1. Evaluate the following limits.

(a) (6 pts) $\displaystyle\lim_{\theta\to\pi/2}\frac{\sin(3\theta)}{\theta}$ (b) (6 pts) $\displaystyle\lim_{x\to\pi/4}\frac{1-\tan x}{\sin x-\cos x}$

(c) (6 pts) $\displaystyle\lim_{x\to 0}|x|\cos(1/x)$ (d) (8 pts) $\displaystyle\lim_{x\to 0}(1-2x)^{1/x}$

Solution:

(a) $\displaystyle\lim_{x\to\pi/2}\frac{\sin(3\theta)}{\theta}=\frac{\sin(3\pi/2)}{\pi/2}=\frac{-1}{\pi/2}=\boxed{-\frac{2}{\pi}}$

(b) $\displaystyle\lim_{x\to\pi/4}\frac{1-\tan x}{\sin x-\cos x}\overset{LH}{=}\lim_{x\to\pi/4}\frac{-\sec^2 x}{\cos x+\sin x}=\frac{-\sec^2(\pi/4)}{\cos(\pi/4)+\sin(\pi/4)}=\frac{-2}{\sqrt{2}}=\boxed{-\sqrt{2}}$

Alternate Solution:

$$\lim_{x\to\pi/4}\frac{1-\tan x}{\sin x-\cos x}=\lim_{x\to\pi/4}\frac{1-\frac{\sin x}{\cos x}}{\sin x-\cos x}=\lim_{x\to\pi/4}\frac{\cos x-\sin x}{\cos x(\sin x-\cos x)}=\lim_{x\to\pi/4}\frac{-1}{\cos x}=-\sqrt{2}$$

(c) Use the Squeeze Theorem.

$$-1\le\cos\left(\tfrac{1}{x}\right)\le 1$$
$$-|x|\le|x|\cos\left(\tfrac{1}{x}\right)\le|x|$$

Since $\displaystyle\lim_{x\to 0}-|x|=0$ and $\displaystyle\lim_{x\to 0}|x|=0$, by the Squeeze Theorem, $\displaystyle\lim_{x\to 0}|x|\cos(1/x)=\boxed{0}$.

(d) Let

$$\begin{aligned}
L&=\lim_{x\to 0}(1-2x)^{1/x}.\\
\ln L&=\lim_{x\to 0}\ln(1-2x)^{1/x}\\
&=\lim_{x\to 0}\frac{\ln(1-2x)}{x}\\
&\overset{LH}{=}\lim_{x\to 0}\frac{\frac{1}{1-2x}(-2)}{1}=-2\\
L&=\boxed{e^{-2}}
\end{aligned}$$

2. (16 pts, 8 pts each) Evaluate the following integrals.

(a) $\displaystyle\int_1^e\frac{1}{x(1+(\ln x)^2)}\,dx$ (b) $\displaystyle\int_0^1 2e^{-x}\cosh x\,dx$

Solution:

(a) Let $u = \ln x$, $du = dx/x$.

$$\int_1^e \frac{1}{x(1+(\ln x)^2)}\,dx = \int_0^1 \frac{du}{1+u^2} = \arctan u\Big|_0^1 = \arctan 1 - \arctan 0 = \boxed{\pi/4}$$

(b)

$$\int_0^1 2e^{-x}\cosh x\,dx = \int_0^1 2e^{-x}\left(\frac{e^x+e^{-x}}{2}\right)dx = \int_0^1 \left(1+e^{-2x}\right)dx$$

Let $u = -2x$, $du = -2\,dx$.

$$\begin{aligned} &= \int_0^{-2} -\frac{1}{2}\left(1+e^u\right)du = \left[-\frac{1}{2}\left(u+e^u\right)\right]_0^{-2} \\ &= -\frac{1}{2}\left(-2+e^{-2}-1\right) = \boxed{\frac{1}{2}\left(3-e^{-2}\right)} = \frac{3}{2}-\frac{1}{2}e^{-2} \end{aligned}$$

3. (12 pts) Let $h(x) = x^{3/2} + \displaystyle\int_1^{x^3} \frac{1}{1+t^3}\,dt$

(a) Find the linearization of $h(x)$ at $a = 1$.

(b) Use the linearization to approximate $h(1.1)$.

Solution:

(a)

$$\begin{aligned} h'(x) &= \frac{3}{2}x^{1/2} + \frac{1}{1+(x^3)^3}\cdot 3x^2 \qquad \text{(by FTC-1)} \\ h'(1) &= \frac{3}{2}+\frac{3}{2} = 3 \\ h(1) &= 1 \\ L(x) &= h(1) + h'(1)(x-1) \\ &= \boxed{1+3(x-1)} = 3x-2 \end{aligned}$$

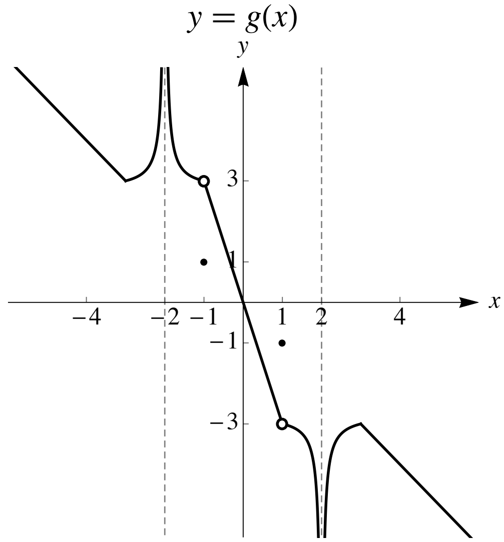
(b) $h(1.1) \approx L(1.1) = 1 + 3(1.1-1) = \boxed{1.3}$

4. (15 pts) Sketch a graph of a single function $y = g(x)$ that satisfies all of the following conditions. No explanation is necessary. Clearly label all important features of the graph.

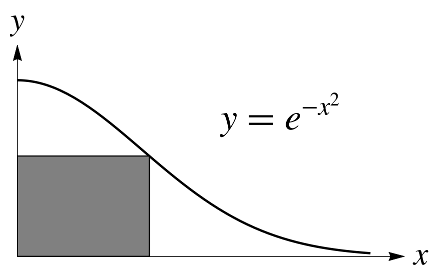
(a) $g(-x) = -g(x)$ (b) $g(-1) = 1$ (c) $\displaystyle\lim_{h\to 0}\frac{g(4+h)-g(4)}{h} < 0$

(d) $\displaystyle\lim_{x\to 2} g(x) = -\infty$ (e) $\displaystyle\lim_{x\to -1} g(x) = 3$

Solution: Here is a possible solution.



5. (15 pts) The rectangle shown has sides along the positive x and y axes and its upper right vertex on the curve $y = e^{-x^2}$. What dimensions give the rectangle its largest area?



Solution: We want to maximize area = (length)(height). Let x = length of the rectangle. Then, the height is given by $y = e^{-x^2}$. So, the area is given by $A(x) = xe^{-x^2}$ for $x \geq 0$.

$$A'(x) = e^{-x^2} - 2x^2e^{-x^2} = e^{-x^2}(1 - 2x^2)$$

$A'(x) = 0$ when $x = \pm 1/\sqrt{2}$. We discard $x = -1/\sqrt{2}$ since it is not in the domain. Hence, the only critical point is $x = 1/\sqrt{2}$. We need to verify that $x = 1/\sqrt{2}$ maximizes the area.

First derivative test: $A'(0) = 1 > 0$ and $A'(1) = -1/e < 0$. So, by the first derivative test, $x = 1/\sqrt{2}$ maximizes the area of the rectangle.

Final answer: The dimensions of the maximum rectangle are $\boxed{x = 1/\sqrt{2}}$ and $\boxed{y(1/\sqrt{2}) = e^{-1/2}}$.

6. The following questions are unrelated.

   (a) (10 pts) Write $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{(3i/n + 2)^2} \frac{3}{n}$ as a definite integral and evaluate.

   (b) (10 pts) Find dy/dx for $xy = \tan(y + 3)$ at the point $(x, y) = (0, -3)$.

   (c) (6 pts) Simplify $\sum_{k=1}^{5} \arcsin\left((-1)^k\right)$.

(d) (10 pts) Let $g(x) = x^{(1/\ln x)}$. (i) What is the domain of $g(x)$? (ii) Find $g'(x)$

Solution:

(a) The definite integral $\int_a^b f(x)\,dx$ corresponds to $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$ for $\Delta x = (b-a)/n$ and $x_i = a + i\Delta x$.

Here is one solution: let $f(x) = 1/(x+2)^2$ and $[a,b] = [0,3]$. Then $\Delta x = 3/n$ and $x_i = 3i/n$.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{(3i/n+2)^2}\frac{3}{n} = \int_0^3 \frac{dx}{(x+2)^2}$$

Now evaluate the integral. Let $u = x+2,\ \ du = dx$.

$$= \int_2^5 u^{-2}\,du = -\frac{1}{u}\bigg|_2^5 = -\frac{1}{5} + \frac{1}{2} = \boxed{\frac{3}{10}}$$

(Note that any integral of the form $\int_a^{a+3} \frac{dx}{(x-a+2)^2}$ also would work.)

(b)

$$\begin{aligned}
xy &= \tan(y+3) \\
x\frac{dy}{dx} + y &= \sec^2(y+3)\frac{dy}{dx} \\
x\frac{dy}{dx} - \sec^2(y+3)\frac{dy}{dx} &= -y \\
\left(x - \sec^2(y+3)\right)\frac{dy}{dx} &= -y \\
\frac{dy}{dx} &= \frac{-y}{x - \sec^2(y+3)} \\
\frac{dy}{dx}\bigg|_{(0,-3)} &= \frac{3}{0-1} = \boxed{-3}
\end{aligned}$$

(c)

$$\sum_{k=1}^{5} \arcsin\left((-1)^k\right) = \arcsin(-1) + \arcsin(1) + \arcsin(-1) + \arcsin(1) + \arcsin(-1)$$

$$= \cancel{-\frac{\pi}{2} + \frac{\pi}{2}} \cancel{- \frac{\pi}{2} + \frac{\pi}{2}} - \frac{\pi}{2} = \boxed{-\frac{\pi}{2}}$$

(d) (i) Since $\ln x$ is defined for $x > 0$ and the denominator $\ln x$ equals 0 at $x = 1$, the domain is $\boxed{(0,1) \cup (1,\infty)}$.

(ii) Let $y = g(x) = x^{(1/\ln x)}$. Use logarithmic differentiation.

$$\begin{aligned} y &= x^{(1/\ln x)} \\ \ln y &= \ln x^{(1/\ln x)} \\ &= \frac{1}{\ln x}(\ln x) = 1 \\ y &= e \\ y' &= g'(x) = \boxed{0} \end{aligned}$$

Note that if we substitute $x = e^{\ln x}$, then $g(x) = x^{(1/\ln x)} = \left(e^{\ln x}\right)^{(1/\ln x)} = e$, a constant function.

7. (15 pts) The graph of a function $f(x)$ is shown below. Suppose $f(x)$ is the **derivative** of $F(x)$. Assume that $F(x)$ is continuous on the interval $[-2, 8]$. No justification is required for the following questions. If the answer to any question is "none", write "none".

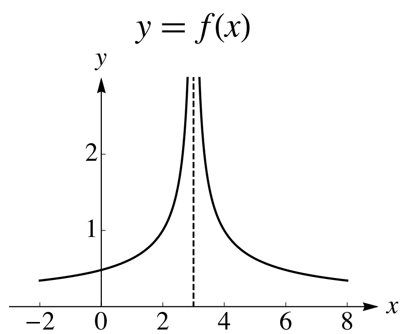
   (a) On what intervals is F increasing?

   (b) On what intervals is F concave up?

   (c) What are the x-coordinates of the absolute maximum and minimum values of F?

   (d) What are the x-coordinates of the inflection points of F?

   (e) Suppose we restrict the domain of f to $(3, 8]$ so that it is one-to-one. Then what is the value of $f^{-1}(1)$?



Solution:

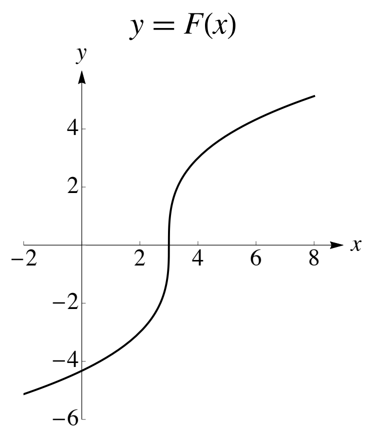
(a) Since $F' = f > 0$, F is increasing on the entire interval $\boxed{(-2, 8)}$.

(b) F is concave up where $F'' = f' > 0$ on $\boxed{(-2, 3)}$.

(c) Since F is increasing throughout the interval, the absolute minimum value occurs at $x = \boxed{-2}$ and the absolute maximum value occurs at $x = \boxed{8}$.

(d) F changes from concave up to concave down at $x = \boxed{3}$.

(e) The graph shows that $f(4) = 1 \implies f^{-1}(1) = \boxed{4}$.



8. (15 pts) A skier glides on flat terrain. His motion is slowed only by friction with the snow. His velocity $v(t)$ obeys the equation:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = kv$$

where k is a constant. His initial velocity is 10 meters per second; after 50 s, his velocity is 5 meters per second.

(a) Find the velocity of the skier at an arbitrary time t.

(b) Find the velocity of the skier after 25 seconds. Simplify your answer.

(c) Let $s(t)$ represent the distance traveled by the skier by time t, where t is measured in seconds. Find an equation for $s(t)$.

Solution:

(a) We are given that $dv/dt = kv$. Hence, $v(t) = v_0 e^{kt}$ where $v_0 = 50$ mps is the initial velocity.

We are also given that $v(50) = 5 = 10e^{50k}$. Solving for k, we obtain $k = (\ln 1/2)/50 = -(\ln 2)/50$. Thus, we have $\boxed{v(t) = 10e^{(-\ln 2/50)t}}$, which can also be written as $v(t) = 10e^{(-\ln 2/50)t} = 10e^{\ln 2^{-t/50}} = \boxed{10 \cdot 2^{-t/50}}$. Either answer is acceptable.

(b) $v(25) = 10e^{(-\ln 2/50)\cdot 25} = 10e^{-(\ln 2)/2} = 10e^{\ln 2^{-1/2}} = \boxed{10/\sqrt{2}} = 5\sqrt{2}$

(c)

$$\begin{aligned}
s(t) &= \int_0^t v(x)\,dx \\
&= \int_0^t 10e^{(-\ln 2/50)x}\,dx \\
&= \frac{-50}{\ln 2} 10e^{(-\ln 2/50)x}|_0^t \\
&= \frac{-500}{\ln 2} e^{(-\ln 2/50)t} + \frac{500}{\ln 2} \\
&= \frac{500}{\ln 2}\left(1 - e^{(-\ln 2/50)t}\right) \\
&= \boxed{\frac{500}{\ln 2}\left(1 - 2^{-t/50}\right)}
\end{aligned}$$