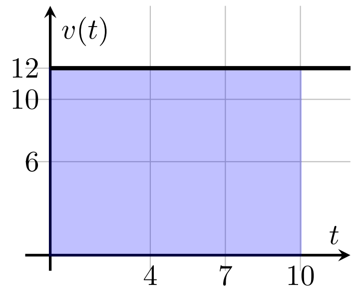
1. A girl is running at a velocity of 12 feet per second for 10 seconds, as shown in the velocity graph below.



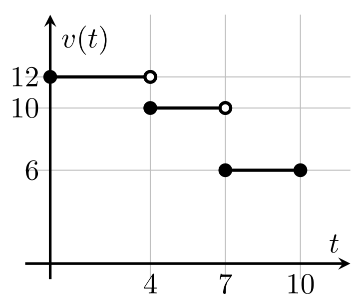
How far does she travel during this time?

Solution: 120 feet

This distance can be depicted graphically as a rectangle. Shade such a rectangle and explain why it gives the distance.

Solution: The area is found by multiplying an x-axis value by a y-axis value. In this case, we multiply 10 ft/sec by 12 sec, which gives $\dfrac{10\text{ ft}}{1\,\cancel{\text{sec}}} \cdot 12\,\cancel{\text{sec}} = 120\text{ ft}$

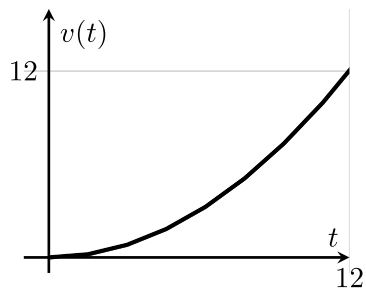
2. Now the girl changes her velocity as she runs. Her velocity graph is approximately as shown:



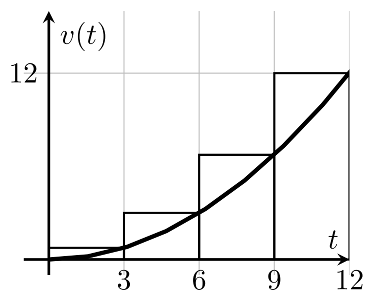
How far does she travel this time?

Solution: 12 ft/sec $\cdot$ 4 sec + 10 ft/sec $\cdot$ 3 sec + 6 ft/sec $\cdot$ 3 ft/sec=96 ft.

3. This time she starts off slowly and speeds up.



The velocity is given by $v(t) = \frac{t^2}{12}$ (time in seconds, velocity in ft/sec). We can no longer exactly find the distance travelled using areas of rectangles. But we can estimate it using areas of rectangles.

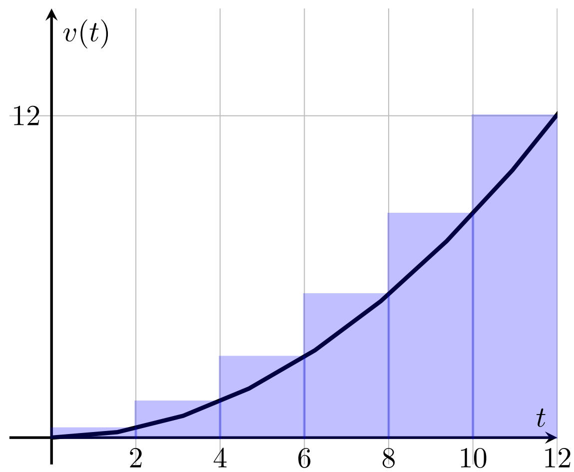


Find her velocity at time $t = 3, 6, 9, 12$ and use it to estimate her distance travelled in the first 12 seconds. In your answer to this problem **use fractions, not decimals.**

Solution: Using $v(t) = \frac{t^2}{12}$, $v(3) = 3/4$, $v(6) = 3$, $v(9) = 27/4$, and $v(12) = 12$. Though her velocity is always changing, we can use each of these velocities as an estimate of the velocities on the entire interval. We'll estimate her velocity as 3/4 ft/sec for $t = 0$ to $t = 3$, 3 ft/sec for $t = 3$ to $t = 6$, 27/4 ft/sec for $t = 6$ to $t = 9$, and 12 ft/sec for $t = 9$ to $t = 12$. This gives us a distance of

$$\frac{3}{4}\text{ ft/sec} \cdot 3\text{ sec} + 3\text{ ft/sec} \cdot 3\text{ sec} + \frac{27}{4}\text{ ft/sec} \cdot 3\text{ sec} + 12\text{ ft/sec} \cdot 3\text{ sec} = \frac{135}{2} = 67.5\text{ ft.}$$

4. Now, for the same velocity function $v(t) = \frac{t^2}{12}$, get a better estimate of how far she travelled using $n = 6$ rectangles. Draw a graph showing the areas, and use their areas to estimate her distance travelled in the first 12 seconds. Again **use fractions, not decimals.**



Solution:

$$\text{Area} \approx \frac{12}{6}\left(\frac{(1\cdot 2)^2}{12} + \frac{(2\cdot 2)^2}{12} + \frac{(3\cdot 2)^2}{12} + \frac{(4\cdot 2)^2}{12} + \frac{(5\cdot 2)^2}{12} + \frac{(6\cdot 2)^2}{12}\right)$$
$$\approx \frac{182}{3} = 60\frac{2}{3}\text{ ft.}$$

5. Now we will estimate the area when there are $n = 37$ rectangles. **Use fractions, not decimals.**

   (a) Width of each rectangle:

   Solution: $\frac{12}{37}$

   (b) List of right-hand endpoint of each rectangle:

   $12/37$, $2\cdot 12/37$, $3\cdot 12/37$, ..., $36\cdot 12/37$, $37\cdot 12/37$

   (c) List of heights of each rectangle:

   $\frac{(1\cdot 12/37)^2}{12}$, $\frac{(2\cdot 12/37)^2}{12}$, $\frac{(3\cdot 12/37)^2}{12}$, ..., $\frac{(36\cdot 12/37)^2}{12}$, $\frac{(37\cdot 12/37)^2}{12}$

   (d) List of areas of rectangles:

   $\frac{12}{37}\cdot\frac{(1\cdot 12/37)^2}{12}$, $\frac{12}{37}\cdot\frac{(2\cdot 12/37)^2}{12}$, $\frac{12}{37}\cdot\frac{(3\cdot 12/37)^2}{12}$, ..., $\frac{12}{37}\cdot\frac{(36\cdot 12/37)^2}{12}$, $\frac{12}{37}\cdot\frac{(37\cdot 12/37)^2}{12}$

   (e) Sum of all areas:

   $\frac{12}{37}\cdot\frac{(1\cdot 12/37)^2}{12} + \frac{12}{37}\cdot\frac{(2\cdot 12/37)^2}{12} + \frac{12}{37}\cdot\frac{(3\cdot 12/37)^2}{12} + \ldots + \frac{12}{37}\cdot\frac{(36\cdot 12/37)^2}{12} + \frac{12}{37}\cdot\frac{(37\cdot 12/37)^2}{12}$

6. Now we will figure out the estimate when there are an arbitrary number of rectangles, or n rectangles.

   (a) Width of each rectangle:

   Solution: $\frac{12}{n}$

   (b) List of right-hand endpoint of each rectangle:

   $1 \cdot 12/n$, $2 \cdot 12/n$, $3 \cdot 12/n$,..., $(n-1) \cdot 12/n$, $n \cdot 12/n$

   (c) List of heights of each rectangle:

   $\frac{(1 \cdot 12/n)^2}{12}$, $\frac{(2 \cdot 12/n)^2}{12}$, $\frac{(3 \cdot 12/n)^2}{12}$,..., $\frac{((n-1) \cdot 12/n)^2}{12}$, $\frac{(n \cdot 12/n)^2}{12}$

   (d) List of areas of rectangles:

   $\frac{12}{n} \cdot \frac{(1 \cdot 12/n)^2}{12}$, $\frac{12}{n} \cdot \frac{(2 \cdot 12/n)^2}{12}$, $\frac{12}{n} \cdot \frac{(3 \cdot 12/n)^2}{12}$,..., $\frac{12}{n} \cdot \frac{((n-1) \cdot 12/n)^2}{12}$, $\frac{12}{n} \cdot \frac{(n \cdot 12/n)^2}{12}$

   (e) Sum of all areas:

   $\frac{12}{n} \cdot \frac{(1 \cdot 12/n)^2}{12} + \frac{12}{n} \cdot \frac{(2 \cdot 12/n)^2}{12} + \frac{12}{n} \cdot \frac{(3 \cdot 12/n)^2}{12} + \dots + \frac{12}{n} \cdot \frac{((n-1) \cdot 12/n)^2}{12} + \frac{12}{n} \cdot \frac{(n \cdot 12/n)^2}{12}$

7. Manipulate the sum algebraically until it is of the form:

$$\textit{stuff} \cdot (1 + 4 + 9 + ... + n^2).$$

Solution:

$$\begin{aligned}
\text{Sum} &= \frac{12}{n} \cdot \frac{(1 \cdot 12/n)^2}{12} + \frac{12}{n} \cdot \frac{(2 \cdot 12/n)^2}{12} + \frac{12}{n} \cdot \frac{(3 \cdot 12/n)^2}{12} + ... + \frac{12}{n} \cdot \frac{(n \cdot 12/n)^2}{12} \\
&= \frac{12}{n} \cdot \frac{(12/n)^2}{12} \cdot (1^2 + 2^2 + 3^2 + ... + n^2) \\
&= \frac{12^2}{n^3} \cdot (1^2 + 2^2 + 3^2 + ... + n^2)
\end{aligned}$$

8. Simplify further by substituting $1 + 4 + 9 + ... + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ into your answer above. Check that it gives the same answer for $n = 6$ that you got in problem 4.

Solution:

$$\frac{12^2}{n^3} \cdot (1^2 + 2^2 + 3^2 + ... + n^2) = \frac{12^2}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{24}{n^2} \cdot (n+1)(2n+1)$$

For $n = 6$ this gives $\dfrac{24}{36} \cdot (6+1)(2 \cdot 6 + 1) \approx 60.667$ which is the same answer we got above.

9. As n approaches infinity we find her exact distance travelled (the exact area under the curve). Take the limit as n goes to infinity for your answer to the previous problem.

Solution:

$$\lim_{n \to \infty} \frac{24(n+1)(2n+1)}{n^2} = \lim_{n \to \infty} \frac{48n^2 + 72n + 24}{n^2} = 48$$

Notice that you just found the area inside a region with a curved edge!