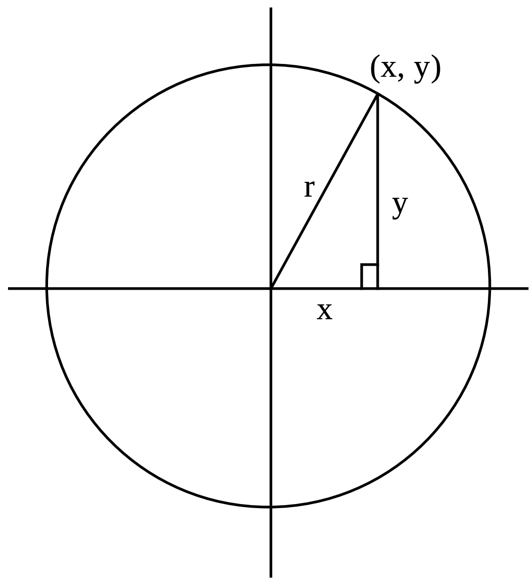
MATH 116 Circles and Circular Functions

A *circle* is defined to be the collection of all points (x, y) that are equidistant from a fixed center point (h, k). The distance to this center point is called the radius r.

A Circle Centered at the Origin

Suppose a circle is centered at the origin (0, 0) and has a radius of length r. Then a point (x, y) on the circle creates a right triangle with sides having lengths x and y, and the hypotenuse having length r. By the *Pythagorean Theorem*, we can say

$$\boxed{x^2 + y^2 = r^2}$$



So $x^2 + y^2 = r^2$ is the equation of a circle centered at the origin with radius r.

Example 1. Give the equation of the circle centered at (0, 0) with radius $9/5$. What points are on the circle for $x = -6/5$?

Solution. Using $r = 9/5$, then $r^2 = 81/25$; so the equation becomes $x^2 + y^2 = \dfrac{81}{25}$.

For $x = -6/5$, there are two possible y-values: $\left(-\dfrac{6}{5}\right)^2 + y^2 = \dfrac{81}{25} \rightarrow \dfrac{36}{25} + y^2 = \dfrac{81}{25} \rightarrow$

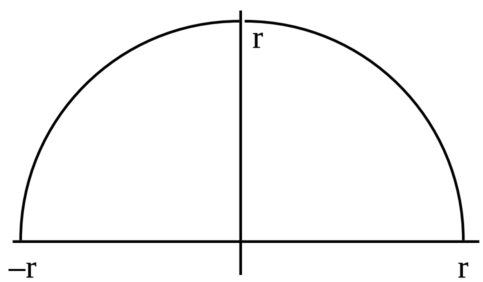
$y^2 = \dfrac{81}{25} - \dfrac{36}{25} \rightarrow y^2 = \dfrac{45}{25} \rightarrow y = \pm\sqrt{\dfrac{45}{25}} = \pm\dfrac{\sqrt{9\times 5}}{5} = \pm\dfrac{3\sqrt{5}}{5}$. So the two points on the

circle are $\left(-\dfrac{6}{5}, \pm\dfrac{3\sqrt{5}}{5}\right)$

Upper and Lower Semicircle Functions

The circle $x^2+y^2=r^2$ defines two semicircle functions, which are the top-half and lower-half of the circle. These functions are obtained y solving for y:

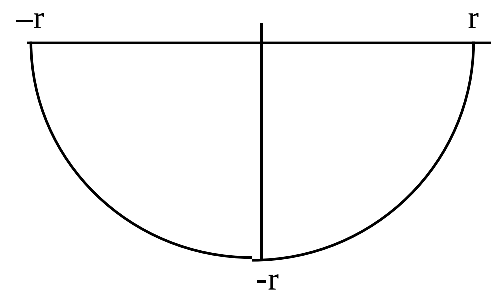
$$x^2+y^2=r^2 \quad\rightarrow\quad y^2=r^2-x^2 \quad\rightarrow\quad y=\pm\sqrt{r^2-x^2}$$



$y=\sqrt{r^2-x^2}$

Domain: $-r\le x\le r$ Range: $0\le y\le r$

Upper Semi-circular Function



$y=-\sqrt{r^2-x^2}$

Domain: $-r\le x\le r$ Range: $-r\le y\le 0$

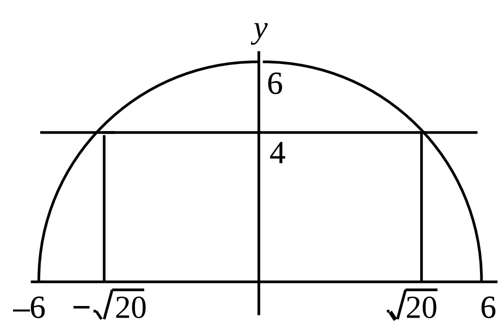
Lower Semi-circular Function

Example 2. (a) Give the equation of the upper-semicircle function centered at the origin with radius 6.
(b) Graph the function and state its domain and range.
(c) Solve for the x that make $y=4$.
(d) For which x is $y\ge 4$ and for which x is $y<4$?

Solution. (a) Using $r = 6$, the entire circle has equation $x^2+y^2=36$. So the upper-semi-circle function is $y=\sqrt{36-x^2}$.

(c) If $y=4$, then $x^2+4^2=36 \rightarrow x^2=20$

$$\rightarrow \quad x=\pm\sqrt{20}=\pm\sqrt{4\times 5}=\pm 2\sqrt{5}$$



Domain: $-6\le x\le 6$ Range: $0\le y\le 6$

(d) From the graph, we see that $y\ge 4$ when $-\sqrt{20}\le x\le\sqrt{20}$. We see that $y<4$ when $-6\le x<-\sqrt{20}$ or when $\sqrt{20}<x\le 6$.

General Equation of a Circle

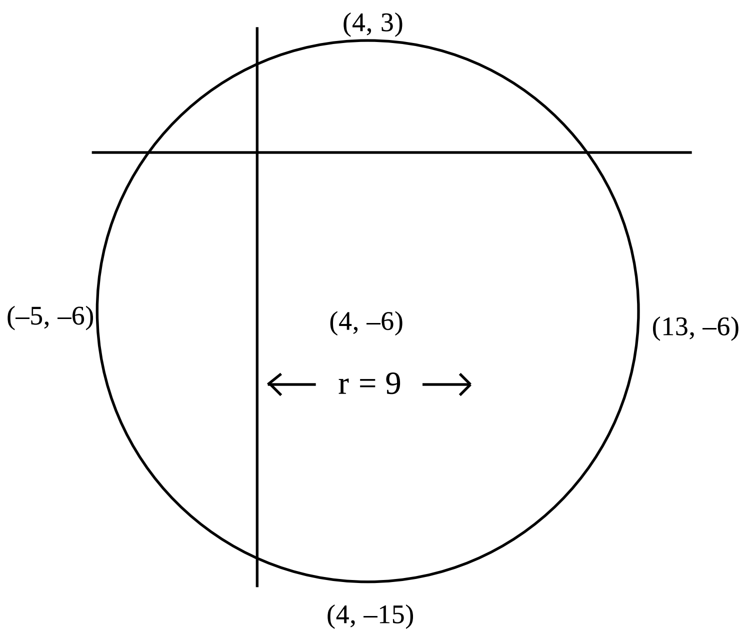
The equation of the circle with center (h, k) and radius of length r is given by

$$\boxed{(x-h)^2+(y-k)^2=r^2}$$

Example 3. (a) Give the equation of the circle having center (4, –6) and radius 9.
(b) Graph the circle by plotting the center and the 4 "directional" points on the circle.
(c) Give the function form of the upper and lower semicircle functions determined by the circle and state the domain and range for each.

Solution. Center: (4, –6) and $r = 9$ $\rightarrow$ (a) Equation: $(x-4)^2+(y+6)^2=81$

(b) To find the four "directional" points, go to the center (4, –6) then add ± 9 to the x-coordinate to obtain the points (–5, 6) and (13, 6). Then go back to the center (4, –6) and add ± 9 to the y-coordinate to obtain the points (4, 3) and (4, –15).



(c) Now we solve for y to obtain the two semi-circle functions:

$$(x-4)^2+(y+6)^2=81 \rightarrow (y+6)^2=81-(x-4)^2 \rightarrow y+6=\pm\sqrt{81-(x-4)^2}$$

$$\rightarrow y=\pm\sqrt{81-(x-4)^2}-6$$

Upper Semi-Circle Function	**Lower Semi-Circle Function**
$y=\sqrt{81-(x-4)^2}-6$	$y=-\sqrt{81-(x-4)^2}-6$
Domain: $-5\le x\le 13$ Range: $-6\le y\le 3$	Domain: $-5\le x\le 13$ Range: $-15\le y\le -6$

Exercises

1. Give the equation of the circle centered at (0, 0) with radius $5/3$. Give the coordinates of the points on the circle that occur when $x = -1/3$.

2. (a) Give the equation of the upper-semicircle function centered at the origin with radius 3.
(b) Graph the function and state its domain and range.
(c) Solve for the x that make $y = 2$.
(d) For which x is $y \geq 2$ and for which x is $y < 2$?

3. (a) Give the equation of the lower-semicircle function centered at the origin with radius 7.
(b) Graph the function and state its domain and range.
(c) Solve for the x that make $y = -3$.
(d) For which x is $y < -3$ and for which x is $y \geq -3$?

4. (a) Find the equation of the circle with center (8, –2) and radius 4.
(b) Graph the circle by plotting the center and the 4 directional points on the circle.
(c) Give the function form of the upper semicircle and state the domain and range.
(d) Give the function form of the lower semicircle and state the domain and range.

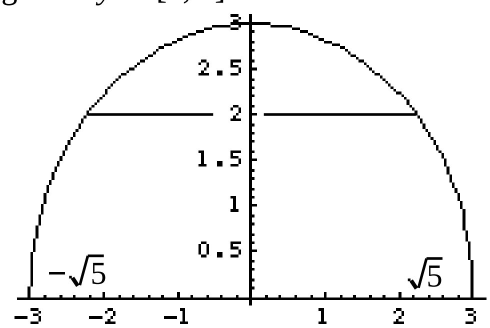
Solutions

1. Using $r = \frac{5}{3}$, the equation becomes $x^2 + y^2 = \frac{25}{9}$.

For $x = -\frac{1}{3}$, then $\left(-\frac{1}{3}\right)^2 + y^2 = \frac{25}{9} \rightarrow y^2 = \frac{25}{9} - \frac{1}{9} = \frac{24}{9} \rightarrow y = \pm\sqrt{\frac{24}{9}} = \pm\frac{\sqrt{4\times 6}}{3} = \pm\frac{2\sqrt{6}}{3}$

So the points on the circle are $\left(-\frac{1}{3}, \pm\frac{2\sqrt{6}}{3}\right)$.

2. $x^2 + y^2 = 9$; so the upper semicircle function is $y = \sqrt{9 - x^2}$ for $-3 \le x \le 3$. The range for y is [0, 3].

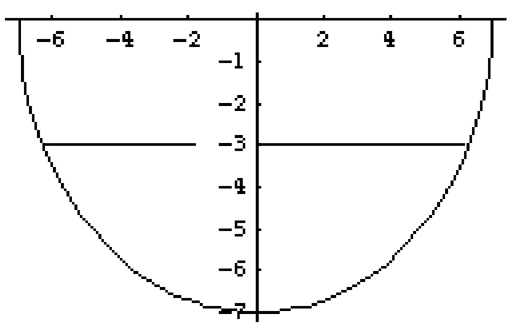


(c) To solve $y = 2$, we have $\sqrt{9 - x^2} = 2$, or $9 - x^2 = 4$. So $x^2 = 5$ and $x = \pm\sqrt{5}$.

(d) $y \ge 2$ for $-\sqrt{5} \le x \le \sqrt{5}$,

and $y < 2$ for x in $[-3, -\sqrt{5})$ or in $(\sqrt{5}, 3]$.

3. $x^2 + y^2 = 49$; so the lower semicircle is $y = -\sqrt{49 - x^2}$, for $-7 \le x \le 7$. The range for y is [–7, 0].

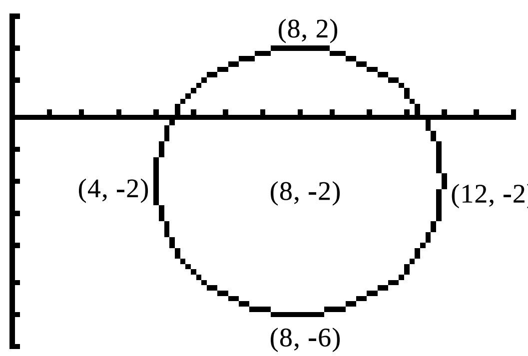


(c) To solve $y = -3$, use $-\sqrt{49 - x^2} = -3$, or $49 - x^2 = 9$. Then $x^2 = 40$ and $x = \pm\sqrt{40} = \pm 2\sqrt{10}$.

(d) $y < -3$ for $-\sqrt{40} < x < \sqrt{40}$,

and $y \ge -3$
for x in $[-7, -\sqrt{40}]$ or in $[\sqrt{40}, 7]$.

4 (a) Use $(x - h)^2 + (y - k)^2 = r^2$.
Here $(x - 8)^2 + (y + 2)^2 = 16$.



(c) $(y + 2)^2 = 16 - (x - 8)^2$; so

$$y = \sqrt{16 - (x - 8)^2} - 2, \text{ for } 4 \le x \le 12.$$

The range for the
upper-semicircle function is $-2 \le y \le 2$.

(d) $y = -\sqrt{16 - (x - 8)^2} - 2$ for $4 \le x \le 12$ with range $-6 \le y \le -2$.