A Method For Fine Resolution Frequency Estimation From Three DFT Samples

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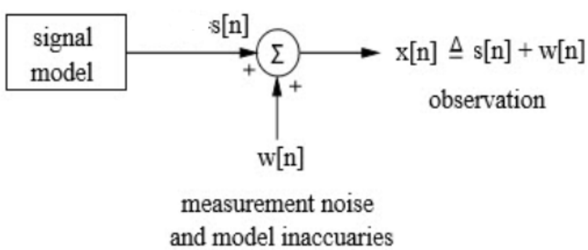
1. Abstract

The standard method of frequency estimation is done in two steps - coarse search to get an approximate frequency and fine search in the vicinity of the approximate frequency to get the accurate frequency of the signal. In this paper we propose an efficient alternative for the second step. The proposed method is to derive a non-linear relation between 3 DFT samples to estimate the frequency of a signal. We will also improve the accuracy of the Jacobian Estimator by providing a Bias Correction Factor.

2. Important concepts and short discussion about the existing method

2.1 Signal Estimation



Signal estimation is the process of extracting useful information from noisy data.
Here the information is the frequency of the signal and data is the signal with noise.

2.2 Estimator

Suppose you have a quantity of interest, say θ whose value is to be calculated. And you have a set of observations say x.
An estimator is a set of rules or an algorithm that can compute the value of the unknown parameter θ using the observed data.

Mathematically it can be defined as,

$$\hat{\theta} = f(x)$$

Where,

θ is the quantity of interest (parameter to be estimated),
$x = x_1, x_2, \dots x_N$ is the observed data,
$f(x)$ is the Estimator and $\hat{\theta}$ is the Estimated Value.

2.3 Bias of an estimator

Bias can be defined as the accuracy of the estimator. It is the difference between sample mean and the expected value obtained from the estimator.

For an unbiased estimator, the sample mean and expected value are equal. Hence, the bias is zero for an unbiased estimator.

$$B_{\hat{\theta}}(\theta) = E\left[\hat{\theta} - \theta\right]$$

Where,

θ is the quantity of interest (parameter to be estimated),
$f(x)$ is the Estimator and $\hat{\theta}$ is the Estimated Value

2.4 Characteristics of a good estimator

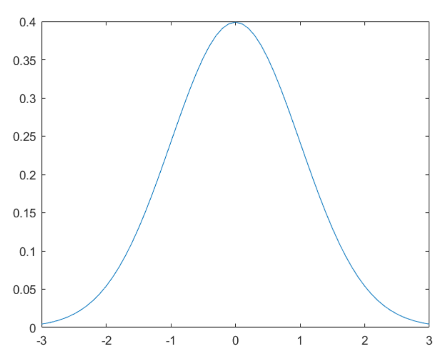
A good estimator must be

- Unbiased
  A good estimator must have zero bias. In this report, we will be comparing different estimators by plotting the bias against varying SNR ratios.

- Minimum Variance and Root Mean Square Error
  The variance of an estimator must be minimal for a good estimator. In this report we have compared the different estimators by plotting the RMSE (Root Mean Square Error) for different values of sound-to-noise ratio.

- Consistent
  The variance must tend to 0, as we increase the number of observations.
  $$lim_{n\rightarrow\infty}\sigma_{\hat{\theta}}{}^2 = 0$$

2.5 Gaussian White Noise



Normal Distribution of values in the range [-3,3] with mean = 0 and variance = 1

A random signal that has a uniform distribution of power (white) and probability density function has normal distribution (Gaussian).

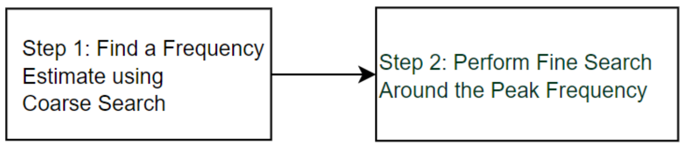
2.6 SNR

It is the ratio of signal power to noise power. It is used to determine how much of the desired frequency is present in the signal when compared to unwanted frequencies.

$$SNR = 10log_{10}\frac{Signal\ Power}{Noise\ Power}$$

2.7 Existing Methods

The standard method of frequency estimation involves the following steps



Step 1: Coarse Search

In coarse searching method, we iterate through the magnitude of the DFT of the signal (|X[k]|) and the index for which we get maximum value of the magnitude of the DFT is taken. Let us call this index as k_p.

Step 2: Fine Search

Since this is DFT, a sampled for DTFT, this frequency at k_p may not be accurate.
For example,

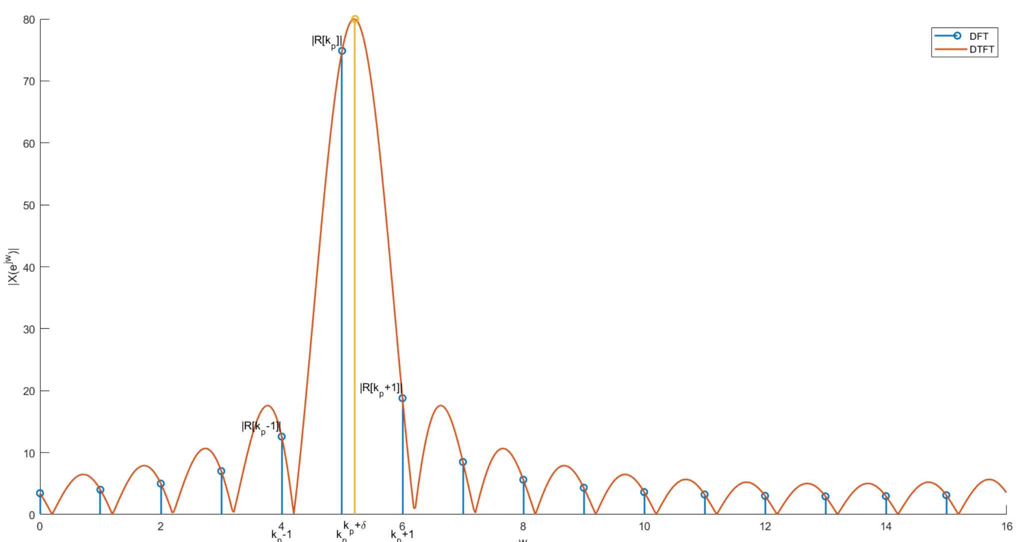


Figure: Signal for which the max frequency of the signal (obtained from DTFT) does not match with the max frequency from DFT.

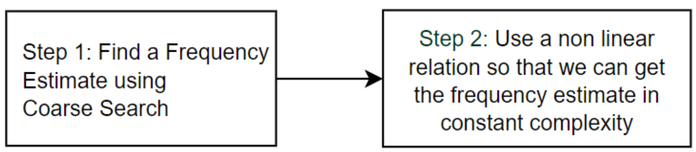
In the given plot, k_p = 5. But we get the maximum frequency when the index is 5.2.

Since the peak frequency lies in the neighborhood of the maximum frequency obtained from DFT, we now perform a fine search in the neighborhood of k_p to get the actual value of frequency of the signal.

Here the neighborhood value lies in the range [k_p – 0.5, k_p + 0.5]. Any value lesser than k_p-0.5 will go to the index k_p – 1. Similarly, any value greater than k_p + 0.5 will go to the index k_p + 1. Hence, we can model the index of the actual frequency of the signal as $k_p + \delta$, where $|\delta| < \frac{1}{2}$.

Frequency of the signal from DFT index $k_p = \frac{2\pi k_p}{N}$.
Since the index is modelled as $k_p + \delta$, the frequency of the signal $\omega = \frac{2\pi}{N}\left(k_p + \delta\right)$. Now our aim is to estimate the value of δ.

3. Proposed Method



Step 1: Coarse Search

The first step in the proposed estimator is the same as the standard frequency estimation technique.

Step 2: Use a Non-Linear Relation

Since the actual frequency of the signal lies in the vicinity of k_p, we can calculate the DFT at the neighboring indices of k_p which are
$k_p \,+\, 1 \; and \; k_p \,-\, 1$.

Now we can derive relation between the 3 DFT samples – R[k_p], R[$k_p + 1$] and R[$k_p - 1$] , to compute δ.

Derived Relationship between the 3 DFT samples to get the estimated δ

$$\hat{\delta} = \frac{\tan(\pi/N)}{\pi/N} Re\left(\frac{R[k-1] - R[k+1]}{2R[k] - R[k-1] - R[k+1]}\right)$$

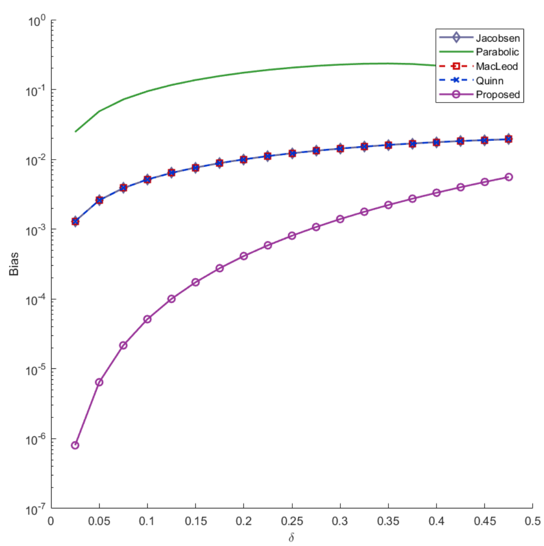
Estimated δ is denoted as $\widehat{\delta}$.

Algorithm to Implement Candan's Estimator (Proposed Estimator):

Step 1: Calculate FFT of signal
[Time Complexity: O(nlogn)]
Step 2: Calculate index for maximum magnitude of R[k], say k_p
Step 3: Compute value of bias using value of FFT at index: k_p, k_p-1, k_p+1
Step 4: Use the relation between the 3 DFT Samples to estimate δ
[Time Complexity: O(1)]

4. Simulation and Results

Figure 1: Bias vs δ Plot for a Noiseless Signal

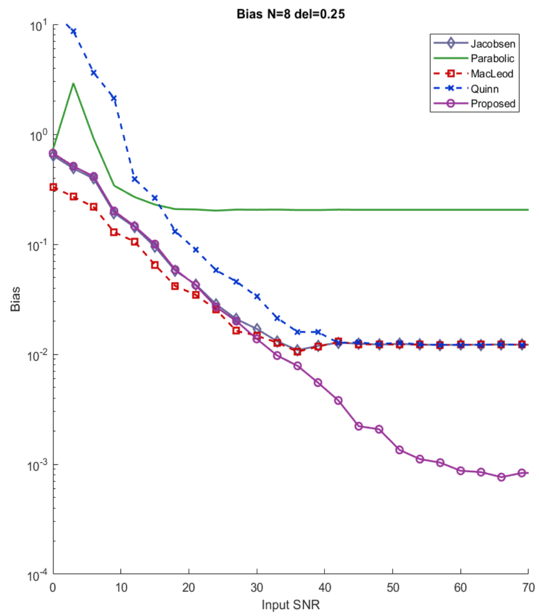


Bias of $\boldsymbol{\delta}$ estimate for a Noiseless Signal

- The estimators behave similarly in the absence of noise.

- At low δ, the bias of the proposed estimator is very less and it increases as δ increases.

Figure 2: Bias vs Input SNR Plot for a Noisy Signal

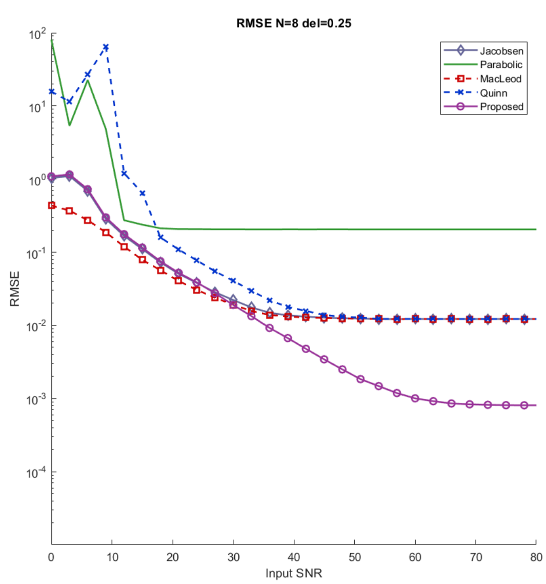


- The bias of the estimators approach 10^{-3}, which coincides with the value for bias for noiseless case when $\boldsymbol{\delta}$= 0.25

- For a very low SNR(<20) Jacobsen's estimator and the proposed estimator have almost same bias, but the proposed estimator gets considerably better as SNR increases.

Bias of $\hat{\delta}$ for a Noisy Signal

Figure 3: RMSE vs Input SNR Plot for a Noisy Signal for N = 8 and $\boldsymbol{\delta}$ = 0.25

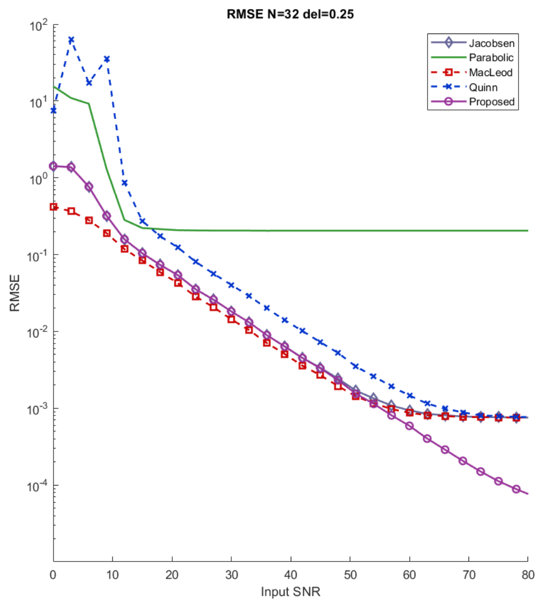


- Proposed estimator follows Cramer Rao Bound Closely in high SNR Region

- RMSE of proposed estimator is almost equal to that of Jacobsen's estimator at low SNR.

RMSE of $\hat{\delta}$ for N = 8 and $\boldsymbol{\delta}$ = 0.25 for varying levels of Input SNR

Figure 4: RMSE vs Input SNR Plot for a Noisy Signal for N = 32 and $\boldsymbol{\delta}$ = 0.25



RMSE of $\boldsymbol{\delta}$ estimate for N = 32 and $\boldsymbol{\delta}$ = 0.25 for varying levels of Input SNR

- The floor of the RMSE value occurs at a Higher SNR value for higher N (N=32)

5. Conclusion

1. From the observations, we can conclude that the proposed estimator and bias correction values have improved the estimator, by decreasing the bias and RMSE.
2. The proposed estimator is more efficient in computing the frequency of signals. The time complexity of the proposed estimator is O(nlogn + c) ≈ O(nlogn) [O(nlogn) for computing FFT and O(c) for computing $\hat{\delta}$ using the derived relation. Here c is a constant]

References

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