Print Name ________________________________

APPM 1350 **Final Exam** **Summer 2016**

On the front of your bluebook, please write: a grading key, your name, student ID, your lecture number and instructor. This exam is worth 100 points and has 5 questions on both sides of this paper.

- Submit this exam sheet with your bluebook. However, nothing on this exam sheet will be graded. Make sure all of your work is in your bluebook.
- **Show all work and simplify your answers!** Answers with no justification will receive no points.
- Please begin each problem on a new page.
- No notes or papers, calculators, cell phones, or electronic devices are permitted.

---

1. Evaluate the following integrals. Show all work to justify your answer and make sure to simplify as much as possible.

   (a) (6 pts) $\displaystyle\int \frac{x+2}{\sqrt{x^2+4x}}\,dx$

   (b) (6 pts) $\displaystyle\int \frac{\sinh x}{e^x}\,dx$

   (c) (6 pts) If f is continuous and $\displaystyle\int_0^9 f(x)\,dx = 4$, find $\displaystyle\int_0^3 xf(x^2)\,dx$.

   Solution:

   (a) Let $u = x^2 + 4x$ and $du = (2x+4)\,dx$.

   Then $\displaystyle\int \frac{x+2}{\sqrt{x^2+4x}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\int u^{-1/2}\,du = u^{1/2} + C = \sqrt{x^2+4x} + C.$

   (b) By definition $\sinh x = \dfrac{e^x - e^{-x}}{2}$ and $\displaystyle\int \frac{\sinh x}{e^x}\,dx = \int \frac{e^x - e^{-x}}{2e^x}\,dx = \frac{1}{2}\int \left(1 - e^{-2x}\right)\,dx = \frac{x}{2} + \frac{e^{-2x}}{4} + C.$

   (c) Let $u = x^2$ then $du = 2x\,dx$. Then $\displaystyle\int_0^3 xf(x^2)\,dx = \frac{1}{2}\int_0^9 f(u)\,du = \frac{1}{2}(4) = 2.$

---

2. Find $\dfrac{dy}{dx}$ for the following. Show all work to justify your answer and make sure to simplify as much as possible.

   (a) (6 pts) $y = (\sin x)^x$

   (b) (6 pts) $ye^{x^2} = \cos^{-1}(e^y)$

   (c) (6 pts) $\displaystyle y = \int_e^{e^x} t^{\ln t}\,dt$

   Solution:

(a) Taking the natural logarithm of both sides $\ln y = x\ln(\sin x)$ and so $\frac{1}{y}\frac{dy}{dx} = \ln(\sin x) + \frac{x}{\sin x}\cos x$ Since $y = (\sin x)^x$ then $\frac{dy}{dx} = (\sin x)^x \ln(\sin x) + x(\sin x)^x \cot x$.

(b) Taking the derivative we get $\frac{dy}{dx}e^{x^2} + 2xye^{x^2} = -\frac{e^y}{\sqrt{1-e^{2y}}}\frac{dy}{dx}$ and so:

$\frac{dy}{dx}e^{x^2}\sqrt{1-e^{2y}} + 2xye^{x^2}\sqrt{1-e^{2y}} = -e^y\frac{dy}{dx}$ and $\frac{dy}{dx} = \frac{-2xye^{x^2}\sqrt{1-e^{2y}}}{e^{x^2}\sqrt{1-e^{2y}}+e^y}$.

(c) Let $u = e^x$ then $\frac{du}{dx} = e^x$. So $\frac{dy}{dx} = \frac{d}{dx}\int_e^{e^x} t^{\ln t}\,dt = \frac{d}{du}\left(\int_e^u t^{\ln t}\,dt\right)\frac{du}{dx} = e^x u^{\ln u}$ by the fundamental theorem of calculus, and we get after substituting back in for u: $e^x e^{x\ln e^x} = e^{(x^2+x)}$.

---

3. Answer the following.

Given $f(x) = \frac{e^x}{x}$ with, $f'(x) = \frac{e^x(x-1)}{x^2}$ and, $f''(x) = \frac{e^x(x^2-2x+2)}{x^3}$, find the following for f.

Make sure to state any rules or theorems you utilize.

(a) (3 pts) State the domain of f.
(b) (8 pts) Find all asymptote(s) for f. Justify your answer(s) using the appropriate limits.
(c) (5 pts) Find the intervals of increase and decrease for the function f. Justify your answer(s).
(d) (5 pts) Find the local maximum and minimum values for the function f. Justify your answer(s).
(e) (6 pts) Find the intervals of concavity and the inflection points for the function f. Justify your answer(s).
(f) (7 pts) Use parts (a) - (e) to sketch the graph of f. LABEL the asymptote(s), maximum(s), minimum(s), and inflection point(s) on your graph.

Solution:

(a) The domain of the function is: $(-\infty, 0) \cup (0, \infty)$.

(b) There is a vertical asymptote at $x = 0$ since $\lim_{x\to 0^-} f(x) = -\infty$. There is a horizontal asymptote at $y = 0$ since $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\frac{e^x}{x} = 0$.

Remark: We see that there is no horizontal asymptote as x goes to infinity since $\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{e^x}{x} = \lim_{x\to\infty} e^x = \infty$ by L'Hospital's rule.

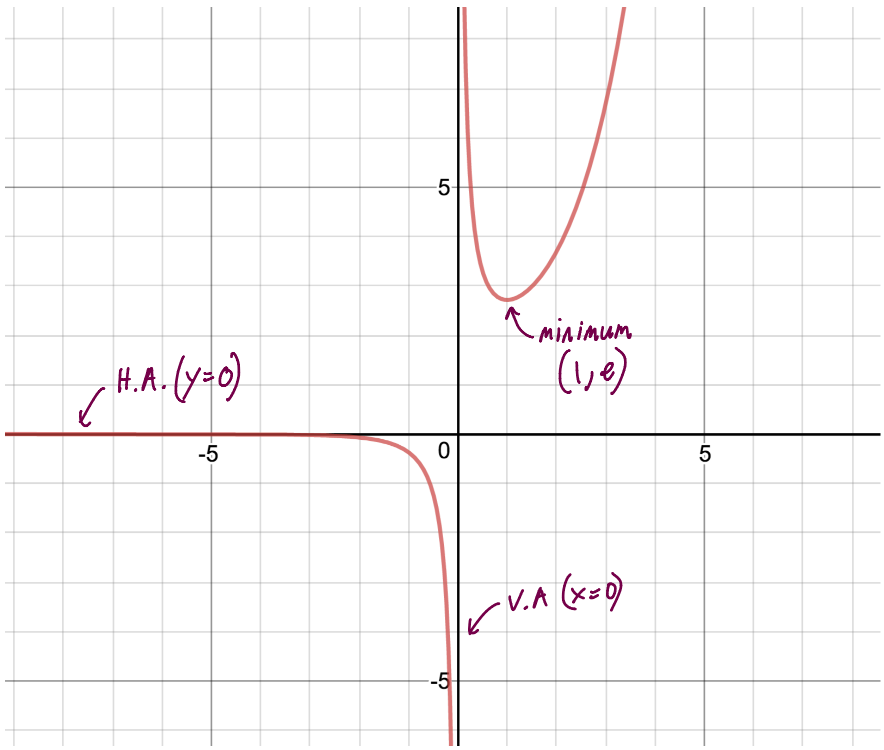
(c) Finding the critical values of $f'(x) = \frac{e^x(x-1)}{x^2}$ we see that $f'(x) = 0$ when $x = 1$ and $f'(x)$ DNE at $x = 0$. The function is decreasing on $(-\infty, 0)$ and $(0, 1)$ and increasing on $(1, \infty)$.

(d) By the first derivative test there is a minimum at $(1, e)$.

(e) $f''(x)$ is never equal to zero. $f''(x)$ DNE at $x = 0$, and we get that $f(x)$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$.

Remark: There are no inflection points.

(f)



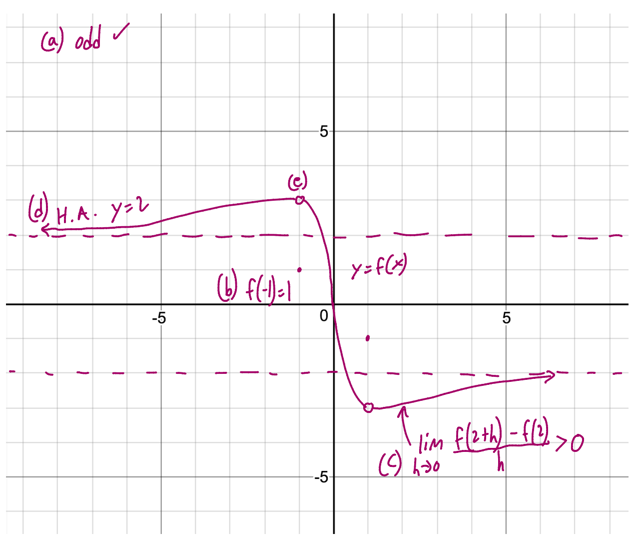
---

4. (12 pts) Sketch a function $y = f(x)$ that satisfies **all** of the following conditions. No explanation is necessary. Clearly label all important features of the graph.

(a) $f(-x) = -f(x)$ (b) $f(-1) = 1$ (c) $\lim_{h \to 0} \frac{f(2+h) - f(2)}{h} > 0$

(d) $\lim_{x \to -\infty} f(x) = 2$ (e) $\lim_{x \to -1} f(x) = 3$

Solution:



5. Some unrelated questions:

   (a) (6 pts) Find the linearization of $f(x) = \sqrt{1-x}$ at $a = -3$ and use the linearization to approximate $\sqrt{5}$. Show all work to justify your answer and make sure to simplify as much as possible.

   (b) (6 pts) Suppose a rectangle is entirely contained in the first quadrant of the xy-plane. The rectangle borders the x-axis and y-axis and its upper right corner touches the curve $y = \frac{2}{x}$. What dimensions minimize the perimeter of the rectangle? Show all work to justify your answer and make sure to simplify as much as possible.

   (c) (6 pts) **True** or **False**: $\displaystyle\int_{-1}^{1} \frac{\sin x}{1+x^2}\, dx = 0$. Justify your answer for full credit.

Solution:

(a) The linearization is given by: $L(x) = f(a) + f'(a)(x-a) = f(-3) + f'(-3)(x+3)$. We see that $f(-3) = \sqrt{4} = 2$ and $f'(-3) = -\frac{1}{2\sqrt{4}} = -\frac{1}{4}$. So $L(x) = 2 - \frac{1}{4}(x+3)$ and $L(x) = -\frac{x}{4} + \frac{5}{4}$. When $1 - x = 5$ then $x = -4$ and $L(-4) = \frac{4}{4} + \frac{5}{4} = \frac{9}{4}$.

(b) The perimeter of a rectangle is $P = 2x + 2y = 2x + 2\left(\frac{2}{x}\right) = 2x + \frac{4}{x}$. Note that the practical domain is $(0, \infty)$. Taking the derivative we get $\frac{dP}{dx} = 2 - \frac{4}{x^2}$. Setting $\frac{dP}{dx} = 0$, then $x = \sqrt{2}$ is the only critical value in the practical domain. To see that this is a minimum we find $\frac{d^2P}{dx^2} = \frac{8}{x^3}$ which is positive at $x = \sqrt{2}$, so we have a minimum by the second derivative test. Thus the dimensions of the rectangle that minimize perimeter are $x = \sqrt{2}$ and $y = \frac{2}{\sqrt{2}} = \sqrt{2}$.

(c) **True**. Note that $\frac{\sin x}{1+x^2}$ is odd since $\frac{\sin(-x)}{1+(-x)^2} = -\frac{\sin x}{1+x^2}$, then $\displaystyle\int_{-1}^{1} \frac{\sin x}{1+x^2}\, dx = 0$.