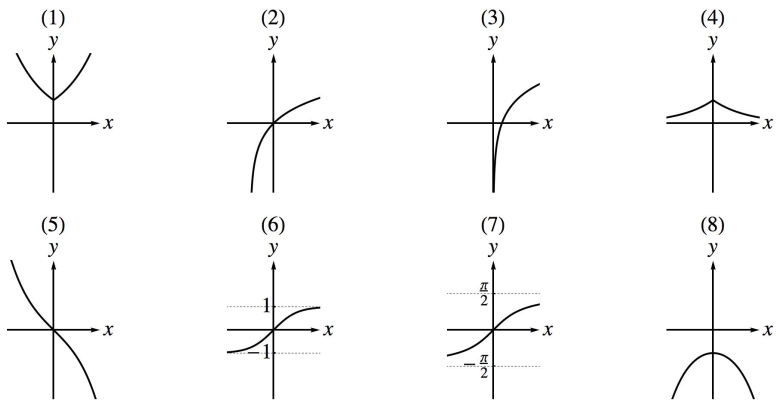
APPM 1345/1350 Final Exam Solutions Spring 2013

1. (8 points) Match the following functions to their graphs below. No explanation is necessary.

   (a) $y = \ln(x+1)$ (b) $y = 2^{|x|}$ (c) $y = \tan^{-1} x$ (d) $y = -\sinh(x)$.



   Solution: (a) 2 (b) 1 (c) 7 (d) 5

2. (38 points) Evaluate the following expressions.

   (a) $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\,dx$ (b) $\displaystyle\int e^{2013\ln x}\,dx$ (c) $\displaystyle\int_{-1}^{1} 2^{|x|}\,dx$

   (d) $\displaystyle\int_0^{\sqrt{3}} \frac{e^{\arctan x}}{1+x^2}\,dx$ (e) $\displaystyle\frac{d}{dx}\left(x^{\cos 3x}\right)$ (f) $\displaystyle\lim_{h\to 0}\frac{\sec^3(x+h)-\sec^3(x)}{h}$

   Solution:

   (a) Let $u = 1-4x^2$. Then $du = -8x\,dx \;\Rightarrow\; -du/8 = x\,dx$.

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx = -\frac{1}{8}\int\frac{du}{\sqrt{u}} = -\frac{1}{8}\int u^{-1/2}\,du$$

$$= -\frac{1}{8}\left(2u^{1/2}\right) + C = \boxed{-\frac{1}{4}\sqrt{1-4x^2}+C}$$

   (b) First simplify the integrand.

$$\int e^{2013\ln x}\,dx = \int e^{\ln x^{2013}}\,dx = \int x^{2013}\,dx = \boxed{\frac{x^{2014}}{2014}+C}$$

   (c) Note that the integrand is an even function.

$$\int_{-1}^{1} 2^{|x|}\,dx = 2\int_0^1 2^x\,dx = 2\cdot\frac{2^x}{\ln 2}\bigg]_0^1 = \frac{2}{\ln 2}(2-1) = \boxed{\frac{2}{\ln 2}}$$

   (d) Let $u = \arctan x$. Then $du = dx/(1+x^2)$ and the u-limits are $u(0) = \arctan(0) = 0$, $u\left(\sqrt{3}\right) = \arctan\left(\sqrt{3}\right) = \pi/3$.

$$\int_0^1 \frac{e^{\arctan x}}{1+x^2}\,dx = \int_0^{\pi/3} e^u\,du = e^u\bigg]_0^{\pi/3} = \boxed{e^{\pi/3}-1}$$

   (e) Let $y = x^{\cos 3x}$ and find dy/dx using logarithmic differentiation.

$$\begin{aligned}
y &= x^{\cos 3x}\\
\ln y &= \ln x^{\cos 3x} = (\cos 3x)(\ln x)\\
\frac{1}{y}\frac{dy}{dx} &= \cos 3x\cdot\frac{1}{x} + (\ln x)(-3\sin 3x)\\
\frac{dy}{dx} &= \boxed{x^{\cos 3x}\left(\frac{\cos 3x}{x} - 3(\sin 3x)(\ln x)\right)}
\end{aligned}$$

   (f) The limit represents the derivative of the function $\sec^3 x$.

$$\begin{aligned}
\lim_{h\to 0}\frac{\sec^3(x+h)-\sec^3(x)}{h} &= [\sec^3 x]' = 3\sec^2 x(\sec x\tan x)\\
&= \boxed{3\sec^3 x\tan x}
\end{aligned}$$

3. (15 points) Consider the function $f(x)$. Is there a value of c that will make f continuous at $x = 0$? Use the definition of continuity to justify your answer.

$$f(x) = \begin{cases} \dfrac{\tan x}{2x} & x < 0 \\ c & x = 0 \\ x^{2x} & x > 0 \end{cases}$$

Solution:

f is continuous at $x = 0$ if $f(0) = \lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x)$. The value of $f(0)$ is c. First evaluate the left-hand limit.

$$\begin{aligned} \lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} \frac{\tan x}{2x} = \lim_{x\to 0^-} \frac{\sin x}{2x\cos x} = \lim_{x\to 0^-} \frac{\sin x}{x} \cdot \frac{1}{2\cos x} \\ &= 1 \cdot \frac{1}{2\cdot 1} = \frac{1}{2} \end{aligned}$$

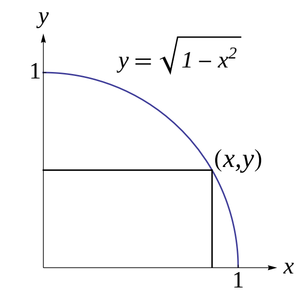
by the theorem $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$.

Next evaluate the right-hand limit which is an indeterminate power. Let L equal the value of the limit.

$$\begin{aligned} L &= \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^{2x} \\ \ln L &= \lim_{x\to 0^+} \ln x^{2x} = \lim_{x\to 0^+} 2x\ln x \\ &= \lim_{x\to 0^+} \frac{2\ln x}{1/x} \overset{L'H}{=} \lim_{x\to 0^+} \frac{2/x}{-1/x^2} = \lim_{x\to 0^+} -2x = 0 \\ L &= e^0 = 1 \end{aligned}$$

Since $\lim\limits_{x\to 0^-} f(x) = 1/2$ and $\lim\limits_{x\to 0^+} f(x) = 1$, there is $\boxed{\text{no value of } c}$ that will make f continuous at $x = 0$.

4. (14 points) Find the area of the largest rectangle that can be inscribed in the first quadrant of the unit circle if one side of the rectangle lies along the x-axis and another lies along the y-axis.
(*Hint:* The equation of a unit circle centered at the origin is $x^2 + y^2 = 1$.)



We wish to maximize the area A of the rectangle given $y = \sqrt{1-x^2}$.

$$\begin{aligned} A &= xy = x\sqrt{1-x^2} \\ A' &= x \cdot \frac{-2x}{2\sqrt{1-x^2}} + \sqrt{1-x^2} \\ &= \frac{-x^2}{\sqrt{1-x^2}} + \sqrt{1-x^2} = \frac{-2x^2+1}{\sqrt{1-x^2}} \end{aligned}$$

Find the critical points where $A' = 0$ given $x > 0$.

$$-2x^2 + 1 = 0 \Rightarrow x^2 = 1/2 \Rightarrow x = 1/\sqrt{2}$$

Confirm that there is a maximum value at the critical point using the first derivative test. Since $A'\left(\frac{1}{2}\right) > 0$ and $A'\left(\frac{4}{5}\right) < 0$, there is a maximum value at $x = 1/\sqrt{2}$, $y = 1/\sqrt{2}$. The area of the largest inscribed rectangle is $\left(1/\sqrt{2}\right)^2 = \boxed{1/2}$.

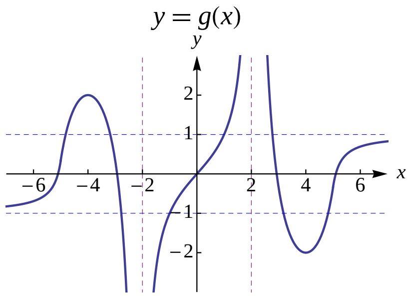
5. (14 points) Sketch a graph of a single function $y = g(x)$ that satisfies all of the following conditions. No explanation is necessary.

(a) $g(-x) = -g(x)$ (b) $\lim\limits_{x\to\infty} g(x) = 1$ (c) $\lim\limits_{x\to 2} g(x) = \infty$

(d) $\lim\limits_{x\to 4} g(x) = -2$ (e) $g'' > 0$ for x in $(2, 5)$ (f) $g'' < 0$ for x in $(5, \infty)$

(g) $\lim\limits_{h\to 0} \dfrac{g(4+h) - g(4)}{h} = 0$

Solution: Here is one possible solution. Note that condition (a) indicates that g is odd and condition (g) indicates that $g'(4) = 0$.



6. (12 points) Let $f(x) = \dfrac{\sinh x}{e^x}$.

   (a) Simplify $f(x)$ using the definition of $\sinh x$.

   (b) Find the value of $f(-\ln 2)$.

   (c) Is f increasing or decreasing at $x = -\ln 2$?

   (d) Is f concave up or down at $x = -\ln 2$?

Solution:

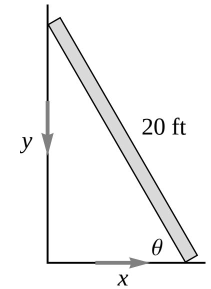
(a) $f(x) = \dfrac{\sinh x}{e^x} = \dfrac{e^x - e^{-x}}{2} \cdot \dfrac{1}{e^x} = \boxed{\dfrac{1 - e^{-2x}}{2}}$

(b) $f(-\ln 2) = \dfrac{1}{2}\left(1 - e^{2\ln 2}\right) = \dfrac{1}{2}\left(1 - 2^2\right) = \boxed{-\dfrac{3}{2}}$

(c) $f'(x) = \dfrac{1}{2}\left(2e^{-2x}\right) = e^{-2x}$. Since $f' > 0$ for all x, f is $\boxed{\text{increasing}}$ at $x = -\ln 2$.

(d) $f''(x) = -2e^{-2x}$. Since $f'' < 0$ for all x, f is $\boxed{\text{concave down}}$ at $x = -\ln 2$.

7. (14 points) A 20-ft ladder is leaning against a wall when its base starts to slide away. At the moment when the angle between the ladder and the ground is $\pi/3$ radians, the top of the ladder is sliding down the wall at a rate of 1/4 ft/sec. How fast is the base of the ladder moving away from the wall then?



Solution:

Let x represent the distance from the base of the ladder to the wall and y represent the distance from the top of the ladder to the ground. When the angle θ between the ladder and the ground is $\pi/3$ radians, then $y = 10\sqrt{3}$ ft and $x = 10$ ft because $\sin\theta = \sqrt{3}/2$ and $\cos\theta = 1/2$. We wish to find dx/dt given $dy/dt = -1/4$ ft/sec. Use the Pythagorean Theorem.

$$\begin{aligned}
x^2 + y^2 &= 20^2 \\
2x\frac{dx}{dt} + 2y\frac{dy}{dt} &= 0 \\
x\frac{dx}{dt} + y\frac{dy}{dt} &= 0 \\
10\frac{dx}{dt} + (10\sqrt{3})\left(-\frac{1}{4}\right) &= 0 \\
\frac{dx}{dt} &= \boxed{\frac{\sqrt{3}}{4}\ \text{ft/sec}}
\end{aligned}$$

8. (10 points) Let $g(x) = \int_0^x \frac{dt}{\sqrt{1-t^2}}$. Find the linearization of g at $x = -1/2$.

Solution:

Note that

$$g(x) = \int_0^x \frac{dt}{\sqrt{1-t^2}} = \arcsin t\Big]_0^x = \arcsin x - \arcsin 0 = \arcsin x$$

and

$$g'(x) = \frac{1}{\sqrt{1-x^2}}.$$

Then $g(-1/2) = \arcsin(-1/2) = -\pi/6$ and $g'(-1/2) = \frac{1}{\sqrt{1-1/4}} = 2/\sqrt{3}$. The linearization is therefore

$$L(x) = g(-1/2) + g'(-1/2)\left(x + \frac{1}{2}\right) = \boxed{-\frac{\pi}{6} + \frac{2}{\sqrt{3}}\left(x + \frac{1}{2}\right)}.$$

9. (10 points) Suppose $h(x)$ is an even, continuous function with roots at $x = \pm 2$. Given $\int_0^2 h(x)\,dx = 7$ and $\int_{-5}^0 h(x)\,dx = -4$, find the values of the following expressions.

(a) $\int_{-5}^{-2} h(x)\,dx$ (b) h_{ave} on $[-5, 5]$ (c) $\int_{-5}^{5} |h(x)|\,dx$

Solution:

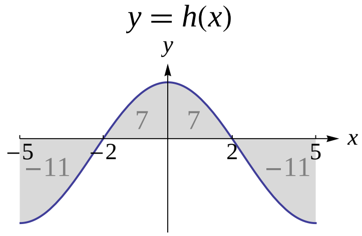
Since h is even, $\int_{-2}^0 h(x)\,dx = \int_0^2 h(x)\,dx = 7$ and $\int_0^5 h(x)\,dx = \int_{-5}^0 h(x)\,dx = -4$.

(a) $\int_{-5}^{-2} h(x)\,dx = \int_{-5}^{0} h(x)\,dx - \int_{-2}^{0} h(x)\,dx = -4 - 7 = \boxed{-11}$

(b) $h_{ave} = \frac{1}{5-(-5)} \int_{-5}^{5} h(x)\,dx = \frac{1}{10}(-4-4) = \boxed{-\frac{4}{5}}$

(c)

$$\int_{-5}^{5} |h(x)|\,dx = 2\int_0^5 |h(x)|\,dx = 2\left(\int_0^2 h(x)\,dx + \left|\int_2^5 h(x)\,dx\right|\right)$$
$$= 2(7 + 11) = \boxed{36}$$



10. (15 points) Zach walks into a casino with \$1000 and steadily loses 20% of his money each hour. Meredith enters the casino with \$500 and steadily loses 15% of her money each hour.

(a) How long will it take before Zach is left with just \$50?

(b) Who will reach \$50 first: Zach or Meredith?

Use the law of exponential decay and the following approximations to compute your answers.

$\ln 0.15 \approx -1.9$	$\ln 0.2 \approx -1.6$	$\ln 0.8 \approx -.22$
$\ln 0.85 \approx -.16$	$\ln 10 \approx 2.3$	$\ln 20 \approx 3.0$

Solution:

(a) Let $z(t)$ represent the amount of money Zach has t hours after he enters the casino. We are given that after one hour, $z(1) = 0.8z_0$.

First find k.

$$\begin{aligned} z(t) &= z_0 e^{kt} \\ z(1) &= z_0 e^k = 0.8 z_0 \\ e^k &= 0.8 \ \Rightarrow \ k = \ln 0.8 \end{aligned}$$

Now find t when $z(t) = 50$.

$$\begin{aligned} z(t) &= z_0 e^{kt} = 50 \\ 1000 e^{kt} &= 50 \\ e^{kt} &= 1/20 \\ kt &= \ln(1/20) = -\ln(20) \\ t &= -\frac{\ln(20)}{k} = \frac{-\ln(20)}{\ln 0.8} \\ &\approx \frac{-3.0}{-.22} = \frac{300}{22} = \frac{150}{11} = \boxed{13\frac{7}{11} \text{ hrs}} \end{aligned}$$

(b) Let $m(t)$ represent the amount of money Meredith has t hours after she enters the casino. We are given that after one hour, $m(1) = 0.85 m_0$. First find k.

$$\begin{aligned} m(t) &= m_0 e^{kt} \\ m(1) &= m_0 e^k = 0.85 m_0 \\ e^k &= 0.85 \ \Rightarrow \ k = \ln 0.85 \end{aligned}$$

Now find t when $m(t) = 50$.

$$\begin{aligned} m(t) &= m_0 e^{kt} = 50 \\ 500 e^{kt} &= 50 \\ e^{kt} &= 1/10 \\ kt &= \ln(1/10) = -\ln(10) \\ t &= -\frac{\ln(10)}{k} = \frac{-\ln(10)}{\ln 0.85} \\ &\approx \frac{-2.3}{-.16} = \frac{230}{16} = \frac{115}{8} = 14\frac{3}{8} \text{ hrs} \end{aligned}$$

$\boxed{\text{Zach}}$ will reach \$50 before Meredith.