Solutions

Math 1300-010 - Fall 2016

Related Rates, Pt. II - 10/18/16

Guidelines: Please work in groups of two or three. This will not be handed in, but is a study resource for Midterm 3. This second worksheet over related rates covers some intermediate examples now that we are used to the process.

For ***each*** of the following related rates problems:

(a) Draw a picture of the situation and assign variables.

(b) Write down the known and unknown quantities in terms of the assigned variables.

(c) Use your picture to write an equation that relates the variables.

(d) Take d/dt of each side of this equation, solve for the unknown quantity, and then plug in the known quantities.

1. The height of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of 2 cm^2/min. At what rate is the base of the triangle changing with the height is 10 cm and the area is 100 cm^2.

(a)

area = A

(b) Known quantities:

$\frac{dh}{dt} = 1 \text{ cm/min}$

$\frac{dA}{dt} = 2\text{cm}^2/\text{min}$

Unknown:

$\frac{db}{dt}$ when $h = 10\text{cm}$

(c) $A = \frac{1}{2}bh$

(d) $\frac{dA}{dt} = \frac{1}{2}\left(\frac{db}{dt}h + b\frac{dh}{dt}\right)$

when $A = 100$ and $h = 10$,

$b = \frac{2A}{h} = 20$

so

$2 = \frac{1}{2}\left(\frac{db}{dt}(10) + 20(1)\right)$

$\Rightarrow 4 = 10\frac{db}{dt} + 20 \rightarrow \frac{db}{dt} = -1.6$

$$\boxed{\frac{db}{dt} = -1.6 \text{ cm/min}}$$

2. If a snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10 cm.

(a)

Surface area = S

$D = 2r$

(b) $\frac{dS}{dt} = -1 \text{ cm}^2/\text{min}$

(c) $S = 4\pi r^2$
$= 4\pi\left(\frac{D}{2}\right)^2$
$= \pi D^2$

(d) $\frac{dS}{dt} = 2\pi D\frac{dD}{dt}$

So when $D = 10\text{cm}$ and $\frac{dS}{dt} = -1 \text{ cm}^2/\text{min}$

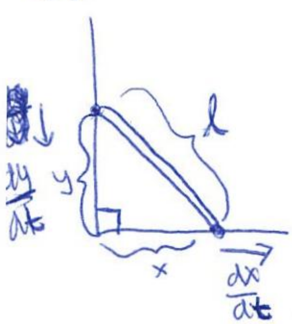
$\frac{dD}{dt} = \frac{dS/dt}{2\pi D} = \frac{-1}{2\pi(10)}$

$$\boxed{\frac{dD}{dt} = -\frac{1}{20\pi} \text{ cm/min}}$$

$$\boxed{\approx -0.02 \text{ cm/min}}$$

3. The top of a ladder slides down a vertical wall at a rate of 0.15 m/s. At the moment when the bottom of the ladder is 3 m from the wall, it slides away from the wall at a rate of 0.2 m/s. How long is the ladder?

(a)



(b) Known:

$\frac{dy}{dt} = -0.15$ m/s

$\frac{d\ell}{dt} = 0$ m/s

when $x = 3$ m,

$\frac{dx}{dt} = 0.2$ m/s

Unknown:

length of ladder when $x = 3$ m

$\boxed{\ell = 5\text{m}}$

(c) $x^2 + y^2 = \ell^2$

If we can figure out y when $x=3$, we will have ℓ.

(d) $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2\ell\frac{d\ell}{dt} = 0$, since ℓ is constant.

So when $x = 3$m, $\frac{dx}{dt} = 0.2$ m/s, and $\frac{dy}{dt} = -0.15$ m/s,

$2(3)(0.2) + 2y(-0.15) = 0$

$\Rightarrow \frac{12}{10} = \frac{3}{10}y$

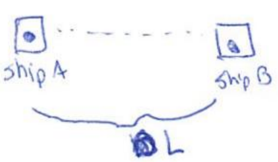
$\Rightarrow 12 = 3y$

so $y = 4$m. Thus

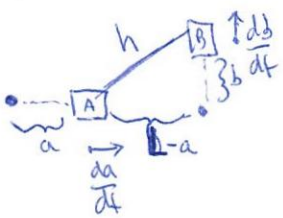
$\ell = \sqrt{x^2+y^2} = \sqrt{3^2+4^2} = 5$ m

4. At noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 km/h and ship B is sailing north at 25km/h. How fast is the distance between the ships changing at 4:00 PM?

(a) Start: Noon



Some time t:



(b) Known:

$\frac{da}{dt} = 35$ km/h

$\frac{db}{dt} = 25$ km/h

$L = 150$ km

Unknown:

$\frac{dh}{dt}$ at 4:00 PM,

ie, when

$a = 35 \text{ km/hr} \cdot 4\text{hr} = 140$ km

$b = 25 \text{ km/hr} \cdot 4\text{hr} = 100$ km

$$\boxed{\frac{dh}{dt} = \frac{215}{\sqrt{101}} \text{ km/h}}$$

$\boxed{\approx 21.4 \text{ km/h}}$

(c) $(L-a)^2 + b^2 = h^2$

0 since L is constant

(d) $\cancel{2}(L-a)\cdot\left(\frac{dL}{dt} - \frac{da}{dt}\right) + \cancel{2}b\frac{db}{dt} = \cancel{2}h\frac{dh}{dt}$

$\Rightarrow (L-a)\cdot\left(-\frac{da}{dt}\right) + b\frac{db}{dt} = h\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{1}{h}\left(b\frac{db}{dt} - (L-a)\frac{da}{dt}\right)$

Now, at 4:00pm, $a = 140$, $b = 100$, $L-a = 10$,

so $h = \sqrt{(L-a)^2 + b^2} = \sqrt{(10)^2 + (100)^2}$

$= 10\sqrt{1+100}$

$= 10\sqrt{101}$ km,

$\frac{dh}{dt} = \frac{1}{10\sqrt{101}}\left(100\cdot 25 - 10\cdot 35\right)$

$= \frac{1}{\sqrt{101}}\left(10\cdot 25 - 35\right)$

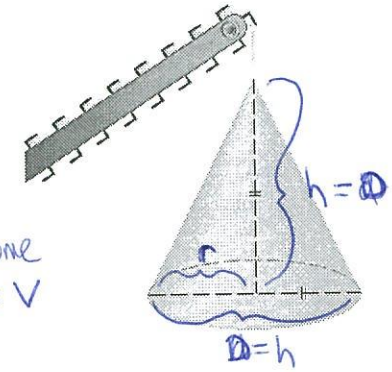
$= \frac{1}{\sqrt{101}}\left(250 - 35\right)$

Solutions

5. Gravel is being dumped from a conveyer belt at a rate of 30 ft^3/min, and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 ft high? The volume of a right cone is $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the base of the cone.

(a)

Volume $= V$



$r = \frac{D}{2} = \frac{h}{2}$ since $d = h$

(b) Known:

$\frac{dV}{dt} = 30 \text{ ft}^3/\text{min}$

$h = D$, so

$\frac{dh}{dt} = \frac{dD}{dt}$

Unknown:

$\frac{dh}{dt}$ when $h = 10$ ft.

(c) $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h$

$= \frac{1}{3}\pi \frac{h^3}{4}$

$= \frac{\pi h^3}{12}$

(d) $\frac{dV}{dt} = \frac{\pi}{12}\left(3h^2 \frac{dh}{dt}\right)$

$= \frac{\pi}{4} h^2 \frac{dh}{dt}$

So $\frac{dh}{dt} = \frac{4}{\pi h^2} \cdot \frac{dV}{dt}$

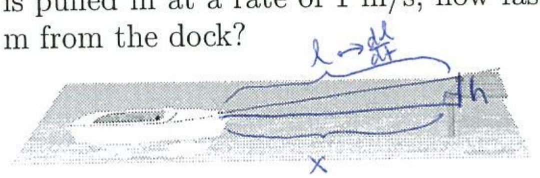
when $h = 10$ ft and $\frac{dV}{dt} = 30 \text{ ft}^3/\text{min}$

$$\frac{dh}{dt} = \frac{4}{\pi (10)^2} \cdot 30 = \frac{4}{100\pi} \cdot 30 = \frac{12}{10}\pi = 1.2\pi$$

$$\boxed{\frac{dh}{dt} = 1.2\pi \text{ ft/min}} = \boxed{3.77 \text{ ft/min}}$$

6. A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s, how fast is the boat approaching the dock when it is 8 m from the dock?

(a)



$\frac{dx}{dt} \rightarrow$

(b) Known:

$\frac{dl}{dt} = 1$ m/s

$h = 1$ m

$\frac{dh}{dt} = 0$ m/s

Unknown:

$\frac{dx}{dt}$ when $x = 8$ m

(c) $x^2 + h^2 = l^2$

(d) $\cancel{2}x\frac{dx}{dt} + \cancel{2}h\cancel{\frac{dh}{dt}}^{0} = \cancel{2}l\frac{dl}{dt}$

$\Rightarrow x\frac{dx}{dt} = l\frac{dl}{dt}$

so $\frac{dx}{dt} = \frac{l}{x}\frac{dl}{dt}$

when $x = 8$,

$l = \sqrt{8^2 + 1^2}$

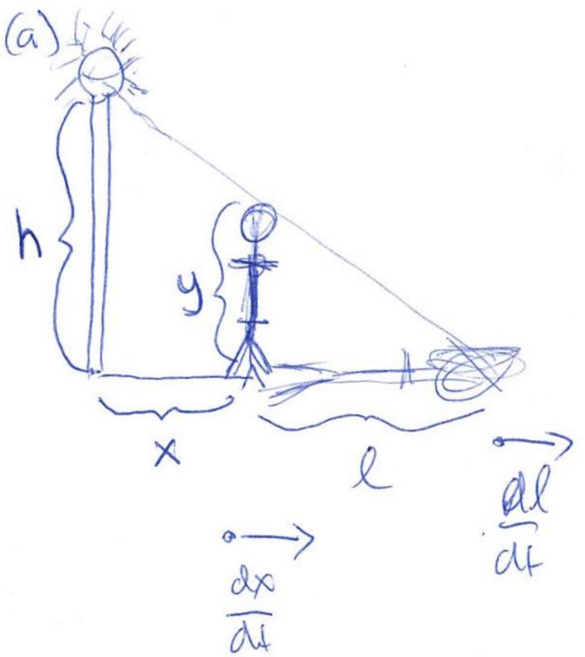
$= \sqrt{65}$ m, so

$\frac{dx}{dt} = \frac{\sqrt{65}}{8}(1)$

$$\boxed{\frac{dx}{dt} = \frac{\sqrt{65}}{8} \text{ m/s} \approx 1.01 \text{ m/s}}$$

7. A street light is mounted at the top of a 15-ft-tall pole. A man 6 ft tall walks away from the pole with a speed of 5 ft/s along a straight path. How fast is the tip of his shadow moving when he is 40 ft from the pole?

(a)



(b) Known:

$h = 15$ ft, $y = 6$ ft

$\frac{dh}{dt} = 0$, $\frac{dy}{dt} = 0$

$\frac{dx}{dt} = 5$ ft/s

Unknown:

$\frac{dl}{dt}$ when $x = 40$ ft.

(c) Similar triangles

$$\frac{h}{x+l} = \frac{y}{l}$$

$\hookrightarrow hl = y(x+l)$.

(d) h and y are constant!

$$h\frac{dl}{dt} = y\left(\frac{dx}{dt} + \frac{dl}{dt}\right)$$

$$h\frac{dl}{dt} - y\frac{dl}{dt} = y\frac{dx}{dt}$$

$$\frac{dl}{dt}(h-y) = y\frac{dx}{dt}$$

$$\frac{dl}{dt} = \frac{y}{h-y}\frac{dx}{dt}$$

So

$$\frac{dl}{dt} = \frac{6}{15-6}(5)$$

$$= \frac{6}{9}(5)$$

$$\boxed{\frac{dl}{dt} = \frac{30}{9} \text{ ft/s} = 3.33 \text{ ft/s}}$$