INSTRUCTIONS: Books, notes, and electronic devices are **not** permitted. Write (1) **your name**, (2) **1350/Test 1**, (3) **lecture number/instructor name** and (4) **SUMMER 2015** on the front of your bluebook. Also make a **grading table** with room for 5 problems and a total score. **Start each problem on a new page.** $\boxed{\textbf{Box}}$ **your answers.** A correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit. **SHOW ALL WORK! JUSTIFY ALL YOUR ANSWERS!**

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1. For this problem, suppose $f(x) = 2\cos x$ and $g(x) = \frac{1}{x^2-1}$.

   (a) (6 pts) Find $(g \circ f)(x)$.

   (b) (6 pts) What is the domain of $(g \circ f)(x)$?

   (c) (8 pts) Suppose we let $h(x) = \begin{cases} f(x), \text{if } x > 2\pi \\ g(x), \text{if } x \leq 2\pi \end{cases}$, are there any values of x for which $h(x)$ is *not* continuous? Justify your answer. What type of discontinuities does $h(x)$ have (i.e. *jump*, *removable*, or *infinite*), if any?

Solution:

(a)(6 pts) $(g \circ f)(x) = g(f(x)) = g(2\cos(x)) = \frac{1}{4\cos^2(x)-1}$

(b)(6 pts) The domain of $f(x)$ is all real x except for values of x where $4\cos^2(x) - 1 = 0$.

$$\cos^2 x = 1/4$$

$$\cos x = \pm 1/2$$

$$\implies x = ... - \frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, ...$$

(c)(8 pts) Note that for $x > 2\pi$, $h(x)$ is continuous since $f(x)$ is well defined and continuous for all x. At $x = 2\pi$, we need check for continuity. Note that

$$\lim_{x\to 2\pi^+} h(x) = \lim_{x\to 2\pi^+} 2\cos x = 2\cos(2\pi) = 2 \text{ and } \lim_{x\to 2\pi^-} h(x) = \lim_{x\to 2\pi^-} \frac{1}{x^2-1} = \frac{1}{(2\pi)^2 - 1} \neq 2$$

Therefore there is a jump discontinuity at $x = 2\pi$. When $x < 2\pi$, $h(x) = \frac{1}{x^2-1}$ and so $h(x)$ has infinite discontinuities at $x = \pm 1$.

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2. Evaluate the following limits and show all supporting work. If a limit does not exist, clearly state that fact and explain your reasoning. (Note: You may not use l'Hopital's Rule.)

   (a) (4 pts) $\displaystyle\lim_{x\to 1} \frac{x^2 + x - 2}{x^2 - 4x + 3}$

   (b) (4 pts) $\displaystyle\lim_{x\to -\infty} 2x - \sqrt{4x^2 - 5x}$

   (c) (4 pts) $\displaystyle\lim_{x\to 0^+} \sqrt{x}\cos\frac{\pi}{x}$

   (d) (4 pts) $\displaystyle\lim_{x\to 0^-} \frac{x}{x - |x|}$

   (e) (4 pts) $\displaystyle\lim_{x\to\infty} \sqrt{\frac{4x^2 - x}{x^2 + 9}}$

Solution:

(a) (4 pts) $\displaystyle\lim_{x\to 1} \frac{x^2 + x - 2}{x^2 - 4x + 3} = \lim_{x\to 1} \frac{(x-1)(x+2)}{(x-3)(x-1)} = \lim_{x\to 1} \frac{x+2}{x-3} = -\frac{3}{2}$

(b) (4 pts) $\lim_{x\to-\infty} 2x - \sqrt{4x^2-5x} = \lim_{x\to-\infty} 2x - |x|\sqrt{4-\frac{5}{x}} = \lim_{x\to-\infty} 2x + x\sqrt{4-\frac{5}{x}}$

$$= \lim_{x\to-\infty} 2x + 2x = \lim_{x\to-\infty} 4x = -\infty$$

(c) (4 pts) $\lim_{x\to 0^+} \sqrt{x}\cos\frac{\pi}{x}$

We may bound cosine: $-1 \le \cos\frac{\pi}{x} \le 1$. Then $-\sqrt{x} \le \sqrt{x}\cos\frac{\pi}{x} \le \sqrt{x}$. Now we can apply the Squeeze Theorem:

$$\lim_{x\to 0^+} -\sqrt{x} \le \lim_{x\to 0^+} \sqrt{x}\cos\frac{\pi}{x} \le \lim_{x\to 0^+} \sqrt{x}$$

$$0 \le \lim_{x\to 0^+} \sqrt{x}\cos\frac{\pi}{x} \le 0$$

$$\implies \lim_{x\to 0^+} \sqrt{x}\cos\frac{\pi}{x} = 0$$

(d) (4 pts) $\lim_{x\to 0^-} \frac{x}{x-|x|} = \lim_{x\to 0^-} \frac{x}{x+x} = \frac{1}{2}$

(e) (4 pts) $\lim_{x\to\infty} \sqrt{\frac{4x^2-x}{x^2+9}} = \sqrt{\lim_{x\to\infty} \frac{4x^2-x}{x^2+9}} = \sqrt{4} = 2$

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3. (a) (5 pts) Given the function $f(x) = 3^{-x}\cos(10x)$. Is f a continuous function of x? Justify why or why not.

(b) (5 pts) Does $f(x) = 3^{-x}\cos(10x)$ have a real root? Justify why or why not.

(c) (5 pts) Use continuity to evaluate: $\lim_{x\to\pi} \sin(x + \sin x)$.

(d) (5 pts) Use the <u>definition of the derivative</u> to show that $b(x) = \sqrt{x} + x - 1$ is an increasing function.

Solution:

(a) (5 pts) Yes. $\frac{1}{3^x}$ is continuous for all $x \in (-\infty,\infty)$ and $\cos(10x)$ is also continuous for all $x \in (-\infty,\infty)$, therefore the product of those two continuous functions is also continuous on $x \in (-\infty,\infty)$.

(b) (5 pts) Since this function is continuous everywhere, we apply the Intermediate Value Theorem:

$$f(\pi/10) = \frac{1}{3^{\pi/10}}\cos(\pi) = -\frac{1}{3^{\pi/10}} < 0$$

$$f(\pi) = \frac{1}{3^{\pi}}\cos(10\pi) = \frac{1}{3^{\pi}} > 0$$

Therefore $f(x)$ must have a root somewhere in the interval $(\frac{\pi}{10}, \pi)$. [Note that $f(x)$ actually has infinitely many roots.]

(c) (5 pts) $\lim_{x\to\pi} \sin(x+\sin x) = \sin(\lim_{x\to\pi}(x+\sin x)) = \sin(\pi+0) = 0$

(d) (5 pts) Using the definition of a derivative:

$$b'(x) = \lim_{h\to 0}\frac{b(x+h)-b(x)}{h} = \lim_{h\to 0}\frac{\sqrt{x+h}+x+h-1-(\sqrt{x}+x-1}{h} = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}+h}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}+h}{h}\cdot\frac{\sqrt{x+h}+(\sqrt{x}-h)}{\sqrt{x+h}+(\sqrt{x}-h)} = \lim_{h\to 0}\frac{x+h-(\sqrt{x}-h)^2}{h(\sqrt{x+h}+\sqrt{x}-h)}$$

$$= \lim_{h\to 0}\frac{1+2\sqrt{x}+h}{\sqrt{x+h}+\sqrt{x}-h} = \frac{1+2\sqrt{x}}{2\sqrt{x}} = 1 + \frac{1}{2\sqrt{x}}$$

So $b'(x) = 1 + \frac{1}{2\sqrt{x}} > 0$ for all x in its domain. Therefore this function must be increasing for all x in its domain.

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4. (a) (7 pts) Use the limit definition of the derivative to find the slope of $f(x) = 3x^2 - 10x - 7$ at any point x.

(b) (7 pts) Find an equation of the tangent line to the parabola $f(x) = 3x^2 - 10x - 7$ whose slope is $m = -8$.

(c) (6 pts) If $s(t) = 3t^2 - 10t - 7$ for $t \geq 0$ describes the position of an object (in feet) at time t, find the average velocity of the object from $t = 1$ second to $t = 2$ seconds.

Solution:

(a) (7 pts)

$$\lim_{h\to 0} \frac{3(x+h)^2 - 10(x+h) - 7 - (3x^2 - 10x - 7)}{h} = \lim_{h\to 0} \frac{6xh + 3h^2 - 10h}{h} = \lim_{h\to 0} 6x + 3h - 10 = 6x - 10$$

. So therefore, $f'(x) = 6x - 10$ describes the slope at any point x.

(b) (7 pts) $-8 = 6x - 10 \implies x = \frac{1}{3}$. Then $f(\frac{1}{3}) = 3(\frac{1}{3})^2 - 10(\frac{1}{3}) - 7 = -10$. To find the equation of the tangent line, we use the point-slope formula:

$$y - (-10) = -8(x - \frac{1}{3}) \implies y = -8x - \frac{22}{3}$$

(c) (6 pts)

$$v_{ave} = \frac{s(2) - s(1)}{2 - 1} = \frac{-15 - (-14)}{1} = -1$$

Therefore, the average velocity between 1 and 2 seconds is -1 ft/sec.

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5. The following parts are *not* related:

   (a) (6 pts) For what values of x does the graph of $f(x) = x + 2\sin x$ have a horizontal tangent?

   (b) (6 pts) Find the first and second derivatives of: $G(r) = \sqrt{r} + \sqrt[3]{r}$.

   (c) (8 pts) Find the n^{th} derivative of each function by calculating the first few derivatives and observing the pattern that occurs:

      i. $f(x) = x^n$

      ii. $f(x) = \frac{1}{x}$

Solution:

(a) (6 pts) We must find when $f'(x) = 0$.

$$f'(x) = 1 + 2\cos x = 0 \implies \cos x = -\frac{1}{2} \implies x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \ldots$$

(b) (6 pts) $G(r) = r^{1/2} + r^{1/3}$

$$G'(r) = \frac{1}{2}r^{-1/2} + \frac{1}{3}r^{-2/3}$$

$$G''(r) = -\frac{1}{4}r^{-3/2} - \frac{2}{9}r^{-5/3}$$

(c) (8pts)

(i)

$$f^{(n)}(x) = n(n-1)(n-2)\cdots(n-(n-1))x^{n-n} = n!$$

(ii)

$$f^{(n)}(x) = (-1)^n n! x^{-(n+1)} = \frac{(-1)^n n!}{x^{n+1}}$$
