Goal: To collect information about the first and second derivatives of a function, then use this information to graph the function without using technology.

1. Consider the function $f(x) = 3x^4 - 8x^3 + 6x^2$.

   (a) Determine the open intervals on which the function is increasing/decreasing.

   Solution: $f'(x) = 12x^3 - 24x^2 + 12x = 12x(x-1)^2$. Solving $0 = 12x^3 - 24x^2 + 12x$ gives us $x = 0$ and $x = 1$. The derivative can only change signs at these two points. To determine the sign within each interval, we substitute a value within each interval. Substituting $x = -1$ into the derivative gives $f'(-1) = -48 < 0$, so f is decreasing on $(-\infty, 0)$. Now $f'(\frac{1}{2}) = \frac{3}{2} > 0$, so f is increasing on $(0, 1)$. Finally, $f'(2) = 24 > 0$, so f is also increasing on $(1, \infty)$. (In fact, by the definition of "increasing on an interval", the last two intervals can be joined and $f(x)$ is actually increasing on the entire interval $(0, \infty)$.)

   (b) Find the local maxima and local minima of $f(x)$, if any. Be sure to find the critical points, classify them using either the first or second derivative test, then substitute the x-values into $f(x)$ to find the local mininum/maximum values.

   Solution: From the previous part, the critical points are at $x = 0$ and $x = 1$. $f''(x) = 36x^2 - 48x + 12$, so $f''(0) = 12$ and $f''(1) = 0$. By the second derivative test, we can say there is a local minimum at $x = 0$. Since $f''(1) = 0$ we must use the first derivative test. From the previous part, the function is increasing on both sides of $x = 1$, and therefore this is neither a local min nor a local max there.

   Evaluating the function at our critical values gives $f(0) = 0$ (the local minimum of f) and $f(1) = 1$ (a stationary point of f).

   (c) Find the inflection points of the function, if any. Be sure to find where the second derivative is zero, use a sign chart to determine whether or not the second derivative changes, then substitute the x-values into $f(x)$ to find the y-value at each inflection point.

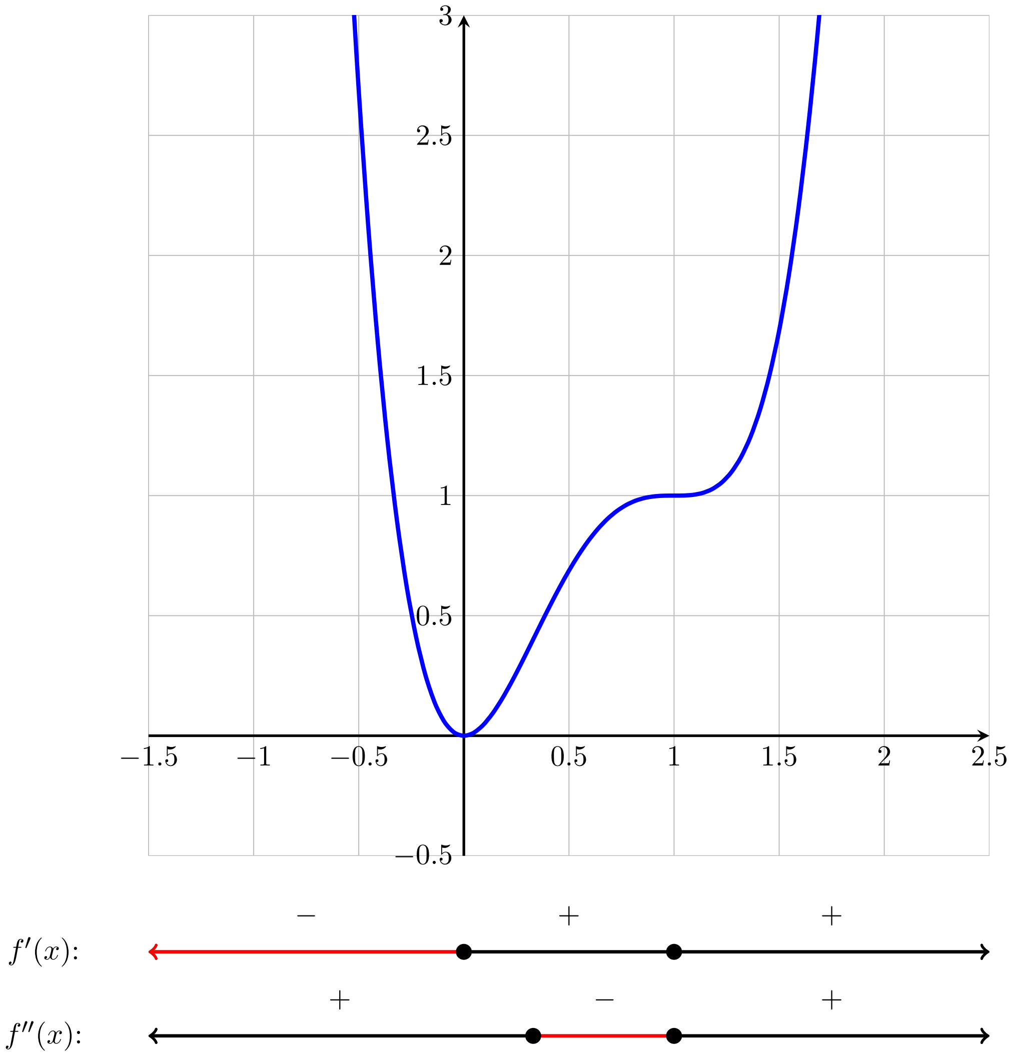
   Solution: $f''(x) = 36x^2 - 48x + 12 = 12(3x-1)(x-1)$. Solving $0 = 12(3x-1)(x-1)$ gives us $x = 1/3$ and $x = 1$.

   Testing on either side of $x = 1/3$. Left: $f''(0) = 12$. Right: $f''(1/2) = -3$. The signs change so this is an inflection point.

   Testing on either side of $x = 1$. Left: $f''(1/2) = -3$. Right: $f''(2) = 60$. The signs change so this is an inflection point.

   Evaluating the function at these x values gives $f(1/3) = 11/27$ and $f(1) = 1$. So the points $(0, 0)$ and $(1/3, 11/27)$ are inflection points.

(d) Plot the local extrema and the inflection points on the graph. Transfer the information from parts (a) and (b) to the number lines for $f'(x)$ and $f''(x)$. Sketch the graph of the function $f(x) = 3x^4 - 8x^3 + 6x^2$, using all of the information.



(e) Now use your graphing calculator to get the graph of $y = f(x)$ on this domain, and compare it to the graph you just drew. How well did you do?

Solution: Great! They look just the same!

2. Using the same process as in the previous problem, graph $f(x) = x^{\frac{1}{3}}(x+4)$ on the next page.

Solution:

$$f'(x) = x^{\frac{1}{3}} + \frac{1}{3}x^{-\frac{2}{3}}(x+4) = x^{\frac{1}{3}} + \frac{x+4}{3x^{\frac{2}{3}}}$$

Combine these terms by finding common denominators:

$$f'(x) = \frac{3x}{3x^{\frac{2}{3}}} + \frac{x+4}{3x^{\frac{2}{3}}} = \frac{4x+4}{3x^{\frac{2}{3}}} = \frac{4(x+1)}{3x^{\frac{2}{3}}}$$

There is a critical point at $x = -1$ (horizontal tangent line) and a critical point at $x = 0$ (vertical tangent line, possibly a cusp). Substituting points $x = -2$, $x = -\frac{1}{2}$ and $x = 1$ into the derivative, we see that $f(x)$ is decreasing on $(-\infty, -1)$ and increasing on $(-1, 0)$ as well as $(0, \infty)$. (In fact, by the definition of "increasing on an interval", the last two intervals can be joined and $f(x)$ is thus increasing on the interval $(-1, \infty)$.)
By the first derivative test, since $f'(x)$ changes signs from negative to positive at $x = -1$, f has a local minimum of $f(-1) = -3$ at $x = -1$. Since $f'(x)$ does not change sign at $x = 0$, f has no local extremum at $x = 0$. The graph has a vertical tangent line there and no cusp.

Now on to the second derivative. We need the quotient rule.

$$f''(x) = \frac{(3x^{\frac{2}{3}}) \cdot 4 - (4x+4) \cdot 2x^{-\frac{1}{3}}}{(3x^{\frac{2}{3}})^2} = \frac{12x^{\frac{2}{3}} - (8x+8) \cdot x^{-\frac{1}{3}}}{9x^{\frac{4}{3}}}$$

Now get rid of negative exponents by multiplying numerator and denominator by $x^{\frac{1}{3}}$:

$$f''(x) = \frac{12x^{\frac{2}{3}} - (8x+8) \cdot x^{-\frac{1}{3}}}{9x^{\frac{4}{3}}} \cdot \frac{x^{\frac{1}{3}}}{x^{\frac{1}{3}}} = \frac{12x - (8x+8)}{9x^{\frac{5}{3}}} = \frac{4x-8}{9x^{\frac{5}{3}}} = \frac{4(x-2)}{9x^{\frac{5}{3}}}$$

The second derivative is 0 at $x = 2$ and undefined at $x = 0$. Substituting $x = -1$, $x = 1$, and $x = 4$ we see that $f(x)$ is concave up on $(-\infty, 0) \cup (2, \infty)$ and concave down on $(0, 2)$, and so there are inflection points at $x = 0$ and $x = 2$.

Graph of $f(x) = x^{\frac{1}{3}}(x+4)$