Solutions

Math 1300-005 - Spring 2017

The Intermediate Value Theorem - 1/31/17

Guidelines: Please work in groups of two or three. Please show all work and clearly denote your answer.

1. Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval. Remember, in solving each problem, you must verify that each of the hypotheses of the IVT are satisfied.

   (a) $x^4 + x - 3 = 0, \quad (1, 2)$

   Let $f(x) = x^4 + x - 3$, which is continuous on $[1,2]$ b/c it is a polynomial. Since
   $f(1) = 1+1-3 < 0$
   $f(2) = 16+2-3 > 0$, 0 is between $f(1)$ & $f(2)$, so by IVT there exists c in $(1,2)$ such that $f(c)=0$, ie, $c^4+c-3=0$.

   (b) $\sqrt[3]{x} = 1 - x, \quad (0, 1)$

   Rearrange first as $\sqrt[3]{x}+x-1=0$. Let $f(x)=\sqrt[3]{x}+x-1$, which is cont. on $[0,1]$ b/c it is a root function plus a polynomial. Since
   $f(0) = -1 < 0$
   $f(1) = 1 > 0$, 0 is between $f(0)$ and $f(1)$. By the IVT there exists c in $(0,1)$ such that $f(c)=0$, ie, $\sqrt[3]{c}+c-1=0$.

   (c) $e^x = 3 - 2x, \quad (0, 1)$

   Rearrange first as $e^x+2x-3=0$. Set $f(x)=e^x+2x-3$, which is cont. on $[0,1]$ b/c it is an exponential function plus a polynomial. Since
   $f(0) = e^0+2(0)-3 = 1-3 < 0$
   $f(1) = e+2-3 = e-1 > 0$ (since $e \approx 2.7$), 0 is between $f(0)$ and $f(1)$. By the IVT, there exists c in $(0,1)$ such that $f(c)=0$, ie, $e^c+2c-3=0$.

   (d) $\sin(x) = x^2 - x, \quad (1, 2)$

   Rearrange first as $\sin(x)+x-x^2=0$. Set $f(x)=\sin(x)+x-x^2$, which is continuous on $[1,2]$ b/c it is a trig function plus a polynomial. Since
   $f(1) = \sin(1)+1-(1)^2 = \sin(1) > 0$
   $f(2) = \sin(2)+2-(2)^2 = \sin(2)-2 < 0$, 0 is between $f(1)$ and $f(2)$. By the IVT, there exists c in $(1,2)$ such that $f(c)=0$, ie, $\sin(c)+c-c^2=0$.

   To get that $\sin(1)>0$, note 1 is between 0 and $\frac{\pi}{2}$ and $\sin(x)>0$ on $(0,\frac{\pi}{2})$ [draw a graph]. $\leftarrow$

   To get $\sin(2)-2<0$, note $-1 \le \sin(2) \le 1$, so at most $\sin(2)=1$, and $1<2$, so $\sin(2)-2<0$

The following problems are review of the material we covered Monday 1/30 over the definition of continuity.

2. State the interval(s) where the following function is continuous.

$$f(x) = \begin{cases} \cos(x) & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 - x^2 & \text{if } x > 0 \end{cases}$$

We need only check continuity at $a=0$.

① $f(0)$ is defined and $f(0)=0$.

② $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \cos(x) = \cos(0) = 1$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 1-x^2 = 1-(0)^2 = 1$ $\rightarrow$ so $\lim_{x \to 0} f(x) = 1$.

③ Since $\lim_{x \to 0} f(x) \neq f(0)$, f is not continuous at 0. Hence

$\boxed{f \text{ is continuous on } (-\infty, 0) \cup (0, \infty)}$

3. For what value of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ e^{2x+c} & \text{if } x \geq 0 \end{cases}$$

We need only check continuity at $a=0$.

① $f(0)$ is defined and $f(0) = e^{2(0)+c} = e^c$

② $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x+2) = 2$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} e^{2x+c} = e^c$

We need ~~RHL = LHL, so we~~ RHL = LHL = $f(0)$, so we need $2 = e^c$ or $\boxed{c = \ln(2)}$