PART 1 SOLUTIONS

1. Sketch a graph of a function $y = g(x)$ that satisfies all of the following conditions. No explanation is necessary.

   (a) $\lim_{x\to\infty} g(x) = 3$ (c) $\lim_{x\to 4^+} g(x) = \infty$ (e) g is an odd function.

   (b) $g(-2) = g(2) = 0$ (d) $\lim_{h\to 0} \frac{g(0+h) - g(0)}{h} = 1$ (f) $\lim_{x\to 4^-} g(x) = -\infty$

Solution:

2. (a) What is the definition of continuity? In other words, what conditions must a function obey in order to be considered continuous at some point a.

   (b) Find the numbers at which the function below is discontinuous. Use the definition of continuity from part (a) to justify your answers.

$$f(x) = \begin{cases} x+2 & \text{if } x < 0 \\ 2x^2 & \text{if } 0 \le x \le 1 \\ 2-x & \text{if } x > 1 \end{cases}$$

Solution:

(a)In order for a function to be continuous at a point a it must obey the condition, $\lim_{x\to a} f(x) = f(a)$. This means that $f(a)$ is defined, $\lim_{x\to a} f(x)$ exists, and $\lim_{x\to a} f(x) = f(a)$.

(b) f is continuous on $(-\infty, 0), (0, 1)$, and $(1, \infty)$ because on each of these intervals, f is a polynomial.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x+2) = 2$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 2x^2 = 0$$

So f is discontinuous at $x = 0$ since the limit does not exist at $x = 0$.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2x^2 = 2$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2-x) = 1$$

So f is discontinuous at $x = 1$, because the limit does not exist at $x = 1$.

3. Find y'' by implicit differentiation:

$$x^3 + y^3 = 1$$

Solution:

$$x^3 + y^3 = 1 \implies 3x^2 + 3y^2y' = 0 \implies y' = -\frac{x^2}{y^2}$$

$$y'' = -\frac{y^2(2x) - x^2 \cdot 2yy'}{(y^2)^2} = -\frac{2xy^2 - 2x^2y(-x^2/y^2)}{y^4} = -\frac{2xy^4 + 2x^4y}{y^6} = -\frac{2xy(y^3 + x^3)}{y^6} = -\frac{2x}{y^5}$$

since x and y must satisfy the original equation, $x^3 + y^3 = 1$,

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4. Boyle's Law states that when a sample of gas is compressed at a constant temperature, the pressure P and volume V satisfy the equation $PV = C$, where C is a constant. Suppose that at a certain instant the volume is $600cm^3$, the pressure is 150 kPa, and the pressure is increasing at a rate of 20 kPa/min. At what rate is the volume decreasing at this instant?

Solution:
Differentiating both sides of $PV = C$ with respect to t and using the product Rule gives us $P\frac{dV}{dt} + V\frac{dP}{dt} = 0$.

$$\implies \frac{dV}{dt} = -\frac{V}{P}\frac{dP}{dt}$$

When $V = 600$, $P = 150$ and $\frac{dP}{dt} = 20$, then we have

$$\frac{dV}{dt} = -\frac{600}{150}(20) = -80$$

Thus, the volume is decreasing at a rate of $80cm^3/min$.

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5. Consider the number: $(1.999)^4$

   (a) What function, $f(x)$, could be used to create a linear approximation to estimate this number? What a value would be used?

   (b) What is the linear approximation, $L(x)$?

   (c) Use this linear approximation to estimate $(1.999)^4$. [You do not need to simplify all the way.]

Solution:
(a) We should find the linearization of $f(x) = x^4$ at $a = 2$.
(b) $f'(x) = 4x^3, f(2) = 16$, and $f'(2) = 32$. So therefore the linearization is

$$L(x) = 16 + 32(x - 2)$$

(c) Therefore, $x^4 \approx 16 + 32(x - 2)$ when x is near 2, so

$$(1.999)^4 \approx 16 + 32(1.999 - 2) = 16 - 0.032 = 15.968$$

PART 2 SOLUTIONS

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1. (15 pts) A farmer has 600 m of fencing with which she plans to enclose a rectangular divided pasture adjacent to a long existing wall. She plans to build one fence parallel to the wall, two to form the ends of the enclosure, and a fourth (parallel to the two ends of the enclosure) to divide it. What is the maximum area that she can enclose in this way?

Solution:
The constraint equation is $y + 3x = 600$, while the equation that we are trying to optimize (i.e. the Area) is $A = xy$. By solving the constraint equation for y, we have $y = 600 - 3x$, therefore the area equation as a function of x is:

$$A(x) = x(600 - 3x) = 600x - 3x^2$$

Now we look for critical points. $A'(x) = 600 - 6x = 0 \implies x = 100$. If $x = 100$, then $y = 600 - 3(100) = 300$. Therefore the maximum area that the divided pasture can enclose is

$$A(x) = 100 \cdot 300 = 30,000m^2$$

We know that this is indeed a maximum, because $A''(100) < 0 \implies x = 100$ is the x value that gives the maximum area.

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2. (a) (5 pts) The interval $[0, 3]$ is partitioned into n subintervals of equal length. Express the integral $\int_0^3 (3x^2+1)dx$ as the limit of a Riemann sum using the right-hand endpoints of each subinterval.

   (b) (5 pts) Given that $a < b$, what values of a and b minimize the value of $\int_a^b ((t^2 + t)(t^2 - 4t + 4))dt$?

   (c) (5 pts) Solve the initial value problem: $\frac{dy}{dx} = x^3 - x$ with $y(1) = -2$ [i.e., Solve for y(x) and C].

Solution:
(a) $x_i = 0 + \Delta x \cdot i = \frac{3}{n}i$, $f(x_i) = 3x_i^2 + 1 = 3(\frac{3i}{n})^2 + 1$. Therefore we may express the integral as the limit of a Riemann sum:

$$\int_0^3 (3x^2 + 1)dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(3\left(\frac{3i}{n}\right)^2 + 1\right)\left(\frac{3}{n}\right)$$

(b) Given $a < b$, we can factor $f(t) = (t^2 + t)(t^2 - 4t + 4)$ to see where the function is positive and where it is negative.

$$f(t) = (t^2 + t)(t^2 - 4t + 4) = t(t + 1)(t - 2)(t - 2) = t(t + 1)(t - 2)^2$$

Therefore the intercepts are $t = -1, 0, 2$. $f(t)$ is positive on $(-\infty, -1) \cup (0, 2) \cup (2, \infty)$. $f(t)$ is negative on $(-1, 0)$. Therefore if we choose $a = -1$ and $b = 0$, this will minimize the value of the integral.

(c) $\dfrac{dy}{dx} = x^3 - x, y(1) = -2$. We wish to find $y(x)$, so to start we take the antiderivative:

$$y(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2 + C$$

Then we use the initial condition, $y(1) = -2$ to solve for C.

$$y(1) = \frac{1}{4} - \frac{1}{2} + C = -2 \implies C = -2 + \frac{1}{4} = -\frac{7}{4}$$

Therefore,

$$y(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2 - \frac{7}{4}$$

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3. Evaluate the following:

(a) (7 pts) $\dfrac{d}{dx}\displaystyle\int_1^{5^x} \frac{1}{\sqrt{1-t^2}}dt =?$

(b) (7 pts) $\displaystyle\int_0^{\sqrt{\ln \pi}} -\frac{1}{\sqrt{1-x^2}}dx =?$

(c) (7 pts) $\displaystyle\int \frac{\text{sech}\sqrt{x}\ \ \tanh\sqrt{x}}{\sqrt{x}}dx$

(d) (7 pts) $\displaystyle\int \sqrt{\cot x}\csc^2 x \ \ dx$

Solution:
(a) By the Fundamental Theorem of Calculus,

$$\frac{d}{dx}\int_1^{5^x} \frac{1}{\sqrt{1-t^2}}dt = \frac{1}{\sqrt{1-(5^x)^2}}(5^x \ln 5) = \frac{(\ln 5)5^x}{\sqrt{1-5^{2x}}}$$

(b)

$$\int_0^{\sqrt{\ln \pi}} -\frac{1}{\sqrt{1-x^2}}dx = -\cos^{-1}(x)|_0^{\sqrt{\ln\pi}} = -\cos^{-1}(\sqrt{\ln\pi}) + \cos^{-1}(0) = -\cos^{-1}(\sqrt{\ln\pi}) + \frac{\pi}{2}$$

(c) Let $u = \sqrt{x}, \quad du = \frac{1}{2\sqrt{x}}dx$. Then

$$\int \frac{\text{sech}\sqrt{x}\ \ \tanh\sqrt{x}}{\sqrt{x}}dx = 2\int \text{sech}u \ \ \tanh u \ \ du = -2\text{sech}u + C = -2\text{sech}\sqrt{x} + C$$

(d) Let $u = \cot x, \quad du = -\csc^2 x \ \ dx$. Then

$$\int \sqrt{\cot x}\csc^2 x \ \ dx = -\int u^{1/2} \ \ du = -\frac{2}{3}u^{3/2} + C = -\frac{2}{3}(\cot x)^{3/2} + C$$

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4. Evaluate the following:

(a) (8 pts) $\displaystyle\lim_{r\to 7} \sin^{-1}(\log_7 \sqrt{r})$

(b) (8 pts) $\displaystyle\lim_{x\to\infty} x\tan(8/x)$

(c) (8 pts) $\displaystyle\lim_{x\to 1^+} x^{1/(1-x)}$

Solution:
(a) $\displaystyle\lim_{r\to 7} \sin^{-1}(\log_7 \sqrt{r}) = \sin^{-1}(\lim_{r\to 7}\log_7 \sqrt{r} = \sin^{-1}(1/2) = \pi/6$
(b) Currently $\displaystyle\lim_{x\to\infty} x\tan(8/x)$ is of indeterminate form $\infty\cdot 0$, so we must rewrite it before L'Hospital's Rule can be applied.

$$\lim_{x\to\infty} x\tan(8/x) = \lim_{x\to\infty}\frac{\tan(8/x)}{1/x}$$

Now we have the case $\frac{0}{0}$, so we may apply L'Hostpital's Rule:

$$\lim_{x\to\infty}\frac{\tan(8/x)}{1/x} = \lim_{x\to\infty}\frac{\sec^2(8/x)\left(-\frac{8}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} 8\sec^2(8/x) = 8(1) = 8$$

(c) Let $y = x^{1/(1-x)} \implies \ln y = \frac{1}{1-x}\ln x$. Then

$$\lim_{x\to 1^+}\ln y = \lim_{x\to 1^+}\frac{\ln x}{1-x} = \lim_{x\to 1^+}\frac{1/x}{-1} = -1$$

$$\implies \lim_{x\to 1^+} x^{1/(1-x)} = \lim_{x\to 1^+} e^{\ln y} = \lim_{x\to 1^+} e^{-1} = \frac{1}{e}$$

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5. Let $f(x) = \frac{\cosh x}{e^x}$.

(a) (4 pts) Simplify $f(x)$ using the definition of $\cosh x$.

(b) (4 pts) Find the value of $f(\ln 3)$.

(c) (5 pts) Is f increasing or decreasing at $x = \ln 3$?

(d) (5 pts) Is f concave up or down at $x = \ln 3$?

Solution:

(a)

$$\cosh x = \frac{e^x + e^{-x}}{2} \implies f(x) = \frac{\cosh x}{e^x} = \frac{e^x + e^{-x}}{2e^x} = \frac{1}{2} + \frac{e^{-2x}}{2}$$

(b)

$$f(\ln 3) = \frac{1}{2} + \frac{e^{-2\ln 3}}{2} = \frac{1}{2} + \frac{e^{\ln 3^{-2}}}{2} = \frac{1}{2} + \frac{1/9}{2} = \frac{5}{9}$$

(c) $f'(x) = \frac{1}{2}(-2)e^{-2x} = -e^{-2x}$ Therefore,

$$f(\ln 3) = -e^{-2\ln 3} = -e^{\ln 3^{-2}} = -\frac{1}{9} < 0$$

Therefore, f is decreasing at $x = \ln 3$.

(d) $f''(x) = 2e^{-2x}$ Therefore,

$$f''(\ln 3) = 2e^{-2\ln 3} = 2e^{\ln 3^{-2}} = 2\left(\frac{1}{9}\right) = \frac{2}{9}$$

Therefore, f is concave up at $x = \ln 3$.