1. The following parts are not related:

(a)(5 pts) Evaluate the limit $\lim_{x\to\infty} \cosh(x)^{1/x}$

(b)(5 pts) Evaluate the limit $\lim_{x\to 0^+} [\ln(\sin(x)) - \ln(x)]$

(c)(7 pts) Prove that $\lim_{x\to 0} x^4 \cos\left(\frac{2}{x}\right) = 0$. Justify your answer.

(d)(8 pts) Suppose $f(x) = \begin{cases} x^2, \, x \le 2 \\ 4, \, x > 2 \end{cases}$. Use the limit definition of the derivative to determine whether or not $f(x)$ is differentiable at the point $x = 2$. You may not use L'Hospital's Rule for this problem. Justify your answer.

Solution:

(a)(5 pts) We have the indeterminate form "∞^0" and so if let $y = \lim_{x\to\infty} \cosh(x)^{1/x}$ then taking the natural log and using continuity yields

$$\begin{aligned} \ln(y) = \ln\left(\lim_{x\to\infty} \cosh(x)^{1/x}\right) &= \lim_{x\to\infty} \ln\left(\cosh(x)^{1/x}\right) \\ &= \lim_{x\to\infty} \frac{\ln(\cosh(x))}{x} \overset{L'H}{=} \lim_{x\to\infty} \frac{\sinh(x)/\cosh(x)}{1} = \lim_{x\to\infty} \tanh(x) = 1 \end{aligned}$$

and so $\lim_{x\to\infty} \cosh(x)^{1/x} = y = e^1 = e$.

(b)(5 pts) Here we have the indeterminate from "$-\infty + \infty$". Combining the natural log terms and using continuity yields

$$\lim_{x\to 0^+} [\ln(\sin(x)) - \ln(x)] = \lim_{x\to 0^+} \ln\left(\frac{\sin(x)}{x}\right) \overset{L'H}{=} \lim_{x\to 0^+} \ln\left(\frac{\cos(x)}{1}\right) = \ln(1) = 0$$

(Note that we could also use continuity and the fact that, using geometry, we previously proved the "special limit" $\lim_{x\to 0} \sin(x)/x = 1$)

(c)(7 pts) Note that for any $x \neq 0$ we have

$$-1 \le \cos(2/x) \le 1 \Longrightarrow -x^4 \le x^4\cos(2/x) \le x^4 \Longrightarrow \lim_{x\to 0} -x^4 \le \lim_{x\to 0} x^4\cos(2/x) \le \lim_{x\to 0} x^4$$

and note that $\lim_{x\to 0} -x^4 = \lim_{x\to 0} x^4 = 0$, and so, by the Squeeze Theorem, we have $\lim_{x\to 0} x^4\cos(2/x) = 0$.

(d)(8 pts) Note that $f(x)$ is differentiable at $x = 2$ if the two-sided limit $\lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$ exists. Now if $h < 0$, then

$$\lim_{h\to 0^-} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0^-} \frac{(2+h)^2 - 4}{h} = \lim_{h\to 0^-} \frac{(4 + 4h + h^2) - 4}{h} = \lim_{h\to 0^-} \frac{\cancel{h}(4+h)}{\cancel{h}} = 4$$

and if $h > 0$

$$\lim_{h\to 0^+} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0^+} \frac{4-4}{h} = 0$$

and so the two-sided limit does not exist and therefore $f(x)$ is not differentiable at $x = 2$. (Note, could have also used the limit definition $f'(2) = \lim_{x\to 2} \frac{f(x) - f(2)}{x-2}$)

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2. The following problems are not related:

(a)(5 pts) Find the derivative of $y = \ln(\arctan(x))$.

(b)(5 pts) Find $f'(x)$ if $f(x) = \int_{2x}^{10} \sin^{-1}(\theta)\, d\theta$.

(c)(7 pts) Use the Intermediate Value Theorem to show that there is a root to the equation $2^x + x = 0$. Justify your answer.

(d)(8 pts) A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizon changing when 200 ft of string has been let out?

Solution:

(a)(5 pts) Using Chain Rule yields

$$y' = \frac{1}{\arctan(x)} \cdot \frac{1}{1+x^2} = \frac{1}{(1+x^2)\arctan(x)}$$

(b)(5 pts) By Fundamental Theorem of Calculus we have

$$f'(x) = \frac{d}{dx}\left[\int_{2x}^{10} \sin^{-1}(\theta)\,d\theta\right] = \frac{d}{dx}\left[-\int_{10}^{2x} \sin^{-1}(\theta)\,d\theta\right] = -\sin^{-1}(2x)\cdot 2 = -2\sin^{-1}(2x)$$

(c)(7 pts) First note that $f(x) = 2^x + x$ is continuous for all x. Now if $x = -1$, we have $f(-1) = 1/2 - 1 = -1/2 < 0$ and if $x = 1$, we have $f(1) = 2 + 1 = 3 > 0$ and so by the Intermediate Value Theorem we have $f(c) = 0$ for some number c in $(-1, 1)$ and so $2^x + x = 0$ has at least one root.

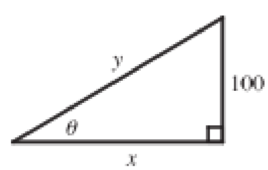
(d)(8 pts) We form a right triangle using the following three points: the person flying the kite, the kite and the point on the ground directly below the kite. Let x denote the horizontal distance from the kite-flyer to the point directly below the kite and let y denote the distance between the kite and the kite-flyer. Then if θ denotes the angle between the string and the horizon, we have

$$\cot(\theta) = \frac{x}{100} \implies -\csc^2(\theta)\frac{d\theta}{dt} = \frac{1}{100}\cdot\frac{dx}{dt} \implies \frac{d\theta}{dt} = -\frac{\sin^2(\theta)}{100}\frac{dx}{dt}$$

Note that when $x = 200$, we have $\sin(\theta) = \dfrac{100}{200} = \dfrac{1}{2}$, and so

$$\frac{d\theta}{dt} = -\frac{\sin^2(\theta)}{100}\frac{dx}{dt} = -\left(\frac{1}{2}\right)^2 \cdot \frac{1}{100}\cdot 8 = -\frac{2}{100} = -\frac{1}{50}$$

so rate of change of the angle between the string and horizon is $-1/50$ rad or it is decreasing at a rate of $1/50$ rad.



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3. The following problems are not related:

(a)(5 pts) Find the instantaneous rate of change of $f(x) = \dfrac{\tanh(x)}{x}$ with respect to x.

(b)(5 pts) Find $\dfrac{dy}{dx}$ given $y = x^{\cos(x)}$.

(c)(7 pts) Use logarithmic differentiation to find y' if $y = \dfrac{e^x(x+1)^3}{\sqrt{\sec(x)}}$.

(d)(8 pts) Classify all discontinuities of $f(x) = \dfrac{2x^2+12x}{x|x+6|}$ as either *jump*, *removable* or *infinite*. Justify your answers.

Solution:

(a)(5 pts) Using the Quotient Rule we have

$$\frac{d}{dx}\left[\frac{\tanh(x)}{x}\right] = \frac{x\,\text{sech}^2(x) - \tanh(x)}{x^2}$$

(b)(5 pts) Using the fact that $a = e^{\ln(a)}$ we have

$$\begin{aligned}\frac{d}{dx}\left[x^{\cos(x)}\right] = \frac{d}{dx}\left[e^{\ln(x^{\cos(x)})}\right] &= \frac{d}{dx}\left[e^{\cos(x)\ln(x)}\right] \\ &= e^{\cos(x)\ln(x)}\left(-\sin(x)\ln(x) + \frac{\cos(x)}{x}\right) = x^{\cos(x)}\left(-\sin(x)\ln(x) + \frac{\cos(x)}{x}\right)\end{aligned}$$

(or one could also use logarithmic differentiation.)

(c)(7 pts) Note that taking the natural log of both sides yields

$$\ln(y) = \ln\left(\frac{e^x(x+1)^3}{\sqrt{\sec(x)}}\right) = \ln\left(e^x(x+1)^3\right) - \ln\left(\sqrt{\sec(x)}\right) = x + 3\ln(x+1) - \frac{1}{2}\ln(\sec(x))$$

and so, differentiation yields

$$\frac{y'}{y} = 1 + \frac{3}{(x+1)} - \frac{1}{2}\cdot\frac{\sec(x)\tan(x)}{\sec(x)} = 1 + \frac{3}{(x+1)} - \frac{\tan(x)}{2}$$

and so

$$y' = y\left(1 + \frac{3}{(x+1)} - \frac{\tan(x)}{2}\right) = \frac{e^x(x+1)^3}{\sqrt{\sec(x)}}\left(1 + \frac{3}{(x+1)} - \frac{\tan(x)}{2}\right)$$

(d)(8 pts) Note that $f(x)$ is undefined at $x = 0$ and $x = -6$. We check continuity at $x = a$ by definition, *i.e.* check if $\lim_{x\to a} f(x) = f(a)$. Now, at $x = 0$ we have

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{2x^2 + 12x}{x|x+6|} = \lim_{x\to 0}\frac{\cancel{x}(2x+12)}{\cancel{x}|x+6|} = \frac{12}{6} = 2$$

so there is a removable discontinuity at $x = 0$. At $x = -6$ we have

$$\lim_{x\to -6^-}\frac{2x^2+12x}{x|x+6|} = \lim_{x\to -6^-}\frac{2x(x+6)}{-x(x+6)} = \lim_{x\to -6^-} -\frac{2\cancel{x(x+6)}}{\cancel{x(x+6)}} = -2 \text{ and } \lim_{x\to -6^+}\frac{2x^2+12x}{x|x+6|} = \lim_{x\to -6^+}\frac{2x(x+6)}{x(x+6)} = 2$$

and so there is a jump discontinuity at $x = -6$.

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4. The following problems are not related:

(a)(5 pts) Find the following antiderivative $\int (x^5 + 5^x)\,dx$

(b)(5 pts) Evaluate the definite integral $\int_0^8 \frac{x}{\sqrt{1+x}}\,dx$

(c)(7 pts) A curve passes through the point $(\ln(2), 8)$ and has the property that the slope of the curve at every point P is 2 times the y-coordinate of P. What is the equation of the curve?

(d)(8 pts) Find the linear approximation of $f(x) = \sqrt[3]{1+x}$ at $a = 0$ and use it to approximate $\sqrt[3]{1.1}$

Solution:

(a)(5 pts) Note that $\int (x^5 + 5^x)\,dx = \frac{x^6}{6} + \frac{5^x}{\ln(5)} + C$

(b)(5 pts) Here, if we let $u = 1 + x$, then $du = dx$ and $x = u - 1$, and so

$$\begin{aligned}\int_0^8 \frac{x}{\sqrt{1+x}}\,dx = \int_1^9 \frac{u-1}{\sqrt{u}}\,du &= \int_1^9 (u^{1/2} - u^{-1/2})\,du \\ &= \left(\frac{2}{3}u^{3/2} - 2u^{1/2}\right)\Big|_1^9 = \left(\frac{2}{3}(9)^{3/2} - 2(9)^{1/2}\right) - \left(\frac{2}{3} - 2\right) = (18-6) + \frac{4}{3} = \frac{40}{3}\end{aligned}$$

(c)(7 pts) Here we have $dy/dx = 2y$ and so by a theorem in class we know $y = Ce^{2x}$ and since the curve passes through the point $(x,y) = (\ln(2), 8)$, we have

$$8 = Ce^{2\ln(2)} \Longrightarrow 8 = Ce^{\ln(4)} \Longrightarrow 8 = C\cdot 4 \Longrightarrow C = 2$$

and so the equation of the curve is $y = 2e^{2x}$.

(d)(8 pts) Recall that the linearization of $f(x)$ at $a = 0$ is the equation $L(x) = f(0) + f'(0)(x - 0)$ and note that

$$f(0) = \sqrt[3]{1} = 1 \text{ and } f'(x) = \frac{1}{3}(1+x)^{-2/3} \Longrightarrow f'(0) = \frac{1}{3}$$

and so the linearization of $f(x)$ at $a = 0$ is $L(x) = 1 + \dfrac{x}{3}$ and so we have the approximation

$$\sqrt[3]{1.1} = f(0.1) \approx L(0.1) = 1 + \frac{0.1}{3} = 1.0\bar{3}$$

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5. The following problems are not related.

(a)(5 pts) Evaluate the integral $\displaystyle\int \frac{e^x}{e^x + 1}\, dx$.

(b)(5 pts) Evaluate the definite integral $\displaystyle\int_0^{\pi/2} \frac{\cos(x)}{1 + \sin^2(x)}\, dx$.

(c)(7 pts) If $f(x) = 3 + x + e^x$ find $(f^{-1})'(4)$.

(d)(8 pts) Use the Mean Value Theorem to show that there exists a number c in $(-1, 1)$ such that $e^c = \sinh(1)$.

Solution:

(a)(5 pts) Note that if we let $u = e^x + 1$ the $du = e^x dx$ and so

$$\int \frac{e^x}{e^x+1}\, dx = \int \frac{du}{u} = \ln|u| + C = \ln(e^x + 1) + C.$$

(b)(5 pts) If we let $u = \sin(x)$ the $du = \cos(x)dx$ and we have

$$\int_0^{\pi/2} \frac{\cos(x)}{1+\sin^2(x)}\, dx = \int_0^1 \frac{1}{1+u^2}\, du = \arctan(u)\Big|_0^1 = \arctan(1) - \arctan(0) = \frac{\pi}{4}$$

(c)(7 pts) First note that $f(x) = 4$ implies $3 + x + e^x = 4$ and we can guess $x = 0$ so $f(0) = 4$ implies $f^{-1}(4) = 0$ and also note that $f'(x) = 1 + e^x$, so

$$(f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(0)} = \frac{1}{1+e^0} = \frac{1}{2}$$

(d)(8 pts) Note that the function $f(x) = e^x$ is continuous in $[-1, 1]$ and differentiable on $(-1, 1)$, so by the Mean Value Theorem there exists a c in $(-1, 1)$ such that

$$f'(c) = \frac{f(1) - f(-1)}{1 - (-1)}$$

but note that $f'(c) = e^c$ and $\dfrac{f(1) - f(-1)}{1-(-1)} = \dfrac{e^1 - e^{-1}}{2} = \sinh(1)$, thus $e^c = \sinh(1)$ for some c in $(-1, 1)$.

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6. For this problem we have $f(x) = \dfrac{x^2 - 4}{x^2 + 4}$, $f'(x) = \dfrac{16x}{(x^2+4)^2}$, and $f''(x) = \dfrac{16(4 - 3x^2)}{(x^2+4)^3}$

(a)(8 pts) Find the intervals of concavity and the inflection points of $f(x)$.

(b)(8 pts) On what intervals is $f(x)$ increasing? decreasing? Find and classify all local extrema of $f(x)$.

(c)(4 pts) Find all Vertical and Horizontal Asymptotes of $f(x)$.

(d)(3 pts) Is $f(x)$ an even function an odd function or neither? Why or why not? Justify your answer.

(e)(2 pts) Sketch the graph of $f(x)$ (Clearly label all the axes, intercepts, asymptote and local extrema).

Solution:

(a)(8 pts) Note that $f''(x)$ is defined for all x and $f''(x) = 0$ if and only if $x = \pm\dfrac{2}{\sqrt{3}} = \pm\dfrac{2\sqrt{3}}{3} = \pm\dfrac{\sqrt{12}}{3}$. Now we test the sign of $f''(x)$. If we pick some $x > 2/\sqrt{3}$, for example note that $x = \sqrt{16}/3 > \sqrt{12}/3 = 2/\sqrt{3}$ and so

- if $x > 2/\sqrt{3}$, *i.e.* if $x = \sqrt{16}/3$ then $f''(\sqrt{16}/3) = \dfrac{12}{3} - \dfrac{16}{3} < 0$ so $f(x)$ is concave down on $(2\sqrt{3}/3, +\infty)$
- if $-2/\sqrt{3} < x < 2/\sqrt{3}$, *i.e.* if $x = 0$ then $f''(0) > 0$ so $f(x)$ is concave up on $(-2\sqrt{3}/3, 2\sqrt{3}/3)$
- if $x < -2/\sqrt{3}$, for example, if $x = -\sqrt{16}/3$ then $f''(-\sqrt{16}/3) < 0$ so $f(x)$ is concave down on $(-\infty, -2\sqrt{3}/3)$

and so $f(x)$ has inflection points when $x = \pm\dfrac{2\sqrt{3}}{3}$.

(b)(8 pts) Note that $f'(x)$ is defined for all x and $f'(x) = 0$ if and only if $x = 0$. Also note that $f'(x) < 0$ if $x < 0$ and $f'(x) > 0$ if $x > 0$ thus $f(x)$ is decreasing on $(-\infty, 0)$ and $f(x)$ is increasing on $(0, +\infty)$ and note that, by the First Derivative Test, we see that there is a local min at $x = 0$.

(c)(4 pts) Note that $f(x)$ is defined for all x so there are no vertical asymptotes and

$$\lim_{x\to\infty} \frac{x^2-4}{x^2+4} \stackrel{L'H}{=} 1 \text{ and } \lim_{x\to-\infty} \frac{x^2-4}{x^2+4} \stackrel{L'H}{=} 1$$

so $y = 1$ is a horizontal asymptote of $f(x)$ (note we could have also evaluated the limited using "Dominance of Powers")

(d)(3 pts) Note that $f(x)$ is even since $f(-x) = \dfrac{(-x)^2-4}{(-x)^2+4} = \dfrac{x^2-4}{x^2+4} = f(x)$.

(e)(2 pts) Note that the x-intercepts occur at $(-2, 0)$ and $(2, 0)$ and there is a local minimum at $(0, -1)$ and a horizontal asymptote at $y = 1$ (and inflection points at $x = \pm 2\sqrt{3}/3$, note that $f(\pm 2\sqrt{3}/3) < 0$). Thus the graph looks like