GEODESIC FLOWS IN SPECIAL LINEAR GROUPS
MATH 440 LAB OF GEOMETRY

HARISH JAISANKAR

January 5, 2026

1. Abstract

This report is intended for a general audience with a foundational understanding of undergraduate-level mathematics, as well as related fields like computer science and physics. Readers are expected to have familiarity with essential concepts in differential geometry, such as vector fields, manifolds, and geodesics. Intuitive, non-rigorous definitions of these concepts will be provided to support accessibility.

The paper focuses on the key steps involved in deriving the geodesic equation for left-invariant structures on the Lie group $\mathrm{SL}(2, \mathbb{R})$. Central to this discussion are the concepts of left-invariant vector fields, the adjoint map, and the momentum function. The primary contributions include the derivation of the left-invariant vector field on $\mathrm{SL}(2, \mathbb{R})$, a numerical approach to solving the geodesic equation, and a visualization of geodesic flow using a spherical parametrization of the initial momentum function. This exploration sheds light on the geometric structure of $\mathrm{SL}(2, \mathbb{R})$ and its role in understanding geodesic flows on Lie groups.

2. Introduction

When a ball rolls down a hill, it seems to follow a path that optimizes the balance of speed and distance. This path is the ”shortest path” from the ball's starting point to where it ends. In mathematics, this idea of the ”shortest path” can be extended to more abstract scenarios, where we refer to it as a **geodesic**.

A fundamental object in mathematics is the group of 2×2 matrices with determinant 1, formally known as $\mathrm{SL}(2, \mathbb{R})$ or the **special linear group**. It's important to note that since these matrices have determinant 1. This means that when they are applied as a transformation on a structure, they preserve the structure's volume. This is like spinning pizza dough - actions are taken that change the shape of the dough but the overall size/volume

Date: January 5, 2026.

stays the same.

The goal of this paper is to understand the geodesic flows on $\mathrm{SL}(2,\mathbb{R})$. It's worth noting that these flows have already been defined (more on this in the *Background* section). Our goal is to offer a more computationally effective approach and a cleaner conceptual view of the geodesic flow.

3. Background

We'll begin this paper by describing the existing problem and solution. The main approach to defining a geodesic relies on identifying the "shortest path" between two points in the target group. However, it's not as simple as a straight line on a flat plane when we deal with $\mathrm{SL}(2,\mathbb{R})$. Geodesics in the special linear group are curved, akin to the way great circles on a sphere represent "straight lines" in spherical geometry.

The answer above uses an exponential map to compute these geodesics. Picture the special linear group as a curved space known as a Riemannian Manifold. If you are not familiar with Riemannian Manifolds, visit Appendix A: Mathematical Foundations.

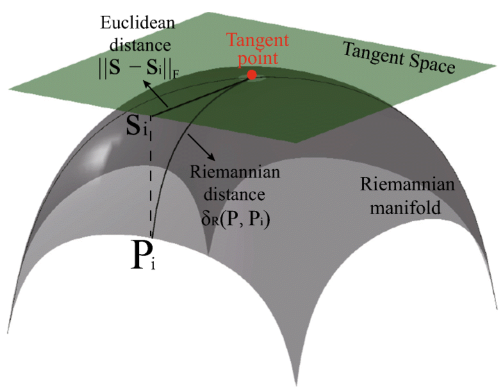


Figure 1. A 3D Visualization of a Riemannian Manifold and its Tangent Plane [6]

The above figure depicts a curved Riemannian manifold and the tangent plane at a point on the manifold. The idea behind an exponential map is to take a point on the tangent space (essentially a "flat version" of the group at the identity element) and "push it forward" onto the group itself. Imagine you have a small arrow sitting on a curved surface. You "nudge" it forward by rolling it along the curve while keeping its direction constant.

One of the key properties of an exponential map is that for each vector $V \in TM$, the tangent space, the geodesic γ_V is given by $\gamma_V(t) = \exp(tV)$ for all t such that either side is defined [4]. Using this formulation to represent a geodesic with an exponential map, the following geodesic formula is attained for the special linear group:

$$\gamma_V(t) = e^{tV} T e^{t(V - V^T)}$$

where V is an element of $\mathrm{SL}(2, \mathbb{R})$'s tangent space, T is a fixed matrix, and t is the time variable.

The existing method of computing geodesics using the exponential map has several shortcomings that limit its practical application. Primarily, the computational complexity of matrix exponentiation required to compute geodesics is significant, especially for larger matrix groups. This makes the process slow and inefficient, hindering real-time calculations.

This leads to the motivation for a new approach. Instead of relying on matrix exponentiation, our approach shifts perspective to Hamiltonian mechanics. By thinking of geodesics as flows generated by energy-like functions (called "Hamiltonians"), we can leverage well-known tools from physics to compute geodesics in a more structured way. This avoids computational bottlenecks and provides deeper geometric insight into how the paths evolve. The next section will provide the necessary definitions to understand this new perspective.

4. Methods

4.1. The Special Linear Group.

We begin discussing our approach to the problem by defining the special linear group $\mathrm{SL}(2, \mathbb{R})$ as a Lie group.

Definition 4.1 (Lie Group)**.** A Lie group is defined as a group that is also a smooth manifold, with the group operations (multiplication and inversion) being smooth maps. This is crucial as Lie groups have both algebraic and differential structures, allowing them to be studied through tools from both abstract algebra and differential geometry [1].

Definition 4.2 (Special Linear Group)**.** The special linear group $\mathrm{SL}(2, \mathbb{R})$ is defined as the group of all invertible 2×2 matrices with real entries and determinant 1. That is,

$$\mathrm{SL}(2, \mathbb{R}) = \{A \in M_{2\times 2}(\mathbb{R}) | \det A = 1\}$$

where the binary operation is matrix multiplication.

4.2. Lie Algebra of $\mathrm{SL}(2, \mathbb{R})$.

The Lie algebra of a Lie group is defined as the tangent space at the identity element of the group. It captures the infinitesimal structure of the group near the identity [3].

Definition 4.3 (Lie Algebra of $\mathrm{SL}(2,\mathbb{R})$)**.** The Lie algebra of the special linear group $\mathrm{SL}(2,\mathbb{R})$, denoted $\mathfrak{sl}(2,\mathbb{R})$, is the set of all 2×2 matrices $X \in M_{2\times 2}(\mathbb{R})$ such that the trace of X is zero:

$$\mathfrak{sl}(2,\mathbb{R}) = \{X \in M_{2\times 2}(\mathbb{R}) \mid \mathrm{Tr}(X) = 0\}.$$

The Lie algebra captures the infinitesimal symmetries of the group. It can be thought of as a vector space of matrices that generates the group through exponentiation. The Lie bracket (also called the commutator) of two matrices X and Y is defined as:

$$[X,Y] = XY - YX,$$

where XY and YX represent the standard matrix multiplication of X and Y [5].

The commutator is bilinear, antisymmetric, and satisfies the Jacobi identity, which are key properties of a Lie algebra. The Jacobi identity states that for any three elements $X, Y, Z \in \mathfrak{g}$, the following holds:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0,$$

where $\mathfrak{g}$ is a Lie algebra. This identity reflects the structure of the Lie algebra and is fundamental for the study of Lie groups and their representations [5].

To compute the commutation relations, consider the standard basis for $\mathfrak{sl}(2,\mathbb{R})$, which consists of the following matrices:

$$H = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The commutation relations for these matrices are:

$$[H,E] = 2E, \quad [H,F] = -2F, \quad [E,F] = H.$$

These relations describe how the Lie algebra elements interact and generate the structure of the Lie group $\mathrm{SL}(2,\mathbb{R})$.

The Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ is important for understanding the local structure of $\mathrm{SL}(2,\mathbb{R})$, and it plays a critical role in the Hamiltonian formalism that we will later discuss in the context of geodesic flows.

4.3. Introduction to Hamiltonian Mechanics.

Hamiltonian mechanics is a powerful framework used to describe the evolution of physical systems, especially those that can be modeled on curved spaces or manifolds, such as Lie groups. It provides a way to track the motion of systems by relating their energy to their position and momentum over time. In simple terms, it helps us understand how things move by focusing on energy instead of forces, using a central function called the Hamiltonian.

Definition 4.4 (Hamiltonian)**.** The Hamiltonian, H, is a function that represents the total energy of a system, combining both kinetic energy (related to the motion of the system) and potential energy (related to the position). In classical mechanics, this is usually written as:

$$H(q_1, p_1, \ldots, q_n, p_n) = T(p) + V(q)$$

where q_i are the generalized coordinates (position variables) and p_i are the generalized momenta (related to velocity). The Hamiltonian allows us to predict how these quantities change over time [1].

Hamilton's equations describe how the system evolves over time. These equations relate the time derivative of the generalized coordinates q_i and momenta p_i to the Hamiltonian H [1]. They are written as:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

In simpler terms, these equations describe how the position and momentum of the system change as time progresses. The first equation tells us how the position evolves based on the momentum, and the second equation tells us how the momentum changes based on the position.

In our study of geodesics on the Lie group $\mathrm{SL}(2, \mathbb{R})$, we rely on a special type of Riemannian metric known as a *left-invariant metric*. To understand why this choice is crucial, let's start by considering the general concept of a metric.

A *metric* is essentially a way to measure distances on a space. Imagine you're standing on a curved surface, like a sphere. The metric would tell you how far apart two points on that surface are, accounting for the curvature of the sphere. Similarly, in the case of Lie groups, the left-invariant metric allows us to measure distances between points on the group manifold while respecting the structure of the group itself.

Now, why is this metric called *left-invariant*? The term refers to the fact that the metric remains unchanged if we "shift" the space by multiplying it with an element from the group. To illustrate this, picture a map of a city. If you take the map and shift it in some direction (representing multiplication by an element of the group), the distances between points on the map stay the same, just like the left-invariant metric preserves distances under such shifts. This property makes the metric very useful for our problem because it allows us to define distances that are consistent across the entire group, without needing to re-compute them every time we "move" in the group.

Definition 4.5 (Left-Invariant Metric of a Lie Group)**.** For any $g \in G$, the metric at a point g is related to the metric at the identity element e of the group via a left translation. In other words, if we translate the group by any element g, the way we measure distances (or angles) doesn't change. Mathematically, the left-invariant metric satisfies

$$\langle X, Y \rangle_g = \langle dL_{g^{-1}}(X), dL_{g^{-1}}(Y) \rangle_e,$$

where L_g represents left multiplication by g, and e is the identity element of the group [2].

The system we're studying, the geodesic motion on the Lie group $SL(2, \mathbb{R})$, can be described using Hamiltonian mechanics. Just like a ball rolling down a hill (where the ball's speed and position depend on its energy), the geodesic flow in our project represents a "path" the system follows on the manifold, where energy (in the form of the Hamiltonian) dictates how the system evolves. This is a key connection because geodesics are curves that minimize energy in a certain sense, and in our case, they are described by the Hamiltonian.

For example, imagine a simple system like a pendulum: its Hamiltonian could be the sum of its kinetic energy (due to its motion) and potential energy (due to its height). For more complex systems, such as the Lie group $SL(2, \mathbb{R})$, the Hamiltonian governs the evolution of the system through its phase space (a space that tracks both position and momentum).

In our case, Hamiltonian mechanics plays a critical role in understanding the dynamics of geodesics in Lie groups like $SL(2, \mathbb{R})$. By applying the Hamiltonian framework to this group, we are able to set up a system of equations that describe the geodesic flow. This approach helps us track the evolution of the system more accurately and potentially addresses some of the issues seen in the existing solution, like computational complexity and difficulties in identifying conjugate points.

In the context of $SL(2, \mathbb{R})$, this framework is particularly useful because the Lie group has a well-defined geometric structure, and the motion of elements on this group can be described in terms of geodesics, much like how the motion of a pendulum can be described by its Hamiltonian. These geodesics are critical in understanding the flow of the system over time.

4.4. **Poisson Bracket and Momentum Maps.**

In the Hamiltonian formalism, the Poisson bracket and momentum map are key tools that help us understand how quantities evolve and interact, especially when the system has symmetries or geometric structures, such as those present in Lie groups.

Definition 4.6 (Poisson Bracket)**.** The Poisson bracket is an operation that measures how two functions on phase space change with respect to each other. Given two functions f and g defined on phase space, the Poisson bracket is defined as:

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$$

where q_i and p_i are generalized coordinates and momenta, respectively [1].

The Poisson bracket measures the infinitesimal change of one function with respect to another. In simpler terms, it shows how the evolution of one quantity affects the evolution of another in a dynamic system. If the Poisson bracket of two functions is zero, it means these quantities do not influence each other as the system evolves—they are independent.

To understand this better, consider two cars moving on a road. Each car has a position and velocity. If the position of one car does not affect the velocity of the other, then their interaction, or Poisson bracket, is zero. This implies the cars are moving independently. However, if the velocity of one car influences the position of the other, the Poisson bracket is non-zero, indicating a dependency between the two.

In our context, the Poisson bracket $\{\cdot,\cdot\} : C^\infty(T^* \mathrm{SL}(2,\mathbb{R})) \times C^\infty(T^* \mathrm{SL}(2,\mathbb{R})) \to C^\infty(T^* \mathrm{SL}(2,\mathbb{R}))$ is a bilinear operation that encodes how two smooth functions on the cotangent bundle $T^* \mathrm{SL}(2,\mathbb{R})$ interact under Hamiltonian dynamics. For two smooth functions $f_1, f_2 \in C^\infty(T^* \mathrm{SL}(2,\mathbb{R}))$, the Poisson bracket is defined as:

$$\{f_1, f_2\} = \frac{\partial f_1}{\partial \theta} \frac{\partial f_2}{\partial p_\theta} - \frac{\partial f_1}{\partial p_\theta} \frac{\partial f_2}{\partial \theta} + \frac{\partial f_1}{\partial x} \frac{\partial f_2}{\partial p_x} - \frac{\partial f_1}{\partial p_x} \frac{\partial f_2}{\partial x} + \frac{\partial f_1}{\partial y} \frac{\partial f_2}{\partial p_y} - \frac{\partial f_1}{\partial p_y} \frac{\partial f_2}{\partial y}.$$

This generalizes the familiar notion of the Poisson bracket for classical systems with position-momentum pairs (q_i, p_i), but now incorporates the coordinates (θ, x, y) and their conjugate momenta (p_θ, p_x, p_y).

A crucial interpretation of this bracket is that it provides a method to compute the time evolution of a function f along a Hamiltonian flow. If H is the Hamiltonian governing the system, then the time evolution of f is given by:

$$\dot{f} = \{f, H\}.$$

This relationship underpins the Hamiltonian formalism and allows us to track how quantities like position, momentum, and energy change over time. The Poisson bracket also satisfies key algebraic properties, including bilinearity, antisymmetry, and the Jacobi identity [1].

Definition 4.7 (Momentum Map)**.** A momentum map captures the symmetries of a system by associating each symmetry transformation with a conserved quantity. Given a Lie group G acting on a manifold M, the momentum map J is defined as:

$$J : M \to \mathfrak{g}^*$$

where $\mathfrak{g}^*$ is the dual of the Lie algebra of G, and J assigns to each point in M a conserved momentum associated with the symmetry of the group action [1].

A momentum map can be thought of as a translation of the "motion" of a system under symmetry transformations (like rotations or translations) into conserved quantities. For example, when an object rotates, its angular momentum remains conserved. The momentum map tells us how this conserved angular momentum is related to the rotational symmetry of the system.

In simpler terms, consider a spinning top. The rotational symmetry (the fact that it can spin without changing the overall system) corresponds to a conserved quantity: the angular momentum. The momentum map provides a mathematical way of relating the spinning (a symmetry transformation) to the angular momentum (the conserved quantity in phase space).

To facilitate the analysis of geodesic flow on $\mathrm{SL}(2, \mathbb{R})$, we introduce a change of coordinates for the momenta from (p_θ, p_x, p_y) to a new set of functions (P_1, P_2, P_3). This transformation is defined as:

$$\begin{aligned}
P_1 &= -\frac{\sinh(2y)}{\cosh(2x)} p_\theta + \cosh(2y) p_x - \sinh(2y)\tan(2x) p_y, \\
P_2 &= \frac{\cosh(2y)}{\cosh(2x)} p_\theta - \sinh(2y) p_x + \cosh(2y)\tan(2x) p_y, \\
P_3 &= p_y.
\end{aligned}$$

The variables (P_1, P_2, P_3) serve as momentum functions and provide a more natural basis for understanding the geodesic flow on $\mathrm{SL}(2, \mathbb{R})$. These functions are inspired by the structure of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$, where the coordinates (P_1, P_2, P_3) play an analogous role to the basis elements of the algebra.

The relationship between the momentum functions (P_1, P_2, P_3) is governed by the following Poisson bracket identities:

$$\{P_1, P_2\} = 2P_3, \quad \{P_2, P_3\} = 2P_1, \quad \{P_1, P_3\} = 2P_2.$$

These identities reflect the algebraic structure of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$, where the brackets satisfy the same commutation relations as the generators of $\mathfrak{sl}(2, \mathbb{R})$. This observation is significant because it highlights a deep relationship between the geometry of geodesic flow on the group $\mathrm{SL}(2, \mathbb{R})$ and the algebraic structure of its Lie algebra.

4.5. **The Geodesic Flow.**

The final step in our methodology is to understand and analyze geodesic flow on the Lie group $\mathrm{SL}(2,\mathbb{R})$. This section builds on all the mathematical structures we've developed so far - Lie groups, Lie algebras, Hamiltonian mechanics, Poisson brackets, momentum maps, and left-invariant metrics - and unifies them into one cohesive framework.

Definition 4.8 (Geodesic Flow)**.** Geodesic flow refers to the evolution of a point along a geodesic curve on a manifold at constant speed. Formally, let $\gamma(t)$ be a geodesic on a Riemannian manifold M. The geodesic flow ϕ_t is a one-parameter family of diffeomorphisms on the tangent bundle TM, such that for $v \in T_pM$, we have

$$\phi_t(v) = (\gamma(t), \dot{\gamma}(t))$$

where $\gamma(t)$ is the geodesic with initial conditions $\gamma(0) = p$ and $\dot{\gamma}(0) = v$.

To understand geodesic flow, we must first define the geodesic equation. Given a Riemannian manifold (M, g), geodesics are curves $\gamma(t)$ that locally minimize distance. These curves satisfy a second-order nonlinear differential equation:

$$\frac{d^2\gamma^k}{dt^2} + \Gamma^k_{ij}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt} = 0$$

where Γ^k_{ij} are the Christoffel symbols of the Riemannian metric g.

The geodesic equation can be interpreted as a generalization of Newton's second law. Just as an object with no external forces moves along a straight line in Euclidean space, a point on a Riemannian manifold "moves straight" in the sense of minimizing its distance, but the curvature of the space changes the notion of "straight."

In the case of a Lie group G, the geodesic flow can be described using the structure of the group's tangent bundle. If G has a left-invariant Riemannian metric, then geodesics on G starting from the identity element are determined by the exponential map of the Lie algebra $\mathfrak{g}$. Specifically, geodesics emanating from the identity can be written as:

$$\gamma(t) = \exp(tX)$$

for some $X \in \mathfrak{g}$. This exponential curve is a geodesic since the Riemannian metric is left-invariant, meaning the geodesic starting from any other point $g \in G$ can be obtained by left-translation:

$$\gamma_g(t) = g \cdot \exp(tX)$$

Thus, understanding geodesic flow on $\mathrm{SL}(2,\mathbb{R})$ reduces to understanding the exponential map of its Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, which we've already explored.

Recall from our discussion of Hamiltonian mechanics that we can describe motion on the cotangent bundle T^*G using a Hamiltonian function H. For geodesic flow, the natural choice of Hamiltonian is the kinetic energy, given by

$$H(q,p) = \frac{1}{2}\|p\|_g^2$$

where p represents the momentum coordinates and the norm $\|\cdot\|_g$ is induced by the left-invariant Riemannian metric g. The Hamiltonian system evolves according to Hamilton's equations:

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

These equations describe how the position q and momentum p evolve over time. Since geodesics trace out paths of least distance on the manifold, their evolution is entirely driven by the kinetic energy of the system. The geodesic flow is thus a special case of Hamiltonian flow.

To build intuition for geodesic flow, consider the concept of a ball rolling on a curved surface like a hilly landscape. If you give the ball a push, it will roll along a "straight" path, but due to the hills and valleys, its trajectory will curve. This path is analogous to a geodesic on a manifold. The speed and direction of the ball are determined by its initial velocity — this is the role of the tangent vector at the start of the flow. The geodesic flow traces out how this position and velocity change over time.

The geodesic flow on $\mathrm{SL}(2,\mathbb{R})$ is governed by a Hamiltonian system. The Hamiltonian function, which encodes the total energy of the system, is given by

$$H(P,g) = \frac{1}{2}\left(P_1^2 + P_2^2 + P_3^2\right).$$

This form reflects the kinetic energy associated with the geodesic motion, with (P_1, P_2, P_3) playing the role of momentum coordinates. The geodesic flow follows Hamilton's equations, and if $(P(t), g(t))$ is a solution, then $g(t)$ traces out a geodesic curve in $\mathrm{SL}(2,\mathbb{R})$. If we normalize the energy to $H = \frac{1}{2}$, then the geodesic $g(t)$ is parameterized by arc length.

To compute the geodesic equations, we apply Hamilton's equations, which state that

$$\dot{P}_i = -\frac{\partial H}{\partial g_i}, \quad \dot{g}_i = \frac{\partial H}{\partial P_i}.$$

To simplify the system, we work in terms of the momentum functions (P_1, P_2, P_3), which respect the underlying Lie algebra structure. The geodesic equations in terms of these functions are

$$\begin{pmatrix} \dot{P}_1 \\ \dot{P}_2 \\ \dot{P}_3 \end{pmatrix} = \begin{pmatrix} 4P_2P_3 \\ 0 \\ -4P_1P_2 \end{pmatrix}.$$

These equations describe the evolution of the momentum functions. Notably, P_2 remains constant along the flow, while P_1 and P_3 interact according to a nontrivial coupling, reflecting the symmetries of the geodesic system.

To compute the evolution of the group variables $(\theta, x, y) \in \mathrm{SL}(2, \mathbb{R})$, we use the following system of ordinary differential equations:

$$\begin{pmatrix} \dot{\theta} \\ \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & \frac{\cosh(2y)}{\cosh(2x)} & -\frac{\sinh(2y)}{\cosh(2x)} \\ 0 & -\sinh(2y) & \cosh(2y) \\ 1 & \cosh(2y)\tanh(2x) & -\sinh(2y)\tanh(2x) \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}.$$

This system governs the evolution of (θ, x, y) in $\mathrm{SL}(2, \mathbb{R})$ as a function of the momenta (P_1, P_2, P_3). The matrix coefficients depend on the geometry of the space, with $\cosh(2x)$ and $\sinh(2x)$ reflecting hyperbolic components. These hyperbolic terms highlight the non-Euclidean nature of the group $\mathrm{SL}(2, \mathbb{R})$ and the geodesic flow within it.

The geodesic flow is a smooth one-parameter family of symplectomorphisms

$$\Phi^t : T^* \mathrm{SL}(2, \mathbb{R}) \to T^* \mathrm{SL}(2, \mathbb{R})$$

that describes the evolution of a point (P, g) in the cotangent bundle. The map Φ^t is defined as

$$\Phi^t(P, g) = (P(t), g(t)),$$

where $(P(t), g(t))$ is a solution to the geodesic equations with initial condition $(P(0), g(0)) = (P, g)$. This formulation provides a global picture of how the geodesics evolve in the phase space $T^* \mathrm{SL}(2, \mathbb{R})$, where both the position (θ, x, y) and the momenta (P_1, P_2, P_3) evolve in tandem. The flow Φ^t preserves the symplectic structure on the cotangent bundle, meaning that it preserves the Poisson bracket $\{f, g\}$ for any smooth functions f and g.

To understand the structure of the geodesic flow, we consider the surface $\mathbb{S}(t, g_0)$, which can be viewed as the set of points that flow to $g_0 \in \mathrm{SL}(2, \mathbb{R})$ under the action of the geodesic flow. This surface is defined as

$$\mathbb{S}(t; g_0) := \left\{ g \in \mathrm{SL}(2, \mathbb{R}) \mid (P, g) = \Phi^t(P(0), g_0) \quad \text{and} \quad P_1^2(0) + P_2^2(0) + P_3^2(0) = 1 \right\}.$$

The condition $P_1^2(0) + P_2^2(0) + P_3^2(0) = 1$ normalizes the initial momenta to lie on the unit sphere $\mathbb{S}^2$. As the system evolves, the flow preserves this norm condition, allowing the surface $\mathbb{S}(t; g_0)$ to be interpreted as a "wavefront" of geodesics emanating from the point g_0 on $\mathrm{SL}(2,\mathbb{R})$.

The geodesic flow has important dynamical properties. Since $\mathrm{SL}(2,\mathbb{R})$ is a non-compact Lie group, the geodesic flow exhibits hyperbolic behavior. The surface $\mathbb{S}(t; g_0)$ provides insight into how geodesics diverge over time. The preservation of the unit norm of (P_1, P_2, P_3) reflects the conservation of energy, and the preservation of the symplectic structure implies that the geodesic flow preserves area in phase space [1].

5. Results

To compute the geodesic flow, we numerically solve the system of ordinary differential equations (ODEs) derived in the Methods section. Using `scipy.integrate.solve_ivp`, a robust solver for initial value problems, we obtain the geodesic curves parameterized by time t over the interval $t \in [0, 2]$. This process allows us to visualize how the geodesic curve evolves in $\mathbb{R}^3$ as t increases.

5.1. **Initial Conditions.**

The system requires both initial positions and initial momenta to be specified. The initial position is set to

$$\begin{pmatrix} \theta(0) \\ x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The initial momentum vector is parameterized as a point on the 2-sphere $\mathbb{S}^2$. Specifically, we define

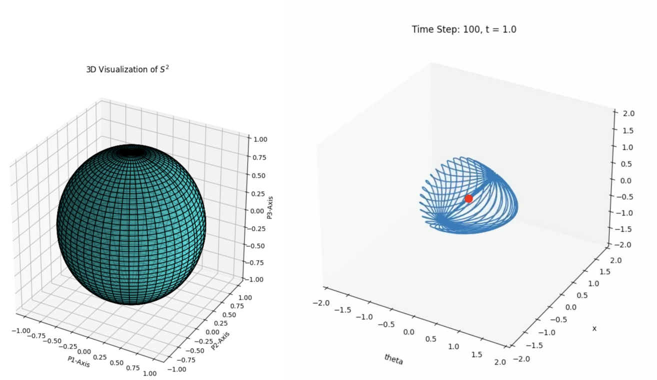
$$\begin{pmatrix} P_1(0) \\ P_2(0) \\ P_3(0) \end{pmatrix} = \begin{pmatrix} \cos(\psi)\cos(\phi) \\ \sin(\psi)\cos(\phi) \\ \sin(\phi) \end{pmatrix}$$

where $\phi \in (-\pi/2, \pi/2)$ and $\psi \in (0, 2\pi)$. This choice ensures that the initial momentum is properly normalized to lie on $\mathbb{S}^2$. Intuitively, each pair (ϕ, ψ) corresponds to a point on the surface of a unit sphere, and this initial point influences the trajectory of the geodesic flow.

5.2. **Visualization of Geodesic Flow.**

To visualize the geodesic flow, we generate two key figures.

- **Figure 1 (left)**: This figure shows a 3D visualization of the Euclidean unit sphere $\mathbb{S}^2$ embedded in $\mathbb{R}^3$. Each triplet $(P_1(0), P_2(0), P_3(0))$ can be understood as a point on this sphere, providing a clear geometric interpretation of our initial momentum configuration.
- **Figure 1 (right)**: This figure captures a snapshot of the geodesic flow at time $t = 1.0$. The visualization illustrates how the initial conditions affect the trajectory, as well as how the geodesic evolves over



time. The path traced by the geodesic clearly reflects the influence of the Hamiltonian formalism, where the momentum and position evolve according to the equations discussed earlier.

5.3. **Generating the Initial Point Grid.**

To generate initial conditions for a variety of geodesics, we use a grid of points in spherical coordinates. The azimuthal angle ϕ is varied over the interval $[-\pi/2, \pi/2]$, while the polar angle ψ is varied over $[0, 2\pi]$. Each pair (ϕ, ψ) generates a unique point on $\mathbb{S}^2$, producing a dense grid of possible initial momenta.

To visualize this grid, we project these points onto the surface of $\mathbb{S}^2$ in $\mathbb{R}^3$ and plot the resulting surface. The visualization reveals the distribution of initial conditions on $\mathbb{S}^2$ and serves as a starting point for all subsequent geodesic flow computations.

6. Conclusions

This report presents a novel approach to deriving and visualizing geodesic flow on $\mathrm{SL}(2, \mathbb{R})$, offering a more complete and accessible framework compared to the existing method. Our approach emphasizes the role of momentum functions and utilizes Hamiltonian equations, providing a structured pathway to obtain the geodesic equations, which are then numerically solved and visualized. This approach has practical applications in fields like robotics, where Lie groups model rigid body motion, and in theoretical physics, where geodesic flow on Lie groups underpins modern studies of symmetry and conservation laws. By streamlining the derivation process and offering a clear computational pathway, this work serves as a valuable resource for those aiming to understand and apply geodesic flow in both theoretical and applied settings.

References

[1] Ruth Armitage Abraham and John E. Marsden. *Foundations of Mechanics*. Benjamin/Cummings, Reading, MA, 1978.

[2] Vladimir I. Arnold. *Mathematical Methods of Classical Mechanics*. Springer-Verlag, New York, 1980.

[3] Sigurdur Helgason. *Differential Geometry, Lie Groups, and Symmetric Spaces*. Academic Press, San Diego, 1978.

[4] Vaughan F. Rannick. *The Riemannian Metric*. 2017.

[5] John Stillwell. *Naive Lie Theory*. Springer, New York, 2008.

[6] Xiaofeng Xie, Zou Xiaokun, Rongnian Tang, Yao Hou, and Feifei Qi. Multiple graph fusion based on riemannian geometry for motor imagery classification - scientific figure. https://www.researchgate.net/figure/Riemannian-manifold-and-its-tangent-space_fig1_357510234, 06 2022.

7. Appendix A: Mathematical Foundations

7.1. Riemannian Manifolds.

A **manifold** is a mathematical space that locally looks like the flat Euclidean space (i.e. $\mathbb{R}^n$) but can have a more complex global structure. Imagine a curved 2D sheet of paper (like a wavy surface) floating in 3D space. If you place your finger on the surface, it feels flat in your local neighborhood — this is a "local Euclidean" property. But if you step back and look at the whole paper, you see it's globally curved. For example, the Earth is a two-dimensional surface (for mathematical purposes, we focus on the surface of the Earth and not depth towards the core, meaning we only need two parameters latitude and longitude to define any point on the surface) floating in three-dimensional space. Still, when we zoom in close enough, it appears flat. When looking at a 2D map of the Earth, it has 'distortion' because the 2D surface of a sphere cannot be perfectly flattened into a 2D plane without distortion. This is a consequence of the Earth being a manifold.

A function is infinitely differentiable if you can take its derivative as many times as you want, and the result is always a continuous function. When you have a manifold, you cover it with coordinate charts — essentially, systems for assigning coordinates to each point on the manifold. For a manifold to be considered a **smooth manifold**, the change of coordinates when moving from one chart to another must be an infinitely differentiable function. This ensures that even if the manifold curves in complicated ways, you can move smoothly between local "views" of it.

Definition 7.1 (Smooth Manifold)**.** A **smooth manifold** of dimension n is a topological space M that is Hausdorff, second-countable, and locally homeomorphic to $\mathbb{R}^n$, together with an atlas $\{(U_i, \varphi_i)\}$ where $\{U_i\}$ is an open cover of M, and $\varphi_i : U_i \to \mathbb{R}^n$ are homeomorphisms.

The critical condition that makes M a **smooth** manifold is that, for any two overlapping charts (U_i, φ_i) and (U_j, φ_j), the transition map

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

is an infinitely differentiable ($\mathcal{C}^\infty$) map.

In other words, the coordinate changes between charts are smooth, ensuring that differentiable functions on M behave as they would on $\mathbb{R}^n$.

Consider a triangle in 2D space. If you walk along its perimeter, you'll hit corners where the direction changes suddenly — these are points where the derivative is not defined. This is why a triangle (with its sharp vertices) is not a smooth manifold. However, if you smooth out the edges (like rounding the corners), you can create a smooth manifold where all transition maps are infinitely differentiable.

Definition 7.2 (Riemannian Metric)**.** A **Riemannian metric** on a smooth manifold M is a 2-tensor field $g \in \mathcal{T}^2(M)$ that is symmetric ($g(X,Y) = g(Y,X)$) and positive definite ($g(X,X) > 0$ if $X \neq 0$). A Riemannian metric thus determines an inner product on each tangent space T_pM, which is typically written $\langle X, Y \rangle := g(X,Y)$ for $X, Y \in T_pM$. [4]

A Riemannian metric is a mathematical tool that lets us measure lengths, angles, and distances on a smooth, curved space (manifold). The 2-tensor field means the Riemannian Metric g is a function that takes in two tangent vectors at each point on the manifold and outputs a number. This number is the inner product $\langle X, Y \rangle$, which we use to compute lengths, angles, and distances. At each point p on the manifold M, there's a specific space of directions we can travel called the tangent space T_pM. The inner product $\langle X, Y \rangle$ is defined for every pair of vectors X and Y in this space and allows us to:

- Measure the length of a vector X as $\sqrt{\langle X, X \rangle}$
- Measure the angle between two vectors X and Y using a generalization of the cosine rule
- Measure the distance between two points on a manifold by finding the shortest connecting curve (geodesic)

A manifold paired together with a given Riemannian metric is called a **Riemannian Manifold**.