1. In this problem, *if you need to find a derivative you must do so using only the limit definition of the derivative*, you may not use the rules of differentiation. Justify your answers.

   In economics, the function $C(x)$ is called a *cost function*, it measures the cost of production. For example, suppose the cost (in dollars) of producing x units of an energy efficient lightbulb is $C(x) = 50+10x+5x^2$.

   (a) (10 pts) What is the average rate of change of the cost when productions levels are changed from $x = 1$ to $x = 2$.

   (b) (10 pts) Although $C(x)$ is not differentiable, economists define the *marginal cost* of production as the instantaneous rate of change of the cost with respect to the number of units produced. Marginal costs can be used to predict future production costs. Find the marginal cost of producing energy efficient lightbulbs when $x = 2$.

   Solution:

   (a) The average rate is $\dfrac{C(2) - C(1)}{2 - 1} = (50+10(2)+5(2)^2)-(50+10+5) = 90-65 = \boxed{\text{25 dollars/lightbulb}}$

   (b) The marginal cost at $x = 2$ is

$$C'(2) = \lim_{x\to 2} \frac{C(x) - C(2)}{x - 2} = \lim_{x\to 2} \frac{5x^2 + 10x + 50 - 90}{x - 2} = \lim_{x\to 2} \frac{5x^2 + 10x - 40}{x - 2} = \lim_{x\to 2} \frac{(x - 2)(5x + 20)}{x - 2} = 30$$

   so the marginal cost is $\boxed{\text{30 dollars/lightbulb.}}$ Note could also use $C'(2) = \lim_{h\to 0} \dfrac{C(2 + h) - C(2)}{h}$.

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2. All parts of this problem refer to the function $f(x)$ given below. In this problem, *if you need to find a derivative you must do so using only the limit definition of the derivative* and, for all parts of the problem, be sure to show all work and justify your answers. Consider the function

$$f(x) = \begin{cases} \dfrac{x+2}{x-1}, & \text{if } x \le -2 \\ -x, & \text{if } -2 < x \le 4 \\ \sqrt{x} - 6, & \text{if } x > 4 \end{cases}$$

   (a) (5 pts) Find all horizontal and vertical asymptotes of $f(x)$.

   (b) (5 pts) Find and classify all discontinuities of $f(x)$ as *jump*, *removable* and/or *infinite*.

   (c) (10 pts) Find the slope of the curve $f(x)$ exactly when $x = 5$.

   (d) (10 pts) Use the limit definition of the derivative to find $f'(4)$ if possible.

   (e) (5 pts) Is $f(x)$ differentiable for all x in its domain? If not, where is it *non-differentiable* and explain why it fails to have a derivative there.

   Solution:

   (a) To determine HAs we need to check $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$, so

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \sqrt{x} - 6 = +\infty, \quad \text{and,} \quad \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x+2}{x-1} = \lim_{x\to-\infty} \frac{x(1 + 2/x)}{x(1 - 1/x)} = 1$$

   or the last step can also be done using a "Dominance of Powers"/"Leading Terms" type argument. So $\boxed{y = 1 \text{ is a HA of } f(x)}$. Note that $f(x)$ is defined for all x and therefore has $\boxed{\text{no vertical asymptotes.}}$

(b) Note that at $x = -2$ we have

$$\lim_{x\to -2^+} f(x) = 2 \neq 0 = \lim_{x\to -2^-} f(x) = f(-2)$$

so we have $\boxed{\text{a jump discontinuity at } x = -2}$, and note that at $x = 4$, $f(x)$ is continuous since

$$\lim_{x\to 4^-} f(x) = -4 = \lim_{x\to 4^+} f(x) = f(4).$$

(c) Note that from the way $f(x)$ is defined at $x = 5$, we can just look at the two-sided limit,

$$f'(5) = \lim_{h\to 0} \frac{f(5+h) - f(5)}{h} = \lim_{h\to 0} \frac{(\sqrt{5+h} - 6) - (\sqrt{5} - 6)}{h} = \lim_{h\to 0} \frac{\sqrt{5+h} - \sqrt{5}}{h}$$

now multiplying by the conjugate yields

$$\lim_{h\to 0} \frac{\sqrt{5+h} - \sqrt{5}}{h} \cdot \frac{\sqrt{5+h} + \sqrt{5}}{\sqrt{5+h} + \sqrt{5}} = \lim_{h\to 0} \frac{h}{h(\sqrt{5+h} + \sqrt{5})} = \lim_{h\to 0} \frac{1}{\sqrt{5+h} + \sqrt{5}} = \frac{1}{2\sqrt{5}}$$

so the slope of $f(x)$ when $x = 5$ is $\boxed{m = \frac{1}{2\sqrt{5}}.}$

(d) First note that $f(x)$ is continuous at $x = 4$. Now, we should look at the one sided limits at $x = 4$, for $x < 4$, we use $f(x) = -x$, and so

$$\lim_{h\to 0^-} \frac{f(4+h) - f(4)}{h} = \lim_{h\to 0^-} \frac{-(4+h) - (-4)}{h} = -1$$

and for $x > 4$, we use $f(x) = \sqrt{x} - 6$, (but $f(4) = -4$) and thus

$$\begin{aligned} \lim_{h\to 0^+} \frac{f(4+h) - f(4)}{h} &= \lim_{h\to 0} \frac{(\sqrt{4+h} - 6) - (-4)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{4+h} - 2}{h} \cdot \frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2} = \lim_{h\to 0} \frac{h}{h(\sqrt{4+h} + 2)} = \frac{1}{\sqrt{4} + 2} = \frac{1}{4}. \end{aligned}$$

and so,

$$\lim_{h\to 0^-} \frac{f(4+h) - f(4)}{h} \neq \lim_{h\to 0^+} \frac{f(4+h) - f(4)}{h}$$

Thus $\boxed{f(x) \text{ is not differentiable at } x = 4}$ since the slope from the left does not agree with the slope from the right.

(e) No, $f(x)$ is not differentiable at $x = -2$ since it is not continuous there and $f(x)$ is not differentiable at $x = 4$ since the derivative from the left does not equal the derivative from the right (i.e. $f'(4) = \text{d.n.e}$).

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3. (a) (10 pts) Show that $\lim_{x\to\infty} 2^{-x} \sin(2\pi x) = 0$. Hint: Use the Squeeze Theorem. Show all work.

(b) (10 pts) Is there a solution to the equation $x2^x = 1$ on the interval $[0, 2]$? Justify your response.

Solution:

(a) First we notice that

$$-1 \leq \sin(2\pi x) \leq 1$$

Then, since 2^{-x} is always positive we have

$$-2^{-x} \leq 2^{-x} \sin(2\pi x) \leq 2^{-x}.$$

But $\lim_{x\to\infty} -2^{-x} = \lim_{x\to\infty} 2^{-x} = 0$ so we have $\lim_{x\to\infty} 2^{-x}\sin(2\pi x) = 0$ by the Squeeze Theorem.

(b) This problem is equivalent to asking if the function $h(x) = x2^x - 1$ has a root in the interval $[0, 2]$. We first note that $g(x)$ is continuous everywhere. Then, we note that $h(0) = -1 < 0$ and $h(1) = 2 - 1 > 0$. Therefore, by the Intermediate Value Theorem there is a solution of $x2^x - 1$ on the interval $[0, 2]$.

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4. (a) Evaluate the following limits, you may <u>not</u> use l'Hospital's Rule, justify your answers:

(i) (7 pts) $\lim_{t\to 0} \dfrac{\tan(10t)}{\sin(6t)}$ (ii) (7 pts) $\lim_{x\to 3} \dfrac{x-3}{|x-3|}$ (iii) (7 pts) $\lim_{x\to -1^-} \dfrac{x-1}{x^4-1}$

(b) (4 pts) Find a number c so that the following limit exists, justify your answer: $\lim_{x\to -2} \dfrac{2x^2+2cx+c-6}{x^2+x-2}$.

Solution:

(a)(i) Note that applying the limit we get " 0/0" and so, applying the definition of the tangent function, we get

$$\lim_{t\to 0} \frac{\tan(10t)}{\sin(6t)} = \lim_{t\to 0} \frac{\sin(10t)}{\cos(10t)} \cdot \frac{1}{\sin(6t)} = \lim_{t\to 0} \frac{1}{\cos(10t)} \cdot \underbrace{\frac{\sin(10t)}{10t}}_{\to 1} \cdot \underbrace{\frac{6t}{\sin(6t)}}_{\to 1} \cdot \frac{10}{6} = \frac{10}{6} = \boxed{5/3}$$

where in the last equality we used the limit $\lim_{\theta\to 0} \dfrac{\sin(\theta)}{\theta} = 1$.

(a)(ii) First note this is yields the indeterminate form " 0/0", looking at one sided limits yields

$$\lim_{x\to 3^-} \frac{x-3}{|x-3|} = \lim_{x\to 3^-} \frac{x-3}{-(x-3)} = -1, \text{ and, } \lim_{x\to 3^+} \frac{x-3}{|x-3|} = \lim_{x\to 3^+} \frac{x-3}{(x-3)} = 1$$

so the limit from the left is not equal to the limit from the right therefore the two sided limit does not exist, i.e. $\boxed{\lim_{x\to 3} \dfrac{x-3}{|x-3|} = \text{d.n.e}}$

(a)(iii) Applying the limit, we get the form " number/zero", so checking the sign of the function as x approaches 1 yields,

$$\lim_{x\to -1^-} \frac{x-1}{x^4-1} = \boxed{-\infty}$$

since the expression $\dfrac{x-1}{x^4-1}$ is negative for $x < -1$.

(b) Note that $x = -2$ makes the denominator zero and is therefore a root of the polynomial $x^2 + x - 2$, now if $x = -2$ were also a root of the polynomial in the numerator then these roots would cancel and the limit would exist (there would be a hole in the graph), so we need the numerator to also be zero when $x = -2$, *i.e.* we need

$$2(-2)^2 + 2c(-2) + c - 6 = 0 \Longrightarrow 8 + (-4)c + c - 6 \Longrightarrow 2 - 3c = 0 \Longrightarrow \boxed{c = 2/3}$$

Not part of the solution, but just FYI: when $c = 2/3$ we get

$$\lim_{x\to -2} \frac{2x^2+2cx+c-6}{x^2+x-2} = \lim_{x\to -2} \frac{2x^2+\frac{4}{3}x-\frac{16}{3}}{x^2+x-2} = \lim_{x\to -2} \frac{(x+2)(2x-8/3)}{(x+2)(x-1)} = \frac{20}{9}$$
