1. The following problems are not related:

(a) (15 pts, 5 pts ea.) Find the following limits or show that they do not exist:

$$(i)\ \lim_{x\to\infty} e^{-x}\sqrt{x} \qquad (ii)\ \lim_{x\to 0}\frac{\arcsin(x)}{x} \qquad (iii)\ \lim_{x\to 0^+}(\sin x)^x$$

(b) (5 pts) Find and classify all relative extrema of $f(x) = \dfrac{x^5 - 5x}{5}$.

(c) (10 pts) A right triangle is formed in the first quadrant by both the x-axis and y-axis and a line through the point $(1, 2)$. Find the vertices of the triangle so that its area is a minimized.

[*Hint:* the slope of the hypotenuse should be the same whether slope is found using the y-intercept and the point (1,2) or the x-intercept and the point (1,2).]

Solution:

(a)(i) Note that, $\displaystyle\lim_{x\to\infty} e^{-x}\sqrt{x} = \lim_{x\to\infty}\frac{\sqrt{x}}{e^x} \overset{L'H}{=} \lim_{x\to\infty}\frac{1}{2e^x\sqrt{x}} = \boxed{0}$.

(a)(ii) Applying L'Hospitals Rule yields,

$$\lim_{x\to 0}\frac{\arcsin(x)}{x} \overset{L'H}{=} \lim_{x\to 0}\frac{\frac{1}{\sqrt{1-x^2}}}{1} = \lim_{x\to 0}\frac{1}{\sqrt{1-x^2}} = \boxed{1}$$

(a)(iii) If we let $y = \lim_{x\to 0^+}(\sin x)^x$, then, using continuity,

$$\ln y = \lim_{x\to 0^+}\ln(\sin(x)^x) = \lim_{x\to 0^+} x\ln(\sin(x)) \overset{0\cdot\infty}{=} \lim_{x\to 0^+}\frac{\ln(\sin x)}{1/x} \overset{L'H}{=} \lim_{x\to 0^+}\frac{\cot x}{\frac{-1}{x^2}} = \lim_{x\to 0^+}\frac{-x^2}{\tan(x)}$$

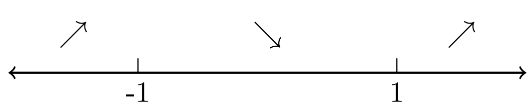
Finally, applying L'Hospitals Rule again yields

$$\ln(y) = \lim_{x\to 0^+}\frac{-x^2}{\tan(x)} \overset{L'H}{=} \lim_{x\to 0^+}\frac{-2x}{\sec^2(x)} = 0$$

so $\ln(y) = 0$ implies $y = e^0 = 1$, that is $\boxed{\displaystyle\lim_{x\to 0^+}(\sin x)^x = 1.}$ Note that we could also do this limit directly using continuity by rewriting the limit as $\displaystyle\lim_{x\to 0^+}(\sin x)^x = \lim_{x\to 0^+} e^{\ln(\sin(x)^x)} = \lim_{x\to 0^+} e^{x\ln(\sin(x))}$.

(b) Taking the derivative yields $f'(x) = \dfrac{5x^4 - 5}{5} = x^4 - 1$, so the critical points are $x = \pm 1$. Now we need to classify the critical points. Using, for example the First Derivative Test, we have

1st Derivative Test for $f(x)$:



Therefore, we have a $\boxed{\text{relative maximum at } (-1, \frac{4}{5})}$, and $\boxed{\text{relative minimum at } (1, -\frac{4}{5})}$. (Note could also use the 2nd Derivative Test.)

(c) Assume the x-intercept and y-intercept of the line are $(0, y)$ and $(x, 0)$ respectively.

Note that we must have $x > 1$, since if $x = 1$ then the line through $(x, 0)$ and $(1, 2)$ is vertical and we don't get a triangle, and if $x < 1$ then the line through $(x, 0)$ and $(1, 2)$ has a negative y-intercept and we don't get a triangle solely in the first quadrant.

Then the slope of the hypotenuse using the two intercepts is $\frac{y-2}{0-1} = \frac{0-2}{x-1}$. This produces the relationship $y = 2 + \frac{2}{x-1}$.

So we wish to $\boxed{\text{minimize the area } A(x,y) = \frac{1}{2}xy, \text{ subject to } y = 2 + \frac{2}{x-1} \text{ and } x > 1.}$ Substituting, yields

$$A(x) = \frac{1}{2}x(2 + \frac{2}{x-1}) = x + \frac{x}{x-1}$$

Then differentiating and finding critical points yields,

$$A'(x) = 1 + \frac{(x-1) - x}{(x-1)^2} = 1 - \frac{1}{(x-1)^2} = 0$$

Thus $(x-1)^2 = 1$ implies $x - 1 = \pm 1$ and so the critical points are $x = 0, 2$. But $x = 0$ is not in the domain $x > 1$ so we may ignore it. Note that $A''(x) = \frac{2}{(x-1)^3}$, and by the 2nd Derivative test $A''(2) > 0$, so we have a relative minimum at $x = 2$ (we may also use the First Derivative Test to determine this). Since $x = 2$ is the only critical point however, it gives an absolute minimum. Thus, select $x = 2$ then $y = 4$ and $A = 4$ and the vertices that yield the minimum area are $\boxed{(0,0), (2,0), (0,4).}$

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2. The following problems are not related:

(a) (15 pts, 5 pts ea.) Find dy/dx if:

$(i)\ y = e^{x+\tan^{-1}(x)}$ $\qquad$ $(ii)\ y = x^2\cosh(\sqrt{x})$ $\qquad$ $(iii)\ xe^y - \tanh(y) = 10,690$

(b) (7 pts) Consider the function $f(x) = 2x^2 - 4x + 1$ on the interval $[0, 2]$.

(i) (3 pts) Does the function $f(x)$ satisfy the hypotheses of the Mean Value Theorem on the given interval?

(ii) (4 pts) If it satisfies the hypotheses, find all numbers c that satisfy the conclusion of the Mean Value Theorem. If it does not satisfy the hypotheses, write DNE.

(c) (8 pts) A kite 100 feet above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal decreasing when 200 ft of string has been let out?

Solution:

(a)(i) Using the Chain Rule, we have $\frac{dy}{dx} = e^{x+\tan^{-1}(x)}\left(1 + \frac{1}{1+x^2}\right) = \boxed{e^{x+\tan^{-1}(x)}\left(\frac{2+x^2}{1+x^2}\right)}$

(a)(ii) Using the Product Rule and Chain Rule, we have

$$\frac{d}{dx}\left(x^2\cosh(\sqrt{x})\right) = 2x\cosh(\sqrt{x}) + x^2\sinh(\sqrt{x})\frac{1}{2\sqrt{x}} = \boxed{2x\cosh(\sqrt{x}) + \frac{x^{3/2}}{2}\sinh(\sqrt{x}).}$$

(a)(iii) Using implicit differentiation yields

$$e^y + xe^y y' - \operatorname{sech}^2(y)y' = 0 \Longrightarrow y' = -\frac{e^y}{xe^y - \operatorname{sech}^2(y)} = \boxed{\frac{e^y}{\operatorname{sech}^2(y) - xe^y}}$$

(b)(i) $\boxed{\text{Yes.}}$ The function is a polynomial, so it is continuous on $[0,2]$ and differentiable on $(0,2)$.

(b)(ii) We see that $c=1$, since

$$f'(c) = \frac{f(b)-f(a)}{b-a} \iff 4c-4 = \frac{1-1}{2-0} = 0 \iff 4c = 4 \iff \boxed{c=1} \quad \text{which is in } (0,2).$$

(c) Let θ denote the angle of the kite string with the ground, let x denote the horizontal distance of the kite from the person holding the string and let y be the amount of string let out. Then

$$\cot(\theta) = \frac{x}{100} \Rightarrow x = 100\cot(\theta) \Rightarrow \frac{dx}{dt} = -100\csc^2(\theta)\frac{d\theta}{dt} \Rightarrow \frac{d\theta}{dt} = -\frac{\sin^2(\theta)}{100}\frac{dx}{dt}$$

Now, $\dfrac{dx}{dt} = 8\text{ft/s}$ and when $y=200$ we have $\sin(\theta) = \dfrac{100}{200} = \dfrac{1}{2}$, so

$$\frac{d\theta}{dt} = -\frac{\sin^2(\theta)}{100}\frac{dx}{dt} = -\frac{(1/2)^2}{100}\cdot 8 = \boxed{-\frac{1}{50}\text{rad/s}}$$

so the angle is $\boxed{\text{decreasing at a rate of } \frac{1}{50}\text{rad/s.}}$

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3. The following problems are not related:

(a) (15 pts, 5 pts ea.) Evaluate the integrals:

$$(i)\ \int_1^2 \frac{e^{1/x^3}}{x^4}\,dx \qquad (ii)\ \int \frac{1}{x^2+4}\,dx \qquad (iii)\ \int \frac{e^{2x}}{\sqrt{1-e^{4x}}}\,dx$$

(b) (5 pts) Find h' in terms of f' and g' if $h(x) = \dfrac{f(x)g(x)}{f(x)+g(x)}$.

(c) (10 pts) The Ideal Gas Law relating the *pressure*, *volume*, and *temperature* of a gas is given by $PV = nRT$ where R is the ideal gas constant. Assuming that the current pressure of the gas, P, is 10 atmospheric pressure units (atm) and assuming that n, R, and T are held fixed, use differentials to estimate the **relative change** in the volume, V, if the pressure increases by 0.1 atms.

Solution:

(a)(i) Let $u = \dfrac{1}{x^3}$ then $du = -\dfrac{3}{x^4}\,dx$ and

$$\int_1^2 \frac{e^{1/x^3}}{x^4}dx = -\frac{1}{3}\int_1^{1/8} e^u du = \frac{1}{3}\int_{1/8}^1 e^u du = \frac{1}{3}\Big[e^u\Big]_{1/8}^1 = \boxed{\frac{1}{3}(e - e^{1/8})}$$

(a)(ii) Factoring out a 4 from the denominator yields

$$\int \frac{dx}{x^2+4} = \frac{1}{4}\int \frac{dx}{\frac{x^2}{4}+1} = \frac{1}{4}\int \frac{dx}{(x/2)^2+1}$$

let $u = x/2$, then $du = \dfrac{dx}{2}$, and so we have

$$\frac{1}{4}\int \frac{dx}{(x/2)^2+1} = \frac{2}{4}\int \frac{du}{u^2+1} = \frac{1}{2}\tan^{-1}(u) + C = \boxed{\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C}$$

(a)(iii) Let $u = e^{2x}$, then $du = 2e^{2x}\,dx$ and

$$\int \frac{e^{2x}dx}{\sqrt{1-e^{4x}}} = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\sin^{-1}(u) + C = \boxed{\frac{1}{2}\sin^{-1}(e^{2x}) + C}$$

(b) Use the Quotient Rule and Product Rule,

$$h' = \frac{(f'g + fg')(f+g) - fg(f'+g')}{(f+g)^2} = \frac{ff'g + f^2g' + f'g^2 + fgg' - ff'g - fgg'}{(f+g)^2} = \boxed{\frac{f^2g' + f'g^2}{(f+g)^2}}$$

(c) We have

$$PV = nRT \quad \Rightarrow \quad V = \frac{nRT}{P} \quad \Rightarrow \quad dV = -\frac{nRT}{P^2}dP$$

Then

$$\frac{dV}{V} = \frac{-\frac{nRT}{P^2}dP}{\frac{nRT}{P}} = -\frac{1}{P}dP = \boxed{-\frac{1}{10}(0.1) = -0.01}$$

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4. The following problems are not related:

(a) (15 pts, 5 pts ea.) Find dy/dx given:

$$(i)\ y = (\sin x)^x \qquad (ii)\ y = \frac{2^x}{1+2^x} \qquad (iii)\ y = e^{u(x)} \text{ where } u(x) = \int_0^{g(x)} \cos(f(t))\,dt$$

(b) (5 pts) In your blue book clearly sketch the graph of a function $f(x)$ that satisfies the following properties (label any <u>extrema</u>, <u>inflection points</u> or <u>asymptotes</u>):

- $f(0) = 0,\ f(-x) = f(x)$
- $\lim_{x\to 1^-} f(x) = -\infty,\ \lim_{x\to -1^-} f(x) = +\infty$
- $\lim_{x\to\infty} f(x) = 2$
- $f'(x) > 0$ if $x < 0$
- $f''(x) < 0$ if $|x| < 1$

(c) (10 pts) A bacteria culture initially contains 106 cells and its population grows at rate proportional to its size. After an hour the population has increased to 420. Find an expression for the number of bacteria after t hours.

Solution:

(a)(i) Using logarithmic differentiation yields,

$$\ln(y) = \ln[\sin(x)^x] \Longrightarrow \ln(y) = x\ln[\sin(x)] \Longrightarrow \frac{y'}{y} = \ln[\sin(x)] + x \cdot \frac{\cos(x)}{\sin(x)}$$

and so $\boxed{y' = (\sin(x))^x \big[\ln(\sin(x)) + x\cot(x)\big].}$

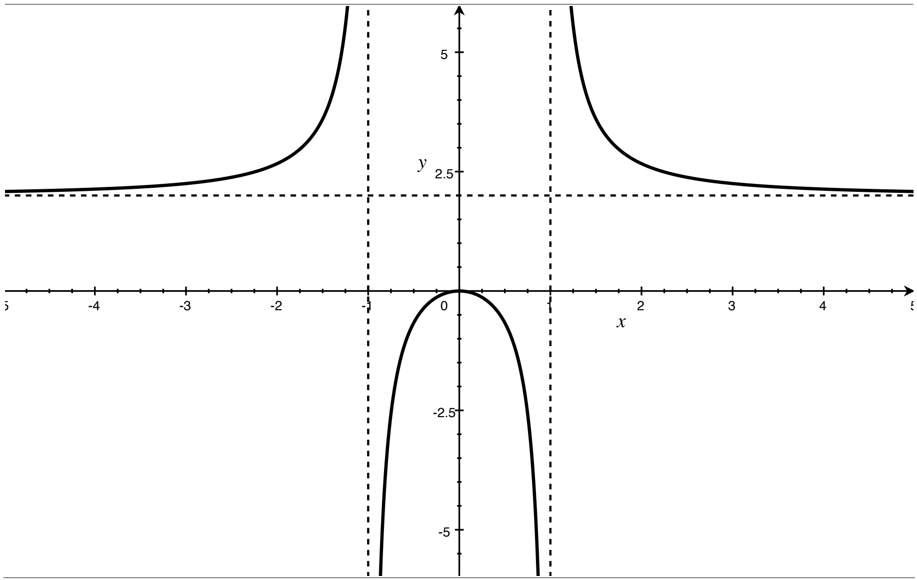
(a)(ii) Applying the Quotient Rule yields,

$$\frac{dy}{dx} = \frac{2^x\ln(2)\cdot(1+2^x) - 2^x \cdot 2^x\ln(2)}{(1+2^x)^2} = \boxed{\frac{2^x\ln(2)}{(1+2^x)^2}}$$

(a)(iii) First note that, $dy/dx = e^{u(x)}u'(x)$ by the Chain Rule, and applying the Fundamental Theorem of Calculus to determine $u'(x)$, we get

$$\frac{dy}{dx} = e^{u(x)}u'(x) = e^{u(x)} \cdot \big(\cos(f(g(x))) \cdot g'(x)\big) = \boxed{e^{u(x)}g'(x)\cos(f(g(x)))}$$

(b) There are vertical asymptotes at $x = \pm 1$, a horizontal asymptote at $y = 2$ and a local maximum at $(0,0)$, the graph could, for example, look like:



(c) Let $y(t)$ represent the population of the bacteria in cells at time t hours, then $\dfrac{dy}{dt} = ky$ for some constant k, and so $y(t) = y_0 e^{kt}$ where $y_0 = y(0) = 106$. Now note that $y(1) = 420$ implies

$$420 = y(1) = 106e^k \implies e^k = \frac{420}{106} \implies k = \ln\left(\frac{420}{106}\right)$$

and so $\boxed{y(t) = 106e^{\ln\left(\frac{420}{106}\right)t} = 106\left(\frac{420}{106}\right)^t}.$

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5. The following problems are not related:

(a) (15 pts, 5 pts ea.) Evaluate the integrals:

$$(i)\ \int_{-4}^{4}\left(\sqrt{16-x^2}+\frac{\sin(x)}{1+x^2}\right)dx \qquad (ii)\ \int \frac{1+x}{1+x^2}\,dx \qquad (iii)\ \int \frac{dx}{x\ln(x^e)}$$

(b) (5 pts) Use the Squeeze Theorem to show $\displaystyle\lim_{x\to 0^+}\sqrt{x}\sin\left(\frac{1}{x}\right) = 0$.

(c) (10 pts) The equation of motion of a particle is $s(t) = \dfrac{t^3}{3} - \dfrac{t^2}{2} - 6t$ ft

(i) Find the velocity at time t.

(ii) When is the particle moving in the positive direction?

(iii) Find the acceleration after 3 seconds.

(iv) Find the total distance traveled during the first 4 seconds.

Solution:

(a)(i) First note that $y = \dfrac{\sin(x)}{1+x^2}$ is an odd function, and using geometry to calculate the first integral we get,

$$\int_{-4}^{4}\left(\sqrt{16-x^2}+\frac{\sin(x)}{1+x^2}\right)dx = \int_{-4}^{4}\sqrt{16-x^2}\,dx + \int_{-4}^{4}\frac{\sin(x)}{1+x^2}\,dx = \int_{-4}^{4}\sqrt{16-x^2}\,dx + 0 = \frac{\pi\cdot 4^2}{2} = \boxed{8\pi}$$

(a)(ii) Breaking up the integral yields,

$$\int \frac{1+x}{1+x^2}\,dx = \int \frac{1}{1+x^2}\,dx + \int \frac{x}{1+x^2}\,dx$$

Now, in the second integral, let $u = 1 + x^2$ then $du = 2x\,dx$, and

$$\int \frac{1}{1+x^2}\,dx + \int \frac{x}{1+x^2}\,dx = \tan^{-1}(x) + \frac{1}{2}\int u\,du = \tan^{-1}(x) + \frac{1}{2}\ln|u| + C = \boxed{\tan^{-1}(x) + \frac{1}{2}\ln(1+x^2) + C}$$

(a)(*iii*) First note that $\displaystyle\int \frac{dx}{x\ln(x^e)} = \frac{1}{e}\int \frac{dx}{x\ln(x)}$, now let $u = \ln(x)$ then $du = \dfrac{dx}{x}$ and so

$$\int \frac{dx}{x\ln(x^e)} = \frac{1}{e}\int \frac{dx}{x\ln(x)} = \frac{1}{e}\int \frac{du}{u} = \frac{1}{e}\ln|u| + C = \boxed{\frac{1}{e}\ln|\ln(x)| + C}$$

Alternately, one could also use the substitution $u = \ln(x^e)$.

(b) Note that $-1 \le \sin\left(\frac{1}{x}\right) \le 1$ implies $-\sqrt{x} \le \sqrt{x}\sin\left(\frac{1}{x}\right) \le \sqrt{x}$ and $\displaystyle\lim_{x\to 0^+} \pm\sqrt{x} = 0$ implies that $\displaystyle\lim_{x\to 0^+} \sqrt{x}\sin\left(\frac{1}{x}\right) = 0$ by the Squeeze Theorem.

(c)(*i*) Here $\boxed{v(t) = s'(t) = t^2 - t - 6.}$

(c)(*ii*) Note, $v(t) = t^2 - t - 6 = (t-3)(t+2)$ and $\boxed{v(t) > 0 \text{ when } t > 3}$, so the particle is moving in a positive direction $\boxed{\text{after exactly 3 seconds.}}$

(c)(*iii*) Note that $a(t) = v'(t) = 2t - 1$ and so $\boxed{a(3) = 6 - 1 = 5 \text{ seconds.}}$

(c)(*iv*) The total distance in the first 4 seconds is

$$\begin{aligned}
\int_0^4 |v(t)|dt &= \int_0^3 -v(t)\,dt + \int_3^4 v(t)\,dt = \int_0^3 -(t^2 - t - 6)dt + \int_3^4 (t^2 - t - 6)dt \\
&= -\left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_0^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_3^4 = \left[\frac{4^3}{3} - \frac{4^2}{2} - 6(4)\right] - 2\left[\frac{3^3}{3} - \frac{3^2}{2} - 6(3)\right] \\
&= \boxed{\frac{10}{3} + 13 = \frac{49}{3}\text{ft.}}
\end{aligned}$$