1. (14 points) Consider the function $f(x) = \cos^2 x$ on the interval $[-\pi, \pi]$.

   (a) On which intervals is $f(x)$ increasing and on which intervals is $f(x)$ decreasing?

   (b) Name any points of inflection. Make sure to **verify** the points are indeed points of inflection.

Solution:

(a) In order to find where the function is increasing and decreasing, first find the critical points, and then check the behavior of the function in-between the critical points in order to name the intervals where the function is increasing and decreasing. Critical points are where the derivative is equal to zero, or is undefined.
$f'(x) = -2\cos x \sin x$
This derivative is defined for all x values. However it equals zero at several x values within the interval of concern. $f'(x) = 0 \Rightarrow x = -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi$
Now we check the behavior of the function between the critical points. $f'(-\frac{3\pi}{4}) < 0$, $f'(-\frac{\pi}{4}) > 0$, $f'(\frac{\pi}{4}) < 0$, $f'(\frac{3\pi}{4}) > 0$
Therefore $f(x)$ is decreasing on $[-\pi, -\frac{\pi}{2}]$, and $[0, \frac{\pi}{2}]$
$f(x)$ is increasing on $[-\frac{\pi}{2}, 0]$, and $[\frac{\pi}{2}, \pi]$.

(b) Possible locations of points of inflection are found where the second derivative equals zero, or is undefined. $f''(x) = -2(\cos^2 x - \sin^2 x) = -2\cos 2x$

$f''(x) = 0$ when $2x = 0$ or when $x = -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$.
These locations are only **possible** x value locations of points of inflection. They need to be verified. Verification is accomplished by checking concavity on either side of these points.
$f''(-\pi) < 0$, $f''(\frac{-\pi}{2}) > 0$, $f''(0) < 0$, $f''(\frac{\pi}{2}) > 0$, $f''(\pi) < 0$
Therefore we have $f(x)$ concave-down on $[-\pi, -\frac{3\pi}{4}]$ and $[-\frac{\pi}{4}, \frac{\pi}{4}]$, and $[\frac{3\pi}{4}, \pi]$
$f(x)$ is concave-up on the intervals $[-\frac{3\pi}{4}, -\frac{\pi}{4}]$, and $[\frac{\pi}{4}, \frac{3\pi}{4}]$.

2. (16 points) For two resistors, R_1 and R_2, connected in parallel, the combined electrical resistance R is given by $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. Further note that R, R_1, and R_2 are all functions of time and are measured in ohms. R_1 and R_2 are each increasing at rates of $\frac{1}{2}$ ohms per second. At what rate is the combined resistance changing when $R_1 = 2$ ohms and $R_2 = 4$ ohms?

Solution:

Write the given relationship in a convenient form for derivation

$$R^{-1} = R_1^{-1} + R_2^{-1}$$

Now implicitly derivate the given relationship in order to find $\frac{dR}{dt}$

$$-\frac{1}{R^2}\frac{dR}{dt} = -\frac{1}{R_1^2}\frac{dR_1}{dt} - \frac{1}{R_2^2}\frac{dR_2}{dt}$$

$$\frac{1}{R^2}\frac{dR}{dt} = \frac{1}{R_1^2}\frac{dR_1}{dt} + \frac{1}{R_2^2}\frac{dR_2}{dt}$$

$$\frac{dR}{dt} = R^2\left[\frac{1}{R_1^2}\frac{dR_1}{dt} + \frac{1}{R_2^2}\frac{dR_2}{dt}\right]$$

Now input the given values and calculate,

$$= \frac{16}{9}\left[\frac{1}{4}\frac{1}{2} + \frac{1}{16}\frac{1}{2}\right]$$

$$= \frac{1}{2}\frac{16}{9}\left[\frac{1}{4} + \frac{1}{16}\right]$$

$$= \boxed{\frac{5}{18}}$$

3. Consider the function $h(x) = \sqrt{25 - x^2}$.

   (a) (12 points) What is an appropriate linear approximation that could be used to estimate $h(2.9)$?

   (b) (2 points) Would the linear approximation of $h(2.9)$ provide an overestimate or an underestimate?

   Solution:

   (a) We are asked to find the equation of a line. In particular, the line used for approximating values for $f(x) = \sqrt{x}$

   This means we need a point and a slope for the point-slope form of an equation. Slope is found using the derivative of the given function.

   $$h'(x) = \frac{-x}{\sqrt{25 - x^2}}$$

   In particular we want the slope at $x = 3$.

   Slope is $h'(3) = -\dfrac{3}{4}$

   point $(3, 4)$

   $$L(x) = -\frac{3}{4}x + \frac{25}{4}$$

   (b) Because the graph is Con-cave down, our approximating line is above the curve and hence this is an over-estimate.

4. (16 points) Consider the function $f(x) = \sqrt{x^2 - 25}$.

   (a) Determine whether Rolle's theorem can be applied to $f(x)$ on $[-13, 13]$.
   If so, then find all values of c in $(-13, 13)$ satisfying the conclusion of the theorem.
   If not, then explain why the theorem does not apply in this instance.

   (b) Determine whether the Mean Value Theorem can be applied to $f(x)$ on $[5, 13]$.
   If so, then find all c guaranteed by the theorem.
   If not, then explain why the theorem does not apply in this instance.

Solution:

(a) Rolle's theorem cannot be applied since the function does not exist between $(-5, 5)$.

(b) The MVT applies in this case because the function is the radical of a continuous polynomial which is not negative on the given interval. Furthermore the function is differentiable on the given interval; note the derivative is defined for all values inside the given interval. Therefore we know the average r.o.c. will equal the instantaneous r.o.c. at some value c inside the interval $[5, 13]$.

$$f'(c) = \frac{f(13) - f(5)}{13 - 5} = \frac{3}{2}$$

$$\frac{c}{\sqrt{c^2 - 25}} = \frac{3}{2}$$

$$\frac{c^2}{c^2 - 25} = \frac{9}{4}$$

$$4c^2 = 9c^2 - 9(25)$$

$$5c^2 = 9(25)$$

$$c^2 = 45$$

$$c = 3\sqrt{5}$$

5. (12 points) Determine the point(s) at which the graph of $y^4 = y^2 - x^2$ has a horizontal tangent. Hint: there is no horizontal tangent at the origin.

Solution: We are asked to find locations where $\frac{dy}{dx} = 0$. Because we cannot solve the equation for y we must find $\frac{dy}{dx}$ implicitly.

$y^4 = y^2 - x^2$

$4y^3 y\prime = 2yy\prime - 2x$

$2x = (2y - 4y^3)y\prime$

$$y\prime = \frac{2x}{2y - 4y^3}$$

In order for this fraction to equal zero, the numerator must equal zero and the denominator must be a non-zero real number. This occurs so long as $2x = 0$ or when $x = 0$ and $y \neq 0$, or $y \neq \pm\sqrt{\frac{1}{2}}$

$$y\prime = \frac{2x}{2y - 4y^3} = 0 \Rightarrow x = 0$$

Using $x = 0$ inside the function $y^4 = y^2 - x^2$ produces $y^4 = y^2$ or $y^2(y^2 - 1) = 0$. This produces y values of $0, 1,$ and -1. The origin has already been excluded.

Therefore, there are horizontal tangents at $(0, 1)$ and $(0, -1)$

6. (28 points) Indicate, in your blue book, the following statements as True or False. No explanation required.

(a) A point c in $(-2, 2)$ is guaranteed to exist such that the instantaneous rate of change of $f(x) = \dfrac{x}{(x^2+1)^2}$ is $\dfrac{1}{25}$.

(b) $g(x) = x^{\frac{2}{3}}(2-x)$ has a local minimum at $x = 0$.

(c) $x = 0$ is a critical point and local extremum of $f(x) = x^4 - 2x^3$.

(d) The origin is a point of inflection for the function $f(x) = x^6$.

(e) The function $g(x) = \begin{cases} x^2, & -1 \le x < 2 \\ 9 - 3x, & 2 \le x \le 3 \end{cases}$ has an absolute maximum of 4.

(f) Every point where a function possesses a horizontal tangent is a local extremum of the function.

(g) The point of inflection of $f(x) = x(x-6)^2$ lies midway between the relative extrema of f.

Solution:
(a) True
(b) True
(c) False
(d) False
(e) False
(f) False
(g) True

$\boxed{\text{END of Exam}}$