1. (2 pts each) **True/False**

   (a) (T/F) If f is undefined at $x = c$, then the limit of $f(x)$ as x approaches c does not exist.

   (b) (T/F) If the limit of $f(x)$ as x approaches c is 0, then there must be a number k such that $f(k) < 0.0001$.

   (c) (T/F) $\lim_{x\to 0} \sin\left(\frac{|x|}{x}\right) = 0$

   (d) (T/F) If f is an even function and $\lim_{x\to 2^-} f(x) = 7$ then $\lim_{x\to -2^-} f(x) = 7$

   Solution:

   (a) **FALSE**

   (b) **TRUE**

   (c) **FALSE**

   (d) **FALSE**

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2. Evaluate the following limits, you may not use l'Hospital's Rule, justify your answers:

   (a) (7 pts) $\lim_{x\to 1} \frac{\sin(2x)}{\sin(3x)}$ (b) (7 pts) $\lim_{x\to 1^+} \frac{\sqrt{2x}(x-1)}{|x-1|}$ (c) (7 pts) $\lim_{x\to 0} \sqrt{x}\sin\left(\frac{1}{x}\right)$

   (d) $f(b^2+1) = ?$ given that $f(x) = \begin{cases} |x|+1, & \text{if } x < 1 \\ -x+1, & \text{if } x \geq 1 \end{cases}$

   Solution:

   (a) $\lim_{x\to 1} \frac{\sin(2x)}{\sin(3x)} = \frac{\sin(2)}{\sin(3)}$

   (b) $\lim_{x\to 1^+} \frac{\sqrt{2x}(x-1)}{|x-1|}$ is an indeterminate form $\frac{0}{0}$ but we can cancel because of the one-sided limit, that is, as $x \to 1^+$, $|x-1| = (x-1)$ and so $\lim_{x\to 1^+} \frac{\sqrt{2x}(x-1)}{|x-1|} = \lim_{x\to 1^+} \frac{\sqrt{2x}(x-1)}{(x-1)} = \lim_{x\to 1^+} \sqrt{2x} = \sqrt{2}$.

   (c) The limit does not exist because the left hand limit does not exist.

   (d) The quantity b^2+1 is always greater or equal to 1 so we simply have $f(b^2+1) = -(b^2+1)+1 = -b^2$.

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3. (18 pts) For what value(s) of k is the function $f(x) = \begin{cases} \sin(kx), & \text{if } x \leq 0 \\ 3x, & \text{if } x > 0 \end{cases}$ continuous at $x = 0$. A complete answer will include the definition of continuity.

   Solution: We have $f(0) = 0$ and so to be continuous both one-sided limits must equal 0 as $x \to 0^{\pm}$. We compute $\lim_{x\to 0^-} \sin(kx) = \sin(0) = 0$ and similarly $\lim_{x\to 0^+} 3x = 0$ and so the function is continuous independent of k, i.e. $k \in \mathbb{R}$.

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4. (20 pts) Show the equation $x + 2\cos(4x) = 0$ has at least one solution. Explain your work.

   Solution: It must be noted that the function $f(x) = x + 2\cos(4x)$ is continuous. We can use the IVT to show existence of roots. Note that $f(-\pi/8) = -\pi/8 < 0$ and $f(\pi/8) = \pi/8 > 0$ and so there must exist a number $c \in (-\pi/8, -\pi/8)$ so that $f(c) = 0$.

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5. Consider the function $f(x) = \dfrac{2}{3x+3}$.

   (a) (14 pts) Find the rate of change of $f(x)$ at $x = a$.

   (b) (3 pts) Using part (a) find the rate of change of $f(x)$ at $x = -1$.

   (c) (3 pts) Using part (a) find the rate of change of $f(x)$ at $x = 0$.

   (d) (20 pts) Using the above information find the equation of two different tangent lines that are parallel to the line that goes through the points $(-2, 4)$ and $(-5, 6)$.

Solution:

(a) We have

$$\begin{aligned}
\lim_{x\to a} \frac{f(x) - f(a)}{x - a} &= \lim_{x\to a} \frac{\frac{2}{3x+3} - \frac{2}{3a+3}}{x - a} \\
&= \lim_{x\to a} \frac{\frac{2}{3(x+1)} - \frac{2}{3(a+1)}}{x - a} \\
&= \frac{2}{3} \lim_{x\to a} \frac{\frac{a+1-(x+1)}{(x+1)(a+1)}}{x - a} \\
&= \frac{2}{3} \lim_{x\to a} \frac{a - x}{(x+1)(a+1)(x-a)} \\
&= \frac{2}{3} \lim_{x\to a} \frac{-1}{(x+1)(a+1)} = \frac{-2}{3(a+1)^2}
\end{aligned}$$

(b) The rate of change at $x = -1$ does not exist. The one-sided limits of the derivative both are $-\infty$ so that is an acceptable answer as well.

(c) Plug in 0... we have $f'(0) = -2/3$.

(d) The line that goes through the points $(-2, 4)$ and $(-5, 6)$ has slope $m = \dfrac{4-6}{-2-(-5)} = \dfrac{-2}{3}$. We need to set the derivative equal to $\dfrac{-2}{3}$ and solve for a,

$$\frac{-2}{3(a+1)^2} = -\frac{2}{3} \iff (a+1)^2 = 1 \iff a = 0 \text{ or } a = -2.$$

We have $f(0) = \frac{2}{3}$ and $f(-2) = -\frac{2}{3}$ and so out two tangent lines are

$$\begin{aligned}
t_1 &: \left(y - \frac{2}{3}\right) = -\frac{2}{3}(x - 0), \text{and} \\
t_2 &: \left(y + \frac{2}{3}\right) = -\frac{2}{3}(x + 2)
\end{aligned}$$

or

$$\begin{aligned}
t_1 &: y = -\frac{2}{3}(x - 1), \text{and} \\
t_2 &: y = -\frac{2}{3}(x + 3)
\end{aligned}$$

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6. The function $g(x) = x|x|$ is differntiable at $x = 0$. Show this by

   (a) (x pts) Define $g(x)$ as a piecewise function.

   (b) (y pts) Using the definition of the derivative consider the left and right hand limits of the difference quotient at 0.

Solution:

(a)

$$g(x) = \begin{cases} -x^2, & \text{if } x \leq 0 \\ x^2, & \text{if } x \geq 0 \end{cases}$$

(b) Left hand limit:

$$\lim_{h \to 0^-} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0^-} \frac{-(h)^2 - 0}{h} = \lim_{h \to 0^-} -h = 0$$

and the right hand limit:

$$\lim_{h \to 0^+} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0^-} \frac{(h)^2 - 0}{h} = \lim_{h \to 0^-} h = 0$$

and so the derivative exists.