Solutions

Math 1300-005 - Spring 2017

Derivatives with Logarithms - 3/13/17

Guidelines: This will not be handed in, but is a study resource for Midterm 3.

The goal of this worksheet is to discover the derivatives of logarithmic functions using the chain rule and implicit differentiation. Then we apply these derivatives in a technique known as *logarithmic differentiation*.

1. **The Derivative of** $\log_a(x)$

   (a) For the first step, let $y = \log_a(x)$. How can we rewrite this expression without logarithms?

   $y = \log_a(x) \iff a^y = x$

   (b) Use implicit differentiation to find y'. Your final answer should not have any y's in it.

   $\frac{d}{dx}(a^y) = \frac{d}{dx}(x) \implies a^y \ln(a) \cdot y' = 1$

   $y' = \frac{1}{a^y \ln(a)} = \frac{1}{x \ln(a)}$

   (c) Conclusion:

   $$\frac{d}{dx}\log_a(x) = \frac{1}{x\ln(a)}$$

   and in particular, taking $a = e$ gives

   $$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

   (d) Let us practice using this result. Find the following derivatives.

   i. $f(x) = \dfrac{\log_7(x)}{x^2}$ Quotient rule

   $$\boxed{f'(x) = \frac{\frac{1}{x\ln(7)} \cdot x^2 - 2x \log_7(x)}{(x^2)^2}}$$

   ii. $g(x) = \ln(\tan(x))$ Chain

   $g'(x) = \frac{1}{\tan(x)} \cdot \frac{d}{dx}\tan(x)$

   $\hookrightarrow \boxed{g'(x) = \frac{1}{\tan(x)} \cdot \sec^2(x)}$

   iii. $h(x) = 3^x \log_2(x)$ Product

   $$\boxed{h'(x) = 3^x \ln(3) \log_2(x) + 3^x \cdot \frac{1}{x\ln(2)}}$$

   iv. $p(x) = \arctan(\ln(x))$ Chain

   $p'(x) = \frac{1}{1+[\ln(x)]^2} \cdot \frac{d}{dx}\ln(x)$

   $$\boxed{p'(x) = \frac{1}{1+[\ln(x)]^2} \cdot \frac{1}{x}}$$

2. **Logarithmic Differentiation**

With the derivatives of a^x and $\log_a(x)$ taken care of, we can at this point derive almost any function that is placed in front of us. However, there is still one tricky case that *none* of our rules apply to. This case involves functions of the form

$$\frac{d}{dx}[f(x)]^{g(x)},$$

where we take some differentiable function f, and raise as its exponent another differentiable function g. However, such derivatives can (and must) be found using a neat method called ***logarithmic differentiation***, which combines our rule for the derivative of $\ln(x)$ with the technique of implicit differentiation. Let us explore this below to find

$$\frac{d}{dx}(2x+3)^x.$$

(a) Let $y = (2x+3)^x$. Taking the natural log of both sides and using log rules, we have

$$\ln(y) = x\ln(2x+3)$$

(b) Apply implicit differentiation to this equation and solve for y'.

$$\frac{d}{dx}\ln(y) = \underbrace{\frac{d}{dx}\left(x\overbrace{\ln(2x+3)}^{\text{chain}}\right)}_{\text{product}} \rightarrow \frac{1}{y}\cdot y' = \ln(2x+3) + x\cdot\frac{1}{2x+3}\cdot\frac{d}{dx}(2x+3)$$

$$\rightarrow \frac{1}{y}\cdot y' = \ln(2x+3) + \frac{2x}{2x+3}$$

$$y' = y\left[\ln(2x+3) + \frac{2x}{2x+3}\right]$$

(c) Since $y = (2x+3)^x$, replace any y in your answer from (c) with $(2x+3)^x$.

$$y' = (2x+3)^x\left[\ln(2x+3) + \frac{2x}{2x+3}\right]$$

(d) Conclusion:

$$\frac{d}{dx}(2x+3)^x = (2x+3)^x\left[\ln(2x+3) + \frac{2x}{2x+3}\right]$$

(e) Let us practice using logarithmic differentiation. Compute the following according to the method on the previous page

i. $\frac{d}{dx}x^x$ Let $y = x^x$, then $\ln(y) = \ln(x^x) = x\ln(x)$

Now implicitly differentiate,

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}(\underbrace{x\ln(x)}_{\text{product}})$$

$$\hookrightarrow \frac{1}{y}y' = \ln(x) + x \cdot \frac{1}{x} \to y' = y(\ln(x) + 1)$$

$$\boxed{y' = x^x(\ln(x) + 1)}$$

ii. $\frac{d}{dx}x^{\sin(x)}$ Let $y = x^{\sin(x)}$, then $\ln(y) = \sin(x)\ln(x)$

Now implicitly differentiate,

$$\frac{d}{dx}\ln(y) = \frac{d}{dx}(\overbrace{\sin(x)\ln(x)}^{\text{Product}})$$

$$\hookrightarrow \frac{1}{y}y' = \cos(x)\ln(x) + \sin(x) \cdot \frac{1}{x} \to y' = y\left(\cos(x)\ln(x) + \frac{\sin(x)}{x}\right)$$

$$\boxed{y' = x^{\sin(x)}\left(\cos(x)\ln(x) + \frac{\sin(x)}{x}\right)}$$

iii. $\frac{d}{dx}(\arctan(x))^{1/x}$ Let $y = (\arctan(x))^{1/x} \to \ln(y) = \frac{1}{x}\ln(\arctan(x))$

Now implicitly differentiate, chain

$$\frac{d}{dx}\ln(y) = \frac{d}{dx}(\underbrace{\tfrac{1}{x}\ln(\arctan(x))}_{\text{product}})$$

$$\hookrightarrow \frac{1}{y} \cdot y' = \frac{-1}{x^2}\ln(\arctan(x)) + \frac{1}{x} \cdot \frac{1}{\arctan(x)} \cdot \frac{1}{1+x^2}$$

$$\hookrightarrow y' = y\left(\frac{-\ln(\arctan(x))}{x^2} + \frac{1}{x(1+x^2)\arctan(x)}\right)$$

$$\boxed{y' = (\arctan(x))^{1/x}\left(\frac{-\ln(\arctan(x))}{x^2} + \frac{1}{x^2(1+x^2)\arctan(x)}\right)}$$

iv. On your own paper and if you have time, find y' if $y^x = x^y$.

$\ln(y^x) = \ln(x^y)$

$\hookrightarrow x\ln(y) = y\ln(x)$

Now imp. diff.

$$\frac{d}{dx}(\underbrace{x\ln(y)}_{\text{product}}) = \frac{d}{dx}(\underbrace{y\ln(x)}_{\text{product}})$$

$$\ln(y) + x \cdot \frac{1}{y} \cdot y' = y'\ln(x) + \frac{y}{x}$$

$$\frac{x}{y}y' - y'\ln(x) = \frac{y}{x} - \ln(y)$$

$$\underbrace{y'\left(\frac{x}{y} - \ln(x)\right)}_{\text{get } y' \text{ to one side, factor out}} = \frac{y}{x} - \ln(y)$$

$$\boxed{y' = \frac{\frac{y}{x} - \ln(y)}{\frac{x}{y} - \ln(x)}}$$