1. (30 points) Evaluate the following expressions. If the limit does not exist then provide a reason why it does not exist:

(a) $\lim_{x\to -1} \sqrt[3]{3x^4+x^3-5x^2+1}$ (b) $\lim_{s\to 0}\left(\frac{\frac{1}{\sqrt{1+s}}-1}{s}\right)$ (c) $\lim_{t\to -\infty}\left(\frac{30t}{200+t}\right)$

(d) $\lim_{x\to 0}\frac{|x+1|-|x-1|}{x}$ (e) $\lim_{x\to 2}\sqrt{4-x^2}$ (f) $\lim_{x\to 2^+}\left(\frac{x-2}{x^2-x-2}\right)$

Solution:

(a) by direct substitution

$$\sqrt[3]{3-1-5+1} = \boxed{\sqrt[3]{-2}}$$

(b)

$$\lim_{s\to 0}\left(\frac{1-\sqrt{1+s}}{s\sqrt{1+s}}\right) = \lim_{s\to 0}\left(\frac{1-\sqrt{1+s}}{s\sqrt{1+s}} * \frac{1+\sqrt{1+s}}{1+\sqrt{1+s}}\right) = \lim_{s\to 0}\left(\frac{-s}{s\sqrt{1+s}\,(1+\sqrt{1+s})}\right) = \boxed{-\frac{1}{2}}$$

(c) $\boxed{30}$ by D.O.P.

(d) For x's near zero we have:

$$\lim_{x\to 0}\frac{|x+1|-|x-1|}{x} = \lim_{x\to 0}\left(\frac{(x+1)-[-(x-1)]}{x}\right) = \lim_{x\to 0}\frac{x+1+x-1}{x} = \lim_{x\to 0}\frac{2x}{x} = \boxed{2}$$

(e) D.N.E. because their is no function to the right of 2. The function domain does not exceed 2.

(f)

$$\lim_{x\to 2^+}\left(\frac{x-2}{x^2-x-2}\right) = \lim_{x\to 2^+}\frac{(x-2)}{(x-2)(x+1)} = \lim_{x\to 2^+}\frac{1}{x+1} = \boxed{\frac{1}{3}}$$

2. (12 points) The following questions are not necessarily related.

   (a) Find and describe any discontinuities of $f(x) = \frac{x^2-3x-4}{x^2+3x+2}$ as removable, jump discontinuity, or infinite discontinuity.

   (b) Is the following statement **always true**, or **not always true**. Provide a **brief** explanation:
   If $f(x)$ is a continuous function and $f(x) < 0$ for $x < 3$, and $f(x) > 0$ for $x > 6$, then $f(x) = 0$ for some $3 < x < 6$.

   (c) Consider the function $f(x) = x\sqrt{7-x}$. Is there a real number x such that $f(x) = 2$?
   If "Yes" then explain why. If "No" then explain why not.

   Solution:

   (a) $\frac{x^2-3x-4}{x^2+3x+2} = \frac{(x+1)(x-4)}{(x+1)(x+2)} = \frac{(x-4)}{(x+2)}$ This implies that $x = -1$ is a removable discontinuity and $x = -2$ is a non-removable, infinite discontinuity.

   (b) Not Always True. Consider the line $y = x - 3$. The given information says nothing about what is going on at $x = 3$ or at $x = 6$.

   (c) This function is continuous on its domain $(-\infty, 7]$. Consider $f(-2) = -6$ and $f(3) = 6$. Since $y = 2$ is between $y = -6$ and $y = 6$, there must be a value for x such that $f(x) = 2$ between $x = -2$ and $x = 3$ by the IVT.

3. (18 points) Indicate, in your blue book, the following statements as True or False. No explanation required.

(a) The functions $\sin 2x$ and x are continuous for all real numbers.

(b) The function $\dfrac{\sin 2x}{x}$ is continuous on $(-\infty, \infty)$.

(c) The function $\dfrac{\sin 2x}{x}$ has a vertical asymptote at $x = 0$.

(d) $\lim\limits_{x \to 0^+} \dfrac{\sin 2x}{x}$ and $\lim\limits_{x \to 0^-} \dfrac{\sin 2x}{x}$ both exist but their values differ.

(e) $\lim\limits_{x \to 0} \dfrac{\sin 2x}{x} = \dfrac{1}{2}$.

(f) $y = 0$ is a horizontal asymptote of the function $\dfrac{\sin 2x}{x}$.

(g) The domain of the function $\dfrac{\sin 2x}{x}$ is $(-\infty, 0) \cup (0, \infty)$.

(h) The function $g(x) = \begin{cases} \dfrac{\sin 2x}{x} & x \neq 0 \\ 2 & x = 0 \end{cases}$ is continuous on the interval $[-\pi, \pi]$.

(i) The function $\dfrac{\sin 2x}{x}$ possesses a removable discontinuity.

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Solution: (a) True (b) False (c) False (d) False (e) False (f) True (g) True (h) True (i) True

4. (16 points) Consider the function $f(x) = \begin{cases} \sqrt{x+4}, & x \leq 5, \\ (x-5)^2, & x > 5. \end{cases}$

(a) Evaluate $f(-5)$. (b) What is the average rate of change of $f(x)$ between 0 and 5?

(c) Evaluate $\lim_{x \to 5} f(x)$. (d) What is the instantaneous rate of change of $f(x)$ at $x = 0$?

Solution:

(a) $f(-5)$ $\boxed{\text{DNE}}$ because -5 is not in the domain of $f(x)$.

(b) $\frac{f(5)-f(0)}{5-0} = \frac{3-2}{5} = \boxed{\dfrac{1}{5}}$

(c) $\boxed{\text{DNE}}$ because the LH and RH limits are different.

(d)

$$f'(0) = \lim_{h \to 0} \left(\frac{\sqrt{0+h+4} - \sqrt{4}}{h} \right) = \lim_{h \to 0} \left(\frac{\sqrt{h+4} - 2}{h} * \frac{\sqrt{h+4}+2}{\sqrt{h+4}+2} \right) = \lim_{x \to 0} \frac{h+4-4}{h\sqrt{h+4}+2}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{h+4}+2} = \frac{1}{2+2} = \boxed{\frac{1}{4}}$$

5. (12 points) Water runs into an initially empty vase of height 30 centimeters at a constant rate. The vase is full after 5 seconds and the water quits running. Use this information and the shape of the vase shown to answer the questions if d is the depth of the water in centimeters and t is the time in seconds.

   (a) Explain, **in one brief sentence**, why the depth of the water is a functional relationship of time.

   (b) Which variable is the independent variable?

   (c) Which variable is the dependent variable?

   (d) Determine the domain and range of the functional relationship.

   (e) Sketch a graph of the relationship between depth and time.

   (f) Sketch a graph of the rate of change of the height of water in the bottle as a function of time.

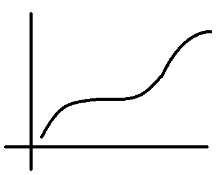
Solution:

(a) There is only one depth (output) at any given time (input).

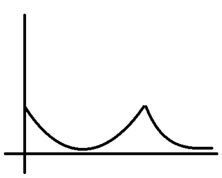
(b) Time, t, is the independent variable.

(c) Depth, d, is the dependent variable.

(d) The domain is times from 0 to 5 seconds. The range is depths from 0 to 30 centimeters.

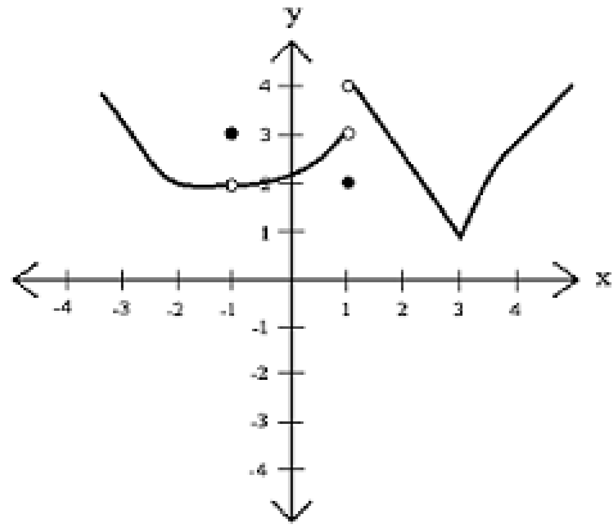


(e)



(f)

6. (12 points) Consider the graph of $g(x)$ shown. Evaluate the following, list numerical answers to the nearest integer, no explanation needed:



(a) $g(1)$

(b) $\lim_{x \to 1^-} g(x)$

(c) $\lim_{x \to 1^+} g(x)$

(d) $\lim_{x \to 1} g(x)$

(e) $\lim_{x \to 3} g(x)$

(f) $\lim_{h \to 0} \dfrac{g(3+h) - g(3)}{h}$

Solution:

(a) $g(1) = 2$

(b) $\lim_{x \to 1^-} g(x) = 3$

(c) $\lim_{x \to 1^+} g(x) = 4$

(d) $\lim_{x \to 1} g(x) =$DNE

(e) $\lim_{x \to 3} g(x) = 1$

(f) $\lim_{h \to 0} \dfrac{g(3+h) - g(3)}{h}$ =DNE.

$\boxed{\text{END of Exam}}$