Solutions

Math 1300-005 - Spring 2017

Applied Optimization, Pt. 1 - 4/5/17

Guidelines: Please work in groups of two or three. This will not be handed in, but is a study resource for Midterm 3.

1. Consider
$$f(x) = \frac{x}{1+x^2}$$
on the open interval $(0, \infty)$. Use the First Derivative Test for Absolute Extrema to determine whether or not f has an absolute maximum or absolute minimum on $(0, \infty)$. Be sure to include full justification.

$f'(x) = \dfrac{1(1+x^2) - x(2x)}{(1+x^2)^2} = \dfrac{1-x^2}{(1+x^2)^2}$. So $f'(x) = 0$ wherever $1 - x^2 = 0$, Hence $x = \pm 1$ are critical #'s.

$x = 1$ is only critical number in $(0, \infty)$ and a sign chart looks like

must include this full statement

So $f(1)$ is a local max and since $x = 1$ is the only critical # on $(0, \infty)$, $f(1)$ must be the absolute max

2. Consider
$$g(x) = 12x - x^3$$
on the open interval $(-\infty, 0)$. Use the First Derivative Test for Absolute Extrema to determine whether or not g has an absolute maximum or absolute minimum on $(-\infty, 0)$. Be sure to include full justification.

$g'(x) = 12 - 3x^2 = 3(4 - x^2)$. So $g'(x) = 0$ when $4 - x^2 = 0$. Hence $x = \pm 2$ are critical #'s.

$x = 2$ is the only critical number in $(-\infty, 0)$ and

So $g(-2)$ is a local min and since $x = -2$ is the only critical # on $(-\infty, 0)$, $g(-2)$ must be the absolute min.

Having practiced the first derivative test for absolute extrema in a general setting, let us now apply it to applied optimization problems. Today we will start off easy.

3. Find two numbers whose difference is 100 and whose product is a minimum.

   (a) Let the two numbers in question be denoted x and y. By the given info we know the difference of x and y is 100, so

   $$x - y = 100.$$

   This equation is called our *constraint equation* as it puts constraints on what x and y can be.

   (b) We are seeking to minimize the product of x and y, denoted

   $$P = xy.$$

   This equation is called our *optimizing equation*, and we will eventually do our calculus here.

   (c) Notice our expression for P involves two variables! To get around this, let us solve our constraint equation $x - y = 100$ for y, giving

   $$y = \cancel{100 - x}\ x - 100$$

   Substituting this into our expression for P gives

   $$P = x(\overset{x-100}{\cancel{100 - x}}) = \cancel{100x - x^2} = x^2 - 100x$$

   (d) Notice P is now a function of a single variable. What is the domain of P? Well x and y can in principle be anything, so our domain is $(-\infty, \infty)$.

   (e) Now use the first derivative test for absolute extrema to find the minimum value of $P = \underset{x^2-100x}{\cancel{100x - x^2}}$ on the interval $(-\infty, \infty)$. Be sure to include full justification.

   ~~P' = 100 - 2x~~ $P' = 2x - 100$, so $x = 50$ is the critical #.

   $-$ $+$ P'

   50

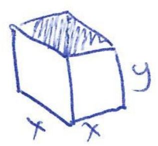
   So $P(50)$ is a local min and since $x = 50$ is the only critical # on $(-\infty, \infty)$, $P(50)$ must be the absolute minimum.

   (f) The step above will give the value of x the satisfies the problem; but we were told to find two numbers. To find y, substitute the value for x back into the constraint. Thus

   $$\boxed{x = 50} \qquad y = x - 100 = (50 - 100) = \boxed{-50}$$

4. If 27 cm^2 of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

(a) Let x denote the length of the sides of the square base of the box, and let y denote the height. We know the surface area of the box is 27 cm^2, so write this as an equation involving x and y. This is our *constraint.* [Hint: a picture helps a ton!]



$$SA = \underbrace{4xy}_{\text{4 faces of are } x\cdot y} + \underbrace{x^2}_{\text{bottom has area } x^2}$$

We need $SA=27$ so

$$27 = 4xy + x^2$$

(b) We are seeking to ~~minimize~~ maximize the volume V of the box. Write an equation involving V, x, and y. This is our *optimizing equation.*

$$V = x^2 y$$

(c) Your expression for V should involve x and y. To get it purely in terms of x, solve your constraint equation from (a) for y and substitute this into your optimizing equation and simplify. V should now be a function of x alone.

Solving the constraint for y gives $y = \frac{27-x^2}{4x}$. So

$$V = x^2 y = x^2\left(\frac{27-x^2}{4x}\right) = \frac{1}{4}x(27-x^2) = \frac{1}{4}(27x - x^3)$$

(d) Since x and y are lengths of sides of a box, what is the domain of V?

Domain: $(0,\infty)$

(e) Apply the first derivative test for absolute extrema to find the maximum value of V on the domain found above. Be sure to include full justification.

$V' = \frac{1}{4}(27-3x^2)$. So $V'=0$ when $27-3x^2=0 \rightarrow x=\pm 3$ are the critical numbers.

$x=3$ is only critical number in $(0,\infty)$ and

+ − V'

0 3 ∞

So $V(3)$ is a local max, and since 3 was the only critical # on $(0,\infty)$, $V(3)$ must be the absolute max.

(f) What is the largest possible volume?

By part (e), the largest volume is $V(3) = \frac{1}{4}\left(27(3)-(3)^3\right)$

$$= \frac{1}{4}(81-27)$$

$$= \frac{54}{4} = \boxed{\frac{27}{2} \text{ cm}^3}$$

5. Now try one on your own. Find two positive numbers whose product is 100 and whose sum is a minimum.

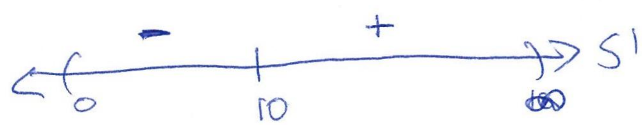
Let x and y denote the two numbers. We need $x \cdot y = 100$ and we want to minimize the sum $S = x + y$.

To get S in terms of 1-variable, we note $xy = 100$ implies $y = \frac{100}{x}$.

So $S = x + \frac{100}{x}$ and the domain is $(0, \infty)$ since we are told to find "two positive numbers".

$S' = 1 - \frac{100}{x^2}$ so $S' = 0$ when $1 - \frac{100}{x^2} = 0$. So $x = \pm 10$ are critical #'s.

Only 10 is in $(0, \infty)$.



So $S(10)$ is a local minimum and since $x = 10$ is the only critical number on $(0, \infty)$, $S(10)$ is the absolute min.

Since $y = \frac{100}{x} = \frac{100}{10} = 10$, $\boxed{x = 10 \text{ and } y = 10}$ are the positive numbers whose product is 100 and whose sum is a minimum.