Solutions

Math 1300-005 - Spring 2017

Derivatives and the Shapes of Curves, Pt. I - 3/22/17

Guidelines: Please work in groups of two or three. This will not be handed in, but is a study resource for Midterm 3.

1. Consider the function $g(x)$ and its first and second derivatives.

$$g(x) = \frac{x^2}{(x-2)^2}, \qquad g'(x) = \frac{-4x}{(x-2)^3}, \qquad g''(x) = \frac{8(x+1)}{(x-2)^4}$$

(a) Find the x-intercept(s) of g, if any. Find the y-intercept(s) of g, if any.

x-int: $0 = \frac{x^2}{(x-2)^2} \iff x^2 = 0$, so $x = 0 \to (0,0)$

y-int: $y = \frac{0^2}{(0-2)^2} = 0$, so $y = 0 \to (0,0)$

same x and y intercept

(b) Find the vertical asymptote(s) of g, if any. Find the horizontal asymptote(s) of g, if any

VA when denom $= 0$, so $x = 2$,

HA: $\lim_{x\to\infty} \frac{x^2}{(x-2)^2} \approx \lim_{x\to\infty} \frac{x^2}{x^2} = 1$, $\lim_{x\to-\infty} \frac{x^2}{(x-2)^2} \approx \lim_{x\to-\infty} \frac{x^2}{x^2} = 1$, so $y = 1$ is HA

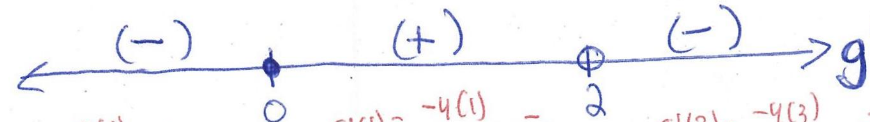
(c) Find all values of x such that $g'(x) = 0$ ***AND*** all values of x such that the denominator of g' is zero. Which of these x-values are critical numbers?

$g'(x) = 0$ when $-4x = 0$, so at $x = 0$.

Denom of $g' = 0$ when $(x-2)^3 = 0$, so $x = 2$.

$x = 0$ is critical since it is in the domain of g.

(d) Plot *all* values from (c) on a sign chart for g'. If an x-value is critical, place it on the sign chart with a solid dot. If an x-value is not critical, place it on the sign chart with an open dot. Fill in your sign chart using test points.



$g'(-1) = \frac{-4(-1)}{(-1-2)^3} = \frac{+}{-} = -$ $\qquad g'(1) = \frac{-4(1)}{(1-2)^3} = \frac{-}{-} = +$ $\qquad g'(3) = \frac{-4(3)}{(3-2)^3} = \frac{-}{+} = -$

(e) Find the intervals of increase or decrease. Justify your answer.

increasing $(0,2)$ since $g' > 0$

decreasing $(-\infty, 0) \cup (2, \infty)$ since $g' < 0$

Remember to justify

(f) When finding local extrema, we only consider numbers on our sign chart for g' with a solid dot...why? Find the x-coordinates of the local maximum and minimum values. Justify your answer.

local min at $x = 0$ since g' goes $(-)$ to $(+)$.

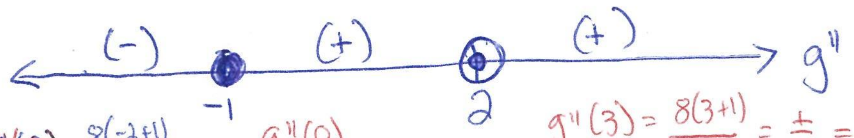
No local max ($x = 2$ is not in domain of g, so can't be a local max)

(g) Find all values of x such that $g''(x) = 0$ ***AND*** all values of x such that the denominator of g'' is zero.

$g''=0$ when $8(x+1)=0$, so $x=-1$ (in domain of g)

denom of $g''=0$ when $(x-2)^4=0$, so $x=2$. (not in domain of g)

(h) Plot all values from (g) on a sign chart for g''. If an x-value is in the domain of g, place it on the sign chart with a solid dot. If an x-value is not in the domain of g, place it on the sign chart with an open dot. Fill in your sign chart using test points.



$g''(-2) = \frac{8(-2+1)}{(-2-2)^4} = \frac{-}{+} = -$

$g''(0) = \frac{8(0+1)}{(0-2)^4} = \frac{+}{+} = +$

$g''(3) = \frac{8(3+1)}{(3-2)^4} = \frac{+}{+} = +$

(i) Find the intervals of concavity. Justify your answer.

Concave down $(-\infty, -1)$ since $g''<0$

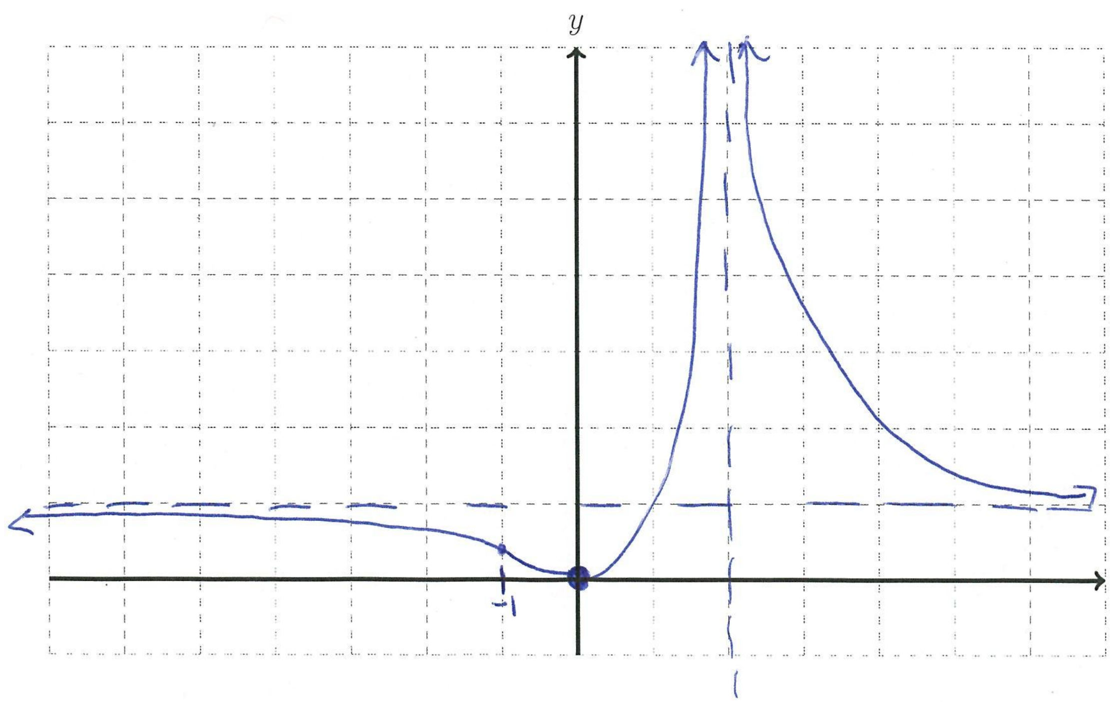
concave up $(-1,2)\cup(2,\infty)$ since $g''>0$.

(j) When finding inflection points, we only consider numbers on our sign chart for g'' with a solid dot...why? Find the x-coordinates of any inflection points. Justify your answer.

~~No inflection~~ Inflection point at $x=-1$ since $x=-1$ has a solid dot and g'' switches (-) to (+).

[At $x=-1$, $g(x) = \frac{(-1)^2}{(-1-2)^2} = \frac{1}{9}$] → so $(-1, \frac{1}{9})$ is inflection

(k) Using all the information from parts (a) through (j), sketch a graph of $g(x)$ below.



Check that I have all the correct features

2. Consider the function $f(x)$ and its first and second derivatives.

$$f(x) = x^{2/3}(3-x)^{1/3}, \qquad f'(x) = \frac{2-x}{x^{1/3}(3-x)^{2/3}}, \qquad f''(x) = \frac{-2}{x^{4/3}(3-x)^{5/3}}$$

(a) Find the x-intercept(s) of f, if any. Find the y-intercept(s) of f, if any.

X-int: $0 = x^{2/3}(3-x)^{1/3}$, so $x=0$ and $x=3$. The points are $(0,0)$ and $(3,0)$

y-int $y = 0^{2/3}(3-0)^{1/3} = 0$. The point is $(0,0)$.

(b) Find the vertical asymptote(s) of f, if any. Find the horizontal asymptote(s) of f, if any.

No denominator for $f(x)$, so no VA.

$\lim_{x\to\infty} x^{2/3}(3-x)^{1/3} = (\infty)(-\infty) = -\infty$, so the function eventually heads to $-\infty$ as $x \to \infty$

$\lim_{x\to-\infty} x^{2/3}(3-x)^{1/3} = (\infty)(\infty) = \infty$, so the function eventually heads to ∞ as $x\to-\infty$.

No HA.

(c) Find all values of x such that $f'(x) = 0$ ***AND*** all values of x such that the denominator of f' is zero. Which of these x-values are critical numbers?

$f'=0$ when $2-x=0$, so $x=2$.
Denom of $f'=0$ when $x=0$, $x=3$.
} all points are in the domain of f, so all get solid dots.

Since f' DNE at 0 and 3, these x-values are cusps or vertical tangents

(d) Plot *all* values from (c) on a sign chart for f'. If an x-value is critical, place it on the sign chart with a solid dot. If an x-value is not critical, place it on the sign chart with an open dot. Fill in your sign chart using test points.

(−) 0 (+) 2 (−) 3 (−)

$f'(-1) = \frac{2-(-1)}{(-1)^{1/3}(3+1)^{2/3}} = \frac{+}{(-)(+)} = (-)$

$f'(1) = \frac{2-1}{(1)^{1/3}(3-1)^{2/3}} = \frac{+}{(+)(+)} = +$

$f'(\frac{5}{2}) = \frac{2-\frac{5}{2}}{(\frac{5}{2})^{1/3}(3-\frac{5}{2})^{2/3}} = \frac{-}{(+)(+)} = (-)$

$f'(4) = \frac{2-4}{(4)^{1/3}(3-4)^{2/3}} = \frac{-}{(+)(+)} = -$

(e) Find the intervals of increase or decrease. Justify your answer.

f increases $(0,2)$ since $f'>0$ there.

f decreases $(-\infty,0)\cup(2,3)\cup(3,\infty)$ since $f'<0$ there.

(f) When finding local extrema, we only consider numbers on our sign chart for f' with a solid dot...why? Find the x-coordinates of the local maximum and minimum values. Justify your answer.

f has a local min at $x=0$, and since $f'(0)$ DNE, this will be a cusp ✓. The point is $(0,0)$

f has a local max at $x=2$. $f'(2)=0$ so this is a normal, smooth maximum. The point is $(2, 2^{2/3})$ $\approx(2,1.5)$

No max or min at $x=3$, but since $f'(3)$ DNE, it will be a vertical tangent.

(g) Find all values of x such that $f''(x) = 0$ ***AND*** all values of x such that the denominator of f'' is zero.

$f'' \neq 0$ since $-2 \neq 0$

f'' DNE at $x=0$, $x=3$ and both are in domain of f. So solid dots.

(h) Plot all values from (g) on a sign chart for f''. If an x-value is in the domain of f, place it on the sign chart with a solid dot. If an x-value is not in the domain of f, place it on the sign chart with an open dot. Fill in your sign chart using test points.

f'': $(-)$ | 0 | $(-)$ | 3 | $(+)$

$f''(-1) = \dfrac{-2}{(-1)^{4/3}(3-(-1))^{5/3}} = \dfrac{(-)}{(+)(+)} = (-)$

$f''(1) = \dfrac{-2}{(1)^{4/3}(3-1)^{5/3}} = \dfrac{(-)}{(+)(+)} = (-)$

$f''(4) = \dfrac{-2}{(4)^{4/3}(3-4)^{5/3}} = \dfrac{(-)}{(+)(-)} = (+)$

(i) Find the intervals of concavity. Justify your answer.

f is concave down $(-\infty, 0) \cup (0, 3)$ since $f'' < 0$

f is concave up $(3, \infty)$ since $f'' > 0$.

(j) When finding inflection points, we only consider numbers on our sign chart for f'' with a solid dot...why? Find the x-coordinates of any inflection points. Justify your answer.

Inflection at $x=3$. The actual point is $(3,0)$

(k) Using all the information from parts (a) through (j), sketch a graph of $f(x)$ below.