Print Name ________________________________

APPM 1350 **Exam 3** **Summer 2016**

On the front of your bluebook, please write: a grading key, your name, student ID, your lecture number and instructor. This exam is worth 100 points and has 5 questions on both sides of this paper.

- Submit this exam sheet with your bluebook. However, nothing on this exam sheet will be graded. Make sure all of your work is in your bluebook.
- **Show all work and simplify your answers!** Answers with no justification will receive no points.
- Please begin each problem on a new page.
- No notes or papers, calculators, cell phones, or electronic devices are permitted.

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1. The following parts are not related.

   (a) (12 pts) A company wants to make a square-bottomed, open-top, cake pan that holds a volume of 108 in^3 while minimizing construction materials. Find the dimensions of the pan that minimize the materials used to make the cake pan. Justify your answer.

   (b) (9 pts) An astronaut on Mars drops a ball from a 50 meter high cliff. Given that the constant of acceleration on Mars is 4 meters per second squared, find both the time it takes for the ball to hit the ground and the velocity when the ball hits the ground. Show all work to justify your answer.

   Solution:

   (a) The amount of material is determined by the surface area of the container: $SA = w^2 + 4wh$ where w is the width of the base in inches and h is the height in inches. We can use the volume equation to relate w and h as: $V = 108 = w^2h$. This gives $h = \dfrac{108}{w^2}$. So the surface area becomes $SA = w^2 + \dfrac{432}{w}$. Finding the critical values we get: $\dfrac{d}{dw}SA = 2w - \dfrac{432}{w^2}$. Note that the practical domain is $w > 0$. Setting $2w - \dfrac{432}{w^2} = 0$ we find one value $w = \sqrt[3]{216} = 2\sqrt[3]{27} = 6$ in. This is a minimum because $\dfrac{d^2}{dw^2}SA = 2 + \dfrac{864}{w^3} > 0$ for $w = 6$. So the dimensions are: 6 inches by 6 inches by 3 inches.

   (b) Let $a(t) = -4$ (with $s(t) = 0$ at the Martian floor). The antiderivative is $v(t) = -4t + C$. Since $v(0) = 0$ then $C = 0$ giving us $v(t) = -4t$. Finding the antiderivative again we get: $s(t) = -2t^2 + D$. We know $s(0) = 50$ and so $D = 50$ and $s(t) = -2t^2 + 50$. We know $s(t_{impact}) = 0$. So $t_{impact} = 5$ s. The velocity when the ball hits the ground is $v(5) = -20$ m/s.

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2. Suppose an object, traveling in a straight line, has a velocity of $v(t) = 3t^2 - 6t$, find the following.

   (a) (5 pts) Find the net distance traveled (i.e. displacement) by the object for $0 \leq t \leq 3$. Show all work to justify your answer.

(b) (7 pts) Find the total distance traveled by the object for $0 \leq t \leq 3$. Show all work to justify your answer.

Solution:

(a) The net distance is given by $\displaystyle\int_0^3 3t^2 - 6t\, dt = t^3 - 3t^2\Big|_0^3 = 0 - 0 = 0.$

(b) To find the total distance, we need to find where $v(t) = 3t^2 - 6t = 0$ this happens when $t = 0, 2$. Since $v(t) \leq 0$ on the interval $[0, 2]$ and $v(t) \geq 0$ on the interval $[2, 6]$, then, we break the integral $\displaystyle\int_0^6 \left|3t^2 - 6t\right|\, dt$ into $\displaystyle\int_0^2 -3t^2 + 6t\, dt + \int_2^3 3t^2 - 6t\, dt = -t^3 + 3t^2\Big|_0^2 + t^3 - 3t^2\Big|_2^3 = 4 + 4 = 8.$

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3. Using right end points, a definite integral is approximated by the Riemann sum:

$$\sum_{i=1}^{n}\left[2\left(\frac{3i}{n}\right)^2 - \frac{3i}{n}\right]\frac{3}{n}$$

(a) (6 pts) Find a definite integral representing this Riemann sum. Show all work to justify your answer.

(b) (6 pts) Evaluate $\displaystyle\sum_{i=1}^{n}\left[2\left(\frac{3i}{n}\right)^2 - \frac{3i}{n}\right]\frac{3}{n}$, in other words, find the sum in terms of n. Show all work to justify your answer.

(c) (6 pts) Find the value of $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left[2\left(\frac{3i}{n}\right)^2 - \frac{3i}{n}\right]\frac{3}{n}$. Show all work to justify your answer.

Formulas: $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ $\qquad\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ $\qquad\displaystyle\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

Solution:

(a) First we note that $\Delta x = \dfrac{b-a}{n} = \dfrac{3}{n}$ and so $b - a = 3$. Now note that $x_i = a + i\Delta x = \dfrac{3i}{n}$ and so we can conclude that $a = 0$ and $b = 3$. Finally we can write the definite integral as: $\displaystyle\int_0^3 \left(2x^2 - x\right)\, dx.$

(b) $\displaystyle\sum_{i=1}^{n}\left[2\left(\frac{3i}{n}\right)^2 - \frac{3i}{n}\right]\frac{3}{n} = \sum_{i=1}^{n}\left[2\left(\frac{3}{n}\right)^3 i^2 - \left(\frac{3}{n}\right)^2 i\right] = 2\left(\frac{3}{n}\right)^3\sum_{i=1}^{n} i^2 - \left(\frac{3}{n}\right)^2\sum_{i=1}^{n} i$

$\displaystyle= 2\left(\frac{3}{n}\right)^3\frac{n(n+1)(2n+1)}{6} - \left(\frac{3}{n}\right)^2\frac{n(n+1)}{2} = \frac{9(n+1)(2n+1)}{n^2} - \frac{9(n+1)}{2n}.$

(c) $\displaystyle\lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[\frac{9(n+1)(2n+1)}{n^2} - \frac{9(n+1)}{2n}\right] = \lim_{n\to\infty}\left[\frac{18n^2 + 27n + 9}{n^2} - \frac{9n+9}{2n}\right]$

$\displaystyle= \lim_{n\to\infty}\left[18 + \frac{27}{n} + \frac{9}{n^2} - \frac{9 + \frac{9}{n}}{2}\right] = 18 - \frac{9}{2} = \frac{27}{2}.$

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4. The following are unrelated. For (a) find the general indefinite integral. For (b) and (c) evaluate the integral. For (d) compute the derivative of the integral. Show all work to justify your answer and make sure to simplify as much as possible.

   (a) (9 pts) $\displaystyle\int \left(\sec x \tan x + \frac{-\cos x \sin^2 x + \cos x}{\cos^2 x}\right)\, dx$

   (b) (9 pts) $\displaystyle\int_0^3 \sqrt{9 - x^2}\, dx$

   (c) (10 pts) $\displaystyle\int_0^6 \left(|t-2| - \frac{3}{2}\sqrt{t}\right)\, dt$

   (d) (9 pts) $\displaystyle\frac{d}{dx}\int_a^{\sqrt{x}} (t + t\sin^2 t)^3\, dt$ where a is a constant.

   Solution:

   (a) First note that: $\displaystyle\frac{-\cos x \sin^2 x + \cos x}{\cos^2 x} = \frac{-\sin^2 x + 1}{\cos x} = \frac{\cos^2 x}{\cos x} = \cos x$ so that:
   $\displaystyle\int \sec x \tan x + \cos x\, dx = \sec x + \sin x + C.$

   (b) $y = \sqrt{9 - x^2}$ is the upper half of a circle, centered at the origin, of radius 3. Therefore, the integral from 0 to 3 is the area of a quarter of a full circle, or $\displaystyle\frac{1}{4}\pi 3^2 = \frac{9\pi}{4}$. So $\displaystyle\int_0^3 \sqrt{9-x^2}\, dx = \frac{9\pi}{4}$.

   (c) $\displaystyle\int_0^6 |t-2| - \frac{3}{2}\sqrt{t}\, dt$. Note that: $|t-2| = \begin{cases} t-2, & t \geq 2, \\ -t+2, & t < 2 \end{cases}$.
   So $\displaystyle\int_0^6 |t-2| - \frac{3}{2}\sqrt{t}\, dt = \int_0^2 -t + 2 - \frac{3}{2}\sqrt{t}\, dt + \int_2^6 t - 2 - \frac{3}{2}\sqrt{t}\, dt = -\frac{t^2}{2} + 2t - t^{3/2}\Big|_0^2 + \frac{t^2}{2} - 2t - t^{3/2}\Big|_2^6 = -2 + 4 - 2\sqrt{2} + 18 - 12 - 6\sqrt{6} - (2 - 4 - 2\sqrt{2}) = 10 - 6\sqrt{6}.$

   (d) For $\displaystyle\frac{d}{dx}\int_a^{\sqrt{x}} (t + t\sin^2 t)^3\, dt$ let $u = \sqrt{x}$ and $\displaystyle\frac{du}{dx} = \frac{1}{2}x^{-1/2}$. Then we get:
   $\displaystyle\frac{d}{du}\left(\int_a^u (t + t\sin^2 t)^3\, dt\right)\frac{du}{dx} = (u + u\sin^2 u)^3 \frac{1}{2}x^{-1/2} = \frac{(\sqrt{x} + \sqrt{x}\sin^2\sqrt{x})^3}{2\sqrt{x}}$ utilizing both the FTC1 and the chain rule.

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5. The following are unrelated. Clearly state "Always True" or "False" for each statement. No justification is required and no partial credit will be given.

   (a) (4 pts) $\displaystyle\int_{x^3}^0 f(t)\, dt - \int_{x^3}^x f(t)\, dt = -\int_0^x f(t)\, dt$ for any real number x and f a continuous function on the real numbers.

   (b) (4 pts) For a velocity function $v(t)$ continuous on an interval $[a, b]$ such that $v(t) < 0$ on $[a, b]$ then $\displaystyle\int_a^b |v(t)|\, dt = \left|\int_a^b v(t)\, dt\right|$.

   (c) (4 pts) $\displaystyle\int_0^a 3x^n\, dx = \frac{3a^{(n+1)}}{n+1} - \frac{3(0)^{(n+1)}}{n+1} = \frac{3a^{(n+1)}}{n+1}$ by the evaluation theorem where n is a real number such that $n \neq -1$ and a is any positive real number.

Solution:

(a) Always True

(b) Always True

(c) False

Remarks:

(a) By rearrangement: $\int_{x^3}^{0} f(t)\,dt - \int_{x^3}^{x} f(t)\,dt = -\int_{0}^{x} f(t)\,dt$ becomes

$-\int_{0}^{x^3} f(t)\,dt + \int_{x}^{x^3} f(t)\,dt = -\int_{0}^{x} f(t)\,dt$ becomes

$\int_{0}^{x^3} f(t)\,dt = \int_{0}^{x} f(t)\,dt + \int_{x}^{x^3} f(t)\,dt$ which is always true.

(b) Since $v(t) < 0$ on $[a, b]$ then $\int\limits_{a}^{b} v(t)\,dt < 0$ and $|v(t)| = -v(t)$. We get:

$\int\limits_{a}^{b} |v(t)|\,dt = \int\limits_{a}^{b} -v(t)\,dt = -\int\limits_{a}^{b} v(t)\,dt = \left|\int\limits_{a}^{b} v(t)\,dt\right|$. So the statement is always true.

(c) Since n can be negative, let $n = -2$. In this case $3x^n$ is discontinuous at $x = 0$ meaning that the evaluation theorem is not applicable and so the left side does not equal the right side.
