Table of Contents

Chapter 1: Problem Statement

Characterize the deformation and the stress distribution of a thick walled cylinder of internal radius 40 mm and external radius 80 mm with an internal pressure of 50 MPa . Use an elastic ideal-Mises-Plastic material of Young's modulus $2*10^5$ MPa, Poisson's ratio $\upsilon = 0.25$ and yield stress $\sigma_y = 70$ MPa. A plane strain condition '$\varepsilon_{zz} = 0$' is assumed. Consider '1' Gauss point for the problem.

The boundary conditions for the problem are :

- σ_{rr} (r = a) = −p
- σ_{rr} (r = b) = 0

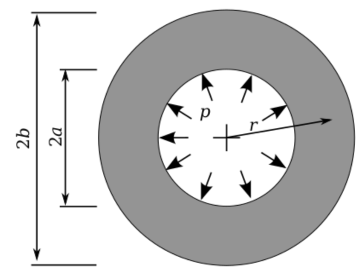

Figure 1: Thick-walled pipe

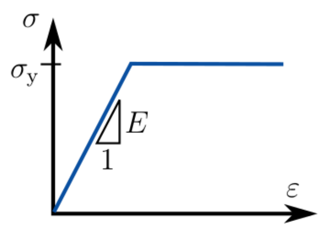

Figure 2: Elastic ideal-plastic material

Fig 1.1 : Problem Statement

Chapter 2: A Brief Overview

2.1 Introduction

Due to axisymmetric conditions, the only non-vanishing equilibrium condition is

$$0=\frac{\partial(r\,\sigma_{rr})}{\partial r}-\sigma_{\varphi\varphi} \tag{2.1}$$

with stresses and strains, written in Voigt notation as

$$\underline{\sigma}=\begin{bmatrix}\sigma_{rr}\\ \sigma_{\phi\phi}\end{bmatrix} \qquad \delta\underline{\varepsilon}=\begin{bmatrix}\delta\varepsilon_{rr}=\frac{\partial\,\delta u_r}{\partial r}\\ \delta\varepsilon_{\phi\phi=\frac{\delta u_r}{r}}\end{bmatrix} \qquad \underline{\varepsilon}=\begin{bmatrix}\varepsilon_{rr}=\frac{\partial u_r}{\partial r}\\ \varepsilon_{\phi\phi=\frac{u_r}{r}}\end{bmatrix} \tag{2.2}$$

The weak form of Eq. (1)

$$0=\delta w=-\int_a^b \delta\underline{\varepsilon}*\underline{\sigma}*r\,dr-[r\,\sigma_{rr}\,\delta u_r]_{r=a}^{b} \tag{2.3}$$

2.2 Discretization

The radial line in the domain can be discretized as iso-parametric line elements. The coordinates r and the radial displacement $u_r(r)$ in the elements are interpolated through the same *Ansatz* function via the nodal coordinates and nodal displacement respectively. The interpolation function N (ξ) is

$$\underline{N(\xi)}=\begin{bmatrix}\frac{1-\xi}{2} & \frac{1+\xi}{2}\end{bmatrix} \tag{2.4}$$

From Eq 1.4, for ξ [−1, 1] in each element,

$$\underline{u_r}=\underline{N}^T*\underline{\tilde{u}}=N_1u_1+N_2u_2 \tag{2.5}$$

$$\underline{r}=\underline{N}^T*\underline{\tilde{r}}=N_1r_1+N_2r_2 \tag{2.6}$$

The Element strain interpolation function is,

$$B=\begin{bmatrix}\frac{-1}{2*J} & \frac{1}{2*J}\\ \frac{N_1}{(N_1*r_1+N_2*r_2)} & \frac{N_2}{(N_1*r_1+N_2*r_2)}\end{bmatrix} \tag{1.7}$$

where the *Jacobian* ‘J’ maps element from the physical coordinate system ‘r’ to the natural coordinate system ξ. ‘r_1’ and ‘r_2’ are the *radial distance* of the *inner* and *outer* nodes of the element respectively.

The *Jacobian* of the element is given by,

$$J = \frac{r_2 - r_1}{2} \qquad (2.8)$$

The discretized form of Eqn.(1.3) for each element in the natural coordinate is

$$G^e = \delta w = \underline{\delta u_r} * \int_{-1}^{1} \underbrace{\underline{B^T} * \underline{\sigma} * \underline{N^T} * \tilde{r}(\xi) * \|J\| \partial \xi}_{F_{internal}} - \underbrace{\underline{p * a}}_{F_{external}} = 0 \qquad (2.9)$$

Since $\underline{\delta u_r}$ is arbitrary, $G^e = F_{internal} - F_{external}$ (2.10)

2.3 Numerical Integration

The term i $\boldsymbol{F_{internal}}$ s calculated via *Gauss Quadrature*. Depending on the accuracy requirements the number of integration points can the chosen. One point quadrature is used in this problem, with $w_{alpha} = 2$ and $\xi_{alpha} = 0$. The internal and external forces are given by,

$$\underline{F_{internal}} = w_\alpha * (\sum_n \underline{B^T}(\xi) * \underline{\sigma} * \underline{N^T}(\xi) * \underline{r} * \|J\|) \qquad (2.11)$$

$$\underline{F_{external}} = p * a \qquad (2.12)$$

Since , $\underline{\varepsilon} = \underline{B} * \underline{u}$ the element stiffness is given by,

$$(K_t) = w_\alpha * (\sum_n \underline{B^T}(\xi) * \underline{C^T} * \underline{B}(\xi) * (\underline{N^T}(\xi) * \underline{r})^T * \|J\|) \qquad (2.13)$$

2.4 Linearization and Incremental Solution

Due to elastic-plastic material type, the problem is non-linear and involves plastic strain contribution. Hence the Eqn. 1.9 cannot be solved directly,for obtaining nodal displacements. So *Newton Raphson Method* can be used, which follows an incremental solution strategy.

$$G(\hat{u} + \Delta \hat{u}) \approx G(\hat{u}) + \frac{\partial G}{\partial \hat{u}} \Delta \hat{u} = 0 \qquad (2.14)$$

2.5 Assembling the Global matrix

The $(K_e)_t$ and the $(F_e)_{int}$ from each element is calculated and assembled into a compatible *Global stiffness matrix* K_t and internal force vector, $F_{internal}$. Using Eqn. 1.10, Δu is calculated and incremented to get the displacement. The load is incremented once the internal force due to the displacement is in equilibrium with the external load in that time step

Chapter 3: An Overview of the Structure of the Program

The Program consists of five *.m files.

The input file ‘*Input.m’* sets the parameters from the user which include the inner radius and the outer radius of the cylinder, material parameters, internal pressure and the number of elements. The input file calls the ‘*elasticconvergence’* or the ‘*Main’* function, based on the *mode* selected by the user .

For each time step, the ‘*Main*’ function calls the ‘*Elementroutine*’ function in order to compute the values of shape matrix (N), B matrix and Jacobian value and calls the ‘*Materialroutine*’ function to get the stress and the material stiffness values. By obtaining these values, the internal forces and the element stiffness for each element is calculated.

In the ‘Elasticconvergence’ function, the radial displacement in the elastic regime ($p < p_{init}$) is calculated.

With the help of the obtained elemental values, the global stiffness and force matrices are formulated. For every time iteration, the values of global displacement matrix keeps updating, until the condition of convergence is reached. The graphs and distributions for displacement, as a function of radial length, is generated.

Chapter 5: Results

5.1 Convergence with the number of elements

The Numerical solution converge to the exact solution as the number of elements increase for the elastic problem. Max elastic load is **30 Mpa** (approx). This can be verified from the generated graph.

5.2 Convergence in single iteration for elastic problem

The Numerical solution converges within a single iteration for the elastic problem (Fig. 4.1).

5.3 Convergence for max Pressure

The convergence study for the load P_{max} shows that the solution converges for 1 elements and a time step $\Delta t = 0.2s$.

5.4 Results

The convergence of u(r) in the Elastic Regime, is shown in Fig 4.1. The displacement history of the pipe at r = b, is shown in Fig 4.2. The distributions of σ_{rr} , $\sigma_{\varphi\varphi}$ and u (r), along the radius 'r' at final loading t = 1, are shown in Fig 4.3, 4.4 and 4.5 respectively.

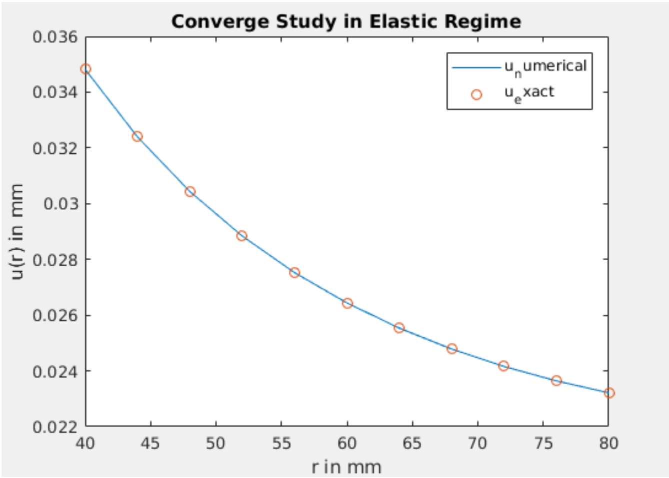

Fig 5.1 : Convergence Study in Elastic Regime

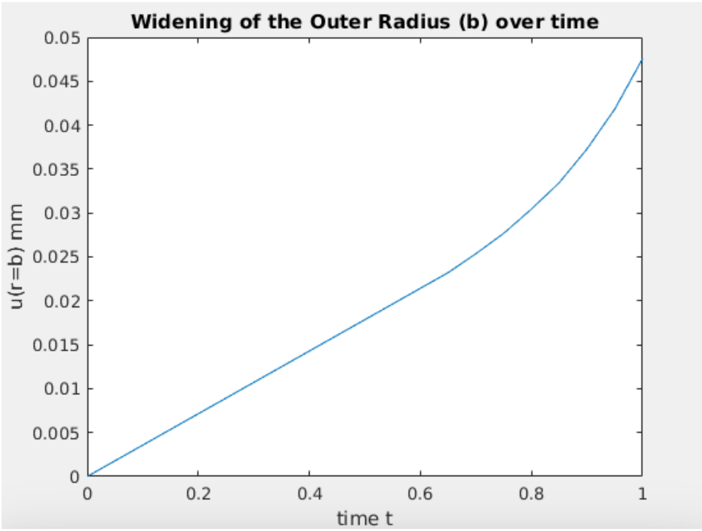

Fig 5.2 : Widening of the Outer Radius (b) over time

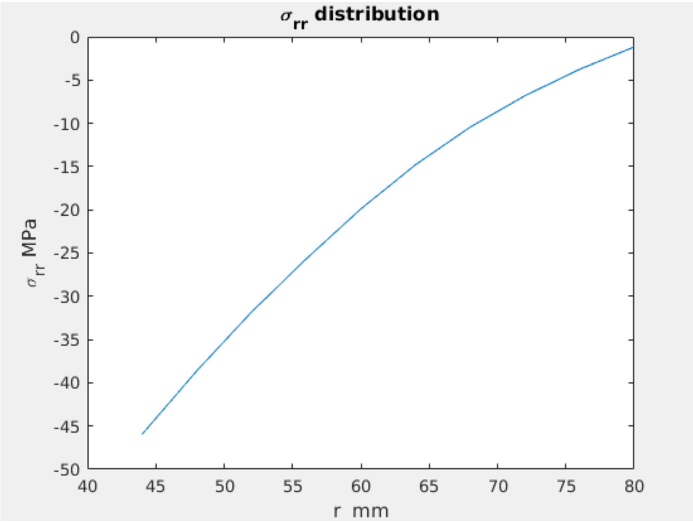

Fig 5.3 : Radial Stress distribution

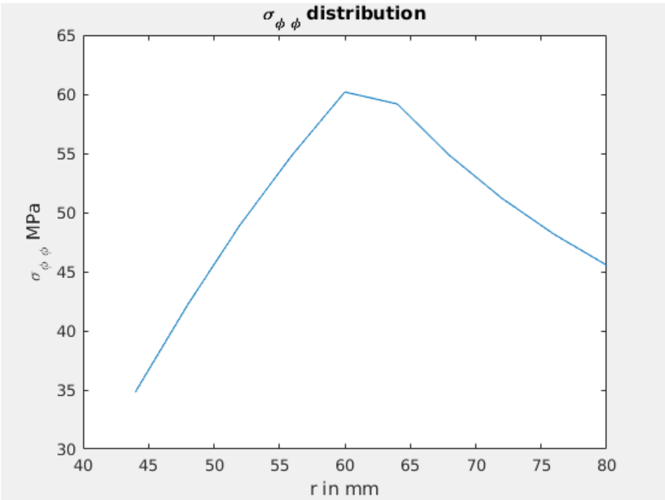

Fig 5.4 : Tangential Stress Distribution

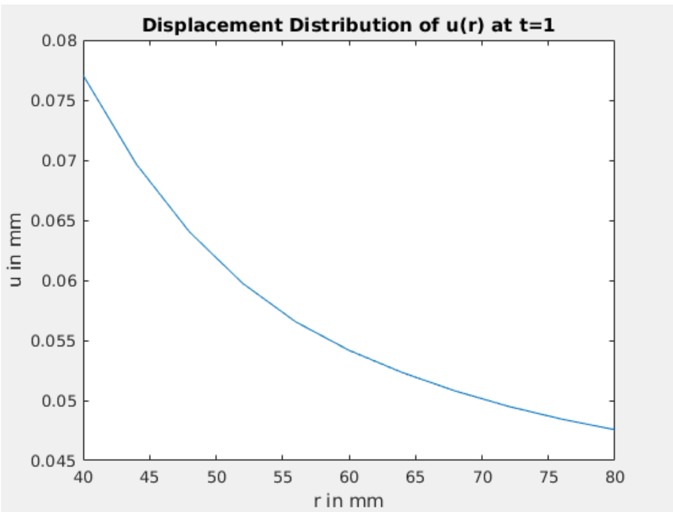

Fig 5.5 : Displacement Distribution of u(r) at t=1

5.5 Behavior at Pressure 60 Mpa

When the pressure is increased to **1.2*p_{max}**, high values of ‘u’ are encountered. The behavior states that all the elements are plastically deformed at this pressure, beyond which the ideal plastic material cannot withstand additional loading.