Chapter 1

Classification of Various Cell

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1.1 Recap of previous lectures and basics

In the previous lectures we have gone through the aspects of limit and continuity along with the concept of homeomorphism, isometry, uniform continuity and lipschits continuity etc. In this section we will gone through the concept of discontinuity, differentiability and related concepts.

1.2 Discontinuity

1.2.1 Definition

Let f be defined on an interval (a, b). Assume c ϵ [a, b). If $f(x) \to$ A as x $\to$ c through values greater than c, we say that A is the right hand limit off at c and we indicate this by writing

$$\lim_{x\to\ c+} f(x) = A$$

The righthand limit A is also denoted by $f(c+)$. Note that f need not be defined at the point c itself. If f is defined at c and if $f(c+) = f(c)$, we say that f is continuous from the right at c.

Left hand limits and continuity from the left at c are similarly defined if c ϵ (a, b].

$$\lim_{x\to\ c-} f(x) = A$$

If $a < c < b$, then f is continuous at c if, and only if,

$$f(c) = f(c+) = f(c-)$$

We say c is a discontinuity of f if f is not continuous at c. In this case one of the following conditions is satisfied:
(a) Either $f(c+)$ or $f(c-)$ does not exist.
(b) Both $f(c+)$ and $f(c-)$ exist but have different values.
(c) Both $f(c+)$ and $f(c-)$ exist and $f(c+) = f(c-) \neq f(c)$.

In case (c), the point c is called a removable discontinuity, since the discontinuity could be removed by redefining f at c to have the value $f(c+) = f(c-)$. In cases (a) and (b), we call c an irremovable discontinuity because the discontinuity cannot be removed by redefining f at c.

1.2.2 Examples

Ex1.

$f(x) = \frac{x}{|x|}$

$\lim_{x\to\ 0+} f(x) = 1 \Rightarrow$ Right limit exist

Also

$\lim_{x\to\ 0-} f(x) = -1 \Rightarrow$ Left limit exist

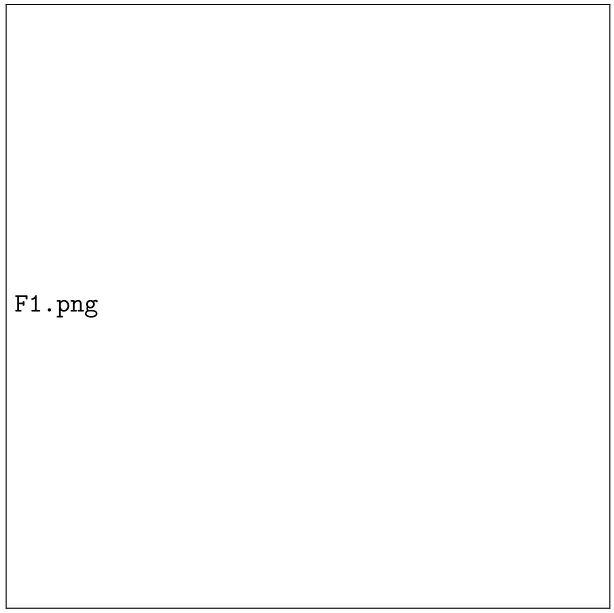

Figure 1.1:

$\lim_{x\to\ 0+} f(x) \neq \lim_{x\to\ 0-} f(x)$, However right and left limit exist so this is jump discontinuity.

Ex2.

$f(x) = 1$; if $x \neq 0$

$f(x) = 0; x = 0$

$\lim_{x\to\ 0+} f(x) = 1 \Rightarrow$ Right limit exist

Also

$\lim_{x\to\ 0-} f(x) = 1 \Rightarrow$ Left limit exist

$\lim_{x\to\ 0+} f(x) = \lim_{x\to\ 0-} f(x)$, However it is not equal to $f(x)$ so this is removal discontinuity.

Ex3.

$f(x) = \frac{1}{x}$;

$\lim_{x\to\ 0+} f(x) = \infty \Rightarrow$ Right limit does not exist

F2.png

Figure 1.2:

F3.png

Figure 1.3:

Also

$\lim_{x\to\ 0-} f(x) = -\infty \Rightarrow$ Left limit does not exist

Since $\lim_{x\to\ 0+} f(x) and \lim_{x\to\ 0-} f(x)$, does not exist therefore function have irremovable discontinuity at 0.

1.3 Differentiability

1.3.1 Definition

Let f be defined on an open interval (a, b), and assume that c ϵ (a,b) Then f is said to be differentiable at c whenever

$$\lim_{x\to\ c} \frac{f(x)-f(c)}{x-c} exist$$

Then the limit, denoted by $f'(c)$, is called the derivative of f at c.

This limit process defines a new function f', whose domain consists of the points in (a, b) at which f is differentiable. The function f' is called the first Derivative of f.

Similarly, the $n^{(th)}$ derivative of f, denoted by $f^{(n)}$, is defined to be the first derivative of $f^{(n-1)}$, for n = 2, 3, ... (By our definition, we do not consider $f^{(n)}$ unless $f^{(n-1)}$ is defined on an open interval.) Other notations with which we are familiar

$$f'(c) = Df(c) = df(c) = \tfrac{df(c)}{dx} = \tfrac{dy}{dx}$$

Process of finding f' from f called differentiation.

1.3.2 Theorem 1

If f is defined on (a, b) and differentiable at a point c in (a, b), then there is a function $f*$ (depending on f and on c) which is continuous at c and which satisfies the equation

$$f(x) - f(c) = (x-c)f^*(x) \quad ..... \ (1)$$
$$\forall x\epsilon \text{ (a,b) with } f^*(x) = f'(c)$$

Conversely, if there is a function f^*, continuous at c, which satisfies above equation, then f is differentiable at c and $f'(c) = f^*(c)$.

Proof: If $f'(c)$ exists, let f^* be defined on (a, b) as follows :

$$f^*(x) = \frac{f(x)-f(c)}{x-c}$$

if x $\neq$ c, and $f^*(x) = f'(c)$

Then f^* is continuous at c and (1) holds for all x in (a, b). Conversely, if (1) holds for some f^* continuous at c, then by dividing by x - c and letting x $\to c$ we see that $f'(c)$ exists and equals $f^*(c)$.

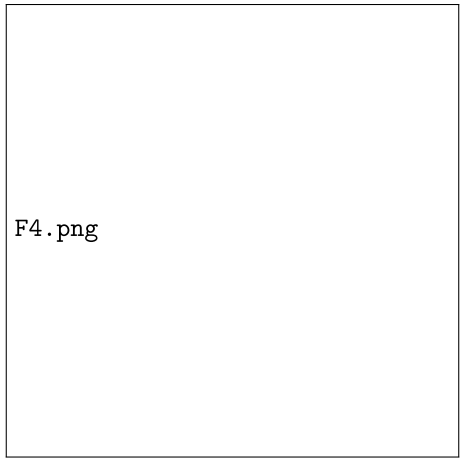

Figure 1.4:

1.3.3 Theorem 2

If f is differentiable at c, then f is continuous at c.

Proof: Let $x \to c$ in Eq (1)

Equation (1) has a geometric interpretation which helps us gain insight into its meaning. Since f^* is continuous at c, $f^*(x)$ is nearly equal to $f^*(c) = f'(c)$. if x is near c. Replacing $f^*(x)$ by $f'(c)$ in (1) we obtain the equation

$$f(x) - f(c) = (x - c)f'(c)$$

which should be approximately correct when x - c is small. In other words, if f is differentiable at c, then f is approximately a linear function near c. (See Fig. 5). Differential calculus continually exploits this geometric property of functions.

1.3.4 Theorem 3

Assume f and g are defined on (a, b) and differentiable at c. Then $f + g, f - g$, and $f - g$ are also differentiable at c. This is also true for f/g if $g(c) \neq 0$. The derivatives at c are given by the following formulas:

a) $(f \pm g)'(c) = f'(c) \pm g'(c)$,

b) $(f * g)'(c) = f(c)g'(c) + f'(c)g(c)$,

c) $(f/g)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{g(c)^2}$ provided $g(c) \neq 0$.

F5.jpg

Figure 1.5:

1.4 One Side Derivative

Let f be defined on a closed interval S and assume that f is continuous at the point c in S. Then f is said to have a righthand derivative at c if the righthand limit

$$\lim_{x\to\ c} \frac{f(x)-f(c)}{x-c}$$

exist as a finite value , or if the limit is $+\infty$ or -∞.

1.5 Maximum and Minimum of function

1.5.1 Local and Global Maximum

Let f be a real-valued function defined on a subset S of a metric space X, and assume a ϵ S. Then f is said to have a local maximum at a if there is a ball U(a) such that

$$f(x) < f(a) \forall x \epsilon \ \mathrm{U(a,r)} \cap S.$$

For global maximum

$$f(x) < f(a) \forall x \epsilon S.$$

This can be understand with the figure no 6.

F6.png

Figure 1.6:

1.5.2 Local and Global Minimum

Let f be a real-valued function defined on a subset S of a metric space X, and assume a ϵ S. Then f is said to have a local minimum at a if there is a ball U(a) such that

$$f(x) > f(a) \forall x \epsilon \text{ U(a,r)} \cap S.$$

For global minimum

$$f(x) > f(a) \forall x \epsilon S.$$

This can be understand with the figure no 7.

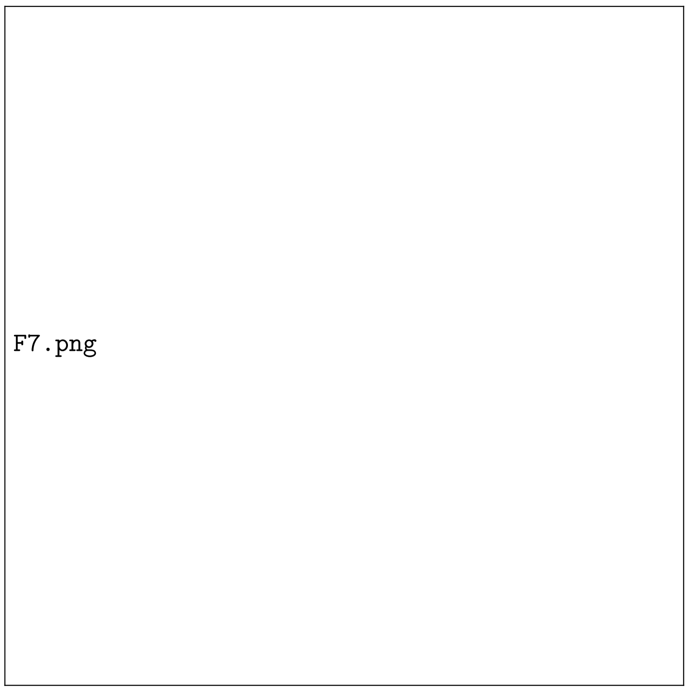

Figure 1.7:

1.5.3 Theorem 4

If there is maximum or minimum between a b then gradient at global/local maximum and minimum will be 0.

This can be seen from the following figure no 8.

1.6 Rolls Theorem

Assume f has a derivative (finite or infinite) at each point of an open interval (a, b), and assume that f is continuous at both endpoints a and b. If $f(a) = f(b)$ there is at least one interior point c at which $f'(c) = 0$.

This can be seen from the following figure no 9.

F8.png

Figure 1.8:

F9.png

Figure 1.9:

1.6.1 Theorem 5

Assume f has a derivative (finite or infinite) at each point of an open interval (a, b) and that f is continuous at the endpoints a and b.

a) If $f'(x) \geq 0$ in (a, b), then f is strictly increasing on [a, b].
b) If $f'(x) \leq 0$ in (a, b), then f is strictly decreasing on [a, b].
c) If $f'(x) = 0$ everywhere in (a, b) then f is constant on [a, b].

1.7 Taylor's Formula

As noted earlier, if f is differentiable at c, then f is approximately a linear function near c. That is, the equation

$$f(x) - f(c) = (x - c)f'(c)$$

is approximately correct when x - c is small. Taylor's theorem tells us that, more generally, f can be approximated by a polynomial of degree $n-1$ if f has a derivative of order n. Moreover, Taylor's theorem gives a useful expression for the error made by this approximation.

1.7.1 Theorem 6: Taylor's Theorem

Let f be a function having finite nth derivative $f^{(n)}$ everywhere in an open interval (a, b) and assume that $f^{(n-1)}$ is continuous on the closed interval [a, b]. Assume that c ϵ [a, b]. Then, for every x in [a, b], x $\neq$ c, there exists a point x_1 interior to the interval joining x and c such that

$$f(x) = f(c) + \sum_{k=1}^{n-1} \frac{f^{(k)}(c))}{k!}(x-c)^k + \frac{f^{(n)}(x_1)}{n!}(x-c)^n$$

Taylor's theorem will be obtained as a consequence of a more general result that is a direct extension of the generalized Mean-Value Theorem.

1.8 Partial Derivative

Let S be an open set in Euclidean space R^n, and let $f : S \to R$ be a real-valued function defined on S. If $x = (x_1, ..., x_n)$ and $c = (c_1, ..., c_n)$ are two points of S having corresponding coordinates equal except for the k^{th}, that is, if $x_i = c_i fori \neq k$ and if $x_k \neq c_k$, then we can consider the limit

$$\lim_{x_k \to c_k} \frac{f(x) - f(c)}{x_k - c_k}$$

In generalizing a concept from R^1 to R^n, we seek to preserve the important properties in the one-dimensional case. For example, in the one-dimensional case, the existence of the derivative at c implies continuity at c. Therefore it seems desirable to have a concept of derivative for functions of several variables which will imply continuity. Partial derivatives do not do this. A function of n variables can have partial derivatives at a point with respect to each of the variables and yet not be continuous at the point.

1.9 Directional Derivative

Let S be a subset of R^n, and let $f : S \to R^{n\circ}$ be a function defined on S with values in R^n. We want how f changes as we move from a point c in S along a line segment to a nearby point $c + u$, where $u \neq 0$. Each point on the segment can be expressed as $c + hu$, where h is real. The vector u describes the direction of the line segment. We assume that c is an interior point of S. Then there is an n-ball U(c; r) lying in S, and, if h is small enough, the line segment joining c to $c + hu$ will lie in U(c; r) and hence in S.

The directional derivative of f at c in the direction u, denoted by the symbol $f'(c; u)$, is defined by the equation

$$f'(c; u) = \lim_{h \to 0} \frac{f(c + hu) - f(c)}{h}$$

whenever the limit on the right exists.

Bibliography

[1] MATHEMATICAL ANALYSIS SECOND EDITION APOSTOL

[2] CLASS NOTES ON MATHEMATICAL FOUNDATION COURSE