1. The following problems are not related.

(a)(8 pts) Approximate the area of the region bounded by the function $f(x) = \begin{cases} \sqrt{4-x^2}, & \text{if } 0 \le x < 2 \\ \frac{1}{2}x - 1, & \text{if } x \ge 2 \end{cases}$ and the x-axis from $x = 0$ to $x = 4$ using a Riemann Sum with 4 subintervals of equal length and left endpoints.

(b)(8 pts) Suppose an object has velocity $v(t) = t^2 - 4$, find the *total distance* travelled by the object for $0 \le t \le 4$.

(c)(8 pts) If $\int_0^4 g(x)\,dx = 3.4$, find the *net area* of the region bounded by the curve $h(x) = 1 - 2g(x)$ for $0 \le x \le 4$.

Solution: (a)(8 pts) Note that $\Delta x = (b-a)/n = (4-0)/4 = 1$ and so $x_i = a + i\Delta x = 0 + i \cdot 1 = i$ for $i = 0, 1, 2, 3, 4$. Thus, using left endpoints, we have

$$L_4 = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x = f(0)\cdot 1 + f(1)\cdot 1 + f(2)\cdot 1 + f(3)\cdot 1 = 2 + \sqrt{3} + 0 + \frac{1}{2} = \boxed{\frac{5}{2} + \sqrt{3}}$$

(b)(8 pts) The total distance is given by

$$\begin{aligned} \int_0^4 |t^2 - 4|\,dt &= \int_0^2 -(t^2-4)\,dt + \int_2^4 (t^2 - 4)\,dx \\ &= \left[4t - \frac{t^3}{3}\right]_0^2 + \left[\frac{t^3}{3} - 4t\right]_2^4 = \left(8 - \frac{8}{3}\right) + \left[\left(\frac{64}{3} - 16\right) - \left(\frac{8}{3} - 8\right)\right] = \frac{16}{3} + \frac{16}{3} + \frac{16}{3} = \boxed{16} \end{aligned}$$

(c)(8 pts) The net area is

$$\int_0^4 h(x)\,dx = \int_0^4 (1 - 2g(x))\,dx = \int_0^4 dx - 2\int_0^4 g(x)\,dx = 4 - 2(3.4) = 4 - 6.8 = \boxed{-2.8}$$

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2. Evaluate the following integrals, remember to show all work and simplify your answer:

(a)(9 pts) $\displaystyle\int \frac{d\theta}{\cos^2(\theta)\sqrt[3]{1+\tan(\theta)}}$ (b)(9 pts) $\displaystyle\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx$ (c)(9 pts) $\displaystyle\int_{-a}^{a}\left[\pi a - \pi|x| + \sqrt{a^2-x^2}\right]dx$, for $a > 0$.

Solution: (a)(9 pts) Let $u = 1 + \tan(\theta)$ then $du = \sec^2(\theta)d\theta = d\theta/\cos^2(\theta)$, thus

$$\int \frac{d\theta}{\cos^2(\theta)\sqrt[3]{1+\tan(\theta)}} = \int \frac{du}{u^{1/3}} = \frac{3}{2}u^{2/3} + C = \boxed{\frac{3}{2}(1+\tan(\theta))^{2/3} + C}$$

(b)(9 pts) Let $u = 2x - 1 \Rightarrow x = (u+1)/2$ and $dx = du/2$ thus $xdx = \frac{(u+1)}{2}\cdot\frac{du}{2} = \frac{(u+1)}{4}du$ and so

$$\begin{aligned} \int_1^5 \frac{xdx}{\sqrt{2x-1}} &= \frac{1}{4}\int_1^9 \frac{u+1}{\sqrt{u}}\,du = \frac{1}{4}\int_1^9 (u^{1/2} + u^{-1/2})\,du \\ &= \frac{1}{4}\left(\frac{2}{3}u^{3/2} + 2u^{1/2}\right)\Big|_1^9 = \frac{1}{4}\left[\left(\frac{2}{3}\cdot 27 + 2\cdot 3\right) - \left(\frac{2}{3} + 2\right)\right] = \frac{1}{4}\left[24 - \frac{8}{3}\right] = 6 - \frac{2}{3} = \boxed{\frac{16}{3}} \end{aligned}$$

(c)(9 pts) We can use symmetry of functions and geometry for this integral, note that

$$\begin{aligned} \int_{-a}^{a} \pi a\,dx = \pi a\cdot[(a - (-a)] &= 2\pi a^2 \text{ (area of a rectangle)} \\ \int_{-a}^{a} -\pi|x|\,dx = -2\pi\int_0^a |x|\,dx = -2\pi\cdot\frac{1}{2}a\cdot a &= -\pi a^2 \text{ } (y = -\pi|x| \text{ is even and area of a triangle)} \\ \int_{-a}^{a}\sqrt{a^2-x^2}\,dx &= \frac{1}{2}\pi a^2 \text{ (area of a semicircle of radius } a) \end{aligned}$$

and so we have $\displaystyle\int_{-a}^{a}\left[\pi a - \pi|x| + \sqrt{a^2-x^2}\right]dx = 2\pi a^2 - \pi a^2 + \frac{1}{2}\pi a^2 = \boxed{\frac{3\pi}{2}a^2}$

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3. The following problems are not related, justify your answers and cite any theorems that you use.

(a)(8 pts) Find dy/dx if $y = \left(\int_0^{3x}\sin(t^4)\,dt\right)^3$ (Justify your answer and cite any theorems that you use.)

(b)(8 pts) The height H (in feet) of a palm tree after growing for t years is given by $H(t) = \sqrt{t+1} + 5t^{1/3}$, find the tree's average height for $0 \le t \le 8$.

(c)(8 pts) Suppose $F(x)$ is an antiderivative of $f(x) = (\sin x)/x$, $x > 0$. Express $\displaystyle\int_1^3 \frac{\sin(2x)}{x}\,dx$ in terms of F. (Justify your answer and cite any theorems that you use.)

Solution: (a)(8 pts) Using the chain rule and the <u>Fundamental Theorem of Calculus *(Part I)*</u> we have

$$dy/dx = \frac{d}{dx}\left(\int_0^{3x}\sin(t^4)\,dt\right)^3 = 3\left(\int_0^{3x}\sin(t^4)\,dt\right)^2\cdot\sin((3x)^4)\cdot 3 = \boxed{9\left(\int_0^{3x}\sin(t^4)\,dt\right)^2\sin(81x^4)}$$

(b)(8 pts) We have, by definition,

$$\begin{aligned} H_{ave} = \frac{1}{8-0}\int_0^8(\sqrt{t+1} + 5t^{1/3}) = \frac{1}{8}\underbrace{\int_1^9 u^{1/2}\,du}_{u=t+1} + \frac{5}{8}\cdot\frac{3}{4}\cdot t^{4/3}\Big|_0^8 &= \frac{1}{8}\cdot\frac{2}{3}u^{3/2}\Big|_1^9 + \frac{15}{32}\cdot 16 \\ &= \frac{1}{12}(27-1) + \frac{15}{2} = \frac{26}{12} + \frac{90}{12} = \frac{116}{12} = \frac{29}{3} \end{aligned}$$

thus the tree's average height over the first 8 years was $\boxed{29/3 \text{ ft.}}$

(c)(8 pts) Using the u-substitution $u = 2x \Rightarrow x = u/2$ and $dx = du/2$ and so $dx/x = \frac{2}{u}\cdot\frac{du}{2} = du/u$ thus

$$\int_1^3 \frac{\sin(2x)}{x}\,dx = \int_2^6 \frac{\sin(u)}{u}\,du = F(u)\Big|_2^6 = \boxed{F(6) - F(2)}$$

where the last equality follows by the <u>Fundamental Theorem of Calculus *(Part Deux)*</u>.

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4. The following problems are not related, remember to justify your answers.

(a)(10 pts) Ralphie has asked you to design an open-top stainless steel box. It is to have square base and a volume of 32 ft^3, to be welded from thin stainless steel and to weigh no more than necessary. Find the dimensions (length, width, and height) of such a box.

(b)(10 pts) Suppose $f(x) = x^4/4 + x^2 - 3x$, **set up** (but do <u>not</u> evaluate) Newton's method to approximate a *critical point* of the function $f(x)$. (In other words, give a formula for calculating the $(n+1)$-st approximation, x_{n+1}, in terms of the n-th approximation, x_n.) Simpify your answer.

(c)(5 pts) Which of the five choices <u>given below</u> is the limit $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi}{n}\sin\left(2\pi + \frac{2\pi i}{n}\right)$ equivalent to? Clearly write down your answer(s) in your blue book, **no justification necessary** - *be sure to copy down the <u>entire answer</u>, don't just write down the roman numeral of your choice(s).*

(*i*) $\displaystyle\int_\pi^{3\pi}\frac{\sin(2x)}{2}\,dx$ (*ii*) $\displaystyle\int_\pi^{2\pi}\sin(2x)\,dx$ (*iii*) $\displaystyle\int_0^{\pi}\sin(2x+\pi)\,dx$ (*iv*) $\displaystyle\int_0^{\pi}\frac{\pi}{x}\sin(2\pi + x)\,dx$ (*v*) $\displaystyle\int_{-\pi}^{\pi}\sin(2x+\pi)\,dx$

Solution: (a)(10 pts) For the vat to weigh the least we need to minimize the amount of material used to construct the vat, so we need to minimize the surface area of the vat, thus if x represents the width of the base and if h represents the height of the vat, then the optimization problem we are solving is

$$\text{minimize } SA = x^2 + 4xh \text{ subject to } V = x^2h = 32.$$

Using the constraint we see that $h = 32/x^2$, $x > 0$ and so we need to minimize

$$SA(x) = x^2 + 4x \cdot \frac{32}{x^2} = x^2 + \frac{128}{x} \Longrightarrow \frac{d}{dx}SA = 2x - \frac{128}{x^2}$$

and so $\frac{d}{dx}SA = 0$ implies $x^3 = 64 \Rightarrow x = 4$ and note that $\frac{d}{dx}SA$ is undefined when $x = 0$ but this is not a feasible answer since this not in the domain of the problem (the volume is positive). Now note that $\frac{d^2}{dx^2}SA\Big|_{x=4} = 2 + \frac{256}{x^3}\Big|_{x=4} > 0$ by the 2nd Derivative Test and so we have a local minimum at $x = 4$, thus the dimensions of the vat that will meet Ralphie's specifications are $\boxed{w \times l \times h = 4 \text{ ft} \times 4 \text{ ft} \times 2 \text{ ft.}}$

(b)(10 pts) Note that if $f(x) = x^4/4 + x^2 - 3x$ then $f'(x) = x^3 + 2x - 3$ and so we wish to set up Newtons's Method to approximate a root of the equation $f'(x) = 0$, thus we have

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)} = x_n - \frac{x_n^3 + 2x_n - 3}{3x_n^2 + 2} = \frac{2x_n^3 + 3}{3x_n^2 + 2} \text{ for } n = 0, 1, 2, \ldots$$

(c)(5 pts) $\boxed{\text{Choice } (ii).}$ Consider the integral $\int_{\pi}^{2\pi} \sin(2x)\,dx$ then $\Delta x = (2\pi - \pi)/n = \pi/n$ and $x_i = a + i\Delta x = \pi + i\pi/n$, now if we use right endpoints, *i.e.* $x_i^* = x_i$ then we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \sin(2(\pi + i\pi/n))\frac{\pi}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{n} \sin\left(2\pi + \frac{2\pi i}{n}\right)$$