1. Consider the curve $y = \dfrac{x-2}{x+2}$.

   (a) (5 pts) Find dy/dx. Simplify your answer.

   (b) (8 pts) Find the (x, y) point(s) on the curve where the tangent line is perpendicular to $x + 4y + 12 = 0$.

   (c) (8 pts) Use a linearization of y to estimate the value of $-\dfrac{2.03}{1.97}$.

   (d) (7 pts) The Mean Value Theorem can be applied to this function on the interval $[a, 2]$.

   i. For what values of a will the hypotheses be met?

   ii. The theorem states that there will be a number c in $(a, 2)$ such that $y'(c)$ equals some expression. Write the expression in terms of a and simplify your answer.

Solution: Parts (a) and (b) of this problem were adapted from Written HW 2.4.48.

(a) $y = \dfrac{x-2}{x+2} \Rightarrow y' = \dfrac{(x+2)-(x-2)}{(x+2)^2} = \boxed{\dfrac{4}{(x+2)^2}}$

(b) The given line has slope $-1/4$ so the perpendicular slope is 4. Solve $y' = 4$ for the points of tangency.

$$y' = \frac{4}{(x+2)^2} = 4 \Rightarrow (x+2)^2 = 1 \Rightarrow x = -3, -1.$$

The points of tangency are $\boxed{(-3, 5)}$ and $\boxed{(-1, -3)}$.

(c) Let the center $a = 0$. Since $y(0) = -1$ and $y'(0) = 1$, the linearization at $a = 0$ is

$$L(x) = y(0) + y'(0)(x - 0) = -1 + x.$$

Then $-\dfrac{2.03}{1.97} = y(-0.03) \approx L(-0.03) = -1 - 0.03 = \boxed{-1.03}$.

(d) i. The function will be continuous on $[a, 2]$ and differentiable on $(a, 2)$ if $\boxed{-2 < a < 2}$.

ii. There will be a number c in $(a, 2)$ such that

$$f'(c) = \frac{f(2) - f(a)}{2 - a} = \frac{0 - \frac{a-2}{a+2}}{2 - a} = \boxed{\frac{1}{a+2}}.$$

2. (12 pts) Find the (x, y) points where the function $y = 2\sin x + \sin^2 x$ on the interval $[0, 2\pi]$ attains its absolute maximum and minimum values.

   Solution: This problem was adapted from WebAssign HW 3.1.6.

   First find the critical numbers.

$$f(\theta) = 2\sin\theta + \sin^2\theta \;\Rightarrow\; f'(\theta) = 2\cos\theta + 2\sin\theta\cos\theta.$$

   Solve $f'(\theta) = 0$.

$$(2\cos\theta)(1 + \sin\theta) = 0 \;\Rightarrow\; \cos\theta = 0 \text{ or } \sin\theta = -1.$$

   In the interval $[0, 2\pi]$, the critical numbers are $\theta = \pi/2, 3\pi/2$. Now compare the critical points and endpoints.

$$\begin{array}{ll} f(\pi/2) = 3 & \qquad\qquad f(0) = 0 \\ f(3\pi/2) = -1 & \qquad\qquad f(2\pi) = 0 \end{array}$$

   The function has an absolute maximum value at $\boxed{(\pi/2, 3)}$ and an absolute minimum value at $\boxed{(3\pi/2, -1)}$.

3. (12 pts) Find all asymptotes (horizontal, vertical and slant) for $y = \dfrac{3x^2}{x+2}$. Justify your answer with the appropriate limits.

   Solution: This problem was adapted from WebAssign HW 3.4.3.

   There is a vertical asymptote at $\boxed{x = -2}$: $\displaystyle\lim_{x\to -2^+} \frac{3x^2}{x+2} = \infty$ since $3x^2 \to 12$ and $x + 2 \to 0$ with positive values.

   There is a slant asymptote at $\boxed{y = 3x - 6}$:

$$\begin{array}{r l}
 & \;\; 3x \quad - 6 \\ \cline{2-2}
x + 2 \big) & \;\; 3x^2 \\
 & -\,3x^2 - 6x \\ \cline{2-2}
 & \qquad -\,6x \\
 & \qquad\;\; 6x + 12 \\ \cline{2-2}
 & \qquad\qquad 12
\end{array}$$

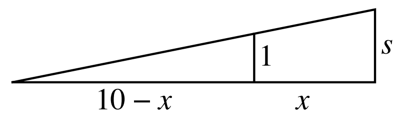
   The function $y = \dfrac{3x^2}{x+2} = 3x - 6 + \dfrac{12}{x+2}$. Verify that $\displaystyle\lim_{x\to\infty} (f(x) - (mx + b)) = 0$.

$$\lim_{x\to\infty} \left(3x - 6 + \frac{12}{x+2} - (3x - 6)\right) = \lim_{x\to\infty} \frac{12}{x+2} = 0$$
$$\lim_{x\to-\infty} \left(3x - 6 + \frac{12}{x+2} - (3x - 6)\right) = \lim_{x\to-\infty} \frac{12}{x+2} = 0$$

   Because there is a slant asymptote, there are no horizontal asymptotes.

4. (15 pts) A spotlight on the ground shines on a wall 10 m away. If a young girl, 1 meter tall, walks from the spotlight toward the building at a speed of $\frac{4}{5}$ m/s, how fast is the length of her shadow on the building decreasing when she is 2 m from the building? Simplify your answer. (Be sure to draw a diagram and clearly label all quantities.)

Solution: This problem was adapted from WebAssign HW 2.7.7.

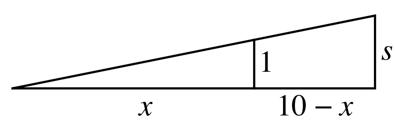


Find ds/dt when $x = 2$ given $dx/dt = -4/5$. Use similar triangles to set up a proportion. At that moment $\frac{1}{8} = \frac{s}{10} \Rightarrow s = \frac{5}{4}$.

$$\begin{aligned} \frac{1}{10-x} &= \frac{s}{10} \\ 10s - sx &= 10 \\ 10\frac{ds}{dt} - \left(s\frac{dx}{dt} + x\frac{ds}{dt}\right) &= 0 \\ 10\frac{ds}{dt} - \left(\frac{5}{4}\left(-\frac{4}{5}\right) + 2\frac{ds}{dt}\right) &= 0 \\ \frac{ds}{dt} &= -\frac{1}{8} \text{ m/s} \end{aligned}$$

The shadow is decreasing at $\boxed{\frac{1}{8} \text{ m/s}}$.

Alternate Solution:



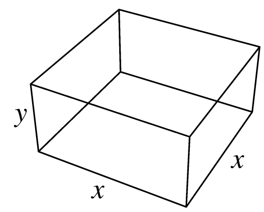
Find ds/dt when $x = 8$ given $dx/dt = 4/5$. Use similar triangles to set up a proportion. At that moment $\frac{1}{8} = \frac{s}{10} \Rightarrow s = \frac{5}{4}$.

$$\begin{aligned} \frac{1}{x} &= \frac{s}{10} \\ sx &= 10 \\ s\frac{dx}{dt} + x\frac{ds}{dt} &= 0 \\ \frac{5}{4} \cdot \frac{4}{5} + 8\frac{ds}{dt} &= 0 \\ \frac{ds}{dt} &= -\frac{1}{8} \text{ m/s} \end{aligned}$$

The shadow is decreasing at $\boxed{\frac{1}{8} \text{ m/s}}$.

5. (15 pts) A rectangular storage container with an open top and square base is to have a volume of $\frac{9}{4}$ m^3. Material for the base costs \$4 per square meter. Material for the sides costs \$3 per square meter. Find the dimensions of the cheapest such container. (Be sure to draw a diagram and clearly label all quantities.)

Solution: This problem was adapted from Written HW 3.5.14.



Let x equal the side length of the square base and y equal the height of the container. We are given that the volume is $V = x^2y = 9/4 \Rightarrow y = 9/(4x^2)$. We wish to minimize the cost C.

$$C = 4(x^2) + 3(4xy) = 4x^2 + 12xy = 4x^2 + 12x\left(\frac{9}{4x^2}\right)$$
$$C = 4x^2 + \frac{27}{x}$$
$$C' = 8x - \frac{27}{x^2}$$

Solve $C' = 0$.

$$8x = \frac{27}{x^2} \Rightarrow x^3 = \frac{27}{8} \Rightarrow x = \frac{3}{2}$$

Since $C'' = 8 + 54/x^3 > 0$ for $x > 0$, there is a minimum value at $x = 3/2,\ \ y = 9/(4x^2) = 1$.

The optimal dimensions are $\boxed{3/2 \times 3/2 \times 1 \text{ m}}$.

6. (18 pts) The graph of the **derivative** $f'(x)$ of some function $f(x)$ is shown below. Assume that $f(x)$ is continuous on the interval $[0, 9]$. Answer the following questions concerning the function $f(x)$. If the answer to any question is "none", write "none". For parts (a) through (d), no justification is required.

   (a) What are the critical numbers of f on $(0, 9)$?

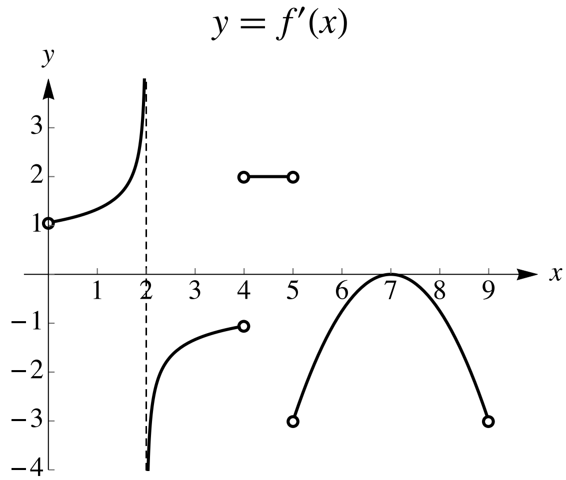
   (b) What are the x-coordinates of the local maximum values of f?

   (c) On what intervals is f concave up?

   (d) What are the x-coordinates of the inflection points of f?

   (e) If $f(0) = -3$ and $f(9) = -5$, is there a value of c in $(0, 9)$ such that $f'(c) = \dfrac{f(9) - f(0)}{9}$? Justify your answer.

   (f) If the linearization of f at $a = 4.5$ is used to approximate the value of $f(4.4)$, would the approximation be an underestimate, overestimate, or neither? Explain.



Solution: Parts (a) to (d) of this problem were adapted from WebAssign HW 3.3.7

(a) $f' = 0$ or is undefined at $x = \boxed{2, 4, 5, 7}$.

(b) By the first derivative test, there are local maximum values at $x = \boxed{2, 5}$.

(c) f is concave up where the slope of f' is positive on $\boxed{(0, 2), (2, 4), (5, 7)}$.

(d) There is an inflection point where the concavity of f changes from concave up to concave down at $x = \boxed{7}$.

(e) Yes, the graph of f' shows two values of c in $(6, 8)$ where $f'(c) = (f(9) - f(0))/9 = -2/9$. (Note that it is not necessary for the function to satisfy the hypotheses of the Mean Value Theorem.)

(f) Neither. Because f is linear on $(4, 5)$, the linearization at $a = 4.5$ would produce an exact value for $f(4.4)$.