1. The following are not related:

   (a) (7 pts) Differentiate $y = \int_{2x}^{2/x} \frac{3+t}{1+5t^2}\,dt$ [Do not simplify.]

   (b) (6 pts) Evaluate the integral $\int (1+\cot^2 x)dx$

   (c) (6 pts) Evaluate the integral $\int \frac{7x^3+5x^2-3}{\sqrt[4]{x}}dx$

   (d) (6 pts) Evaluate the integral $\int_1^3 |x^2-1|dx$

Solution:
(a)

$$y = \int_{2x}^{1} \frac{3+t}{1+5t^2}dt + \int_{1}^{2/x} \frac{3+t}{1+5t^2}dt = -\int_{1}^{2x} \frac{3+t}{1+5t^2}dt + \int_{1}^{2/x} \frac{3+t}{1+5t^2}dt$$

$$\frac{dy}{dx} = -\frac{3+2x}{1+5(2x)^2}(2) + \frac{3+2/x}{1+5(2/x)^2}(-\frac{2}{x^2})$$

(b) $\int (1+\cot^2 x)dx = \int \csc^2 x dx = -\cot x + C$

(c)

$$\int \frac{7x^3}{x^{1/4}} + \frac{5x^2}{x^{1/4}} - \frac{3}{x^{1/4}}dx = \int 7x^{3-1/4} + 5x^{2-1/4} - 3x^{-1/4}dx = \int 7x^{11/4} + 5x^{7/4} - 3x^{-1/4}dx$$

$$= 7(\frac{4}{15})x^{15/4} + 5(\frac{4}{11})x^{11/4} - 3(\frac{4}{3})x^{3/4} + C$$

$$= \frac{28}{15}x^{15/4} + \frac{20}{11}x^{11/4} - 4x^{3/4} + C$$

(d) Since x^2-1 is only negative on the interval $(-1,1)$ and this integral is being evaluated from $(1,3)$, we may drop the absolute value symbols:

$$\int_1^3 x^2 - 1 \quad dx = \frac{1}{3}x^3 - x|_1^3 = \frac{20}{3}$$

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2. The following problems are not related:

   (a) (12 pts) The function $f(x) = \frac{1}{3}x^3 - 2x^2 + 3x + 5$ has two local extrema. Estimate the value of x where one of the local extreme values of $f(x)$ occur using one iteration of Newton's method (in other words, find x_2). Use $x_1 = 0$ as an initial approximation.

   (b) (12 pts)

      i. Find the area of the largest rectangle that can be inscribed in a right triangle with legs of lengths 3 cm and 4 cm respectively, if two sides of the rectangle lie along the legs.

      ii. How do you know your answer is a maximum? Justify your answer based on the theories of this class.

Solution:
(a) The local extreme values of $f(x)$ occur where $f'(x) = 0$. Therefore, Newton's method must be applied to $f'(x) = x^2 - 4x + 3 = (x-3)(x-1)$.
Newton's method tells us that $x_2 = x_1 - \frac{f'(x_1)}{f''(x_1)}$.
$f''(x) = 2x - 4$. So then:

$$x_2 = x_1 - \frac{x_1^2 - 4x_1 + 3}{2x_1 - 4} = 0 - \frac{3}{-4} = \frac{3}{4}$$

(b) i. The rectangle has area xy. Using similar triangles, $\frac{3-y}{x} = \frac{3}{4} \implies -4y + 12 = 3x \implies y = -\frac{3}{4}x + 3$. Therefore the area of the rectangle can be written as:

$$A(x) = x(-\frac{3}{4}x + 3) = -\frac{3}{4}x^2 + 3x$$

where $0 \leq x \leq 4$.

$$0 = A'(x) = -\frac{3}{2}x + 3 \implies x = 2, y = \frac{3}{2}.$$

ii. Since $A(0) = A(4) = 0$, the maximum area is $A(2) = 2(\frac{3}{2}) = 3cm^2$. This may also be justifued by recognizing that $A''(x) = -\frac{3}{2} < 0$ for all x $\implies$ we have found a maximum by the second derivative test.

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3. (a) (6 pts) Using the definition for area using right hand endpoints,

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}[f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_n)\Delta x]$$

find an expression for the area under the curve $y = -3x^2 + 6x$ from 0 to 2 as a limit.
Note: The following formulas may be useful:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \quad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

(b) (6 pts) Evaluate the limit.
(c) (6 pts) Now express the area as an integral and find the average value, f_{avg}.
(d) (6 pts) Find all c between $x = 0$ and $x = 2$ so that $f(c) = f_{avg}$.

Solution:
(a) $\Delta x = \frac{2}{n}, x_i = \frac{2i}{n}$. So we can express the area as:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left[-3\left(\frac{2i}{n}\right)^2 + 6\left(\frac{2i}{n}\right)\right]\left(\frac{2}{n}\right)$$

(b) Evaluating the limit in part (a) gives us: $\lim_{n\to\infty} \sum_{i=1}^{n} \left[-\frac{24i^2}{n^3} + \frac{24i}{n^2}\right] = \lim_{n\to\infty} -\frac{24}{n^3}\sum_{i=1}^{n} i^2 + \frac{24}{n^2}\sum_{i=1}^{n} i$

$$= \lim_{n\to\infty}\left[-\frac{24}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{24}{n^2}\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\left[-\frac{4}{n^3}(2n^3 + 3n^2 + n) + 12 + \frac{12}{n}\right]$$

$$= \lim_{n\to\infty}\left[-8 - \frac{12}{n} - \frac{4}{n^2} + 12 + \frac{12}{n}\right] = -8 + 12 = 4$$

(c) $f(x) = \int_0^2 -3x^2 + 6x \;\; dx$. So the average value of the integral is given by:

$$f_{ave} = \int_0^2 -3x^2 + 6x \;\; dx = \frac{1}{2}(-x^3 + 3x^2|_0^2) = \frac{1}{2}(-8 + 12) = 2$$

(d) $f(c) = -3c^2 + 6c$. So we need $-3c^2 + 6c = 2 \implies -3c^2 + 6c - 2 = 0 \implies 3c^2 - 6c + 2 = 0$. Using the quadratic formula we have $c = 1 \pm \frac{\sqrt{3}}{3}$.

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4. Let the function f be defined by $f(x) = \int_1^x \frac{1}{t}dt$ for $x > 0$.

(a) (5 pts) What is $f(1)$? What is $f'(x)$? What is $f'(1)$?

(b) (5 pts) f is differentiable. Why?

(c) (6 pts) Show that $f'(5x) = f'(x)$.

(d) (5 pts) Using the definition of f, show that $f(x+h) - f(x) = \int_x^{x+h} \frac{1}{t}dt$

(e) (6 pts) Now suppose $h(x) = \int_0^{\cos(x-2)} 3t^2 dt$ and $f(s) = \int_\pi^{4s} h(x)dx$.
Find $f''(1/2)$.

Solution:
(a)

$$f(1) = \int_1^1 \frac{1}{t}dt = 0$$

$$f'(x) = \frac{1}{x}$$

$$f'(1) = \frac{1}{1} = 1$$

(b) f is differentiable because $\frac{1}{t}$ is continuous on its domain.

(c)

$$f(5x) = \int_1^{5x} \frac{1}{t}dt$$

$$\implies f'(5x) = \frac{1}{5x}(5) = \frac{1}{x} = f'(x)$$

(d) $f(x+h) = \int_1^{x+h} \frac{1}{t}dt, \quad f(x) = \int_1^x \frac{1}{t}dt$. So therefore,

$$f(x+h) - f(x) = \int_1^{x+h} \frac{1}{t}dt - \int_1^x \frac{1}{t}dt = \int_1^{x+h} \frac{1}{t}dt + \int_x^1 \frac{1}{t}dt$$

$$\implies f(x+h) - f(x) = \int_x^{x+h} \frac{1}{t}dt$$

(e) $f'(s) = 4h(4s)$.
Then $f''(s) = 4h'(4s) = 4 \cdot 3(\cos^2(4s-2)(-4\sin(4s-2)) = -48\cos^2(4s-2)\sin(4s-2)$

$$f''(1/2) = -48\cos(0)\sin(0) = 0$$
