Solutions

Math 1300-005 - Spring 2017

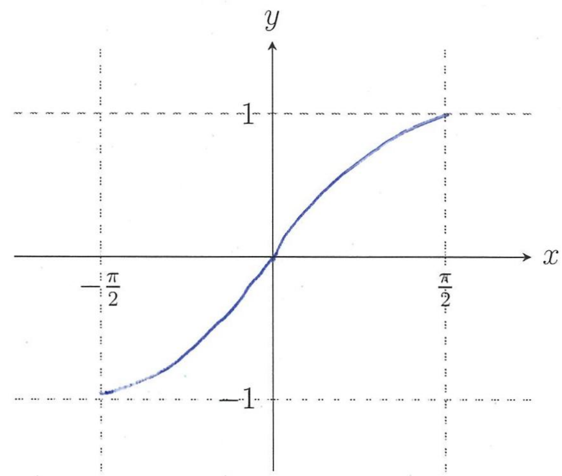
Inverse Trig Derivatives - 3/8/17

Guidelines: Please work in groups of two or three. This will not be handed in, but is a study resource for Midterm 2.

The goal of this worksheet is to discover the derivatives of arctangent, arcsine, and arccosine using implicit differentiation.

1. **The Derivative of Arcsine**

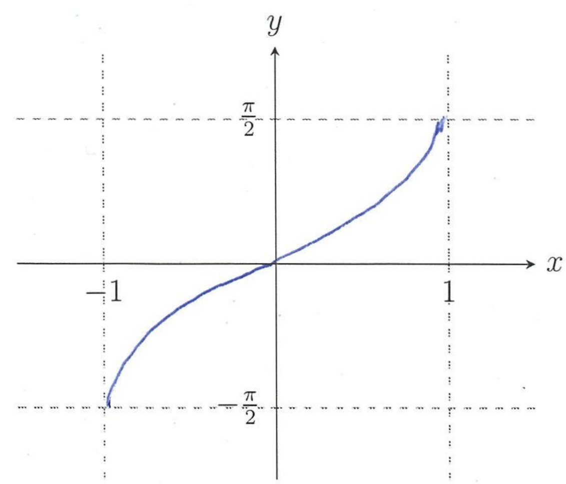
   (a) On the axes below, sketch a graph of $y = \sin(x)$ from $-\pi/2 < x < \pi/2$.



   (b) Why does $f(x) = \sin(x)$, when restricted to $-\pi/2 < x < \pi/2$, have an inverse?

   On $-\frac{\pi}{2} < x < \frac{\pi}{2}$, $\sin(x)$ is one-to-one.

   (c) Sketch the graph of the inverse $y = f^{-1}(x) = \arcsin(x)$ on the axes below.



(d) By properties of inverse functions, we have the following identity.

$$\sin(\arcsin(x)) = x$$

Differentiate both sides of this equation to find a formula for the derivative of $\arcsin(x)$. Express your answer in terms of $\cos(\arcsin(x))$.

$$\frac{d}{dx}\sin(\arcsin(x)) = \frac{d}{dx}x$$

$$\cos(\arcsin(x)) \cdot \frac{d}{dx}\arcsin(x) = 1$$

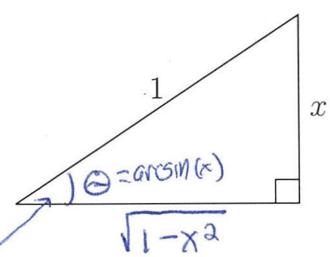
$$\boxed{\frac{d}{dx}\arcsin(x) = \frac{1}{\cos(\arcsin(x))}}$$

(e) Referring to the triangle below, explain why $\cos(\arcsin(x)) = \sqrt{1-x^2}$.

Let $\arcsin(x) = \theta$.

Then $\sin(\theta) = \frac{x}{1} = \frac{\text{opp}}{\text{hyp}}$,

so θ is



Then $\cos(\arcsin(x)) = \cos(\theta) = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$

(f) Combining the results of part (d) and part (e), we conclude

$$\boxed{\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}}$$

2. **The Derivative of Arccosine**

   Here I will just state the result. You can arrive at this by a method very similar to that for arctangent and arcsine.

$$\frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}}.$$

   What is the difference between the derivatives of $y = \arcsin(x)$ and $y = \arccos(x)$?

$$\frac{d}{dx}\arccos(x) = -\frac{d}{dx}\arcsin(x).$$

3. Derivative practice using inverse trig. Find dy/dx for the following.

   (a) $y = (\arctan(x))^2$

$$\frac{dy}{dx} = 2\arctan(x)\cdot\left(\frac{1}{1+x^2}\right)$$

   (b) $y = \arcsin(2x + e^x)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(2x+e^x)^2}}\cdot(2+e^x)$$

   (c) $y = \arccos(\arcsin(x))$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-(\arcsin(x))^2}}\cdot\frac{1}{\sqrt{1-x^2}}$$

(d) $\arccos(xy) = 1 + x^2y$ [This involves implicit differentiation]

$$\frac{d}{dx}\arccos(xy) = \frac{d}{dx}(1+x^2y)$$

$$\frac{-1}{\sqrt{1-(xy)^2}} \cdot \frac{d}{dx}(xy) = 2xy + x^2y'$$

$$\frac{-1}{\sqrt{1-x^2y^2}} \cdot (y + xy') = 2xy + x^2y'$$

multiply by $\sqrt{1-x^2y^2}$ on both sides

$$-y - xy' = 2xy\sqrt{1-x^2y^2} + x^2y'\sqrt{1-x^2y^2}$$

move y' terms to one side

$$-y - 2xy\sqrt{1-x^2y^2} = xy' + x^2y'\sqrt{1-x^2y^2}$$

$$-y - 2xy\sqrt{1-x^2y^2} = y'(x + x^2\sqrt{1-x^2y^2})$$

$$\boxed{y' = \frac{-y-2xy\sqrt{1-x^2y^2}}{x + x^2\sqrt{1-x^2y^2}}}$$

(e) $y = x\arcsin(x) + \sqrt{1-x^2}$

$$\frac{dy}{dx} = \arcsin(x) + \frac{x}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{1-x^2}} \cdot (-2x)$$

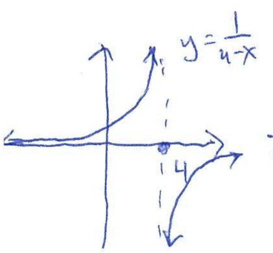
$$= \arcsin(x) + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}}$$

$$= \boxed{\arcsin(x)}$$

$\rightarrow$ this says $F(x) = x\arcsin(x) + \sqrt{1-x^2}$ is the antiderivative of $f(x) = \arcsin(x)$!

4. Compute the following limit.

$$\lim_{x \to 4^-} \arctan\left(\frac{1}{4-x}\right)$$



Let $t = \frac{1}{4-x}$. As $x \to 4^-$, we see $t \to \infty$ (from the graph).

So $$\lim_{x \to 4^-} \arctan\left(\frac{1}{4-x}\right) = \lim_{t \to \infty} \arctan(t) = \boxed{\frac{\pi}{2}}$$