APPM 1350 Exam 3 Solutions Spring 2013

1. (32 points) Evaluate the following expressions.

(a) $\displaystyle \frac{d}{dx}\int_0^{1/x}\left(2t^3-t^2\right)dt$

(b) $\displaystyle \int \frac{\cos x}{(1+2\sin x)^2}\,dx$

(c) $\displaystyle \int_{-6}^{0}\sqrt{36-x^2}\,dx$

(d) $\displaystyle \int_2^{16}\frac{5}{3x}\,dx$

Solution:

(a) Use the Fundamental Theorem of Calculus and the Chain Rule.

$$\begin{aligned}\frac{d}{dx}\int_0^{1/x}\left(2t^3-t^2\right)dt &= \left(2\left(\frac{1}{x^3}\right)-\left(\frac{1}{x}\right)^2\right)\frac{d}{dx}\left(\frac{1}{x}\right)\\ &= \left(\frac{2}{x^3}-\frac{1}{x^2}\right)\left(-\frac{1}{x^2}\right)\\ &= \boxed{-\frac{2}{x^5}+\frac{1}{x^4}}\end{aligned}$$

(b) Let $u=1+2\sin x$. Then $du=2\cos x\,dx$ and $\frac{1}{2}du=\cos x\,dx$.

$$\begin{aligned}\int\frac{\cos x}{(1+2\sin x)^2}\,dx &= \frac{1}{2}\int\frac{du}{u^2}=\frac{1}{2}\left(-\frac{1}{u}\right)+C\\ &= \boxed{\frac{-1}{2(1+2\sin x)}+C}\end{aligned}$$

(c) The integral equals the area of a quarter-circle with radius 6.

$$\int_{-6}^{0}\sqrt{36-x^2}\,dx=\frac{1}{4}\pi r^2=\frac{1}{4}\pi(36)=\boxed{9\pi}.$$

(d)
$$\begin{aligned}\int_2^{16}\frac{5}{3x}\,dx &= \frac{5}{3}\int_2^{16}\frac{dx}{x}=\frac{5}{3}\Big[\ln|x|\Big]_2^{16}\\ &= \frac{5}{3}(\ln 16-\ln 2)=\frac{5}{3}(\ln 8)=\frac{5}{3}(3\ln 2)=\boxed{5\ln 2}\end{aligned}$$

2. (14 points) Let $p(x)=x^3+2x^2$.

(a) Estimate the area under the curve on the interval $[0,2]$ using n evenly spaced subintervals and right endpoints. (Find R_n.) Leave your answer unsimplified.

(b) Find the exact area under the curve by evaluating the limit as $n\to\infty$ of the expression you found in part (a).

(c) Check your answer by calculating $\displaystyle\int_0^2 p(x)\,dx$ using the Evaluation Theorem.

Solution:

(a) $\displaystyle R_n=\sum_{i=1}^{n}p(x_i)\,\Delta x=\boxed{\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^3+2\left(\frac{2i}{n}\right)^2\right]\frac{2}{n}}$

(here $\Delta x=(b-a)/n=2/n$ and $x_i=a+i\Delta x=2i/n$).

(b)
$$\begin{aligned}A &= \lim_{n\to\infty}R_n\\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^3+2\left(\frac{2i}{n}\right)^2\right]\frac{2}{n}\\ &= \lim_{n\to\infty}\frac{2}{n}\left[\sum_{i=1}^{n}\left(\frac{2i}{n}\right)^3+2\sum_{i=1}^{n}\left(\frac{2i}{n}\right)^2\right]\\ &= \lim_{n\to\infty}\frac{2}{n}\left[\sum_{i=1}^{n}\left(\frac{8i^3}{n^3}\right)+2\sum_{i=1}^{n}\left(\frac{4i^2}{n^2}\right)\right]\\ &= \lim_{n\to\infty}\frac{2}{n}\left[\left(\frac{8}{n^3}\right)\sum_{i=1}^{n}i^3+2\left(\frac{4}{n^2}\right)\sum_{i=1}^{n}i^2\right]\\ &= \lim_{n\to\infty}\frac{2}{n}\left[\left(\frac{8}{n^3}\right)\left(\frac{n(n+1)}{2}\right)^2+2\left(\frac{4}{n^2}\right)\left(\frac{n(n+1)(2n+1)}{6}\right)\right]\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty}\left[4\left(\frac{n}{n}\right)\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\left(\frac{n+1}{n}\right)\right.\\
&\qquad\qquad\left.+\frac{16}{6}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)\right]\\
&= \left[4(1\times1\times1\times1)+\frac{16}{6}(1\times1\times2)\right] \quad \text{(by DOP)}\\
&= 4+\frac{16}{3} = \boxed{\frac{28}{3}}.
\end{aligned}
$$

(c)

$$
\begin{aligned}
\int_0^2 p(x)\,dx &= \int_0^2 (x^3+2x^2)\,dx\\
&= \left[\frac{1}{4}x^4+\frac{2}{3}x^3\right]_0^2\\
&= \left[\frac{16}{4}+\frac{16}{3}\right] = 4+\frac{16}{3} = \boxed{\frac{28}{3}}.
\end{aligned}
$$

3. (12 points) A particle is moving along a straight line with velocity $v(t) = t^2 - t$ (in m/s).

(a) What is the total displacement of the particle over the interval $0 \le t \le 4$?

(b) What is the total distance traveled over the same interval?

Solution:

(a) Total displacement is

$$
\begin{aligned}
\int_0^4 v(t)dt &= \int_0^4 (t^2-t)dt\\
&= \left[\frac{1}{3}t^3-\frac{1}{2}t^2\right]_0^4\\
&= \frac{64}{3}-\frac{16}{2} = \frac{64}{3}-8 = \boxed{\frac{40}{3}\text{ m}}.
\end{aligned}
$$

(b) Total distance traveled is $\int_0^4 |v(t)|dt$. Here $v(t) = t^2 - t = t(t-1)$ so $v(t) < 0$ on $(0,1)$ and $v(t) > 0$ on $(1,\infty)$.
Thus,

$$
\begin{aligned}
\int_0^4 |v(t)|dt &= \int_0^1 -v(t)dt + \int_1^4 v(t)dt\\
&= \int_0^1 (t-t^2)dt + \int_1^4 (t^2-t)dt\\
&= \left[\frac{1}{2}t^2-\frac{1}{3}t^3\right]_0^1 + \left[\frac{1}{3}t^3-\frac{1}{2}t^2\right]_1^4\\
&= \left[\frac{1}{2}-\frac{1}{3}\right] + \left[\left(\frac{64}{3}-\frac{16}{2}\right)-\left(\frac{1}{3}-\frac{1}{2}\right)\right]\\
&= \left[\frac{1}{6}\right] + \left[\frac{40}{3}-\left(\frac{-1}{6}\right)\right]\\
&= \frac{1}{6}+\frac{40}{3}+\frac{1}{6} = \boxed{\frac{41}{3}\text{ m}}.
\end{aligned}
$$

4. (10 points) Use one iteration of Newton's Method to approximate $\sqrt[5]{3}$ starting with an initial guess of $x_1 = 1$.

Solution:
$x = \sqrt[5]{3} \Rightarrow x^5 = 3 \Rightarrow x^5 - 3 = 0$. We wish to approximate the root of $f(x) = x^5 - 3$. Differentiating yields $f'(x) = 5x^4$.

$$
\begin{aligned}
x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)}\\
&= 1 - \frac{1^5-3}{5(1^4)}\\
&= 1 - \frac{-2}{5} = \boxed{\frac{7}{5}}.
\end{aligned}
$$

5. (10 points) Given that $f(x)$ is odd, $\int_0^1 f(2x)dx = 1$, and $\int_7^2 f(x)dx = 14$, find $\int_{-7}^0 f(x)dx$.

Solution:

- $\int_7^2 f(x)\,dx = 14 \Rightarrow \int_2^7 f(x)\,dx = -14$
- $\int_0^1 f(2x)\,dx = 1$. Choosing $u = 2x$, ($du = 2dx$, $u(1) = 2$, $u(0) = 0$), we get $\int_0^1 f(2x)\,dx = \frac{1}{2}\int_0^2 f(u)\,du = 1 \Rightarrow \int_0^2 f(x)\,dx = 2$.
- f is odd so $\int_{-7}^7 f(x)\,dx = 0$.

Thus,

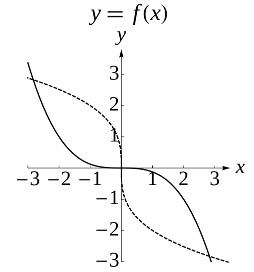
$$\begin{aligned} 0 &= \int_{-7}^7 f(x)\,dx \\ &= \int_{-7}^0 f(x)\,dx + \int_0^2 f(x)\,dx + \int_2^7 f(x)\,dx \\ &= \int_{-7}^0 f(x)\,dx + 2 - 14 \end{aligned}$$

so $\int_{-7}^0 f(x)\,dx = \boxed{12}$.

6. (12 points) Let f be a differentiable, one-to-one function.

(a) Copy the graph of f and add a sketch of the inverse function f^{-1}.

Solution:



(b) Given

$$f(1) = -\tfrac{1}{8} \qquad f'(2) = -\tfrac{3}{2}$$
$$f(2) = -1 \qquad \left(f^{-1}\right)'\left(-\tfrac{1}{8}\right) = -\tfrac{8}{3}$$

find the following values.

i. $f^{-1}(-1)$
ii. $f\left(f^{-1}(8)\right)$
iii. $\left(f^{-1}\right)'(-1)$

Solution:

i. Since $f(2) = -1$, then $f^{-1}(-1) = \boxed{2}$
ii. The cancellation equation for inverse functions is $f\left(f^{-1}(x)\right) = x$ so $f\left(f^{-1}(8)\right) = \boxed{8}$.
iii. The slope of f^{-1} at $(-1, 2)$ is the reciprocal of the slope of f at $(2, -1)$ so $\left(f^{-1}\right)'(-1) = 1/f'(2) = \boxed{-2/3}$.

7. (10 points) Suppose that the function $f(x)$ has a positive derivative for all x and that $f(1) = 0$. Let

$$g(x) = \int_0^x f(t)\,dt.$$

Answer TRUE (if always true) or FALSE (if not always true) for the following statements. No explanation is necessary.

(a) $g(1)$ is negative.

(b) g is increasing on $(0, 1)$.

(c) g has a local maximum at $x = 1$.

(d) g has an inflection point at $x = 1$.

(e) The average value of g on $[0, 1]$ is negative.

Solution:

(a) $\boxed{\text{TRUE}}$. Since f is an increasing function and $f(1) = 0$, then f is negative on the interval $[0, 1)$. Therefore $g(1) = \int_0^1 f(t)\,dt$ is negative.

(b) $\boxed{\text{FALSE}}$. Since $g'(x) = f(x)$ and f is negative on $(0, 1)$, g decreases on $(0, 1)$.

(c) $\boxed{\text{FALSE}}$. f is positive on $(1, \infty)$. Since g is decreasing on $(0, 1)$ and increasing on $(1, \infty)$, g has a local minimum at $x = 1$.

(d) $\boxed{\text{FALSE}}$. $g'(x) = f(x)$ and $g''(x) = f'(x)$. Since f' is positive for all x, g'' is also positive so the graph of g is concave up and does not change concavity.

(e) $\boxed{\text{TRUE}}$. Since g is negative on $(0, 1)$, $g_{ave} = \int_0^1 g(x)\,dx$ is also negative.