Solutions

Math 1300-005 - Spring 2017

The Mean Value Theorem - 3/21/17

Guidelines: Please work in groups of two or three. This will not be handed in, but is a study resource for Midterm 3.

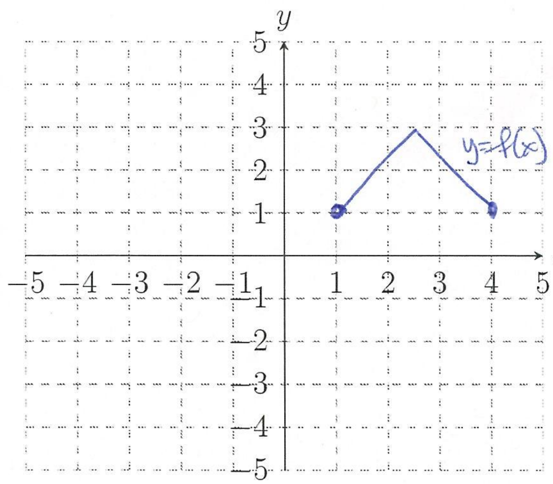
The purpose of this worksheet is to explore the ***Mean Value Theorem***, which states that if f is continuous on $[a, b]$ and differentiable on $(a.b)$, then there exists a number c between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{1}$$

or equivalently,

$$f(b) - f(a) = f'(c)(b - a). \tag{2}$$

1. The MVT requires f to be continuous on $[a, b]$ *and* differentiable on (a, b). On the axes below, sketch an example of a function f, continuous on $[a, b]$, but *not* differentiable on (a, b) such that the conclusion of the MVT is false. That is, such that there is no c between a and b such that (1) is true.



Note: $\dfrac{f(4)-f(1)}{4-1} = 0$

But there is no point c where $f'(c)=0$.

2. Let $f(x) = (x - 4)^2 - 1$ on $[3, 6]$.

(a) Does f satisfy the hypotheses of the MVT on $[3, 6]$? That is, is f continuous on $[3, 6]$ and differentiable on $(3, 6)$? Please explain.

Yes... f is a polynomial so continuous on $[3,6]$. And $f'(x) = 2(x-4)$, which is a line, so f' exists for all x in $(3,6)$.

(b) Determine all numbers c which satisfy the conclusion of the MVT for f on $[3, 6]$.

MVT says there exists c in $(3,6)$ with $f'(c) = \dfrac{f(6)-f(3)}{6-3}$.

$f'(c) = 2(c-4)$

$f(6) = (6-4)^2 - 1 = 3$

$f(3) = (3-4)^2 - 1 = 0$

$2(c-4) = \dfrac{3-0}{6-3}$

$2c - 8 = 1$

$2c = 9$

$\boxed{c = \frac{9}{2}}$

3. A corollary of the Mean Value Theorem is known as **Rolle's Theorem**. In this problem, we will derive the result of this theorem. Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Suppose as well that $f(a) = f(b)$.

(a) Write down the conclusion of the MVT for f (this is (1) on the first page).

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

(b) What is the value of the right hand side of (1) in this case?

Well, $f(b) = f(a)$, so $f(b) - f(a) = 0$.

Hence $f'(c) = \frac{0}{b-a} = 0$

(c) Finish the statement of **Rolle's Theorem**: If f is continuous on $[a, b]$, differentiable on (a, b) and if $f(a) = f(b)$, then there exists a number c between a and b such that

$$f'(c) = \underline{\ 0\ }$$

4. Let $f(x) = x^2 - 2x - 8$ on $[-1, 3]$.

(a) Does f satisfy the hypotheses of Rolle's theorem on $[-1, 3]$? That is, is f continuous on $[-1, 3]$ and differentiable on $(-1, 3)$ and does $f(-1) = f(3)$? Please explain.

f is a polynomial so is continuous on $[-1,3]$.

$f'(x) = 2x - 2$ so f' exists on $(-1,3)$, ... ie f is differentiable on $(-1,3)$.

$f(-1) = (-1)^2 - 2(-1) - 8 = 1 + 2 - 8 = -5$

$f(3) = (3)^2 - 2(3) - 8 = 9 - 6 - 8 = -5$

so $f(-1) = f(3)$.

(b) Determine all numbers c which satisfy the conclusion of Rolle's theorem for f on $[-1, 3]$.

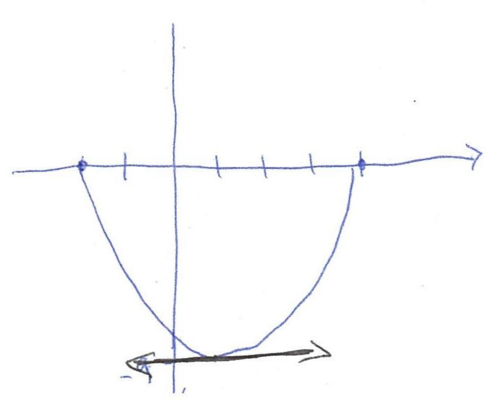
$f'(c) = 0$

$2c - 2 = 0$

$2c = 2$

$\boxed{c = 1}$

Picture: $f(x) = (x-4)(x+2)$



5. Let $f(x) = x^3 + 6x^2 + 6x$ on $[-6, 0]$.

(a) Does f satisfy the hypotheses of the MVT on $[-6, 0]$? That is, is f continuous on $[-6, 0]$ and differentiable on $(-6, 0)$? Please explain.

f is a polynomial, so is continuous on $[-6,0]$.

$f'(x) = 3x^2 + 12x + 6$ is a polynomial and is defined on $(-6,0)$, so f is differentiable on $(-6,0)$.

(b) Determine all numbers c which satisfy the conclusion of the MVT for f on $[-6, 0]$.

MVT says there is a c between -6 and 0 with

$$f'(c) = \frac{f(0) - f(-6)}{0 - (-6)}$$

$f'(c) = 3c^2 + 12c + 6$

$f(0) = 0$

$f(-6) = (-6)^3 + 6(-6)^2 + 6(-6)$

$= -216 + 216 - 36 = -36$

$$3c^2 + 12c + 6 = \frac{0 - (-36)}{0 - (-6)} = \frac{36}{6} = 6$$

$3c^2 + 12c + \cancel{6} = \cancel{6}$

$3c^2 + 12c = 0$

$3c(c+4)$

$\cancel{c = 0}$ → not in $(-6,0)$

$\boxed{c = -4}$

6. Suppose that $f(0) = -3$ and $f'(x) \le 5$ for all values of x. The inequality gives a restriction on the rate of growth of f, which then imposes a restriction on the possible values of f. Use the MVT to determine how large $f(4)$ can possible be. [Hint: setup the MVT using (2) on the first page, and solve for $f(4)$.]

MVT ~~says~~ on $[0,4]$ says $f(4) - f(0) = f'(c)(4 - 0)$ for some c in $(0,4)$.

So $f(4) = f'(c) \cdot 4 + f(0)$

$= 4f'(c) - 3.$

But $f'(c) \le 5$, so

$f(4) = 4f'(c) - 3 \le 4(5) - 3 = 17.$

So $\boxed{f(4) \le 17, \text{ ie } 17 \text{ is the largest } f(4) \text{ could possibly be.}}$