Print Name ________________________________

APPM 1350 **Exam 2** **Summer 2016**

On the front of your bluebook, please write: a grading key, your name, student ID, your lecture number and instructor. This exam is worth 100 points and has 5 questions on both sides of this paper.

- Submit this exam sheet with your bluebook. However, nothing on this exam sheet will be graded. Make sure all of your work is in your bluebook.
- **Show all work and simplify your answers!** Answers with no justification will receive no points.
- Please begin each problem on a new page.
- No notes or papers, calculators, cell phones, or electronic devices are permitted.

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1. The following parts are not related:

   (a) (9 pts each) Find $\dfrac{dy}{dx}$ given:

   i. $y = \dfrac{\tan x + 1}{\csc x}$

   ii. $x^2 = \sqrt{xy} + 2y^2$

   (b) (9 pts) A function f and its derivative have values shown in Table 1 below.
   Let $g(x) = -x^2 f(2x)$. Use the values in the table to compute $g'(1)$.

Table 1

x	$f(x)$	$f'(x)$
0	1	2
1	1	-2
2	3	-1

Solution:

(a) Find $\dfrac{dy}{dx}$ given:

i. $y = \dfrac{\tan x + 1}{\csc x}$

$$\frac{dy}{dx} = \frac{\sec^2 x \csc x - (\tan(x) + 1)(-\csc x \cot x)}{\csc^2 x} = \frac{\sec^2 x + 1 + \cot x}{\csc x}$$

or:

$$y = \frac{\tan x + 1}{\csc x} = \sin x(\tan x + 1) \text{ and } \frac{dy}{dx} = \cos x(\tan x + 1) + \sin x(\sec^2 x)$$

$$= \cos x \tan x + \cos x + \sin x \sec^2 x$$

ii. $x^2 = \sqrt{xy} + 2y^2$

Differentiating implicitly we get:

$$\begin{aligned}
& x^2 = \sqrt{xy} + 2y^2 \\
\Rightarrow\ & 2x = \frac{1}{2}(xy)^{-1/2}\left(y + x\frac{dy}{dx}\right) + 4y\frac{dy}{dx} \\
\Rightarrow\ & 4x\sqrt{xy} = y + x\frac{dy}{dx} + 8y\sqrt{xy}\frac{dy}{dx} \\
\Rightarrow\ & 4x\sqrt{xy} - y = (x + 8y\sqrt{xy})\frac{dy}{dx} \\
\Rightarrow\ & \frac{dy}{dx} = \frac{4x\sqrt{xy} - y}{x + 8y\sqrt{xy}}
\end{aligned}$$

(b) The derivative of $g(x)$ is given by $g'(x) = -2xf(2x) - x^2 f'(2x)2 = -2\left[xf(2x) + x^2 f'(2x)\right]$ At $x = 1$ we get $g'(1) = -2\left[f(2) + f'(2)\right] = -2\left[3 - 1\right] = -4$.

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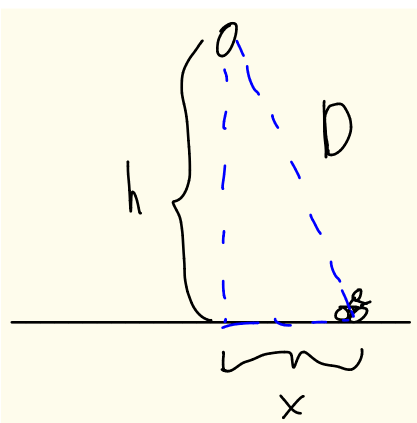
2. (14 pts) A balloon is rising at a constant speed of 2 meters per second. A girl is cycling along a straight road at a constant speed of 1 meter per second. When she passes under the balloon, it is 3 meters above her. How fast is the distance between the balloon and the girl increasing 3 seconds later?

Solution:



Given: $\frac{dh}{dt} = 2\frac{\text{m}}{\text{s}}$, $\frac{dx}{dt} = 1\frac{\text{m}}{\text{s}}$ and when $t = 0$, $h = 3$ meters. Find $\frac{dD}{dt}$.

By the Pythagorean theorem: $x^2 + h^2 = D^2$. Implicitly differentiating: $2x\frac{dx}{dt} + 2h\frac{dh}{dt} = 2D\frac{dD}{dt}$ or $x\frac{dx}{dt} + h\frac{dh}{dt} = D\frac{dD}{dt}$. After 3 seconds have passed, $h = 9$ m and $x = 3$ m. $D = \sqrt{x^2 + h^2} = \sqrt{3^2 + 9^2} = \sqrt{90} = 3\sqrt{10}$. Thus $\frac{dD}{dt} = \frac{x\frac{dx}{dt} + h\frac{dh}{dt}}{D} = \frac{3(1) + 9(2)}{3\sqrt{10}} = \frac{7}{\sqrt{10}}\frac{\text{m}}{\text{s}}$ or $\frac{7\sqrt{10}}{10}\frac{\text{m}}{\text{s}}$.

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3. The local dice company has a machine that creates six-sided dice with a volume of $V(x) = (2x - 1)^3$.

(a) (9 pts) Use a linearization to compute $V(2.01)$.

(b) (9 pts) If a particular die is made with a value of $x = 5$ mm with maximum error of 0.01 mm in the measurement of x, compute the percent error in the volume.

Solution:

(a) Recall $L(x) = f(a) + f'(a)(x - a)$. We want $V(2.01)$ so let $a = 2$ and $f(x) = (2x - 1)^3$.

Then $f'(x) = 3(2x-1)^2(2) = 6(2x-1)^2$ and $f'(2) = 54$. Thus, $L(x) = 27+54(x-2) = 54x-81$. So, $L(2.01) = 54(2.01) - 81 = 27.54$ mm.

(b) $\dfrac{dV}{V} = \dfrac{6(2x-1)^2}{(2x-1)^3}dx = \dfrac{6}{2x-1}dx$. When $x = 5$ and $dx = 0.01$ then $\dfrac{dV}{V} = \dfrac{0.02}{3}$. So the percent change in V is given by: $\dfrac{dV}{V} \cdot 100\% = \dfrac{2}{3}\%$ or approximately 0.67%.

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4. Answer the following.

Given $f(x) = \dfrac{-x^2+1}{(x-2)^2}$ with, $f'(x) = \dfrac{4x-2}{(x-2)^3}$ and, $f''(x) = \dfrac{-8x-2}{(x-2)^4}$, where the intercepts of f are $(1, 0)$, $(-1, 0)$, and $\left(0, \dfrac{1}{4}\right)$, find the following for f.

(a) (10 pts) Find all asymptote(s) for f. Justify you answer(s) using the appropriate limits.

(b) (4 pts) Find the intervals of increase and decrease for the function f. Justify your answer(s).

(c) (4 pts) Find the local maximum and minimum values for the function f. Justify your answer(s).

(d) (7 pts) Find the intervals of concave up and down and the inflection points for the function f. Justify your answer(s).

(e) (6 pts) Use parts (a) - (d) to sketch the graph of f. LABEL the asymptote(s), maximum(s), minimum(s), and inflection point(s) on your graph.

Solution:

(a) Find the vertical, horizontal, and slant asymptote(s). Justify your answer(s) by using the appropriate limits.
Note that the domain is $(-\infty, 2) \cup (2, \infty)$.

VA: There is a vertical asymptote at $x = 2$ since $\displaystyle\lim_{x\to 2^{\pm}} \frac{-x^2+1}{(x-2)^2} = -\infty$

HA: There is a horizontal asymptote at $y = -1$ since $\displaystyle\lim_{x\to\pm\infty} \frac{-x^2+1}{(x-2)^2} = \lim_{x\to\pm\infty} \frac{-x^2+1}{x^2-4x+4}$

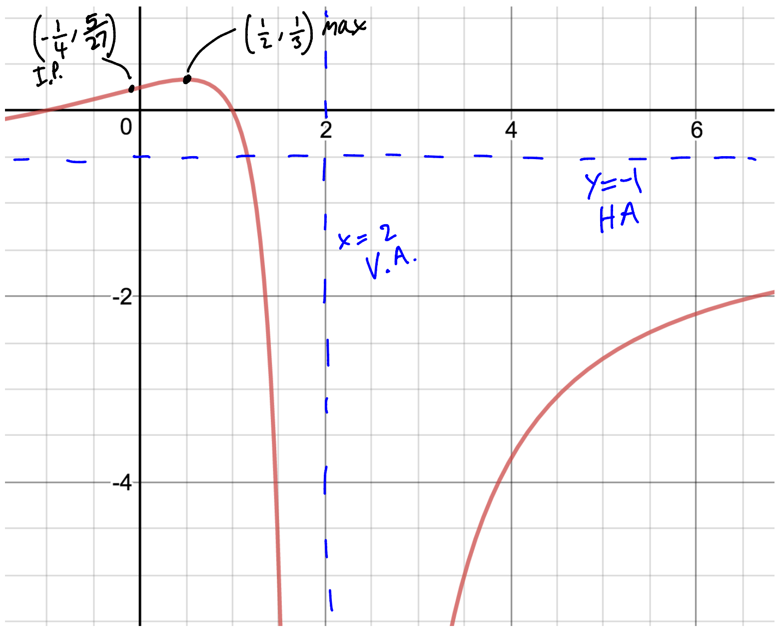
$$= \lim_{x\to\pm\infty} \frac{-1+\frac{1}{x^2}}{1-\frac{4}{x}+\frac{4}{x^2}} = -1.$$

(b) Finding the critical values: $f'(x) = 0$ when $x = \dfrac{1}{2}$ and $f'(x)$ does not exist when $x = 2$. So the critical values are $x = \dfrac{1}{2}$ and $x = 2$. Using a sign chart we get the interval of decrease: $\left(\dfrac{1}{2}, 2\right)$ and intervals of increase $\left(-\infty, \dfrac{1}{2}\right)$ and $(2, \infty)$.

(c) By the first derivative test we see that there is a maximum at $\left(\frac{1}{2}, \frac{1}{3}\right)$. There is no minimum at $x = 2$ since this value is not in the domain of f.

(d) Note that $f''(x) = 0$ when $x = -\frac{1}{4}$ and $f''(x)$ DNE when $x = 2$. The interval where f is concave up is: $\left(-\infty, -\frac{1}{4}\right)$. The intervals where f is concave down are: $\left(-\frac{1}{4}, 2\right)$ and $(2, \infty)$. There is an inflection point at $\left(-\frac{1}{4}, \frac{5}{27}\right)$ due to the change in concavity.

(e)



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5. (a) (5 pts) State the mean value theorem.

   (b) (5 pts) Suppose that $f(x)$ is an even function and is differentiable everywhere. Use the mean value theorem to show that for every positive number b, there exists a number c in $(-b, b)$ such that $f'(c) = 0$.

Solution:

(a) State the mean value theorem.
Let f be a function that is continuous on an interval $[a, b]$ and differentiable on (a, b). Then there exists at least one c in (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

(b) Since f is a function that is differentiable everywhere then it is continuous everywhere, and so the Mean Value Theorem is applicable. By the Mean Value Theorem there exists a c in the interval $(-b, b)$ such that $f'(c) = \frac{f(b) - f(-b)}{b - (-b)} = \frac{f(b) - f(b)}{b + b} = \frac{0}{2b}$. The quantity $\frac{0}{2b} = 0$ since b is a positive value.
