Spectra Math Module:

Day 1: Euler

Narrative: It is winter in Russia, and Leonhard Euler has just arrived at St. Petersburg Academy of Sciences. He is flooded by requests from the townsmen, local merchants, and councils to help solve their problems. Participants are enlisted to help solve them for him.

- Euler's identity:

  Russian mapmakers are cataloguing new territories and ask Euler how to keep track of regions, borders, and junctions.

  Each team gets:

  - Drinking straws + connectors (or toothpicks + clay)
  - Worksheets with simple planar graphs

  Tasks:

  1. Build one of the polyhedral frames physically (cube, tetrahedron, octahedron).
  2. Count vertices, edges, and faces.
  3. Verify Euler's Formula holds.
  4. Then solve a puzzle: Given V and E, predict F for a new "territory map".

- Bridge of Königsberg:

  The city council demands Euler solve whether a messenger can traverse all city bridges once without repeating any.

  But instead of Königsberg, its St. Petersburg's river islands:

  - Vasilyevsky Island
  - Admiralty Island
  - Petrograd Side
  - Fortress Island

  They relate to bridges.

  1. Each team must physically attempt a walk across all bridges without repeating.
  2. After trying and failing, they must deduce Euler's rule:

     "A network has a Eulerian path iff exactly 0 or 2 vertices have odd degree."

  3. They classify each vertex's degree and conclude it's impossible.

- Probability:

  Merchants ask Euler to resolve a gambling game happening on the frozen river. The stakes grow exponentially, and they fear bankruptcy.

  Each team plays the “St. Petersburg game” physically:

  1. Flip a coin.
  2. If heads appears on flip n, reward = 2^n
  3. They record outcomes for 4–5 rounds.

  Teams must then:

  - Observe the average payoff is volatile.
  - Explain why the expected value diverges.
  - Propose a practical cap on payout the merchants could adopt.

- Theorem on Partition:

  The Tsar imposes an emergency tax requiring gold coin to be grouped into exact partitions. Euler must show all possible ways to distribute N coins.

  Each group gets:

  - A small pile of identical coins (or counters).
  - A target number, 6, 7, or 8.

  Teams must:

  1. List every partition of the number.
  2. Physically arrange them (4+2, 3+3, 2+2+2).
  3. Prove that partitions into odd parts = partitions into distinct parts.

  To make it physical:

  - They group coins into only odd piles.
  - Then regroup into distinct-sized piles.

- Recurrence Relations:

  Ships entering the St. Petersburg harbor rely on a lighthouse that uses a repeating light pattern. But the pattern must be redesigned to avoid confusion.

Euler proposes sequences defined by recurrence.

- a0 =1
- a1 =2
- n =an−1 +an−2

Tasks:

1. Generate the first 10 terms.
2. Build the pattern using:
    a. Flashlights
    b. Hand signals
    c. Paper cards
3. Re-encode their sequence as:
    a. A rhythm
    b. A sequence of steps
    c. A physical march pattern

Each team must present their new lighthouse signal.

- Multidisciplinary:

  The Admiralty at St. Petersburg warns that only by aligning Euler's collected insights can the floodgates be released.

  Each team receives a final cryptic map, containing:

  - A mini graph with odd/even degree clues
  - A sequence with missing terms
  - A treasure box requiring the number of partitions of 5
  - A polyhedron outline missing V or E

  They must use all the ideas used before. When they solve all 5, the day ends.

Other notable mathematicians from the 18th century: Goldbach, Laplace, Bernoulli, Lagrange, Lambert
Thinking of activities around:

- Goldbach's conjecture
- Law of Large Numbers by Bernoulli
- Lagrange's Four-Square Theorem

- Bernoulli trials
- Irrationality of pi (Lambert)
- Rule of Succession in Bayesian Probability by Laplace

Day 2: Goldbach and Bernoulli

Narrative: Euler meets Christian Goldbach in St. Petersburg, and both go on a walk around the capital observing the recent developments in the city and discussing letters from Daniel Bernoulli who is writing from Switzerland.

- Goldbach's Strong Conjecture

  Goldbach and Euler visit the newly founded Hermitage Museum for art. At the entrance there is a code in numeric form that needs to be deciphered to enter the building. Goldbach suggests that each of the even numbers can be expressed as sum of pair of primes.

  Teams get different codes of even numbers for different entrances of the building

  - e.g. 16,22,18 -> 11 + 5, 13 +3 (2), 19 + 3, 17 + 5, 11 + 11 (3), 13 + 5, 11 + 7 (2) -> 232
  - e.g. 14, 10, 28,etc.

- Goldbach's Weak Conjecture
  After the visit to the museum, both aim to go to the post office but reach a confusing intersection when walking down the Nevsky Prospect with three different paths. Euler notices the street number they are currently on is greater than 7, and Goldbach conjectures that it can be expressed by the sum of three odd primes, which would help determine the street to take.

  Teams get a small map, each having a different intersection of the same big version.
  - One path value (prime) is given, and they need to figure out the two and the smallest prime of the three, given one of them.

- e.g. Euler says they need to go to Street 32. Map shows intersection with left as 2, straight and right unknown.
- Solved to be 2 (left) + 7 + 23 = 32 -> take the 7th street.
- Repeat this multiple times for successive intersections to solve the map and arrive at the post office.

- Bernoulli Trials

Day 3: Pierre-Simon Laplace

Narrative: A long journey from Paris brings Laplace, who asks for assistance in his work.

- Rule of Succession:
  An old sailor folktale entails the end of the world is near, and that the sun will not rise tomorrow. Laplace introduces a mathematical estimation to estimate the probability that something will happen after being observed many times.

  Teams get:

  - A bag of colored balls
  - A chart

  Teams first apply the rule of succession on random colored balls to estimate the probability of the next ball they may draw.
  Then they use the same rule to explain their calculation of the probability that the sun will rise tomorrow and submit their findings to the sailors.

- Bayesian Inference:

  While walking to the Senate, Laplace misplaces a letter from the French Academy. There are three rooms the letter could possibly be in. He has partial information from witnesses and wants to update the probabilities of finding the letter in one of these rooms.

  Teams will get:

- Envelope
- Probability Boards
- Clue Cards
- Colored chips

Tasks:

- Starting with equal probabilities P(A) = P(B) = P(C) = 1/3
- Participants receive clue cards like:
  A servant said: “I saw someone entering Room A with probability 0.6”
- They then update their probabilities until they can make a final guess about the room.

- Laplace Transform (with discrete weights):
  Ships entering the harbor rely on sending flashes of light as coded signals to identify themselves. Recently, enemy ships have begun using the same patterns as decoy signals. Participants are asked to use Laplace transform to identify whether a signal is an impostor or not depending on its signature.
  Participants are given:
  - Signal Cards (of binary sequence, with 1 for a flash of light and 0 for none)
  - Weight Cards (1, 1/2, 1/3, 1/4, ... that mimic the exponential decay factor)
  - Transformation Table
  - Signature of valid signals (Key)

  Tasks:

  - Participants are given several signals, and the signature of a valid signal.
  - They must transform the signals with the given weights and ascertain which signals are impostor signals.
  - The transformation is done by multiplying each signal value with its corresponding weight at time t and adding the values to get a transform score.

- Probability Carnival: Laplace visits a carnival where some games are fair, and others might be rigged. Teams must play, observe, and calculate probabilities to determine which games are biased.
  Participants are given:
  - Score sheets to record results and calculate probabilities
  - Game stations or Coins, dice, or small spinners

  Tasks:

  - Rotate through the games and play each 5–10 times.
  - Record outcomes and calculate experimental probabilities.

- Compare with theoretical probabilities.
- Decide whether each game is fair or rigged.

Game examples:

- Coin Toss:
  - Fair: Toss a standard coin; heads = win (P = 0.5).
  - Rigged: Use a double-headed coin to bias results.
- Dice Roll:
  - Fair: Roll a 6-sided die; win if a 6 comes up (P = 1/6).
  - Rigged: Mark one side of the die or use a biased die.