APPM 1350 **Exam 1 Solutions** **Spring 2013**

1. (12 points) Consider the function $y = f(x)$. Use transformations to match the following functions to the graphs shown. No explanation is necessary.

   (a) $y = f(x+1)$ (c) $y = f(-x) - 1$

   (b) $y = f(2x)$ (d) $y = |-f(x) - 1|$

   Solution:

   (a) 8 (b) 1 (c) 4 (d) 2

2. (10 points) Let $f(x) = \sin x$ and $g(x) = \dfrac{x}{x^2+2}$.

   (a) Find $(g \circ f)(x)$.

   (b) What is the domain of $g \circ f$?

   (c) Is $g \circ f$ even, odd, or neither? Justify your answer.

   Solution:

   (a) $(g \circ f)(x) = g(f(x)) = g(\sin x) = \boxed{\dfrac{\sin x}{\sin^2 x + 2}}$

   (b) Observe that the function $\sin x$ is defined for all x, and since $\sin^2 x \geq 0$, the denominator cannot equal 0. The domain is $\boxed{(-\infty, \infty)}$.

   (c) Check $(g \circ f)(-x)$. Note that $\sin x$ is an odd function and therefore $\sin(-x) = -\sin x$.

   $$(g \circ f)(-x) = \frac{\sin(-x)}{\sin^2(-x) + 2} = \frac{-\sin x}{\sin^2 x + 2} = -(g \circ f)(x)$$

   Since $(g \circ f)(-x) = -(g \circ f)(x)$, then $g \circ f$ is an $\boxed{\text{odd}}$ function.

3. (14 points) Let $f(x) = \sqrt{5-4x}$.

   (a) Use the definition of the derivative to find $f'(x)$.

   (b) Find an equation of the normal line to the curve $y = f(x)$ at $x = -1$.

   Solution:

   (a)
   $$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{5-4(x+h)} - \sqrt{5-4x}}{h} \end{aligned}$$

   Multiply by the conjugate of the numerator.

   $$\begin{aligned} &= \lim_{h\to 0} \frac{\sqrt{5-4x-4h} - \sqrt{5-4x}}{h} \cdot \frac{\sqrt{5-4x-4h} + \sqrt{5-4x}}{\sqrt{5-4x-4h} + \sqrt{5-4x}} \\ &= \lim_{h\to 0} \frac{5-4x-4h-(5-4x)}{h(\sqrt{5-4x-4h} + \sqrt{5-4x})} \\ &= \lim_{h\to 0} \frac{-4h}{h(\sqrt{5-4x-4h} + \sqrt{5-4x})} \\ &= \lim_{h\to 0} \frac{-4}{\sqrt{5-4x-4h} + \sqrt{5-4x}} \\ &= \frac{-4}{2\sqrt{5-4x}} = \boxed{\frac{-2}{\sqrt{5-4x}}} \end{aligned}$$

   (b) The tangent slope is $f'(-1) = -2/\sqrt{5+4} = -2/3$. The normal slope is the negative reciprocal of the tangent slope, or $3/2$. The point of tangency is $x = -1, y = f(-1) = 3$. An equation of the normal line is therefore $\boxed{y = 3 + \frac{3}{2}(x+1)}$ or $y = \frac{3}{2}x + \frac{9}{2}$.

4. (32 points) Evaluate the following limits.

   (a) $\displaystyle\lim_{x\to 3^-} \frac{x^2 + x - 12}{9 - x^2}$ (b) $\displaystyle\lim_{x\to 0^-} \sqrt[3]{\frac{5x^3 - 3|x|}{x}}$ (c) $\displaystyle\lim_{x\to 0+} \sqrt{x} \cos\frac{\pi}{x}$

(d) $\displaystyle\lim_{x\to-\infty}\frac{7x-\sqrt{49x^2-8x}}{7x+\sqrt{x^2-6x}}$

Solution:

(a)
$$\begin{aligned}\lim_{x\to3^-}\frac{x^2+x-12}{9-x^2}&=\lim_{x\to3^-}\frac{(x+4)(x-3)}{(3+x)(3-x)}\\&=\lim_{x\to3^-}-\frac{x+4}{3+x}\\&=\boxed{-\frac{7}{6}}\end{aligned}$$

(b) Observe that $\displaystyle\lim_{x\to0^-}|x|=-x$.

$$\begin{aligned}\lim_{x\to0^-}\sqrt[3]{\frac{5x^3-3|x|}{x}}&=\left[\lim_{x\to0^-}\frac{5x^3-3|x|}{x}\right]^{1/3}\\&=\left[\lim_{x\to0^-}\frac{5x^3+3x}{x}\right]^{1/3}\\&=\left[\lim_{x\to0^-}(5x^2+3)\right]^{1/3}\\&=\boxed{\sqrt[3]{3}}\end{aligned}$$

(c) Use the Squeeze Theorem. First note that

$$\begin{gathered}-1\le\cos\frac{\pi}{x}\le1\\-\sqrt{x}\le\sqrt{x}\cos\frac{\pi}{x}\le\sqrt{x}.\end{gathered}$$

Next we calculate

$$\lim_{x\to0^+}-\sqrt{x}=0=\lim_{x\to0^+}\sqrt{x}\text{ (via direct substitution).}$$

Hence, $\displaystyle\lim_{x\to0^+}\sqrt{x}\cos\frac{\pi}{x}=\boxed{0}$.

(d) Note that $\sqrt{x^2}=|x|=-x$ for $x<0$.

$$\begin{aligned}\lim_{x\to-\infty}\frac{7x-\sqrt{49x^2-8x}}{7x+\sqrt{x^2-6x}}&=\lim_{x\to-\infty}\frac{7x-\sqrt{x^2\left(49-\frac{8}{x}\right)}}{7x+\sqrt{x^2\left(1-\frac{6}{x}\right)}}\\&=\lim_{x\to-\infty}\frac{7x-|x|\sqrt{49-\frac{8}{x}}}{7x+|x|\sqrt{1-\frac{6}{x}}}\\&=\lim_{x\to-\infty}\frac{7x+x\sqrt{49-\frac{8}{x}}}{7x-x\sqrt{1-\frac{6}{x}}}\\&=\lim_{x\to-\infty}\frac{7+\sqrt{49-\frac{8}{x}}}{7-\sqrt{1-\frac{6}{x}}}\\&=\frac{7+\sqrt{49}}{7-\sqrt{1}}=\frac{14}{6}=\boxed{\frac{7}{3}}\end{aligned}$$

5. (10 points) Show that the equation $\sqrt{x}=\sin x+\frac{1}{2}$ has at least one real root.

Solution:

We wish to show that $f(x)=\sqrt{x}-\sin x=\frac{1}{2}$ has at least one real root. Since $\sin x$ is a continuous function and $\sqrt{x}$ is continuous on $[0,\infty)$, then f is continuous on $[0,\infty)$. We use the Intermediate Value Theorem.

Note that

$$\begin{aligned}f(0)&=0-0=0,\\f(\pi)&=\sqrt{\pi}-0=\sqrt{\pi}.\end{aligned}$$

Since $f(0)<\frac{1}{2}$ and $f(\pi)>\frac{1}{2}$, by the Intermediate Value Theorem, $f(x)=\frac{1}{2}$ has a solution in the interval $(0,\pi)$.

6. (12 points) Use the definition of continuity to determine whether the following function g is continuous at $x = 0$.

$$g(x) = \begin{cases} 6\tan(2x)\csc(3x), & x < 0 \\ \sec^4(x + \frac{\pi}{4}), & x \geq 0 \end{cases}$$

Solution:

The function g is continuous at $x = 0$ if

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^+} g(x) = g(0).$$

Evaluate the one-sided limits and find the value of $g(0)$. We use the theorem $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$.

$$\begin{aligned} \lim_{x\to 0^-} g(x) &= \lim_{x\to 0^-} 6\tan(2x)\csc(3x) \\ &= \lim_{x\to 0^-} 6 \cdot \frac{\sin 2x}{\cos 2x} \cdot \frac{1}{\sin 3x} \\ &= \lim_{x\to 0^-} \frac{6}{\cos 2x} \cdot \frac{\sin 2x}{1} \cdot \frac{2x}{2x} \cdot \frac{1}{\sin 3x} \cdot \frac{3x}{3x} \\ &= \lim_{x\to 0^-} \frac{6}{\cos 2x} \cdot \frac{\sin 2x}{2x} \cdot \frac{2x}{3x} \cdot \frac{3x}{\sin 3x} \\ &= \frac{6}{1} \cdot 1 \cdot \frac{2}{3} \cdot 1 = 4 \\ \lim_{x\to 0^+} g(x) &= \lim_{x\to 0^+} \sec^4\left(x + \frac{\pi}{4}\right) \\ &= \sec^4\left(\frac{\pi}{4}\right) = \left(\sqrt{2}\right)^4 = 4 \\ g(0) &= \sec^4\left(\frac{\pi}{4}\right) = 4 \end{aligned}$$

Since $\lim_{x\to 0^-} g(x) = \lim_{x\to 0^+} g(x) = g(0)$, g is $\boxed{\text{continuous}}$ at $x = 0$.

7. (10 points) Find a parabola with equation $y = ax^2 + bx + c$ that has slope 1 at $x = 6$, slope -3 at $x = -2$ and passes through the point $(0, 5)$.

Solution:

Since the parabola passes through the point $(0, 5)$, we can use the coordinates to solve for constant c.

$$\begin{aligned} y(0) = a(0) + b(0) + c = 5 \\ c = 5 \end{aligned}$$

Next use y' to solve for a and b. We are given that $y'(6) = 1$ and $y'(-2) = -3$.

$$\begin{aligned} y' &= 2ax + b \\ y'(6) &= 12a + b = 1 \\ y'(-2) &= -4a + b = -3 \end{aligned}$$

Now subtract the two equations to eliminate b.

$$\begin{aligned} 16a &= 4 \\ a &= \frac{1}{4} \\ b &= 1 - 12a = -2 \end{aligned}$$

The parabola is $\boxed{y = \frac{x^2}{4} - 2x + 5}$.