac{d}{dx}(\sqrt{x}(1+x^2)^{10})) \\
&= e^{x^2}(\frac{d}{dx}(\sqrt{x}(1+x^2)^{10}) + e^{x^2}\frac{d}{dx}(x^2)\sqrt{x}(1+x^2)^{10} \\
&= 2e^{x^2}x^{\frac{3}{2}}(1+x^2)^{10} + e^{e^2}(\frac{d}{dx}(1+x^2)^{10}) \\
&= 2e^{x^2}x^{\frac{3}{2}}(1+x^2)^{10} + e^{x^2}(\frac{(1+x^2)^{10}}{2\sqrt{x}} + 2x10\sqrt{x}(1+x^2)^9) \\
&= \boxed{2e^{x^2}x^{\frac{3}{2}}(1+x^2)^{10} + e^{x^2}(20x^{\frac{2}{3}}(1+x^2)^9 + \frac{(1+x^2)^{10}}{2\sqrt{x}}}
\end{aligned}$$

38. Use logarithmic differentiation to find the derivative of the function $y = x^{\cos x}$.

$$\begin{aligned}
y &= x^{\cos x} \\
\ln y &= \cos x \ln x \\
\frac{1}{y}dy &= (-\sin x \ln x + \frac{\cos x}{x})dx \\
\frac{dy}{dx} &= (y)(-\sin x \ln x + \frac{\cos x}{x}) \\
\frac{dy}{dx} &= \boxed{x^{\cos x}(-\sin x \ln x + \frac{\cos x}{x})}
\end{aligned}$$

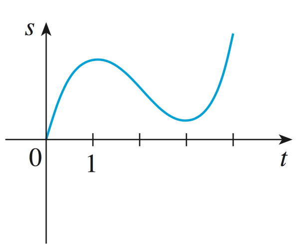
45. Find a formula for $(f^{(n)})(x)$ if $f(x) = \ln(x-1)$.

$$\begin{aligned}
\frac{d}{dx}f^{(n)}(x) &= \frac{d}{dx}\left(\frac{(-1)^{n-1}(n-1)}{(x-1)^n}\right) \\
&= (-1)^{n-1}(n-1)\frac{d}{dx}\left((x-1)^{-n}\right) \\
&= (-1)^{n-1}(n-1)\left((-n)(x-1)^{-n-1}\right) \\
&= \boxed{\frac{(-1)^n n}{(x-1)^{n+1}}}
\end{aligned}$$

Section 3.8

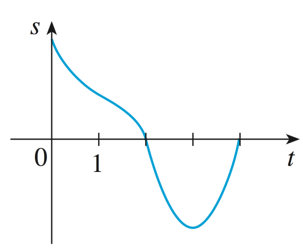
6. Graphs of the *position* functions of two particles are shown, where t is measured in seconds. When is each particle speeding up? When is it slowing down? Explain.

(a)



The cup is speeding up at $(0,1)\cup(3,4)$ because $t'(s)$ is positive, and slowing down at $(1,3)$ because the tangent is negative.

(b)



The cup is speeding up at $(3,4)$ because $t'(s)$ is positive, and slowing down at $(0,3)$ because the tangent is negative.

10. If a ball is thrown vertically upward with a velocity of 80 ft/s, then its height after t seconds is $s = 80t - 16t^2$.

    (a) What is the maximum height reached by the ball?

$$\begin{aligned} s'(t) &= 80 - 32t \\ 0 &= 80 - 32t \\ -80 &= -32t \\ t &= \frac{5}{2} \\ s &= 80\left(\frac{5}{2}\right) - 16\left(\frac{5}{2}\right)^2 \\ &= 200 - 16\left(\frac{25}{4}\right) \\ &= 200 - 100 \\ &= \boxed{100 \text{ ft.}} \end{aligned}$$

    (b) What is the velocity of the ball when it is 96 ft above the ground on its way up? On its way down?

$$\begin{aligned} 96 &= 80t - 16t^2 \\ t &= 2 \text{ or } 3 \\ s'(2) &= 80 - 32(2) \\ s'(2) &= 16 \\ s'(3) &= 80 - 32(3) \\ s'(3) &= -16 \end{aligned}$$

    On the way up, the velocity is 16 ft/min, on the way down the velocity is -16 ft/min.

24. The number of yeast cells in a laboratory culture increases rapidly initially but levels off eventually. The population is modeled by the function

$$n = f(t) = \frac{a}{1 + be^{-0.7t}}$$

    where t is measured in hours. At time $t = 0$ the population is 20 cells and is increasing at a rate of 12 cells/hour. Find the values of a and b. According to this model, what happens to the yeast population in the long run?

$$\begin{aligned} f'(t) &= \frac{0.7abe^{-0.7t}}{(1 + be^{-0.7t})^2} \\ f'(0) &= 12 \\ 12 &= \frac{0.7ab}{(1+b)^2} \\ 0.7ab &= 12(1+b)^2 \\ a &= 140 \\ b &= 6 \\ \lim_{t\to\infty} f(t) &= \frac{a}{1+b \lim_{t\to\infty} e^{-0.7t}} \\ \lim_{t\to\infty} f(t) &= \frac{a}{1 + b \cdot 0} \\ \lim_{t\to\infty} f(t) &= \frac{a}{1} \\ \lim_{t\to\infty} f(t) &= a \end{aligned}$$

According to the model, as time goes on, the number of cells will get infinitely closer to 140, but never actually reach 140.

30. The cost function for production of a commodity is

$$C(x) = 339 + 25x - 0.09x^2 + 0.0004x^3$$

(a) Find and interpret $C'(200)$.

$$\begin{aligned} C'(x) &= 25 - 0.18x + 0.0012x^2 \\ C'(200) &= 25 - 0.18(200) + 0.0012(200)^2 \\ C'(200) &= 37 \end{aligned}$$

The number 37 tells us that the cost for the production of a commodity is is increasing at \$37/item. This means producing one more item (for a total of 201) would cost \$37.

(b) Compare $C'(100)$ with the cost of producing the 101st item.

$$\begin{aligned} C'(100) &= 25 - 0.18(100) + 0.0012(100)^2 \\ C'(100) &= 19 \\ C(101) &= 339 + 25(101) - 0.09(101)^2 + 0.0004(101)^3 \\ C(101) &= \$2358.0304 \end{aligned}$$

This means that using $C'(x)$, you can estimate that the cost of producing the 100th item as $2358.0304 - 19 = 2339.034$.

Section 3.9

6. The table shows the population of Nepal (in millions) as of June 30th of the given year. Use linear approximation to estimate the population at midyear in 1989. Use another linear approximation to predict the population in 2010.

t	1985	1990	1995	2000	2005
$N(t)$	17.04	19.33	21.91	24.70	27.68

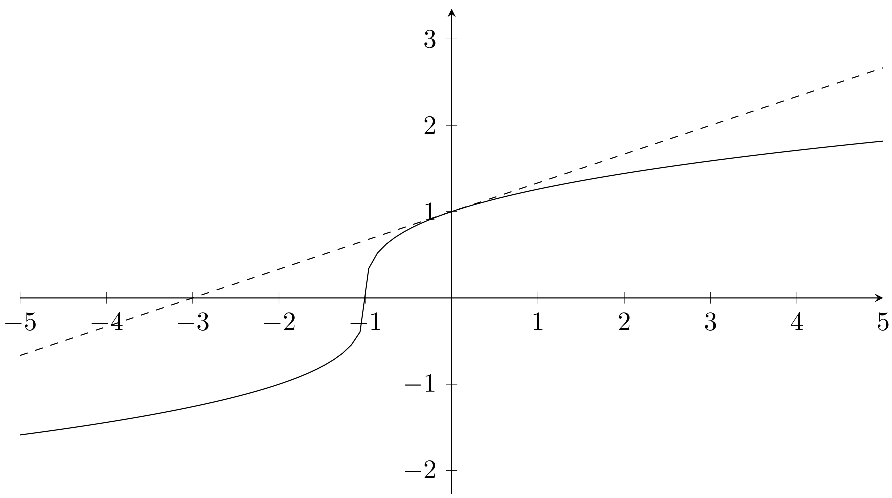
$$
\begin{aligned}
L(t) &= N(1990) + N'(1990)(t-1990) \\
&= 19.33 + N'(1990)(t-1990) \\
N'(1990) &\approx \frac{1}{2}\left(\left(\frac{21.91-19.33}{1995-1990}\right) + \left(\frac{19.33-17.04}{1990-1985}\right)\right) \\
N'(1990) &\approx \frac{1}{2}(0.5160 + 0.4580) \\
N'(1990) &\approx 0.4870 \\
L(t) &= 19.33 + 0.4870(t-1990) \\
L(1989.5) &\approx 19.33 + 0.4870(1989.5-1990) \\
L(1989.5) &\approx \boxed{19.1 \text{ million people}}
\end{aligned}
$$

$$
\begin{aligned}
L(t) &= N(2005) + N'(2005)(t-2005) \\
&= 27.68 + N'(2005)(t-2005) \\
N'(2005) &\approx \left(\frac{27.68-24.70}{2005-2000}\right) \\
N'(2005) &\approx 0.596 \\
L(t) &= 27.68 + 0.596(t-2005) \\
L(2010) &= 27.68 + 0.596(2010-2005) \\
L(2010) &= \boxed{30.66 \text{ million people}}
\end{aligned}
$$

10. Find the linear approximation of the function $g(x) = \sqrt[3]{1+x}$ at $a = 0$ and use it to approximate the numbers $\sqrt[3]{0.95}$ and $\sqrt[3]{1.1}$. Illustrate by graphing g and the tangent line.

$$
\begin{aligned}
f'(x) &= \frac{1}{3\sqrt[3]{(1+x)^2}} \\
L(x) &= f(0) + f'(0)(x-0) \\
L(x) &= 1 + \frac{x}{3}
\end{aligned}
$$

$$
\begin{aligned}
L(\sqrt[3]{0.95}) &\approx 1 + \frac{-0.05}{3} \approx 0.9833 \\
L(\sqrt[3]{1.1}) &\approx 1 + \frac{0.1}{3} \approx 1.0333
\end{aligned}
$$



20. Explain, in terms of linear approximation or differentials, why the approximation is reasonable for $(1.01)^6 \approx 1.06$.

$$\begin{aligned} L(x) &= f(a) + f'(a)(x-a) \\ &= f(1) + f'(1)(x-1) \\ &= 1^6 + 6(1)^5(x-1) \\ &= 1 + 6(x-1) \\ L(1.01) &= 1 + 6(1.01-1) \\ &= 1 + 6(0.01) \\ &= \boxed{1.06} \end{aligned}$$

30. Use differentials to estimate the amount of paint needed to apply a coat of paint 0.05 cm thick to a hemispherical dome with diameter 50 m.

$$\begin{aligned} V &= \frac{2}{3}\pi r^3 \\ dV &= \frac{2}{3}3\pi r^2 dr \\ dV &= 2\pi r^2 dr \\ dV &= 2\pi(25)^2(0.05) \\ dV &= \boxed{\frac{125\pi}{2}\ \text{cm}^3} \end{aligned}$$