1. (a) (5 pts. each) Find the following derivatives, $f'(x)$, for the f given:

$$\text{(a) } f(x) = x^2 \sin^{-1}(x^2) \qquad \text{(b) } f(x) = \frac{1}{1+x^2} \qquad \text{(c) } f(x) = x \ln x \qquad \text{(d) } f(x) = x^x$$

(b) (15 pts) Using rules for differentiation it is easy to show that $\dfrac{d}{dx}(3x^2+1) = 6x$. Show that this is true directly from the definition of differentiation. No credit will be given for quoting rules of differentiation.

Solution:

(a) $f'(x) = 2x\sin^{-1}(x^2) + x^2 \frac{1}{1-x^4} 2x = \boxed{2x\sin^{-1}(x^2) + \dfrac{2x^3}{\sqrt{1-x^4}}}$

(b) $\boxed{f'(x) = \dfrac{-2x}{(1+x^2)^2}}$

(c) $f'(x) = (x)' \ln x + x(\ln x)' = \boxed{\ln x + 1}$

(d) $\ln y = x \ln x \implies \frac{1}{y} y' = \ln x + 1 \implies \boxed{y' = x^x(\ln x + 1)}$

(e) $\boxed{\dfrac{d}{dx}(3x^2+1) = \lim_{h\to 0} \dfrac{3(x+h)^2 + 1 - (3x^2+1)}{h} = \lim_{h\to 0} \dfrac{3x^2 + 6xh + 3h^2 + 1 - 3x^2 - 1}{h} = \lim_{h\to 0}(6x+3h) = 6x}$

2. (5 pts each) In answering the following questions, justify each part. Given $f(x) = \dfrac{x|x|}{2+x}$, for f:

(a) Find the vertical and horizontal asymptotes.

(b) Find the intervals of increase or decrease.

(c) Find the local maximum and minimum values.

(d) Find the intervals of concavity and the inflection points.

(e) Use parts (a) - (d) to the sketch the graph of f. LABEL your sketch (Intercepts, asymptotes, etc.).

Solution:

(a) For the HA:

$$\lim_{x\to\infty} \frac{x|x|}{2+x} = \lim_{x\to\infty} \frac{x^2}{2+x} = \infty$$

and

$$\lim_{x\to-\infty} \frac{x|x|}{2+x} = \lim_{x\to-\infty} \frac{-x^2}{2+x} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to-\infty} \frac{-2x}{1} = \infty$$

For VA:

$$\lim_{x\to-2^+} \frac{x|x|}{2+x} \approx \frac{-4}{0^+} = -\infty$$

and

$$\lim_{x\to-2^-} \frac{x|x|}{2+x} \approx \frac{-4}{0^-} = \infty$$

(b) Chop up into two regions, $x > 0$ and $x < 0$. Then

$$(x<0),\quad f'(x) = \left(\frac{-x^2}{2+x}\right)' = \frac{-2x(2+x)+x^2}{(2+x)^2} = \frac{-4x-x^2}{(2+x)^2}$$

and so we look at $-4x - x^2 = 0 \Rightarrow x = -4, 0$. For

$$(x>0),\quad f'(x) = \left(\frac{x^2}{2+x}\right)' = \frac{2x(2+x)-x^2}{(2+x)^2} = \frac{4x+x^2}{(2+x)^2}$$

and so we look at $4x + x^2 = 0 \Rightarrow x = 0, -4$. Either way, we have three regions. $f'(-5) < 0$ so we are decreasing on $(-\infty, -4)$ and $f'(-1) > 0$ so we are increasing on $(-4, 0)$. On $(0, \infty)$ we have $f'(1) > 0$ so it is increasing on $(0, \infty)$.

(c) From part (b) we know that $f(-4) = \dfrac{-16}{(2-4)} = 8$ is a local min and that is all.

(d) Here,

$$(x<0)\ \ f''(x) = -\frac{(4+2x)(2+x)^2 - 2(4x+x^2)(2+x)}{(2+x)^4} = -\frac{(2x+4)(x+2) - 2x^2 - 8x}{(2+x)^3} = -\frac{8}{(2+x)^3}$$

and

$$(x>0)\ \ f''(x) = \frac{8}{(2+x)^3}$$

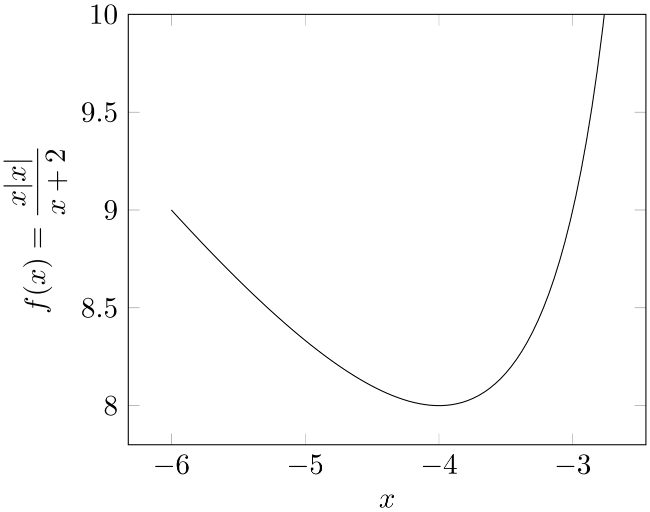
The inflection points then are $x = -2$ and $x = 0$. (no justification needed). We compute

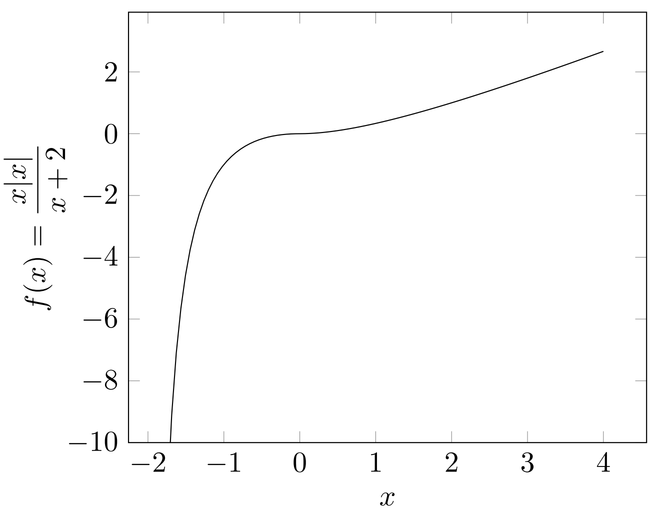
$$f''(-3) = -\frac{8}{(-1^3)} > 0$$

and so it is CU on $(-\infty, -2)$. Also,

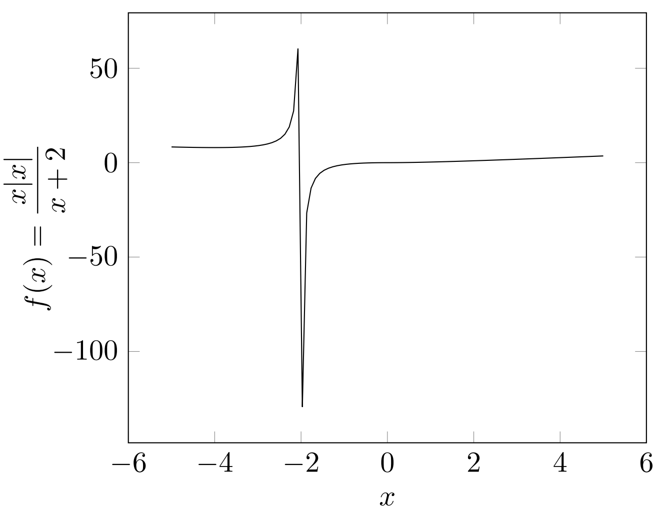
$$f''(-1) = -\frac{8}{1^3} < 0$$

so it is CD on $(-2, 0)$ and lastly, $f''(1) > 0$ so it is CU on $(0, \infty)$.


10
9.5
9
8.5
8
−6
−5
−4
−3
x
$f(x) = \frac{x|x|}{x+2}$


2
0
−2
−4
−6
−8
−10
−2
−1
0
1
2
3
4
x
$f(x) = \frac{x|x|}{x+2}$

(e)


50
0
−50
−100
−6
−4
−2
0
2
4
6
x
$f(x) = \frac{x|x|}{x+2}$

3. (15 points)You are designing a mural and you would like to have a margin of 1 feet on the left, right, and top of the artwork, but none on the bottom. If you allow 32 ft^2 for the area containing the artwork itself, what dimensions should the artwork have if you want to minimize the total area of the mural (i.e., of the artwork and the margins.)

   Solution: Draw a picture. If we call the total length of the base x and the total length of the height y then the total area is $A = xy$. The restriction is that $(x-2)(y-1) = 32$ and upon solving for y we have

$$y = \frac{32}{x-2} + 1$$

   Substituting this into the original equation we get

$$A = x\frac{32}{x-2} + x$$

   . It follows

$$A' = -\frac{64}{(x-2)^2} + 1 = 0 \Rightarrow x = 10.$$

   Then $y = 5$. This is a min because

$$A'' = \frac{128}{(x-2)^3}, A''(10) > 0$$

   and so the area is CU at the critical point.

4. (6 points each)

(a) The interval $[-1, 3]$ is partitioned into n subintervals of equal length. Let r_k denote the right-hand endpoint of the k^{th} subinterval. Express the integral $\int_{-1}^{3}(3x^2 - 2x + 5)dx$ as the limit of a Riemann sum using the right-hand endpoints of each subinterval.

(b) Given that $a < b$, what values of a and b minimize the value of $\int_a^b (t^4 - 2t^2)dt$?

(c) Solve the initial value problem: $\frac{dy}{dx} = x\sqrt{1 + x^2}$ with $y(1) = -2$.

(d) $\displaystyle \frac{d}{dx}\int_1^{10^x} t^t dt = ?$

(e) $\displaystyle \int_0^{\sqrt{ln\pi}} 2xe^{x^2}\cos(e^{x^2})dx = ?$

Solution:

(a) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} (3r_k^2 - 2r_k + 5)\frac{4}{n}$

(b) $\int_a^b t^2(t^2 - 2)dt$ integrand equals 0 $\implies t = 0, \sqrt{2}, -\sqrt{2}$ The polynomial is non-positive only between $\sqrt{2}$ and $-\sqrt{2}$, therefore $a = \sqrt{2}$ and $b = -\sqrt{2}$.

(c) $\frac{dy}{dx} = x\sqrt{1 + x^2} \implies \int 1dy = \int x\sqrt{1 + x^2}dx$. u-sub produces: $u = 1 + x^2, du = 2xdx, \frac{1}{2}du = xdx$

$y = \frac{1}{2}\int u^{\frac{1}{2}}du \implies y = \frac{1}{2}[\frac{2}{3}u^{\frac{3}{2}} + c] \implies y = \frac{1}{3}(1 + x^2)^{\frac{3}{2}} + c$

$y(1) = -2 \implies -2 = \frac{1}{3}(1+1)^{\frac{3}{2}} + c \implies c = -2 - \frac{1}{3}\sqrt{8} = \frac{-6-2\sqrt{2}}{3} \implies \boxed{y = \dfrac{1}{3}(1 + x^2)^{\frac{3}{2}} - \dfrac{6 + 2\sqrt{2}}{3}}$.

(d) $u = 10^x, \frac{du}{dx} = 10^x \ln 10$

$\frac{d}{dx}\int_1^u t^t dt = u^u 10^x \ln 10 = (10^x)^{10^x} 10^x \ln 10 + c = \boxed{(10^x)^{10^x+1}\ln 10 + c}$.

(e) With $u = e^{x^2}, du = e^{x^2}2xdx$ we get $\int_0^{\sqrt{\ln\pi}} 2xe^{x^2}\cos(e^{x^2})dx = \int_?^{??} \cos u du = \sin(e^{x^2})\Big|_0^{\sqrt{\ln\pi}} = \sin(e^{\ln\pi}) - \sin(e^0) = \sin\pi - \sin 1 = \boxed{-\sin 1}$

5. (6 points each) Evaluate the following:

(a) $\lim_{t\to 3} \cos^{-1}\left(\log_3 \sqrt{t}\right)$

(b) $\lim_{x\to\infty} x\tan(5/x)$

(c) $\lim_{x\to\infty} (e^x + x)^{3/x}$

(d) $\displaystyle\int \frac{\sinh x}{1+\cosh x}\,dx$

(e) $\displaystyle\int \frac{\sin(2x)}{\cos^2(2x)+1}\,dx$

Solution:

(a)

$$\lim_{t\to 3} \cos^{-1}\left(\log_3 \sqrt{t}\right) = \cos^{-1}\left(\lim_{t\to 3} \log_3 \sqrt{t}\right) = \cos^{-1}(\log_3 \sqrt{3}) = \cos^{-1}(1/2) = \boxed{\pi/3}$$

(b) This is an indeterminate product. Apply L'Hopital's Rule.

$$\lim_{x\to\infty} x\tan(5/x) = \lim_{x\to\infty} \frac{\tan(5/x)}{1/x} \overset{LH}{=} \lim_{x\to\infty} \frac{\sec^2(5/x)(-5/x^2)}{-1/x^2} = \lim_{x\to\infty} 5\sec^2(5/x) = 5(1) = \boxed{5}$$

(c) This is an indeterminate power. Let y equal the limit value.

$$\begin{aligned}
y &= \lim_{x\to\infty} (e^x + x)^{3/x} \\
\ln y &= \lim_{x\to\infty} \ln (e^x + x)^{3/x} = \lim_{x\to\infty} \frac{3\ln(e^x + x)}{x} \overset{LH}{=} \lim_{x\to\infty} \frac{3}{(e^x + x)} \cdot (e^x + 1) \\
&\overset{LH}{=} \lim_{x\to\infty} \frac{3e^x}{e^x + 1} = \lim_{x\to\infty} \frac{3}{1+e^{-x}} = 3 \\
y &= \boxed{e^3}
\end{aligned}$$

(d) Let $u = 1 + \cosh x, \quad du = \sinh x\, dx$.

$$\int \frac{\sinh x}{1+\cosh x}\,dx = \int \frac{du}{u} = \ln|u| + C = \boxed{\ln|1+\cosh x| + C}$$

(e) Let $u = \cos(2x), \quad du = -2\sin(2x)\,dx$.

$$\int \frac{\sin(2x)}{\cos^2(2x)+1}\,dx = -\frac{1}{2}\int \frac{du}{u^2+1} = -\frac{1}{2}\tan^{-1} u + C = \boxed{-\frac{1}{2}\tan^{-1}(\cos(2x)) + C}$$

6. The intensity $L(x)$ of light x feet beneath the surface of the ocean satisfies the differential equation $dL/dx = kL$.

   (a) (5 points) Use the law of exponential decay to find an expression for $L(x)$ in terms of k.

   (b) (5 points) If diving to 18 ft cuts the light intensity in half, what is the rate constant k?

   (c) (5 points) Once the intensity falls below one-tenth of the surface value, you will not be able to work without artificial light. How deep can you work without artificial light?

Solution:

(a) The solution to $dy/dt = ky$ is $y(t) = y(0)e^{kt}$ so the solution to $dL/dx = kL$ is $L(x) = \boxed{L(0)e^{kx}}$.

(b) We are given that $L(18) = L(0)/2$. Note that $\ln(1/a) = -\ln a$.

$$L(18) = L(0)e^{18k} = \frac{L(0)}{2} \Rightarrow e^{18k} = \frac{1}{2} \Rightarrow 18k = \ln\frac{1}{2} \Rightarrow k = \boxed{-\frac{\ln 2}{18}}$$

(c) Solve for x when $L(x) = L(0)/10$.

$$L(x) = L(0)e^{kx} = \frac{L(0)}{10} \Rightarrow e^{kx} = \frac{1}{10} \Rightarrow kx = \ln\frac{1}{10} \Rightarrow x = -\frac{\ln 10}{k} \Rightarrow x = \boxed{\frac{18\ln(10)}{\ln 2}\text{ ft}} \approx 59.8\text{ ft}$$