1. For this problem, suppose $f(x) = \sqrt{x-6}$ and $g(x) = |2x-1|$.

   (a) (6 pts) Write down the domain and range of $(f \circ g)(x)$. Show all work.

   (b) (6 pts) Evaluate the limit: $\lim\limits_{x\to 0.5} \dfrac{g(x)}{1-2x}$.

   (c) (6 pts) Suppose we let $h(x) = \begin{cases} f(x), \text{if } x > 6 \\ g(x), \text{if } x \le 6 \end{cases}$, are there any values of x for which $h(x)$ is *not* continuous? Justify your answer. What type of discontinuities does $h(x)$ have (i.e. *jump*, *removable*, or *infinite*), if any?

   (d) (7 pts) Use the limit definition of the derivative to find $f'(106)$.

Solution:

(a)(6 pts) Note that $(f \circ g)(x) = \sqrt{|2x-1|-6}$, and we need $|2x-1| - 6 \ge 0$, thus

$$|2x-1| - 6 \ge 0 \iff |2x-1| \ge 6 \iff 2x-1 \ge 6 \text{ or } 2x-1 \le -6 \iff x \ge 7/2 \text{ or } x \le -5/2$$

so the $\boxed{\text{domain is all } x \text{ such that } x \le -5/2 \text{ or } x \ge 7/2}$ and $\boxed{\text{the range is all } y \text{ such that } y \ge 0.}$

(b)(6 pts) Note that this is a two sided limit and, in this case, we must look at the one sided limits:

$$\lim_{x\to 0.5^-} \frac{g(x)}{1-2x} = \lim_{x\to 0.5^-} \frac{|2x-1|}{1-2x} = \lim_{x\to 0.5^-} \frac{-(2x-1)}{1-2x} = 1 \text{ and } \lim_{x\to 0.5^+} \frac{|2x-1|}{1-2x} = \lim_{x\to 0.5^+} \frac{(2x-1)}{1-2x} = -1$$

and so $\boxed{\lim\limits_{x\to 0.5} \dfrac{g(x)}{1-2x} = \text{does not exist.}}$

(c)(6 pts) Note that for $x \ne 6$, $h(x)$ is continuous since both $f(x)$ and $g(x)$ are well defined and continuous. At $x = 6$, we need $\lim\limits_{x\to 6^+} h(x) = \lim\limits_{x\to 6^-} h(x) = h(6)$ for continuity. Now note that

$$\lim_{x\to 6^+} h(x) = \lim_{x\to 6^+} \sqrt{x-6} = 0 \text{ and } \lim_{x\to 6^-} h(x) = \lim_{x\to 6^-} |2x-1| = 11 = h(6)$$

so $\lim\limits_{x\to 6^+} h(x) \ne \lim\limits_{x\to 6^-} h(x)$ and so $\boxed{h(x) \text{ is not continuous at } x = 6 \text{ since there is a jump discontinuity at } x = 6.}$

(d)(7 pts) Note that

$$f'(106) = \lim_{h\to 0} \frac{f(106+h) - f(106)}{h} \text{ or } f'(106) = \lim_{x\to 106} \frac{f(x) - f(106)}{x - 106}$$

using the first limit:

$$\begin{aligned} f'(106) = \lim_{h\to 0} \frac{f(106+h) - f(106)}{h} &= \lim_{h\to 0} \frac{\sqrt{(106+h)-6} - \sqrt{100}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{100+h} - 10}{h} \cdot \frac{\sqrt{100+h}+10}{\sqrt{100+h}+10} \\ &= \lim_{h\to 0} \frac{(100+h) - 100}{h(\sqrt{100+h}+10)} = \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{100+h}+10)} = \frac{1}{20} \end{aligned}$$

or, using the second limit:

$$\begin{aligned} f'(106) = \lim_{x\to 106} \frac{f(x) - f(106)}{x-106} &= \lim_{x\to 106} \frac{\sqrt{x-6} - \sqrt{100}}{x-106} \\ &= \lim_{x\to 106} \frac{\sqrt{x-6} - 10}{x-106} \cdot \frac{\sqrt{x-6}+10}{\sqrt{x-6}+10} \\ &= \lim_{x\to 106} \frac{(x-6) - 100}{(x-106)(\sqrt{x-6}+10)} = \lim_{x\to 106} \frac{\cancel{(x-106)}}{\cancel{(x-106)}(\sqrt{x-6}+10)} = \frac{1}{20} \end{aligned}$$

in both cases $\boxed{f'(106) = 1/20.}$

2. (a)(7 pts) Evaluate the limit or explain why it doesn't exist: $\lim\limits_{x\to -3} \dfrac{x^2-9}{2x^2+4x-6}$

(b)(6 pts) Evaluate the limit or explain why it doesn't exist: $\lim\limits_{x\to 1^-} \dfrac{x^2-9}{2x^2+4x-6}$

(c)(6 pts) Find all horizontal asymptotes of $f(x) = \dfrac{x^2-9}{2x^2+4x-6}$. Justify your answer, show all work.

(d)(6 pts) Find and classify all the discontinuities of $f(x) = \dfrac{x^2-9}{2x^2+4x-6}$, if any. Justify your answer.

Solution:

(a)(7 pts) Note that,

$$\lim_{x\to -3} \frac{x^2-9}{2x^2+4x-6} = \lim_{x\to -3} \frac{(x-3)(x+3)}{2(x-1)(x+3)} = \lim_{x\to -3} \frac{(x-3)\cancel{(x+3)}}{2(x-1)\cancel{(x+3)}} = \frac{-6}{-8} = \frac{3}{4}$$

So the limit exists and converges to 3/4.

(b)(6 pts) Here we have,

$$\lim_{x\to 1^-} \frac{x^2-9}{2x^2+4x-6} = \lim_{x\to 1^-} \frac{(x-3)}{2(x-1)} = +\infty$$

Thus the limit does not exist, it diverges to $+\infty$.

(c)(6 pts) To find horizontal asymptotes, we take the limit of $f(x)$ as $x\to\infty$ and $x\to -\infty$:

$$\lim_{x\to \infty} \frac{x^2-9}{2x^2+4x-6} = \lim_{x\to \infty} \frac{x^2(1-9/x^2)}{x^2(2+4/x-6/x^2)} = \lim_{x\to\infty} \frac{\cancel{x^2}(1-\cancelto{0}{9/x^2})}{\cancel{x^2}(2+\cancelto{0}{4/x}-\cancelto{0}{6/x^2})} = \frac{1}{2}$$

and, similarly,

$$\lim_{x\to -\infty} \frac{x^2-9}{2x^2+4x-6} = \lim_{x\to -\infty} \frac{\cancel{x^2}(1-9/x^2)}{\cancel{x^2}(2+4/x-6/x^2)} = \frac{1}{2}$$

thus, $\boxed{y = 1/2 \text{ is a horizontal asymptote of } f(x).}$

(d)(6 pts) From the work done in parts (a) and (b), we see that $\boxed{x=-3 \text{ is a removable discontinuity}}$ of $f(x)$ and the vertical asymptote $\boxed{x=1 \text{ is an infinite discontinuity}}$ of $f(x)$ (note that it suffices to just check one of the 1-sided limits to classify $x=1$).

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3. (a)(10 pts) Given the function $f(x) = \begin{cases} x^3\cos\left(\dfrac{1}{x^2}\right), & \text{if } x\neq 0 \\ a-4, & \text{if } x=0 \end{cases}$, find the value of the parameter a that makes $f(x)$ continuous for every real number.

(b)(5 pts) Evaluate the following limit if possible: $\lim\limits_{x\to 0}\big(x^3+11\cos(2x)\big)$.

(c)(5 pts) If $b(x) = x^3+11\cos(2x)$, show that there is at least one solution to the equation $b(x)=0$.

(d)(5 pts) Evaluate $\lim\limits_{\theta\to 0}\dfrac{1-\cos(\theta)}{\theta^2}$. (*Hint:* recall that $\sin^2(\theta)+\cos^2(\theta)=1$ for any real number θ).

Solution:

(a)(10 pts) At $x=0$, we need $\lim\limits_{x\to 0} f(x) = f(0)$. Since the limit is indeterminate, we use the Squeeze Theorem and because of the way the function is defined, we will look at the 1-sided limits.

Now if $x<0$ then $x^3<0$ and

$$-1 \le \cos\left(\frac{1}{x^2}\right) \le 1 \Rightarrow -x^3 \ge x^3\cos\left(\frac{1}{x^2}\right) \ge x^3 \text{ and note } \lim_{x\to 0^-} x^3 = \lim_{x\to 0^-} -x^3 = 0$$

and so by the Squeeze Theorem we have that $\lim_{x\to 0^-} x^3 \cos\left(\frac{1}{x^2}\right) = 0$. Now, similarly, if $x > 0$ then $x^3 > 0$ and

$$-1 \le \cos\left(\frac{1}{x^2}\right) \le 1 \Rightarrow -x^3 \le x^3 \cos\left(\frac{1}{x^2}\right) \le x^3 \text{ and note } \lim_{x\to 0^+} x^3 = \lim_{x\to 0^+} -x^3 = 0$$

and so by the Squeeze Theorem we have that $\lim_{x\to 0^+} x^3 \cos\left(\frac{1}{x^2}\right) = 0$ and thus for the 2-sided limit we have $\boxed{\lim_{x\to 0} x^3 \cos\left(\frac{1}{x^2}\right) = 0.}$ Finally, for continuity, we need

$$f(0) = \lim_{x\to 0} f(x) \Rightarrow f(0) = 0 \Rightarrow a - 4 = 0 \Rightarrow \boxed{a = 4}$$

Thus if $a = 4$ then $f(x)$ will be continuous.

For $x \neq 0$, note that $f(x)$ is continuous since polynomials, trigonometric functions and rational functions are continuous on their domains and products and compositions of continuous functions are again continuous on their domains.

(b)(5 pts) Here, by continuity, we can evaluate the limit by plugging $x = 0$ into the function, thus

$$\lim_{x\to 0} \left(x^3 + 11\cos(2x)\right) = 0^3 + 11\cos(0) = 11$$

(c)(5 pts) First note that $b(x)$ is continuous for all real numbers since it is the sum of continuous functions, so we will attempt to use the Intermediate Value Theorem. Note that, from part (b), we know $b(0) = 11 > 0$ and if $x = -\pi/4$ then

$$b(-\pi/4) = \left(-\frac{\pi}{4}\right)^3 + 11\cos\left(-\frac{\pi}{2}\right) = -\frac{\pi^3}{4^3} < 0 \Longrightarrow \boxed{b(-\pi/4) < 0 < b(0)}$$

and since $b(x)$ is continuous on $[-\pi/4, 0]$, by the Intermediate Value Theorem there exists at least one number c in $(-\pi/4, 0)$ such that $b(c) = 0$.

(d)(5 pts) From the hint, we see that $\sin^2(\theta) = 1 - \cos^2(\theta) = (1-\cos(\theta))(1+\cos(\theta))$, and since this limit is of the type "0/0", let us try to multiply top and bottom by $1 + \cos(\theta)$:

$$\lim_{\theta\to 0} \frac{1-\cos(\theta)}{\theta^2} = \lim_{\theta\to 0} \frac{1-\cos(\theta)}{\theta^2} \cdot \frac{1+\cos(\theta)}{1+\cos(\theta)} = \lim_{\theta\to 0} \frac{1-\cos^2(\theta)}{\theta^2} \cdot \frac{1}{1+\cos(\theta)} = \lim_{\theta\to 0} \frac{\sin^2(\theta)}{\theta^2} \cdot \frac{1}{1+\cos(\theta)}$$

now recall the special limit $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1$, the properties of limits implies $\lim_{\theta\to 0} \left[\frac{\sin(\theta)}{\theta}\right]^2 = (1)^2 = 1$ and so,

$$\lim_{\theta\to 0} \frac{1-\cos(\theta)}{\theta^2} = \lim_{\theta\to 0} \left[\frac{\sin(\theta)}{\theta}\right]^2 \cdot \frac{1}{1+\cos(\theta)} = 1 \cdot \frac{1}{1+1} = \boxed{\frac{1}{2}}$$

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4. (a)(10 pts) Use the limit definition of the derivative to find the slope of $f(x) = 2x^2 - 13x + 5$ at any point x.

(b)(5 pts) Find an equation of the tangent line to the parabola $f(x) = 2x^2 - 13x + 5$ whose slope is $m = -9$.

(c)(5 pts) If $s(t) = 2t^2 - 13t + 5$ for $t \ge 0$ describes the position of an object (in feet) at time t, find the average velocity of the object from $t = 1$ second to $t = 2$ seconds.

(d)(5 pts) Note that if $g(x) = x^2$, then $g'(x) = 2x$. Now, suppose L is the tangent line to $y = g(x)$ at the point $(1, 1)$. The *angle of inclination* of L is the angle ϕ that L makes with the positive x-axis, find $\tan(\phi)$.

Solution:

(a)(10 pts) Recall that the slope of $f(x)$ at any point x is $f'(x)$ and note that we have two formulas, namely:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \quad \text{or} \quad f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

using the first limit:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{2(x+h)^2-13(x+h)+5-(2x^2-13x+5)}{h}\\
&= \lim_{h\to 0}\frac{(\cancel{2x^2}+4xh+2h^2)+(\cancel{-13x}-13h)+\cancel{5}-(\cancel{2x^2}-\cancel{13x}+\cancel{5})}{h} = \lim_{h\to 0}\frac{4xh+2h^2-13h}{h}\\
&= \lim_{h\to 0}\frac{\cancel{h}(4x+2h-13)}{\cancel{h}} = 4x-13 \Rightarrow f'(x) = 4x-13
\end{aligned}$$

or, using the second limit:

$$\begin{aligned}
f'(a) &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{(2x^2-13x+\cancel{5})-(2a^2-13a+\cancel{5})}{x-a}\\
&= \lim_{x\to a}\frac{2(x^2-a^2)-13(x-a)}{x-a} = \lim_{x\to a}\frac{\cancel{(x-a)}\big(2(x+a)-13\big)}{\cancel{x-a}} = 4a-13 \Rightarrow f'(a) = 4a-13
\end{aligned}$$

in both cases $\boxed{f'(x) = 4x-13.}$

(b)(5 pts) From part (a), we know the slope of the tangent line is $f'(x) = 4x-13$ and $f'(x) = m$ implies $4x-13 = -9$ which yields $x = 1$, thus the equation of the tangent line with slope $m = -9$ is

$$y = f(1) + f'(1)(x-1) \Longrightarrow y = -6-9(x-1) \Longrightarrow \boxed{y = -9x+3}$$

(c)(5 pts) The average velocity is

$$\boxed{\frac{s(2)-s(1)}{2-1} = \frac{(-13)-(-6)}{1} = -7 \text{ ft/s}}$$

(d)(5 pts) Here the equation of the tangent line at $(1,1)$ is $y = 2x-1$ and the tangent line intersects the x-axis when $x = 1/2$. Note that the tangent line, the x-axis and the line $x = 1$ form a triangle with one angle equal to ϕ (see graph below). Thus, we can see from the graph that $\tan(\phi)$ is the same as the slope of the tangent line at $x = 1$, i.e. $\tan(\phi) = g'(1) = 2$ or, using trigonometry, we can calculate $\tan(\phi)$:

$$\tan(\phi) = \frac{1}{1/2} = 2$$