1. (a)(8 pts) At what point on the curve $y = 1 + 2e^x - 3x$ is the tangent line parallel to the line $3x - y = 5$? Specify both the x and y coordinates.

(b)(8 pts) Use the Squeeze Theorem to evaluate the following limit: $\lim_{x\to 0^+} \sqrt{x}e^{\sin(\pi/x)}$.

(c)(8 pts) Find the absolute extrema of $g(x) = \ln(x^2 + x + 1)$ for $-1 \le x \le 1$. Specify both the x and y coordinates of all extrema.

(d)(6 pts) Which of the five choices given below is equivalent to y' if $y = (\sin x)^{\ln(x)}$? Pick only one answer, **no justification necessary** - *be sure to copy down the entire answer, don't just write down the roman numeral of your choice*:

$(i)\ \ln(x)\sin(x)^{\ln(x)-1} \qquad (ii)\ -\sin(x)^{\ln(x)}\cos(x)^{1/x} \qquad (iii)\ \sin(x)^{\ln(x)}\left[\frac{\ln(\sin x)}{x} + \ln(x)\tan(x)\right]$

$(iv)\ \sin(x)^{\ln(x)}\left[\frac{\ln(\sin(x))}{x} + \ln(x^{\cot x})\right] \qquad (v)\ \frac{\ln(\sin(x))}{x} + \ln(x)\tan(x)$

Solution: (a) The slope of the tangent line is $dy/dx = 2e^x - 3$ and the slope of the line $3x - y = 5 \Rightarrow y = 3x - 5$ is $m = 3$ and so

$$2e^x - 3 = 3 \Rightarrow e^x = \frac{6}{2} = 3 \Rightarrow x = \ln(3)$$

and note that when $x = \ln(3)$ we have $y = 1 + 2e^{\ln 3} - 3\cdot\ln(3) = 7 - 3\ln(3)$ thus the curve $y = 1 + 2e^x - 3x$ has a tangent line parallel to the line $3x - y = 5$ at the point $\boxed{(x, y) = (\ln(3), 7 - 3\ln(3)).}$

(b) First note that $y = e^x$ is an increasing function and, now, for any $x > 0$ we have

$$-1 \le \sin\left(\frac{\pi}{x}\right) \le 1 \Longrightarrow e^{-1} \le e^{\sin(\pi/x)} \le e^1 \Longrightarrow \sqrt{x}e^{-1} \le \sqrt{x}e^{\sin(\pi/x)} \le \sqrt{x}e^1$$

finally note that $\lim_{x\to 0^+}\sqrt{x}e^{-1} = \lim_{x\to 0^+}\sqrt{x}e = 0$ and so $\boxed{\text{by the Squeeze Theorem we see that } \lim_{x\to 0^+}\sqrt{x}e^{\sin(\pi/x)} = 0.}$

(c) First note that

$$g'(x) = \frac{2x+1}{x^2+x+1} \Longrightarrow g'(x) = 0 \text{ if } x = -1/2$$

and note that $x^2 + x + 1$ has no real roots and so $x = -1/2$ is the only critical point. Now we test for the absolute extrema, note that $g(-1) = 0$, $g(1) = \ln(3)$ and $g(-1/2) = \ln(3/4)$ and note that $\ln(3/4) < 0$ and so we conclude that we have an $\boxed{\text{absolute minimum at } (x, y) = (-1/2, \ln(3/4)) \text{ and an absolute maximum at } (x, y) = (1, \ln(3)).}$

(d) $\boxed{\text{Choice } (iv)}$ Discussion: We use implicit differentiation, note that if $y = \sin(x)^{\ln x}$ then

$$\ln(y) = \ln(x)\ln(\sin x) \Longrightarrow \frac{y'}{y} = \frac{1}{x}\cdot\ln(\sin x) + \ln(x)\cdot\frac{\cos(x)}{\sin(x)} \Longrightarrow y' = y\left(\frac{\ln(\sin x)}{x} + \cot(x)\ln(x)\right)$$

and note that $\cot(x)\ln(x) = \ln(x^{\cot(x)})$, thus $y = \sin(x)^{\ln(x)}\left[\frac{\ln(\sin(x))}{x} + \ln(x^{\cot x})\right]$.

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2. (a)(10 pts) Find the linearization of $f(x) = e^{-2x}$ at $a = 0$ and use it to approximate $e^{0.1}$.

(b)(10 pts) Is $f(x) = \begin{cases} e^x + 1, & x < 0 \\ \log_2(x+1) + 2\cosh(x), & x \ge 0 \end{cases}$ continuous at $x = 0$? Use limits to answer this question.

(c)(10 pts) If a snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter changes when the diameter is 10 cm. (Recall that if r is the radius then the surface area is $SA = 4\pi r^2$.)

Solution: (a) The linearization is $L(x) = f(0) + f'(0)(x - 0)$ where $f(0) = e^{-2\cdot 0} = 1$ and $f'(0) = -2e^{-2x}\big|_{x=0} = -2$ thus we have the linearization $\boxed{L(x) = 1 - 2x}$ and so

$$e^{0.1} = e^{-2(-0.05)} = f(-0.05) \approx L(-0.05) = 1 - 2(-0.05) = 1 + 0.1 = \boxed{1.1}$$

(b) Using one-sided limits we see that

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} e^x + 1 = e^0 + 1 = 2 \text{ and } \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\log_2(x+1) + 2\cosh(x) = \log_2(1) + 2\cosh(0) = 0 + 2 = 2$$

and thus we see that $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 2 = f(0)$ and so we see that $\boxed{f(x) \text{ is continuous at } x = 0.}$

(c) Note that if r denotes the radius of the snowball then $D = 2r$ is the diameter and we can write the surface area as $SA = 4\pi r^2 = 4\pi(D/2)^2 = \pi D^2$ and now using the fact that $\frac{d}{dt}SA = -1\text{cm}^2/\text{min}$ we see that

$$SA = \pi D^2 \Longrightarrow \frac{d}{dt}SA = \pi\cdot 2D\cdot\frac{dD}{dt} \Longrightarrow \frac{dD}{dt} = \frac{(-1\ \text{cm}^2/\text{min})}{2\pi D}\bigg|_{D=10\,\text{cm}} = -\frac{1}{20\pi}\ \text{cm/min}$$

thus $\boxed{\text{the diameter is changing at a rate of } -1/20\pi \text{ cm/min or the diameter is } \textit{decreasing} \text{ at a rate of } 1/20\pi \text{ cm/min.}}$

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3. (a)(10 pts) Water leaks slowly from the bottom of a large storage tank at a rate of $r(t) = 100 - e^{2t}$ gallons per minute for $t > 0$. Find the amount of water that leaks from the tank during the first 10 minutes.

(b)(10 pts) Find the area of the region bounded by the curve $y = \frac{x+1}{x^2+1}$ and the x-axis for $0 \le x \le \sqrt{3}$.

(c)(10 pts) Evaluate the definite integral $\int_0^3 |x^2 - 4x + 3|\,dx$.

Solution: (a) Note that $r(t)$ is the rate of change of the amount of water (in gallons) with respect to time (in minutes) that is, if $R(t)$ denotes the amount of water in the storage tank as a function of time t in minutes then $R'(t) = r(t)$ and we wish to find the net change in the amount of water in the first 10 minutes, *i.e.* we wish to find $R(10) - R(0)$, thus (by the FTOC2) the amount of water that leaks from the storage tank in the first 10 minutes is

$$R(10) - R(0) = \int_0^{10} R'(t)\,dt = \int_0^{10}(100 - e^{2t})\,dt = 100t - \frac{e^{2t}}{2}\bigg|_0^{10} = \left(1000 - \frac{e^{20}}{2}\right) + \frac{1}{2} = \boxed{\frac{1}{2}\left(2001 - e^{20}\right)\ \text{gallons}}$$

(b) The given curve is positive for $x \ge 0$ and so the area of the region is

$$\begin{aligned}\int_0^{\sqrt{3}}\frac{x+1}{x^2+1}\,dx &= \underbrace{\int_0^{\sqrt{3}}\frac{x}{x^2+1}\,dx}_{\substack{\text{let } u = x^2+1\\ \text{then } du = 2x dx}} + \int_0^{\sqrt{3}}\frac{1}{x^2+1}\,dx \\ &= \frac{1}{2}\int_1^4\frac{du}{u} + \arctan(x)\bigg|_0^{\sqrt{3}} = \frac{1}{2}\ln|u|\bigg|_1^4 + \left(\arctan(\sqrt{3}) - \arctan(0)\right) = \boxed{\frac{1}{2}\ln(4) + \frac{\pi}{3}}\end{aligned}$$

(c) Note that $x^2 - 4x + 3 = (x-1)(x-3)$ and so, by the definition of the absolute value we have

$$\begin{aligned}\int_0^3 |x^2 - 4x + 3|\,dx &= \int_0^1 (x^2 - 4x + 3)\,dx + \int_1^3 -(x^2 - 4x + 3)\,dx \\ &= \left(\frac{x^3}{3} - \frac{4x^2}{2} + 3x\right)\bigg|_0^1 - \left(\frac{x^3}{3} - \frac{4x^2}{2} + 3x\right)\bigg|_1^3 \\ &= \left(\frac{1}{3} - 2 + 3\right) - \left[(9 - 18 + 9) - \left(\frac{1}{3} - 2 + 3\right)\right] = \boxed{8/3}\end{aligned}$$

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4. (a)(10 pts) Evaluate the limit: $\lim_{x\to\infty}(e^x + x)^{1/x}$.

(b)(10 pts) Use logarithmic differentiation to find the derivative of $y = \frac{\sqrt{x}e^{x^2}}{(x^2+1)^{10}}$.

(c)(10 pts) Suppose $f(x) = \int_e^{2x} e^t\ln(t+2)\,dt$, for $x > 0$. Find $(f^{-1})'(0)$.

(d)(5 pts) In your blue book clearly sketch the graph of a function $h(x)$ that satisfies all the following properties (label all extrema, inflection points and asymptotes):

- $h(x)$ is odd, $h(0)=0$ and $\lim_{x\to\infty} h(x) = -2$
- $h'(x) < 0$ if $0 < x < 2$ and $h'(x) > 0$ if $x > 2$,
- $h''(x) > 0$ if $0 < x < 3$ and $h''(x) < 0$ if $x > 3$.

Solution: (a) Note that this is an indeterminate form of type "∞^0". Now, using continuity, we have

$$\ln(\lim_{x\to\infty}(e^x+x)^{1/x}) = \lim_{x\to\infty}\ln[(e^x+x)^{1/x}] = \lim_{x\to\infty}\frac{\ln(e^x+x)}{x} \overset{L'H}{=} \lim_{x\to\infty}\frac{e^x+1}{e^x+x} \overset{L'H}{=} \lim_{x\to\infty}\frac{e^x}{e^x+1} \overset{L'H}{=} \lim_{x\to\infty}\frac{e^x}{e^x} = 1$$

and thus $\boxed{\lim_{x\to\infty}(e^x+x)^{1/x} = e^{\ln(\lim_{x\to\infty}(e^x+x)^{1/x})} = e^1 = e.}$

(b) First note that

$$y = \frac{\sqrt{x}e^{x^2}}{(x^2+1)^{10}} \Longrightarrow \ln(y) = \ln\left(\frac{\sqrt{x}e^{x^2}}{(x^2+1)^{10}}\right) = \frac{1}{2}\ln(x) + x^2 - 10\ln(x^2+1)$$

now differentiating both sides yields

$$\frac{y'}{y} = \frac{1}{2x} + 2x - 10\left(\frac{2x}{x^2+1}\right) \Longrightarrow y' = y\left[\frac{1}{2x} + 2x - 10\left(\frac{2x}{x^2+1}\right)\right]$$

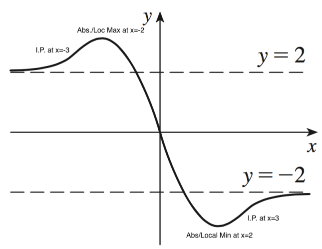
and so we see that

$$\boxed{y' = \frac{\sqrt{x}e^{x^2}}{(x^2+1)^{10}}\left[\frac{1}{2x} + 2x - \frac{20x}{x^2+1}\right]}$$

(c) Note that by a theorem in the book $(f^{-1})'(0) = \dfrac{1}{f'(f^{-1}(0))}$ and note that $f(e/2) = 0$ and so $e/2 = f^{-1}(0)$ and also note that $f'(x) = e^{2x}\ln(2x+2)\cdot 2$ by the FTOC1 and so

$$(f^{-1})'(0) = \frac{1}{f'(f^{-1}(0))} = \frac{1}{f'(e/2)} = \frac{1}{2e^{2e/2}\ln(2e/2+2)} = \boxed{\frac{1}{2e^e\ln(e+2)}}$$

(d) The the graph could look like, for example, the following:



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5. (25 pts) Answer either **ALWAYS TRUE** or **FALSE**. You do NOT need to justify your answer. (*Don't just write down "A.T." or "F", completely write out the words "ALWAYS TRUE" or "FALSE" depending on your answer.*)

(a)(5 pts) If we use a Riemann Sum with right endpoints and subintervals of equal length then

$$\int_0^1 x^2 + x\,dx = \lim_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n}(i^2+in)$$

(b)(5 pts) A bacteria culture initially contains 140 cells and grows at a rate proportional to its size and, after an hour, the population is 420. Based on this information the population will be 2800 bacteria at $t = \log_3(20)$ hours.

(c)(5 pts) According to the limit definition of the derivative $\dfrac{d}{dx}3^x = \lim_{h\to 0}\dfrac{3^x(3^h-1)}{h}$.

(d)(5 pts) $\displaystyle\sum_{n=1}^{4}\frac{1}{5}\left(\frac{1}{2}\right)^n = \frac{3}{8}$.

(e)(5 pts) Suppose a particle moves on a vertical line so that its coordinate at time t is $y = t^3 - 12t + 3$ for $t \geq 0$ then the particle starts moving upward after 1 seconds.

Solution: (a) AT (b) AT (c) AT (d) F (e) F

Discussion:

(a) Here $\Delta x = (b-a)/n = 1/n$ and $x_i = a + i\Delta x = i/n$ and thus if we let $x_i^* = x_i$ then

$$\int_0^1 (x^2+x),\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}(x_i^2+x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i^2}{n^2}+\frac{i}{n}\right)\frac{1}{n} = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i^2+in}{n^3}\right) = \lim_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n}(i^2+in) \quad ✔$$

(b) Here if y is the population at time t then $y = y_0e^{kt} = 140e^{kt}$ and we know when $t=1$ we have $420 = 140e^k$ which implies $k = \ln(3)$ and thus $y = 140e^{\ln(3)t} = 140(3)^t$ and if $t = \log_3(20)$ then $y = 140(3)^{\log_3(20)} = 140\cdot 20 = 2800$. ✔

(c) By definition

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{3^{x+h}-3^x}{h} = \lim_{h\to 0}\frac{3^x(3^h-1)}{h} \quad ✔$$

(d) Note that

$$\sum_{n=1}^{4}\frac{1}{5}\left(\frac{1}{2}\right)^n = \frac{1}{5}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}\right) = \frac{1}{5}\left(\frac{8}{16}+\frac{4}{16}+\frac{2}{16}+\frac{1}{16}\right) = \frac{1}{5}\cdot\frac{15}{16} = \frac{3}{16} \neq \frac{3}{8} \quad ✗$$

(e) The particle moves with velocity $dy/dt = 3t^2 - 12 = 3(t-2)(t+2)$ and we see that $dy/dt > 0$ for $t > 2 \neq 1$ ✗.
