Solutions

Math 1300-005 - Spring 2017

Applied Optimization, Pt. 1 - 4/5/17

Guidelines: Please work in groups of two or three. This will not be handed in, but is a study resource for Midterm 3.

1. Consider
$$f(x) = \frac{x}{1+x^2}$$
on the open interval $(0, \infty)$. Use the First Derivative Test for Absolute Extrema to determine whether or not f has an absolute maximum or absolute minimum on $(0, \infty)$. Be sure to include full justification.

$f'(x) = \frac{1(1+x^2) - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$. So $f'(x)=0$ wherever $1-x^2=0$,
Hence $x = \pm 1$ are critical #'s.

$x=1$ is only critical number in $(0, \infty)$ and a sign chart looks like

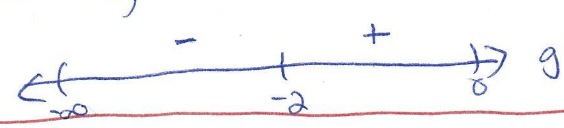
must include this full statement

$\boxed{\text{So } f(1) \text{ is a local max and since } x=1 \text{ is the only critical \# on } (0,\infty),\ f(1) \text{ must be the absolute max}}$

2. Consider
$$g(x) = 12x - x^3$$
on the open interval $(-\infty, 0)$. Use the First Derivative Test for Absolute Extrema to determine whether or not g has an absolute maximum or absolute minimum on $(-\infty, 0)$. Be sure to include full justification.

$g'(x) = 12 - 3x^2 = 3(4-x^2)$. So $g'(x)=0$ when $4-x^2=0$.
Hence $x = \pm 2$ are critical #'s.

$x=-2$ is the only critical number in $(-\infty, 0)$ and



$\boxed{\text{So } g(-2) \text{ is a local min and since } x=-2 \text{ is the only critical \# on } (-\infty,0),\ g(-2) \text{ must be the absolute min.}}$

Having practiced the first derivative test for absolute extrema in a general setting, let us now apply it to applied optimization problems. Today we will start off easy.

3. Find two numbers whose difference is 100 and whose product is a minimum.

(a) Let the two numbers in question be denoted x and y. By the given info we know the difference of x and y is 100, so

$$x - y = 100.$$

This equation is called our *constraint equation* as it puts constraints on what x and y can be.

(b) We are seeking to minimize the product of x and y, denoted

$$P = xy.$$

This equation is called our *optimizing equation*, and we will eventually do our calculus here.

(c) Notice our expression for P involves two variables! To get around this, let us solve our constraint equation $x - y = 100$ for y, giving

$$y = \cancel{100 - x}\ x - 100$$

Substituting this into our expression for P gives

$$P = x(\overset{x-100}{\cancel{100 - x}}) = \cancel{100x - x^2} = x^2 - 100x$$

(d) Notice P is now a function of a single variable. What is the domain of P? Well x and y can in principle be anything, so our domain is $(-\infty, \infty)$.

(e) Now use the first derivative test for absolute extrema to find the minimum value of $P = \overset{x^2-100x}{\cancel{100x - x^2}}$ on the interval $(-\infty, \infty)$. Be sure to include full justification.

$\cancel{P' = 100 - 2x}$ $P' = 2x - 100$, so $x = 50$ is the critical #.

$$\underset{50}{\xleftarrow{\quad - \quad | \quad + \quad}} \ P'$$

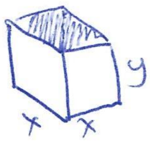
$\boxed{\text{So } P(50) \text{ is a local min and since } x = 50 \text{ is the only critical \# on } (-\infty, \infty), P(50) \text{ must be the absolute minimum.}}$

(f) The step above will give the value of x the satisfies the problem; but we were told to find two numbers. To find y, substitute the value for x back into the constraint. Thus

$$\boxed{x = 50} \qquad y = x - 100$$
$$= (50 - 100)$$
$$\boxed{= -50}$$

4. If 27 cm^2 of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

(a) Let x denote the length of the sides of the square base of the box, and let y denote the height. We know the surface area of the box is 27 cm^2, so write this as an equation involving x and y. This is our *constraint.* [Hint: a picture helps a ton!]



$SA = \underbrace{4xy}_{\text{4 faces of are } x\cdot y} + \underbrace{x^2}_{\text{bottom has area } x^2}$

We need $SA=27$ so

$$27 = 4xy + x^2$$

(b) We are seeking to ~~minimize~~ maximize the volume V of the box. Write an equation involving V, x, and y. This is our *optimizing equation.*

$$V = x^2 y$$

(c) Your expression for V should involve x and y. To get it purely in terms of x, solve your constraint equation from (a) for y and substitute this into your optimizing equation and simplify. V should now be a function of x alone.

Solving the constraint for y gives $y = \frac{27-x^2}{4x}$. So

$$V = x^2y = x^2\left(\frac{27-x^2}{4x}\right) = \frac{1}{4}x(27-x^2) = \frac{1}{4}(27x - x^3)$$

(d) Since x and y are lengths of sides of a box, what is the domain of V?

Domain: $(0, \infty)$

(e) Apply the first derivative test for absolute extrema to find the maximum value of V on the domain found above. Be sure to include full justification.

$V' = \frac{1}{4}(27 - 3x^2)$. So $V' = 0$ when $27 - 3x^2 = 0 \rightarrow x = \pm 3$ are the critical numbers.

$x = 3$ is only critical number in $(0, \infty)$ and

So $V(3)$ is a local max, and since 3 was the only critical # on $(0, \infty)$, $V(3)$ must be the absolute max.

(f) What is the largest possible volume?

By part (e), the largest volume is $V(3) = \frac{1}{4}(27(3) - (3)^3)$

$$= \frac{1}{4}(81 - 27)$$

$$= \frac{54}{4} = \boxed{\frac{27}{2} \text{ cm}^3}$$

5. Now try one on your own. Find two positive numbers whose product is 100 and whose sum is a minimum.

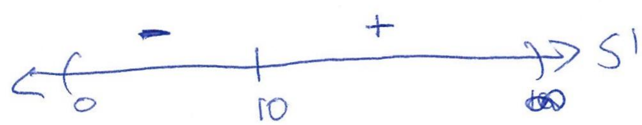
Let x and y denote the two numbers. We need $x \cdot y = 100$ and we want to minimize the sum $S = x + y$.

To get S in terms of 1-variable, we note $xy = 100$ implies $y = \frac{100}{x}$.

So $S = x + \frac{100}{x}$ and the domain is $(0, \infty)$ since we are told to find "two positive numbers".

$S' = 1 - \frac{100}{x^2}$ so $S' = 0$ when $1 - \frac{100}{x^2} = 0$. So $x = \pm 10$ are critical #'s.

Only 10 is in $(0, \infty)$.



So $S(10)$ is a local minimum and since $x = 10$ is the only critical number on $(0, \infty)$, $S(10)$ is the absolute min.

Since $y = \frac{100}{x} = \frac{100}{10} = 10$, $\boxed{x = 10 \text{ and } y = 10}$ are the positive numbers whose product is 100 and whose sum is a minimum.