1. (15 pts) Given that $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$, and $\displaystyle\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$,

use the <u>limit definition of the integral</u>, with **left** endpoints and an equally spaced partition, to find the exact value of the definite integral $\displaystyle\int_0^1 x^2\,dx$. Show all work.

Solution: Note that here $\Delta x = \dfrac{1}{n}$ and $x_i^* = x_{i-1} = \dfrac{i-1}{n}$ and so

$$\int_0^1 x^2\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i-1}{n}\right)^2\frac{1}{n}$$

and

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i-1}{n}\right)^2\frac{1}{n} = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{i^2-2i+1}{n^2}\right]\frac{1}{n} = \lim_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n} i^2 - \frac{2}{n^3}\sum_{i=1}^{n} i + \frac{1}{n^3}\sum_{i=1}^{n} 1$$

thus,

$$\begin{aligned}\int_0^1 x^2\,dx &= \lim_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n} i^2 - \frac{2}{n^3}\sum_{i=1}^{n} i + \frac{1}{n^3}\sum_{i=1}^{n} 1\\ &= \lim_{n\to\infty}\frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{2}{n^3}\cdot\frac{n(n+1)}{2} + \frac{n}{n^3}\\ &= \lim_{n\to\infty}\frac{2n^3 + (\text{l.o.t})}{6n^3} - \frac{2n^2+2n}{2n^3} + \frac{n}{n^3} \overset{DOP}{=} \frac{2}{6} = \boxed{\frac{1}{3}}\end{aligned}$$

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2. (25 pts) The following parts are **<u>not related</u>**:

(a) Evaluate the integrals:

(i) (6 pts) $\displaystyle\int \frac{\sec^2(4+1/x^2)}{4x^3}\,dx$ (ii) (6 pts) $\displaystyle\int_{-2}^{2} |x^3+x|\,dx$ (iii) (6 pts) $\displaystyle\int_1^{\sqrt{2}} 2x^3\sqrt{x^2-1}\,dx$

(b) (7 pts) Find $\displaystyle\int_0^{\pi} f'(x)\,dx$ if the antiderivative of $f(x)$ satisfies $F'(x)+\sin(x)-4x-1=0$, $F(0)=1$.

Solution:

(a) (i) Let $u = 4+1/x^2$, then $du = -2/x^3\,dx$.

$$\begin{aligned}\int \frac{\sec^2(4+1/x^2)}{4x^3}\,dx &= -\frac{1}{8}\int \sec^2 u\,du\\ &= -\frac{1}{8}\tan u + C\\ &= \boxed{-\frac{1}{8}\tan(4+1/x^2)+C}\end{aligned}$$

(a) (*ii*) First we notice that the integrand is an even function. So we have

$$\begin{aligned}
\int_{-2}^{2} |x^3 + x|\, dx &= 2\int_0^2 |x^3 + x|\, dx \\
&= 2\int_0^2 (x^3 + x)\, dx \\
&= 2\left(\frac{x^4}{4} + \frac{x^2}{2}\right)\Bigg|_0^2 = 2\left(\frac{2^4}{4} + \frac{2^2}{2}\right) - 0 = 2(4+2) = \boxed{12}
\end{aligned}$$

(a) (*iii*) Let $u = x^2 - 1$, then $du = 2x\, dx$ and $x^2 = u + 1$, so

$$\int_1^{\sqrt{2}} 2x^3\sqrt{x^2-1}\, dx = \int_0^1 (u+1)\sqrt{u}\, du = \int_0^1 (u^{3/2} + u^{1/2})\, du = \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\Bigg|_0^1 = \frac{2}{3} + \frac{2}{5} = \boxed{\frac{16}{15}}$$

(b) If we denote the antiderivative of $f(x)$ as $F(x)$, then $f(x) = F'(x) = -\sin(x) + 4x + 1$ and so

$$\int_0^{\pi} f'(x)\, dx = f(x)\Bigg|_0^{\pi} = -\sin(x) + 4x + 1\Bigg|_0^{\pi} = (4\pi + 1) - 1 = \boxed{4\pi}$$

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3. (20 pts) We wish to construct a box with a square base and an open top, let x denote the length of the edge of the square base of this box and let h denote the height. If 1200 cm^2 of material is available to make this box then:

(a) (10 pts) Find the dimensions of the box that give the largest possible volume of the box.

(b) (3 pts) Justify that your answer in part (a) does indeed yield a maximum volume.

(c) (3 pts) Suppose it costs 10 cents per cm^2 to construct the base of the box and 5 cents per cm^2 to construct the sides of the box. Write down a function, in terms of x and h, that describes how much it costs (in dollars) to construct the box.

(d) (4 pts) Now, set up, **but do not solve**, an optimization problem to minimize the cost of constructing this box. Be sure to specify the mathematical function that is being optimized, specify whether this function is being minimized or maximized and specify what condition(s) need to be satisfied.

Solution:

(a) Let x be the length of the base of the box and h the height.

We wish to $\boxed{\text{maximize the volume } V = x^2h \text{ subject to the condition } x^2 + 4xh = 1200,\ x, h > 0.}$

Note that $x^2 + 4xh = 1200$ implies $h = \dfrac{1200 - x^2}{4x}$ and so

$$V = x^2h = x^2\left(\frac{1200 - x^2}{4x}\right) = 300x - \frac{x^3}{4} \text{ and } V'(x) = 300 - \frac{3x^2}{4}$$

and so $V'(x) = 0$ and $x > 0$ implies $x = \sqrt{400} = 20$. Note that $V''(20) = -\dfrac{6\cdot(20)}{4} < 0$ and, so, by the 2nd Derivative Test, we have a maximum at $x = 20$, so the dimensions that yield the maximum volume are $\boxed{x \times x \times h = 20 \times 20 \times 10.}$

(b) We can use the 1st Derivative Test or the 2nd Derivative Test to justify our answer, see work done in part (a).

(c) The cost function, in dollars, to construct such a box is

$$\boxed{C(x,h) = (0.10)x^2 + (0.05)4xh = \frac{x^2}{10} + \frac{xh}{5}}$$

(d) We wish to $\boxed{\text{minimize } C(x,h) = (.10)x^2 + (0.05)4xh \text{ subject to } x^2 + 4xh = 1200,\ x, h > 0.}$

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4. (20 pts) The following parts are **not related**:

(a) Let $f(x) = x\int_0^x \cos(t)\,dt$, then:

(*i*) (5 pts) Find $f'(\pi)$.

(*ii*) (5 pts) Show that for nonzero x, we have $\dfrac{f(x)}{x} - f'(x) = -x\cos x$.

(b) (10 pts) Find $g(2)$ if $\int_0^x g(t)dt = x^2(1+x)$.

Solution:

(a) (*i*) Compute the derviative using the product rule,

$$f'(x) = \frac{d}{dx}\left(x\int_0^x \cos t dt\right) = \int_0^x \cos(t)\,dt + x\cos x = \sin(x) + x\cos x$$

and so $\boxed{f'(\pi) = 0 + \pi\cdot(-1) = -\pi.}$

(a) (*ii*) Using the dervative from part (*i*), we have

$$\begin{aligned}\frac{f(x)}{x} - f'(x) &= \frac{x\int_0^x \cos(t)\,dt}{x} - \left(\int_0^x \cos(t)\,dt + x\cos x\right)\\ &= \int_0^x \cos(t)\,dt - \int_0^x \cos(t)\,dt - x\cos x = -x\cos x\end{aligned}$$

(b) Taking the derivative of both sides yields,

$$\frac{d}{dx}\int_0^x f(t)dt = \frac{d}{dx}x^2(1+x) \Longrightarrow f(x) = 2x(1+x) + x^2 = 2x + 3x^2$$

and so $\boxed{f(2) = 4 + 12 = 16.}$

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5. (20 pts, 5 ea.) Answer either **Always True** or **False**. Do *NOT* justify your answer. Do *NOT* abbreviate your answer.

(a) By the property of integrals, we have $1 \leq \int_{-1}^{1} \frac{dx}{1+x^2} \leq 2$.

(b) Using Riemann sums, one can show that $\lim_{n\to\infty} \frac{1}{n^5}\left[1^4 + 2^4 + 3^4 + 4^4 + \cdots + n^4\right] = \frac{1}{5}$

(c) We can show that $\int_0^2 \sqrt{4-x^2}\,dx - \int_0^1 \sqrt{1-x^2}\,dx = \frac{3\pi}{4}$

(d) Using Newtons Method, we can derive the algorithm

$$x_{n+1} = \frac{5}{6}x_n + \frac{5}{3x_n^5},\ n = 1, 2, 3, \ldots$$

for approximating $\sqrt[6]{10}$.

Solution:

(a) A.T. (b) A.T. (c) A.T. (d) A.T.

Comments:

(a) Note that since $\frac{1}{2} \leq \frac{1}{1+x^2} \leq 1$ for $-1 \leq x \leq 1$, we have $1 \leq \int_{-1}^{1} \frac{dx}{1+x^2} \leq 2$.

(b) Note that using an equal partition of $[0, 1]$ with right endpoints, we get

$$\lim_{n\to\infty} \frac{1}{n^5} \sum_{i=1}^{n} i^4 = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^4 \frac{1}{n} = \int_0^1 x^4\, dx = \frac{x^5}{5}\bigg|_0^1 = \frac{1}{5}$$

(c) Note that by interpreting each integral geometrically, we have

$$\int_0^2 \sqrt{4-x^2}\, dx - \int_0^1 \sqrt{1-x^2}\, dx = \frac{\pi \cdot 2^2}{4} - \frac{\pi \cdot 1^2}{4} = \frac{3\pi}{4}$$

where we used the area formula for a circle from geometry to evaluate the integrals.

(d) If we let $x = \sqrt[6]{10}$ then this implies $x^6 - 10 = 0$, so letting $f(x) = x^6 - 10$ and applying Newtons Method yields,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^6 - 10}{6x_n^5} = x_n - \frac{x_n^6}{6x_n^5} + \frac{10}{6x_n^5} = \frac{5}{6}x_n + \frac{5}{3x_n^5}$$
