Model No-3.5: Binomial Distribution

The Binomial distribution is a discrete distribution Expression the probability Of a set of (distribution) dichotomous alternative i.e. Success or failure

Eg: * Tossing of a Coin

* Performance Of Student in an Examination (Pass or Fail)

→ A Random Variable 'x' is Said to follow the Binomial distribution, If it's Probability Mass Function is Given by:

$$P(X=x) = \begin{cases} n_{C_x} p^x q^{n-x} & , x = 0,1,2,3,\dots n \\ 0 & , \text{Otherwise} \end{cases}$$

Simply, $B(x; n, p) = n_{C_x} p^x q^{n-x}$

Here 'n' and 'p' are known as the parameters

x - No. of Successes

n - No. of Trails

p - Probability Of Success

q - Probability Of failure

* $\boxed{p+q=1}$

Conditions For Binomial Distribution:

1, The No. of Trails 'n' is Finite.

2, The Trails are Independent Of each Other.

3, The probability Of Success 'p' is Constant for each Trail

4, Each Trail must result in a Success or a failure

Characteristics Of a Binomial Distribution:

1, Mean $\boxed{\mu = np}$

2, Variance $\boxed{\sigma^2 = npq}$

3, Standard deviation $\boxed{\sigma = \sqrt{npq}}$

4, Mode: i, $(n+1)p$ is an Integer then the Modes are $(n+1)p, (n+1)p-1$.

ii, If $(n+1)p$ is not an Integer, then Mode is the Integral part of $(n+1)p$.

Mean:

$E(x) = \sum x p(x)$

$= \sum_{x=0}^{n} x \, {}^nC_x p^x q^{n-x}$

$= {}^nC_1 p q^{n-1} + {}^nC_2 p^2 q^{n-2} + {}^nC_3 p^3 q^{n-3} + \cdots + {}^nC_n p^n q^{n-n}$

$= npq^{n-1} + 2\frac{n(n-1)p^2}{2!}q^{n-2} + 3\frac{(n-1)(n-2)}{3!}p^3 q^{n-3} + \cdots + np^n$

$= np[q+p]^{n-1}$

$\boxed{\mu = np}$

Variance:

$\sigma^2 = \sum_{x=0}^{n} x^2 p(x) - \mu^2$

$= \sum_{x=0}^{n} x^2 \, {}^nC_2 p^x q^{n-x} - (np)^2$

$= \sum_{x=0}^{n} [x(x-1)+x] \, {}^nC_x p^x q^{n-x} - n^2p^2$

$= \sum_{x=0}^{n} x(x-1) \, {}^nC_x p^x q^{n-x} + \sum_{x=0}^{n} x \, {}^nC_x p^x q^{n-x} - n^2p^2$

$= 0 + 2\,{}^nC_2 p^2 q^{n-2} + 6\,{}^nC_3 p^3 q^{n-3} + \cdots + n(n-1)\,{}^nC_n p^n + np - n^2p^2$

$= n(n-1)p^2[q+p]^{n-2} + np - n^2p^2$

$= (n^2-n)p^2 + np - n^2p^2$

$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$ $\quad \boxed{\therefore \sigma^2 = npq}$

$= np(1-p)$

Note:

If 'n' Independent Trails are repeated 'N' times then the Expected Frequency of 'x' Success is $N \times {}^nC_x p^x q^{n-x}$

* Atleast Means $\geq$
* Atleast 6 Means: ≥ 6
* Atmost 6 Means: ≤ 6

Problems:

1. A fair coin is tossed 6 times. Find the probability of getting
   i. exactly 2 heads
   ii. atleast 4 heads
   iii. atmost 4 heads
   iv. No Heads
   v. Atleast 1 Head

Sol: The No. of trails $n = 6$

X - Random Variable - No of Heads

p - P(success) = p(Getting Head) = $1/2$

q - q(failure) = p(Getting failure) = $1-p = 1/2$

i. P[exactly 2 Heads] $= p(x=2)$

$$= nc_x p^x q^{n-x} = 6c_2 \left(\tfrac{1}{2}\right)^2 \left(\tfrac{1}{2}\right)^{6-2}$$

$$= 15 \times \tfrac{1}{4} \times \tfrac{1}{16} = 0.2343$$

ii. P[Atleast 4 Heads] $= p(x=4) + p(x=5) + p(x=6)$

$$= 6c_4 \left(\tfrac{1}{2}\right)^4 \left(\tfrac{1}{2}\right)^2 + 6c_5 \left(\tfrac{1}{2}\right)^5 \left(\tfrac{1}{2}\right)^1 + 6c_6 \left(\tfrac{1}{2}\right)^6 \left(\tfrac{1}{2}\right)^0$$

$$= [6c_4 + 6c_5 + 6c_6]\left(\tfrac{1}{2}\right)^6 = \frac{22}{64} = 0.34375$$

iii. P[Atmost 4 Heads] $= P(x \le 4)$

$$= p(x=0) + p(x=1) + p(x=2) + p(x=3) + p(x=4)$$

or

$$= 1 - p(x > 4)$$

$$= 1 - [p(x=5) + p(x=6)]$$

$$= 1 - \left[6c_5 \left(\tfrac{1}{2}\right)^5 \left(\tfrac{1}{2}\right)^1 + 6c_6 \left(\tfrac{1}{2}\right)^6 \left(\tfrac{1}{2}\right)^0\right]$$

$$= 1 - \left[(6c_5 + 6c_6)\left(\tfrac{1}{2}\right)^6\right] = \frac{57}{64} = 0.8906$$

iv. P[NO Heads] $= p(x=0)$

$$= 6c_0 \left(\tfrac{1}{2}\right)^0 \left(\tfrac{1}{2}\right)^6 = \frac{1}{64} = 0.0156$$

v. P[Atleast 1 Head] $= p(x \ge 1)$

$$= 1 - p(x < 1)$$

$$= 1 - p(x=0)$$

$$= 1 - \frac{1}{64} = \frac{63}{64} = 0.9843$$

2, Out of 800 Families, 5 Children each How many would expect to have: i, 3 boys

ii, 5 Girls

iii, either 2 or 3 boys

Assume Equal Probabilities for boys & Girls

Sol: Given that:

Total Frequency $N = 800$ $n=5$

$P(\text{success}) = P(\text{boy}) = \frac{1}{2}$ x = Random Variable

$q = 1-p = 1-\frac{1}{2} = \frac{1}{2}$ = No. of boys

i, 3 boys: $P(3 \text{ boys})$

$= P(x=3) = 5C3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^{5-3} = 10 \times \frac{1}{2^5} = \frac{10}{32} = \frac{5}{16}$

Frequency:

Expected No. of Families having exactly 3 boys $= NP(x=3)$

$= 800 \times \frac{5}{16} = 250$

∴ 250 Families having exactly 3 boys

ii, $P(5 \text{ Girls}) = P(\text{no Boy})$

$= P(x=0)$

$= 5C0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^5$

$= \frac{1}{32} = 0.03125$

Expected No. of Families Having Exactly 5 Girls

$= NP(x=0)$

$= 800 \times \frac{1}{32} = 25$

iii, $P(2 \text{ or } 3 \text{ Boys}) = P(x=2) + P(x=3)$

$= 5C2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3 + 5C3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2$

$= \frac{10}{2^5} + \frac{10}{2^5}$

$= \frac{20}{32}$

Expected No of Families:

Having Either 2 or 3 boys $= N(P(x=2)+P(x=3))$

$= 800 \times \frac{20}{32}$

$= 25 \times 20$

$= 500.$

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Answers

1) $p = \frac{20}{100} = \frac{1}{5}$, $q = \frac{4}{5}$, $n=5$

i) $P(X=0) = 5C0 \left(\frac{1}{5}\right)^0 \left(\frac{4}{5}\right)^5 = \left(\frac{4}{5}\right)^5 = 0.32768$

$P(X=1) = 5C1 \left(\frac{1}{5}\right)^1 \left(\frac{4}{5}\right)^4 = \frac{256}{625} = 0.4096$

$P(1<x<4) = P(x=2)+P(x=3)$

$= 5C_2 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^3 + 5C3 \left(\frac{1}{5}\right)^3 \left(\frac{4}{5}\right)^2$

$= 10 \times \frac{1}{25} \times \left(\frac{4}{5}\right)^3 + 10 \times \frac{1}{125} \left(\frac{4}{5}\right)^2 = 0.256$

2) $P(5 \le x \le 7) = P(x=5)+P(x=6)+P(x=7)$

$= 9C5 \left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right)^4 + 9C6 \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right)^3 + 9C7 \left(\frac{2}{3}\right)^7 \left(\frac{1}{3}\right)^2$

$= 0.7120865$

3) Given, Mean $= np = 3$

i) Variance $npq = \frac{9}{4}$ $\quad 3q = \frac{9}{4}$, $\boxed{q = \frac{3}{4}}$, $p = 1 - \frac{3}{4} = \frac{1}{4}$ $\boxed{p = 1/4}$

$np = 3$

$n\left(\frac{1}{4}\right) = 3$ $\boxed{n = 12}$

ii) $P(x \ge 7) = P(x=7)+P(x=8)+P(x=9)+P(x=10)+P(x=11)+P(x=12)$

$= 12C7 \left(\frac{1}{4}\right)^7 \left(\frac{3}{4}\right)^5 + 12C8 \left(\frac{1}{4}\right)^8 \left(\frac{3}{4}\right)^4 + 12C9 \left(\frac{1}{4}\right)^9 \left(\frac{3}{4}\right)^3 +$

$12C10 \left(\frac{1}{4}\right)^{10} \left(\frac{3}{4}\right)^2 + 12C11 \left(\frac{1}{4}\right)^{11} \left(\frac{3}{4}\right)^1 + 12C12 \left(\frac{1}{4}\right)^{12} \left(\frac{3}{4}\right)^0$

$= 0.01425$

4) Sample Space = {(1,5), (2,4), (3,3), (4,2), (5,1)} → Getting Sum '6' $\boxed{P(S)=5}$

Exhaustive cases

Real Sample Space S = {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),
(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),
$n(S) = 36$ (3,1), (3,2), (3,3), (3,4), (3,5), (3,6),
(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),
(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),
(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)}

$\therefore\ p = \frac{5}{36},\ q = \frac{31}{36}\ \boxed{n=7}$

$P(x=3) = 7c_3\left(\frac{5}{36}\right)^3\left(\frac{31}{36}\right)^4 = 0.051559$

5) Given a die is thrown '6' times

$n(S) = 6^6 \Rightarrow \boxed{n=6}$ $p = \frac{3}{6} = \frac{1}{2}$ {2,4,6} Getting odd $q = \frac{3}{6} = \frac{1}{2}$ {1,3,5}

i) $P(X\geq 1) = 1 - P(X=0)$

$= 1 - 6c_0\left(\frac{1}{2}\right)^6 = 1 - \frac{1}{26} = \frac{63}{64} = 0.984375$

ii) $P(X\leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$= 6c_0\left(\frac{1}{2}\right)^6 + 6c_1\left(\frac{1}{2}\right)^6 + 6c_2\left(\frac{1}{2}\right)^6 + 6c_3\left(\frac{1}{2}\right)^6 + \cancel{6c_4}$

$= [6c_0 + 6c_1 + 6c_2 + 6c_3]\left(\frac{1}{26}\right)$ $6c_3 = \frac{\cancel{6}\times 5\times 4}{\cancel{3}\times 2\times 1}$

$= [1 + 6 + 15 + 20]\left(\frac{1}{64}\right)$

$= \frac{42}{64} = 0.65625$ $6c_4 = \frac{\cancel{6}\times 5\times \cancel{4}\times 3}{\cancel{4}\times \cancel{3}\times 2\times 1}$

iii) $P(X=4) = 6c_4\left(\frac{1}{2}\right)^6 = 0.234375$

6) Given that $p = \frac{3}{20},\ n=4,\ q = 1 - \frac{3}{20} = \frac{17}{20}$

$P(X=1) = 4c_1\left(\frac{3}{20}\right)^1\left(\frac{17}{20}\right)^3 = 0.368475$

7) Given: $p = \frac{1}{2}$ $q = \frac{1}{2}$ $n = ?$

$P(X=0) > 0.1$ $n=1 \Rightarrow \frac{1}{2} = 0.5 > 0.1$

$nc_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^n > 0.1$ $n=2 \Rightarrow \left(\frac{1}{2}\right)^2 = \frac{1}{4} = 0.25 > 0.1$

$= \left(\frac{1}{2}\right)^n > 0.1$

$n=3 \Rightarrow \left(\frac{1}{2}\right)^3 = \frac{1}{8} = 0.125 > 0.1$

$\boxed{\text{'n' max} = 3}$

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1) 20% of the Items produced from a Factory are Defective. Find the probability that in a Sample of 5 Choosen at Random
   i) None is defective
   ii) One is defective
   iii) $P(1<x<4)$

(Hint: x is the random Variable, No. of defectives $P = 20\% = \frac{20}{100} = \frac{1}{5}$
$n=5$)

2) A Discrete Random Variable 'x' has the Mean 6 and Variance 2. It is Assumed that the distribution is Binomial, then find $P(5 \le x \le 7)$

3) The Mean of a Binomial Distribution is 3 and Variance is $\frac{9}{4}$. Find: i) Value of 'n' ii) $P(x \geq 7)$

4) Determine the probability Of Getting the Sum = 6 Exactly 3 times in 7 throws with a pair of fair dice

5) A die is thrown 6 times. If Getting an Even Number is a Success. Find the probabilities Of:
   i) Atleast 1 Success.
   ii) Less than or Equal to 3 Successes.
   iii) 4 Successes.

6) If 3 Of 20 toys are defective & 4 Of them are Randomly Choosen for Inspection, What is the probability that only One Of the Toy is defective will be Included. Find $P(x=1)$

7) Find the Maximum Value Of 'n', Such that the probability Of Getting NO HEAD and Tossing of a Fake Coin 'n' times is Greater than 0.1?