1. (20 pts) Consider the function $f(x) = \sqrt{2-x}$.
   (a) Find the linear approximation for $f(x)$ at $x = -2$.
   (b) Use your approximation from part (a) to estimate the value of $\sqrt{4.1}$.
   (c) Compute $f''(x)$. Use f'' to explain whether your approximation from (b) is an overestimate or an underestimate.

Solution:

(a) Notice that $f'(x) = -\frac{1}{2}(2-x)^{-1/2} = -\frac{1}{2\sqrt{2-x}}$, $f'(-2) = -\frac{1}{2\sqrt{4}} = -\frac{1}{4}$, and $f(-2) = \sqrt{4} = 2$.
The linearization of $f(x)$ at $x = -2$ is given by

$$\begin{aligned} L(x) &= f(-2) + f'(-2)\,(x-(-2)) \\ &= 2 - \frac{1}{4}\,(x+2)\,. \end{aligned}$$

(b) $\sqrt{4.1} = \sqrt{2-(-2.1)} = f(-2.1) \approx L(-2.1) = 2 - \frac{1}{4}\,(-2.1+2) = 2 - \frac{1}{4}\,(-0.1) = 2 + \frac{1}{4}\cdot\frac{1}{10} = \frac{81}{40}.$

(c) $f''(x) = -\frac{1}{4}(2-x)^{-3/2} = -\frac{1}{4(2-x)^{3/2}}$. Since, $f''(x) < 0$ for all x in the domain, the funtion is concave down on its whole domain. That is, the graph of $y = f(x)$ lies below its tangent lines, so the approximation in (b) must be an overestimate.

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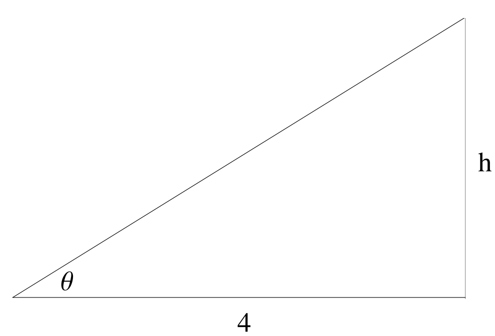
2. (24 pts) A rocket is launched vertically and is tracked by a radar station located on the ground 4 miles from the launch pad. If the angle between the ground and the line of sight from the radar station to the rocket is increasing at a rate of 0.05 radians per second, what is the speed of the rocket when the angle is $\frac{\pi}{3}$ radians? Assume the ground is horizontal and flat. A complete answer should include a labeled diagram and the correct units.

Solution:



We know that $\frac{d\theta}{dt} = 0.05$ radians per second. We want to find $\frac{dh}{dt}$ when $\theta = \frac{\pi}{3}$.
Notice that θ and h are related by

$$\tan\theta = \frac{h}{4}.$$

Implicitly differentiating, we find that

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{4}\frac{dh}{dt} \quad \Longrightarrow \quad \frac{dh}{dt} = \frac{4}{\cos^2\theta}\frac{d\theta}{dt}.$$

When $\theta = \frac{\pi}{3}$, $\cos\theta = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$. Thus we have

$$\frac{dh}{dt} = \frac{4}{\left(\frac{1}{2}\right)^2} \cdot 0.05 = 16 \cdot 0.05 = 0.8.$$

That is, the rocket is moving at a speed of 0.8 miles per second, or $0.8\ \frac{\text{mi}}{\text{sec}} \times 3600\ \frac{\text{sec}}{\text{hr}} = 2880$ miles per hour.

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3. (35 pts) Consider the function $f(x) = \dfrac{\sin x}{\cos x + 2}$, whose second derivative is $f''(x) = \dfrac{2\sin x(\cos x - 1)}{(\cos x + 2)^3}$. You must justify your answers for each part of this problem.

(a) Show that $f'(x) = \dfrac{2\cos x + 1}{(\cos x + 2)^2}$.

(b) What is the domain of f? Is $f(x)$ even, odd, or neither?

(c) Find the x and y intercepts of $f(x)$.

(d) Find the vertical and horizontal asymptotes of $f(x)$, if they exist.

(e) Find the intervals of increase and decrease of $f(x)$ for $0 \le x \le 2\pi$. What are the coordinates of any local extrema of $f(x)$ in this interval?

(f) Where is $f(x)$ concave up for $0 \le x \le 2\pi$? Where is $f(x)$ concave down?

(g) Sketch $f(x)$ for $0 \le x \le 2\pi$.

Solution:

(a) To find $f'(x)$, we use the quotient rule:

$$\begin{aligned} f'(x) &= \frac{(\cos x + 2)(\cos x) - (\sin x)(-\sin x)}{(\cos x + 2)^2} \\ &= \frac{\cos^2 x + 2\cos x + \sin^2 x}{(\cos x + 2)^2} \\ &= \frac{2\cos x + 1}{(\cos x + 2)^2}, \end{aligned}$$

where we used $\sin^2 x + \cos^2 x = 1$.

(b) The domain of $f(x)$ is all real numbers since the demoninator can never be zero.

$f(x)$ is an odd function: $f(-x) = \dfrac{\sin(-x)}{\cos(-x) + 2} = \dfrac{-\sin x}{\cos x + 2} = -f(x)$, where we used the fact that $\sin x$ is odd (so $\sin(-x) = -\sin x$) and $\cos x$ is even (so $\cos(-x) = \cos x$).

(c) The x intercepts are found by setting $y = 0$ (where $y = f(x)$):

$$\frac{\sin x}{\cos x + 2} = 0 \quad \Longrightarrow \quad \sin x = 0 \quad \Longrightarrow \quad x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots.$$

In other words, the x intercepts are $x = n\pi$, where n is an integer.
The y intercepts are found by setting $x = 0$:

$$y = \frac{\sin 0}{\cos 0 + 2} = \frac{0}{3} \quad \Longrightarrow \quad y = 0.$$

That is, the y interecept is $y = 0$.

(d) There are no vertical or horizontal asymptotes.

(e) The critical points of $f(x)$ occur where $f'(x) = 0$ or $f'(x)$ DNE. Since $f'(x) = \dfrac{2\cos x + 1}{(\cos x + 2)^2}$, we have that $f'(x) = 0$ when $2\cos x + 1 = 0$:

$$2\cos x + 1 = 0 \quad \Longrightarrow \quad \cos x = -\frac{1}{2} \quad \Longrightarrow \quad x = \frac{2\pi}{3} \text{ and } x = \frac{4\pi}{3}.$$

$f'(x)$ always exists, so $x = \frac{2\pi}{3}, \frac{4\pi}{3}$ are the only critical points in $[0, 2\pi]$.
We check the sign of f' in the intervals $[0, \frac{2\pi}{3}), (\frac{2\pi}{3}, \frac{4\pi}{3})$, and $(\frac{4\pi}{3}, 2\pi]$:

$$\begin{aligned} \left[0, \frac{2\pi}{3}\right) &: \quad f' > 0 \\ \left(\frac{2\pi}{3}, \frac{4\pi}{3}\right) &: \quad f' < 0 \\ \left(\frac{4\pi}{3}, 2\pi\right] &: \quad f' > 0 \end{aligned}$$

It follows that f is increasing on $[0, \frac{2\pi}{3}) \cup (\frac{4\pi}{3}, 2\pi]$ and decreasing on $(\frac{2\pi}{3}, \frac{4\pi}{3})$. There is a local maximum at $x = \frac{2\pi}{3}$, with value

$$f\left(\frac{2\pi}{3}\right) = \frac{\sin\left(\frac{2\pi}{3}\right)}{\cos\left(\frac{2\pi}{3}\right) + 2} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2} + 2} = \frac{\sqrt{3}}{3}.$$

There is a local minimum at $x = \frac{4\pi}{3}$, with value

$$f\left(\frac{4\pi}{3}\right) = \frac{\sin\left(\frac{4\pi}{3}\right)}{\cos\left(\frac{4\pi}{3}\right) + 2} = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2} + 2} = -\frac{\sqrt{3}}{3}$$

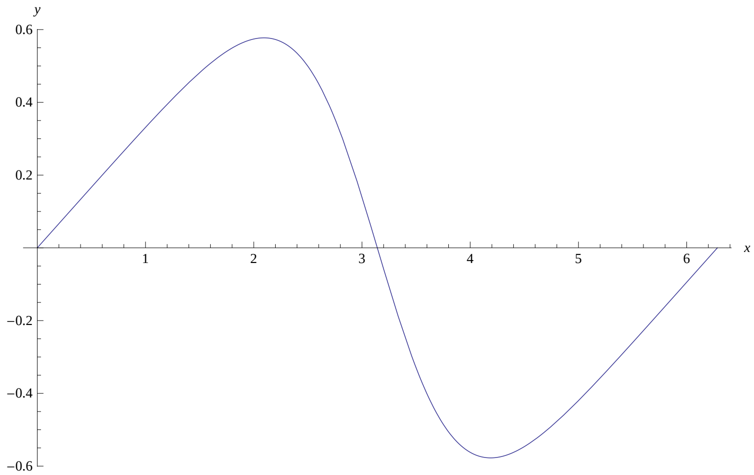
(f) Since $f''(x) = \dfrac{2\sin x(\cos x - 1)}{(\cos x + 2)^3}$, we see that $f''(x)$ always exists and is zero when

$$2\sin x(\cos x - 1) = 0 \quad \Longrightarrow \quad \sin x = 0 \text{ or } \cos x - 1 = 0 \quad \Longrightarrow \quad x = 0, \pi, 2\pi.$$

We check the sign of f'' in the intervals $(0, \pi)$ and $(\pi, 2\pi)$:

$$\begin{aligned} (0, \pi) &: \quad f'' < 0 \\ (\pi, 2\pi) &: \quad f'' > 0 \end{aligned}$$

It follows that f is concave down on $(0, \pi)$ and concave up on $(\pi, 2\pi)$.

(g) 

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4. (15 points) Let $y = x^5 + 2x^3 + 5x + 2$ on the interval $(-1, 1)$. Show that at least one tangent line to the curve is parallel to the line $y = 8x + 3$. If you use any theorems, you must state them and show that their conditions are satisfied.

   Solution:
   We want to show that there is a tangent line to $y = f(x)$ at some point in $(-1, 1)$ with slope $m = 8$ (since the line $y = 8x + 3$ has slope 8). We use the Mean Value Theorem, which states that if $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then there is a number c in (a, b) such that

   $$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

   In our case, $f(x)$ is a polynomial, so it is continuous and differentiable for all real numbers. In particular, f is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$. From MVT, there is a c in $(-1, 1)$ such that

   $$f'(c) = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{10 - (-6)}{2} = 8.$$

   That is, there is a c in $(-1, 1)$ for which $f'(c) = 8$, meaning that there is a point in $(-1, 1)$ at which the tangent line has slope $m = 8$.

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5. (6 points) For this question, answer with the word **True** or **False**. Do not write T or F. You do not need to show any work for this problem.

   (a) If $f'(c) = 0$, then there is a local maximum or a local minimum at $x = c$.

   (b) If $f(x)$ and $g(x)$ are increasing on an interval I, then the product $f(x)g(x)$ is also increasing on I.

   (c) If $f'(x) = g'(x)$ for $-1 < x < 1$, then $f(x) = g(x)$ for $-1 < x < 1$.

   Solution:

   (a) False. Consider $f(x) = x^3$. Then $f'(x) = 3x^2$, and $f'(x) = 0$ when $x = 0$. However, $f(x)$ has neither a local maximum nor a local minimum at $x = 0$.

   (b) False. Suppose $f(x) = x + 1$ and $g(x) = x - 1$ on [-1,0]. Then $f'(x) = g'(x) = 1$, so both functions are increasing on $[-1, 0]$. However, $f(x)g(x) = x^2 - 1$, so $(fg)' = 2x$. It follows that $f(x)g(x)$ must be decreasing when $x < 0$, and in particular on $[-1, 0]$.

   (c) False. If $f'(x) = g'(x)$, then $f(x)$ and $g(x)$ can differ by a constant. (See Corollary 7 in Section 3.2.)