1. (5 points each) Evaluate each of the following integrals

$$\text{(a)} \int_{-3}^{3} \frac{t|t|}{t^4+2}\,dt \qquad \text{(b)} \int t^3\sqrt{t-4}\,dt \qquad \text{(c)} \int_0^{3\pi/2} |\sin x|\,dx \qquad \text{(d)} \int \cos^3\theta \sin\theta\,d\theta$$

Solution:

(a) The function $f(t) = \dfrac{t|t|}{t^4+2}$ is odd so the integral around $[-3,3]$ is 0.

(b) Let $u = t-4$ so that $du = dt$. This substitution yields $t = u+4$ so $t^3 = (u+4)^5$. With these substitutions we have

$$\begin{aligned}
\int t^3\sqrt{t-4}\,dt &= \int (u+4)^3\sqrt{u}\,du \\
&= \int (u^3 + 3u^2(4) + 3u(16) + 64)u^{1/2}\,du \\
&= \int u^{7/2} + 12u^{5/2} + 48u^{3/2} + 64u^{1/2}\,du \\
&= \frac{2}{9}u^{9/2} + 12\frac{2}{7}u^{7/2} + 48\frac{2}{5}u^{5/2} + 64\frac{2}{3}u^{3/2} + C \\
&= \frac{2}{9}u^{9/2} + \frac{24}{7}u^{7/2} + \frac{96}{5}u^{5/2} + \frac{128}{3}u^{3/2} + C
\end{aligned}$$

(c) The function $\sin x$ becomes negative at $x = \pi$ so we can break up the integral into two pieces,

$$\int_0^{3\pi/2} |\sin x|\,dx = \int_0^{\pi} \sin x\,dx + \int_{\pi}^{3\pi/2} -\sin x\,dx$$

Calculating:

$$\begin{aligned}
\int_0^{\pi} \sin x\,dx + \int_{\pi}^{3\pi/2} -\sin x\,dx &= -\cos x\Big|_0^{\pi} + \cos x\Big|_{\pi}^{3\pi/2} \\
&= (-\cos(\pi) + \cos(0)) + (\cos(3\pi/2) - \cos(\pi)) \\
&= (1+1) + (0-(-1)) \\
&= 3
\end{aligned}$$

Therefore,

$$\int_0^{3\pi/2} |\sin x|\,dx = 3$$

(d) Using the substitution $u = \cos\theta$, $du = -\sin\theta\,d\theta$, we have

$$\int \cos^3\theta \sin\theta\,d\theta = -\int u^3, du = -\frac{u^4}{4} + C = -\frac{\cos^4\theta}{4} + C$$

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2. (20 points) The profit P (in thousands of dollars) for a company spending an amount s (in thousands of dollars) on advertising is $P = -\frac{1}{10}s^3 + 6s^2 + 400$. Find the amount of money the company should spend on advertising in order to yield a maximum profit.

Solution: To maximize find the derivative and look for critical points. $P' = -\frac{3}{10}s^2 + 12s$ and so $P' = 0$ when

$$-\frac{3}{10}s^2 + 12s = 0$$

or when

$$\frac{3s}{10}(-s+40)=0$$

so there are critical points at $s=0$ and $s=40$. The second derivative is $P''=-\frac{3}{5}s+12$ and $P''(0)=12$ which means the function is concave up at $s=0$ and $P''(40)=-12$ so the function is concave down at $s=40$. Therefore, $s=40$ is a maximum and so they should spend $\$40,000$.

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3. (a) (6 points) Write the integral which gives the area of the region between $x=0$ and $x=1$, above the x-axis, and below the curve $y=x-x^2$.
   (b) (8 points) Evaluate your integral exactly to find the area.
   (c) (6 points) Find all c between $x=0$ and $x=1$ so that $f(c)=f_{avg}$.

Solution:

(a) The function has zeroes at $x=0,1$. The function is above the $x-$axis between 0 and 1 and so the integral is

$$\int_0^1 x-x^2\,dx$$

(b) We compute

$$\int_0^1 x-x^2\,dx=\left(\frac{x^2}{2}-\frac{x^3}{3}\right)\Big|_0^1=\left(\frac{1}{2}-\frac{1}{3}\right)-(0-0)=\frac{1}{6}$$

(c) The average value of a function is defined to be

$$f_{avg}=\frac{1}{b-a}\int_a^b f(x)\,dx$$

In this case we have

$$f_{avg}=\frac{1}{1}\int_0^1 x-x^2\,dx=\frac{1}{6}$$

from part (b). Now, we need to find c such that $f(c)=\frac{1}{6}$. In other words solve

$$c-c^2=\frac{1}{6}$$

This is a quadratic equation.

$$c^2-c+\frac{1}{6}=0\Leftrightarrow c=\frac{1\pm\sqrt{1-4(1/6)}}{2}$$

Or,

$$c=\frac{1}{2}\pm\frac{1}{2\sqrt{3}}=\frac{1}{2}\left(1\pm\frac{1}{\sqrt{3}}\right).$$

Both of these c are in the interval.

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4. (20 points) Using the definition for area using right hand endpoints,

$$A=\lim_{n\to\infty}R_n=\lim_{n\to\infty}\left[f(x_1)\Delta x+f(x_2)\Delta x+f(x_3)\Delta x+\ldots+f(x_n)\Delta x\right]$$

find an expression for the area under the curve $y=x^3$ from 0 to 1 as a limit.

Solution: Let $\Delta x=\frac{b-a}{n}=\frac{1-0}{n}=\frac{1}{n}$, this is the width of each rectangle. Using a right hand endpoint on each subinterval we have $x_1^*=\frac{1}{n}$, $x_2^*=\frac{2}{n}$, $x_3^*=\frac{3}{n}$, $\ldots$, $x_n^*=\frac{n}{n}$. It follows that

$$A=\lim_{n\to\infty}\sum_{i=1}^{n}f(x_i^*)\Delta x=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^3\frac{1}{n}.$$

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5. (5 points each) Let the function f be defined by $f(x) = \int_1^x \frac{1}{t}\, dt$ for $x > 0$.

(a) What is f(1)? What is $f'(x)$? What is $f'(1)$?

(b) f is differentiable. Why?

(c) Show that $f'\left(\frac{1}{x}\right) = -f'(x)$.

(d) Using the definition of f, show that $f(x+h) - f(x) = \int_x^{x+h} \frac{1}{t} dt$.

Solution:

(a)

$$f(1) = \int_1^1 \frac{1}{t}\, dt = 0$$

By the Fundamental Theorem of Calculus,

$$f'(x) = \frac{1}{x}.$$

It follows that

$$f'(1) = \frac{1}{1} = 1.$$

(b) By the FTOC part (1). Since $\frac{1}{t}$ is continuous on its domain its antiderivative exists and so we can take it's derivative.

(c) From part (a) we know $-f'(x) = -\frac{1}{x}$. Since

$$\frac{d}{dx}\int_0^{1/x} \frac{1}{t}\, dt = \frac{1}{1/x} \cdot \left(\frac{1}{x}\right)' = x \cdot \left(\frac{-1}{x^2}\right) = -\frac{1}{x}$$

we have that $\frac{d}{dx}\left(f\left(\frac{1}{x}\right)\right) = -f'(x)$.

(d)

$$\begin{aligned} f(x+h) - f(x) &= \int_1^{x+h} \frac{1}{t}\, dt - \int_1^x \frac{1}{t}\, dt \\ &= \left(\int_1^x \frac{1}{t}\, dt + \int_x^{x+h} \frac{1}{t}\, dt\right) - \int_1^x \frac{1}{t}\, dt \\ &= \int_x^{x+h} \frac{1}{t}\, dt \end{aligned}$$