Math 1300-010 - Fall 2016

Quotient Rule, Product Rule, and Trig Derivatives - 9/26/16

Guidelines: Please work in groups of two or three. Please show all work and clearly denote your answer. You are encouraged to call me over to check on your progress.

The goal of this worksheet is to use the **quotient rule** to derive all of the trig derivatives. Recall, the **quotient rule** states

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

Before moving on to finding the derivative of the trig functions, let's practice this rule. Compute the following derivatives. You do not need to simplify.

(a) $\frac{d}{dx}\left[\frac{3x^2-2}{4x+9}\right]$

$$= \frac{(3x^2-2)'(4x+9) - (3x^2-2)(4x+9)'}{(4x+9)^2}$$

$$= \boxed{\frac{6x(4x+9) - 4(3x^2-2)}{(4x+9)^2}}$$

(b) $\frac{d}{dx}\left[\frac{e^x - 2x}{\sqrt{x}}\right]$

$$= \frac{(e^x-2x)'\sqrt{x} - (e^x-2x)(x^{1/2})'}{(\sqrt{x})^2}$$

$$= \frac{(e^x-2)\sqrt{x} - (e^x-2x)\frac{1}{2}x^{-\frac{1}{2}}}{x}$$

$$= \boxed{\frac{2(e^x-2)\sqrt{x} - (e^x-2x)x^{-\frac{1}{2}}}{2x}}$$

To find the trig derivatives, we'll need two basic facts:

$$\frac{d}{dx}\sin(x) = \cos(x)$$
$$\frac{d}{dx}\cos(x) = -\sin(x).$$

Notice the negative sign on the derivative of cosine; it is easy to forget! Let us practice using these two derivatives. Compute the following derivatives.

(a) $\frac{d}{dx}[x^2 - \cos(x) + 2\sin(x)]$

$$= 2x - (-\sin(x)) + 2\cos(x)$$

$$= \boxed{2x + \sin(x) + 2\cos(x)}$$

(b) $\frac{d}{dx}[e^x \sin(x)]$ (Product Rule!)

$$= (e^x)'\sin(x) + e^x(\sin(x))'$$

$$= e^x\sin(x) + e^x\cos(x)$$

$$= \boxed{e^x(\sin(x) + \cos(x))}$$

Now we are ready to compute the derivatives of tangent, cotangent, secant, and cosecant.

1. ***The Derivative of Tangent***

   (a) Rewrite $\tan(x)$ as a quotient involving $\sin(x)$ and $\cos(x)$.

   $$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

   (b) Compute the derivative of $\tan(x)$ using the result from part (a) and the quotient rule.

   $$\frac{d}{dx}\tan(x) = \frac{d}{dx}\frac{\sin(x)}{\cos(x)} = \frac{\cos(x)\cos(x) - \sin(x)\cdot(-\sin(x))}{\cos^2(x)} = \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)}$$
   $$= \frac{1}{\cos^2(x)}.$$

   (c) Rewrite your answer in terms of $\sec(x)$ only. The result should no longer be a quotient. $\cos(x) = \frac{1}{\sec(x)}$, so $\frac{d}{dx}\tan(x) = \frac{1}{\left(\frac{1}{\sec(x)}\right)^2} = \sec^2(x)$

   (d) In conclusion

   $$\frac{d}{dx}\tan(x) = \sec^2(x)$$

2. ***The Derivative of Cotangent***

   (a) Rewrite $\cot(x)$ as a quotient involving $\sin(x)$ and $\cos(x)$.

   $$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

   (b) Compute the derivative of $\cot(x)$ using the result from part (a) and the quotient rule.

   $$\frac{d}{dx}\cot(x) = \frac{d}{dx}\frac{\cos(x)}{\sin(x)} = \frac{-\sin(x)\sin(x) - \cos(x)\cos(x)}{\sin^2(x)} = \frac{-1}{\sin^2(x)}$$

   (c) Rewrite your answer in terms of $\csc(x)$ only. The result should no longer be a quotient. $\sin(x) = \frac{1}{\csc(x)}$, so $\frac{d}{dx}\cot(x) = \frac{-1}{\left(\frac{1}{\csc(x)}\right)^2} = -\csc^2(x)$

   (d) In conclusion

   $$\frac{d}{dx}\cot(x) = -\csc^2(x)$$

3. ***The Derivative of Secant***

(a) Rewrite $\sec(x)$ as a quotient involving $\cos(x)$.

$$\sec(x) = \frac{1}{\cos(x)}$$

(b) Compute the derivative of $\sec(x)$ using the result from part (a) and the quotient rule.

$$\frac{d}{dx}\sec(x) = \frac{d}{dx}\frac{1}{\cos(x)} = \frac{0 \cdot \cos(x) - 1(-\sin(x))}{\cos^2(x)} = \frac{\sin(x)}{\cos^2(x)}$$

(c) Rewrite your answer in terms of $\sec(x)$ and $\tan(x)$. The result should no longer be a quotient.

$$\frac{\sin(x)}{\cos^2(x)} = \frac{1}{\cos(x)}\frac{\sin(x)}{\cos(x)} = \sec(x)\tan(x)$$

(d) In conclusion

$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$

4. ***The Derivative of Cosecant***

(a) Rewrite $\csc(x)$ as a quotient involving $\sin(x)$.

$$\csc(x) = \frac{1}{\sin(x)}$$

(b) Compute the derivative of $\csc(x)$ using the result from part (a) and the quotient rule.

$$\frac{d}{dx}\csc(x) = \frac{d}{dx}\frac{1}{\sin(x)} = \frac{0 \cdot \sin(x) - 1 \cdot \cos(x)}{\sin^2(x)} = \frac{-\cos(x)}{\sin^2(x)}$$

(c) Rewrite your answer in terms of $\csc(x)$ and $\cot(x)$ only. The result should no longer be a quotient.

$$\frac{-\cos(x)}{\sin^2(x)} = \frac{-1}{\sin(x)} \cdot \frac{\cos(x)}{\sin(x)} = -\csc(x)\cot(x)$$

(d) In conclusion

$$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$$

Here is some further practice using the quotient rule, product rule, and the trig derivatives.

(a) Find f' if $f(x) = \sin^2(x) + \cos^2(x)$ [Product Rule for Each Term]

Rewrite $f(x) = \overbrace{\sin(x)\sin(x)}^{PR} + \overbrace{\cos(x)\cos(x)}^{PR}$

So $f'(x) = \sin(x)\cos(x) + \cos(x)\sin(x) - \cos(x)\sin(x) - \sin(x)\cos(x)$
$= 0.$

Alternatively, $f(x) = \sin^2(x) + \cos^2(x) = 1$, so $f'(x) = \frac{d}{dx}(1) = 0.$

(b) Find g' if $g(x) = xe^x \csc(x)$ [Product Rule Twice]

$$g'(x) = \underbrace{\frac{d}{dx}(xe^x)}_{PR}\csc(x) + xe^x\frac{d}{dx}\csc(x)$$

$$= \boxed{[e^x + xe^x]\csc(x) - xe^x\csc(x)\cot(x)}$$

(c) Find h' if $h(x) = \dfrac{1 + \sec(x)}{x + \cot(x)}$ → quotient rule

$$h'(x) = \frac{(1+\sec(x))' \cdot (x+\cot(x)) - (1+\sec(x))(x+\cot(x))'}{(x+\cot(x))^2}$$

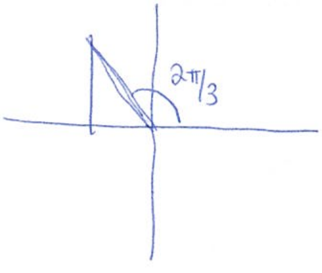
$$= \boxed{\frac{\sec(x)\tan(x)(x+\cot(x)) - (1+\sec(x))(1-\csc^2(x))}{(x+\cot(x))^2}}$$

(d) For what values of x does the graph of

$$f(x) = x + 2\sin(x)$$

have a horizontal tangent?

$f'(x) = 1 + 2\cos(x)$. We need points x such that $f'(x) = 0$



$$0 = 1 + 2\cos(x)$$

$$\Leftrightarrow \cos(x) = \frac{-1}{2} \rightarrow \boxed{\underbrace{x = \frac{2\pi}{3} + 2n\pi}_{\text{an } \infty\text{-number of solutions}}, \ n \text{ an integer}}$$