$\kappa \rightarrow$ curvature of the section ($1/m$)

$\varepsilon \rightarrow$ axial strain of the section (unitless)

$M \rightarrow$ bending moment (Nm)

$N \rightarrow$ axial force (N)

$$\begin{bmatrix} N \\ M \end{bmatrix} = f(\varepsilon, \kappa)$$

$\rightarrow$ axial force and bending moment are functions of κ and ε

$$K(\varepsilon, \kappa) = \begin{bmatrix} \frac{\partial N}{\partial \varepsilon} & \frac{\partial N}{\partial \kappa} \\ \frac{\partial M}{\partial \varepsilon} & \frac{\partial M}{\partial \kappa} \end{bmatrix}$$

$\downarrow$

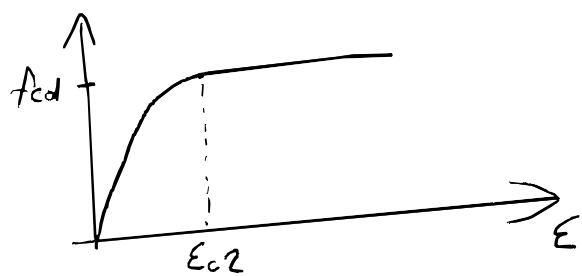
section tangent stiffness matrix

From Eurocode, concrete $\sigma(\varepsilon)$ relation:

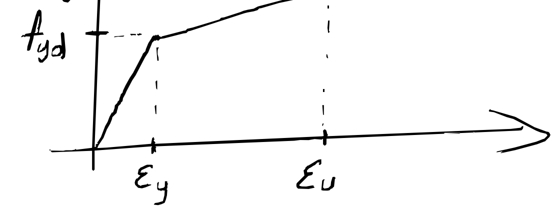
$$\sigma = f_{cd}\left(1-\left(1-\frac{\varepsilon_c}{\varepsilon_{c2}}\right)^n\right)$$

$$\frac{d\sigma}{d\varepsilon} = -f_{cd}\, n\left(1-\frac{\varepsilon_c}{\varepsilon_{c2}}\right)^{n-1} \times \left(-\frac{1}{\varepsilon_{c2}}\right) =$$

$$= \frac{n f_{cd}}{\varepsilon_{c2}}\left(1-\frac{\varepsilon_c}{\varepsilon_{c2}}\right)^{n-1}$$



From eurocode, rebar $\sigma(\varepsilon)$ relation:



$$K = \begin{bmatrix} \frac{dN}{da} & \frac{dN}{dc_3} & \frac{dN}{dc_2} \\ \frac{dM_3}{da} & \frac{dM_3}{dc_3} & \frac{dM_3}{dc_2} \\ \frac{dM_2}{da} & \frac{dM_2}{dC_3} & \frac{dM_2}{dc_2} \end{bmatrix}$$

$a \to$ axial strain
ε

$c_3 \to$ curvature around axis 3
M_3

$c_2 \to$ curvature around axis 2
M_2

$$N = \int_Y \int_X \sigma(a + c_3 Y - c_2 X)\, dx\, dy$$

$$\sigma' = K = \frac{d\sigma}{d\varepsilon}$$

$$\frac{dN}{da} = \iint K(a + c_3 y - c_2 X)\, dx\, dy$$

$$\frac{dN}{dc_3} = \iint K(a + c_3 Y - c_2 X)\, Y\, dx\, dy$$

$$\frac{dN}{dc_2} = \iint K(a + c_3 Y - c_2 X)(-X)\, dx\, dy$$

$$M_3 = \iint \sigma(a + c_3 y - c_2 x)\, y \, dx\, dy$$

$$\frac{dM_3}{da} = \iint K(a + c_3 y - c_2 x)\, y \, dx\, dy$$

$$\frac{dM_3}{dc_3} = \iint K(a + c_3 y - c_2 x)\, y^2 \, dx\, dy$$

$$\frac{dM_3}{dc_2} = \iint K(a + c_3 y - c_2 x)(-xy) \, dx\, dy$$

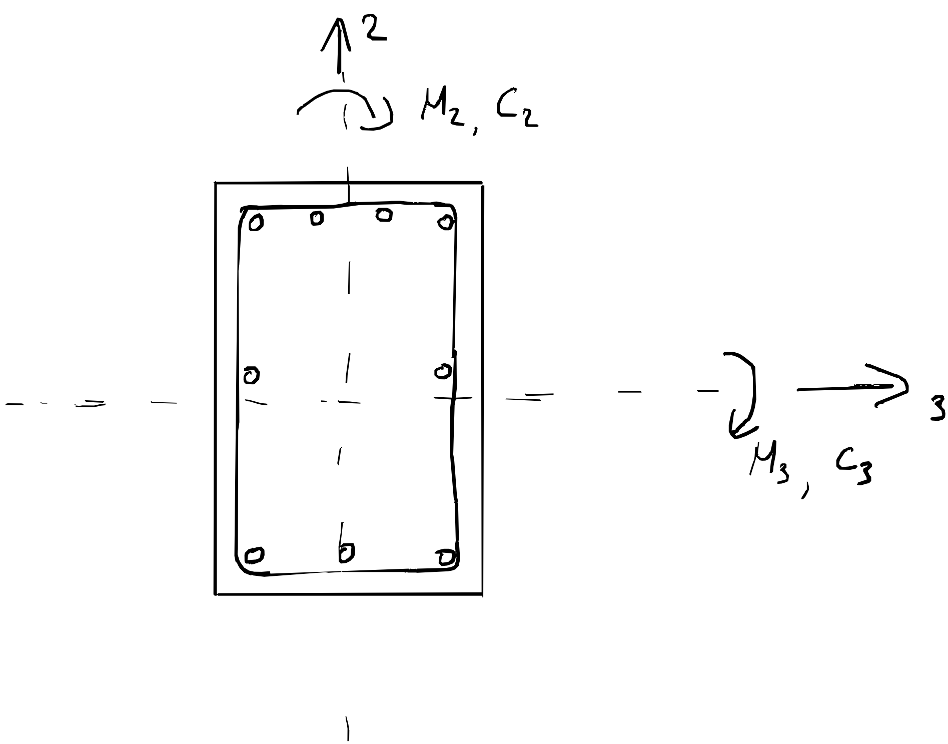
$$M_2 = \iint \sigma(a + c_3 y - c_2 x)(-x) \, dx\, dy$$

$$\frac{dM_2}{da} = \iint K(a + c_3 y - c_2 x)(-x) \, dx\, dy$$

$$\frac{dM_2}{dc_3} = \iint K(a + c_3 y - c_2 x)(-xy) \, dx\, dy$$

$$\frac{dM_2}{dc_2} = \iint K(a + c_3 y - c_2 x)\, x^2 \, dx\, dy$$

$$K = \begin{bmatrix} \frac{dN}{da} & \frac{dN}{dc_3} & \frac{dN}{dc_2} \\ \frac{dM_3}{da} & \frac{dM_3}{dc_3} & \frac{dM_3}{dc_2} \\ \frac{dM_2}{da} & \frac{dM_2}{dc_3} & \frac{dM_2}{dc_2} \end{bmatrix}$$



Displacement-controlled Newton-Raphson method

$$\vec{f}_{int}^{\,n,0}(\vec{d}) = \vec{f}_{ext}^{\,n,0}$$ → balance of internal and external forces in n'th step, zero'th iteration

$$\cancel{\vec{f}_{int}^{\,n,0}} + \bar{K}^{n,0}\,\delta\vec{d}^{\,n,0} = \cancel{\vec{f}_{ext}^{\,n,0}} + \delta\lambda^{n,0}\,\vec{f}_p$$

$$\Delta\vec{d}^{\,n,i} = \Delta\vec{d}^{\,n,i-1} + \vec{\delta}^{\,n,i-1}$$

$$\vec{d}^{\,n} = \vec{d}^{\,n-1} + \Delta\vec{d}^{\,n-1}$$

if d_c is component of $\vec{d}$ that is controlled, in zeroth iteration of n'th step:

$$\begin{bmatrix} K_{tt}^{n,0} & K_{tc}^{n,0} \\ K_{ct}^{n,0} & K_{cc}^{n,0} \end{bmatrix} \begin{bmatrix} \delta d_t^{n,1} \\ \delta d_c^{n,1} \end{bmatrix} = \delta\lambda^{n,1} \begin{bmatrix} f_{pt} \\ f_{pc} \end{bmatrix}$$

$$K_{tt}\,\delta d_t - \delta\lambda\, f_{pt} = -K_{tc}\,\delta d_c$$

$$K_{ct}\,\delta d_t - \delta\lambda\, f_{pc} = -K_{cc}\,\delta d_c$$

In m'th iteration of n'th step:

$$\begin{bmatrix} f_{int,t}^{n,m} \\ f_{int,c}^{n,m} \end{bmatrix} + \begin{bmatrix} K_{tt}^{n,m} & K_{tc}^{n,m} \\ K_{ct}^{n,m} & K_{cc}^{n,m} \end{bmatrix} \begin{bmatrix} \delta d_t^{n,m} \\ \delta d_c^{n,m} \end{bmatrix} = \delta\lambda^{n,m} \begin{bmatrix} f_{pt} \\ f_{pc} \end{bmatrix} + \begin{bmatrix} f_{ext,t} \\ f_{ext,c} \end{bmatrix}^{n,m}$$

$$K_{tt}^{n,m}\delta dt^{n,m} - \delta\lambda^{n,m} f_{pt} = -K_{tc}^{n,m}\delta d_c^{n,m} + f_{ext,t}^{n,m} - f_{int_t}^{n,m}$$

$$K_{ct}^{n,m}\delta dt^{n,m} - \delta\lambda^{n,m} f_{pc} = -K_{cc}^{n,m}\delta dc^{n,m} + f_{ext,c}^{n,m} - f_{int,c}^{n,m}$$

$$\begin{bmatrix} K_{tt}^{n,m} & f_{pt} \\ K_{ct}^{n,m} & f_{pc} \end{bmatrix} \begin{bmatrix} \delta dt \\ \delta\lambda^{n,m} \end{bmatrix}^{n,m} = \begin{bmatrix} -K_{tc}^{n,m}\delta dc^{n,m} + f_{ext,t}^{n,m} - f_{int\,t}^{n,m} \\ -K_{cc}^{n,m}\delta dc^{n,m} + f_{ext,c}^{n,m} - f_{int\,c}^{n,m} \end{bmatrix}$$

$\vec{f}_{int}^{\,n,m}$ is evaluated in each iteration according to the current $\vec{d}^{\,n,m}$ displacement vector $([\varepsilon, \kappa_3, \kappa_2]^T)$

$\vec{d}^{\,n,m} = \vec{d}^{\,n,m-1} + \delta d^{\,n,m-1}$. $\vec{f}_{int}$ is a known function of $\vec{d}$

$\vec{f}_{ext}$ is incremented by $\delta\lambda \vec{f}_p$ in each iteration

$$\vec{f}_{ext}^{\,n,m} = \vec{f}_{ext}^{\,n,m-1} + \delta\lambda^{n,m-1}\vec{f}_p$$

$\vec{f_p}$ is applied force shape $([N, M_3, M_2]^T)$.

In case of bending (moment-curvature diagram for constant N) $\vec{f_p} = [0, 1, 0]^T$

$\delta\lambda^{n,m}$ is force increment of m'th iteration

$\delta d_c^{n,m}$ is increment of controlled displacement component in m'th iteration. Since it is controlled, it is only incremented at beginning of step n, when $m=0$. When $m>0$, $\delta d_c^{n,m} = 0$. In case of moment-curvature diagram d_c is M_3 component (curvature)

d_t are other components of displacement ($[\varepsilon, M_2]^T$)

$\bar{K}^{n,m}$ is section tangent stiffness matrix of m'th iteration, evaluated according to $\vec{d}^{n,m}$ deformation state.

Iterations are run until balance is approximately achieved $f_{int}^{n,m} \cong f_{ext}^{n,m}$. n is then incremented and new step initiated.