Systematic Cyclic Code

- In a systematic code, the first k digits are bits and the last m = n – k digits are the parity check bits.
- For systematic code, the codeword polynomial c(x) corresponding to the data polynomial d(x) is given by

$$c(x) = x^{n-k} d(x) + \rho(x)$$

- Where $\rho(x)$ is the remainder from dividing $x^{n-k}d(x)$ by g(x).

$$\rho(x) = \operatorname{Rem} \frac{x^{n-k} d(x)}{g(x)}$$

- To prove this we observe that

$$\frac{x^{n-k}d(x)}{g(x)} = q(x) + \frac{\rho(x)}{g(x)}$$

- Where q(x) is of degree k – 1 or less. We add $\rho(x)/g(x)$ to both sides, and because of f(x) + f(x) = 0 under modulo-2 operation, we have

$$\frac{x^{n-k}d(x) + \rho(x)}{g(x)} = q(x)$$

Or

$$q(x)g(x) = x^{n-k}d(x) + \rho(x)$$

Example

- Construct a systematic (7,4) cyclic code using a generator polynomial.

Solution

As we know $g(x) = x^3 + x^2 + 1$

Consider a data vector $d = 1010$

$$d(x) = x^3 + x$$

so $$x^{n-k}d(x) = x^6 + x^4$$

- $$\begin{array}{rl}
 & \overline{x^3 + x^2 + 1} \quad \leftarrow q(x) \\
x^3 + x^2 + 1 \,\big) & x^6 + x^4 \\
 & \underline{x^6 + x^5 + x^3} \\
 & \qquad x^5 + x^4 + x^3 \\
 & \qquad \underline{x^5 + x^4 + x^2} \\
 & \qquad\qquad x^3 + x^2 \\
 & \qquad\qquad \underline{x^3 + x^2 + 1} \\
 & \qquad\qquad\qquad\quad 1 \;\leftarrow \rho(x)
\end{array}$$

Hence

$$c(x) = x^3 d(x) + \rho(x)$$
$$= x^3(x^3 + x) + 1$$
$$= x^6 + x^4 + 1$$

and

C = 1010001

- There is one shortcut to make code table for cyclic codes by making generator matrix G.
- Find out code words only for four combinations of inputs 1000, 0100, 0010, 0001, these are 1000110, 0100011, 0010111, 0001101.
- Now recognize these four code words are the four rows of G.

$$G = \begin{bmatrix} 1\,0\,0\,0\,1\,1\,0 \\ 0\,1\,0\,0\,0\,1\,1 \\ 0\,0\,1\,0\,1\,1\,1 \\ 0\,0\,0\,1\,1\,0\,1 \end{bmatrix}$$

- Once we make generator matrix , then code table can be create using equation

c = d.G

data	Code
1111	1111111
1110	1110010
1101	1101000
1100	1100101
1011	1011100
1010	1010001
1001	1001011
1000	1000110
0111	0111001
0110	0110100
0101	0101110
0100	0100011
0011	0011010
0010	0010111
0001	0001101
0000	0000000

Generator Polynomial and Generator Matrix of Cyclic Codes

- Once the generator matrix G = [I p] by determining the parity submatrix P

$$1^{st} \text{ row of P: } \operatorname{Re} m \frac{x^{n-1}}{g(x)}$$

$$2^{nd} \text{ row of P: } \operatorname{Re} m \frac{x^{n-2}}{g(x)}$$

$$K^{th} \text{ row of P: } \operatorname{Re} m \frac{x^{n-k}}{g(x)}$$

- For example $g(x) = x^3 + x + 1$
- 1st row of P: $\operatorname{Rem} \dfrac{x^6}{g(x)} = x^2 + 1$
- 2nd row of P: $\operatorname{Rem} \dfrac{x^5}{g(x)} = x^2 + x + 1$
- 3rd row of P: $\operatorname{Rem} \dfrac{x^4}{g(x)} = x^2 + x$
- 4th row of P: $\operatorname{Rem} \dfrac{x^3}{g(x)} = x + 1$

$$G = \begin{bmatrix} 1\,0\,0\,0\,1\,0\,1 \\ 0\,1\,0\,0\,1\,1\,1 \\ 0\,0\,1\,0\,1\,1\,0 \\ 0\,0\,0\,1\,0\,1\,1 \end{bmatrix}$$

$$H = \begin{bmatrix} 1\,1\,1\,0\,1\,0\,0 \\ 0\,1\,1\,1\,0\,1\,0 \\ 1\,1\,0\,1\,0\,0\,1 \end{bmatrix}$$

Decoding

- Every valid code polynomial c(x) is a multiple of g(x). In other words c(x) is divisible by g(x). When an error occurs during the transmission the received polynomial r(x) will not be a multiple of g(x). If the number of errors in r is correctable. Thus

$$\frac{r(x)}{g(x)} = m_1(x) + \frac{s(x)}{g(x)}$$

$$s(x) = \operatorname{Rem} \frac{r(x)}{g(x)}$$

Where the syndrome polynomial s(x) has a degree n – k – 1 or less.

If e(x) is the error polynomial then

$$r(x) = c(x) + e(x)$$

Remembering that c(x) is a multiple of g(x)

$$s(x) = \operatorname{Rem} \frac{r(x)}{g(x)}$$

$$= \operatorname{Rem} \frac{c(x) + e(x)}{g(x)}$$

$$= \operatorname{Rem} \frac{e(x)}{g(x)}$$

Example

- Construct the decoding table for the single error correcting (7,4) code. Determine the data vectors transmitted for the following received vectors r: (a) 1101101 (b) 0101000 (c) 0001100
- **Solution**:

The first step is to construct the decoding table.

- Because n – k – 1 , the syndrome polynomial of the second order, and there are seven possible nonzero syndroms.

- There are seven possible correctable single-error patterns because n = 7, we can use to find error

$$s = e.\ H^T$$

- s = 001,010,011,100,101,110,111

$$H = \begin{bmatrix} 1\,0\,1\,1\,1\,0\,0 \\ 1\,1\,1\,0\,0\,1\,0 \\ 0\,1\,1\,1\,0\,0\,1 \end{bmatrix}$$

- For example s = [1 1 0] then e = [1 0 0 0 0 0 0]

error (e)	syndrome (s)
1000000	110
0100000	011
0010000	111
0001000	101
0000100	100
0000010	010
0000001	001

- $r = [1\ 1\ 0\ 1\ 1\ 0\ 1]$

$s = r.\ H^T$

$$= [1\ 1\ 0\ 1\ 1\ 0\ 1] \begin{bmatrix} 1011100 \\ 1110010 \\ 0111001 \end{bmatrix}$$

$= [\ 1\ 0\ 1]$

$e = [\ 0\ 0\ 0\ 1\ 0\ 0\ 0]$ for $s = [1\ 0\ 1]$

$$c = r \oplus e = 1101101 \oplus 0001000$$

$$= 1100101$$

d = 1100