MATH 1300: HW #4

Due on February 9, 2017 at 10:00am

Professor Braden Balentine Section 005

John Keller

Interpretation of Derivatives

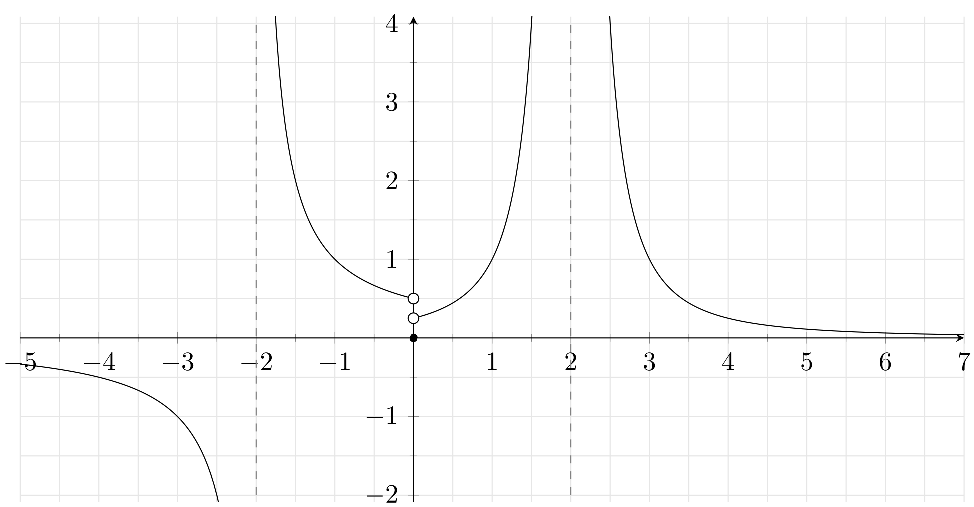
1. Let $V(t)$ be the volume of water in a tank (in liters), at time t (in seconds).
   (a) What are the meaning and units of $\frac{dV}{dt}$?
   This means the derivative on the curve of liters vs. seconds of water in the tank. The units are liters per second.
   (b) The tank is full at time t_0, so that $V(t_0) > 0$. At some later time t_1 a drain is opened 20 cm above the bottom of the tank (which is taller than 20 cm), emptying water from side of the tank. Is $\frac{dV}{dt}$ positive, negative, or zero
      i. at times t with $t_0 < t < t_1$?
      Negative.
      ii. after the drain has been opened at t_1, but before the water has dropped to 20 cm above the bottom of the tank?
      Negative.
      iii. after the water drops to the drain hole 20 cm above the bottom of the tank?
      Zero.

2. Let $f(t)$ be the amount of rain in cm, that has fallen since midnight, with t measured in hours. Interpret the following in practical terms, giving units:
   (a) $f(7) = 2.4$
   At 7 hours past midnight, there has been 2.4 cm total of rain that has fallen.
   (b) $f'(7) = 0.21$
   At exactly 7 hours past midnight, the rain is falling at approximately 0.21 cm per hour.
   (c) $f^{-1}(3) = 12.3$
   At $\frac{1}{3}$ hour past midnight, 12.3 cm of total rain has fallen.
   (d) $(f^{-1})'(3) = 86$
   At exactly $\frac{1}{3}$ hour past midnight, the rain is falling at approximately 86 cm per hour.

Section 2.5

6. Sketch a graph of an example of a function f that satisfies all of the given conditions

$$\lim_{x\to 2} f(x) = \infty, \quad \lim_{x\to -2^+} f(x) = \infty, \quad \lim_{x\to -2^-} f(x) = -\infty, \quad \lim_{x\to -\infty} f(x) = 0, \quad \lim_{x\to \infty} f(x) = 0, \quad f(0) = 0$$



26. Find the limit:

$$\lim_{x\to\infty} \frac{x+2}{\sqrt{9x^2+1}}$$

$$\lim_{x\to\infty} \frac{1+\frac{2}{x}}{\sqrt{\frac{1}{x^2}+9}}$$

$$\frac{\lim_{x\to\infty}(1+\frac{2}{x})}{\lim_{x\to\infty}(\sqrt{\frac{1}{x^2}+9})} = \frac{\lim_{x\to\infty}(1)+\lim_{x\to\infty}(\frac{2}{x})}{\sqrt{\lim_{x\to\infty}(\frac{1}{x^2})+\lim_{x\to\infty}(9)}} = \frac{1}{\sqrt{9}} = \frac{1}{3}$$

32. Find the limit:

$$\lim_{x\to\infty} \frac{\sin^2 x}{x^2}$$

$$-1 \le \sin(x) \le 1$$

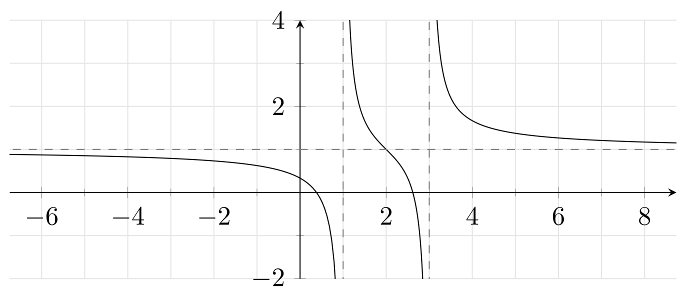
$$\lim_{x\to\infty}\left(\frac{0}{x^2}\right) \le \lim_{x\to\infty}\left(\frac{\sin^2 x}{x^2}\right) \le \lim_{x\to\infty}\left(\frac{1}{x^2}\right)$$

$$0 \le \lim_{x\to\infty}\left(\frac{\sin^2 x}{x^2}\right) \le 0$$

Therefore, by the **squeeze theorem**, because $\lim_{x\to\infty}\left(\frac{0}{x^2}\right) = 0$, then $\lim_{x\to\infty}\frac{\sin^2 x}{x^2}$ must be 0.

48. Find a formula for a function that has vertical asymptotes $x = 1$ and $x = 3$ and a horizontal asymptote at $y = 1$.

$$f(x) = \frac{x-2}{(x-2)^2-1} + 1$$



55. (a) A tank contains 5000 L of pure water brine that contains 30 g of salt per liter of water is pumped into the tank at a rate of 25 L/min. Show that concentration of salt after t minutes (in grams per liter) is

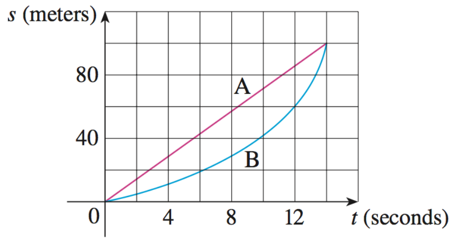
$$C(t) = \frac{30t}{200+t}$$

$$C(t) = 5000 \cdot \left(\frac{30 \text{ g of salt}}{1 \text{ L of water}}\right) \cdot \left(\frac{25 \text{ L of water}}{1 \text{ minute}}\right)$$

(b) What happens to the concentration as $t \to \infty$?
The concentration approaches 30 grams per liter.

Section 2.6

12. Shown are graphs of the position of functions of two runners, A and B, who run a 100-m race and finish in a tie.



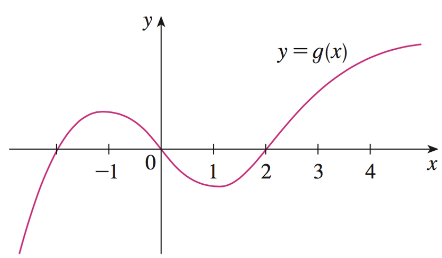
(a) Describe and compare how the runners run the race.
Both runners had different strategies, but got to the 100-m finish at the same time. Runner A decided to keep the exact same speed throughout the entire race, which is pretty unbelievable. Runner B was much more realistic, and started at a slower speed, steadily increasing it throughout. It is likely that Runner A began before the start line, as it is impossible to go from 0 to full speed instantaneously.

(b) At what time is the distance between the runners the greatest?
The time at which the distance between the runners is greatest is at approximately 9 seconds, when the runners are about 40-m apart.

(c) At what time do they have the same velocity?
The runners have the same velocity the second they cross the finish line, at 14 seconds into the race.

17. For the function g whose graph is given, arrange the following numbers in increasing order and explain your reasoning:

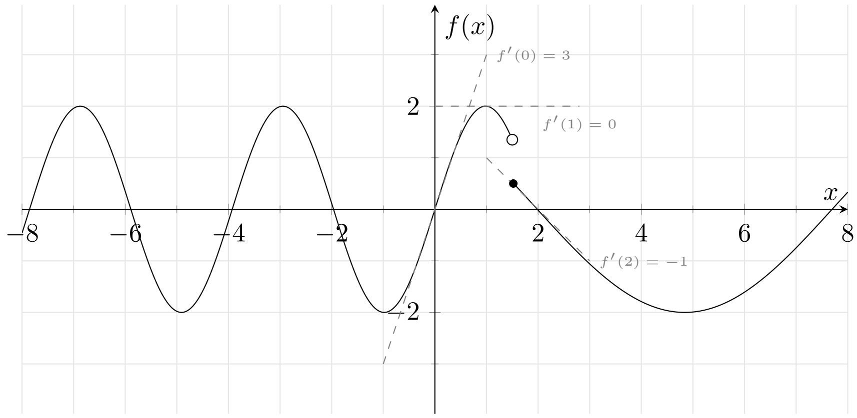
$$0 \qquad g'(-2) \qquad g'(0) \qquad g'(2) \qquad g'(4)$$



$$g'(0) < 0 < g'(4) < g'(2) < g'(-2)$$

To arrange the above values by order, I first determined which prime values were less than 0, and it turned out only $g'(0)$ is negative. Then I determined which lines have a greater tangent slope by visualizing a tangent line at each of the points listed.

21. Sketch the graph of a function f for which $f(0) = 0, f'(0) = 3, f'(1) = 0,$ and $f'(2) = -1$



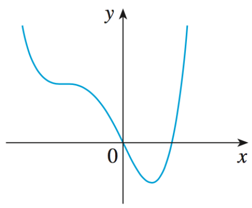
30. Find $f'(a)$ using the limit definition of derivative:

$$f(x) = x^{-2}$$

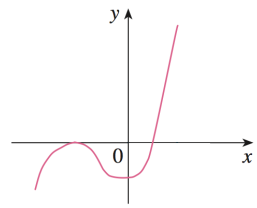
$$\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{(x+h)^{-2} - (x^{-2})}{h} \\
&= \lim_{h\to 0} \frac{\frac{2}{x+h} - \frac{2}{x}}{h} \\
&= \lim_{h\to 0} \frac{\frac{2}{x+h} - \frac{2+h}{x+h}}{h} \\
&= \lim_{h\to 0} \frac{\frac{-h}{x+h}}{h} \\
&= \lim_{h\to 0} \frac{2h}{xh(x+h)} \\
&= \lim_{h\to 0} \frac{2}{x(x+h)} \\
&= \frac{2}{x(x+0)} \\
&= \boxed{\frac{2}{x^2}}
\end{aligned}$$

Section 2.7

10. Trace or copy the graph of the given function f. (Assume that the axes have equal scales.) Then use the method of Example 1 to sketch the graph f' below it.



$f(x)$



$f'(x)$

34. Let $P(t)$ be the percentage of Americans under the age of 18 at time t. The table gives values of this function in census years from 1950 to 2000.

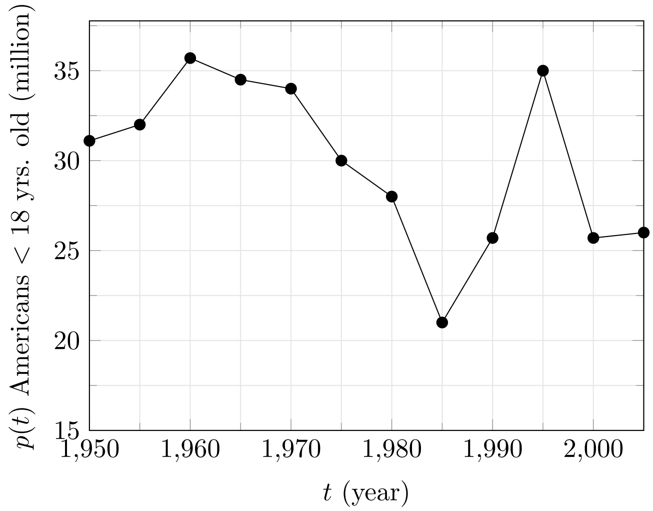
t	$P(t)$
1950	31.1
1960	35.7
1970	34.0
1980	28.0
1990	25.7
2000	25.7

(a) What is the meaning of $P'(t)$? What are its units?
The meaning of $P'(t)$ is simply the rate at which the number of people under 18 is decreasing or increasing every year. The units would likely be t in years (considering the census is done annually), and $P(t)$ in millions of kids.

(b) Construct a table of estimated values for $P'(t)$.

t	$P(t)$
1955	32
1965	34.5
1975	30
1985	21
1995	35
2005	26

(c) Graph P and P'.



(d) How would it be possible to get more accurate values for $P'(t)$?
The obvious method would be to look up the data on the census website, but practically speaking the best way to get more accurate data would be to increase the frequency for which the data is recorded (for example every year instead of every 5).