Solutions

Math 1300-005 - Spring 2017

Implicit Differentiation Intro - 2/27/17

Guidelines: Please work in groups of two or three. This will not be handed in, but is a study resource for Midterm 2.

1. Our goal in this problem is to use implicit differentiation to find an equation of the tangent line to the curve
$$x^2 + xy = 3 - y^2$$
at the point $(1, 1)$.

(a) First, apply d/dx to both sides of $x^2 + xy = 3 - y^2$ and use the chain rule and the guidelines from lecture. Notice the second term on the left-hand side is a product.

$$\frac{d}{dx}(x^2 + \underbrace{xy}_{PR}) = \frac{d}{dx}(3 - y^2)$$

$$2x + y + xy' = -2yy'$$

(b) Move all terms that have a y' to the left-hand side of the equation from (a) and move terms that do NOT have a y' to the right-hand side of the equation from (a). Solve for y'.

$$xy' + 2yy' = -2x - y$$

$$y'(x + 2y) = -2x - y \rightarrow \boxed{y' = \frac{-2x - y}{x + 2y}}$$

(c) Plug in $x = 1$ and $y = 1$ to your expression for y' to get the slope of the tangent line to the curve at $(1, 1)$.

$$y' \text{ at } (1,1): \quad y' = \frac{-2(1) - (1)}{1 + 2(1)} = \frac{-3}{3} = \boxed{-1}$$

(d) Write an equation of the tangent line based on your work in (a), (b), and (c).

$$\boxed{y - 1 = -1(x - 1)}$$

2. Following the same procedure outline in problem 1, find an equation of the tangent line to the curve

$$y\sin(2x) = x\cos(2y)$$

at the point $(\pi/2, \pi/4)$. Notice that each side of the equation above involves a product.

(a) $\frac{d}{dx}(y\sin(2x)) = \frac{d}{dx}(x\cos(2y))$ [Product rule to each side, then chain rule as needed]

$y'\sin(2x) + y\cos(2x)\cdot 2 = \cos(2y) - x\sin(2y)\cdot 2y'$

(b) Rearranging: $y'\sin(2x) + 2xy'\sin(2y) = \cos(2y) - 2y\cos(2x)$

$y'(\sin(2x) + 2x\sin(2y)) = \cos(2y) - 2y\cos(2x)$

$$\boxed{y' = \frac{\cos(2y) - 2y(\cos(2x))}{\sin(2x) + 2x\sin(2y)}}$$

(c) Plug in $(\pi/2, \pi/4)$:

$$y' = \frac{\cos\left(\frac{2\pi}{4}\right) - 2\left(\frac{\pi}{4}\right)\cos\left(2\left(\frac{\pi}{2}\right)\right)}{\sin\left(2\left(\frac{\pi}{2}\right)\right) + 2\left(\frac{\pi}{2}\right)\sin\left(2\left(\frac{\pi}{4}\right)\right)} = \frac{0 - \frac{\pi}{2}(-1)}{0 + \pi(1)} = \frac{\pi/2}{\pi} = \boxed{\frac{1}{2}}$$

(d) $\boxed{y - \pi/4 = \frac{1}{2}(x - \pi/2)}$

3. Find dy/dx by implicit differentiation according to steps (a) and (b) in problem 1.

$$\frac{d}{dx}\underbrace{(\tan(x-y))}_{\text{Chain}} = \frac{d}{dx}\underbrace{\left(\frac{y}{1+x^2}\right)}_{\text{QR}}$$

$\sec^2(x-y)\cdot\frac{d}{dx}(x-y) = \frac{y'(1+x^2) - y\cdot 2x}{(1+x^2)^2}$

$\sec^2(x-y)\cdot(1-y') = \frac{y'(1+x^2) - 2xy}{(1+x^2)^2}$

$\sec^2(x-y) - y'\sec^2(x-y) = \frac{y'(1+x^2) - 2xy}{(1+x^2)^2}$ → multiply both sides

$(1+x^2)^2[\sec^2(x-y)] - y'(1+x^2)^2\sec^2(x-y) = y'(1+x^2) - 2xy$

$-y'(1+x^2)^2\sec^2(x-y) - y'(1+x^2) = -(1+x^2)^2\sec^2(x-y) - 2xy$

$y'(-(1+x^2)^2\sec^2(x-y) - (1+x^2)) = -(1+x^2)^2\sec^2(x-y) - 2xy$

$$\boxed{y' = \frac{-(1+x^2)^2\sec^2(x-y) - 2xy}{-(1+x^2)^2\sec^2(x-y) - (1+x^2)}} \quad \text{or} \quad \boxed{y' = \frac{(1+x^2)^2\sec^2(x-y) + 2xy}{(1+x^2)^2\sec^2(x-y) + (1+x^2)}}$$

4. Find dy/dx by implicit differentiation according to steps (a) and (b) in problem 1.

$$\overbrace{e^y\cos(x)}^{\text{PR}} = 1 + \sin(\overbrace{xy}^{\text{PR}})$$

chain

$\frac{d}{dx}(e^y\cos(x)) = \frac{d}{dx}(1 + \sin(xy))$

$e^y\cdot y'\cos(x) + e^y(-\sin(x)) = \cos(xy)\cdot\frac{d}{dx}\overbrace{(xy)}^{\text{PR}}$

$e^y\cdot y'\cos(x) - e^y\sin(x) = \cos(xy)(y + xy')$

$e^y\cdot y'\cos(x) - e^y\sin(x) = y\cos(xy) + xy'\cos(xy)$

$e^y\cdot y'\cos(x) - xy'\cos(xy) = y\cos(xy) + e^y\sin(x)$

$y'(e^y\cos(x) - x\cos(xy)) = y\cos(xy) + e^y\sin(x)$

$$\boxed{y' = \frac{y\cos(xy) + e^y\sin(x)}{e^y\cos(x) - x\cos(xy)}}$$