1. (28 pts, 7pts each) Evaluate each of the following.

   (a) $\displaystyle\int_0^{2\pi/3} \left(\sin\theta\cos^2\theta + \sin^3\theta\right) d\theta$

   (b) $\displaystyle\int \frac{(2y-1)^{3/2}-1}{\sqrt{2y-1}}\, dy$

   (c) $\displaystyle\int_e^5 \frac{1}{x\ln x}\, dx$

   (d) Use logarithmic differentiation to find dy/dx for $y = \dfrac{\ln x}{(x+1)(x^2+2)}$.
   Leave your answer unsimplified.

Solution:

(a) This problem is similar to WebAssign HW 4.3.5.

$$\int_0^{2\pi/3} \left(\sin\theta\cos^2\theta + \sin^3\theta\right) d\theta = \int_0^{2\pi/3} \sin\theta\left(\cos^2\theta + \sin^2\theta\right) d\theta = \int_0^{2\pi/3} \sin\theta\, d\theta$$
$$= -\cos\theta\Big]_0^{2\pi/3} = -\left(-\frac{1}{2} - 1\right) = \boxed{\frac{3}{2}}$$

(b) **Solution 1:**

$$\int \frac{(2y-1)^{3/2}-1}{\sqrt{2y-1}}\, dy = \int \left(2y - 1 - \frac{1}{\sqrt{2y-1}}\right) dy$$

Let $u = 2y-1,\ \ du = 2\,dy$.

$$= y^2 - y - \frac{1}{2}\int u^{-1/2}\, du = y^2 - y - \sqrt{u} + C$$
$$= \boxed{y^2 - y - \sqrt{2y-1} + C}$$

Solution 2:
Let $u = 2y-1,\ \ du = 2\,dy$.

$$\int \frac{(2y-1)^{3/2}-1}{\sqrt{2y-1}}\, dy = \frac{1}{2}\int \frac{u^{3/2}-1}{u^{1/2}}\, du = \frac{1}{2}\int \left(u - u^{-1/2}\right) du$$
$$= \frac{1}{2}\left(\frac{u^2}{2} - 2u^{1/2}\right) + C = \frac{1}{4}u^2 - u^{1/2} + C = \boxed{\frac{1}{4}(2y-1)^2 - \sqrt{2y-1} + C}$$

Solution 3:
Let $u = \sqrt{2y-1},\ \ du = 1/\sqrt{2y-1}\,dy$.

$$\int \frac{(2y-1)^{3/2}-1}{\sqrt{2y-1}}\, dy = \int \left(u^3 - 1\right) du = \frac{u^4}{4} - u + C = \boxed{\frac{1}{4}(2y-1)^2 - \sqrt{2y-1} + C}$$

(c) This problem was adapted from Written HW 5.2.60.
Let $u = \ln x, \quad du = dx/x$.

$$\int_e^5 \frac{1}{x \ln x}\, dx = \int_1^{\ln 5} \frac{du}{u} = \ln |u| \Big]_1^{\ln 5} = \ln(\ln 5) - \ln 1 = \boxed{\ln(\ln 5)}$$

(d)

$$y = \frac{\ln x}{(x+1)(x^2+2)}$$

$$\ln y = \ln |\ln x| - \ln |x+1| - \ln\left(x^2+2\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x \ln x} - \frac{1}{x+1} - \frac{2x}{x^2+2}$$

$$\frac{dy}{dx} = \boxed{\frac{\ln x}{(x+1)(x^2+2)}\left(\frac{1}{x \ln x} - \frac{1}{x+1} - \frac{2x}{x^2+2}\right)}$$

2. (15 pts) Using right hand endpoints, a definite integral is approximated by the Riemann sum :

$$\sum_{i=1}^{n}\left[\left(\frac{4i}{n}\right)^2 - 3\right]\frac{4}{n}.$$

(a) Find a definite integral represented by this Riemann sum.

(b) Evaluate $\displaystyle\sum_{i=1}^{n}\left[\left(\frac{4i}{n}\right)^2 - 3\right]\frac{4}{n}$, that is, find the sum in terms of n. Simplify your answer.

(c) Use either part (a) or part (b) to find the value of $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{4i}{n}\right)^2 - 3\right]\frac{4}{n}$.

Solution:

(a) One solution is $\boxed{\displaystyle\int_0^4 \left(x^2-3\right) dx}$. Then $\Delta x = (b-a)/n = 4/n$ and $x_i = a + i\Delta x = 4i/n$. Note that another solution is $\displaystyle\int_a^{a+4} \left((x-a)^2 - 3\right) dx$ for any constant a.

(b) This problem is similar to Written HW App.B.32.

$$\begin{aligned}
\sum_{i=1}^{n}\left[\left(\frac{4i}{n}\right)^2 - 3\right]\frac{4}{n} &= \frac{4}{n}\sum_{i=1}^{n}\left(\frac{16i^2}{n^2} - 3\right) \\
&= \frac{4}{n}\left[\frac{16}{n^2}\sum_{i=1}^{n} i^2 - \sum_{i=1}^{n} 3\right] \\
&= \frac{4}{n}\left[\frac{16}{n^2}\cdot\frac{n(n+1)(2n+1)}{6} - 3n\right] \\
&= \frac{32}{3}\frac{(n+1)(2n+1)}{n^2} - 12 \\
&= \frac{32}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) - 12 \\
&= \boxed{\frac{28}{3} + \frac{32}{n} + \frac{32}{3n^2}}
\end{aligned}$$

(c) Evaluating the limit from part (b) we get

$$\lim_{n\to\infty}\left(\frac{28}{3}+\frac{32}{n}+\frac{32}{3n^2}\right)=\boxed{\frac{28}{3}}.$$

Evaluating the definite integral from part (a) we get

$$\int_0^4 \left(x^2-3\right)dx=\left[\frac{x^3}{3}-3x\right]_0^4=\frac{64}{3}-12=\boxed{\frac{28}{3}}.$$

3. The following problems are not related.

(a) (5 pts) Write the sum in sigma notation. (Note: Do not try to find the value.)

$$\frac{3}{7}-\frac{4}{8}+\frac{5}{9}-\frac{6}{10}+\cdots+\frac{23}{27}.$$

(b) (7 pts) Water is flowing into a tub at $3t+\dfrac{1}{t+1}$ gallons per minute. Assuming the tub started with 10 gallons of water at time $t=0$, how much water is in the tub after 2 minutes?

(c) (7 pts) Use Newton's Method to find a root of the equation $x^3-7x-6=0$. Start with an initial guess of $x_1=1$ and find x_2 and x_{100}.

(d) (7 pts) Let $f(x)=\displaystyle\int_2^x \sqrt{1+t^3}\,dt$. Show that f is one-to-one (i.e. so it has an inverse) and find $\left(f^{-1}\right)'(0)$.

(e) (7 pts) Find the average value of the function $f(x)=x\left(\sqrt[3]{x}+\sqrt[5]{x}\right)$ on $[-1,1]$.

Solution:

(a) This problem was adapted from WebAssign HW App.B.3.

Here are two possible solutions: $\boxed{\displaystyle\sum_{k=3}^{23}(-1)^{k-1}\frac{k}{k+4}}$ or $\displaystyle\sum_{k=7}^{27}(-1)^{k-1}\frac{k-4}{k}$.

(b) This problem is similar to Written HW 4.3.54.

By the Net Change Theorem, the amount of water after 2 minutes is

$$10+\int_0^2\left(3t+\frac{1}{t+1}\right)dt=10+\left[\frac{3}{2}t^2+\ln|t+1|\right]_0^2=10+6+\ln 3-0=\boxed{16+\ln 3\text{ gallons}}.$$

(c) Let $f(x)=x^3-7x-6,\quad f'(x)=3x^2-7$. Then

$$x_2=1-\frac{f(1)}{f'(1)}=1-\frac{-12}{-4}=\boxed{-2}\quad\text{and}\quad x_3=-2-\frac{0}{5}=-2.$$

Because $f(-2)=0$, then $x=-2$ is a root of f and x_{100} also will equal $\boxed{-2}$.

(d) By FTC-1 $f'(x)=\sqrt{1+x^3}>0$ so f is an increasing function and therefore one-to-one.

Note that $f(2)=\displaystyle\int_2^2\sqrt{1+t^3}\,dt=0\implies f^{-1}(0)=2$. It follows that

$$\left(f^{-1}\right)'(0)=\frac{1}{f'(2)}=\frac{1}{\sqrt{1+8}}=\boxed{\frac{1}{3}}.$$

(e) This problem was adapted from WebAssign HW 4.3.3.

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{2}\int_{-1}^{1} x\left(\sqrt[3]{x}+\sqrt[5]{x}\right)dx = \frac{1}{2}\int_{-1}^{1}\left(x^{4/3}+x^{6/5}\right)dx$$

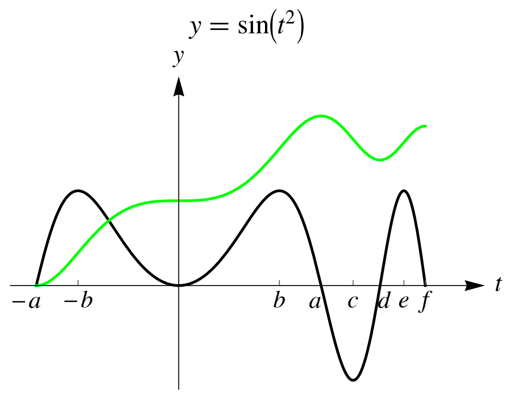
Because the integrand is an even function, we can make use of the symmetry.

$$= \frac{1}{2}(2)\int_0^1\left(x^{4/3}+x^{6/5}\right)dx = \left[\frac{3}{7}x^{7/3}+\frac{5}{11}x^{11/5}\right]_0^1 = \frac{3}{7}+\frac{5}{11} = \boxed{\frac{68}{77}}$$

4. (24 pts, 4 pts each) Consider the function $y = \sin\left(t^2\right)$, shown below.

Let $g(x) = \displaystyle\int_{-a}^{x} \sin\left(t^2\right)dt$, $-a \le x \le f$. Answer the following questions about $g(x)$. Your answers to parts (iii), (iv), and (v) will be in terms of a, b, c, d, e, and f. No justification is needed for this problem.

Solution:



(i) Find $g'(x)$.

(ii) Find $g''(x)$.

(iii) On which interval(s) is g decreasing?

(iv) At what value(s) of x does g have local minimum values?

(v) Suppose we wish to estimate the value of $g(f)$. Calculate the lower and upper sums using $n = 1$ subinterval.

(vi) Now find the numerical value of a and use it to find the numerical value of $g''(a)$.

Solution: This problem is similar to Written HW 4.4.26.

(i) $g'(x) = \boxed{\sin\left(x^2\right)}$ by FTC-1.

(ii) $g''(x) = \boxed{2x\cos\left(x^2\right)}$.

(iii) g is decreasing where $g'(x) = \sin\left(x^2\right) < 0$ on $\boxed{(a, d)}$.

(iv) By the first derivative test, g has a local minimum at $\boxed{x = d}$ where $g'(x) = 0$ and g' changes from negative to positive.

(v) Because $\sin\left(x^2\right)$ has a maximum value of 1 and a minimum value of -1, the upper sum is $U = (1)(f+a) = \boxed{f+a}$ and the lower sum is $L = (-1)(f+a) = \boxed{-(f+a)}$.

(vi) $\sin\left(x^2\right) = 0$ where $x^2 = 0, \pi, 2\pi, \ldots$, or $x = 0, \pm\sqrt{\pi}, \pm\sqrt{2\pi}, \ldots$. It follows that $a = \boxed{\sqrt{\pi}}$ and $g''\left(\sqrt{\pi}\right) = 2\sqrt{\pi}\cos(\pi) = \boxed{-2\sqrt{\pi}}$.