Solutions

Math 1300-010 - Fall 2016

Indeterminate Forms and l'Hospital's Rule - 10/31/16

Guidelines: Please work in groups of two or three. This will not be handed in, but is a study resource for Midterm 3.

The purpose of this activity is to explore the various indeterminate form limits and how to use l'Hospital's Rule to solve them.

A. Recall ***l'Hospital's Rule***, which says that if

$$\lim_{x\to a}\frac{f(x)}{g(x)}$$

is *indeterminate* of form $0/0$ or ∞/∞, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)},$$

provided the limit on the right hand side exists. Note the limit on the right hand side has nothing to do with the quotient rule; we simply take the ratio of derivatives and then compute the limit. This only works for the $0/0$ or ∞/∞ indeterminate forms.

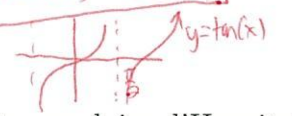
Let us practice using l'Hospital's rule by computing the following limits:

1. $\lim_{x\to\pi/2}\frac{\cos(x)}{1-\sin(x)}$ → form $\frac{0}{0}$

$\overset{l'H}{=} \lim_{x\to\pi/2} \frac{-\sin(x)}{-\cos(x)}$

$\boxed{= \lim_{x\to\pi/2} \tan(x) \text{ DNE}}$

b/c



2. $\lim_{x\to\infty}\frac{\ln(x)}{x}$ → form $\frac{\infty}{\infty}$ (note $\ln(\infty)=\infty$, $\ln(0^+)=-\infty$

$\overset{l'H}{=} \lim_{x\to\infty} \frac{1/x}{1}$

$\boxed{=0}$

Sometimes after applying l'Hospital's rule, the limit is still indeterminate of form $0/0$ or ∞/∞. In this case apply l'Hospital's rule again. This can be repeated indefinitely so long as the resulting limit is still indeterminate. Give this a try below:

3. $\lim_{t\to 0}\frac{e^t-1-t}{t^2}$ → form $\frac{0}{0}$

$\overset{l'H}{=} \lim_{t\to 0} \frac{e^t-1}{2t}$ → form $\frac{0}{0}$

$\overset{l'H}{=} \lim_{t\to 0} \frac{e^t}{2} = \boxed{\frac{1}{2}}$

There are several more indeterminate forms aside from $0/0$ and ∞/∞. l'Hospital's rule is not directly applicable to these other indeterminate forms, but they can be manipulated using various tricks so that a $0/0$ or ∞/∞ type limit appears.

B. **The Indeterminate Form** $0 \cdot \infty$

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$, then it is not clear what happens with

$$\lim_{x\to a} f(x)g(x).$$

There is a struggle as the function f wants trys to pull the limit to 0 whereas the function g trys to pull the limit to infinity.

l'Hospital's Rule does not apply to this indeterminate form directly, but if we rewrite the product $f(x)g(x)$ as either

$$fg = \frac{f}{1/g} \quad \text{or} \quad fg = \frac{g}{1/f},$$

then we notice that $\lim_{x\to a} f(x)g(x)$ has been converted into a limit of the form $0/0$ or ∞/∞, respectively.

Let's see how this works with an example. Consider

$$\lim_{x\to\infty} e^{-x}\ln(x),$$

which is indeterminate of form $0 \cdot \infty$. Then

$$\lim_{x\to\infty} e^{-x}\ln(x) = \lim_{x\to\infty} \frac{\ln(x)}{1/e^{-x}} = \lim_{x\to\infty} \frac{\ln(x)}{e^x} \overset{\textbf{l'H}}{=} \lim_{x\to\infty} \frac{1/x}{e^x} = \lim_{x\to\infty} \frac{1}{xe^x} = 0,$$

where l'Hospital's rule was performed at the step with the "**l'H**" over the equal sign.
Give this a try below:

bad: $\lim_{x\to 0^+} \frac{x^3}{1/\ln(x)}$ → only gets worse as you do l'H

1. $\lim_{x\to 0^+} x^3\ln(x)$ → form $0 \cdot \infty$

$= \lim_{x\to 0^+} \frac{\ln(x)}{1/x^3} \overset{l'H}{=} \lim_{x\to 0^+} \frac{1/x}{-3/x^4} = \lim_{x\to 0} \frac{x^3}{-3} = \boxed{0}$

2. $\lim_{x\to -\infty} x^2 e^x$ → form $\infty \cdot 0$

bad: $\lim_{x\to -\infty} \frac{e^x}{1/x^2}$ → only gets worse as you do l'H

$= \lim_{x\to -\infty} \frac{x^2}{1/e^x} = \lim_{x\to -\infty} \frac{x^2}{e^{-x}} \overset{l'H}{=} \lim_{x\to -\infty} \frac{2x}{-e^{-x}} \overset{l'H}{=} \lim_{x\to -\infty} \frac{2}{e^{-x}}$

$= \lim_{x\to -\infty} 2e^x = \boxed{0}$

C. **Indeterminate Powers**

If we consider

$$\lim_{x \to a} [f(x)]^{g(x)}$$

then several indeterminate forms can arise:

- If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ form 0^0.
  Here f is trying to make the limit 0, whereas g is trying to make the limit 1, since any nonzero number raised to the power 0 is equal to 1.

- If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = 0$ form ∞^0.
  Here f is trying to make the limit infinite, whereas g is trying to make the limit 1.

- If $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \infty$ form 1^∞.
  Here f is trying to make the limit equal to 1, whereas g is trying to make the limit infinite.

In each case, we solve the limit using a method quite similar to what we used in taking the derivative of functions of the form $[f(x)]^{g(x)}$, and that is to use the inverse relationship of e^x and $\ln(x)$ as well as properties of logarithms to write

$$[f(x)]^{g(x)} = e^{g(x) \ln(f(x))}.$$

$0^0 \to g(x)\ln(f(x))$ of form $0 \cdot \infty$

$\infty^0 \to g(x)\ln(f(x))$ of form [illegible] $\cdot \infty$

$1^\infty \to g(x)\ln(f(x)) = \infty \cdot 0$

Thus we are led to the product

$$\lim_{x \to a} g(x) \ln(f(x)),$$

which is indeterminate of form $0 \cdot \infty$. Briefly think about and discuss why each of the three cases above leads to this limit being indeterminate of form $0 \cdot \infty$.

To see how this works, consider

$$\lim_{x \to 0^+} x^{(x^2)}$$

which is indeterminate of form 0^0. Applying the trick discussed above,

$$x^{(x^2)} = e^{x^2 \ln(x)}.$$

Since $\lim_{x \to 0^+} x^2 \ln(x)$ is of form $0 \cdot \infty$, we have

$$\lim_{x \to 0^+} x^2 \ln(x) = \lim_{x \to 0^+} \frac{\ln(x)}{1/x^2} = \lim_{x \to 0^+} \frac{\ln(x)}{x^{-2}} \overset{\text{l'H}}{=} \lim_{x \to 0^+} \frac{1/x}{-2/x^3} = \lim_{x \to 0^+} \frac{x^2}{-2} = 0.$$

Therefore,

$$\lim_{x \to 0^+} x^{(x^2)} = \lim_{x \to 0^+} e^{x^2 \ln(x)} = e^0 = 1.$$

Try this for yourself on this next page.

1. $\lim_{x\to 0^+} (2x)^{(x^3)}$

Rewrite $(2x)^{x^3} = e^{x^3 \ln(2x)}$. Now,

$\lim_{x\to 0^+} x^3 \ln(2x) \to$ form $0 \cdot \infty$, so

$= \lim_{x\to 0^+} \frac{\ln(2x)}{1/x^3} \overset{\text{l'H}}{=} \lim_{x\to 0^+} \frac{2/2x}{-3/x^4} = \lim_{x\to 0^+} \frac{x^3}{-3} = 0.$

Thus

$\lim_{x\to 0^+} (2x)^{(x^3)}$

$= \lim_{x\to 0^+} e^{x^3 \ln(2x)}$

$= e^0 = \boxed{1}$

2. $\lim_{x\to\infty} x^{(1/x)}$ Rewrite $x^{(1/x)} = e^{(1/x)\cdot\ln(x)}$. Now,

$\lim_{x\to\infty} (1/x)\ln(x) = \underbrace{\lim_{x\to\infty} \frac{\ln(x)}{x}}_{\frac{\infty}{\infty}} \overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0.$

Thus $\lim_{x\to\infty} x^{(1/x)} = \lim_{x\to\infty} e^{(1/x)\cdot\ln(x)} = e^0 = \boxed{1}$

3. $\lim_{x\to 0^+} (4x+1)^{(\cot(x))}$ Rewrite $(4x+1)^{\cot(x)} = e^{\cot(x)\ln(4x+1)}$. Now

$\lim_{x\to 0^+} \cot(x)\ln(4x+1) \to$ form $\infty \cdot 0$,

$= \lim_{x\to 0^+} \frac{\ln(4x+1)}{1/\cot(x)} = \lim_{x\to 0^+} \frac{\ln(4x+1)}{\tan(x)} \to$ form $\frac{0}{0}$

$\overset{\text{l'H}}{=} \lim_{x\to 0^+} \frac{4/(4x+1)}{\sec^2(x)} \to$ now plug in

$= \frac{4/0+1}{\sec^2(0)} = 4$

Thus

$\lim_{x\to 0^+} (4x+1)^{\cot(x)}$

$= \lim_{x\to 0^+} e^{\cot(x)\ln(4x+1)}$

$= \boxed{e^4}$