1. The following parts are **not related**:

   (a) (12 pts) Find y' given:

$$(i)\ y = \frac{x}{\sqrt{1-x}} \qquad (ii)\ y = \sec(x^2-1) - x\tan(x) \qquad (iii)\ (x^2+y^2)^2 = 2x^2 - 2y^2$$

   (b) (8 pts) Let $f(x)$ be a function such that $f(2x+1) - xf(2x) = 0$ and $f'(1) = f''(1) = 4$. Find $f''(2)$.

   Solution:

   (a) (i) Using the quotient rule we have,

$$\frac{d}{dx}\left[\frac{x}{\sqrt{1-x}}\right] = \frac{1\cdot\sqrt{1-x} - x\cdot[(1/2)(1-x)^{-1/2}(-1)]}{(\sqrt{1-x})^2} = \frac{(1-x)^{1/2} + \dfrac{x(1-x)^{-1/2}}{2}}{1-x}$$

   so factoring out the $\dfrac{(1-x)^{-1/2}}{2}$ term from the numerator yields

$$\frac{d}{dx}\left[\frac{x}{\sqrt{1-x}}\right] = \frac{(1-x)^{-1/2}[2(1-x)+x]}{2(1-x)} = \boxed{\frac{2-x}{2(1-x)^{3/2}}}$$

   (a) (ii) Taking the derivative yields,

$$[\sec(x^2-1) - x\tan(x)]' = \sec(x^2-1)\tan(x^2-1)(2x) - [1\cdot\tan(x) + x\sec^2(x)]$$

   so, simplifying yields

$$[\sec(x^2-1) - x\tan(x)]' = \boxed{2x\sec(x^2-1)\tan(x^2-1) - \tan(x) - x\sec^2(x)}$$

   (a) (iii) Using implicit differentiation, we have

$$\frac{d}{dx}\left[(x^2+y^2)^2 = 2x^2 - 2y^2\right] \Longrightarrow 2(x^2+y^2)(2x+2yy') = 4x - 4yy'$$

   Now distributing the $2(x^2+y^2)$ term and collecting all the derivatives on one side yields

$$\begin{aligned} 2(x^2+y^2)(2yy') + 4yy' = 4x - 2(x^2+y^2)(2x) \quad &\Rightarrow \quad (x^2+y^2)(4yy') + 4yy' = 4x - 4x(x^2+y^2) \\ &\Rightarrow \quad (1+x^2+y^2)4yy' = 4x(1-x^2-y^2) \end{aligned}$$

   and so,

$$y' = \frac{4x(1-x^2-y^2)}{4y(1+x^2+y^2)} = \boxed{\frac{x(1-x^2-y^2)}{y(1+x^2+y^2)}}$$

   (b) First we take the derivative implicitly,

$$\frac{d}{dx}\left[f(2x+1) - xf(2x) = 0\right] \Rightarrow 2f'(2x+1) - f(2x) - 2xf'(2x) = 0$$

   The second derivative will be $4f''(2x+1) - 4f'(2x) - 4xf''(2x) = 0$. Now, let $x = 1/2$, we find

$$4f''(2) - 4f'(1) - 2f''(1) = 0$$

   And thus $\boxed{f''(2) = 6}$

2. The following parts are **<u>not related</u>**:

(a) (10 pts) When a certain polyatomic gas undergoes adiabatic expansion, its pressure, P, and volume, V, satisfy the equation $PV^{1.3} = k$, where k is a constant. Find the relationship between the related rates dP/dt and dV/dt. (In other words find dP/dt in terms of dV/dt, P, and V.)

(b) (10 pts) A particle is moving to the left along the line $y = 3$. When $x = -4$, the x-coordinate of the particle's position is decreasing at the rate of 0.5 cm/sec. At what rate is the distance of the particle from the origin changing at this moment?

Solution: (a) Method 1: If we assume k is a nonzero constant, then $V \neq 0$ and solving for P explicitly yields $P = kV^{-1.3}$, thus $\boxed{\dfrac{dP}{dt} = -1.3kV^{-2.3}\dfrac{dV}{dt}.}$

Method 2: Note that differentiating $PV^{1.3} = k$ yields

$$V^{1.3}\frac{dP}{dt} + 1.3V^{0.3}P\frac{dV}{dt} = 0 \Longrightarrow V^{1.3}\frac{dP}{dt} = -1.3V^{0.3}P\frac{dV}{dt}$$

solving for dP/dt yields

$$\frac{dP}{dt} = -1.3\frac{V^{0.3}}{V^{1.3}}P\frac{dV}{dt} = -1.3V^{-1}P\frac{dV}{dt} = -1.3\frac{P}{V}\frac{dV}{dt} \Rightarrow \boxed{\frac{dP}{dt} = -1.3\frac{P}{V}\frac{dV}{dt}}$$

$\boxed{\text{Note:}}$ In the solution from Method 1, if we substitute $k = PV^{1.3}$ we get the solution from Method 2:

$$\frac{dP}{dt} = -1.3kV^{-2.3}\frac{dV}{dt} = -1.3(PV^{1.3})V^{-2.3}\frac{dV}{dt} = -1.3PV^{-1}\frac{dV}{dt} = -1.3\frac{P}{V}\frac{dV}{dt}$$

(b) Note that the distance of any point on the line $y = 3$ from the origin is given by $D = \sqrt{x^2 + 9}$, taking the derivative of both sides with respect to t yields

$$\frac{dD}{dt} = \frac{1}{2}(x^2 + 9)^{-1/2} \cdot 2x \cdot \frac{dx}{dt} = \frac{x}{\sqrt{x^2 + 9}} \cdot \frac{dx}{dt}$$

substituting $x = -4$ and $dx/dt = -0.5$ cm/sec yields

$$\frac{dD}{dt} = \frac{-4}{\sqrt{(-4)^2 + 9}} \cdot \left(-\frac{1}{2}\right) = -\frac{4}{5} \cdot \left(-\frac{1}{2}\right) = \boxed{\frac{2}{5}\text{ cm/sec}}$$

that is, the distance is increasing at 0.4 cm/sec when $x = -4$.

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3. The following parts are **<u>not related</u>**:

(a) (10 pts) One leg of a right triangle is known to have length 4 cm. The other leg is measured to be 3 cm with a maximum error of $+0.1$ cm. Use differentials to estimate the maximum error in the calculation of the angle θ between the measured leg and the hypotenuse.

(b) (10 pts) If possible, find all numbers c in the interval $[0.5, 2]$ that satisfy the conclusion of the Mean Value Theorem for the function $f(x) = \dfrac{x+1}{x}$, justify your answer.

Solution:

(a) We have a number of relationships between the side lengths and the angle θ, but only one that uses the lengths of the known leg and the one with measured error, namely

$$\tan\theta = \frac{4}{x}$$

Taking derivatives we have

$$\sec^2\theta \, d\theta = -\frac{4}{x^2}\, dx$$

Solving for $d\theta$ we have

$$d\theta = -\cos^2\theta \frac{4}{x^2} dx$$

Using the fact that the triangle, as measured, is a 3-4-5 triangle we have that $\cos(\theta) = 3/5$ which implies that $\cos^2\theta = 9/25$. Plugging in the values for x and dx we have

$$d\theta = -\frac{9}{25}\cdot\frac{4}{9}\cdot 0.1 = \boxed{-\frac{4}{250} = -\frac{16}{1000} = -0.016}$$

(b) The function f is continuous and differentiable at every point except $x = 0$, which is not in the interval, the MVT can be applied. We need to find the point c such that

$$f'(c) = \frac{f(2) - f(1/2)}{2 - (1/2)}$$

We first write f as $f(x) = 1 + x^{-1}$. Then $f'(x) = \frac{-1}{x^2}$. We have

$$\frac{-1}{c^2} = \frac{1 + 1/2 - (1+2)}{2 - 1/2} \implies \frac{-1}{c^2} = -1 \implies c^2 = 1 \implies c = \pm 1$$

Since only $\boxed{c = 1}$ is in $[\frac{1}{2}, 2]$, it is the point guaranteed to exist by the MVT.

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4. The following parts are **<u>not related</u>**:

(a) (10 pts) Find all local extrema of $y = x\sqrt{1-x^4}$ on the interval $(-1, 1)$, justify your answer without graphing the function. (Just give the x-value of the extrema, if any.)

(b) (10 pts) In your blue book clearly sketch the graph of a function $f(x)$ that satisfies the following properties (label any <u>extrema</u>, <u>inflection points</u> or <u>asymptotes</u>):

- $f'(x) > 0$ if $|x| < 2$
- $f'(x) < 0$ if $|x| > 2$
- $f'(-2) = 0$
- $\lim_{x\to 2} |f'(x)| = +\infty$
- $f''(x) > 0$ if $x \neq 2$

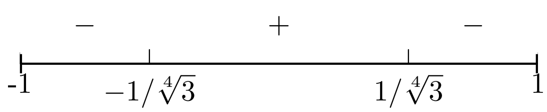
Solution:

(a) Note that

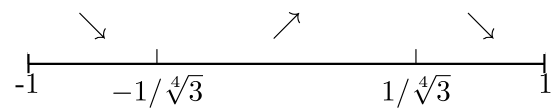
$$f'(x) = \sqrt{1-x^4} + x \cdot \frac{1}{2}(1-x^4)^{-1/2} \cdot (-4x^3) = \frac{1-x^4-2x^4}{\sqrt{1-x^4}} = \frac{1-3x^4}{\sqrt{1-x^4}}$$

and so $f'(x) = 0$ when $x = \pm\frac{1}{\sqrt[4]{3}}$ and $f'(x)$ is undefined when $x = \pm 1$ but these points are not in the domain. So $x = \pm\frac{1}{\sqrt[4]{3}}$ are the critical points of $f(x)$. **We now need to determine if they are local extrema or just critical points.** So we consider the sign chart of $f'(x) = \frac{1-3x^4}{\sqrt{1-x^4}}$,

Sign Chart for $f'(x)$



Increasing/Decreasing behavior for $f(x)$



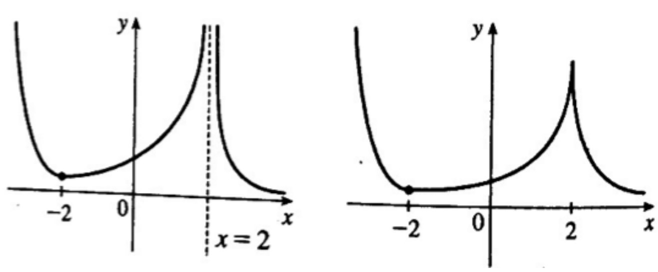
so $\boxed{x = -1/\sqrt[4]{3} \text{ is a local minimum}}$ and $\boxed{x = 1/\sqrt[4]{3} \text{ is a local maximum}}$.

Note: One could also use the 2nd Derivative test to determine whether or not the critical points of $f(x)$ are local extrema or not.

(b) Note that

- $f'(x) > 0$ if $|x| < 2 \Longrightarrow f$ is increasing on $(-2, 2)$
- $f'(x) < 0$ if $|x| > 2 \Longrightarrow f$ is decreasing on $(-\infty, -2)$ and $(2, \infty)$
- $f'(-2) = 0 \Longrightarrow$ horizontal tangent at $x = -2$
- $\lim_{x\to 2} |f'(x)| = +\infty \Longrightarrow$ there is a vertical asymptote or vertical tangent (cusp) at $x = 2$
- $f''(x) > 0$ if $x \neq 2 \Longrightarrow f$ is concave up on $(-\infty, 2)$ and $(2, \infty)$

implies that the graph could, for example, look like either of the following graphs below,



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5. (20 pts, 4 ea.) Answer either <u>**Always True**</u> or <u>**False**</u>. Do <u>*NOT*</u> justify your answer. Do <u>*NOT*</u> abbreviate your answer.

   (a) Using a linearization of $\sqrt{x}$ at $a = 4$, we can estimate $\sqrt{3.6} \approx 1.06$.

   (b) If $f(x)$ is continuous on (a, b), then $f(x)$ attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some points c and d in (a, b).

   (c) If $m(x)$ is differentiable for all x then $m'(x)$ is continuous.

(d) If $k(x) = \lim_{t \to x} \dfrac{\sec(t) - \sec(x)}{t - x}$, then $k(\pi/4) = \sqrt{2}$

(e) If $b(x)$ is differentiable and $b(-1) = b(1)$, then there is a number c such that $|c| < 1$ and $b'(c) = 0$.

Solution:

(a) F (b) F (c) F (d) A.T. (e) A.T.

Comments:

(a) Linearizing $\sqrt{x}$ at $a = 4$, yields $L(x) = 2 + \frac{1}{4}(x - 4)$, thus $\sqrt{3.6} \approx 1.9$

(b) Consider $f(x) = x$ on the interval $(0, 1)$, then $f(x)$ does not attain an absolute max or min in $(0, 1)$.

(c) Consider $m(x) = \begin{cases} x^2 \sin(1/x), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ then $m'(x)$ exists for all x but is not continuous at $x = 0$.

(d) Note $\lim_{t \to x} \dfrac{\sec(t) - \sec(x)}{t - x} = \dfrac{d}{dt}[\sec(t)]\Big|_{t=x} = \sec(x)\tan(x) = f(x)$, so $k(\pi/4) = \sec(\pi/4)\tan(\pi/4) = \sqrt{2}$

(e) Use the Mean Value Theorem on $b(x)$ over the interval $[-1, 1]$.