Competitive Pricing in the Presence of Manipulable Information in Online Platforms

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Abstract

Problem Definition: To entice customers to purchase, sellers on online platforms often misrepresent the quality of their goods/services, e.g., by manipulating consumer opinion. Manipulation increases consumers' perceived valuation, thereby allowing sellers to set higher prices, but also has short- and long-term ill-effects (e.g., higher product returns and loss of goodwill). We analyze an oligopoly where sellers, heterogeneous in their true quality, compete by jointly choosing their prices and their perceived quality (that is manipulable).

Methodology: Non-Cooperative Game Theory, Choice Models, and Optimization.

Results: We solve for the unique equilibrium when price-setting firms can manipulate their perceived quality and characterize the set of sellers that manipulate in equilibrium. We identify two indices – the *antipathy* and the *propensity* to manipulate – based on model primitives to identify the set of sellers who have greater dis-/incentive to manipulate. The extant literature has been mixed in its findings on which sellers have greater incentive to manipulate. Our work helps reconcile the differing viewpoints in the extant literature by providing a unified perspective.

Managerial Implications: We demonstrate the practical relevance of our model by mapping it to an environment consisting of sellers who are differentiated in a star-rating system based on their true rating and the volume of ratings. Depending on a seller's rating and volume of ratings, we identify three distinct regions that arise: a cost-prohibitive region, a cost-dominant region, and a benefit-dominant region. The ability to map a seller to one of these regions allows platform managers to understand a seller's tendency to manipulate consumer opinion as their product ratings evolve.

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... I went to buy a pair of wireless earbuds. After I purchased them I got an email ... telling me that they would give me a free wireless charger if (and only if) I gave a 5 star review. I contacted Amazon about it and they said it was against their policy to do that but they were not going to investigate the matter.

— Customer reports on sellers' efforts to manipulate ratings (Crockett, 2019).

The seller is obviously incentivizing people to leave positive reviews. Does Amazon even care? I'm pretty sure nothing will happen to him and he'll keep outranking me because I guess I'm dumb enough to play by the rules.

— Seller complains on Amazon Seller Forums (Amazon Seller Central, 2019).

1 Introduction

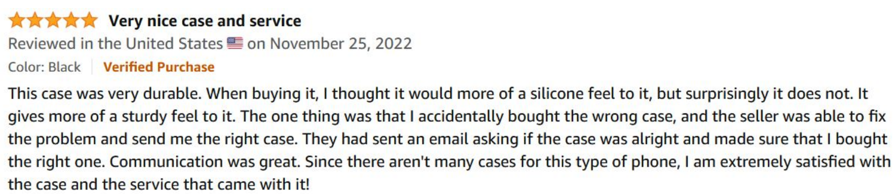
Internet-enabled marketplaces, e.g., retail platforms like Amazon and Ebay, provide consumers with not only the ability to engage in trade with sellers, but also a vast amount of information to guide their purchasing decisions. Information on sellers' performance is typically user-generated in the form of consumer opinion or feedback, consisting of reviews and ratings, either on the platform or other product review forums. A vast literature, both in Marketing and Economics, has shown that consumers are influenced by such information in their purchase decisions (Chevalier and Mayzlin, 2006; Chintagunta et al., 2010; Mayzlin et al., 2014). Recent estimates by World Economic Forum (2021) suggest that consumer opinions via online reviews influence $3.8 trillion of global commerce. In the context of restaurants, Luca (2011) estimates that a 1-star increase on Yelp rating leads to a 5-9% increase in revenues. Besides affecting consumers' purchase decisions, information on sellers' performance plays a critical role in the platform's listing strategy, e.g., in their search rankings. For instance, Amazon ranks sellers on various performance metrics, and awards the "buy-box" to their best performing sellers (Chen and Wilson, 2017).[1] This virtual word-of-mouth effect can form a reinforcing feedback loop that sets the sellers apart: those that succeed and those that fail.

Due to the competitive advantage that superior consumer opinion bestows on sellers, it is no surprise that sellers resort to manipulating these opinions via unfair means. A leading example through which sellers affect consumer opinion is *fake post-for-pay reviews*. In its simplest form, sellers solicit positive opinions that promote their products in exchange for a monetary transfer. While such manipulation of consumer opinion is illegal, in a recent paper, He et al. (2022) show the existence of a large and active market for fake reviews. Recent estimates by certain large platforms show that 4% of online reviews are fake (World Economic Forum, 2021).[2]

[1] The buy-box refers to the white box on the right side of the Amazon product detail page, where customers add items for purchase to their cart. If left unchanged, Amazon assigns the default seller of a product to a top performing seller. Shoppers rarely browse a product's other sellers. Being awarded the buy-box is arguably one of the biggest perks a seller can get on the Amazon marketplace. It is estimated that 82% of a product's sales go through the buy-box. See https://www.bigcommerce.com/blog/win-amazon-buy-box/.

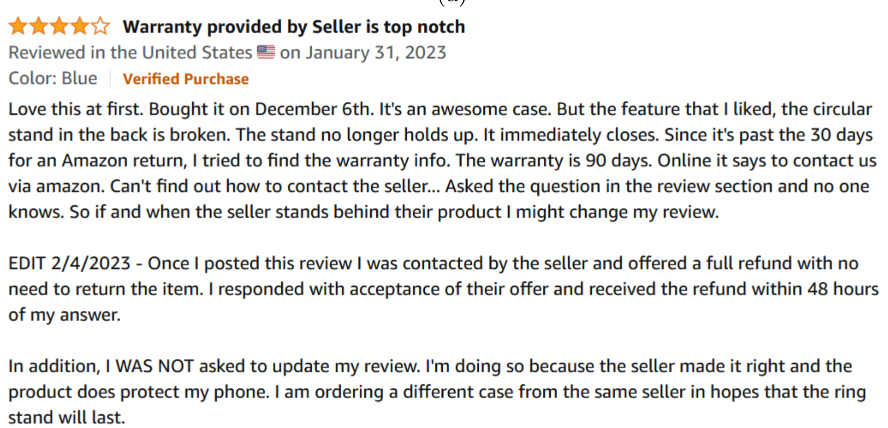
[2] These estimates are based on self-reported data from Trip Advisor, Yelp, TrustPilot and Amazon (World Economic Forum, 2021).

Besides fake reviews, manipulation may be less brazen, e.g., through *incentivized reviews*, where a customer is incentivized to provide a positive opinion; common examples of such incentives include entry into a sweepstake, coupon, or a discount. While incentivized reviews are banned on certain large platforms, e.g., Yelp and Amazon, other platforms allow for incentivized reviews (Techcrunch, 2017; Yelp, 2017; Federal Trade Commission, 2022). In other cases, manipulation may be completely innocuous, e.g., by providing additional after-sale services and care. For example, a seller of phone cases helped customers who ordered an incorrect product to fix the problem proactively, triggering a high-rating review from the customer (Figure 1a). In another example, a seller offered a full refund without requiring product return to a buyer posting a quality complaint, and the buyer subsequently revised the poor review voluntarily without seller request (Figure 1b).


Very nice case and service
Reviewed in the United States on November 25, 2022
Color: Black | Verified Purchase

This case was very durable. When buying it, I thought it would more of a silicone feel to it, but surprisingly it does not. It gives more of a sturdy feel to it. The one thing was that I accidentally bought the wrong case, and the seller was able to fix the problem and send me the right case. They had sent an email asking if the case was alright and made sure that I bought the right one. Communication was great. Since there aren't many cases for this type of phone, I am extremely satisfied with the case and the service that came with it!

(a)


Warranty provided by Seller is top notch
Reviewed in the United States on January 31, 2023
Color: Blue | Verified Purchase

Love this at first. Bought it on December 6th. It's an awesome case. But the feature that I liked, the circular stand in the back is broken. The stand no longer holds up. It immediately closes. Since it's past the 30 days for an Amazon return, I tried to find the warranty info. The warranty is 90 days. Online it says to contact us via amazon. Can't find out how to contact the seller... Asked the question in the review section and no one knows. So if and when the seller stands behind their product I might change my review.

EDIT 2/4/2023 - Once I posted this review I was contacted by the seller and offered a full refund with no need to return the item. I responded with acceptance of their offer and received the refund within 48 hours of my answer.

In addition, I WAS NOT asked to update my review. I'm doing so because the seller made it right and the product does protect my phone. I am ordering a different case from the same seller in hopes that the ring stand will last.

(b)

Figure 1. High-Rating Review for After Sale Care
Source: https://www.amazon.com

Irrespective of the nature of the manipulation, in all cases, sellers find it costly to manipulate consumer opinion. In addition, sellers may face platform filters or sanctions that further drive up the cost of manipulation. For example, Yelp uses automated software tools to identify and remove reviews that are suspected to have been solicited (Yelp, 2017). Similarly, Amazon uses technology to identify and delete consumer opinions that are deemed fraudulent on its platform (He et al., 2022). Beyond the direct costs of manipulation, sellers face additional short- and long-term ill-effects from manipulation, e.g., through product returns and loss of customer loyalty/goodwill. In online retail, if what customers get does not match with what they expect, they are more likely to return the product. Such returns exacerbate sellers' losses.

In this paper, we study the competitive landscape for online sellers that sell differentiated yet substitutable products on a platform. Sellers simultaneously determine their product price and their manipulation strategy, characterized by the extent to which sellers artificially inflate consumer opinion. Each consumer chooses the product that yields the highest (*perceived*) utility among the available options. Consumer utility depends on product price and the *perceived* product quality – that consists of the *true* quality from the product's features, and the extent to which the seller inflates consumer opinion. There are competing arguments relating to which sellers have a greater incentive to manipulate consumer opinion. Dellarocas (2006) considers a market where a seller signals their true quality to uninformed consumers via manipulation. They show that manipulation is increasing in the true quality of the seller if the marginal benefit from higher perceived quality is increasing in its true quality. That is, if sellers stand to gain more from being perceived as high quality, then higher quality sellers manipulate more. In contrast, He et al. (2022) find that manipulation is predominantly employed by lower quality sellers. They argue that, while sellers of all qualities benefit from manipulation, the higher quality sellers find it a lot harder to manipulate, as opposed to the lower quality sellers. In this paper, we examine the equilibrium pricing and manipulation strategy of the sellers in an oligopoly. In particular, when sellers are heterogeneous in their true qualities, how does the equilibrium price and manipulation effort vary based on the true product quality? In light of the contradicting findings of Dellarocas (2006) and He et al. (2022), we explore the dynamics that drive sellers' manipulation incentives, both in their tendency to manipulate and in the extent of manipulation.

We also analyze how manipulation affects sellers and consumers. Sellers' decision to manipulate consumer opinion affects their perceived product quality and their subsequent sales. The equilibrium product prices and manipulation effort affect the sellers' revenue. Sellers resort to manipulation because they expect other sellers also engage in manipulation. Are sellers better off if manipulation was preventable altogether? Furthermore, in the presence of manipulation, consumers' true utility from a product (ex-post purchase) differs from that drives their purchase decision (ex-ante expectation). Are consumers always worse-off due to the presence of manipulation? In this paper, we build a model that encompasses these considerations when competing sellers jointly choose prices in the presence of manipulation. We shed light on the effect of consumer opinion manipulation on market outcomes, and implications for sellers' and consumers' welfare.

Before presenting our work, we emphasize that studying manipulation does not imply that we consent to or endorse such practices. Rather, we believe understanding its impact and its effect on sellers' decisions helps platforms and policymakers design strategies and policies that are effective in curbing unethical practices, reduce trade frictions, and improve market efficiency. Indeed, there has been significant interest in understanding the effects of manipulation by practitioners (The Wall Street Journal, 2023) and regulatory agencies, e.g., the FTC (Federal Trade Commission, 2023). Furthermore, the recent growth in Generative AI technologies has made it easier to create novel and detailed reviews with minimal effort, thereby making it easier for sellers to manipulate consumer opinion. A recent report by the Transparency Company that analyzed millions of reviews in home, legal and medical services found that 14% of the reviews were likely fake, and a quarter of these reviews were partly or entirely generated by AI (The Fast Company, 2024). The rapid growth of these technologies have further exacerbated concerns among policymakers (Federal Trade Commission, 2025). Our work in this paper, on understanding the effects of manipulation on competition, is crucial to policymakers and managers.

2 Related Literature

This paper is closely related to two streams of literature: (a) models of competition using the MNL choice model and (b) empirical and theoretical models on firms' manipulation of consumer opinion.

2.1 A Brief Background on The MNL Choice Model

Discrete choice models are widely used in Economics, Marketing, and Operations Management (OM) to describe and analyze how consumers choose among a collection of alternatives. These models assume that consumers are random utility maximizers. The simplest and most studied discrete choice model is the multinomial logit (MNL) model (McFadden et al., 1973; Berry, 1994). Arguably, one of the most attractive features of the MNL model is in its empirical support to estimate model parameters with data. In their pioneering work, McFadden et al. (1973) establish the concavity of the log-likelihood function in the model parameters. Vulcano et al. (2012) propose an expectation-maximization (EM) algorithm to incorporate incomplete data (e.g., the "no-purchase" option) with the MNL model. We borrow these techniques in estimating our consumer choice model in Section 6 to apply our framework to real-world data.

2.2 Models of Price Competition under the MNL Choice Model

The MNL model has been extensively employed for understanding firms' pricing decisions in oligopolistic competition in an economy. Due to the extensive nature of this research stream, we only mention papers within OM that are closely related to our work. One of the earliest papers in this stream is Anderson and De Palma (1992). They show the existence of an equilibrium when symmetric multiproduct firms compete in prices under the MNL demand and conclude that when all products have equal quality, the equilibrium prices are a fixed markup over the production cost. Besanko et al. (1998) and Besanko et al. (2003) propose a framework to empirically estimate logit demand systems where prices are assumed to be the equilibrium outcomes of Nash competition among manufacturers and retailers. Their work explains the bias that arises in model estimates when the endogeneity of prices is ignored. Berry et al. (1995), and subsequently, Berry et al. (2004), conduct empirical analyses of the US auto industry, where they identify equilibrium prices under competition and obtain estimates of their demand and cost parameters. Earlier work, e.g., by Gallego et al. (2006), Bernstein and Federgruen (2004) and Allon et al. (2011), analyze competitive pricing under the MNL (and MNL-like) demand model, and identify conditions for the existence and uniqueness of a Nash equilibrium. Farahat and Perakis (2011) study models of competition for differentiated products, where firms compete either in prices (Bertrand) or quantity (Cournot), and demand follows the MNL model. They show that the outcomes under Bertrand and Cournot competition are respectively equivalent to outcomes when decisions are made sequentially: the Cournot outcome arises when the production decision precedes the pricing decision, while the Bertrand outcome arises when the pricing decision precedes the production decision. Li and Huh (2011) extend these models of competition to the case of the nested logit model and provide quasi-closed form expressions for the equilibrium market share and markups of firms. Gallego and Wang (2014) identify conditions that ensure a unique equilibrium under the nested logit model with product-specific price sensitivities. Aksoy-Pierson et al. (2013) and Lee and Çakanyildirim (2021) study price competition under the mixed MNL model and identify conditions for a unique Nash equilibrium.

It is important to note that all the aforementioned papers focus exclusively on settings where firms compete on a single attribute, namely price. We emphasize that the analysis (e.g., properties of firms' best-responses) and subsequently the equilibrium outcome of price competition, does not extend in a straightforward manner to settings where firms compete in multiple attributes. In this paper, we build on this literature to analyze the competition among multiple sellers on a platform via *both* price and manipulation. In this setting, we study the effect of consumer opinion manipulation on market outcomes. In our context, the pricing decision affects the manipulation decision, which leads to implicit functional dependencies between one seller's pricing and manipulation decisions and another seller's pricing and manipulation decisions. The existence and uniqueness of equilibrium of multi-attribute competition (price-quality or price-manipulation) is thus not a straightforward consequence of the price competition analysis in the existing literature. For the same reason, analyzing the comparative statics in multi-attribute equilibrium poses a significant technical challenge.

In the extant literature, works involving both price and quality decisions focus mainly on a monopoly. For example, Li et al. (2020) study a product line design problem in which a firm simultaneously decides on the price and quality of all products in a product line. In their work, the decision maker – a monopolist – responds to a constant (i.e., non-strategic) outside option; consequently, their work does not involve an equilibrium analysis.

More recently, Wang et al. (2022) examine product line design with warranty services using an MNL choice model. Their model and analysis largely focuses on a monopolist that decides price, quality, and warranty duration, although they offer one result for a competitive setting where each firm offers one product. Assuming a specific functional form for the unit costs of production for the products, they identify a unique Nash equilibrium, but show that the equilibrium quality and duration are the same as that of a monopolist. In this regard, the multi-attribute competition reduces to single-attribute competition. In contrast to Wang et al. (2022), we assume a generic total cost function in our base model. More importantly, the key differences lie in our results and findings: (a) we show that the multi-attribute competition does not reduce to a single-attribute one. In the joint pricing and manipulation equilibrium, there are complex inter-dependencies between pricing and manipulation decisions, and between the manipulation decisions of one seller and those of other sellers. The equilibrium manipulation of a seller under an oligopoly distinctly differs from the optimal manipulation under a monopoly; (b) we quantify such inter-dependencies through two carefully chosen indices: an index that measures a seller's antipathy to manipulation and an index that measures their propensity to manipulate. This allows us to distinguish sellers that manipulate in equilibrium and those that do not, and show how the set of sellers that manipulate changes with model parameters; (c) we go beyond just price comparison of the oligopoly setting with the monopoly setting, and examine how the intensity of competition affects the extent of manipulation. We present, via analytical results, that competition may induce or prohibit manipulation, and may increase or decrease the equilibrium extent of manipulation, revealing insights not attainable with existing models.

2.3 Empirical and Theoretical Models on Manipulation of Consumer Opinion

Since the dawn of e-commerce, one of the most important roles of platforms that match buyers and sellers has been the provision of information about products and sellers via consumer opinion/feedback, typically absent in offline environments. Such consumer opinion arises via ratings and review comments that are viewed by subsequent shoppers and influences their purchasing decisions. For example, Chevalier and Mayzlin (2006) and Luca (2011) quantify the marginal benefit from an increase in review rating in the context of books and restaurants, respectively. Beyond influencing other shoppers' purchasing behavior, consumer opinion plays an important role in platform's listing strategy (Chen and Wilson, 2017). As a result, sellers may intentionally manipulate their ratings in order to be perceived more attractive to entice more consumers. One of the earliest papers in this stream, Dellarocas (2003), discuss the challenges and opportunities brought by such feedback mechanisms.

Theoretical work in this stream spans multiple disciplines including OM, Information Systems (IS), Marketing, and Economics. We discuss papers closest to our work. Dellarocas (2006) analyze a market where a seller signals its quality via manipulation. Consumers update their beliefs on the seller's true quality based on observed signal (the sum of the true quality, the extent of manipulation, and a noise term). They show that the extent of manipulation depends on the marginal benefit from quality: If the marginal benefit is increasing in quality, then the extent of manipulation is increasing in true quality. Mayzlin (2006) analyzes an environment where sellers use promotional chat and consumers learn about the seller's quality. They show that in equilibrium, sellers with inferior products spend more resources purchasing promotional reviews. Relatedly, Sun (2012) analyzes the effect of variance in product ratings, and posits that a higher average rating corresponds to a higher quality, while a higher variance corresponds to a niche product (i.e., extreme in fit). Empirically, Luca and Zervas (2016) and He et al. (2022) test the economic incentives for firms to purchase fraudulent reviews and show the presence of a large and active market for manipulation. Luca and Zervas (2016) show that a restaurant on Yelp is more likely to manipulate if its reputation is weak. Further, restaurants are more likely to manipulate when the intensity of competition is strong. He et al. (2022) reach a similar finding that low quality sellers on Amazon are more likely to manipulate.

Within OM, there has been a growing interest in understanding the impact of consumer opinion manipulation in platform-enabled marketplaces, predominantly using stylized game-theoretic models. Jin et al. (2023) analyze a phenomenon called "brushing", where sellers may place (fake) orders to boost their ranking in search results. They find that improvements in technology, either due to improvements in the platform's search algorithms, or in preventing brushing may harm consumers. Relatedly, Chen and Papanastasiou (2021) analyze how a platform's defensive measures against manipulation affect a seller's pricing and manipulation strategy. They find that better defensive measures may not reduce seller manipulation, and lead to market inefficiency. Finally, Papanastasiou et al. (2023) analyze how disputes (e.g., a malicious review) should be resolved by online platforms. Relative to intervention by the platform (to delete malicious reviews), they find that interventions by sellers coupled with a penalty for removing true reviews leads to greater market efficiency. All these papers address measures to curb manipulation and they adopt stylized game-theoretic models. Our work differs because we study sellers' manipulation decisions themselves along with their pricing decisions in a competitive environment and the resulting market outcomes. Also, modeling competition using the MNL choice model is empirically well-supported and enables us to apply our framework to real-world data.

Finally, our work incorporates the short- and long-term ill-effects of manipulation, e.g., through product returns and loss of customer goodwill. Sahoo et al. (2018) analyze the impact of consumer opinion on product returns. They show that when products are displayed with a higher average rating than their true rating, they are returned more often. This shows that mismatches in what customers expect to get vs. what customers actually get lead to greater returns, a phenomenon we explicitly incorporate in our model. Further, they show that if a product has a low volume of ratings, customers buy substitutes in conjunction with the focal product to hedge the fit uncertainty, which further leads to greater returns. This observation highlights the sellers' need for manipulation in its nascent stage, which is also reflected in our model as a special case, while, more generally, we identify a comprehensive set of conditions for manipulation.

The main results from the extant literature show the polarity in the types of sellers engaged in consumer opinion manipulation. That is, either high-quality sellers or low-quality sellers choose to manipulate, which appears to be contradicting. In addition, little is known about how sellers "in the middle" react. In this paper, we analyze how sellers' decisions to manipulate affects their prices under multi-seller competition when demand follows the MNL model. First, we identify a unique Nash equilibrium in an oligopoly, deriving the (quasi) closed-form expressions for the equilibrium markup and manipulation level. We then identify an index to measure a seller's propensity to manipulate and an index to measure antipathy to manipulate and show how firms' tendency to manipulate may change with firms' true rating, depending on the cumulative volume of true reviews. Eventually, we investigate the conditions where firms and consumers may gain benefits or be hurt by review manipulation. We apply our theoretical results using real-world data from Wang et al. (2014) to better understand firms' manipulation strategy.

3 Model

Consider a platform-enabled marketplace, consisting of n competing sellers, indexed by $i \in [n]$,[3] and a mass of potential consumers, normalized to 1. Each seller markets and sells a product with true quality a_i, unit cost c_i, and chooses price p_i.[4] A representative consumer purchases exactly one product from the $[n] \cup \{0\}$ products, where 0 represents the no-purchase option. The consumer's *perceived* utility from purchasing product i depends on the following: the perceived quality of the product – the sum of true product quality a_i and the extent of manipulation x_i – and the price. Specifically, the perceived consumption utility from product i is as follows:

$$u_i = a_i + x_i - bp_i + \epsilon_i, \qquad (\underline{\text{Perceived}}\text{ Consumption Utility}) \tag{1}$$

where ϵ_i is an i.i.d standard Gumbel random variable and b is a price sensitivity parameter.[5] We normalize $\mathbb{E}[u_0] = 0$.

From our model of the perceived consumption utility in (1), the purchase incidence for seller i, denoted by q_i^0, follows from the standard MNL model. That is,

$$q_i^0 = \frac{e^{u_i}}{1 + \sum_{j \in [n]} e^{u_j}}. \qquad \text{(Purchase Incidence)}$$

Next, to model the short- and long-term negative effects of manipulation – e.g, through loss of reputation, impact on future purchase incidence, return of purchased goods, etc. – on the market share of seller i, we adopt the following multiplicative model as a "reduced form" approach for the effective market-share of firm i:

$$q_i = \underbrace{q_i^0}_{\text{Purchase Incidence}} \times \underbrace{e^{-dx_i}}_{\text{1 - Return Incidence}}. \qquad \text{(Market Share)} \tag{2}$$

Henceforth, we refer to any downside/ill-effects of manipulation by "return incidence". In (2), the quantity d signifies the degree to which manipulation affects customers' return incidence, with $d = 0$ representing the extreme case that manipulation has no return incidence and $d = \infty$ the case when any manipulation leads to guaranteed return incidence. It is easy to verify that if $d > 1$, the sellers have no incentive to manipulate, thus we focus on the case when $d \in [0, 1]$. Let $h_i(x_i)$ denote the cost of manipulation for seller i. The profit of seller i, denoted by π_i, is as follows:

$$\pi_i = \underbrace{(p_i - c_i)q_i}_{\substack{\text{Profit from} \\ \text{Direct Sales}}} - \underbrace{h_i(x_i)}_{\text{Cost of Manipulation}} \tag{3}$$

[3] We denote the set $\{1, 2, \ldots, n\}$ by $[n]$.

[4] We refer to the product of seller i by product i. We use the terms *seller* and *firm* interchangeably.

[5] Indeed, among the consumers of seller i, the realized consumption/material utility is:

$$u_i^c = a_i - bp_i + \epsilon_i. \qquad \text{(Realized Consumption Utility)}$$

We comment on this distinction in our analysis of consumer surplus in Section 7.3.

An equilibrium of this game consists of the pair price p_i and the extent of manipulation x_i for each seller $i \in [n]$ such that no seller has an incentive to deviate. That is, $(p_i^{\mathsf{PM}}, x_i^{\mathsf{PM}})_{i\in[n]}$ is an equilibrium if the following holds:[6]

$$(p_i^{\mathsf{PM}}, x_i^{\mathsf{PM}}) \in \arg \max_{(p_i,x_i)\in\Re^{+2}} \pi_i \left(p_i, x_i \middle| \mathbf{p}_{-i}^{\mathsf{PM}}, \mathbf{x}_{-i}^{\mathsf{PM}}\right) \text{ for each } i \in [n].$$

For ease of notation and to simplify our exposition, we denote the profit margin (*markup*) of seller i by m_i, i.e., $m_i = p_i - c_i$. Since c_i, $i \in [n]$, is fixed and common knowledge, we employ m_i as seller i's decision (instead of p_i). Next, define A_i as $A_i = e^{a_i - bc_i}$. We refer to A_i – the cost-adjusted quality of seller i – as the *type* of seller i, and refer to a seller with a higher value of A_i as a higher type. We rewrite seller i's market-share from (2) as follows:

$$q_i = \frac{e^{a_i+(1-d)x_i-b(m_i+c_i)}}{1+\sum_{j\in[n]} e^{a_j+x_j-b(m_j+c_j)}} = \frac{A_i e^{(1-d)x_i-bm_i}}{1+\sum_{j\in[n]} A_j e^{x_j-bm_j}} \tag{4}$$

Together, we express seller i's profit in (2) as follows:

$$\pi_i = m_i \left(\underbrace{\frac{A_i e^{(1-d)x_i-bm_i}}{1+\sum_{j\in[n]} A_j e^{x_j-bm_j}}}_{=q_i} \right) - h_i(x_i). \tag{5}$$

The unit production cost c_i and manipulation x_i do not need to be observable (public information), as the market share of each seller depends on the observed price (i.e., the sum $(m_i + c_i)$) and the perceived quality (i.e., the sum $(a_i + x_i)$) of other sellers but not their cost and manipulation directly. Each seller responds to the observed prices and perceived quality of other sellers by choosing its markup and manipulation to maximizes its own profit; once no seller can increase their profit by unilaterally deviating, an equilibrium is reached.[7]

We make the following assumption on the cost of manipulation.

Assumption 1 (Cost of Manipulation)**.** *The cost of manipulation $h_i(x)$, $i \in [n]$ satisfies the following:*

(a) $h_i(x)$ is smooth, non-negative, increasing and strictly convex in $x \in \Re^+$, i.e., $h_i(x) \geq 0, h_i'(x) \geq 0, h_i''(x) > 0$ for $x \geq 0$ with $h_i(0) = 0$.

(b) $h_i''(x) \geq h'(x)$ for all $x \in \Re^+$.

[6] We use the superscript PM to denote the equilibrium outcome in the presence of manipulation.

[7] Essentially, each seller responds to the observed values $p_i = m_i + c_i$ and $\hat{a}_i = a_i + x_i$ of other sellers without directly observing m_i and x_i despite that q_i is written as a function of the markup vector and manipulation vector.

Part (a) is straightforward and assumes that it becomes increasingly more difficult to manipulate. Part (b) states that the cost function is *sufficiently* convex, a regularity condition that ensures π_i is well-behaved (this condition helps us identify a unique equilibrium through f.o.c's). In particular, part (b) can be simplified to $h_i(x) \geq h_i'(0)\,(e^x - 1)$.[8]

To begin with, we present two benchmarks: (a) the absence of any seller-manipulation (denoted by AM), and (b) the absence of competition, i.e., a monopoly (denoted by AC). Subsequently, we analyze the outcomes in the presence of seller-manipulation (denoted by PM).

Absence of Seller Manipulation (AM): In the absence of manipulation ($x_i = 0$ for all $i \in [n]$), sellers compete only on prices. It has been established in the literature that this game admits a unique equilibrium (Li and Huh, 2011). We present the equilibrium outcome in this setting below.

Theorem 1 (Equilibrium Outcome under AM)**.** *Let $\hat{\gamma}^{\mathsf{AM}} \in (1, \infty)$ be the unique solution to the following equation:*

$$1 - \frac{1}{\hat{\gamma}^{\mathsf{AM}}} = \sum_{j \in [n]} f^{-1}\left(\frac{A_i}{\hat{\gamma}^{\mathsf{AM}}}\right), \tag{6}$$

where $f(z) \triangleq z e^{\frac{1}{1-z}}$. The equilibrium market-share and markup of seller i is:

$$q_i^{\mathsf{AM}} = f^{-1}\left(\frac{A_i}{\hat{\gamma}^{\mathsf{AM}}}\right) \text{ and } m_i^{\mathsf{AM}} = \frac{1}{b(1 - q_i^{\mathsf{AM}})}. \tag{7}$$

Absence of Competition (AC) Another useful benchmark is the absence of competition, i.e., the scenario of a monopolist seller ($n = 1$), which we denote by AC. The seller faces the following two-dimensional optimization problem:[9]

$$\max_{m,x} \pi, \text{ where } \pi = m\left(\frac{Ae^{(1-d)x - bm}}{1 + Ae^{x - bm}}\right) - h(x). \tag{8}$$

For notational convenience, we define the following. Suppose $h'(0) < \frac{(1-\sqrt{d})^2}{b}$.

(a) Let $\underline{q}$ and $\overline{q}$ (with $\underline{q} \leq \overline{q}$) denote the roots to the following equation:

$$\frac{q}{b}\left(1 - \frac{d}{1-q}\right) = h'(0) \implies \underline{q}, \overline{q} = \frac{1 - d + bh'(0)}{2} \pm \frac{1}{2}\sqrt{(1 - d + bh'(0))^2 - 4bh'(0)}. \tag{9}$$

We illustrate the roots in Figure 2.

(b) Let $\underline{\gamma}$ and $\overline{\gamma}$ denote the following:

$$\underline{\gamma} = \frac{A}{f(\overline{q})} \text{ and } \overline{\gamma} = \frac{A}{f(\underline{q})}. \tag{10}$$

[8] While part (b) might appear restrictive at first sight, we note that this assumption applies to cost as a function of the *resulting increment* in perceived quality x, not that of the manipulation input. In Section 5, we will show that a quadratic cost function in terms of the number of solicited reviews (i.e., when a seller faces linear marginal cost to solicit fake reviews) satisfies the above assumption.

[9] We drop the subscript i since $n = i = 1$.

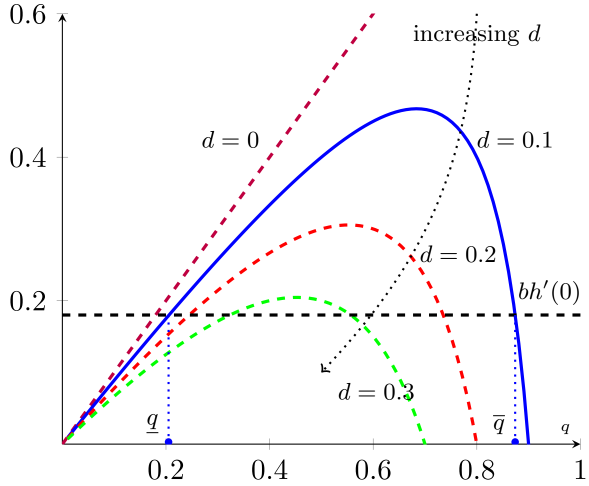


Figure 2. Plot of $q\left(1-\frac{d}{1-q}\right)$ for $d \in \{0, 0.1, 0.2, 0.3\}$. For the case of $d = 0.1$ and a given value of $bh'(0)$ (=0.18), the values of $\underline{q}$ and $\overline{q}$ (resp., 0.205 and 0.874) are also shown.

Next, we define two functions $\hat{h}(\cdot)$ and $g(\cdot)$ below.[10]

$$\begin{aligned} \hat{h}(x) &= h'(x)e^{dx} && (11) \\ g(z) &= \begin{cases} \begin{cases} f(z) & \text{if } z \in [0, \underline{q}) \cup (\overline{q}, 1] \\ f(z)e^{-\hat{h}^{-1}\left(\frac{z}{b}\left(1-\frac{d}{1-z}\right)\right)} & \text{if } z \in [\underline{q}, \overline{q}] \end{cases} & \text{if } h'(0) < \frac{(1-\sqrt{d})^2}{b}. \\ f(z) & \text{if } h'(0) > \frac{(1-\sqrt{d})^2}{b} \end{cases} && (12) \end{aligned}$$

It is straightforward that $\hat{h}(x)$ is increasing and convex in x by Assumption 1. In Lemma B1 in Appendix B, we show that $g(z)$ is continuous and increasing in z. The optimal pricing and manipulation for the monopolist seller is presented in the result below.

Theorem 2 (Optimal Decisions under AC). *Let $\hat{\gamma}^{\mathsf{AC}} \in (1, \infty)$ be the unique solution to the following equation.*

$$1 - \frac{1}{\hat{\gamma}^{\mathsf{AC}}} = g^{-1}\left(\frac{A}{\hat{\gamma}^{\mathsf{AC}}}\right).$$

The seller manipulates if and only if the following holds:

$$h'(0) < \frac{(1-\sqrt{d})^2}{b} \text{ and } \hat{\gamma}^{\mathsf{AC}} \in [\underline{\gamma}, \overline{\gamma}]. \tag{13}$$

[10] The quantities in (9)-(10) are defined for the seller if and only if $h'(0) < \frac{(1-\sqrt{d})^2}{b}$. The quantities in (11)-(12) are always well-defined.

The optimal markup and manipulation is:

$$m^{\mathsf{AC}} = \frac{1}{b\left(1 - g^{-1}\left(\frac{A}{\hat{\gamma}^{\mathsf{AC}}}\right)\right)}, \ \ \textit{and} \ x^{\mathsf{AC}} = \begin{cases} 0 & \textit{if } (13) \textit{ does not hold} \\ \hat{h}^{-1}\left(\frac{g^{-1}\left(\frac{A}{\hat{\gamma}^{\mathsf{AC}}}\right)}{b}\left(1 - \frac{d}{1-g^{-1}\left(\frac{A}{\hat{\gamma}^{\mathsf{AC}}}\right)}\right)\right) & \textit{if } (13) \textit{ holds} \end{cases}$$

The resulting market-share is $q^{\mathsf{AC}} = g^{-1}\left(\frac{A}{\hat{\gamma}^{\mathsf{AC}}}\right) e^{-dx^{\mathsf{AC}}}$.

The first part of Theorem 2 highlights two deterrent forces – the cost of manipulation and the risk of returns – that make manipulation unappealing. The second part identifies the optimal decisions for the monopolist. Below, we build on these observations to identify the sellers' equilibrium decisions in a competitive oligopoly.

4 Presence of Seller Manipulation (PM)

From Theorem 1, we have that in the absence of manipulation, a seller with a higher type has a higher equilibrium margin, market share and profit. The result aligns well with the typical observation that stronger sellers do well in a competitive market. Now, suppose that sellers can manipulate their perceived quality. Are the high-type sellers less inclined to engage in quality manipulation because they are doing well, or are they compelled to dominate the market even more when given the chance to further elevate the market's perception of their quality, barring legal and moral obstacles? Recall from the discussion in Section 2.3 that the findings in the extant literature has been limited but mixed. Dellarocas (2006) argue that higher quality sellers have a greater incentive to manipulate while He et al. (2022) show empirical evidence that manipulation is predominantly employed by lower quality sellers. While these insights are derived either from a stylized theoretical setting or obtained from evidence in a particular data set, we examine the same question by evaluating a multi-seller price competition under the empirically-supported MNL demand model. We present two measures – a seller's "propensity to manipulate" that identifies their upside/benefits from manipulation, and their "antipathy to manipulate" that marks their downside due to manipulation through higher return incidence – that unify the theoretical and empirical observations in the literature and shed new insights.

When a seller begins to manipulate, their perceived quality increases, affecting customer choice and disturbing any market equilibrium. Others will respond and their response is two-pronged - they may manipulate their own perceived quality and/or they may adjust their prices. These actions will in turn trigger a new round of responses until an equilibrium, if one exists, is reached. Consider seller i. Recall seller i's profit in (5). Fix the decisions of all sellers other than i, i.e., $(\mathbf{m}_{-i}, \mathbf{x}_{-i})$. We analyze seller i's best response (m_i, x_i).

Lemma 1 (Best Response of Seller i)**.** *Fix* $(\mathbf{m}_{-i}, \mathbf{x}_{-i})$*. Let* $m_i(x_i)$ *denote the unique solution to the following univariate (fixed-point) equation in* m_i*:*

$$m_i = \frac{1}{b}\left(1 + \frac{A_i e^{x_i - bm_i}}{1 + \sum_{j \neq i} A_j e^{x_j - bm_j}}\right). \tag{14}$$

Let $q_i^0(x_i, m_i(x_i)) \equiv q_i^0(x_i)$*.*

(a) ***Optimal Manipulation*** x_i*: The optimal* $x_i^\star$ *is unique and as follows:*

(i) If $\frac{q_i^0(0)}{b}\left(1 - \frac{d}{1-q_i^0(0)}\right) \leq h_i'(0)$*, then,* $x_i^\star = 0$*.*

(ii) Otherwise, $x_i^\star$ *is the unique solution to the following equation:*

$$\underbrace{\frac{q_i^0(x_i)e^{-dx_i}}{b}\left(1 - \frac{d}{1 - q_i^0(x_i)}\right)}_{\text{Marginal Benefit from Manipulation}} = \underbrace{h_i'(x_i)}_{\text{Marginal Cost of Manipulation}}. \tag{15}$$

(b) ***Optimal Markup*** m_i*: The optimal markup* $m_i^\star$ *can be identified by substituting* $x_i^\star$ *(from above) to solve for* m_i *in* (14)*.*

Underlying (15) are tensions between two sets of competing forces: (a) the tension between the cost and benefit from manipulation identified in (15) that is relatively straightforward; (b) more subtly, the tension in the marginal benefit from manipulation between the purchase incidence and the return incidence. As shown in the l.h.s. of (15), the benefit from manipulation is partly nullified from a higher value of d. We remark that the cost of manipulation may not be homogeneous, and might depend on the seller's type. All of these forces make it hard to predict which sellers manipulate. In what follows, we identify the equilibrium outcome and, subsequently, develop two indices that help identify the set of sellers that manipulate.

4.1 Equilibrium Outcome

Let $\mathbf{x}^{\mathsf{PM}}$ and $\mathbf{m}^{\mathsf{PM}}$ denote the equilibrium manipulation and markups, respectively. Let $\mathcal{X}$ denote the set of sellers that manipulate in equilibrium, i.e.,

$$\mathcal{X} = \{i : x_i^{\mathsf{PM}} > 0\}. \tag{16}$$

If the marginal cost of manipulation $h_i'(0)$ for all $i \in [n]$ is too large, then no seller chooses to manipulate (i.e., the set $\mathcal{X}$ is empty), and the equilibrium outcome under PM is identical to that under AM. For a non-trivial outcome under PM, we make the following assumption.

Assumption 2 (Non-Trivial Outcomes)**.** *There exists some seller $i \in [n]$ s.t.*

$$\underbrace{h_i'(0)}_{\substack{\textit{marginal cost of manipulation}\\ \textit{at } x_i = 0}} < \underbrace{\frac{q_i^{\mathsf{AM}}}{b}\left(1 - \frac{d}{1 - q_i^{\mathsf{AM}}}\right)}_{\substack{\textit{marginal benefit from manipulation to}\\ \textit{seller } i \textit{ at } x_i = 0,\ \mathbf{x}_{-i} = \mathbf{0}}} . \tag{17}$$

The l.h.s. in (17) is the marginal cost of manipulation at $x_i = 0$, while the r.h.s. is the marginal benefit from manipulation at $x_i = 0$, $\mathbf{x}_{-i} = \mathbf{0}$, where q_i^{AM} is the equilibrium market share in the absence of manipulation. Assumption 2 implies that, there exists at least one seller i such that, if no other seller were to manipulate, seller i has a strict incentive to manipulate. Consequently, under this assumption, the absence of manipulation does not constitute an equilibrium outcome.

Analogous to (9), (10), (11), and (12), define the quantities $\underline{q}_i$, $\overline{q}_i$, $\underline{\gamma}_i$, $\overline{\gamma}_i$, $\hat{h}_i$ and g_i for each firm $i \in [n]$.[11] In Theorem 3 below, we identify the equilibrium outcome of the price-manipulation game. Subsequently, we interpret and demonstrate the usefulness of the two indices – $\underline{\gamma}_i$ and $\overline{\gamma}_i$ – in Section 4.1.

Theorem 3 (Equilibrium Outcome under PM)**.** *Let $\hat{\gamma}^{\mathsf{PM}} \in (1, \infty)$ denote the unique solution to the following equation:*

$$1 - \frac{1}{\hat{\gamma}^{\mathsf{PM}}} = \sum_{i \in [n]} g_i^{-1}\left(\frac{A_i}{\hat{\gamma}^{\mathsf{PM}}}\right). \tag{18}$$

*(a) (**Equilibrium** $\mathcal{X}$) In equilibrium, seller i manipulates (i.e., $i \in \mathcal{X}$) if and only if the following holds:*

$$h_i'(0) < \frac{(1-\sqrt{d})^2}{b} \textit{ and } \hat{\gamma}^{\mathsf{PM}} \in \left(\underline{\gamma}_i, \overline{\gamma}_i\right). \tag{19}$$

*(b) (**Equilibrium** m_i **and** x_i) The equilibrium markup and manipulation of seller i are*

$$\begin{aligned} m_i^{\mathsf{PM}} &= \frac{1}{b\left(1 - g_i^{-1}\left(\frac{A_i}{\hat{\gamma}^{\mathsf{PM}}}\right)\right)}, \textit{ and} \\ x_i^{\mathsf{PM}} &= \begin{cases} 0, & \textit{if } (19) \textit{ does not hold} \\ \hat{h}_i^{-1}\left(\frac{g_i^{-1}\left(\frac{A_i}{\hat{\gamma}^{\mathsf{PM}}}\right)}{b}\left(1 - \frac{d}{1 - g_i^{-1}\left(\frac{A_i}{\hat{\gamma}^{\mathsf{PM}}}\right)}\right)\right), & \textit{if } (19) \textit{ holds} \end{cases} \end{aligned} \tag{20}$$

The equilibrium market share of seller i is

$$q_i^{\mathsf{PM}} = g_i^{-1}\left(\frac{A_i}{\hat{\gamma}^{\mathsf{PM}}}\right) e^{-dx_i^{\mathsf{PM}}}. \tag{21}$$

Theorem 3 makes two observations. Part (a) identifies the necessary and sufficient conditions for a seller to manipulate, while part (b) identifies the equilibrium decisions of each seller and their

[11] These quantities depend on i through the seller type A_i and the cost of manipulation $h_i(\cdot)$.

resulting market-share. Depending on the cost of manipulation and their respective antipathy and propensity to manipulate, we categorize a seller into one of the following:

(a) **Cost-Prohibitive**: It is prohibitively expensive for seller i to manipulate if $h_i'(0) > \frac{(1-\sqrt{d})^2}{b}$. To see this, observe that the r.h.s. in (17) is bounded from above by $\frac{(1-\sqrt{d})^2}{b}$. In this case, it is a dominant strategy for seller i to never manipulate (independent of other sellers' strategies).

(b) **Cost-Dominant**: Suppose $h_i'(0) < \frac{(1-\sqrt{d})^2}{b}$. Seller i does not manipulate if either $\hat{\gamma}^{\mathsf{PM}} \leq \underline{\gamma}_i$ or $\hat{\gamma}^{\mathsf{PM}} \geq \overline{\gamma}_i$. The former arises due to greater downsides of manipulation through greater returns, while the latter arises due to low upside benefits of manipulation. Observe that both cases arise due to the market environment/microstructure, i.e., it depends on the exact competitive environment a seller operates in.

(c) **Benefit-Dominant**: Suppose $h_i'(0) < \frac{(1-\sqrt{d})^2}{b}$ and $\hat{\gamma}^{\mathsf{PM}} \in (\underline{\gamma}_i, \overline{\gamma}_i)$, i.e., (19) holds. Seller i has a high benefit from manipulation and low downsides from manipulation, typical for moderate-sized sellers. Part (a) of Theorem 3 shows that only the benefit-dominant sellers manipulate.

In what follows, we demonstrate how the two indices $\underline{\gamma}_i$ and $\overline{\gamma}_i$ can be interpreted as a seller's (relative) antipathy and propensity to manipulate, and help demonstrate its importance in understanding market outcomes.

4.2 Interpreting the Indices: Antipathy and Propensity to Manipulate

From part (a) of Theorem 3, observe that for seller i to manipulate in equilibrium, we require that $\underline{\gamma}_i < \hat{\gamma}^{\mathsf{PM}} < \overline{\gamma}_i$. We interpret $\overline{\gamma}_i$ as seller i's propensity to manipulate due to the upside/benefits of manipulation, while $\underline{\gamma}_i$ as seller i's antipathy to manipulation due to the downside/ill-effects of manipulation. If $\underline{\gamma}_i$ and $\overline{\gamma}_i$ are both high, or both low, part (c) of Theorem 3 states that seller i does not manipulate. On the other hand, if $\overline{\gamma}_i$ is high, and $\underline{\gamma}_i$ is low, then seller i chooses to manipulate.

Further, Lemma 2 below shows how the two indices change with underlying changes in the parameters of the environment.

Lemma 2 (Comparative Statics of $\underline{\gamma}$ and $\overline{\gamma}$)**.** *All else equal, we have the following:*

(a) ***(Effect of Manipulation-Driven Returns)*** *The antipathy (resp., propensity) to manipulate is increasing (resp., decreasing) in the likelihood of return incidence, i.e., $\underline{\gamma}_i$ (resp., $\overline{\gamma}_i$) is increasing (resp., decreasing) in d.*

(b) ***(Effect of Seller Type)*** *The antipathy and propensity to manipulate are increasing in the cost-adjusted quality (type) of a seller, i.e., $\underline{\gamma}_i$ and $\overline{\gamma}_i$ are increasing in A_i.*

4.3 Effect of Competition

To highlight the effect of competition, we analyze the equilibrium outcome under a symmetric oligopoly with n sellers. The analysis of this symmetric oligopoly allows for several useful comparative statics. We interpret the number of sellers n as the intensity of competition. Since all sellers are identical, we drop the subscript i and index the scenarios with n.

Corollary 1 (Symmetric Oligopoly)**.** *Let $\hat{\gamma}^{\mathsf{Sym}}$ denote the value of $\hat{\gamma}^{\mathsf{PM}}$ (defined in (18)) under a symmetric oligopoly with n sellers, i.e., $\hat{\gamma}^{\mathsf{Sym}} \in (1, \infty)$ is the unique solution to the following:*

$$1 - \frac{1}{\hat{\gamma}^{\mathsf{Sym}}} = n g^{-1}\left(\frac{A}{\hat{\gamma}^{\mathsf{Sym}}}\right).$$

(a) $\hat{\gamma}^{\mathsf{Sym}}$ is increasing in n.

(b) Sellers manipulate in equilibrium if and only if the following holds:

$$h'(0) \leq \frac{(1-\sqrt{d})^2}{b} \text{ and } \frac{1}{\overline{q}}\left(1 - \frac{f(\overline{q})}{A}\right) < n < \frac{1}{\underline{q}}\left(1 - \frac{f(\underline{q})}{A}\right). \tag{22}$$

(c) The equilibrium outcome follows from substituting $\hat{\gamma}^{\mathsf{Sym}}$ in (20)-(21) in Theorem 3.

If $n = 1$, Corollary 1 reduces to Theorem 2. From part (b) of Corollary 1, observe that greater competition may encourage or discourage manipulation. If competition is too weak (resp., too intense), i.e., $n \leq \frac{1}{\overline{q}}\left(1 - \frac{f(\overline{q})}{A}\right)$ (resp., $n \geq \frac{1}{\underline{q}}\left(1 - \frac{f(\underline{q})}{A}\right)$), then sellers do not manipulate in equilibrium. In the former case, the downside/ill-effects from manipulation is too large for the seller to manipulate (equivalently, the antipathy index $\underline{\gamma}$ is too high); in the latter case, the upside/benefits from manipulation is too low for the seller to manipulate (equivalently, the propensity index $\overline{\gamma}$ is too low). The scope of manipulation decreases with d. That is, the lower bound in (22), $\frac{1}{\overline{q}}\left(1 - \frac{f(\overline{q})}{A}\right)$, is increasing in d while the upper bound, $\frac{1}{\underline{q}}\left(1 - \frac{f(\underline{q})}{A}\right)$, is decreasing in d.

Further, the effect of an increase in the intensity of competition on seller manipulation depends on the type of the seller, and is in general, not monotone. As an illustration, we contrast the outcomes when the number of sellers goes from $n = 1$ to $n = 2$ (a monopoly to a symmetric duopoly).

- If $A \in \left(\frac{f(\underline{q})}{1-\underline{q}}, \min\left\{\frac{f(\overline{q})}{1-\overline{q}}, \frac{f(\underline{q})}{1-2\underline{q}}\right\}\right)$, then competition *discourages* manipulation in that a monopolist seller manipulates, but the seller does not do so in the presence of an identical competitor.
- If $A \in \left(\max\left\{\frac{f(\underline{q})}{1-2\underline{q}}, \frac{f(\overline{q})}{1-\overline{q}}\right\}, \frac{f(\overline{q})}{1-2\overline{q}}\right)$, then competition *encourages* manipulation in that a monopolist seller does not manipulate, but the seller manipulates in the presence of an identical competitor.

We formalize this insight in the result below.

Lemma 3 (Comparison of an Oligopoly with a Monopoly)**.** *Consider a symmetric oligopoly with n sellers and suppose that $h'(0) < \frac{(1-\sqrt{d})^2}{b}$ and $A > \frac{f(\underline{q})}{1-\underline{q}}$.*[12]

[12] These two conditions ensure that the absence of manipulation is not the equilibrium; see (22).

(a) ***(Competition Discourages Manipulation)*** *Suppose* $A < \frac{f(\overline{q})}{1-\overline{q}}$*. Then, manipulation occurs under a monopoly (*$n = 1$*), while under a symmetric oligopoly, manipulation occurs iff* $n < \frac{1}{\underline{q}}\left(1 - \frac{f(\underline{q})}{A}\right)$*.*

(b) ***(Competition Encourages Manipulation)*** *Suppose* $A > \frac{f(\overline{q})}{1-\overline{q}}$*. Then, manipulation does not occur under a monopoly (*$n = 1$*), while under a symmetric oligopoly, manipulation occurs iff the number of sellers satisfies* $n \in \left(\frac{1}{\overline{q}}\left(1 - \frac{f(\overline{q})}{A}\right), \frac{1}{\underline{q}}\left(1 - \frac{f(\underline{q})}{A}\right)\right)$*.*

The conditions described above explain how competition affects sellers' equilibrium decision to manipulate. The next result further establishes how the extent of manipulation, i.e., the magnitude of x, is affected by competition.

Lemma 4 (Extent of Manipulation with n)**.** *Suppose* $h'(0) < \frac{(1-\sqrt{d})^2}{b}$ *and* $A > \frac{f(\underline{q})}{1-\underline{q}}$*. As the number of sellers* n *increases, the equilibrium extent of manipulation* x^{PM} *is increasing (resp. decreasing) in* n *if and only if* $\frac{1}{\overline{q}}\left(1 - \frac{f(\overline{q})}{A}\right) < n < \frac{1-g(1-\sqrt{d})/A}{1-\sqrt{d}}$ *(resp.* $\frac{1-g(1-\sqrt{d})/A}{1-\sqrt{d}} < n < \frac{1}{\underline{q}}\left(1 - \frac{f(\underline{q})}{A}\right)$*).*

In other words, as competition intensifies (n increases), manipulation intensifies initially and then declines. As n increases, each seller expects to earn a smaller market share. Consequently, while the benefit from manipulation dwindles, the ill-effects of manipulation also become more muted due to the smaller market share. The former effect dominates if n is smaller, while the latter dominates if n is larger.

4.4 Asymmetric Sellers

In this section, we expand on our main results by analyzing asymmetric settings, where firms differ based on their types, their costs of manipulation, or both. Specifically, in the last part, we analyze a log-separable cost of manipulation. The results in these special cases encompass a variety of findings espoused in the extant literature, as well as providing new insights made possible in our unified framework.

Homogeneous Cost of Manipulation: Under a homogeneous cost of manipulation, say $h_i(x) = h(x)$ for all i, observe that the denominators of $\underline{\gamma}_i$ and $\overline{\gamma}_i$ are fixed for all i. To identify the equilibrium $\mathcal{X}$, from part (a) of Theorem 3, it follows that $i \in \mathcal{X}$ if and only if $f(\underline{q})\hat{\gamma}^{\mathsf{PM}} \le A_i \le f(\overline{q})\hat{\gamma}^{\mathsf{PM}}$. Accordingly, we have the following result.

Lemma 5 (Sellers that Manipulate are Contiguous)**.** *Suppose that the cost of manipulation is identical for all sellers, i.e.,* $h_i(x) = h(x)$ *for all* i*. Suppose* $A_1 \le A_2 \le \ldots \le A_n$*. Then, the set* $\mathcal{X}$ *is contiguous, i.e.,* $\mathcal{X}$ *is of the form* $\{i, i+1, \ldots, j\}$ *for some* $1 \le i \le j \le n$*.*

Observe, from Lemma 5, that under a homogeneous cost of manipulation, a seller with a higher (resp., lower) type is more (resp., less) inclined to manipulate, but also faces a higher (resp., lower) antipathy to manipulate. Consequently, the set of sellers who manipulate in equilibrium are "in the middle". Nonetheless, if $i = 1$, then, $\mathcal{X}$ is downward-closed, which is consistent with the findings in He et al. (2022); if $j = n$, then, $\mathcal{X}$ is upward-closed, which is consistent with the findings in Dellarocas (2006). Even our special case encompasses both the perspectives proposed in the literature thus far.

Homogeneous Types of Sellers: Under homogeneous seller types, say $A_i = A$ for all i, recall from the definition of $\underline{\gamma}_i$, $\overline{\gamma}_i$ that the numerator is identical for all i. It suffices to rank order the marginal manipulation cost.

Lemma 6 (Sellers with Lower Manipulation Costs Manipulate More)**.** *Suppose that the types of sellers are identical, i.e.,* $A_i = A$ *for all* i*. Suppose* $h'_1(x) \geq h'_2(x) \geq \ldots \geq h'_n(x)$ *for all* $x \geq 0$*. The set of sellers that manipulate is of the form* $\{i, i+1, \ldots, n\}$ *for some* $i \in [n]$*.*

Lemmas 5 and 6 are special cases under which we glean, respectively, how seller quality and cost affect their equilibrium manipulation behavior. Lemma 6 brings the effect of manipulation cost into focus. In practice, sellers vary both in cost and in quality. More generally, their cost and quality also interact. That is, their cost of manipulation can depend on their true quality. In online retailing, for example, customer review ratings are usually capped at 5-star, and if high quality has already helped a seller achieve a high rating, further improvement through manipulation is increasingly difficult and expensive. Below, we consider a log-separable cost function that is quality-dependent.

Log-Separable Cost Functions: Suppose that the cost of manipulation depends on a firm's cost-adjusted quality A in the following log-separable form:

$$h(x; A) = \mathscr{H}(A)h(x), \tag{23}$$

where $h(\cdot)$ satisfies Assumption 1, and $\mathscr{H}(\cdot)$ is continuous and non-negative over $\Re^+$. Consequently, $h_i(x) = \mathscr{H}(A_i)h(x)$. Define the *type-elasticity* of the cost of manipulation as follows:

$$\varepsilon_A = \frac{\partial \log \mathscr{H}(A)}{\partial \log A}.$$

The type elasticity of the cost of manipulation determines the increase in the cost of manipulation with an increase in the type of the seller. To isolate the effect of quality A and cost h, we let $d = 0$.

Lemma 7. *Suppose* $d = 0$*, and* $A_1 \leq A_2 \leq \cdots \leq A_n$*. Under the log-separable cost function in* (23)*, we have the following distinct outcomes:*[13]

(a) $\overline{\gamma}_i$ *is increasing in* A_i *iff the following condition holds:* $\varepsilon_A < \frac{1}{1+\frac{z}{(1-z)^2}}$*, where* $z = bh'(0)\mathscr{H}(A)$*. Consequently,* $\mathcal{X}$ *is upward-closed in* $[n]$ *iff the above condition holds.*

(b) $\overline{\gamma}_i$ *is decreasing in* A_i *if* $\varepsilon_A > 1$*. Consequently,* $\mathcal{X}$ *is downward-closed in* $[n]$*.*

While higher quality sellers gain more from manipulation, the type elasticity of the cost of manipulation, ε_A, determines the increase in the cost of manipulation with an increase in the seller's type.

[13]Recall that if $d = 0$, $\underline{\gamma}_i = 0$.

Part (a) of Lemma 7 shows that if ε_A is small, then sellers with higher types manipulate. Part (b) of Lemma 7 shows that if ε_A is large, then sellers with lower types manipulate. For $d > 0$, an analytical evaluation of the equilibrium outcome under the log-separable cost function is challenging. In Figure 3 below, we demonstrate the equilibrium outcome using numerical experiments.

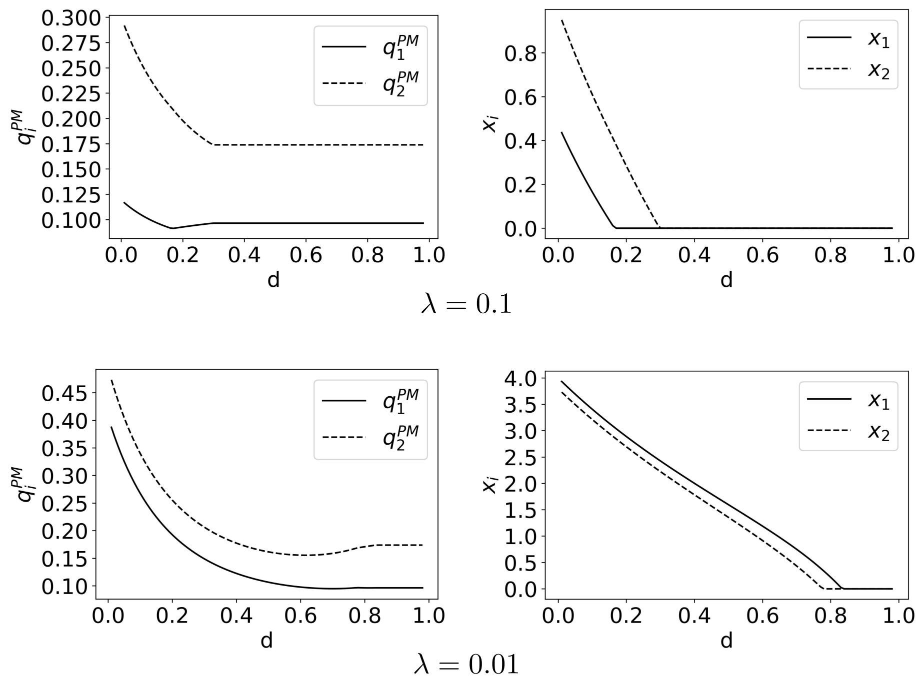


Figure 3. Post-Return Market-Share and Manipulation in a Duopoly under a Log-Separable Cost of Manipulation: $h(x; A) = \lambda e^A(e^x - 1)$. Values of Parameters: $A_1 = 0.4, A_2 = 0.8, b = 0.5$.

In what follows, we apply the framework we developed thus far to the star-rating system commonly observed in retail platforms that provide ratings and reviews. We tailor many elements of the model in Section 4, e.g., the cost of manipulation (where sellers can "buy" fake reviews), perceived quality of a seller (their observed star-rating), etc., to this setting to demonstrate the usefulness of our general framework. Subsequently, in Section 6, we apply the findings in Section 5 to a real-world platform that employs such a star-rating system using publicly available data.

5 A Model of Manipulation Through Online Reviews

To illustrate our results, we examine an environment where sellers in an online platform manipulate their perceived value by soliciting fake reviews. Consider a star rating system employed by the platform on a scale of 0 to R, i.e., each seller is associated with a (true) star-rating between 0 and R, that the seller may manipulate by soliciting promotional (fake) reviews. Often, online star-rating systems adopt a *one-star* to *five-star* scale (e.g., Amazon) where the lowest star rating is one, not zero. In this case, we can map *five-star* to a value of $R = 4$ and *one-star* to the value 0 or use an alternative affine transformation, without loss of generality; similar technique applies to alternatively scaled, e.g., three-star or ten-star rating systems.

In the absence of any manipulation, let seller i's true rating be denoted by r_i^{tr} where $r_i^{\mathsf{tr}} \in [0, R]$. Let v_i^{tr} denote the volume of true ratings for seller i on the platform. Seller i manipulates their perceived rating to be higher than r_i^{tr}. Suppose the seller purchases v_i^{f} fake reviews with rating R.[14] Then, the observed rating for seller i is:

$$r_i^{\mathsf{ob}} \;=\; \frac{v_i^{\mathsf{tr}}}{v_i^{\mathsf{tr}} + v_i^{\mathsf{f}}} r_i^{\mathsf{tr}} + \frac{v_i^{\mathsf{f}}}{v_i^{\mathsf{f}} + v_i^{\mathsf{tr}}} R \;=\; r_i^{\mathsf{tr}} + \frac{v_i^{\mathsf{f}}}{v_i^{\mathsf{tr}} + v_i^{\mathsf{f}}} \left(R - r_i^{\mathsf{tr}}\right).$$

Empirically, we observe r_i^{ob} and $v_i^{\mathsf{tr}} + v_i^{\mathsf{f}}$. Following (1), let a consumer's perceived consumption utility from seller i be denoted as follows:

$$\begin{aligned} u_i &= \beta_0 + \beta_r r_i^{\mathsf{ob}} + \beta_p p_i + \epsilon_i \qquad (24) \\ &= \underbrace{\beta_0 + \beta_r r_i^{\mathsf{tr}}}_{a_i} + \underbrace{\beta_r \frac{v_i^{\mathsf{f}}}{v_i^{\mathsf{tr}} + v_i^{\mathsf{f}}} \left(R - r_i^{\mathsf{tr}}\right)}_{x_i} + \underbrace{\beta_p p_i}_{-bp_i} + \epsilon_i. \end{aligned}$$

The quantities a_i, x_i and $-bp_i$ in (1) correspond to the quantities shown above. The seller's type corresponds to $A_i = e^{\beta_0 + \beta_r r_i^{\mathsf{tr}} + \beta_p c_i}$. To purchase y fake reviews, let the cost incurred by a seller be the following:[15]

$$\text{Cost to purchase } y \text{ fake reviews} = k_1 y + k_2 y^2 \qquad (25)$$

where $k_1, k_2 > 0$. Since seller i purchases v_i^{f} fake reviews, their cost of manipulation is $k_1 v_i^{\mathsf{f}} + k_2 {v_i^{\mathsf{f}}}^2$. Using (25) and the expression for x_i, the cost of manipulation expressed in terms of x_i is:

$$h_i(x_i) = k_1 \left(v_i^{\mathsf{tr}} \frac{\frac{x_i}{\beta_r}}{R - r_i^{\mathsf{tr}} - \frac{x_i}{\beta_r}} \right) + k_2 \left(v_i^{\mathsf{tr}} \frac{\frac{x_i}{\beta_r}}{R - r_i^{\mathsf{tr}} - \frac{x_i}{\beta_r}} \right)^2. \qquad (26)$$

The cost of manipulation in (26) satisfies Assumption 1 if k_2 is sufficiently larger than k_1.[16] Observe that the cost function above depends on both the true rating r_i^{tr} and the volume of true reviews v_i^{tr}.

Recall from (17) that if $h_i'(0) > \frac{\left(1 - \sqrt{d}\right)^2}{b}$, it is a dominant strategy for seller i to never manipulate. In this context, (17) simplifies to:

$$h'(0) > \frac{\left(1 - \sqrt{d}\right)^2}{b} \implies v_i^{\mathsf{tr}} > \overline{\overline{v}} \left(1 - \frac{r_i^{\mathsf{tr}}}{R}\right), \qquad (27)$$

[14] We focus on manipulation by acquiring *additional* fake reviews instead of modifying existing poor reviews.

[15] While the cost to purchase y fake reviews is identical across sellers, the effect of y fake reviews is asymmetric across sellers. Specifically, from a fixed number of fake reviews purchased, seller i experiences a small (resp., large) x_i if v_i^{tr} is large (resp., small) or r_i^{tr} is large (resp., small).

[16] See Appendix E for a detailed proof

where $\overline{\overline{v}} = \frac{\beta_r R}{(-\beta_p)k_1}\left(1-\sqrt{d}\right)^2$ (note that $b = -\beta_p$). Equation (27) states that if the true volume of ratings is large (shown in the r.h.s.), then seller i never manipulates. Further, this threshold is decreasing in the true rating of the seller.

Suppose (27) does not hold, i.e., $v_i^{\mathsf{tr}} \leq \overline{\overline{v}}\left(1-\frac{r_i^{\mathsf{tr}}}{R}\right)$. Recall the definition of $\underline{q}_i$, $\overline{q}_i$ in (9). In the context of our model, let $\underline{q}_i$, $\overline{q}_i$ be the roots of the following equation:

$$\frac{q\left(1-\frac{d}{1-q}\right)}{(1-\sqrt{d})^2} = \frac{v_i^{\mathsf{tr}}}{\overline{\overline{v}}\left(1-\frac{r_i^{\mathsf{tr}}}{R}\right)}.$$

Finally, recall the definitions of $\underline{\gamma}_i$ and $\overline{\gamma}_i$ from (10). In the context of our model, these quantities are:

$$\underline{\gamma}_i = \frac{e^{\beta_0+\beta_r r_i^{\mathsf{tr}}+\beta_p c_i}}{f(\overline{q}_i)}, \text{ and } \overline{\gamma}_i = \frac{e^{\beta_0+\beta_r r_i^{\mathsf{tr}}+\beta_p c_i}}{f(\underline{q}_i)}. \tag{28}$$

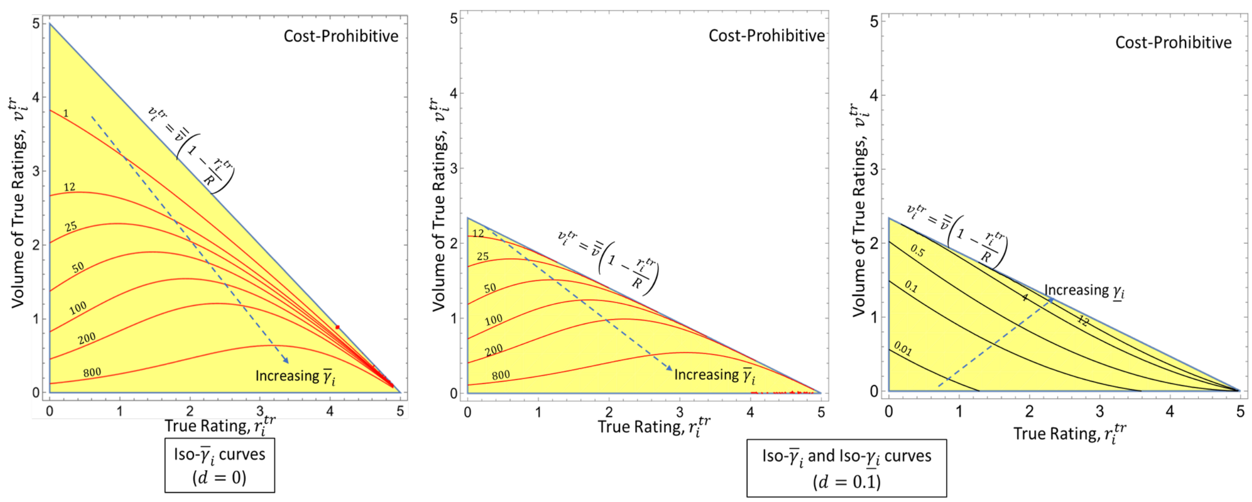


Figure 4. Iso-$\overline{\gamma}$ curves when $d = 0$ (left), and iso-$\overline{\gamma}$ and iso-$\underline{\gamma}$ curves when $d = 0.1$ (right). The respective values of $\overline{\gamma}$ and $\underline{\gamma}$ (for each contour) are also stated.

In Figure 4, we plot the iso-curves for the antipathy to manipulation ($\underline{\gamma}_i$) and the propensity to manipulation ($\overline{\gamma}_i$) in the $(r_i^{\mathsf{tr}}, v_i^{\mathsf{tr}})$ space and how they change with d. Based on Figure 4, we observe the following:

(a) **Effect of the Volume of True Ratings**: The propensity to manipulate $\overline{\gamma}_i$ is strictly decreasing, while the antipathy to manipulate is strictly increasing in the volume of ratings v_i^{tr}. This suggests that a "mature" seller is less likely to manipulate, while a new seller is more likely to manipulate.

(b) **Effect of the True Rating**: Fix v_i^{tr}. The antipathy to manipulate $\underline{\gamma}_i$ is strictly increasing in r_i^{tr}. The propensity to manipulate $\overline{\gamma}$ is initially increasing and then decreasing in r_i^{tr}. This suggests that the seller most likely to manipulate is "in the middle" (in terms of their true rating/quality).

(c) **Effect of Manipulation-Driven Returns**: Using (28), observe that the cost-prohibitive region is increasing in d, i.e., it is more likely that it is a dominant strategy for the seller to never manipulate. Further, a seller's propensity to manipulate decreases with d while their antipathy to manipulate decreases with d (identical to Lemma 2; recall that antipathy is zero for $d = 0$).

5.1 Illustrations of Regions Based on Seller Characteristics

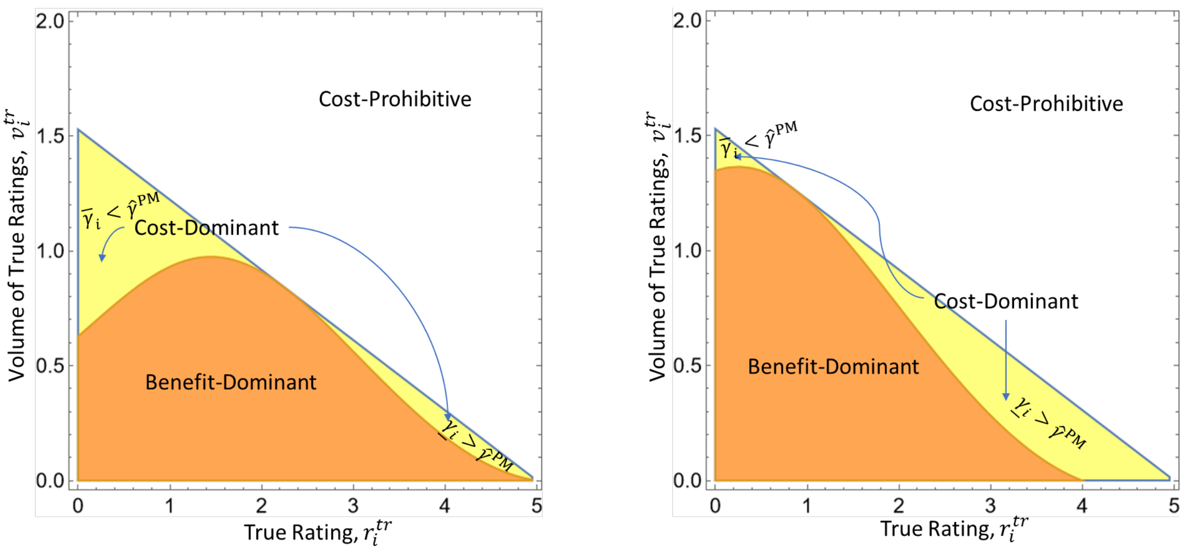


Figure 5. Two illustrative figures demonstrating the three regions (cost-prohibitive, cost-dominant, and benefit-dominant) in the true rating-true volume space. Values of Parameters: $\beta_0 = 5$, $\beta_r = 1$, $\beta_p = -1$, $c = 1$, $k_1 = 1$, $R = 5$, $d = 0.2$, $\hat{\gamma}^{\mathsf{PM}} = 100$ (left), 25 (right).

Given a collection of sellers and their characteristics, we illustrate the two indices – the antipathy and propensity to manipulation – divide the true ratings and the volume of ratings into distinct segments that distinguish seller behavior. Consider Figure 5 as an illustrative example.

- **Cost-Prohibitive Region**: Recall the definition of the cost-prohibitive region from (27), where it is prohibitively expensive for a seller to manipulate and hence a dominant strategy for the seller to never manipulate. In Figure 5, this condition corresponds to the white region.

- **Cost-Dominant Region**: If (27) does not hold, the quantities $\underline{\gamma}_i$ and $\overline{\gamma}_i$ in (28) are well-defined. Recall from Theorem 3 that a seller does not manipulate if either the antipathy to manipulate is too high ($\underline{\gamma}_i \geq \hat{\gamma}^{\mathsf{PM}}$) or the propensity to manipulate is too low ($\overline{\gamma}_i \leq \hat{\gamma}^{\mathsf{PM}}$). Using the iso-$\underline{\gamma}$ and iso-$\overline{\gamma}$ curves, one can identify this region (given $\hat{\gamma}^{\mathsf{PM}}$). In Figure 5, the cost-dominant region is shown in yellow. The yellow region at the left top corresponds to $\overline{\gamma}_i \leq \hat{\gamma}^{\mathsf{PM}}$ while that at the bottom right corresponds to $\underline{\gamma}_i \geq \hat{\gamma}^{\mathsf{PM}}$. The plot on the left and right in Figure 5 differ in the values of $\hat{\gamma}^{\mathsf{PM}}$ (which is determined in equilibrium).

- **Benefit-Dominant Region**: Finally, for a seller to manipulate in equilibrium, we require that (27) not hold, and $\underline{\gamma}_i < \hat{\gamma}^{\mathsf{PM}} < \overline{\gamma}_i$. This region is referred to as the benefit-dominant region and is shown in orange.

One of the unique contributions of our work is that a manager can identify the iso-$\underline{\gamma}$ and iso-$\overline{\gamma}$ curves in the volume-rating coordinates based solely on model parameters. For any given competing set of sellers in a market, we can locate each individual seller on the graph and identify the region they belong to. This ability provides not only a managerial tool for understanding sellers' tendency to manipulate, but is also instructive to a manager to monitor and predict how a seller's dynamically changing status of $(r_i^{\mathsf{tr}}, v_i^{\mathsf{tr}})$ shifts its propensity to manipulate as time evolves. The above can be accomplished without computing the equilibrium. Nonetheless, identifying the exact set of sellers that manipulate requires computing the equilibrium (specifically, $\hat{\gamma}^{\mathsf{PM}}$) as shown in Theorem 3.

Recall the contradicting findings of Dellarocas (2006) and He et al. (2022) on the relationship between quality and manipulation, the former suggesting that high quality (i.e., high r_i^{tr}) sellers are more likely to manipulate whereas the latter concluding that low quality (low r_i^{tr}) sellers are more likely to manipulate. The following result is a consequence of Lemma 2 and (28), and sheds light on the contradicting views in the existing literature, thereby providing a unified perspective.

Corollary 2. *Suppose $d = 0$, $v_i^{\mathsf{tr}} = v$ for all $i \in [n]$, and $r_1^{\mathsf{tr}} \leq r_2^{\mathsf{tr}} \leq \dots r_n^{\mathsf{tr}}$. Then, $\mathcal{X}$ is contiguous. Further, there exists $\overline{r}(v) \in [0, R]$ s.t.*

(a) The set $\mathcal{X}$ is downward-closed in r_i^{tr} if $r_1^{\mathsf{tr}} \geq \overline{r}(v)$.

(b) The set $\mathcal{X}$ is upward-closed in r_i^{tr} if $r_n^{\mathsf{tr}} \leq \overline{r}(v)$.

Corollary 2 shows the various outcomes that emerge in equilibrium. On the one extreme, $\mathcal{X}$ can be downward-closed in the true-rating, e.g., in markets with mature sellers. This conforms with the insights in He et al. (2022), who show that low quality sellers are likely to manipulate. On the other extreme, $\mathcal{X}$ can be upward closed, e.g., in markets with nascent sellers. This conforms with the predictions in Dellarocas (2006). In general, the set $\mathcal{X}$ may be neither upward- nor downward-closed; however, it is contiguous. Because all of the above scenarios are likely to occur in practice, one must take precaution in generalizing the observed trend. For example, when a dataset indicates a negative association between review ratings and manipulation, one cannot extend the association to sellers in other markets or even to extrapolate the trend to draw conclusions regarding other sellers in the same market.

6 Application to Real-World Data

To demonstrate the practical applications of our model, we assemble data from three datasets published by Wang et al. (2014) pertaining to electronic products, where the authors scrape `amazon.com` over a span a period of 24 weeks beginning February 1st, 2012. The first dataset comprises of transaction data for 2,163 unique products, the second dataset contains detailed product characteristics for 794 products, such as Operating System, RAM, processor, processor brand, storage size, average battery life (in hours), screen size, screen resolution, item weight, wireless type, mobile broadband, and webcam resolution and the third dataset comprises of customer reviews and provides information at the reviewer level, including review contents, post date, and review ratings.

6.1 Model Calibration

First, we select a set of $n = 11$ products that are close substitutes, with similar product features such as storage size and screen resolution. These products also have complete transactional information spanning the entire duration of $T = 24$ weeks. We infer the marginal production cost for each product from its selling price and the profit margin from the firms' financial statements if they are publicly listed, or the profit margin of their public competitors as a proxy for products sold by private firms. As Amazon provides information only on sales rank and not sales for each product in their data, we use a mapping from sales rank to sales rate to infer product sales in each time period. This approach has been widely employed in the literature, e.g., see Chevalier and Goolsbee (2003) and He et al. (2022). The mapping between sales s_{it} of product $i \in [n]$ in period $t \in [T]$ and its sales rank R_{it} is as follows:

$$s_{it} = e^{\frac{\beta}{\theta}} \frac{1}{(R_{it} - 1)^{\frac{1}{\theta}}} \tag{29}$$

Prior research report estimates of $\theta = 1.2$ and $\beta = 9.6$ (Chevalier and Goolsbee, 2003; He et al., 2022). We use these estimates to calculate s_{it} based on R_{it}. Consumer utility u_{it} for product $i \in [n]$ in period $t \in [T]$ is modeled as shown in (24):

$$u_{it} = \beta_0 + \beta_r r_{it}^{\mathsf{ob}} + \beta_p p_{it} + \epsilon_{it}, \tag{30}$$

where r_{it}^{ob} is the observed rating for product i at the start of period t, p_{it} is the price for product i in period t, and ϵ_{it} is i.i.d. Gumbel distributed. The price p_{it} is directly observed from the data, but r_{it}^{ob} is inferred as the average rating amongst all posted ratings until the start of period t. That is, we assume that the observed rating in period t is the average of the ratings "thus far" (i.e., until period $t-1$). Let $\mathfrak{r}_{it}$ denote the set of ratings posted for product i in period t. For any period $t \geq 2$, we calculate r_{it}^{ob} as follows:

$$r_{it}^{\mathsf{ob}} = \frac{\sum_{t' \in [t-1]} \sum_{r \in \mathfrak{r}_{it'}} r}{| \cup_{t' \in [t-1]} \mathfrak{r}_{it'} |}.$$

From the above consumer choice model, we have q_{it}^0 (the probability that a representative consumer purchases product i at time t) as follows:

$$q_{it}^0 = \frac{e^{\beta_0 + \beta_p p_{it} + \beta_r r_{it}^{\mathsf{ob}}}}{1 + \sum_{j \in [n]} e^{\beta_0 + \beta_p p_{jt} + \beta_r r_{jt}^{\mathsf{ob}}}}. \tag{31}$$

Since our data does not contain information about no-purchase (s_{0t}), we employ the Expectation-Maximization (EM) algorithm (McLachlan and Krishnan, 2007) to estimate the parameters $\beta_0, \beta_p, \beta_r$.[17]

[17] A detailed explanation of our estimation procedure is provided in Appendix C.

Further, since our data does not include returns, we assume that $d = 0$; hence $q_{it} = q_{it}^0$. The results from our EM estimation procedure are as follows:[18]

Table 1. Estimation Results

	β_p	β_r	β_0
Estimate	-0.0017***	0.103***	0.0109
Std. Error	0.000098	0.004	0.54

$^{*}p < 0.1$, $^{**}p < 0.05$, $^{***}p < 0.01$

6.2 Application and Results

Using the results from our estimation in Section 6.1 above and the analysis in Section 5, we predict the extent of manipulation in equilibrium for each of the n firms in our data. However, the dataset does not indicate whether each review is true or fake. We train a Long Short-Term Memory (LSTM) recurrent neural network model on a separate dataset provided by Salminen et al. (2022), where each review is labeled true or fake. While the nature of products across the two datasets is different, previous studies in Computer Science and Natural Language Processing report commonalities among fake reviews (Fang et al., 2020; Mohawesh et al., 2021).

From Table 1, $\beta_r = 0.103 < \frac{1}{R} = 0.25$; further, recall the assumption that $d = 0$. Therefore, in this application, the benefit-dominant region vanishes and sellers are located in either the cost-prohibitive region or cost-dominant region (where $\overline{\gamma} \leq 1$). In Figure 6, we plot the regions in the volume-rating space and illustrate how the regions shift as the cost parameter changes. Sellers located below the $\overline{\gamma} = \hat{\gamma}^{\mathsf{PM}} (= 1/q_0^{\mathsf{PM}})$ curve (i.e., satisfy $\overline{\gamma} > 1/q_0^{\mathsf{PM}}$) manipulate in equilibrium and those above the $\overline{\gamma} = 1/q_0^{\mathsf{PM}}$ curve (i.e., satisfy $\overline{\gamma} < 1/q_0^{\mathsf{PM}}$) do not manipulate. When k_1 is low ($k_1 = 0.1$; the left plot in Figure 6), all eleven sellers are in the cost-dominant region, and all but one manipulate in equilibrium. As k_1 increases ($k_1 = 1$; the middle plot in Figure 6), three sellers shift from the cost-dominant region to cost-prohibitive regions (one with $\overline{\gamma} = 0$ and two with $\overline{\gamma} \in (0, 1)$) and eight remains in the cost-dominant region, of which five manipulate in equilibrium; when k_1 increases to 3, six sellers shift to cost-prohibitive regions, and five remains in the cost-dominant region, of which only two manipulate. We remark that in these graphs, only the iso-$\overline{\gamma}$ curve and $\overline{\gamma} = 1/q_0^{\mathsf{PM}}$ requires equilibrium computation, whereas the rest are computed directly from model parameters. Therefore, the empirical versatility of the MNL model and our approach enable an easy-to-understand tool for market analysis that is both theoretically sound and managerially appealing, as illustrated in Figure 6.

Finally, the absence of the benefit-dominant region leads to the observation that, *ceteris paribus*, low quality sellers tend to manipulate, which ties back to the phenomenon emphasized in He et

[18]In addition to the consumer utility model in (30), we also consider an alternate model for consumer utility with a product specific constant-term, i.e., $u_{it} = \beta_{0i} + \beta_r r_{it}^{\mathsf{ob}} + \beta_p p_{it} + \epsilon_{it}$, in Appendix C. The estimation results of this alternate utility model are presented in Table C2. We find that the product-specific coefficients β_{0i} (for all products except the reference product) are not significant, most likely due to overfitting (with 10 additional parameters).

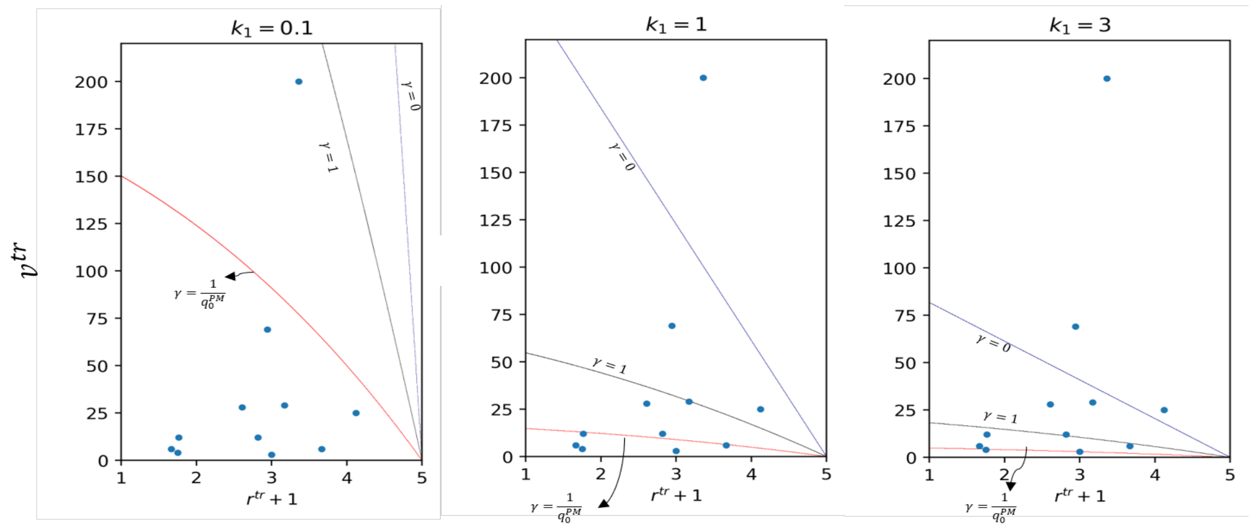


Figure 6. Seller Behavior with Changes in Cost of Manipulation.

al. (2022). Our analysis indicates that, such phenomenon could occur for a specific market under certain conditions on model parameters but should not be viewed as universal truth.

7 Implications of Manipulation: Seller Profits and Consumer Welfare

In an ideal world, sellers do not manipulate for ethical or legal considerations. In practice, such compliance is not guaranteed. Measures to prohibit manipulation and efforts to screen manipulated reviews are costly to platforms. To better understand the economic implications of manipulation, we answer several practical questions. First, is manipulation profitable to sellers who choose to manipulate? Second, how does platform technology to deter manipulation (e.g., flagging and removing manipulated reviews) affect market outcomes? Third, from a policy maker's standpoint, how does manipulation affect consumer welfare? To isolate the effects on each of these quantities of interest, throughout this section, we analyze the outcome when $d = 0$.

7.1 Can Manipulation Hurt All Sellers?

We answer two fundamental questions. First, can the ability to manipulate hurt all sellers in a market? Second, do sellers benefit if manipulation can be made harder (e.g., by a platform that hosts these sellers)? We answer both these questions in the affirmative below.

Consider the case where $d = 0$ and sellers face a homogeneous cost function $h(x) = \lambda(e^x - 1)$, where $\lambda > 0$ is a cost-multiplier.[19] Assumptions 1 and 2 imply that $\lambda < \frac{q_n^{\text{AM}}}{b}$. We analyze the impact of an increase in the cost of manipulation – specifically, λ – on the market outcome.

[19]We illustrate the result in the case of $d = 0$ because the effect of manipulation is extreme in this case; any $d > 0$ softens the effect of manipulation and moves the outcome closer to the absence of manipulation. Our result in Theorem 4 shows that even if $d = 0$, all sellers benefit if manipulation is entirely preventable. Thus, this result will only be strengthened if $d > 0$.

Theorem 4. *Suppose $A_1 \leq A_2 \leq \ldots \leq A_n$. Let $\tau = \frac{\left(\sum_{i\in\mathcal{X}} \frac{A_i}{g'(q_i^{\mathsf{PM}})}\right)}{\left(1+\sum_{i\in\mathcal{X}} \frac{A_i}{g'(q_i^{\mathsf{PM}})}+\sum_{i\in\mathcal{X}^{\mathsf{C}}} \frac{A_i}{f'(q_i^{\mathsf{PM}})}\right)}$. Suppose the following condition holds:*

$$(1-\tau)(2-q_n^{\mathsf{PM}})\frac{q_n^{\mathsf{PM}}}{b} < \lambda < \frac{q_n^{\mathsf{PM}}}{b}. \tag{32}$$

Then, for any $i \in [n]$, seller i's profit is increasing in λ.

Theorem 4 shows the effect an increase in the cost of manipulation can have on a seller's equilibrium profit. In particular, the profit of *all* sellers increases as it becomes harder to manipulate, i.e., as the cost of manipulation increases. Stated differently, *all sellers benefit* from an increase in the cost of manipulation.

As an illustration of Theorem 4, consider the case where $n = 2$, and the sellers are identical, i.e., $A_1 = A_2$. Let $\iota(q) = q(2-q)\left(1 - \frac{2(1-q)^2}{2q^2-6q+3}\right)$. Let $q_{(1)}$ be the first real root of the equation $2q^3 - 10q^2 + 12q - 3 = 0$; $q_{(1)} \approx 0.339$. Equation (32) can be written as follows.

$$\underbrace{\frac{\iota(q_{(1)})}{b}}_{\approx \frac{0.152}{b}} < \lambda < \frac{q_i^{\mathsf{AM}}}{b} \implies \frac{d\pi_i^{\mathsf{PM}}}{d\lambda} > 0 \text{ for } i \in \{1,2\}.$$

Since q_i^{AM} is increasing in A_i, the above condition holds for high values of A_i. This result illustrates a paradoxical example akin to the *prisoner's dilemma*. Although sellers may be forced to manipulate in order to compete with others, every seller is better off had manipulation been preventable altogether. Since individual sellers do not benefit by unilaterally deviating from their equilibrium decision, all sellers are worse-off.

It is often argued that policing manipulation by sellers is an important activity of the platform. Often times, a platform can take measures to make manipulation more costly, and may hold the power to solve the dilemma. If, empowered by a platform's technology, the consumers become sophisticated and are able to (partially) discern manipulation by sellers, how does this ability influence the seller behavior?

7.2 Effect of Consumer Sophistication and Platform Technology

In this section, we extend our main analysis to allow for consumer sophistication and the platform to be able to hinder manipulation. To isolate the effect of (pre-purchase) consumer sophistication, we set $d = 0$. Suppose seller i manipulates by an amount x_i. Instead of an increment of x_i in the (perceived) utility, suppose that their perceived utility increases by an amount δx_i, where $0 < \delta \leq 1$. That is, the consumer's (pre-purchase perceived) utility from seller i is:

$$u_i = a_i + \delta x_i - bp_i + \epsilon_i$$

δ can be interpreted in several ways. For example, δ corresponds to consumers' sophistication in detecting a seller's manipulation ($\delta = 1$ signifies complete naivety and $\delta = 0$ implies complete sophistication). δ can also be interpreted as a platform's technology that hinders a seller's manipulation. For example, several online platforms actively monitor the reviews posted for a seller and detect/flag potential fake reviews. The result below illustrates the effect of δ.

Theorem 5. *A seller's propensity to manipulate $\overline{\gamma}_i$ is increasing in consumer naivety δ. Consequently, for any δ, δ' such that $\delta \leq \delta'$, the set of sellers that manipulate under δ are contained in δ', i.e., $\mathcal{X}^{\delta} \subseteq \mathcal{X}^{\delta'}$.*

While it can be verified that the market-share, markup and profits of sellers that do not manipulate decrease in consumer naivety, the same cannot be said about each seller that manipulates, due to a similar effect as discussed in Section 7.1. Intuitively, greater sophistication among consumers or a better platform technology is equivalent to an increase in the cost of manipulation. Consequently, as discussed in Section 7.1, an increase in consumer sophistication or a better platform technology to hinder manipulation may, paradoxically, benefit all sellers.

7.3 Implications for Consumer Surplus

It can be shown that the expected consumer surplus in the absence of manipulation is:

$$CS^{\mathsf{AM}} = \left(\gamma_e - \log q_0^{\mathsf{AM}}\right)/b$$

where γ_e is the Euler constant. A detailed derivation of consumer surplus under the MNL model under AM and PM is provided in Appendix D. We focus on the case where $d = 0$. In the presence of manipulation, we assume that the manipulation x_i influences only the perceived quality but is not "consumed" (consumption utility; see Footnote 5), and hence it does not add to the true consumer surplus. We derive the expected consumer surplus based on the *true* quality of the products:

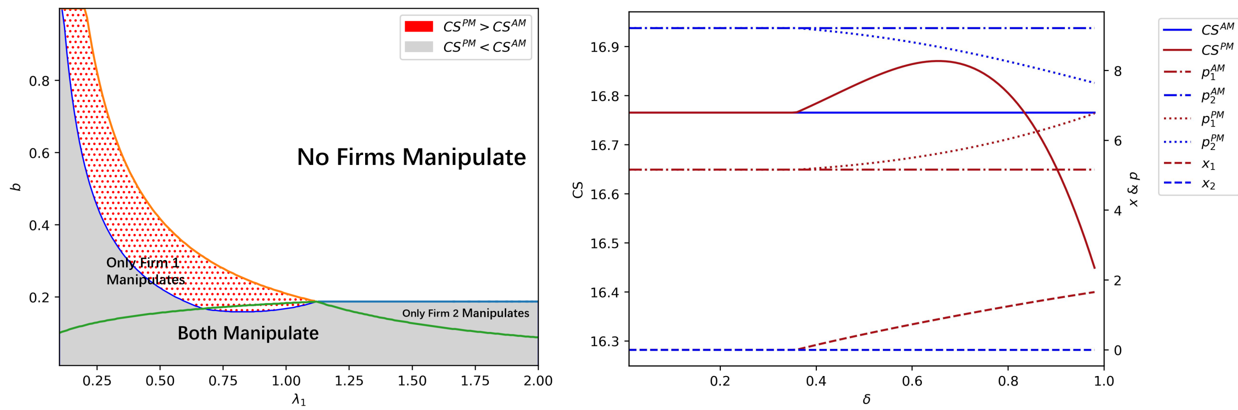
$$CS^{\mathsf{PM}} = \left(\gamma_e - \log q_0^{\mathsf{PM}} - \overline{x}^{\mathsf{PM}}\right)/b.$$

where $\overline{x}^{\mathsf{PM}} \triangleq \sum_{i\in[n]} x_i^{\mathsf{PM}} q_i^{\mathsf{PM}}$ is proportional to the average level of manipulation in the market. From the expressions of consumer surplus, we have

$$CS^{\mathsf{AM}} < CS^{\mathsf{PM}} \Leftrightarrow \overline{x}^{\mathsf{PM}} < \log\left(q_0^{\mathsf{AM}}/q_0^{\mathsf{PM}}\right).$$

Although it seems natural to predict that consumer surplus is negatively affected when manipulation is present, it is not always the case in the equilibrium. We find that manipulation may either increase or decrease the expected consumer surplus.

In Figure 7, we present the outcomes in a duopoly with asymmetric firms that face heterogeneous costs of manipulation $h_i(x) = \lambda_i(e^x - 1)$, and the comparison of equilibrium consumer surplus under AM and PM. We vary the price-sensitivity parameter b and the consumer naivety parameter δ, plot the regions with different market outcomes and denote the region in which $CS^{\mathsf{PM}} > CS^{\mathsf{AM}}$ occurs (red dotted region) in Figure 7(a). Contrary to intuition, under certain circumstances, the presence of manipulation may improve consumer surplus.



(a) $\delta = 1$ (b) $\lambda_1 = 0.3, b = 0.25$

Figure 7. Comparison of consumer surplus in an asymmetric duopoly under AM and PM with a heterogeneous cost of manipulation $h_i(x) = \lambda_i(e^x - 1)$. Values of Parameters: $a_1 = 1.2, a_2 = 3.2, c_1 = c_2 = 0.1, \lambda_2 = 3$.

Recall that sellers' response to competition is two-pronged: they may manipulate their perceived quality and/or adjust their price, depending on the cost of manipulation and consumer price sensitivity. In the example depicted, the two sellers have a notable difference in quality ($a_2 >> a_1$), so a dominant share of the consumer surplus is derived from seller 2. When price sensitivity is very low, both sellers tend to compete more via manipulation than via prices, and a greater extent of manipulation leads to higher prices by both sellers. Consequently, consumers are hurt (as shown in the gray region). On the other hand, when consumers are sufficiently price sensitive and λ_1 is small, while it is more efficient for seller 1 to compete via manipulation than price because of its low manipulation cost, seller 2 finds it expensive to manipulate, so they respond by choosing a lower price. Since the consumer surplus is predominantly derived from seller 2, a lower price by seller 2 contributes to an overall enhancement in total consumer welfare. In essence, the manipulation by the low-quality-low-manipulation-cost firm (seller 1) drives the high-quality-high-manipulation-cost firm (seller 2) to depress its price, leading to an increase in consumer surplus. Figure 7(b) focuses on this scenario and illustrates the compounding effect of δ (consumer naivety). Note that, in this specific instance, seller 2 refrains from engaging in manipulation due to a high cost of manipulation. Greater naivety of consumers allows for a higher extent of manipulation by seller 1, and leads to lower prices by seller 2. This drives up consumer surplus until seller 2's contribution to consumer surplus becomes less dominant, in which case we observe a trend reversal.

The examples above illustrate the mechanism for a possible positive effect of manipulation on consumer surplus – that is, manipulation by some sellers can create pricing pressure on other sellers and benefit consumers. We emphasize that, this is not to say that manipulation is beneficial to the market when consumers are not directly hurt; rather, sellers who do not manipulate are the ones who are unfairly placed at a disadvantage and forced to foot all damages of manipulation by others.

Before we conclude, we comment on the implications of seller manipulation on platforms. It is tempting to infer that manipulation may benefit the platform under a revenue-based commission scheme. However, we remark that the above comparison ignores several additional considerations, e.g., the losses from handling returns. Hence, the extent to which a platform should actively intervene to police manipulation on a platform depends on the trade-off between the benefit from higher revenues from softer competition (due to manipulation) and the short- and long-term losses from the repercussions of manipulation, e.g., loss of reputation and loss of customer loyalty (beyond those captured by the quantity d). A formal model and analysis of these forces is outside the scope of this manuscript and could serve as a useful direction for future research.

8 Conclusion

As consumers place greater emphasis on online product reviews in purchasing decisions, sellers face strong pressure to elevate the rating of their product in order to compete with others. Contradicting views exist in the literature regarding the sellers' tendency to manipulate consumer opinion vis-à-vis the strength/quality/type of sellers. One view argues that high quality sellers have more to gain from manipulation and are more likely to manipulate whereas others present empirical evidence that sellers with low ratings exhibit stronger tendency to manipulate. We construct a model of multi-seller competition in which each seller sets their own price and extent of manipulation to maximize profit, correctly anticipating how sellers act. We solve for the unique equilibrium and present a comprehensive characterization of the set of sellers that manipulate in equilibrium and the resulting market outcomes.

We make several unique contributions to this literature: (i) by identifying two indices $\underline{\gamma}$ and $\overline{\gamma}$ directly computable from model parameters that measure each seller's relative antipathy and propensity to manipulate, (ii) by partitioning the volume-rating space into regions that exhibit distinctive patterns of the iso-$\underline{\gamma}$ and iso-$\overline{\gamma}$ curves, or equivalently, how antipathy and propensity to manipulation is affected by sellers' true quality, and (iii) by mapping our model of review manipulation to a star-rating system where sellers are heterogeneous in their volume of ratings and their true rating (quality), and (iv) by illustrating how to apply it to a real-world data set. A key takeaway is that the two contradicting views regarding the relationship between seller quality and tendency to manipulate can be reconciled through our model and results: We establish the separation of the benefit-dominant region and cost-dominant region. Observations of which types of sellers manipulate in a given application or market may only reflect a censored snapshot view of a market. Decision makers need to be cautious in making generalizations.

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A Proofs of Technical Results

This appendix provides the proofs of all technical results in the main paper. Besides this appendix, we also provide a supplementary appendix, Appendix B, that provides some helpful results.

A.1 Proof of Results in Section 3

Proof of Theorem 1. For completeness, we first show seller i's best-response. We then solve for the equilibrium outcome. In the absence of manipulation ($x_i = 0$ for all i), seller i's market share is:

$$q_i = \frac{A_i e^{-bm_i}}{1 + \sum_{j\in[n]} A_j e^{-bm_j}}. \tag{A.1}$$

Since $\pi_i = m_i q_i$, the derivative of π_i w.r.t m_i is:

$$\frac{d\pi_i}{dm_i} = q_i + m_i \left(\frac{dq_i}{dm_i}\right)$$

Using (A.1), we have

$$\frac{dq_i}{dm_i} = \frac{A_i e^{-bm_i}(-b)\left(1 + \sum_{i\in[n]} A_i e^{-bm_i}\right) - A_i e^{-bm_i} A_i e^{-bm_i}(-b)}{\left(1 + \sum_{i\in[n]} A_i e^{-bm_i}\right)^2} = -bq_i(1-q_i).$$

Substituting this back in the r.h.s. of $\frac{d\pi_i}{dm_i}$, we have:

$$\frac{d\pi_i}{dm_i} = q_i \left(\underbrace{1 - bm_i(1-q_i)}_{\text{decreasing in } m_i}\right).$$

Consider the term inside the brackets in the r.h.s. above: this term is decreasing in m_i. To see this, observe that

$$bm_i(1-q_i) = bm_i \left(\frac{1 + \sum_{j\neq i} A_j e^{-bm_j}}{1 + \sum_{j\neq i} A_j e^{-bm_j} + A_i e^{-bm_i}}\right)$$

Both terms in the r.h.s. above are increasing in m_i; therefore, $1 - bm_i(1-q_i)$ is decreasing in m_i. For given $\mathbf{m}_{-i}$, let $m_i(\mathbf{m}_{-i})$ denote the unique value of m_i s.t. $1 - bm_i(1-q_i) = 0$. Then, for fixed $\mathbf{m}_{-i}$, $\frac{d\pi_i}{dm_i} > 0$ iff $m_i < m_i(\mathbf{m}_{-i})$. That is, π_i is unimodal in m_i; at optimality, $m_i = m_i(\mathbf{m}_{-i}) = \frac{1}{b(1-q_i)}$.

To solve for $m_i(\mathbf{m}_{-i})$, we can write the $m_i(\mathbf{m}_{-i})$ as:

$$bm_i = \frac{1}{1-q_i} \quad \Longrightarrow \quad bm_i - 1 = \frac{q_i}{\underbrace{1-q_i}_{q_0 + \sum_{j\neq i} q_j}} = \frac{A_i e^{-bm_i}}{1 + \sum_{j\neq i, j\in[n]} A_j e^{-bm_j}}$$

$$\Longrightarrow \quad m_i(\mathbf{m}_{-i}) = \frac{1}{b}\left(1 + \frac{A_i e^{-bm_i}}{1 + \sum_{j\neq i, j\in[n]} A_j e^{-bm_j}}\right). \tag{A.2}$$

The r.h.s. of (A.2) is strictly decreasing in m_i. Thus, a unique fixed point to the r.h.s. of (A.2) exists. Further,

$$(bm_i - 1)e^{bm_i - 1} = \frac{A_i e^{-1}}{1 + \sum_{j\neq i, j\in[n]} A_j e^{-bm_j}} \quad \Longrightarrow \quad bm_i - 1 = \mathcal{W}\left(\frac{A_i e^{-1}}{1 + \sum_{j\neq i, j\in[n]} A_j e^{-bm_j}}\right)$$

$$\Longrightarrow \quad m_i(\mathbf{m}_{-i}) = \frac{1}{b}\left(1 + \mathcal{W}\left(\frac{A_i e^{-1}}{1 + \sum_{j\neq i, j\in[n]} A_j e^{-bm_j}}\right)\right),$$

where $\mathcal{W}(\cdot)$ denotes the Lambert-W function.[20]

Next, to identify the equilibrium market-share, we rewrite (A.1) as follows:

$$q_i = \frac{A_i}{\hat{\gamma}} e^{-bm_i},$$

where $\hat{\gamma} = \frac{1}{q_0} = 1 + \sum_{j \in [n]} A_j e^{-bm_j}$. Substituting for $m_i = \frac{1}{b(1-q_i)}$ in the r.h.s. above, we have:

$$q_i e^{\frac{1}{1-q_i}} = \frac{A_i}{\hat{\gamma}} \implies q_i = f^{-1}\left(\frac{A_i}{\hat{\gamma}}\right).$$

Using the above and the identity that $q_0 = 1 - \sum_{j \in [n]} q_j$, we have

$$1 - \frac{1}{\hat{\gamma}} = \sum_{j \in [n]} f^{-1}\left(\frac{A_i}{\hat{\gamma}}\right)$$

Recall the definition of $f(z) = ze^{\frac{1}{1-z}}$ in the statement of Theorem 1. Since $f(x)$ is increasing in x, with $f(0) = 0$ and $f(1) = \infty$, the l.h.s. is increasing in $\hat{\gamma}$, while the r.h.s. is decreasing in $\hat{\gamma}$; thus, the solution to $\hat{\gamma} \in [1, \infty)$, denoted by $\hat{\gamma}^{\mathsf{AM}}$ exists and is unique. Combining the solution to $\hat{\gamma}^{\mathsf{AM}}$ and (A.2), we have the required result. $\square$

Proof of Theorem 2. In the absence of competition, the monopolist seller's market share is:

$$q = \frac{Ae^{(1-d)x-bm}}{1 + Ae^{x-bm}}. \tag{A.3}$$

The seller faces a bivariate optimization problem in the markup and manipulation as shown in (8): $\max_{x,m} \pi(x,m) = mq - h(x)$. We identify the optimal markup (as a function of x) and then identify the optimal extent of manipulation.

For fixed x, we have

$$\frac{d\pi}{dm} = q + m \underbrace{\frac{dq}{dm}}_{\text{using (A.3)}} = q\left(1 - bm(1-q^0)\right).$$

From the r.h.s. above, it follows that for fixed x, the seller's profit is unimodal in m. Therefore, the seller's optimal markup is $m^\star = \frac{1}{b(1-q^0)}$. Using straightforward algebra, $m^\star(x)$ can be stated as:

$$m^\star(x) = \frac{1}{b}\left(1 + \mathcal{W}\left(Ae^{x-1}\right)\right).$$

Let $q(x) \equiv q(x, m^\star(x))$, $q^0(x) = q^0(x, m^\star(x))$ and $\pi(x) = \pi(x, m^\star(x)) = m^\star(x)q(x) - h(x)$. We identify the optimal $x^\star$ below.

$$\begin{aligned}
\frac{d\pi}{dx} &= \underbrace{\frac{dm^\star(x)}{dx}q}_{\substack{\text{effect of manipulation on}\\ \text{the inframarginal units}}} + \underbrace{\frac{dq(x)}{dx}m^\star(x)}_{\text{effect on the marginal unit}} - h'(x) \\
&= \frac{q^0(x)}{b}q(x) + \frac{1}{b(1-q^0(x))}q(x)\left((1-q^0(x))^2 - d\right) - h'(x) \\
&= \underbrace{\frac{q(x)}{b}\left(1 - \frac{d}{1-q^0(x)}\right)}_{\substack{\text{marginal benefit from}\\ \text{manipulation}}} - \underbrace{h'(x)}_{\substack{\text{marginal cost from}\\ \text{manipulation}}}
\end{aligned} \tag{A.4}$$

[20] Fix $k > 0$. The equation $ze^z = k$ has a unique solution at $z = \mathcal{W}(k)$. Further,

$$\frac{dz}{dk} = \left.\frac{1}{e^z(1+z)}\right|_{z=\mathcal{W}(k)} = \frac{1}{e^{\mathcal{W}(k)} + k}.$$

At the critical points (if any; denoted by $x^\star$), we have:

$$\frac{d^2\pi}{dx^2}\bigg|_{x=x^\star} = \left(\underbrace{h'(x^\star)\left(\left(1-q^0\Big|_{x=x^\star}\right)^2 - d\right) - h''(x^\star)}_{\text{negative from Assumption 1}}\right) - \frac{d}{b}\left(q^0\Big|_{x=x^\star}\right)^2.$$

The r.h.s. is negative at any critical points. Therefore, there is at most one critical point; further, if a critical point exists, the second derivative at this point is negative.

From (A.4), observe that

$$\frac{q(0)}{b}\left(1-\frac{d}{1-q^0(0)}\right) > h'(0) \implies x^\star = 0.$$

Otherwise, $x^\star$ solves

$$\begin{aligned}\frac{q(x)}{b}\left(1-\frac{d}{1-q^0(x)}\right) = h'(x) \quad &\implies \quad \frac{q^0(x)}{b}\left(1-\frac{d}{1-q^0(x)}\right) = \hat{h}'(x)\\ &\implies \quad x = \hat{h}^{-1}\left(\frac{q^0(x)}{b}\left(1-\frac{d}{1-q^0(x)}\right)\right)\end{aligned}$$

Next, recall that $q^0 = \frac{A^{x-bm(x)}}{1+Ae^{x-bm(x)}}$. Substituting for the optimal quantities, we have:

$$\begin{aligned} q^0 \quad &= \quad \frac{Ae^{\hat{h}^{-1}\left(\frac{q^0}{b}\left(1-\frac{d}{1-q^0}\right)\right)-\frac{1}{1-q^0}}}{\underbrace{1+Ae^{\hat{h}^{-1}\left(\frac{q^0}{b}\left(1-\frac{d}{1-q^0}\right)\right)-\frac{1}{1-q^0}}}_{=\hat{\gamma}}} = \frac{Ae^{\hat{h}^{-1}\left(\frac{q^0}{b}\left(1-\frac{d}{1-q^0}\right)\right)-\frac{1}{1-q^0}}}{\hat{\gamma}} \\ \implies \underbrace{q^0 e^{\frac{1}{1-q^0}-\hat{h}^{-1}\left(\frac{q^0}{b}\left(1-\frac{d}{1-q^0}\right)\right)}}_{g(q^0)} \quad &= \quad \frac{A}{\hat{\gamma}} \implies q^0 = g^{-1}\left(\frac{A}{\hat{\gamma}}\right).\end{aligned}$$

Since $q_0^0 = \frac{1}{\hat{\gamma}}$, we have that

$$q^0 = 1 - q_0^0 \implies g^{-1}\left(\frac{A}{\hat{\gamma}}\right) = 1 - \frac{1}{\hat{\gamma}}$$

The solution to the above equation is denoted by $\hat{\gamma}^{\mathsf{AC}}$. The markup and the extent of manipulation can be identified by substituting $q^0 = g^{-1}\left(\frac{A}{\hat{\gamma}^{\mathsf{AC}}}\right)$. $\square$

A.2 Proofs of Results in Section 4

Proof of Lemma 1. Fix $(\mathbf{x}_{-i}, \mathbf{m}_{-i})$. To identify seller i's best-response, we proceed in two steps, i.e., solve m_i and then x_i, as follows:

$$\max_{x_i}\left\{\max_{m_i} \pi_i\left((m_i, x_i)\Big|\mathbf{m}_{-i}, \mathbf{x}_{-i}\right)\right\} \tag{A.5}$$

Optimal Markup: For fixed x_i, for the "inner" optimization problem in (A.5), we show the following.

(i) π_i is unimodal in m_i.

(ii) Seller i's best-response $m_i(x_i; \mathbf{x}_{-i}, \mathbf{m}_{-i})$ satisfies $m_i = \frac{1}{b(1-q_i^0)}$.[21] That is, $m_i(x_i; \mathbf{x}_{-i}, \mathbf{m}_{-i})$ is the unique solution to the following univariate equation of m_i:

$$m_i = \frac{1}{b}\left(1 + \frac{A_i e^{x_i - bm_i}}{1+\sum_{j\neq i} A_j e^{x_j - bm_j}}\right). \tag{A.6}$$

(iii) $m_i(x_i; \mathbf{x}_{-i}, \mathbf{m}_{-i})$ is increasing in x_i, increasing in $\mathbf{m}_{-i}$ and decreasing in $\mathbf{x}_{-i}$. Further, $x_i - bm_i(x_i; \mathbf{x}_{-i}, \mathbf{m}_{-i})$ is increasing in x_i.

[21] We omit the arguments of m_i and q_i^0 for brevity.

Recall the expression for q_i from (4):

$$q_i = \frac{A_i e^{(1-d)x_i - bm_i}}{1 + \sum_{j\in[n]} A_j e^{x_j - bm_j}}.$$

From (5), seller i's profit is:

$$\pi_i = m_i q_i - h_i(x_i) \implies \frac{\partial \pi_i}{\partial m_i} = q_i + m_i \frac{\partial q_i}{\partial m_i}$$

We write $\frac{\partial q_i}{\partial m_i}$ below.

$$\begin{aligned}
\frac{\partial q_i}{\partial m_i} &= \frac{\left(A_i e^{(1-d)x_i - bm_i}(-b)\left(1 + \sum_{j\in[n]} A_j e^{x_j - bm_j}\right) - A_i e^{(1-d)x_i - bm_i} A_i e^{x_i - bm_i}(-b)\right)}{\left(1 + \sum_{j\in[n]} A_j e^{x_j - bm_j}\right)^2} \\
&= -b\left(\frac{A_i e^{(1-d)x_i - bm_i}}{1 + \sum_{j\in[n]} A_j e^{x_j - bm_j}}\right)\left(\frac{1 + \sum_{j\in[n], j\neq i} A_j e^{x_j - bm_j}}{1 + \sum_{j\in[n]} A_j e^{x_j - bm_j}}\right) \\
&= -bq_i\left(1 - q_i^0\right) \qquad \text{(A.7)}
\end{aligned}$$

Substituting the r.h.s. from above in the r.h.s. of $\frac{\partial \pi_i}{\partial m_i}$, we have:

$$\frac{\partial \pi_i}{\partial m_i} = q_i\left(1 - bm_i(1 - q_i^0)\right)$$

Since q_i^0 is decreasing in m_i, the r.h.s. single-crosses 0 from above, and hence f.o.c's identify the optimal m_i. At optimality, we have:

$$m_i = \frac{1}{b(1 - q_i^0)} \implies m_i = \frac{1}{b}\left(\underbrace{1 + \frac{A_i e^{x_i - bm_i}}{1 + \sum_{j\neq i} A_j e^{x_j - bm_j}}}_{\frac{1}{1-q_i^0}}\right).$$

The above fixed-point equation is shown in the statement of the result in (14). For fixed $(\mathbf{m}_{-i}, \mathbf{x})$, the r.h.s. is decreasing in m_i, and hence (14) has a unique solution for m_i. The above prove (i) and (ii).

Next, (14) can be written as:

$$(bm_i - 1)e^{bm_i - 1} = \frac{A_i e^{x_i - 1}}{\left(1 + \sum_{j\neq i} A_j e^{x_j - bm_j}\right)} \implies m_i = \frac{1}{b}\left[1 + \mathcal{W}\left(\frac{A_i e^{x_i - 1}}{1 + \sum_{j\neq i} A_j e^{x_j - bm_j}}\right)\right]. \qquad \text{(A.8)}$$

From (A.8), m_i is increasing in x_i, increasing in $\mathbf{m}_{-i}$, and decreasing in $\mathbf{x}_{-i}$. Further, $x_i - bm_i(x_i)$ is increasing in x_i. To see this, consider the derivative of $x_i - bm_i(x_i)$:

$$\frac{d}{dx_i}(x_i - bm_i) = 1 - b\frac{dm_i}{dx_i},$$

We show that the r.h.s. is positive. Consider the second term. Using (14),

$$b\frac{dm_i}{dx_i} = \frac{A_i e^{x_i - bm_i}}{1 + \sum_{j\neq i} A_j e^{x_j - bm_j}}\left(1 - b\frac{dm_i}{dx_i}\right) \implies b\frac{dm_i}{dx_i} = \frac{A_i e^{x_i - bm_i}}{1 + \sum_{j\in[n]} A_j e^{x_j - bm_j}} = q_i^0. \qquad \text{(A.9)}$$

Combining the above, we have $\frac{d}{dx_i}(x_i - bm_i) = (1 - q_i^0) > 0$; hence, $x_i - bm_i$ is increasing in x_i. The quantity $\frac{q_i^0}{1-q_i^0} = \frac{A_i e^{x_i - bm_i}}{1+\sum_{j\neq i} A_j e^{x_j - bm_j}}$; since the r.h.s. is increasing in x_i, it follows that the $\frac{q_i^0}{1-q_i^0}$ is increasing in x_i. Since $\frac{q_i^0}{1-q_i^0}$ is a monotone transformation of q_i^0, it follows that $q_i^0 = q_i^0(x_i, m_i(x_i))$ is increasing in x_i. This proves (iii).

Optimal Manipulation: We show the following:

(i) $\pi_i(x_i)$ is quasi-concave in x_i.

(ii) The optimal x_i^* is unique and as follows:

   (a) If $\frac{q_i^0(0)}{b}\left(1 - \frac{d}{1-q_i^0(0)}\right) \le h_i'(0)$, then, $x_i^* = 0$.

(b) Otherwise, x_i^* is the unique solution to the following equation:

$$\underbrace{\frac{q_i^0(x_i)e^{-dx_i}}{b}\left(1-\frac{d}{1-q_i^0(x_i)}\right)}_{\text{Marginal Benefit from Manipulation}} = \underbrace{h_i'(x_i)}_{\text{Marginal Cost of Manipulation}}. \tag{A.10}$$

Since $\pi_i = m_i q_i - h(x_i)$, the marginal effect of manipulation on seller i's profits (i.e., derivative of π_i w.r.t x_i) is:

$$\frac{d\pi_i}{dx_i} = \underbrace{q_i\frac{dm_i}{dx_i}}_{\substack{\text{effect of manipulation on}\\ \text{infra-marginal units}}} + \underbrace{m_i\frac{dq_i}{dx_i}}_{\substack{\text{effect of manipulation on the}\\ \text{marginal unit}}} - \underbrace{h_i'(x_i)}_{\text{marginal cost of manipulation}}. \tag{A.11}$$

The first term in the r.h.s. of (A.11) is the effect on the inframarginal units, while the second term is the effect on the marginal unit. First, from (A.9), it follows that

$$\frac{dm_i}{dx_i} = \frac{q_i^0}{b}.$$

Next, we can write $\frac{dq_i}{dx_i}$ as follows:

$$\frac{dq_i}{dx_i} = \frac{\partial q_i}{\partial x_i} + \frac{\partial q_i}{\partial m_i}\frac{dm_i}{dx_i}.$$

Using (4),

$$\begin{aligned}
\frac{\partial q_i}{\partial x_i} &= \frac{(1-d)(A_i e^{(1-d)x_i - bm_i})(1+\sum_j A_j e^{x_j - bm_j}) - (A_i e^{(1-d)x_i-bm_i})(A_i e^{x_i - bm_i})}{(1+\sum_j A_j e^{x_j-bm_j})^2} \\
&= q_i(1-d-q_i^0). \\
\frac{\partial q_i}{\partial m_i} &= -bq_i(1-q_i^0). \quad \text{(from (A.7))}
\end{aligned}$$

Therefore,

$$\frac{dq_i}{dx_i} = q_i\left(\left(1-q_i^0\right)^2 - d\right). \tag{A.12}$$

Combining the above, (A.11) simplifies to:

$$\frac{d\pi_i}{dx_i} = \left(\frac{q_i q_i^0}{b} + m_i q_i\left(\left(1-q_i^0\right)^2 - d\right)\right) - h_i'(x_i)$$

Substituting for m_i from (14), we have:

$$\frac{d\pi_i}{dx_i} = \underbrace{\frac{q_i}{b}\left(1-\frac{d}{1-q_i^0}\right)}_{\substack{\text{marginal benefit from}\\ \text{manipulation}}} - \underbrace{h_i'(x_i)}_{\substack{\text{marginal cost of}\\ \text{manipulation}}}. \tag{A.13}$$

First, if $\frac{q_i}{b}\left(1-\frac{d}{1-q_i^0}\right)\Big|_{x_i=0} < h_i'(0)$, then $x_i^\star = 0$. Otherwise, if $\frac{q_i}{b}\left(1-\frac{d}{1-q_i^0}\right)\Big|_{x_i=0} > h_i'(0)$, then, the critical point(s) (if they exist), denoted by x_i^*, solve:

$$q_i\left(1-\frac{d}{1-q_i^0}\right)\Big|_{x_i=x_i^\star} = bh_i'(x_i^\star). \tag{A.14}$$

At the critical point(s), the second derivative is:

$$\begin{aligned}
\frac{d^2\pi}{dx_i^2}\Big|_{x_i=x_i^*} &= \frac{d}{dx_i}\left(\frac{d\pi_i}{dx_i}\right)\Big|_{x_i=x_i^*} = \frac{d}{dx_i}\left(\frac{q_i}{b}\left(1-\frac{d}{1-q_i^0}\right)\right)\Big|_{x_i=x_i^*} - h_i''(x_i^*) \\
&= \frac{1}{b}\left(\underbrace{\frac{dq_i}{dx_i}}_{\text{from (A.12)}}\left(1-\frac{d}{1-q_i^0}\right) + q_i\left(-\frac{d}{(1-q_i^0)^2}\right)\underbrace{\frac{dq_i^0}{dx_i}}_{\text{from (A.16)}}\right)\Bigg|_{x_i=x_i^*} - h_i''(x_i^*) \qquad \text{(A.15)}
\end{aligned}$$

Recall, from (A.12), that $\frac{dq_i}{dx_i} = q_i\left((1-q_i^0)^2 - d\right)$. Further, using an identical procedure, $\frac{dq_i^0}{dx_i}$ can be evaluated to the following:

$$\frac{dq_i^0}{dx_i} = q_i^0(1-q_i^0)^2 \tag{A.16}$$

Substituting (A.12) and (A.16) in the r.h.s. of (A.15), we get

$$\begin{aligned}
\left.\frac{d^2\pi}{dx_i^2}\right|_{x_i=x_i^*} &= \frac{1}{b}\left(q_i\left((1-q_i^0)^2-d\right)\left(1-\frac{d}{1-q_i^0}\right) + q_i\left(-\frac{d}{(1-q_i^0)^2}\right)q_i^0(1-q_i^0)^2\right)\Bigg|_{x_i=x_i^*} - h_i''(x_i^*) \\
&= \underbrace{\frac{q_i}{b}\left(1-\frac{d}{1-q_i^0}\right)}_{=h_i'(x_i^*)\text{ from (A.14)}}\left(\left((1-q_i^0)^2-d\right) - q_i^0 d\underbrace{\left(\frac{1}{1-\frac{d}{1-q_i^0}}\right)}_{=\frac{q_i}{bh_i'(x_i^*)}\text{ from (A.14)}}\right)\Bigg|_{x_i=x_i^*} - h_i''(x_i^*) \\
&= h_i'(x_i^*)\left(\left(1-q_i^0\right)^2 - d\left(1+\frac{q_i^0 q_i}{bh_i'(x_i^\star)}\right)\right)\Bigg|_{x_i=x_i^*} - h_i''(x_i^*) \\
&< h_i'(x_i^\star) - h_i''(x_i^\star) \\
&< 0. && \text{(due to Assumption 1(b))}
\end{aligned}$$

Since the second derivative is negative at the critical points, we have that $\pi_i(x_i)$ is quasi-concave. This proves (i).

As a consequence of the quasi-concavity of $\pi_i(x_i)$, the following hold:

1. If $\frac{q_i(0)}{b}\left(1-\frac{d}{1-q_i^0(0)}\right) < h_i'(0)$, then, $\frac{d\pi_i}{dx_i} < 0$ for all $x_i > 0$.
2. If $\frac{q_i(0)}{b}\left(1-\frac{d}{1-q_i^0(0)}\right) > h_i'(0)$, then, f.o.c.'s identify the unique interior maximum.

For convenience, define $\breve{f}(z)$, $z \in [0,1)$, as follows:

$$\breve{f}(z) \triangleq \left(\frac{z}{1-z}\right)e^{\frac{1}{1-z}}. \tag{A.17}$$

$\breve{f}(z)$ is increasing in z. The solution to the f.o.c. in (A.14) can be expressed as follows.

$$\underbrace{\breve{f}^{-1}\left(\frac{A_i e^{x_i}}{1+\sum_{j\neq i}A_j e^{x_j - bm_j}}\right)\left(1-\frac{d}{1-\breve{f}^{-1}\left(\frac{A_i e^{x_i}}{1+\sum_{j\neq i}A_j e^{x_j-bm_j}}\right)}\right)}_{=q_i^0(x_i)\left(1-\frac{d}{1-q_i^0(x_i)}\right)} = bh_i'(x_i)e^{dx_i}. \tag{A.18}$$

While the l.h.s. and r.h.s. are both increasing in x_i, since π_i is quasi-concave in x_i, it follows that the above equation has a unique solution for x_i. This proves (ii). □

Proof of Theorem 3. From Lemma 1, we have seller i's choice of m_i and x_i. Depending on x_i^{PM}, one of the following applies to seller i:

- If $x_i^{\mathsf{PM}} = 0$ (i.e., $i \in \mathcal{X}^{\mathsf{C}}$), then, using (14), we have the following:

$$q_i^0 = \frac{A_i e^{-\frac{1}{1-q_i^0}}}{\left(\underbrace{1+\sum_j A_j e^{x_j^{\mathsf{PM}}-bm_j^{\mathsf{PM}}}}_{\hat{\gamma}}\right)} = \frac{A_i e^{-\frac{1}{1-q_i^0}}}{\hat{\gamma}} \implies \underbrace{q_i^0 e^{\frac{1}{1-q_i^0}}}_{f(q_i^0)} = \frac{A_i}{\hat{\gamma}} \implies q_i^0 = f^{-1}\left(\frac{A_i}{\hat{\gamma}}\right).$$

- If $x_i^{\mathsf{PM}} > 0$ (i.e., $i \in \mathcal{X}$), using (A.14) and the definition of $\hat{h}(\cdot)$ in (11), we have:

$$x_i^{\mathsf{PM}} = \hat{h}_i^{-1}\left(\frac{q_i^0}{b}\left(1 - \frac{d}{1-q_i^0}\right)\right).$$

  Using (14) and (15), we have the following:

$$\begin{aligned} q_i^0 &= \frac{A_i e^{\hat{h}_i^{-1}\left(\frac{q_i^0}{b}\left(1-\frac{d}{1-q_i^0}\right)\right)-\frac{1}{1-q_i^0}}}{\left(\underbrace{1+\sum_j A_j e^{x_j^{\mathsf{PM}}-bm_j^{\mathsf{PM}}}}_{\hat{\gamma}}\right)} = \frac{A_i e^{\hat{h}_i^{-1}\left(\frac{q_i^0}{b}\left(1-\frac{d}{1-q_i^0}\right)\right)-\frac{1}{1-q_i^0}}}{\hat{\gamma}} \\ \implies \underbrace{q_i^0 e^{\frac{1}{1-q_i^0}-\hat{h}_i^{-1}\left(\frac{q_i^0}{b}\left(1-\frac{d}{1-q_i^0}\right)\right)}}_{g_i(q_i^0)} &= \frac{A_i}{\hat{\gamma}} \implies q_i^0 = g_i^{-1}\left(\frac{A_i}{\hat{\gamma}}\right). \end{aligned}$$

We solve for the equilibrium value of q_0^0. Since $\sum_{i\in[n]} q_i^0 = 1 - q_0^0$, let $\hat{\gamma}^{\mathsf{PM}}$ be the solution to the following:

$$1 - \frac{1}{\hat{\gamma}} = \sum_{i\in\mathcal{X}} g_i^{-1}\left(\frac{A_i}{\hat{\gamma}}\right) + \sum_{i\in\mathcal{X}^{\mathsf{C}}} f^{-1}\left(\frac{A_i}{\hat{\gamma}}\right).$$

The above equation is identical to (18) in the statement of the result. Since $f(\cdot)$ and $g_i(\cdot)$, $i \in [n]$ are increasing functions, the r.h.s. is decreasing in $\hat{\gamma}$, while the l.h.s. is increasing in $\hat{\gamma}$; hence, a unique solution to $\hat{\gamma}$ exists. Further, $\hat{\gamma}^{\mathsf{PM}} \in [1, \infty)$. Substituting $\hat{\gamma}$ in the expression for q_i^0, we have

$$q_i^0 = \begin{cases} f^{-1}\left(\frac{A_i}{\hat{\gamma}^{\mathsf{PM}}}\right), & \text{if } i \in \mathcal{X}^{\mathsf{C}}; \\ g_i^{-1}\left(\frac{A_i}{\hat{\gamma}^{\mathsf{PM}}}\right), & \text{if } i \in \mathcal{X}. \end{cases} \tag{A.19}$$

Substituting q_i^0 in (14) and (15), we have:

$$m_i^{\mathsf{PM}} = \frac{1}{b(1-q_i^0)} \text{ and } x_i^{\mathsf{PM}} = \begin{cases} 0, & \text{if } i \in \mathcal{X}^{\mathsf{C}}; \\ \hat{h}_i^{-1}\left(\frac{q_i^0}{b}\right), & \text{if } i \in \mathcal{X}. \end{cases}$$

The market share of seller i is:

$$q_i^{\mathsf{PM}} = \begin{cases} q_i^0, & \text{if } i \in \mathcal{X}^{\mathsf{C}}; \\ q_i^0 e^{-dx_i^{\mathsf{PM}}}, & \text{if } i \in \mathcal{X}. \end{cases}$$

Observe that in the above, the set $\mathcal{X}$ has still not been identified. For $i \in \mathcal{X}^{\mathsf{C}}$, we require the following:

$$\frac{q_i^0}{b}\left(1 - \frac{d}{1-q_i^0}\right) \leq h_i'(0)$$

The above inequality simplifies to $q_i^0 < \underline{q}_i^0$ or $q_i^0 > \overline{q}_i^0$ (the quantities $\underline{q}_i^0$ and $\overline{q}_i^0$ are defined in (9)). Substituting for q_i^0 from (A.19), we get:

$$i \in \mathcal{X}^{\mathsf{C}} \Leftrightarrow \overline{\gamma}_i \leq \hat{\gamma}^{\mathsf{PM}} \text{ or } \underline{\gamma}_i \geq \hat{\gamma}^{\mathsf{PM}}$$

Conversely,

$$i \in \mathcal{X} \Leftrightarrow \underline{\gamma}_i < \hat{\gamma}^{\mathsf{PM}} < \overline{\gamma}_i.$$

□

Proof of Lemma 2. Recall that $\underline{\gamma} = \frac{A}{f(\overline{q})}$ and $\overline{\gamma} = \frac{A}{f(\underline{q})}$. To show part (a) (comparative statics of $\underline{\gamma}$ and $\overline{\gamma}$ with d), observe from (9) that $\underline{q}$ (resp., $\overline{q}$) is increasing (resp., decreasing) in d. Since $f(\cdot)$ is increasing, the result follows. To show part (b), (comparative statics of $\underline{\gamma}$ and $\overline{\gamma}$ with A), the denominator is independent of A; hence, the result follows. □

Proof of Corollary 1. Under a symmetric oligopoly with n sellers, using (18), $\hat{\gamma}^{\mathsf{Sym}}$ is the solution to the following:

$$1-\frac{1}{\hat{\gamma}}=ng^{-1}\left(\frac{A}{\hat{\gamma}}\right)$$

To show part (a), the l.h.s. is increasing in $\hat{\gamma}$. From Lemma B1, we have that $g(\cdot)$ is increasing and independent of n. Consequently, the r.h.s. is decreasing in $\hat{\gamma}$ and is increasing in n. Consequently, $\hat{\gamma}^{\mathsf{Sym}}\in(1,\infty)$ exists, is unique, and is increasing in n.

Since sellers are symmetric, it follows from Theorem 3 that either all sellers manipulate or no seller manipulates. To show part (b), using part (a) of Theorem 3, the seller(s) manipulate iff $h'(0)<\frac{(1-\sqrt{d})^2}{b}$ and $\hat{\gamma}^{\mathsf{Sym}}\in(\underline{\gamma},\overline{\gamma})$. For $\hat{\gamma}^{\mathsf{Sym}}\in(\underline{\gamma},\overline{\gamma})$, we require the following:

$$1-\frac{1}{\underline{\gamma}}<ng^{-1}\left(\frac{A}{\underline{\gamma}}\right)\text{ and }1-\frac{1}{\overline{\gamma}}<ng^{-1}\left(\frac{A}{\overline{\gamma}}\right).$$

Using the definition of $\underline{\gamma}$ and $\overline{\gamma}$ in (10), we have that $g^{-1}\left(\frac{A}{\underline{\gamma}}\right)=f^{-1}\left(\frac{A}{\underline{\gamma}}\right)=\overline{q}$ and $g^{-1}\left(\frac{A}{\overline{\gamma}}\right)=f^{-1}\left(\frac{A}{\overline{\gamma}}\right)=\underline{q}$. Combining these observations, we have that

$$\hat{\gamma}^{\mathsf{Sym}}\in(\underline{\gamma},\overline{\gamma})\Leftrightarrow A\in\left(\frac{f(\underline{q})}{1-n\underline{q}},\frac{f(\overline{q})}{1-n\overline{q}}\right).$$

□

Proof of Lemma 3. First, from the conditions in part (b) of Corollary 1, it follows that for sellers to manipulate for any $n\geq1$, it must hold that $h'(0)\leq\frac{(1-\sqrt{d})^2}{b}$ and $A>\frac{f(\underline{q})}{1-\underline{q}}$ (otherwise, manipulation does not arise for any $n\geq1$).

Suppose $A>\frac{f(\overline{q})}{1-\overline{q}}$. It follows from part (b) of Corollary 1 that if $n=1$ (monopolist), then manipulation does not arise. For $n\geq2$, manipulation arises iff the condition in part (b) of Corollary 1 holds. This condition can be stated as $n\in\left(\frac{1}{\overline{q}}\left(1-\frac{f(\overline{q})}{A}\right),\frac{1}{\underline{q}}\left(1-\frac{f(\underline{q})}{A}\right)\right)$.

Suppose $\frac{f(\underline{q})}{1-\underline{q}}<A<\frac{f(\overline{q})}{1-\overline{q}}$. It follows from part (b) of Corollary 1 that manipulation arises if $n=1$ (monopolist). Further, manipulation arises in the presence of identical sellers in an oligopoly iff $n<\frac{1}{\underline{q}}\left(1-\frac{f(\underline{q})}{A}\right)$. □

Proof of Lemma 4. Suppose that $h'(0)\leq\frac{(1-\sqrt{d})^2}{b}$ and $A>\frac{f(\underline{q})}{1-\underline{q}}$. From Theorem 3 and Corollary 1, the equilibrium extent of manipulation in a symmetric oligopoly x^{PM} is:

$$x^{\mathsf{PM}}=\hat{h}^{-1}\left(\frac{q^0}{b}\left(1-\frac{d}{1-q^0}\right)\right),\tag{A.20}$$

where $q^0=g^{-1}\left(\frac{A}{\hat{\gamma}^{\mathsf{Sym}}}\right)$ and $\hat{\gamma}^{\mathsf{Sym}}$ is the solution to the equation shown in Corollary 1. Recall from part (a) of Corollary 1 that $\hat{\gamma}^{\mathsf{Sym}}$ is increasing in n. Since $q^0=g^{-1}\left(\frac{A}{\hat{\gamma}^{\mathsf{Sym}}}\right)$ and $g(\cdot)$ is increasing, it follows that q^0 is decreasing in n. Next, since $\hat{h}(\cdot)$ is increasing, from (A.20), we have the following:

$$q^0>1-\sqrt{d}\Leftrightarrow x^{\mathsf{PM}}\text{ is increasing in }n.$$

Substituting for $q^0=g^{-1}\left(\frac{A}{\hat{\gamma}^{\mathsf{Sym}}}\right)$ and using Corollary 1, it follows that

$$q^0>1-\sqrt{d}\Leftrightarrow n<\frac{1}{1-\sqrt{d}}\left(1-\frac{g(1-\sqrt{d})}{A}\right).$$

Combining this with the condition for manipulation to arise, i.e., $n\in\left(\frac{1}{\overline{q}}\left(1-\frac{f(\overline{q})}{A}\right),\frac{1}{\underline{q}}\left(1-\frac{f(\underline{q})}{\underline{q}}\right)\right)$, we have the required result. □

Proof of Lemma 5. From Theorem 3, recall that a seller manipulates in equilibrium if $\underline{\gamma}_i<\hat{\gamma}^{\mathsf{PM}}<\overline{\gamma}_i$. Under homogeneous costs of manipulation, this condition can be stated as $\hat{\gamma}^{\mathsf{PM}}f(\underline{q})<A_i<\hat{\gamma}^{\mathsf{PM}}f(\overline{q})$. Consequently, the set of sellers that manipulate is contiguous. □

Proof of Lemma 6. Suppose that sellers are homogeneous in their types. We will show that $i \in \mathcal{X} \implies i+1 \in \mathcal{X}$. Since $h_1'(0) \geq h_2'(0) \geq \dots h_n'(0)$, it follows that for any i s.t. $h_i'(0) < \frac{(1-\sqrt{d})^2}{b}$, we have $(\underline{\gamma}_i, \overline{\gamma}_i) \subset (\underline{\gamma}_{i+1}, \overline{\gamma}_{i+1})$. Therefore, $i \in \mathcal{X} \implies \hat{\gamma}^{\mathsf{PM}} \in (\underline{\gamma}_i, \overline{\gamma}_i) \implies \hat{\gamma}^{\mathsf{PM}} \in (\underline{\gamma}_{i+1}, \overline{\gamma}_{i+1}) \implies i+1 \in \mathcal{X}$. $\square$

Proof of Lemma 7. Fix $d = 0$. Substituting for $h_i(x, A) = \mathcal{H}(A)h(x)$ in the definition of $\overline{\gamma}_i$ from (10), we have:

$$\overline{\gamma}_i = \frac{A_i}{f\left(b\mathcal{H}(A_i)h'(0)\right)}.$$

Therefore,

$$\frac{d\overline{\gamma}_i}{dA_i} = \frac{A_i}{f(z)}\left(\frac{1}{A_i} - \frac{f'(z)}{f(z)}bh'(0)\mathcal{H}'(A_i)\right)$$

where $z = bh'(0)\mathcal{H}(A_i)$. Further, $0 < z < 1$. In the r.h.s., the term outside the bracket is positive. It suffices to f.o.c.us on the term within the brackets. The term inside the brackets is positive iff the following holds:

$$\begin{aligned}
\left(\frac{1}{z} + \frac{1}{(1-z)^2}\right) z \frac{\mathcal{H}'(A_i)}{\mathcal{H}(A_i)} < \frac{1}{A_i} \quad &\Leftrightarrow \quad \frac{\frac{\mathcal{H}'(A_i)}{\mathcal{H}(A_i)}}{\frac{1}{A_i}} < \frac{f(z)}{zf'(z)} \\
&\Leftrightarrow \quad \underbrace{\frac{\partial \log \mathcal{H}(A_i)}{\partial \log A_i}}_{\varepsilon_A} < \frac{1}{1 + \frac{z}{(1-z)^2}}
\end{aligned}$$

Since the r.h.s. is strictly less than 1, it holds that if $\varepsilon_A > 1$, then, $\overline{\gamma}_i$ is decreasing in A_i. $\square$

A.3 Proofs of Results in Section 7

Proof of Theorem 4. From Theorem 3, recall that the equilibrium market share of seller i is as follows:

$$q_i^{\mathsf{PM}} = g_i^{-1}\left(\frac{A_i}{\hat{\gamma}^{\mathsf{PM}}}\right) \tag{A.21}$$

The following algebraic expressions are useful: Since $h(x) = \lambda(e^x - 1)$, we have:

$$\begin{aligned}
h'(x) &= h''(x) = \lambda e^x, \\
h'^{-1}(z) &= \log\left(\frac{z}{\lambda}\right).
\end{aligned}$$

Since $q_0^{\mathsf{PM}} = 1 - \sum_{i\in[n]} q_i^{\mathsf{PM}}$, we have:

$$\frac{dq_0^{\mathsf{PM}}}{d\lambda} = -\sum_{i\in[n]} \frac{dq_i^{\mathsf{PM}}}{d\lambda} \tag{A.22}$$

- Consider $i \in \mathcal{X}^{\mathsf{C}}$: Since $f(q_i^{\mathsf{PM}}) = A_i q_0^{\mathsf{PM}}$, differentiating both sides w.r.t. λ, we have:

$$\begin{aligned}
\frac{d}{d\lambda}\left(f(q_i)\right) &= \frac{d}{d\lambda}\left(A_i q_0^{\mathsf{PM}}\right) \\
\implies f'(q_i^{\mathsf{PM}})\frac{dq_i^{\mathsf{PM}}}{d\lambda} &= A_i \frac{dq_0^{\mathsf{PM}}}{d\lambda} \implies \frac{dq_i^{\mathsf{PM}}}{d\lambda} = \frac{1}{f'(q_i^{\mathsf{PM}})}\left(A_i \frac{dq_0^{\mathsf{PM}}}{d\lambda}\right).
\end{aligned}$$

- Consider $i \in \mathcal{X}$: Since $g(q_i^{\mathsf{PM}}) = A_i q_0^{\mathsf{PM}}$,

$$\begin{aligned}
\frac{d}{d\lambda}\left(g(q_i^{\mathsf{PM}})\right) &= \frac{d}{d\lambda}\left(A_i q_0^{\mathsf{PM}}\right) \\
\implies g'(q_i^{\mathsf{PM}})\frac{dq_i^{\mathsf{PM}}}{d\lambda} + \frac{\partial g(q_i^{\mathsf{PM}})}{\partial \lambda} &= A_i \frac{dq_0^{\mathsf{PM}}}{d\lambda} \\
\implies \frac{dq_i^{\mathsf{PM}}}{d\lambda} &= \frac{1}{g'(q_i^{\mathsf{PM}})}\left(A_i \frac{dq_0^{\mathsf{PM}}}{d\lambda} - \frac{\partial g(q_i^{\mathsf{PM}})}{\partial \lambda}\right).
\end{aligned}$$

Substituting the above in the r.h.s. of (A.22) and using the identities, we have:

$$\frac{dq_i^{\mathsf{PM}}}{d\lambda} = \begin{cases} \frac{A_i}{f'(q_i^{\mathsf{PM}})}\frac{dq_0^{\mathsf{PM}}}{d\lambda}, & \text{if } i \in \mathcal{X}^{\mathsf{C}}; \\ \frac{1}{g'(q_i^{\mathsf{PM}})}\left(A_i \frac{dq_0^{\mathsf{PM}}}{d\lambda} - \frac{\partial g(q_i^{\mathsf{PM}})}{\partial \lambda}\right), & \text{if } i \in \mathcal{X}. \end{cases}$$

In the r.h.s. above, using algebraic manipulation, we have $\frac{\partial g(q_i^{\mathsf{PM}})}{\partial \lambda} = \frac{g(q_i^{\mathsf{PM}})}{\lambda} = \frac{A_i q_0^{\mathsf{PM}}}{\lambda}$. Substituting the above in (A.22), we have:

$$\begin{aligned} \frac{dq_0^{\mathsf{PM}}}{d\lambda} &= \frac{\sum_{i\in\mathcal{X}} \frac{\frac{\partial g(q_i^{\mathsf{PM}})}{\partial\lambda}}{g'(q_i^{\mathsf{PM}})}}{1 + \sum_{i\in\mathcal{X}^{\mathsf{C}}} \frac{A_i}{f'(q_i^{\mathsf{PM}})} + \sum_{i\in\mathcal{X}} \frac{A_i}{g'(q_i^{\mathsf{PM}})}} \\ &= \frac{q_0^{\mathsf{PM}}\tau}{\lambda}. \end{aligned} \tag{A.23}$$

Since $\tau \in (0,1)$, the r.h.s. of (A.23) is positive. Therefore, q_0^{PM} is increasing in λ. Consequently, q_i^{PM}, $i \in \mathcal{X}^{\mathsf{C}}$ is also increasing in λ.

Now, consider q_i^{PM}, $i \in \mathcal{X}$. Using (A.21), we have:

$$\begin{aligned} \frac{dq_i^{\mathsf{PM}}}{d\lambda} &= \frac{1}{g'(q_i^{\mathsf{PM}})}\left(\underbrace{A_i \frac{dq_0^{\mathsf{PM}}}{d\lambda} - \frac{\partial g(q_i^{\mathsf{PM}})}{\partial\lambda}}_{<0}\right) \\ &= -\frac{1}{g'(q_i^{\mathsf{PM}})}\frac{q_0^{\mathsf{PM}}}{\lambda}(1-\tau). \end{aligned} \tag{A.24}$$

Since $\tau \in (0,1)$, the r.h.s. above is negative. Therefore, q_i^{PM} for $i \in \mathcal{X}$ is decreasing in λ.

To show that π_i^{PM}, $i \in [n]$, is increasing in λ, we first show the result for $i \in \mathcal{X}^{\mathsf{C}}$. We then show the result for $i \in \mathcal{X}$. Consider seller $i \in \mathcal{X}^{\mathsf{C}}$. Since $x_i = 0$, seller i's profit is

$$\pi_i = m_i q_i = \frac{q_i^{\mathsf{PM}}}{b(1-q_i^{\mathsf{PM}})},$$

which is monotone in q_i^{PM}. From part (a) of this result, since q_i^{PM} is increasing in λ, it follows that π_i^{PM} is increasing in λ. Now, consider seller $i \in \mathcal{X}$. Recall that seller i's equilibrium profit is

$$\pi_i = \underbrace{m_i q_i}_{\text{direct profit from sales}} - \underbrace{h\left(h'^{-1}\left(\frac{q_i}{b}\right)\right)}_{\text{cost of manipulation}}.$$

Substituting for $h(\cdot)$ and $h'^{-1}(\cdot)$, we have:

$$\begin{aligned} \pi_i^{\mathsf{PM}} &= \frac{q_i^{\mathsf{PM}}}{b(1-q_i^{\mathsf{PM}})} - \lambda\left(\frac{q_i^{\mathsf{PM}}}{b\lambda} - 1\right) \\ \Longrightarrow \frac{d\pi_i^{\mathsf{PM}}}{d\lambda} &= \frac{(2-q_i)q_i}{b(1-q_i)^2}\frac{dq_i}{d\lambda} + 1. \end{aligned}$$

For the r.h.s. to be positive, we require the following condition to hold:

$$1 > \left(-\frac{dq_i^{\mathsf{PM}}}{d\lambda}\right)\left(\frac{(2-q_i^{\mathsf{PM}})q_i^{\mathsf{PM}}}{b(1-q_i^{\mathsf{PM}})^2}\right). \tag{A.25}$$

Substituting for $\frac{dq_i^{\mathsf{PM}}}{d\lambda}$ from (A.24), the condition in (A.25) simplifies to:

$$\lambda > \frac{q_i^{\mathsf{PM}}(2-q_i^{\mathsf{PM}})}{b}(1-\tau).$$

In the r.h.s., $q_i(2-q_i)$ is increasing in $q_i \in [0,1]$. If $A_1 \le A_2 \le \ldots A_n$, and hence, q_i is an increasing sequence, an upper bound on the r.h.s. is $\frac{q_n^{\mathsf{PM}}(2-q_n^{\mathsf{PM}})}{b}(1-\tau)$ □

Proof of Theorem 5. Consider the consumer utility model below

$$u_i = a_i + \delta x_i - bp_i + \epsilon_i$$

Since $d = 0$, $\underline{\gamma}_i = 0$. Recall the definition of $\overline{\gamma}_i$ in (10):

$$\gamma_i = \begin{cases} \frac{A_i}{f\left(bh_i'(0)\right)}, & \text{if } h_i'(0) < \frac{1}{b}; \\ 0, & \text{otherwise.} \end{cases}$$

Let $\hat{x}_i = \delta x_i$. We can rewrite the game in this setting as a game where sellers choose $(\hat{x}_i, p_i)$, and face the following cost of manipulation $h_i^{(\delta)}(\cdot)$:

$$h_i^{(\delta)}(x) = h_i\left(\frac{x}{\delta}\right)$$

Fix δ. In this game, we have:

$$\gamma_i^{(\delta)} = \begin{cases} \frac{A_i}{f\left(\frac{b}{\delta}h_i'(0)\right)}, & \text{if } h_i'(0) < \frac{\delta}{b}; \\ 0, & \text{otherwise.} \end{cases}$$

Observe that $\gamma_i(\delta)$ is (weakly) increasing in δ. Next, we will show that $q_0^{\mathsf{PM}(\delta)}$ is decreasing in δ. Let $g_i^{(\delta)}(z)$ denote the following:

$$g_i^{(\delta)}(z) = ze^{\frac{1}{1-z}}e^{-\delta h_i'^{-1}\left(\frac{\delta}{b}z\right)} = \frac{f(z)}{e^{\delta h_i'^{-1}\left(\frac{\delta}{b}z\right)}}.$$

Observe that $g_i^{(\delta)}(z)$ is decreasing in δ. For fixed δ, $g_i^{(\delta)}(\cdot)$ is a monotone increasing function. We can show that q_0^{PM} is the solution to the following fixed point equation:

$$q_0 = 1 - \sum_{i\in\mathcal{X}} g_i^{(\delta)^{-1}}(A_i q_0) - \sum_{i\in\mathcal{X}^{\mathsf{C}}} f^{-1}(A_i q_0).$$

For any δ, following the same approach as in Theorem 3, a solution to q_0 exists and is unique. Let $q_0^{\mathsf{PM}(\delta)}$ denote this solution. Since $g_i^{(\delta)}(z)$ is decreasing in δ and $g_i^{(\delta)}(\cdot)$ is a monotone increasing function, it follows that $q_0^{\mathsf{PM}(\delta)}$ is decreasing in δ. Combining the observations that ${\gamma_i}^{(\delta)}$ is increasing in δ and $q_0^{\mathsf{PM}(\delta)}$ is decreasing in δ, we have the result that for any $\delta \leq \delta'$, $\mathcal{X}^{(\delta)} \subseteq \mathcal{X}^{(\delta')}$. $\square$

B Helpful Results

Recall the definition of g from (12):

$$\begin{aligned}
\text{If } bh'(0) \leq (1-\sqrt{d})^2 \text{, then } g(z) &= \begin{cases} f(z), & \text{if } z \in [0, \underline{q}^0) \cup (\overline{q}, 1] \\ f(z) e^{-\hat{h}^{-1}\left(\frac{z}{b}\left(1-\frac{d}{1-z}\right)\right)}, & \text{if } z \in \left[\underline{q}^0, \overline{q}^0\right] \end{cases} \\
\text{If } bh'(0) > (1-\sqrt{d})^2 \text{, then } g(z) &= f(z).
\end{aligned}$$

Lemma B1. *$g(z)$ is continuous and increasing in $z \in [0, 1)$.*

Proof of Lemma B1. First, it is straightforward from the definition that $g(z)$ is continuous and increasing in $z \in [0, 1)$ if $bh'(0) > (1-\sqrt{d})^2$. We focus on i s.t. $bh'(0) \leq (1-\sqrt{d})^2$.

To demonstrate continuity, we have that $g(z)$ is continuous and increasing in $z \in [0, \underline{q}^0) \cup (\overline{q}, 1]$. Second, $\lim_{z \uparrow \underline{q}^0} g(z) = \lim_{z \downarrow \underline{q}^0} g(z) = f(\underline{q}^0)$ and $\lim_{z \uparrow \overline{q}^0} g(z) = \lim_{z \downarrow \overline{q}^0} g(z) = f(\overline{q}^0)$. Therefore, $g(z)$ is continuous in $z \in [0, 1)$.

Next, to demonstrate that $g(z)$ is increasing, it suffices to focus on the interval $[\underline{q}_i^0, \overline{q}_i^0]$ (since $f(\cdot)$ is monotone). Taking log on both sides of (12) and differentiating w.r.t. z, we have:

$$\frac{g_i'(z)}{g_i(z)} = \underbrace{\frac{1}{z}}_{\text{positive}} \left(\underbrace{1 - \frac{\frac{z}{b}\left(1 - \frac{d}{(1-z)^2}\right)}{\hat{h}_i'\left(\hat{h}_i^{-1}\left(\frac{z}{b}\left(1 - \frac{d}{1-z}\right)\right)\right)}}_{\text{term to analyze}} \right) + \underbrace{\frac{1}{(1-z)^2}}_{\text{positive}}.$$

The r.h.s. consists of two terms. The second term is positive. In the first term, the term outside the brackets is positive. We show that the term inside the brackets is positive. First, if $z > 1 - \sqrt{d}$, it is straightforward that the term inside the brackets is positive because the second term is negative. For $z < 1 - \sqrt{d}$, we show that a lower bound of this term is positive. Observe that:

$$1 - \frac{d}{(1-z)^2} < 1 - \frac{d}{1-z}$$

Consequently, it suffices to show that $1 - \frac{\frac{z}{b}\left(1-\frac{d}{1-z}\right)}{\hat{h}_i'\left(\hat{h}_i^{-1}\left(\frac{z}{b}\left(1-\frac{d}{1-z}\right)\right)\right)} > 0$ if $z < 1 - \sqrt{d}$. Substituting $\hat{h}_i^{-1}\left(\frac{z}{b}\left(1 - \frac{d}{1-z}\right)\right) \to y$, it suffices to show that $1 - \frac{\hat{h}_i(y)}{\hat{h}_i'(y)} > 0$, where $0 \leq y < \hat{h}^{-1}\left(\frac{(1-\sqrt{d})^2}{b}\right)$. We have

$$1 - \frac{\hat{h}_i(y)}{\hat{h}_i'(y)} = \frac{h_i''(y) - (1-d)h_i'(y)}{h_i''(y) + dh_i'(y)}$$

In the r.h.s. above, the denominator is positive. From Assumption 1(b), since $h_i''(y) \geq h_i(y)$, the numerator is positive. This concludes the proof of the result. $\square$

Recall the definition of $\iota(q)$ in Section 7: $\iota(q) = q(2-q)\left(1 - \frac{2(1-q)^2}{2q^2-6q+3}\right)$.

Lemma B2. *For $0 \leq q < \frac{1}{2}$, the following hold.*

(a) *$\iota(q)$ is concave in q.*

(b) *$\iota(q) < \frac{1}{2}$.*

Proof of Lemma B2. To show part (a), that $\iota(q)$ is concave in $q \in \left[0, \frac{1}{2}\right]$, we will show that $\iota''(q) < 0$ in $q \in \left[0, \frac{1}{2}\right]$. $\iota''(q)$ is as follows:

$$\iota''(q) = \frac{\overbrace{2(8q^3 - 18q^2 + 18q - 9)}^{P(q),\text{negative in } [0,\, \frac{1}{2}]}}{\underbrace{\left(2\left(\frac{3}{2} - q\right)^2 - \frac{3}{2}\right)^3}_{\text{positive in } [0,\, \frac{1}{2}]}}$$

The denominator in the r.h.s. above is decreasing in q for $q \in \left[0, \frac{1}{2}\right]$ and is positive at $q = \frac{1}{2}$. Therefore, the denominator is positive for $q \in \left[0, \frac{1}{2}\right]$. For convenience, denote the numerator in the r.h.s. by $P(q)$. We show that $P(q) < 0$ for $q \in \left[0, \frac{1}{2}\right]$. We have:

$$P'(q) = 3\left(16\left(q - \frac{3}{4}\right)^2 + \frac{3}{4}\right)$$

Observe that $P'(q) > 0$. Therefore, $P(q)$ is increasing in q for $q \in \left[0, \frac{1}{2}\right]$. Through straightforward algebra, we can verify that $P\left(\frac{1}{2}\right) = -7$. Therefore, $P(q) < 0$ for $q \in \left[0, \frac{1}{2}\right]$. So, we can conclude that $\iota''(q) < 0$ for $q \in \left[0, \frac{1}{2}\right]$.

To show part (b), the first derivative of $\iota(q)$ is as follows:

$$\iota'(q) = \frac{2(1-q)}{(2q^2 - 6q + 3)^2} \underbrace{(-2q^3 + 10q^2 - 12q + 3)}_{\triangleq M(q)} \tag{B.26}$$

From part (a) of this result, f.o.c.'s are necessary and sufficient to find a global maximizer of $\iota(q)$ for $q \in [0, 0.5]$. Setting the r.h.s. of (B.26) to 0 is equivalent to setting the last term in the r.h.s. above, denoted by $M(q)$, to 0. Let Δ as the discriminant of $M(q)$; $\Delta = 564$. This implies that we have three distinct real roots for $M(q) = 0$. Let $q_{(i)}$ denote the i^{th} root for $M(q) = 0$. We have:

$$\begin{aligned}
q_{(1)} &= \frac{5}{3} + \frac{7}{9\left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)\sqrt[3]{\frac{41}{108} + \frac{\sqrt{47}i}{12}}} + \left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)\sqrt[3]{\frac{41}{108} + \frac{\sqrt{47}i}{12}} \\
&\approx 0.33956 \\
q_{(2)} &= \frac{5}{3} + \left(-\frac{1}{2} - \frac{\sqrt{3}i}{2}\right)\sqrt[3]{\frac{41}{108} + \frac{\sqrt{47}i}{12}} + \frac{7}{9\left(-\frac{1}{2} - \frac{\sqrt{3}i}{2}\right)\sqrt[3]{\frac{41}{108} + \frac{\sqrt{47}i}{12}}} \\
&\approx 1.3240 \\
q_{(3)} &= \frac{5}{3} + \frac{7}{9\sqrt[3]{\frac{41}{108} + \frac{\sqrt{47}i}{12}}} + \sqrt[3]{\frac{41}{108} + \frac{\sqrt{47}i}{12}} \\
&\approx 3.3364
\end{aligned}$$

Correspondingly, $\iota(q_{(1)}) \approx 0.152$. $\square$

C Maximum Likelihood Estimation of the MNL Choice Model with Unobserved No Purchase Data

Consider the maximum likelihood estimation procedure where the consumer utility model is as follows:

$$u_i = \beta_0 + \beta_p p_i + \beta_r r_i + \epsilon_i,\ i \in [n],\ u_0 = 0, \tag{C.27}$$

ϵ_i are i.i.d. Gumbel random variables, and consumer purchase model follows

$$q_i = \frac{e^{u_i}}{1 + \sum_{j\in[K]} e^{u_j}},\ i \in [n] \cup \{0\}.$$

The likelihood function, given purchase and no-purchase data ($\mathbf{s} = (s_i)_{i\in[n]}, s_0$), is as follows:

$$\mathcal{L}(\boldsymbol{\beta}|\mathbf{s}, s_0) = \frac{\overline{s}!}{\prod_{i\in[n]\cup\{0\}} s_i!} \prod_{i\in[n]\cup\{0\}} q_i^{s_i}.$$

where $\overline{s} = \sum_{i\in[n]\cup\{0\}} s_i$. The log-likelihood is as follows:

$$\log \mathcal{L}(\boldsymbol{\beta}|\mathbf{n}, n_0) = \sum_{i\in[n]} s_i \log q_i + s_0 \log q_0 + \mathsf{SL}(\overline{s}) - \sum_{i\in[n]\cup\{0\}} \mathsf{SL}(s_i),$$

where, for any $j \in \mathbb{I}^+$, $\mathsf{SL}(j) = \sum_{j'=1}^{j} \log(j')$.

Let $\hat{q}_i$ denote the empirical market share, i.e., $\hat{q}_i = \frac{s_i}{\overline{s}}$. Corresponding to any vector $\mathbf{y} = \{y_i\}_{i\in[n]}$, consider the random variable:

$$\tilde{y} = q_0 \circ 0 + \sum_{i\in[n]} q_i \circ y_i.$$

Define the following operators:

$$\begin{aligned}
\hat{\mathbb{E}}[\mathbf{y}] &= \mathbb{E}[\tilde{y}] = \sum_{i\in[n]} q_i y_i, \\
\hat{\mathrm{Var}}[\mathbf{y}] &= \mathrm{Var}[\tilde{y}] = \sum_{i\in[n]} q_i y_i^2 - \left(\sum_{i\in[K]} q_i y_i\right)^2, \text{ and} \\
\hat{\mathrm{Cov}}[\mathbf{y}_1, \mathbf{y}_2] &= \mathbb{E}[\tilde{y}_A \tilde{y}_B] - \mathbb{E}[\tilde{y}_A]\mathbb{E}[\tilde{y}_B].
\end{aligned}$$

We have the following:

$$\begin{aligned}
\nabla \log \mathcal{L}(\boldsymbol{\beta}|\mathbf{s}, s_0) &= \overline{n} \begin{bmatrix} \sum_{i\in[n]} (\hat{q}_i - q_i) & \sum_{i\in[n]} p_i (\hat{q}_i - q_i) & \sum_{i\in[n]} r_i (\hat{q}_i - q_i) \end{bmatrix} \\
\text{and } H \triangleq \nabla^2 \log \mathcal{L}(\boldsymbol{\theta}|\mathbf{s}, s_0) &= -\overline{n} \begin{bmatrix} q_0 \sum_j q_j & q_0\hat{\mathbb{E}}[\mathbf{p}] & q_0\hat{\mathbb{E}}[\mathbf{r}] \\ q_0\hat{\mathbb{E}}[\mathbf{p}] & \hat{\mathrm{Var}}[\mathbf{p}] & \hat{\mathrm{Cov}}[\mathbf{p}, \mathbf{r}] \\ q_0\hat{\mathbb{E}}[\mathbf{r}] & \hat{\mathrm{Cov}}[\mathbf{p}, \mathbf{r}] & \hat{\mathrm{Var}}[\mathbf{r}] \end{bmatrix}
\end{aligned}$$

For given s_0, H is negative semi-definite.[22] Hence, f.o.c.'s identify the MLE of $\boldsymbol{\beta}$. Since s_0 is unknown, we identify the maximum likelihood estimates for $\boldsymbol{\beta}$, denoted by $\boldsymbol{\beta}^{\mathsf{MLE}}$ using the EM approach.

C.1 EM-Algorithm

1: Input: $\mathbf{s}, \mathbf{p}, \mathbf{r}, n, \varepsilon$
2: $\boldsymbol{\beta}_0 \leftarrow \mathbf{0}$, $\omega \leftarrow 0$
3: **repeat**

[22] This follows from the fact that the variance-covariance matrix of a multivariate distribution is positive definite.

$\omega \leftarrow \omega + 1$ ▷ E-Step

for $i \in [n]$ **do**

$u_i \leftarrow \beta_0^\omega + \beta_p^\omega p_i + \beta_r^\omega r_i$

$q_i \leftarrow \frac{e^{u_i}}{1+\sum_{j\in[n]} e^{u_j}}$

$s_0 \leftarrow \frac{q_0}{1-q_0}\overline{s}$

end for ▷ M-Step

$\boldsymbol{\beta}_\omega \leftarrow \arg\max_{\boldsymbol{\beta}\in\Re^3} \log \mathcal{L}(\boldsymbol{\beta}|\mathbf{s}, \mathbf{s}_0)$

until $\|\boldsymbol{\beta}_\omega - \boldsymbol{\beta}_{\omega-1}\| < \epsilon$

We now substitute $\boldsymbol{\beta}^{\mathsf{MLE}}$ (obtained from the above approach) in H. In particular, $\mathbf{q} \equiv \mathbf{q}(\boldsymbol{\beta}^{\mathsf{MLE}})$. We identify the inverse of H, denoted by H^{-1}. The standard errors of $\boldsymbol{\beta}^{\mathsf{MLE}}$ corresponds to the diagonal elements of H^{-1}.

C.2 Estimation Results

To estimate the model coefficients in (C.27), we use data across 24 weeks for 11 products from Wang et al. (2014). We pool the data across the 24 weeks to estimate $\boldsymbol{\beta}$. Our estimation results are shown in Table 1 and reproduced below for easier reference.

	β_p	β_r	β_0
Estimate	-0.0017***	0.103***	0.0109
Std. Error	0.000098	0.004	0.54

$^{*}p < 0.1$, $^{**}p < 0.05$, $^{***}p < 0.01$

Appendix Table C1. Estimation Results

C.3 Alternate Consumer Utility Model: Product-Specific β_0

Consider the maximum likelihood estimation procedure where the consumer utility model is as follows:

$$u_i = \left(\beta_0 + \sum_{j\in[n-1]} \beta_{0j}\mathbb{I}_{\{j=i\}}\right) + \beta_p p_i + \beta_r r_i + \epsilon_i,\ i \in [n],\ u_0 = 0, \tag{C.28}$$

ϵ_i are i.i.d. Gumbel random variables. β_{0j} is a product-specific intercept, and $\mathbb{I}_{\{j=i\}}$ is the product-specific dummy variables, with product n as the reference product.

We follow an identical procedure as explained in Section C.1 to estimate the coefficients in (C.28). The estimates are presented in the table below.

	Estimate	Std.Error
β_0	1.42×10^{-5}	0.0895
β_p	$-1.6 \times 10^{-3***}$	0.0003
β_r	-1.06×10^{-4}	0.0161
β_1	-3.01×10^{-4}	0.1126
β_2	2.28×10^{-4}	0.1147
β_3	7.01×10^{-5}	0.1158
β_4	2.43×10^{-5}	0.1127
β_5	2.05×10^{-4}	0.1130
β_6	-5.90×10^{-5}	0.1165
β_7	1.49×10^{-4}	0.1147
β_8	1.77×10^{-4}	0.1164
β_9	-1.05×10^{-4}	0.1120
β_10	-5.01×10^{-5}	0.1162

$^*p < 0.1$, $^{**}p < 0.05$, $^{***}p < 0.01$

Appendix Table C2. Estimation Results: Product-Specific β_0

D Consumer Surplus in the MNL Model

Consider the canonical MNL choice model, where the utility from consuming product $i \in [n]$, and 0 (the no-purchase option) is:

$$\begin{aligned} u_0 &= \epsilon_0, \\ u_i &= v_i + \epsilon_i \text{ for } i \in [n] \end{aligned}$$

where ϵ_0 and ϵ_i, $i \in [n]$ are i.i.d Gumbel r.v.'s. A representative consumer chooses the no purchase option or one of the products $i \in [n]$ as follows:

$$\begin{aligned} \text{consumer chooses } i \text{ if: } v_i + \epsilon_i &> \max\{0, \max_{j \neq i}\{v_j + \epsilon_j\}\} \\ \text{consumer chooses 0 if: } \epsilon_0 &> \max_{i \in [n]}\{v_i + \epsilon_i\}. \end{aligned}$$

Recall that the consumer purchases product i w.p. q_i as follows:

$$\begin{aligned} q_0 &= \frac{1}{1 + \sum_{j \in [n]} e^{v_j}} \\ q_i &= \frac{e^{v_i}}{1 + \sum_{j \in [n]} e^{v_j}} \text{ for } i \in [n] \end{aligned}$$

That is, the consumer purchase follows the following discrete distribution over $[n] \cup \{0\}$:

$$\text{Consumer Purchase } \sim q_0 \circ 0 + \sum_{i \in [n]} q_i \circ i.$$

The expected consumer surplus (where the expectations are taken over $\boldsymbol{\epsilon}$) is as follows:

$$\begin{aligned} \text{Expected CS} &= q_0 \mathbb{E}_{\epsilon_0, \boldsymbol{\epsilon}}[\epsilon_0 | \epsilon_0 > \max_{i \in [n]}(v_i + \epsilon_i)] + \\ & \sum_{i \in [n]} q_i \mathbb{E}_{\epsilon_0, \boldsymbol{\epsilon}}[(v_i + \epsilon_i) | v_i + \epsilon_i > \max\{\epsilon_0, \max_{j \in [n] \setminus \{i\}}(v_j + \epsilon_j)\}] \end{aligned}$$

Recall that

$$\begin{aligned} q_0 &= \mathbb{E}_{\epsilon_0, \boldsymbol{\epsilon}}\left[\mathbf{1}\left(\epsilon_0 | \epsilon_0 > \max_{i \in [n]} v_i + \epsilon_i\right)\right] \\ q_i &= \mathbb{E}_{\epsilon_0, \boldsymbol{\epsilon}}\left[\mathbf{1}\left(v_i + \epsilon_i > \max\{\epsilon_0, \max_{j \in [n], j \neq i} v_j + \epsilon_j\}\right)\right]. \end{aligned}$$

$$\begin{aligned} &= \mathbb{E}_{\boldsymbol{\epsilon}}\left[\int_{y_0 \in \Re} y_0 \mathbf{1}\left(y_0 > \max_{i \in [n]}(v_i + \epsilon_i)\right) f(y_0) dy_0\right] + \\ & \left[\sum_{i \in [n]} \mathbb{E}_{\epsilon_0, \boldsymbol{\epsilon}_{-i}} \int_{y_i \in \Re} (v_i + y_i) \mathbf{1}\left(v_i + y_i > \max\{\epsilon_0, \max_{j \in [n] \setminus \{i\}}(v_j + \epsilon_j)\}\right) f(y_i) dy_i\right] \\ &= \int_{y_0 \in \Re} y_0 \prod_{i \in [n]} F(y_0 - v_i) f(y_0) dy_0 + \\ & \sum_{i \in [n]} \int_{y_i \in \Re} (v_i + y_i) F(v_i + y_i) \prod_{j \in [n] \setminus \{i\}} F(v_i - v_j + y_i) f(y_i) dy_i. \end{aligned}$$

where $f(\cdot)$ and $F(\cdot)$ represent the p.d.f. and c.d.f. of the standard Gumbel distribution. Expanding the above,

$$\begin{aligned}\text{Expected CS} &= \int_{y_0\in\Re} y_0 e^{-(y_0+\sum_{j\in[n]} e^{-y_0}\beta_j^0)} dy_0 \\ &+ \sum_{i\in[n]} \left(v_i q_i + \int_{y_i\in\Re} y_i e^{-(y_i+\sum_{j\neq i} e^{-y_i}\beta_j^i + e^{-y_i}\beta_0^i)} dy_i \right)\end{aligned}$$

where $\beta_j^0 = e^{v_j - v_0} = e^{v_j}$ and $\beta_j^i = e^{v_j - v_i}$.

$$\begin{aligned}\text{Expected CS} &= q_0\left(\gamma - \log(q_0)\right) + \sum_{i\in[n]} q_i \left(\gamma - \log(q_0)\right) \\ &= \underbrace{\gamma}_{\text{Euler's constant, } \approx 0.577} \underbrace{- \log(q_0)}_{=\log\left(1+\sum_{i\in[n]} e^{v_i}\right)} .\end{aligned}$$

In the absence of manipulation, observe that

$$\begin{aligned}v_0^{\mathsf{AM}} &= 0 \\ v_i^{\mathsf{AM}} &= a_i - bp_i^{\mathsf{AM}} = \log(A_i) - bm_i^{\mathsf{AM}}.\end{aligned}$$

Therefore,

$$\text{Expected CS under } \mathsf{AM} = \gamma - \log(q_0^{\mathsf{AM}}).$$

In the presence of manipulation, prior to purchase, consumers' anticipate the following:

$$v_i^{\mathsf{PM}} = a_i + x_i^{\mathsf{PM}} - bp_i^{\mathsf{PM}} = \log(A_i) + x_i^{\mathsf{PM}} - bm_i^{\mathsf{PM}}.$$

Indeed, the market shares q_i^{PM} result from the above values of v_i. However, post-purchase, consumers realize the *true* utility as follows:

$$\tilde{v}_i^{\mathsf{PM}} = a_i - bp_i^{\mathsf{PM}} = \log(A_i) - bm_i^{\mathsf{PM}}.$$

The post-purchase consumer surplus can be written as follows:

$$\begin{aligned}\text{Consumer Surplus} &= \underbrace{\mathbb{E}_{\boldsymbol{\epsilon}}\left[\int_{y_0\in\Re} y_0 \mathbf{1}\left(y_0 > \max_{i\in[n]}(v_i^{\mathsf{PM}} + \epsilon_i)\right) f(y_0) dy_0\right]}_{CS_0} + \\ & \sum_{i\in[n]} \underbrace{\mathbb{E}_{\epsilon_0, \boldsymbol{\epsilon}_{-i}}\left[\int_{y_i\in\Re} (\tilde{v}_i^{\mathsf{PM}} + y_i) \mathbf{1}\left(v_i^{\mathsf{PM}} + y_i > \max\{\epsilon_0, \max_{j\in[n], j\neq i}(v_j^{\mathsf{PM}} + \epsilon_j)\}\right) f(y_i) dy_i\right]}_{CS_i}\end{aligned}$$

In the second term, add and subtract x_i to $\tilde{v}_i^{\mathsf{PM}} + y_i$. We have:

$$\text{Consumer Surplus} = \gamma - \log\left(q_0^{\mathsf{PM}}\right) - \underbrace{\sum_{i\in[n]} x_i^{\mathsf{PM}} q_i^{\mathsf{PM}}}_{=\overline{x}^{\mathsf{PM}}, \text{ the average level of manipulation}} .$$

Therefore,

$$CS^{\mathsf{AM}} > CS^{\mathsf{PM}} \Leftrightarrow \overline{x}^{\mathsf{PM}} > \log\left(\frac{q_0^{\mathsf{AM}}}{q_0^{\mathsf{PM}}}\right).$$

E Verification: Equation (26) Satisfies Assumption 1

To save notation, we denote $\tilde{x}_i = \frac{x_i}{\beta_r}$, and $y_i = v_i^{\mathsf{tr}} \frac{\tilde{x}_i}{R - r_i^{\mathsf{tr}} - \tilde{x}_i}$. Then, we re-write (26) as

$$h_i\left(y(\tilde{x}_i)\right) = k_1 y_i(\tilde{x}_i) + k_2 y_i^2(\tilde{x}_i). \tag{E.29}$$

To verify Assumption 1(a), we derive $h_i'(x_i)$ and $h_i''(x_i)$ as

$$\begin{aligned} h_i'(x_i) = &\quad k_1 \frac{dy_i}{dx_i} + 2k_2 y_i(\tilde{x}_i) \frac{dy_i}{dx_i} \\ = &\quad \frac{dy_i}{dx_i} \left(k_1 + 2k_2 y_i(\tilde{x}_i)\right) \end{aligned} \tag{E.30}$$

$$h_i''(x_i) = \frac{d^2 y_i}{dx_i^2} \left(k_1 + k_2 y_i(\tilde{x}_i)\right) + 2k_2 \left(\frac{dy_i}{dx_i}\right)^2 \tag{E.31}$$

By direct calculation, we have:

$$\begin{aligned} \frac{dy_i}{dx_i} &= \frac{1}{\beta_r} \frac{dy_i}{d\tilde{x}_i} \\ &= \frac{v_i^{\mathsf{tr}}(R - r_i^{\mathsf{tr}})}{\beta_r (R - r_i^{\mathsf{tr}} - \tilde{x}_i)^2} \\ \frac{d^2 y_i}{dx_i^2} &= \frac{2 v_i^{\mathsf{tr}}(R - r_i^{\mathsf{tr}})}{\beta_r^2 (R - r_i^{\mathsf{tr}} - \tilde{x}_i)^3} \end{aligned} \tag{E.32}$$

Since $\tilde{x}_i \in [0, R - r_i^{\mathsf{tr}})$ and $r_i^{\mathsf{tr}} \in [0, R)$, Assumption 1$(a)$ is satisfied. Next, we verify Assumption 1(b) by showing that $\frac{h_i''(x_i)}{h_i'(x_i)} \geq 1$. Recall from (E.30), (E.31) and (E.32), we have:

$$\begin{aligned} \frac{h_i''}{h_i'} &= \frac{y_i''(k_1 + k_2 y_i) + 2k_2 (y_i')^2}{y_i'(k_1 + 2k_2 y_i)} \\ &= \frac{y_i''}{y_i'} + \frac{2k_2 y_i'}{k_1 + 2k_2 y_i} \\ &\geq 2\sqrt{\frac{2k_2 y_i''}{k_1 + 2k_2 y_i}} \\ &= 2\sqrt{\frac{4k_2 v_i^{\mathsf{tr}}(R - r_i^{\mathsf{tr}})}{\beta_r^2 \underbrace{(R - r_i^{\mathsf{tr}} - \tilde{x}_i)^2}_{<(R - r_i^{\mathsf{tr}})^2} \underbrace{\left(k_1(R - r_i^{\mathsf{tr}}) + (2k_2 v_i^{\mathsf{tr}} - k_1)\tilde{x}_i\right)}_{<\left(\left(2v_i^{\mathsf{tr}}+1\right)k_2 - k_1\right)(R - r_i^{\mathsf{tr}})}}} \\ &> \underbrace{\frac{4k_2 - v_i^{\mathsf{tr}}}{\beta_r^2 \left(R - r_i^{\mathsf{tr}}\right)^2 \left(\left(2v_i^{\mathsf{tr}} + 1\right) k_2 - k_1\right)}}_{\diamondsuit} \end{aligned}$$

For any seller $i \in [n]$ and $k_1 > 0$, $\diamondsuit$ is a bijection: $k_2 \in [0, +\infty] \to [-\infty, +\infty]$. That is, $\forall i \in [n] \wedge k_1 > 0, \exists k_2 \in [0, +\infty] : \diamondsuit > 1$. This completes our verification for Assumption 1.