Model No 4.4: Central limit theorem

Central Limit Theorem: If $\bar{x}$ be the mean of a sample size n drawn from a population mean μ and standard deviation σ then the standardized normal variate $z = \dfrac{\bar{x} - \mu}{\left(\dfrac{\sigma}{\sqrt{n}}\right)}$ is asymptotically normal.

Here $\bar{x}$ = Sample Mean
μ = Population Mean
σ = S.D of the population
n = Sample Size

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The central limit theorem (CLT) states that the distribution of sample means approximates a normal distribution as the sample size gets larger, regardless of the population's distribution.

Note: Sample sizes equal to or greater than 30 are often considered sufficient for the CLT to hold.

Problem 11: Determine the mean and standard deviation of the sampling distribution of means of 300 random samples each of size n=36 are drawn from the population N=1500 which is normally distributed with $\mu = 22.4$, $\sigma = 0.048$. If the sampling distribution is done

i) With replacement

ii) Without replacement

a) Between 22.39 and 22.41

b) Greater than 22.42

c) Less than 22.37

d) Less than 22.38 and greater than 22.41

Solution: Sample size $n = 36$, Population size $N = 1500$, Population Mean $\mu = 22.4$, Population S.D $\sigma = 0.048$

i) WITH REPLACEMENT

Mean Of the Sampling Distribution of Means = Mean of the Population $= 22.4$

Standard Deviation Of Sampling distribution Of Means = Standard Error $= \frac{\sigma}{\sqrt{n}} = \frac{0.048}{\sqrt{36}}$
$= 0.008$

ii) WITHOUT REPLACEMENT

We know that; Mean Of the Sampling distribution Of Means = Mean of the Population $= 22.4$

S.D of Sampling distribution of Means = Standard Error =

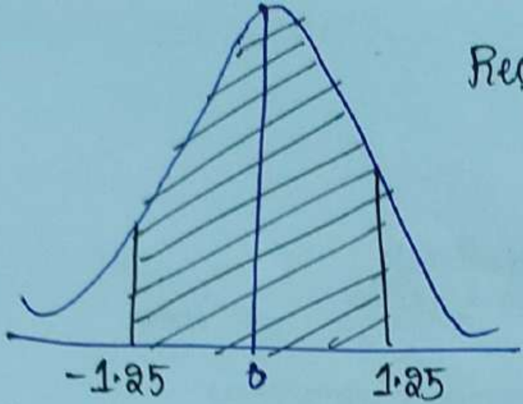
$$\frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}} = \frac{0.048}{\sqrt{36}}\sqrt{\frac{1500-36}{1500-1}} = 0.0079$$

a) Between 22.39 and 22.41

$P(22.39 < \bar{x} < 22.41)$:

At $\bar{x} = 22.39$ $\quad z = \frac{\bar{x}-\mu}{\left(\frac{\sigma}{\sqrt{n}}\right)} = \frac{22.39 - 22.40}{(0.008)} = -1.25$

At $\bar{x} = 22.41$ $\quad z = \frac{\bar{x}-\mu}{\left(\frac{\sigma}{\sqrt{n}}\right)} = \frac{22.41 - 22.4}{(0.008)} = 1.25$



Required Area :

$= A(0 \text{ to } 1.25) + A(0 \text{ to } 1.25)$

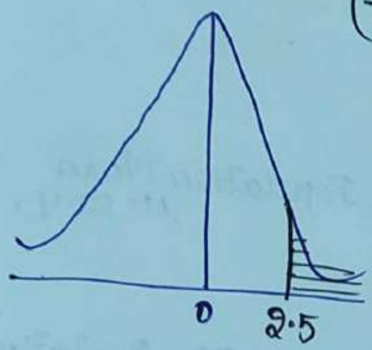
$= 0.3944 + 0.3944$

$= 0.7888 \times \underbrace{300}_{\text{Random Samples}} = \boxed{236.64}$

b, Greater than 22.42

$[P(\bar{x} > 22.42)]$

At $\bar{x} = 22.42$ $\quad z = \dfrac{\bar{x} - \mu}{\left(\frac{\sigma}{\sqrt{n}}\right)} = \dfrac{22.42 - 22.4}{(0.008)} = 2.50$



$RA = 0.5 - A(0 \text{ to } 2.50)$

$= 0.5\cancel{938} - 0.4938$

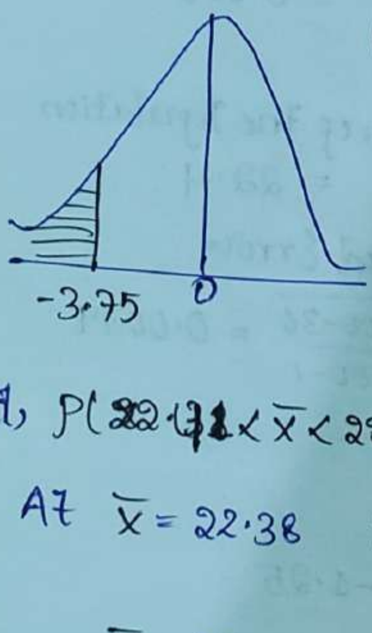
$= 0.5 - 0.4938$

$= 0.0062 \times 300 = \boxed{1.86}$

c, < 22.37

$P(\bar{x} < 22.37)$

At $\bar{x} = 22.37$ $\quad z = \dfrac{\bar{x} - \mu}{\left(\frac{\sigma}{\sqrt{n}}\right)} = \dfrac{22.37 - 22.4}{0.008} = -3.75$



$RA = 0.5 - A(0 \text{ to } 3.75)$

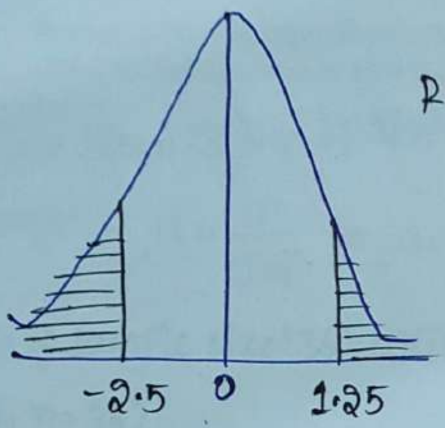
$= 0.5 - 0.4999$

$= 0.0001 \times 300 = \boxed{0.03}$

d, $P(22.41 < \bar{x} < 22.38)$ (or) $P(\bar{x} < 22.38 \text{ \& } \bar{x} > 22.41)$

At $\bar{x} = 22.38$ $\quad z = \dfrac{\bar{x} - \mu}{\left(\frac{\sigma}{\sqrt{n}}\right)} = \dfrac{22.38 - 22.4}{0.008} = -2.5$

$\bar{x} = 22.41$ $\quad z = \dfrac{\bar{x} - \mu}{\left(\frac{\sigma}{\sqrt{n}}\right)} = \dfrac{22.41 - 22.4}{0.008} = 1.25$



$RA = \cancel{A(0\text{ to}}$

$= 0.5 - A(0 \text{ to } 2.5) + 0.5 - A(0 \text{ to } 1.25)$

$= (0.5 - 0.4938) + (0.15 - 0.3944)$

$= 0.1118 \times 300$

$= \boxed{33.54}$

11.05.2022 — Wednesday

Problem 12: A random sample of size 144 is taken from an infinite population having mean $\mu = 75$, $\sigma^2 = 225$. What is the probability that $\bar{x}$ will lies between 72 and 77. 0.9370 $P(72 < \bar{x} < 77)$

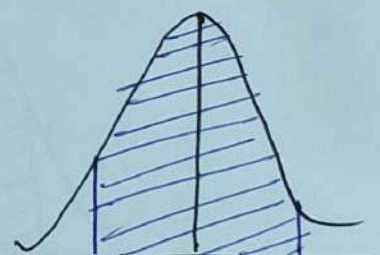
Sol:- Mean Of the Population $\mu = 75$

Variance Of Population $\sigma^2 = 225$

SD $\sigma = 15$

Sample Size $n = 144$

$P(72 < \bar{x} < 77) =$



At $\bar{x} = 72$; $z = \dfrac{\bar{x} - \mu}{\left(\frac{\sigma}{\sqrt{n}}\right)} = \dfrac{72 - 75}{\left(\frac{15}{\sqrt{144}}\right)} = \dfrac{-3 \times 12}{15} = -2.4$

At $\bar{x} = 77$; $z = \dfrac{\bar{x} - \mu}{\left(\frac{\sigma}{\sqrt{n}}\right)} = \dfrac{2 \times 12}{15} = 1.6$

$P(-2.4 < z < 1.6)$

$RA = A(0 \text{ to } 2.4) + A(0 \text{ to } 1.6)$

$= 0.4918 + 0.4452$

$\boxed{RP = 0.9370}$

Problem 13: The mean of certain normal population is equal to the standard error of the mean of the samples of 64 from that distribution. Find the probability that the mean of the sample size 36 will be negative.

Solution: The Mean Of the Population = Standard Error Of Mean Of the Samples of Size '64'

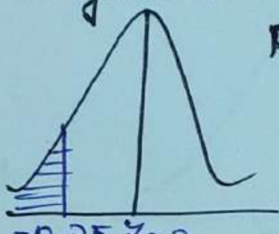
1st Sample : $\mu = \frac{\sigma}{\sqrt{n}} \Rightarrow \mu = \frac{\sigma}{\sqrt{64}} \Rightarrow \boxed{\mu = \frac{\sigma}{8}}$

P(Mean of sample size '36' will be Negative :

$P(\bar{x} < 0, n = 36)$

At $\bar{x} = 0$ $z = \frac{\bar{x} - \mu}{\left(\frac{\sigma}{\sqrt{n}}\right)} = \frac{0 - \frac{\sigma}{8}}{\left(\frac{\sigma}{\sqrt{36}}\right)} = \frac{\frac{-\sigma}{8}}{\frac{\sigma}{6}}$

$z = \frac{-6}{8} = \frac{-3}{4} = -0.75$



$RP = 0.5 - A(0 \text{ to } 0.75)$

$= 0.5 - 0.2734$

$\boxed{RP = 0.2266}$

Problem 14: A random sample of size 100 is taken from an infinite population (With Replacement) having the mean μ=76 and the variance σ^2=256. What is the probability that $\bar{x}$ will be between 75 and 78. 0.6268 $P(75 < \bar{x} < 78)$

Solution: Given $n = 100, \mu = 76, \sigma^2 = 256 \Rightarrow \sigma = 16$

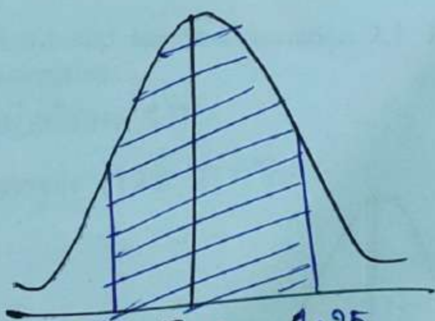
$P(75 < \bar{x} < 78)$:

At $\bar{x} = 75$;

$z = \frac{\bar{x} - \mu}{\left(\frac{\sigma}{\sqrt{n}}\right)} = \frac{75 - 76}{\left(\frac{16}{\sqrt{100}}\right)} = -0.625 \approx -0.62$

At $\bar{x} = 78$;

$z = \frac{\bar{x} - \mu}{\left(\frac{\sigma}{\sqrt{n}}\right)} = \frac{78 - 76}{\left(\frac{16}{\sqrt{100}}\right)} = 1.25$



$P(-0.62 < z < 1.25)$

$RA = A(0 \text{ to } 0.62) + A(0 \text{ to } 1.25)$

$= 0.2324 + 0.3944$

$\boxed{RP = 0.6268}$

Problem 15: A random sample of size 64 is taken from an infinite population having the mean 45 and the Standard deviation 8. What is the probability that $\bar{x}$ will be between 46 and 47.5.

$P(46 < \bar{x} < 47.5) = 0.1525$

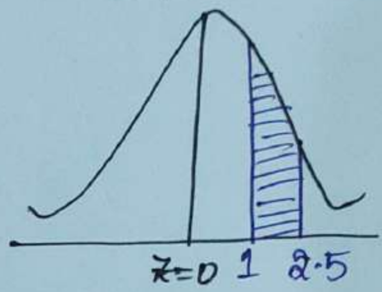
Solution:

Given $n = 64, \mu = 45, \sigma = 8$

$P(46 < \bar{x} < 47.5)$

At $\bar{x} = 46$; $z = \dfrac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \dfrac{46 - 45}{\left(\frac{8}{\sqrt{64}}\right)} = 1$

At $\bar{x} = 47.5$ $z = \dfrac{47.5 - 45}{\left(\frac{8}{\sqrt{64}}\right)} = 2.5$



$RP = A(0 \text{ to } 2.5) - A(0 \text{ to } 1)$

$= 0.4938 - 0.3413$

$\boxed{RP = 0.1525}$

Problem 16: A normal population has a mean of 0.1 and standard deviation 2.1. Find the probability that mean of a sample of size 900 will be negative.

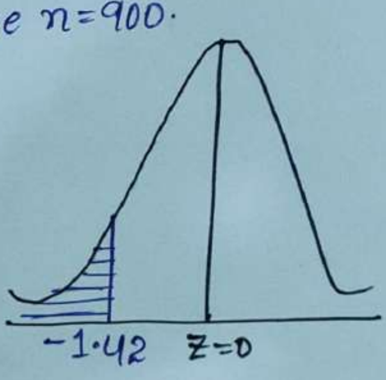
0.0764

Solution: Given $\mu = 0.1$, $\sigma = 2.1$ = Population S.D

Mean of Population $\mu = 0.1$, Sample size $n = 900$.

$P(\bar{x} < 0, n = 900)$

At $\bar{x} = 0$ $z = \dfrac{\bar{x} - \mu}{\left(\frac{\sigma}{\sqrt{n}}\right)} = \dfrac{0 - 0.1}{\left(\frac{2.1}{\sqrt{900}}\right)} = -1.42$



$P(z < -1.42)$

$= 0.5 - A(0 \text{ to } 1.42)$

$= 0.5 - 0.4222$

$= 0.0778$

Problem 17: A random sample of size 64 is taken from a normal population with $\mu=51.4$ and $\sigma=68$. What is the probability that the mean of the sample will (a) exceed 52.9 (b) fall between 50.5 and 52.3 (c) be less than 50.6.

Solution:

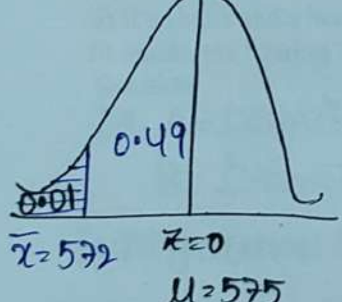
Problem 18: If the mean of breaking strength of copper wire is 575 lbs, with a standard deviation of 8.3 lbs. How large a sample must be used in order that there will be one chance in 100 that the mean breaking strength of the sample is less than 572 lbs?

Solution: Population Mean $\mu=575$, Population S.D $\sigma=8.3$

Given:

$P(\bar{x}<572)=\frac{1}{100}=0.01$

At $\bar{x}=572$, we know: $z=\frac{\bar{x}-\mu}{\left(\frac{\sigma}{\sqrt{n}}\right)}$



RP: $-2.33=\frac{572-575}{\left(\frac{8.3}{\sqrt{n}}\right)}$

$\cancel{-}2.33\left(\frac{8.3}{\sqrt{n}}\right)=\cancel{-}3$

$\sqrt{n}=6.4463$

S.O.B.S

$n=41.55\approx 42$

Search for 0.49 we Get

$\boxed{z=-2.33}$

$\boxed{\therefore n=42}$