1. (28 pts, 7 pts each) Evaluate the following limits and show all supporting work (please do not use l'Hospital's Rule). If a limit does not exist, clearly state that fact and explain your reasoning. Make sure to simplify your answer completely.

   (a) $\lim_{\theta \to 0} \cos\left(3\sqrt{\frac{\pi}{4}(\pi + \theta\pi)}\right)$

   (b) $\lim_{x \to 0} \frac{x^2}{\tan^2(2x)}$

   (c) $\lim_{x \to 7} \frac{2x-14}{|x-7|}$

   (d) $\lim_{x \to -\infty} \frac{12x}{\sqrt{2x^2-7}}$

   Solution:

   (a) $\lim_{\theta \to 0} \cos\left(3\sqrt{\frac{\pi}{4}(\pi + \theta\pi)}\right) = \cos\left(3\sqrt{\frac{\pi}{4}(\pi + 0\pi)}\right) = \cos\left(3\sqrt{\frac{\pi^2}{4}}\right) = \cos\left(\frac{3\pi}{2}\right) = 0$

   (b) $\lim_{x \to 0} \frac{x^2}{\tan^2(2x)} = \lim_{x \to 0} \frac{\cos^2(2x)x^2}{\sin^2(2x)} = \lim_{x \to 0} \frac{\cos^2(2x)}{4} \lim_{x \to 0} \frac{(2x)^2}{\sin^2(2x)} = \frac{1}{4} \cdot (1)^2 = \frac{1}{4}$

   (c) $\lim_{x \to 7} \frac{2x-14}{|x-7|}$

   Note that: $\begin{cases} \frac{2x-14}{x-7}, & x \geq 7 \\ \frac{2x-14}{-(x-7)}, & x < 7 \end{cases} = \begin{cases} 2, & x \geq 7 \\ -2, & x < 7 \end{cases}$. So that:

   $\lim_{x \to 7^+} \frac{2x-14}{|x-7|} = 2$ and $\lim_{x \to 7^-} \frac{2x-14}{|x-7|} = -2$

   Therefore, $\lim_{x \to 7} \frac{2x-14}{|x-7|} =$ DNE, since the left hand limit does not equal the right hand limit.

   (d) $\lim_{x \to -\infty} \frac{12x}{\sqrt{2x^2-7}} = \lim_{x \to -\infty} \frac{12x}{|x|\sqrt{2-\frac{7}{x^2}}}$. Since the limit is approaching negative infinity, then we replace $|x|$ by $-x$

   $$\lim_{x \to -\infty} \frac{12x}{|x|\sqrt{2-\frac{7}{x^2}}} = \lim_{x \to -\infty} \frac{12x}{-x\sqrt{2-\frac{7}{x^2}}} = \lim_{x \to -\infty} \frac{12}{-\sqrt{2-\frac{7}{x^2}}} = \frac{12}{-\sqrt{2}} = -6\sqrt{2}$$

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2. (19 pts) Let $f(x) = \sqrt{-x} - 2$, $k(x) = \sin(x)$

   (a) (4 pts) State the domain of f.

   (b) (5 pts) Sketch the graph of f.

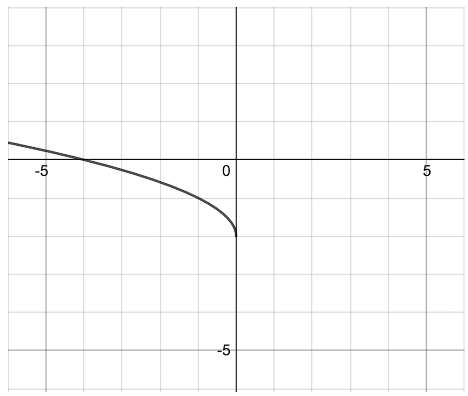
(c) (4 pts) State the range of f.

(d) (6 pts) Does the function $\dfrac{k(x)}{f(x)}$ have vertical asymptotes? Use appropriate limits to justify.

Solution:

(a) Domain: $-x \geq 0$ so $x \leq 0$ or $(-\infty, 0]$.

(b)



(c) Range: From the graph: $[-2, \infty)$.

(d) $\dfrac{k(x)}{f(x)} = \dfrac{\sin(x)}{\sqrt{-x}-2}$. Note that for $x = -4$, $\sin(-4) > 0$ and the denominator is zero. This implies that $x = -4$ may be a vertical asymptote. To show that $x = -4$ is a vertical asymptote we need to show that $\lim\limits_{x\to -4^+} \dfrac{\sin(x)}{\sqrt{-x}-2}$ or $\lim\limits_{x\to -4^-} \dfrac{\sin(x)}{\sqrt{-x}-2}$ is either ∞ or $-\infty$. We see that $\lim\limits_{x\to -4^+} \dfrac{\sin(x)}{\sqrt{-x}-2} = -\infty$ and $\lim\limits_{x\to -4^-} \dfrac{\sin(x)}{\sqrt{-x}-2} = \infty$. So that, by definition, $x = -4$ is a vertical asymptote.

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3. (14 pts) Consider the function $g(x) = \begin{cases} bx^2 + \frac{7\sqrt{2}}{8}, & x > -1, \\ \sqrt{1-x}, & x \leq -1 \end{cases}$.

(a) (10 pts) For what value of b is $g(x)$ <u>differentiable</u> at $x = -1$? Use the <u>limit definition of the derivative</u> for this problem. Justify your answer.

(b) (4 pts) Use your answer to part (a) to find the tangent line of $g(x)$ at $x = -1$.

Solution:

(a) We need to find a b value such that, $\lim\limits_{x\to -1^-} \dfrac{g(x)-g(-1)}{x-(-1)} = \lim\limits_{x\to -1^+} \dfrac{g(x)-g(-1)}{x-(-1)}$. Looking at the right hand limit of the derivative:

$$\lim_{x\to -1^+} \frac{g(x)-g(-1)}{x-(-1)} = \lim_{x\to -1^+} \frac{bx^2 + \frac{7\sqrt{2}}{8} - \sqrt{2}}{x+1} = \lim_{x\to -1^+} \frac{bx^2 - \frac{\sqrt{2}}{8}}{x+1}.$$

Since the denominator has a limit of zero and we want the limit to exist then the numerator must also have a limit of zero. So $\lim_{x\to -1^+} bx^2 - \frac{\sqrt{2}}{8} = 0$ which gives us that $b = \frac{\sqrt{2}}{8}$. Plugging this value in to the right hand limit of the derivative we see $\lim_{x\to -1^+} \frac{\frac{\sqrt{2}}{8}x^2 - \frac{\sqrt{2}}{8}}{x+1} = \lim_{x\to -1^+} \frac{\sqrt{2}}{8}(x-1) = -\frac{\sqrt{2}}{4}$.

And the left hand limit of the derivative gives:

$\lim_{x\to -1^-} \frac{g(x) - g(-1)}{x-(-1)} = \lim_{x\to -1^-} \frac{\sqrt{1-x} - \sqrt{2}}{x+1} = \lim_{x\to -1^-} \frac{1-x-2}{(x+1)(\sqrt{1-x}+\sqrt{2})}$ by multiplying top and bottom by the conjugate. We now get: $\lim_{x\to -1^-} \frac{-(x+1)}{(x+1)(\sqrt{1-x}+\sqrt{2})} =$ $\lim_{x\to -1^-} \frac{-1}{(\sqrt{1-x}+\sqrt{2})} = -\frac{1}{2\sqrt{2}} = -\frac{\sqrt{2}}{4}$. So we see the derivative exists at $x = -1$ with a value of $-\frac{\sqrt{2}}{4}$.

(b) The derivative of g at $x = -1$ in part (a) is $g'(-1) = -\frac{\sqrt{2}}{4}$. To find the tangent line to $g(x)$ at $x = -1$, note that $g(-1) = \sqrt{2}$ and the tangent line is given by: $y - \sqrt{2} = -\frac{\sqrt{2}}{4}(x-(-1))$ and $y = -\frac{\sqrt{2}}{4}x + \frac{3\sqrt{2}}{4}$.

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4. (18 pts) Consider the function $s(x) = \frac{|5x|}{x^2+2x}$

(a) (8 pts) For what value(s) of x is this function discontinuous? Justify your answer(s) by showing how the definition of continuity fails for each value.

(b) (4 pts) Label each discontinuity in part (a) as: *removable*, *jump*, or *infinite* discontinuity. Justification is not necessary for this part.

Solution:

(a) $s(x)$ is discontinuous at $x = -2$ and $x = 0$ since $s(-2)$ and $s(0)$ are undefined so that the definition of continuity $\lim_{x\to a} s(x) = s(a)$ cannot be true for $a = -2, 0$.

(b) $x = -2$ is an infinite discontinuity and $x = 0$ is a jump discontinuity. To determine why, first note that $s(x) = \frac{|5x|}{x^2+2x} = \begin{cases} \frac{5x}{x^2+2x}, & x \geq 0 \\ \frac{-5x}{x^2+2x}, & x < 0 \end{cases} = \begin{cases} \frac{5}{x+2}, & x \geq 0 \\ \frac{-5}{x+2}, & x < 0 \end{cases}$. Then at $x = -2$, we see that $\lim_{x\to -2^-} s(x) = \lim_{x\to -2^-} \frac{-5}{x+2} = \infty$ indicating an infinite discontinuity. For the discontinuity at $x = 0$, $\lim_{x\to 0^-} s(x) = \lim_{x\to 0^-} \frac{-5}{x+2} = -\frac{5}{2}$ and $\lim_{x\to 0^+} s(x) = \lim_{x\to 0^+} \frac{5}{x+2} = \frac{5}{2}$. Since both the left hand limit and the right hand limit exist but are not equal, this indicates the discontinuity at $x = 0$ is a jump discontinuity.

(c) (6 pts) Is the function $s(x)$ odd, even, or neither? Justify your answer.

Using the definition of odd and even functions: $s(-x) = \frac{|5(-x)|}{(-x)^2 + 2(-x)} = \frac{|5x|}{x^2 - 2x}$. Since $s(-x) \neq -s(x)$ and $s(-x) \neq s(x)$ then $s(x)$ is neither odd nor even. We can see this more clearly by providing an example to show s is neither odd nor even. Consider the evaluation of s at $x = -1$: $s(-1) = \frac{|5(-1)|}{(-1)^2 + 2(-1)} = \frac{5}{1-2} = -5$. Since $s(-1) \neq -1$ and $s(-1) \neq 1$ then s is neither odd nor even.

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5. (21 pts, 7 pts each) Some unrelated short answer questions:

(a) The limit, $\lim_{h \to 0} \frac{\sqrt[4]{16+h} - 2}{h}$, represents the derivative of some function f at some number a. State both the function f and the number a.

(b) Does $\sqrt[3]{x} = x^2 - \frac{5}{2}$ have a solution? Justify your answer.

(c) Either sketch or formulate a function, f, with the following properties: f is an even function, $\lim_{x \to 3^+} f(x) = a$ where a is a nonzero value, and the domain of f is $(-\infty, -3) \cup (3, \infty)$.

Solution:

(a) $f(x) = \sqrt[4]{x}$ and $a = 16$ work.

(b) Let $f(x) = \sqrt[3]{x} - x^2 + \frac{5}{2}$. Since $f(0) = \frac{5}{2}$, $f(8) = \frac{-119}{2}$, and f is continuous on the real numbers then by the Intermediate Value Theorem there is a number c that satisfies $0 \leq c \leq 8$ such that $f(c) = 0$. Therefore, c is also a solution of $\sqrt[3]{x} = x^2 - \frac{5}{2}$.

(c) $f(x) = \frac{x^2 - 9}{\sqrt{x^2 - 9}} + x^2$ satisfies all three conditions.