1. For these problems, justify and cite any theorems you have used.

   (a) (9 pts) Verify that the function $f(x) = \sqrt{x} - \dfrac{x}{3}$ satisfies the three hypotheses of Rolle's Theorem on the interval $[0, 9]$, then find all numbers c that satisfy the conclusion of Rolle's Theorem.

   (b) (8 pts) Let g be a differentiable odd function defined on $(-\infty, \infty)$. Show that for every positive number x there exists a number c in $(-x, x)$ such that $g(x) = xg'(c)$.

   (c) (8 pts) Find the linearization of $h(x) = \sqrt[3]{1 + 3x}$ at $a = 0$ and use it to give an approximate value for $\sqrt[3]{1.03}$.

Solution:

(a)(9 pts) Note that $f(x)$ being the difference of a root function and a polynomial, is continuous and differentiable on $[0, \infty)$, so it is $\underline{(i) \text{ continuous on } [0, 9]}$, and $\underline{(ii) \text{ differentiable on } (0, 9)}$ also, note $\underline{(iii)\ f(0) = (9) = 0}$ thus it suffices to use Rolle's Theorem. Here we have,

$$f'(c) = 0 \iff \frac{1}{2\sqrt{c}} - \frac{1}{3} = 0 \iff \sqrt{c} = \frac{3}{2} \iff c = \frac{9}{4} \text{ and note } 0 < \frac{9}{4} < 9$$

thus, necessarily, $c = 9/4$.

(b)(8 pts) We apply the Mean Value Theorem to $g(x)$ on the interval $[-b, b]$ where b is any positive real number. Note that, by assumption, $g(x)$ is $\underline{(i) \text{ continuous on } [-b, b]}$, and $\underline{(ii) \text{ differentiable on } (-b, b)}$, so the Mean Value Theorem is justified. Now note that since g is odd, we have $g(-b) = -g(b)$ and so

$$g'(c) = \frac{g(b) - g(-b)}{b - (-b)} \iff g'(c) = \frac{\cancel{2}g(b)}{\cancel{2}b} \iff bg'(c) = g(b)$$

and so, if we denote b as x, then we have shown for any positive number x, there exists a number c in $(-x, x)$ such that $g(x) = xg'(c)$ by the Mean Value Theorem.

(c)(8 pts) Here we have

$$h'(x) = \frac{1}{3}(1 + 3x)^{-2/3} \cdot 3 = (1 + 3x)^{-2/3} \implies L(x) = h(0) + h'(0)(x - 0) = 1 + 1 \cdot x = 1 + x$$

so $\sqrt[3]{1 + 3x} \approx 1 + x$ and

$$\sqrt[3]{1.03} = \sqrt[3]{1 + 0.03} = \sqrt[3]{1 + 3(0.01)} = h(0.01) \approx L(0.01) = 1 + 0.01 = 1.01$$

thus $\boxed{\sqrt[3]{1.03} \approx 1.01}$

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2. Some useful formulas: Volume of a sphere, $V = \dfrac{4}{3}\pi r^3$, and surface area of a sphere, $SA = 4\pi r^2$, where r = radius.

(a)(10 pts) Consider a spherical raindrop falling through a layer of dry air. As the raindrop falls, it starts to evaporate and the volume decreases. Suppose the volume decreases at a rate of 1 mm^3/s, find the rate at which the diameter of the raindrop decreases when the radius is 5 mm.

(b)(10 pts) The radius of the spherical raindrop is initially measured to be 6 mm, with a possible error of ± 0.01 mm. Approximate the maximum possible percentage error in calculating the surface area of the sphere.

(c)(5 pts) Now consider the volume of the spherical raindrop from part (a) as the radius changes from $r = a$ to $r = b$ (where a and b are some real numbers, $0 < a < b$). (i) Prove that the average rate of change of the volume (with respect to the radius) as the radius changes from $r = a$ to $r = b$ is equal to the surface area of the raindrop with radius equal to some number c where $a < c < b$. (ii) Find c in terms of a and b. Justify your answers and cite any theorems you have used. $\big($*Hint:* $x^3 - y^3 = (x - y)(x^2 + xy + y^2)\big)$

Solution:

(a)(10 pts) Note that if we denote the diameter of the raindrop as D then $D = 2r$ and we wish to find $\dfrac{dD}{dt} = 2\dfrac{dr}{dt}$. Now,

$$V = \frac{4}{3}\pi r^3 \implies \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \implies \frac{dr}{dt} = \frac{dV/dt}{4\pi r^2}$$

and so

$$\frac{dD}{dt} = 2\cdot\frac{dr}{dt} = 2\cdot\left.\frac{dV/dt}{4\pi r^2}\right|_{dV/dt=-1,r=5} = \boxed{-\frac{1}{50\pi}\text{ mm/s}}$$

(b)(10 pts) Here let the surface area be denoted as $S(r) = 4\pi r^2$ and note $dr = \Delta r = \pm 0.01$ mm, so, using differentials, we have $dS = S'(r)dr = 8\pi r dr$ and so

$$\begin{aligned}\frac{\Delta S}{S}\times 100\% \approx \frac{dS}{S}\times 100\% &= \left.\frac{8\pi r dr}{4\pi r^2}\right|_{r=6,dr=\Delta r}\times 100\% \\ &= \frac{2\Delta r}{r}\times 100\% = \frac{2(\pm 0.01)}{6}\times 100\% = \pm\frac{1}{3}\% = \boxed{\pm 0.\bar{3}\%}\end{aligned}$$

and so the maximum possible percentage error in calculating the surface area of the sphere is $\pm 0.\bar{3}\%$.

(c)(*i*)(3 pts) Note that $V = \frac{4}{3}\pi r^3$ is continuous and differentiable for all real numbers r, so $V(r)$ is $\underline{(i)\text{ continuous on }[a,b]}$, and $\underline{(ii)\text{ differentiable on }(a,b)}$, so by the Mean Value Theorem there exists a number c in (a,b) such that

$$V'(c) = \frac{V(b)-V(a)}{b-a} \Longrightarrow 4\pi c^2 = \frac{V(b)-V(a)}{b-a}$$

and note that the surface area of a raindrop with radius $r = c$ is $S = 4\pi c^2$, so

$$\text{Average rate of change of Volume} = \frac{V(b)-V(a)}{b-a} = 4\pi c^2 = \text{ Surface Area of raindrop with radius equal to } c.$$

(c) (*ii*)(2 pts) From part (c)(*i*) above we see that

$$\begin{aligned}V'(c) = \frac{V(b)-V(a)}{b-a} \quad &\Longrightarrow \quad 4\pi c^2 = \frac{\left(\frac{4}{3}\pi b^3 - \frac{4}{3}\pi a^3\right)}{b-a} \\ &\Longrightarrow \quad 4\pi c^2 = \frac{4}{3}\pi\cdot\frac{(b^3-a^3)}{b-a} = \frac{4}{3}\pi\cdot\frac{(b-a)(b^2+ab+a^2)}{b-a}\end{aligned}$$

and so

$$c^2 = \frac{(b^2+ab+a^2)}{3} \Longrightarrow \boxed{c = \sqrt{\frac{(b^2+ab+a^2)}{3}}}$$

(note that $c > 0$ since c is the radius of a raindrop).

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3. (a)(9 pts) Find the local and absolute extreme values of $f(x) = \dfrac{3x-4}{x^2+1}$ on $[-2,2]$. Justify your answer.

(b)(9 pts) Find all local maximum and minimum values of $g(x) = x\sqrt{6-x}$. Justify your answer.

(c)(7 pts) In your blue book clearly sketch the graph of a function $h(x)$ that satisfies the following properties (label any <u>extrema</u>, <u>inflection points</u> or <u>asymptotes</u>):

- $h(0) = 0,\ h'(-2) = h'(1) = h'(9) = 0$
- $\lim_{x\to\infty} h(x) = 0,\ \lim_{x\to 6} h(x) = -\infty$
- $h'(x) < 0$ on the intervals $(-\infty,-2)$, $(1,6)$ and $(9,\infty)$
- $h'(x) > 0$ on the intervals $(-2,1)$ and $(6,9)$
- $h''(x) > 0$ on the intervals $(-\infty,0)$ and $(12,\infty)$
- $h''(x) < 0$ on the intervals $(0,6)$ and $(6,12)$

Solution:

(a)(9 pts) Note that

$$\begin{aligned}f'(x) = \frac{(x^2+1)(3)-(3x-4)(2x)}{(x^2+1)^2} &= \frac{3x^2+3-6x^2+8x}{(x^2+1)^2} \\ &= \frac{-3x^2+8x+3}{(x^2+1)^2} = \frac{-(3x^2-8x-3)}{(x^2+1)^2} = \frac{-(3x+1)(x-3)}{(x^2+1)^2}\end{aligned}$$

and note that $f'(x)$ is defined for all x and $f'(x) = 0$ if $x = -1/3$ or $x = 3$ but 3 is not in the interval. Now note that $f'(x) < 0$ if $x < -\dfrac{1}{3}$ and $f'(x) > 0$ for $x > -\dfrac{1}{3}$ and so $f(-1/3) = \dfrac{-5}{10/9} = -\dfrac{9}{2}$ is a <u>local minimum value.</u> (Note could have also used the Second Derivative Test.)

To check for absolute extrema, checking the endpoints we see that $f(-2) = -2$ and $f(2) = \dfrac{2}{5}$. Thus $f(-1/3) = -\dfrac{9}{2}$ is the <u>absolute minimum value</u> and $f(2) = \dfrac{2}{5}$ is the <u>absolute maximum value.</u>

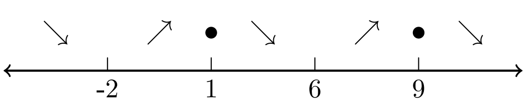
(b)(9 pts) Taking the derivative of $g(x)$, we have,

$$\begin{aligned} g'(x) &= 1 \cdot (6-x)^{1/2} + x \cdot \frac{1}{2}(6-x)^{-1/2}(-1) \\ &= (6-x)^{1/2} - \frac{x}{2(6-x)^{1/2}} = \frac{2(6-x)-x}{2\sqrt{6-x}} = \frac{-3x+12}{2\sqrt{6-x}} \end{aligned}$$

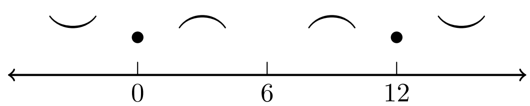
and note that the domain of $g(x)$ is $(-\infty, 6]$, so $g'(x) = 0$ implies $x = 4$ and $g'(x)$ is undefined when $x = 6$ but since we cannot have local extrema at an endpoint we do not need to consider $x = 6$. Now, applying the First Derivative Test (or could also use the Second Derivative Test), we see that $g'(x) > 0$ if $x < 4$ and $g'(x) < 0$ if $4 < x < 6$ so we have a local maximum at $x = 4$ with $\boxed{\text{local maximum value of } g(4) = 4\sqrt{2}.}$

(c)(7 pts) Note that $h(0) = 0$, $h'(-2) = h'(1) = h'(9) = 0$, $\lim\limits_{x\to\infty} h(x) = 0$, $\lim\limits_{x\to 6} h(x) = -\infty$ and

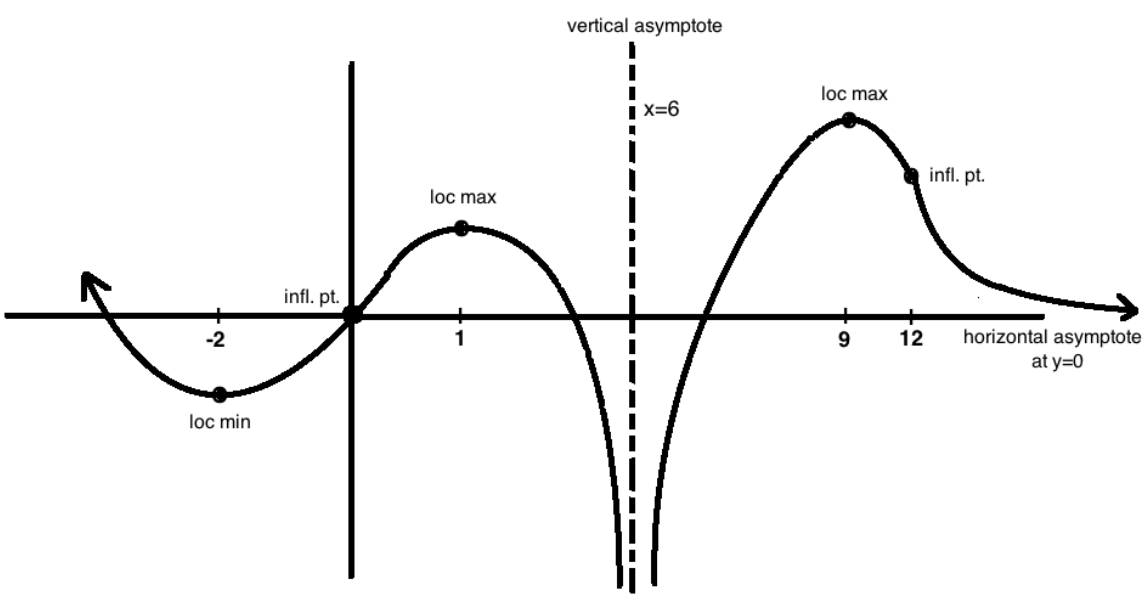
Sign Chart for $h'(x)$



Sign Chart for $h''(x)$



implies that the graph could, for example, look like,



4. (a)(8 pts) The curve $x^3y + y^3 = 1$ defines y implicitly as one or more functions of x. Find the slope of this curve at the point $(0, 1)$. Show all work.

(b)(8 pts) Given that p is a differentiable function for all t, find $q'(t)$ in terms of $p'(t)$ if $q(t) = \dfrac{p(p(t))}{\sqrt{t}}$, for $t > 0$.

(c)(9 pts) A function f and its first two derivatives have values as shown in Table 1 below:

Table 1

x	$f(x)$	$f'(x)$	$f''(x)$
0	0	1	2
1	1	1	1
2	3	2	1
4	6	3	0

Table 2: **Copy this table in your bluebook**

x	$g'(x)$	$g''(x)$
0		
1		
2		

Let $g(x) = xf(x^2)$. Copy the table for $g'(x)$ and $g''(x)$ above in your bluebook and fill it in with *fully simplified values.* Only the final answers in your table in your bluebook will be graded – no partial credit will be given.

Solution:

(a)(8 pts) Using implicit differentiation we see that

$$\begin{aligned} x^3y + y^3 = 1 \quad &\Longrightarrow \quad 3x^2y + x^3y' + 3y^2y' = 0 \\ &\Longrightarrow \quad (x^3 + 3y^2)y' = -3x^2y \Longrightarrow y' = \left.\frac{-3x^2y}{x^3 + 3y^2}\right|_{(x,y)=(0,1)} = \boxed{0} \end{aligned}$$

(b)(8 pts) Using the Quotient Rule and the Chain Rule, we have,

$$\begin{aligned} q'(t) = \left[\frac{p(p(t))}{\sqrt{t}}\right]' \quad &= \quad \frac{[p'(p(t)) \cdot p'(t)]t^{1/2} - \frac{1}{2}t^{-1/2}p(p(t))}{t} \\ &= \quad \frac{1}{2t^{1/2}} \cdot \frac{2tp'(p(t)) \cdot p'(t) - p(p(t))}{t} = \boxed{\frac{2tp'(p(t)) \cdot p'(t) - p(p(t))}{2t^{3/2}}} \end{aligned}$$

(c)(9 pts) Note that

$$g'(x) = f(x^2) + 2x^2f'(x^2)$$

and

$$g''(x) = 2xf'(x^2) + 4xf'(x^2) + 4x^3f''(x^2) = 6xf'(x^2) + 4x^3f''(x^2)$$

and so

x	$g'(x)$ (1pt ea)	$g''(x)$ (2pts ea)
0	0	0
1	3	10
2	30	36