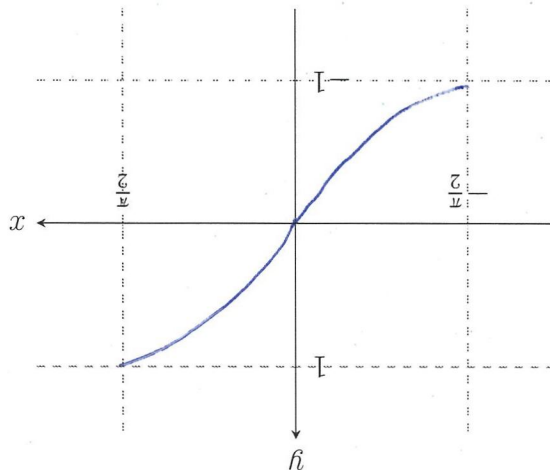


*Guidelines:* Please work in groups of two or three. This will not be handed in, but is a study resource for Midterm 2.

The goal of this worksheet is to discover the derivatives of arctangent, arcsine, and arccosine using implicit differentiation.

### 1. The Derivative of Arcsine

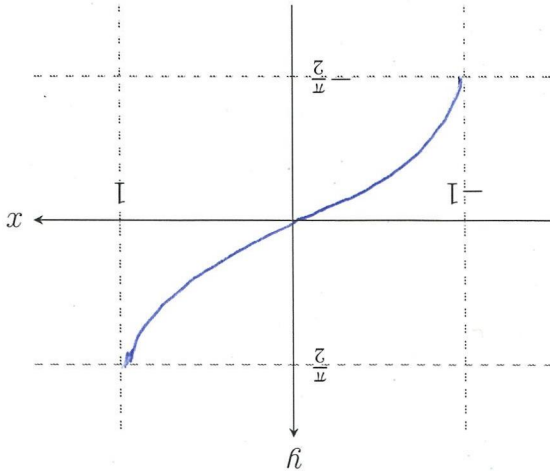
(a) On the axes below, sketch a graph of  $y = \sin(x)$  from  $-\pi/2 < x < \pi/2$ .



(b) Why does  $f(x) = \sin(x)$ , when restricted to  $-\pi/2 < x < \pi/2$ , have an inverse?

on  $-\pi/2 < x < \pi/2$ ,  $\sin(x)$  is one-to-one.

(c) Sketch the graph of the inverse  $y = f^{-1}(x) = \arcsin(x)$  on the axes below.



(d) By properties of inverse functions, we have the following identity.

$$\sin(\arcsin(x)) = x$$

Differentiate both sides of this equation to find a formula for the derivative of  $\arcsin(x)$ . Express your answer in terms of  $\cos(\arcsin(x))$ .

$$\frac{d}{dx} \sin(\arcsin(x)) = \frac{d}{dx} x$$

$$\cos(\arcsin(x)) \cdot \frac{d}{dx} \arcsin(x) = 1$$

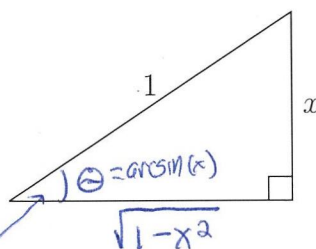
$$\boxed{\frac{d}{dx} \arcsin(x) = \frac{1}{\cos(\arcsin(x))}}$$

(e) Referring to the triangle below, explain why  $\cos(\arcsin(x)) = \sqrt{1-x^2}$ .

Let  $\arcsin(x) = \theta$ .

Then  $\sin(\theta) = \frac{x}{1} = \frac{\text{opp}}{\text{hyp}}$ ,

so  $\theta$  is



Then  $\cos(\arcsin(x)) = \cos(\theta) = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$

(f) Combining the results of part (d) and part (e), we conclude

$$\boxed{\frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1-x^2}}}$$

## 2. The Derivative of Arccosine

Here I will just state the result. You can arrive at this by a method very similar to that for arctangent and arcsine.

$$\frac{d}{dx} \arccos(x) = -\frac{1}{\sqrt{1-x^2}}.$$

What is the difference between the derivatives of  $y = \arcsin(x)$  and  $y = \arccos(x)$ ?

$$\frac{d}{dx} \arccos(x) = -\frac{d}{dx} \arcsin(x).$$

3. Derivative practice using inverse trig. Find  $dy/dx$  for the following.

(a)  $y = (\arctan(x))^2$

$$\frac{dy}{dx} = 2\arctan(x) \cdot \left(\frac{1}{1+x^2}\right)$$

(b)  $y = \arcsin(2x + e^x)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(2x+e^x)^2}} \cdot (2+e^x)$$

(c)  $y = \arccos(\arcsin(x))$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-(\arcsin(x))^2}} \cdot \frac{1}{\sqrt{1-x^2}}$$

(d)  $\arccos(xy) = 1 + x^2y$  [This involves implicit differentiation]

$$\frac{d}{dx} \arccos(xy) = \frac{d}{dx} (1 + x^2y)$$

$$\frac{-1}{\sqrt{1-(xy)^2}} \cdot \frac{d}{dx} (xy) = 2xy + x^2y'$$

$$\frac{-1}{\sqrt{1-x^2y^2}} \cdot (y + xy') = 2xy + x^2y'$$

multiply by  $\sqrt{1-x^2y^2}$  on both sides

$$-y - xy' = 2xy\sqrt{1-x^2y^2} + x^2y'\sqrt{1-x^2y^2}$$

move  $y'$  terms to one side

$$-y - 2xy\sqrt{1-x^2y^2} = xy' + x^2y'\sqrt{1-x^2y^2}$$

$$-y - 2xy\sqrt{1-x^2y^2} = y'(x + x^2\sqrt{1-x^2y^2})$$

$$y' = \frac{-y - 2xy\sqrt{1-x^2y^2}}{x + x^2\sqrt{1-x^2y^2}}$$

(e)  $y = x \arcsin(x) + \sqrt{1-x^2}$

$$\frac{dy}{dx} = \arcsin(x) + \frac{x}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{1-x^2}} \cdot (-2x)$$

$$= \arcsin(x) + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}}$$

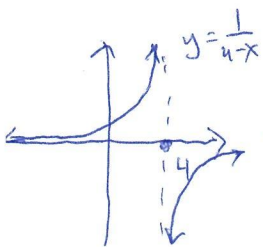
$$= \boxed{\arcsin(x)}$$

→ this says

$F(x) = x \arcsin(x) + \sqrt{1-x^2}$   
is the antiderivative of  
 $f(x) = \arcsin(x)$ !

4. Compute the following limit.

$$\lim_{x \rightarrow 4^-} \arctan\left(\frac{1}{4-x}\right)$$



Let  $t = \frac{1}{4-x}$ . As  $x \rightarrow 4^-$ , we see  $t \rightarrow \infty$  (from the graph).

$$\text{so } \lim_{x \rightarrow 4^-} \arctan\left(\frac{1}{4-x}\right) = \lim_{t \rightarrow \infty} \arctan(t) = \boxed{\frac{\pi}{2}}$$