1. Consider the function $h(x) = \dfrac{4x}{\sqrt{x^2 - 25}}$

   (a) (4 points) Give the domain of this function in interval notation.

   (b) (6 points) Use the appropriate limits to identify any vertical asymptotes. If none exist, write "None" and explain why.

   (c) (6 points) Use the appropriate limits to identify any horizontal asymptotes. If none exist, write "None" and explain why.

   (d) (4 points) Is the function's symmetry even, odd, or neither?

   Solution:

   (a) The expression under the radical needs to be positive: $x^2 - 25 > 0 \;\Rightarrow\; x < -5$ or $x > 5$. The domain is $\boxed{(-\infty, -5) \cup (5, \infty)}$.

   (b) There are possible vertical asymptotes where the denominator equals 0 at $x = -5, 5$.
   $\lim\limits_{x \to 5^+} \dfrac{4x}{\sqrt{x^2 - 25}} = \infty$ since $4x \to 20$ and $\sqrt{x^2 - 25} \to 0$ with positive values.
   $\lim\limits_{x \to -5^-} \dfrac{4x}{\sqrt{x^2 - 25}} = -\infty$ since $4x \to -20$ and $\sqrt{x^2 - 25} \to 0$ with positive values.
   There are vertical asymptotes at $\boxed{x = 5}$ and $\boxed{x = -5}$.

   Grading Comments: Observe that the limit of $h(x)$ is undefined as $x \to 5^-$ and $x \to -5^+$.)

   (c) Note that $\sqrt{x^2} = x$ for $x \geq 0$ and $\sqrt{x^2} = -x$ for $x < 0$.

   $$\lim_{x \to \infty} \frac{4x}{\sqrt{x^2 - 25}} = \lim_{x \to \infty} \frac{4x}{\sqrt{x^2}\sqrt{1 - 25/x^2}} = \lim_{x \to \infty} \frac{4\cancel{x}}{\cancel{x}\sqrt{1 - 25/x^2}} = \frac{4}{\sqrt{1 - 0}} = 4$$
   $$\lim_{x \to -\infty} \frac{4x}{\sqrt{x^2 - 25}} = \lim_{x \to -\infty} \frac{4x}{\sqrt{x^2}\sqrt{1 - 25/x^2}} = \lim_{x \to -\infty} \frac{4\cancel{x}}{-\cancel{x}\sqrt{1 - 25/x^2}} = \frac{-4}{\sqrt{1 - 0}} = -4$$

   There are horizontal asymptotes at $\boxed{y = 4}$ and $\boxed{y = -4}$.

   (d) Since $h(-x) = \dfrac{-4x}{\sqrt{x^2 - 25}} = -h(x)$, then h is $\boxed{\text{odd}}$.

2. (21 points, 7 points each) Evaluate the following limits

   (a) $\lim\limits_{t \to 0} \left(\dfrac{3}{t} - \dfrac{3}{t^2 + t} \right)$

   (b) $\lim\limits_{x \to 0} \dfrac{|\sin x|}{\sin x}$

   (c) $\lim\limits_{\theta \to 0} \dfrac{\sin \theta}{\theta + \tan \theta}$

   Solution:

(a) (HW 1.4.24) $\lim_{t\to 0}\left(\frac{3}{t}-\frac{3}{t^2+t}\right)=\lim_{t\to 0}\frac{3(t+1)-3}{t(t+1)}=\lim_{t\to 0}\frac{3\cancel{t}}{\cancel{t}(t+1)}=\boxed{3}$

(b) Note that $|\sin x| = \begin{cases}\sin x, & 0\le x\le \pi\\ -\sin x, & -\pi\le x\le 0.\end{cases}$

$\lim_{x\to 0^+}\frac{|\sin x|}{\sin x}=\lim_{x\to 0^+}\frac{\sin x}{\sin x}=1$

$\lim_{x\to 0^-}\frac{|\sin x|}{\sin x}=\lim_{x\to 0^-}\frac{-\sin x}{\sin x}=-1$

$\lim_{x\to 0}\frac{|\sin x|}{\sin x}$ is $\boxed{\text{undefined}}$ since the left-hand limit and right-hand limit are not equal.

(c) (HW 1.4.55) $\lim_{\theta\to 0}\frac{\sin\theta}{\theta+\tan\theta}=\lim_{\theta\to 0}\frac{\sin\theta}{\theta+\frac{\sin\theta}{\cos\theta}}\cdot\frac{\frac{1}{\theta}}{\frac{1}{\theta}}=\lim_{\theta\to 0}\frac{\frac{\sin\theta}{\theta}}{1+\frac{\sin\theta}{\theta}\cdot\frac{1}{\cos\theta}}=\frac{1}{1+1\cdot 1}=\boxed{\frac{1}{2}}$

Grading Comments: The fraction $\frac{\sin\theta}{\theta+\tan\theta}$ <u>does not</u> equal $\frac{\sin\theta}{\theta}+\frac{\sin\theta}{\tan\theta}$. It is true that $\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}$. It is <u>not true</u> that $\frac{a}{b+c}$ equals $\frac{a}{b}+\frac{a}{c}$.

3. Consider the function $g(x)=\frac{\sqrt{x}-\sqrt{5}}{x^2-6x+5}$.

(a) (6 points) Give the domain of this function in interval notation.

(b) (6 points) Evaluate $\lim_{x\to 5} g(x)$.

(c) (8 points) The function g has a removable discontinuity at $x=a$. The discontinuity can be removed by creating a new function $h(x)$.

$$h(x)=\begin{cases} g(x) & x\neq a\\ b & x=a\end{cases}$$

Use the definition of continuity of a function to find the values of the constants a and b.

Solution:

(a) The domain of $\sqrt{x}$ is $[0,\infty)$. The function g is undefined when the denominator $x^2-6x+5=(x-1)(x-5)$ equals zero at $x=1,5$. The domain is therefore $\boxed{[0,1)\cup(1,5)\cup(5,\infty)}$.

Grading Comments: When finding the domain of the function, the square root expression must be considered. For this problem that means x cannot be negative.

(b) $\lim_{x\to 5}\frac{\sqrt{x}-\sqrt{5}}{x^2-6x+5}\cdot\frac{\sqrt{x}+\sqrt{5}}{\sqrt{x}+\sqrt{5}}=\lim_{x\to 5}\frac{\cancel{x-5}}{(x-1)\cancel{(x-5)}\left(\sqrt{x}+\sqrt{5}\right)}=\frac{1}{4(2\sqrt{5})}=\boxed{\frac{1}{8\sqrt{5}}}$

(c) Since $g(5)$ is undefined and $\lim_{x\to 5} g(x)$ exists, the function g has a removable discontinuity at $x=5$.

The function h is continuous at a if $h(a)=\lim_{x\to a}h(x)$. Let $\boxed{a=5, b=\frac{1}{8\sqrt{5}}}$. Then $h(5)=\lim_{x\to 5}h(x)=\lim_{x\to 5}g(x)=\frac{1}{8\sqrt{5}}$.

Grading Comments: Note that the problem asked for the definition of continuity. A solution that lacked the definition of continuity was considered incomplete.

4. (15 points) Let $g(x) = \dfrac{x-1}{x+1}$.

(a) Use the definition of derivative to find the slope of the tangent line to $y = g(x)$ at $x = 0$.

(b) Find the equation of the tangent line to $y = g(x)$ at $x = 0$.

(c) Find the equation of the normal line to $y = g(x)$ at $x = 0$.

Solution:

(a)

$$\begin{aligned}
g'(a) &= \lim_{h\to 0} \frac{g(a+h)-g(a)}{h} \\
g'(0) &= \lim_{h\to 0} \frac{g(h)-g(0)}{h} = \lim_{h\to 0} \frac{\frac{h-1}{h+1}-(-1)}{h} = \lim_{h\to 0} \frac{1}{h}\cdot\frac{h-\cancel{1}+h+\cancel{1}}{h+1} \\
&= \lim_{h\to 0} \frac{2\cancel{h}}{\cancel{h}(h+1)} = \boxed{2}
\end{aligned}$$

Alternate Solution.

$$\begin{aligned}
g'(a) &= \lim_{h\to 0} \frac{g(a+h)-g(a)}{h} \\
&= \lim_{h\to 0} \frac{\frac{a+h-1}{a+h+1}-\frac{a-1}{a+1}}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\cdot\frac{(a+h-1)(a+1)-(a-1)(a+h+1)}{(a+h+1)(a+1)} \\
&= \lim_{h\to 0} \frac{1}{h}\cdot\frac{(\cancel{a^2}+\cancel{ah}-\cancel{a}+\cancel{a}+h-\cancel{1})-(\cancel{a^2}-\cancel{a}+\cancel{ah}-h+\cancel{a}-\cancel{1})}{(a+h+1)(a+1)} \\
&= \lim_{h\to 0} \frac{1}{\cancel{h}}\cdot\frac{2\cancel{h}}{(a+h+1)(a+1)} \\
&= \frac{2}{(a+1)^2} \\
g'(0) &= \boxed{2}
\end{aligned}$$

(b) At $x = 0,\ y(0) = -1$ and $y'(0) = 2$. Using point-slope form, the tangent line is

$$y - y_1 = m(x - x_1) \ \Rightarrow\ \boxed{y + 1 = 2x} \text{ or } y = 2x - 1.$$

(c) The normal line passes through the same point and has slope $-1/2$. The equation of the normal line is

$$\boxed{y + 1 = -\frac{1}{2}x} \text{ or } y = -\frac{1}{2}x - 1.$$

5. (24 points, 6 points each) Some unrelated, short answer questions.

(a) (HW 2.1.31) The limit $\displaystyle\lim_{h\to 0} \frac{(1+h)^{10}-1}{h}$ represents the derivative of some function f at some number a.

(i) Find a function f and a value for a. (ii) What is the value of the limit?

(b) Does $x + \tan x = 1$ have a solution? Justify your answer.

(c) Sometimes a function f is not continuous on its domain but $|f|$ is continuous, on the same domain. Find an example of such a function f (i.e. f is not continuous at a point in its domain but $|f|$ is). Either sketch the graph of both $|f|$ and f or find a formula that illustrates this.

(d) A factory manufactures metal cubes of volume $V = 8000\ \text{cm}^3$. An error tolerance of $\pm 5\ \text{cm}^3$ is allowed, which corresponds to a side length s between 19.996 and 20.004 cm. In terms of the formal definition of $\lim_{x \to a} f(x) = L$, identify $x, a, f(x), L, \delta$, and ϵ. No further explanation is necessary for this problem.

Solution:

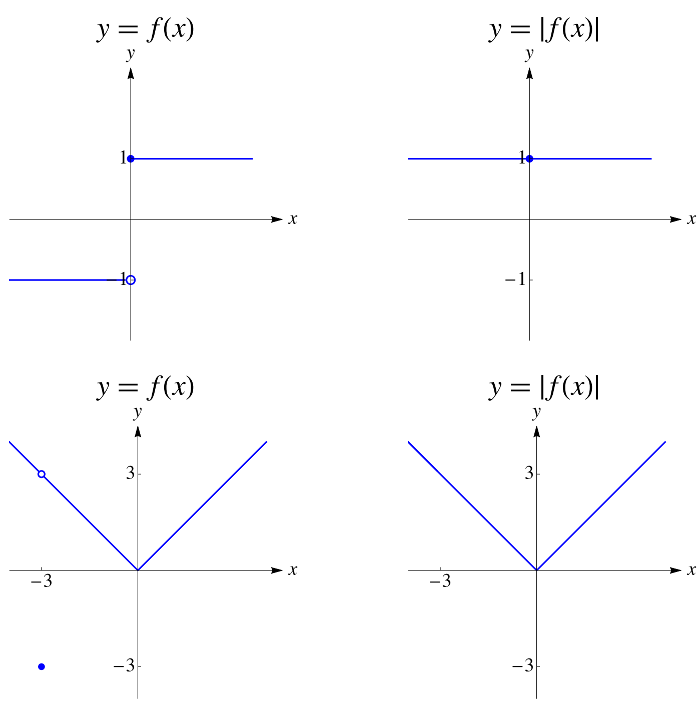
(a) (HW 2.1.31) (i) $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \dfrac{(1+h)^{10} - 1}{h}$. One possible solution is $\boxed{f(x) = x^{10}}$ and $\boxed{a = 1}$.

(ii) Since $f'(x) = 10x^9$, the limit value equals $f'(1) = \boxed{10}$.

(b) Let $f(x) = x + \tan x - 1$. Then $f(0) = 0 + 0 - 1 = -1 < 0$ and $f(\frac{\pi}{4}) = \frac{\pi}{4} + 1 - 1 = \frac{\pi}{4} > 0$. By the Intermediate Value Theorem, since f is continuous in the interval $[0, \frac{\pi}{4}]$, there is a value of c in the interval $\left(0, \frac{\pi}{4}\right)$ such that $f(c) = 0$.

Grading comments: The domain of $\tan(x)$ is not $\mathbb{R}$. Indeed, $\tan x$ is not defined for $\pi/2 + n\pi$, with n an integer. Note that the IVT can be applied only on a *closed* interval included in the domain of the function. The function must be continuous over the whole interval: hence it is not correct to use the IVT on $[0, \pi]$ as this interval contains $\frac{\pi}{2}$, where $\tan x$ is discontinuous.

(c) Below are two possible solutions.



Grading comments: Note that the graph of a piecewise function must clearly indicate what the value of the function is at any jump discontinuity by using clearly distinguishable *open* and *closed* circles. Two closed circles corresponding to one value of x does not result in a function.

(d) The formal definition of limit states that $\lim_{x \to a} f(x) = L$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$. For this problem, the function is $V(s) = s^3$. Since $V(20) = 8000$, the corresponding limit is $\lim_{s \to 20} V(s) = 8000$ so $\boxed{x = s, a = 20, f(x) = V(s)}$, and $\boxed{L = 8000}$. For $\boxed{\epsilon = 5}$, the value of δ is $|20.004 - 20| = |20 - 19.996|$ or $\boxed{\delta = 0.004}$.

Grading comments: The values for ε and δ are positive quantities, by definition. Do not include any $\pm$ symbol.