Solutions

Math 1300-005 - Spring 2017

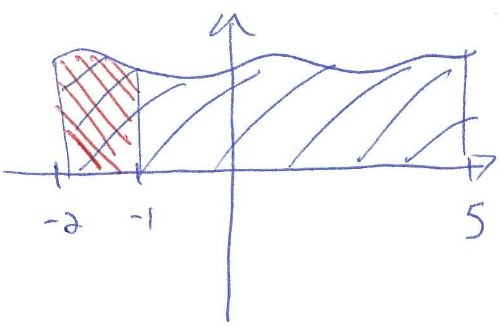
The Definite Integral - 4/19/17

Guidelines: Please work in groups of two or three. This will not be handed in, but is a study resource for the final exam.

1. These first exercises explore the properties of the definite integral found on pgs. 350-351.

   (a) Write as a single integral in the form $\int_a^b f(x)\,dx$:

$$\underbrace{\underbrace{\int_{-2}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx}_{\int_{-2}^{5} f(x)dx} - \int_{-2}^{-1} f(x)\,dx}$$



$$\boxed{\int_{-1}^{5} f(x)dx \rightarrow \text{see graph to the left.}}$$

   (b) If $\int_0^8 f(x)\,dx = 6$ and $\int_6^8 f(x)\,dx = -13$, find $\int_0^6 f(x)\,dx$.

$$\int_0^6 f(x)dx + \int_6^8 f(x)dx = \int_0^8 f(x)dx$$

so $$\int_0^6 f(x)dx = \int_0^8 f(x)dx - \int_6^8 f(x)dx$$

$$= 6 - (-13)$$

$$= \boxed{19}$$

   (c) If $\int_{-4}^0 f(x)\,dx = 25$ and $\int_{-4}^0 g(x)\,dx = -12$, find $\int_{-4}^0 [2f(x) - 3g(x) + 7]\,dx$.

$$\int_{-4}^0 (2f(x) - 3g(x) + 7)dx = 2\int_{-4}^0 f(x)dx - 3\int_{-4}^0 f(x)dx + \int_{-4}^0 7dx$$

$$= 2(25) - 3(-12) + 7(0-(-4))$$

$$= 50 + 36 + 28$$

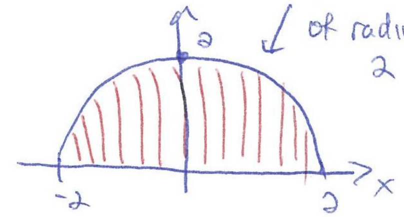
$$= \boxed{114}$$

2. Sometimes we can compute $\int_a^b f(x)\,dx$ by recognizing the bounded area as a known geometric object such as a circle or triangle.

(a) Evaluate the integral by interpreting it in terms of areas. (Sketch a graph!)

semicircle of radius 2

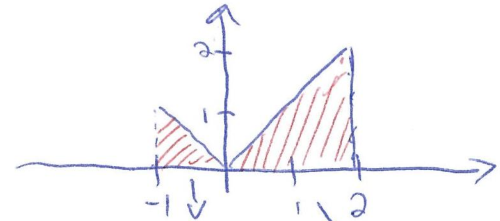
$$\int_{-2}^{2} \sqrt{4-x^2}\,dx \longrightarrow$$



$$= \frac{1}{2}\left(\pi (2)^2\right) = \frac{4\pi}{2} = \boxed{2\pi}$$

(b) Evaluate the integral by interpreting it in terms of areas. (Sketch a graph!)

$$\int_{-1}^{2} |x|\,dx \longrightarrow$$



area $= \frac{1}{2}(1\cdot 1) = \frac{1}{2}$

Area $= \frac{1}{2}(2\cdot 2) = 2$

$$= \frac{1}{2} + 2 = \boxed{\frac{5}{2}}$$

3. Now let us practice some Riemann sums.

(a) If $f(x) = x^2 - 2x$, $0 \le x \le 6$, evaluate the Riemann sum with $N = 6$, taking the sample points to be right endpoints. $\Delta x = \frac{6-0}{6} = 1$, $x_i = 0 + i\Delta x = i$

$$R_6 = \sum_{i=1}^{6} f(x_i)\,\Delta x$$

$$= \sum_{i=1}^{6} (i^2 - 2i)\cdot 1 = (1^2 - 2) + (2^2 - 2(2)) + (3^2 - 2(3)) + \dots + (6^2 - 2(6))$$

$$= -1 + 0 + 3 + 8 + 15 + 24 = \boxed{49}$$

(b) If $f(x) = e^x - 2$, $0 \le x \le 2$, find the Riemann sum with $N = 4$, taking sample points to be midpoints. Our sub intervals are

$[0, \frac{1}{2}], [\frac{1}{2}, 1], [1, \frac{3}{2}], [\frac{3}{2}, 2]$. $\Delta x = \frac{1}{2}$

midpoint $\frac{1}{4}$, midpoint $\frac{3}{4}$, midpoint $\frac{5}{4}$, midpoint $\frac{7}{4}$

$$M_4 = \frac{1}{2}\left(e^{\frac{1}{4}} - 2 + e^{\frac{3}{4}} - 2 + e^{\frac{5}{4}} - 2 + e^{\frac{7}{4}} - 2\right)$$

$$= \boxed{\frac{1}{2}\left(e^{\frac{1}{4}} + e^{\frac{3}{4}} + e^{\frac{5}{4}} + e^{\frac{7}{4}} - 8\right)}$$

4. If f is integrable on $[a, b]$ then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$. Use this form of the definition to compute

$$\int_0^3 (x^2 - 3)\,dx.$$

You will need to use that

$$\sum_{i=1}^{n} c = nc \text{ and } \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

$\Delta x = \frac{3-0}{n} = \frac{3}{n}, \quad x_i = a + i\Delta x = 0 + i\left(\frac{3}{n}\right) = \frac{3i}{n}.$

$$\int_0^3 (x^2-3)\,dx = \int_0^3 x^2\,dx - \int_0^3 3\,dx$$

can compute directly

$$= \int_0^3 x^2\,dx - 3(3-0)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{3i}{n}\right)^2 \cdot \frac{3}{n} - 9$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{27 i^2}{n^3} - 9$$

$$= \lim_{n\to\infty} \frac{27}{n^3} \sum_{i=1}^{n} i^2 - 9$$

$$= \lim_{n\to\infty} \frac{27}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) - 9$$

only largest power of n matters as $n \to \infty$

$$\approx \lim_{n\to\infty} \frac{27}{\cancel{n^3}}\left(\frac{2\cancel{n^3}}{6}\right) - 9$$

$$= \lim_{n\to\infty} \frac{54}{6} - 9$$

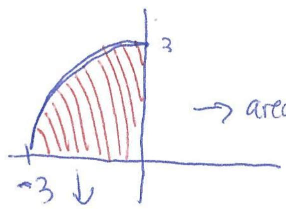
$$= 9 - 9$$

$$= \boxed{0}$$

5. Back to some area stuff:

(a) Evaluate the integral by interpreting it in terms of areas. (Sketch a graph!)

Let's look at $\int_{-3}^{0} \sqrt{9-x^2}\,dx$



$\rightarrow$ area $= \dfrac{\pi(3)^2}{4}$

quarter circle, radius 3

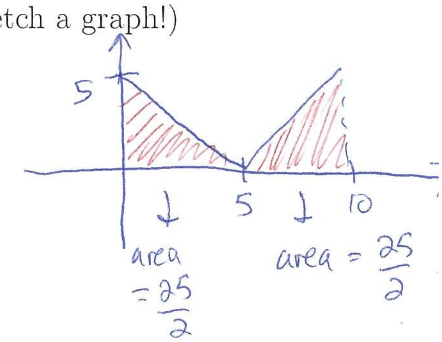
$$\int_{-3}^{0} (1 + \sqrt{9 - x^2})\, dx = \int_{-3}^{0} 1\,dx + \int_{-3}^{0} \sqrt{9-x^2}\,dx$$

$$= 1(0-(-3)) + \frac{\pi(3)^2}{4}$$

$$= \boxed{3 + \frac{9\pi}{4}}$$

(b) Evaluate the integral by interpreting it in terms of areas. (Sketch a graph!)

$$\int_{0}^{10} |x - 5|\, dx \rightarrow$$



$$= \frac{25}{2} + \frac{25}{2}$$

$$= \boxed{50}$$

6. Using $N = 4$ and midpoints as sample points, estimate the given definite integrals:

(a) $\int_{0}^{10} \sqrt{x^3 + 1}\, dx$ $\quad \Delta x = \frac{10-0}{4} = \frac{5}{2}$, so our intervals are

$[0, \frac{5}{2}]$, $[\frac{5}{2}, 5]$, $[5, \frac{15}{2}]$, $[\frac{15}{2}, 10]$

midpoint $\frac{5}{4}$, midpoint $\frac{15}{4}$, midpoint $\frac{25}{4}$, midpoint $\frac{35}{4}$

$$\int_0^{10} \sqrt{x^3+1}\,dx \approx M_4 = \boxed{\frac{5}{2}\left(\sqrt{\left(\tfrac{5}{4}\right)^3+1} + \sqrt{\left(\tfrac{15}{4}\right)^3+1} + \sqrt{\left(\tfrac{25}{4}\right)^3+1} + \sqrt{\left(\tfrac{35}{4}\right)^3+1}\right)}$$

(b) $\int_{0}^{\pi} \cos^2(x)\, dx$ $\quad \Delta x = \frac{\pi - 0}{4} = \frac{\pi}{4}$, so our intervals are

$[0, \frac{\pi}{4}]$, $[\frac{\pi}{4}, \frac{\pi}{2}]$, $[\frac{\pi}{2}, \frac{3\pi}{4}]$, $[\frac{3\pi}{4}, \pi]$

midpoint $\frac{\pi}{8}$, midpoint $\frac{3\pi}{8}$, midpoint $\frac{5\pi}{8}$, midpoint $\frac{7\pi}{8}$

$$\int_0^{\pi} \cos^2(x)\,dx \approx M_4 = \boxed{\frac{\pi}{4}\left(\cos^2\left(\tfrac{\pi}{8}\right) + \cos^2\left(\tfrac{3\pi}{8}\right) + \cos^2\left(\tfrac{5\pi}{8}\right) + \cos^2\left(\tfrac{7\pi}{8}\right.\right.}$$