1. The following problems are not related. Show all work, simplify your answers.

(a)(8 pts) Find $\dfrac{d^2y}{dx^2}$ if $y = x\tan(x)$.

(b)(8 pts) Suppose y is a function of x, find y' if $\cos(xy) = 1$.

(c)(8 pts) Suppose $f(x) = \sqrt[3]{x+|x|}$, find $f'(x)$.

(d)(5 pts) Which of the five choices given below is equivalent to $\dfrac{d}{dx}\left[\dfrac{f(x^2)}{x}\right]$? Clearly write down your answer(s) in your blue book, **no justification necessary** - *be sure to copy down the entire answer, don't just write down the roman numeral of your choice(s)*:

$(i)\ 2xf'(x^2)$ $\quad (ii)\ \dfrac{2x^2f'(2^x)-f(x^2)}{x^2}$ $\quad (iii)\ \dfrac{2x^2f'(x^2)-f(2x)}{x^2}$ $\quad (iv)\ 2f'(x^2)-f(x^2)x^{-2}$ $\quad (v)\ 2x^{-2}f'(x^2)-f(x^2)$

Solution: (a)(8 pts) By the product rule, $y' = \tan(x) + x\sec^2(x)$ and thus

$$y'' = \sec^2(x) + \sec^2(x) + 2x\sec(x)\cdot\sec(x)\tan(x) = \boxed{2\sec^2(x) + 2x\sec^2(x)\tan(x)} = 2\sec^2(x)(1 + x\tan(x))$$

(b)(8 pts) Using implicit differentiation we have

$$\cos(xy) = 1 \Rightarrow -\sin(xy)[y + xy'] = 0 \Rightarrow y' = -\frac{y}{x}$$

(c)(8 pts) By definition, we have

$$f(x) = \sqrt[3]{x+|x|} = \begin{cases} (x+x)^{1/3}, & \text{if } x \geq 0 \\ (x-x)^{1/3}, & \text{if } x < 0 \end{cases} = \begin{cases} (2x)^{1/3}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases} \text{ thus, } f'(x) = \begin{cases} \frac{2}{3}(2x)^{-2/3}, & \text{if } x > 0 \\ 0, & \text{if } x < 0 \end{cases}$$

(d)(5 pts) Choice (iv) only. By the quotient rule,

$$\frac{d}{dx}\left[\frac{f(x^2)}{x}\right] = \frac{x\cdot f'(x^2)2x - f(x^2)}{x^2} = \frac{2x^2f'(x^2)-f(x^2)}{x^2} = 2f'(x^2) - f(x^2)x^{-2}$$

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2. (a)(10 pts) A vertical cylindrical tank with constant diameter of 10 meters contains a certain viscous fluid. At what rate will the fluid level inside the cylindrical tank change if we pump the fluid out at 3000 cubic meters per minute?

(b) The Boulder Ball Bearing Company (BBBCO) produces steel ball bearings (spheres) with a volume of $32\pi/3\,\text{cm}^3$. (Recall that the volume of a sphere is $V = 4\pi r^3/3$ where r represents the radius.)

(i)(10 pts) Use differentials to estimate the change in volume if the radius varies from $r = 2\,\text{cm}$ to $r = 1.9\,\text{cm}$.

(ii)(5 pts) Show that the relative error of the volume of the ball bearing produced is 3 times the relative error of the radius of the ball bearing.

Solution: (a)(10 pts) Note that the radius of the tank is $r = 10/2 = 5\,\text{m}$ and if we let h denote the height of the fluid in the tank then the volume of fluid in the cylindrical tank is $V = \pi(5)^2h = 25\pi h$. Since the fluid is being pumped out of the tank at 3000 cubic meters per minute we have $dV/dt = -3000\,\text{m}^3/\text{min}$ and we wish to find dh/dt. Note

$$V = 25\pi h \Rightarrow \frac{dV}{dt} = 25\pi\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{dV/dt}{25\pi} = \frac{-3000}{25\pi} = -120/\pi \text{ m/min},$$

that is, the change in the fluid level is $-120/\pi$ m/min or the fluid level will *decrease* at $120/\pi$ m/min.

(b)(i) (10 pts) Note that if $V = 4\pi r^3/3$ then

$$\Delta V \approx dV = \frac{4}{3}\pi\cdot 3r^2dr = 4\pi r^2dr\Big|_{\substack{r=2,\\ dr=-0.1}} = 16\pi(-0.1) = -1.6\pi\,\text{cm}$$

that is, the volume decreases by approximately 1.6π cm.

(b)(i) (5 pts) Note that

$$\frac{dV}{V} = \frac{4\pi r^2\,dr}{4\pi r^3/3} = 3\frac{dr}{r}$$

thus the relative error of the volume of the ball bearing being produced is 3 times the relative error of the radius of the ball bearing.

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3. The following problems are not related, remember justify your answers and cite any theorems you use.

(a) Let $f(x) = \sqrt{x} - x/3$. (i)(6pts) Verify that $f(x)$ satisfies the three hypotheses of Rolle's Theorem on $[0,9]$ and (ii)(6 pts) find all numbers c that satisfy the conclusions of Rolle's Theorem for $f(x)$ on $(0,9)$.

(b)(10 pts) Let $p(x) = (1+x)^k$ for any number k. Use the linearization of $p(x)$ at $a = 0$ to establish the most important linear approximation for roots and powers, namely, $(1+x)^k \approx 1 + kx$ for $x \approx 0$ and any number k.

Solution: (a)(i)(6 pts) Note that $f(x)$ is continuous on $[0,9]$ since $f(x)$ is the difference of a root function and polynomial (both known to be continuous in their domains from a theorem in the textbook) and since $x \geq 0$. Note that $f'(x) = \dfrac{1}{2\sqrt{x}} - \dfrac{1}{3}$ which is well defined for $x > 0$ and so $f(x)$ is differentiable on $(0,9)$. Finally, note that at the endpoints we have $f(0) = 0 = f(9)$.

(a)(ii)(6 pts) Now, according to Rolle's Theorem, we have that

$$f'(c) = 0 \Rightarrow \frac{1}{2\sqrt{c}} - \frac{1}{3} = 0 \Rightarrow 2\sqrt{c} = 3 \Rightarrow c = \frac{9}{4} = 2.25 \text{ (which is in the interval } (0,9)).$$

(b)(10 pts) Note that $p'(x) = k(1+x)^{k-1}$ for any number k and so the linearization of $p(x)$ at $a = 0$ is

$$L(x) = p(0) + p'(0)(x-0) = 1^k + k(1+0)^{k-1}\cdot x = 1 + kx$$

and thus for x close to 0, i.e. $x \approx 0$ we have $p(x) \approx L(x)$, i.e. $(1+x)^k \approx 1 + kx$.

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4. The following problems are not related, remember to show all work and justify your answers.

(a)(8 pts) Find all the absolute extreme values of $f(x) = \sqrt{4-x^2}$ for $-2 \leq x \leq 1$. (Be sure to write down the x-coordinate and the y-coordinate of all absolute extrema.)

(b)(8 pts) Find all local extreme values of $g(x) = x^{4/3} - 4x^{1/3}$. (Be sure to write down the x-coordinate and the y-coordinate of all local extrema.)

(c)(8 pts) In your blue book clearly sketch the graph of a function $h(x)$ that satisfies all the following properties (label all extrema, inflection points and asymptotes):

- $h(-2) = 2$, $h(0) = 0$, and $h(5) = 1$,
- $\lim_{x\to-\infty} h(x) = 0$, $\lim_{x\to 2^-} h(x) = -\infty$, and $\lim_{x\to 2^+} h(x) = +\infty$,
- $h'(x) < 0$ if $-2 < x < 2$ or $2 < x < 5$ and $h'(x) > 0$ if $x < -2$ or $x > 5$,
- $h''(x) > 0$ if $x < -3$ or $x > 2$ and $h''(x) < 0$ if $-3 < x < 2$.

Solution: (a)(8 pts) Note that $f'(x) = \dfrac{1}{2}(4-x^2)^{-1/2}\cdot(-2x) = \dfrac{-x}{\sqrt{4-x^2}}$ and so $f'(x)=0$ implies $x=0$ and $f'(x)$ is undefined when $x=\pm 2$ but note that $x=2$ is not in the interval $[-2,1]$ and so the only critical points of $f(x)$ are $x=-2,0$.

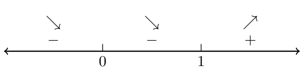
Now evaluating $f(x)$ at the critical points and end points yields $f(-2)=0$, $f(0)=\sqrt{4}=2$ and $f(1)=\sqrt{3}$ and note that $\sqrt{4}>\sqrt{3}>0$. Thus, we see that $g(x)$ has an absolute maximum value at the point $(0,2)$ and an absolute minimum at $(-2,0)$.

(b)(8 pts) Note that

$$g'(x) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} = \frac{4}{3x^{2/3}}(x-1)$$

and note that $g'(x)=0$ implies $x=1$ and $g'(x)$ is undefined if $x=0$. Thus the critical points of $g(x)$ are $x=0,1$. Now using the first derivative test we see that

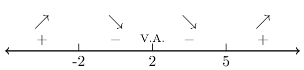
Sign Chart for $g'(x)$



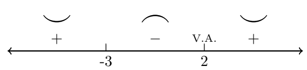
and so there is a local minimum value at the point $(1,g(1))=(1,-3)$.

(c)(8 pts) Note that $y=0$ is a horizontal asymptote (as $x\to-\infty$) and $x=2$ is a vertical asymptote, furthermore there is a local max at $x=-2$ and local min at $x=5$ and there is an inflection point at $x=-3$, all these facts follow from the following sign charts for $h'(x)$ and $h''(x)$:

Sign Chart for $h'(x)$



Sign Chart for $h''(x)$



and also note that the graph crosses its own horizontal asymptote at $(0,0)$. Thus the graph could look like, for example, the following: