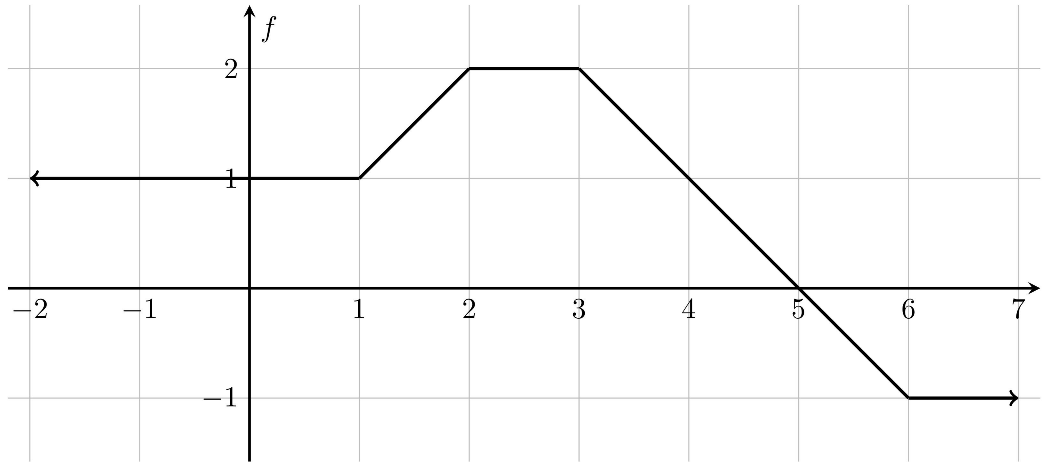
Area accumulation functions – an introduction

Given a function $f(x)$, we create a new function $F(x)$ by evaluating how much area is accumulated under $f(x)$.

1. Example:



(a) Define $F(x) = \int_0^x f(t)\,dt$. Evaluate the following:

$F(0) = 0$ $F(2) = 2.5$
$F(1) = 1$ $F(-1) = -1$

(b) Shade in and find the area represented by $F(3) - F(1)$.
It is the area under the curve between $x = 1$ and $x = 3$, an area of 3.5.

(c) Find a formula for $F(x)$ between $x = 0$ and $x = 1$
x

(d) Give two values at which $F(x) = 0$. (Hint: assume the graph continues to the right.)
$x = 0$ and $x = 12$

(e) Which is larger: $F(3)$ or $F(4)$? Explain.
$F(4)$ is larger because $F(x)$ accumulates area and all the area between $x = 3$ and $x = 4$ counts as positive since $f(x)$ is positive there.

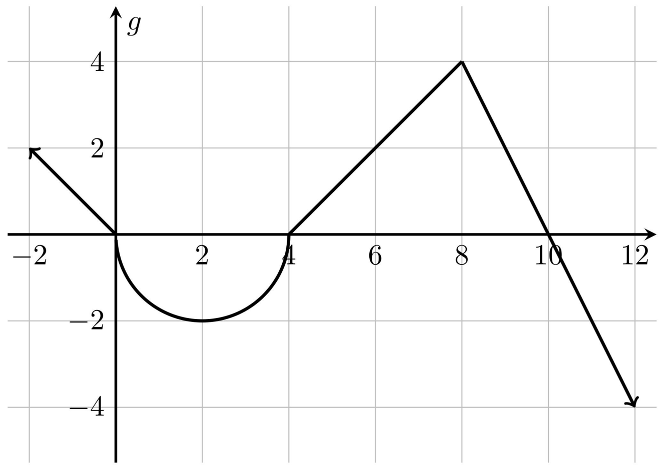
(f) Which is larger: $F(5)$ or $F(6)$? Explain.
$F(5)$ is larger because the area accumulated between $x = 5$ and $x = 6$ counts negatively, since $f(x)$ is negative there.

(g) Give open intervals on which $F(x)$ is increasing. Explain.
$F(x)$ is increasing from $x = 0$ to $x = 5$. On this interval the area is all positive, so as it accumulates, the value of $F(x)$ must increase. To the left of $x = 0$, $F(x)$ is also increasing because the value of the integrals becomes less negative as the value of x moves to the right.

(h) $F(x)$ has a local extremum at $x = 5$. Is it a maximum or a minimum? Explain.
It is a local maximum because $F(x)$ is increasing up to $x = 5$ and decreasing after.

(i) $F(x)$ is increasing at both $x = 1$ and $x = 2$. At which value is $F(x)$ increasing faster? Explain.
$F(x)$ is increasing faster at $F(2)$ because the value of $f(x)$ is larger, so as x grows, the area is accumulating at a faster rate.

2. g is a piecewise function composed of line segments and a semi-circle.



(a) $G(x) = \int_4^x g(t)\,dt$

$G(4) = 0$

$G(12) = 8$

$G(10) = 12$

$G(0) = 2\pi$

(b) On what open intervals is $G(x)$ increasing? decreasing?

$G(x)$ is decreasing on $(0, 4) \cup (10, \infty)$. $G(x)$ is increasing on $(-\infty, 0) \cup (4, 10)$.

(c) Find all local extreme values of $G(x)$ by determining where $G(x)$ switches from increasing to decreasing and from decreasing to increasing.

$G(x)$ has local maximum values at $x = 0$ and $x = 10$ (the area accumulation function switches from increasing to decreasing there), and the local max values are $G(0) = 2\pi$ and $G(10) = 12$. The area accumulation function switches from decreasing to increasing at $x = 4$, so G has a local minimum there. The local minimum is $G(4) = 0$.

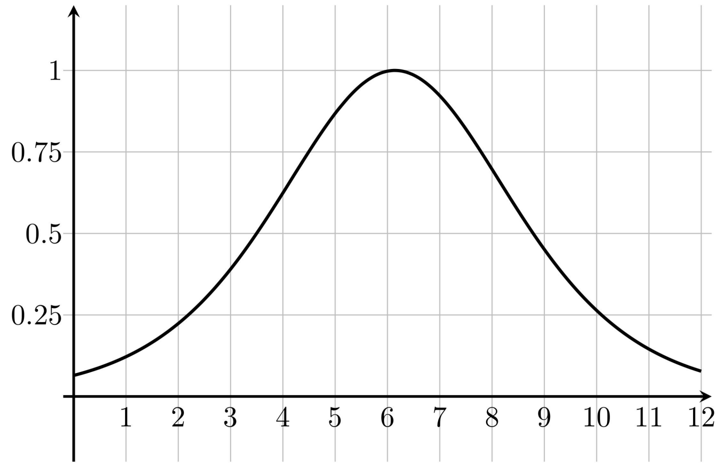
(d) What are the critical numbers of $G(x)$?

$G(x)$ has critical numbers at $x = 0, 4, 10$.

(e) On the interval from $[0, 12]$ where is $G(x)$ increasing fastest?

At $x = 8$.

3. The peak of Boulder's epic rainstorm of 2013 occured between 4pm, Sept 12, and 4am, Sept 13. During those 12 hours the rate of rainfall can be modelled by $r(t) = \dfrac{240e^{2t/3}}{\left(60 + e^{2t/3}\right)^2}$ in inches per hour, where $t = 0$ represents 4pm on Sept 12.



Let $R(x) = \displaystyle\int_0^x \frac{240e^{2t/3}}{\left(60 + e^{2t/3}\right)^2}\,dt$.

(a) Use the graph to estimate $R(4)$. What does it represent? (include units)
At $R(4) \approx 1$. This represents the total number of inches of rain that have fallen between 4pm, Sept 12, and 4 hours later, at 8pm, Sept 12.

(b) Use technology to calculate $R(12)$. What does it represent? (include units)
At $R(12) \approx 5.78326$. This represents the total number of inches of rain that have fallen between between 4pm, Sept 12, and 12 hours later, at 4am, Sept 13.

(c) What does $R(x)$ represent?
The amount of rain that has fallen between 4pm, Sept 12 and x hours later.

(d) Where is $R(x)$ changing the fastest?
At $x = 6$.