Assignment 1

EE5121 Convex Optimization

Deadline: Sept 24, 2025

Important points

1. **Objective:** Formulate selected nonconvex real-world problems and demonstrate how they can be recast as convex optimization problems.
2. **Theoretical foundation:** For each problem, the theoretical questions justify/provide the convex reformulation
3. **Computational component (CVXPY):** In each problem, solve the required subproblem using CVXPY. Report the numerical solution. Detailed analysis is not required for grading, though you are encouraged to comment on any insights.
4. **Deliverables:** Provide a link to your CVXPY code for each solved subproblem. The code must be well commented on and sufficient for the full reproducibility of the reported solution.
5. **Academic integrity:** You may consult LLM-generated examples for reference, but the submitted code must be your own original work. Submissions will be checked for AI-generated code.

Question 1 [25 marks]

Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Assume the linear system $Ax = b$ is consistent and admits multiple solutions.

(a) [**3 marks**] (**ℓ_0-sparse solution**) Among all solutions of $Ax = b$, consider the problem of maximizing sparsity (most elements are zero):

$$\min_{x\in\mathbb{R}^n} \|x\|_0 \quad \text{s.t.} \quad Ax = b,$$

where $\|x\|_0$ counts the number of nonzero components of x. Is this a convex optimization problem? If yes, prove. If no, give a counterexample.

(b) [**3 marks**] (**ℓ_1 proxy**) Consider the ℓ_1-relaxation of the above problem

$$\min_{x\in\mathbb{R}^n} \|x\|_1 \quad \text{s.t.} \quad Ax = b,$$

where $\|x\|_1 = \sum_{i=1}^n |x_i|$. Is this a convex optimization problem? If yes, prove. If no, give a counterexample.

(c) [**5 marks**] (**ℓ_1 as a linear program**) Show that the problem in (b) can be written as a linear program via auxiliary variables $u \in \mathbb{R}^n$:

$$\min_{x,u} \mathbf{1}^\top u \quad \text{s.t.} \quad Ax = b, \quad -u \le x \le u, \quad u \ge 0,$$

In particular, prove the equivalence of problem (b) and the above linear program.

(d) [**7 marks**] (**Application**) A color-matching system measures reflectance at three wavelengths (R, G, B). You can synthesize a target reflectance vector by mixing three laboratory pigments P_1, P_2, P_3. The measured response of each pigment at the three wavelengths is the column of a 3×3 matrix A; the target reflectance is $b \in \mathbb{R}^3$. Because two pigments have overlapping spectra, the responses are linearly dependent (so A is singular).

Given data.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 0.2 \\ 0.8 \\ 1.0 \end{bmatrix}.$$

$Ax = b$ is feasible and has many solutions.

1) Pose the fewest-pigments formulation as a sparse recovery problem and solve.
2) Solve with CVXPY and report: the optimizer $x^\star$, its support $\{i : x_i^\star \neq 0\}$

(e) [**7 marks**] Same laboratory setup, but the target reflectance arises from a slightly different device, so an exact match may not exist.

Given data.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \qquad b = \begin{bmatrix} 0.2 \\ 0.8 \\ 0.95 \end{bmatrix}.$$

$Ax = b$ is infeasible.

1) Modify the formulation intelligently to account for model error/measurement noise.
2) Solve the chosen convex program and report the optimizer, its support, and the achieved residual $\|Ax^\star - b\|_2$.

Fact check: We can prove that epigraph of $\|x\|_1$ is convex hull of epigraph of $\|x\|_0$ under some more technical restrictions. Technically, such functions are called convex envelopes. Here, $\|x\|_1$ is convex envelope of $\|x\|_0$.

Question 2 [13 marks]

Let $A \in \mathbb{R}^{m\times p}$, $X \in \mathbb{R}^{p\times n}$, and $B \in \mathbb{R}^{m\times n}$. Assume the linear matrix equation $AX = B$ is consistent and admits multiple solutions (A is rank-deficient or $p > m$ so that the solution set is an affine space of positive dimension).

(a) [**3 marks**] **(Rank minimization)**
Among all matrices X satisfying $AX = B$, consider

$$\min_{X\in\mathbb{R}^{p\times n}} \ \operatorname{rank}(X) \quad \text{s.t.} \quad AX = B.$$

Is this a convex optimization problem? If yes, prove. If no, give a counterexample.

(b) [**3 marks**] **(Nuclear norm minimization)**
Consider the following nuclear-norm minimization problem

$$\min_{X\in\mathbb{R}^{p\times n}} \ \|X\|_* \quad \text{s.t.} \quad AX = B,$$

where $\|X\|_* = \sum_i \sigma_i(X)$ (sum of singular values). Prove that $\|X\|_*$ is convex by using the representation

$$\|X\|_* \ = \ \sup_{\|Y\|_2\le 1} \ \langle X, Y\rangle,$$

where $\|\cdot\|_2$ is the spectral norm and $\langle X, Y\rangle = \operatorname{Trace}(X^\top Y)$.

(c) [**7 marks**] **Multi-experiment system identification** You conduct $k = 5$ experiments on the same unknown, discrete-time, linear time-invariant (LTI) system, each driven by the same input sequence of length 5. Let $x_j \in \mathbb{R}^5$ denote the (length-5) FIR impulse response of the j-th experiment (e.g., different sensors/outputs or slightly different operating points). Stack the impulse responses as columns of $X = [x_1\ x_2\ \cdots\ x_5] \in \mathbb{R}^{5\times5}$. With the common input Toeplitz matrix $A \in \mathbb{R}^{5\times5}$ built from the input $u = (u_0, \ldots, u_4)$, the measured output records form $B \in \mathbb{R}^{5\times5}$ via

$$B = AX, \quad \text{where} \quad A = \begin{bmatrix} u_0 & 0 & 0 & 0 & 0 \\ u_1 & u_0 & 0 & 0 & 0 \\ u_2 & u_1 & u_0 & 0 & 0 \\ u_3 & u_2 & u_1 & u_0 & 0 \\ u_4 & u_3 & u_2 & u_1 & u_0 \end{bmatrix}.$$

In many such settings, experiments for different sensors or different operating conditions can have similar responses, so X is (approximately) low rank. This motivates estimating X by rank minimization and understanding which experiments could be avoided in the future.

Given data Use the fixed input $u = (1,\ 0.8,\ -0.2,\ 0.5,\ 0)$, so

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.8 & 1 & 0 & 0 & 0 \\ -0.2 & 0.8 & 1 & 0 & 0 \\ 0.5 & -0.2 & 0.8 & 1 & 0 \\ 0 & 0.5 & -0.2 & 0.8 & 1 \end{bmatrix}.$$

The measured outputs (five experiments, five time samples each) are

$$B = \begin{bmatrix} 1.0 & -1.0 & 0.0 & 0.0 & 1.0 \\ 2.8 & -0.8 & 1.0 & -1.0 & 0.8 \\ 2.4 & 1.2 & 1.8 & -1.8 & -1.2 \\ 2.9 & -1.7 & 0.6 & -0.6 & 1.7 \\ 0.4 & -1.8 & -0.7 & 0.7 & 1.8 \end{bmatrix}.$$

1) Solve with CVXPY and report the optimizer.
2) Now replace B as $B + N$, where $N \in \mathbb{R}^{5\times5}$ and $N[i,j] \sim \mathcal{N}(0, 0.1)$ if $i = j$ and 0 if $i \neq j$. Modify your formulation to account for $AX \neq B$ and solve using CVXPY and report the optimizer.

To ponder: Why matrix A has special Toeplitz structure in this application?
Fact check: We can prove that the epigraph of the nuclear norm is the convex hull of the epigraph of the rank under some more technical restrictions.

Question 3 [22 marks]

Let $G = (V, E)$ be a simple undirected graph with $|V| = n$. A K-coloring assigns to each vertex $v \in V$ a color in $\{1, \ldots, K\}$ so that adjacent vertices receive different colors. The smallest K is the chromatic number $\chi(G)$.

(a) [**4 marks**] **Feasibility via a 0–1 formulation.** Introduce variables $x_{vk} \in \{0,1\}$ indicating that vertex v uses color k. Consider the feasibility system for a fixed K:

$$\sum_{k=1}^{K} x_{vk} = 1 \quad (\forall v \in V), \qquad x_{uk} + x_{vk} \le 1 \quad (\forall (u,v) \in E,\ \forall k), \qquad x_{vk} \in \{0,1\}.$$

(1) Is the above constraint set convex? If yes, prove. If not, give a counterexample.

(2) If you drop the binary requirement and only require $0 \le x_{vk} \le 1$, is the set convex? If yes, prove. If not, give a counterexample.

(b) [**8 marks**] **SDP relaxation (vector K-coloring).** Associate each vertex v with a unit vector u_v and let $G \succeq 0$ be the Gram matrix with $G_{uv} = u_u^\top u_v$. For a fixed $K \ge 2$, the vector K-coloring feasibility SDP is

$$\text{find } G \in \mathbb{S}^n$$
$$\text{s.t. } G \succeq 0, \qquad G_{vv} = 1 \ \ (\forall v \in V), \qquad G_{uv} \le -\frac{1}{K-1} \ \ (\forall (u,v) \in E).$$

(1) Explain in one line what it means if the set is feasible? What does it mean if it is not feasible?

(2) Show how the above feasibility problem is equivalent to the optimization problem minimizing the edge-wise inner-product bound ρ:

$$\min_{G,\rho} \ \rho \quad \text{s.t.} \quad G \succeq 0, \ G_{vv} = 1, \ G_{uv} \le \rho \ \ (\forall (u,v) \in E),$$

(3) Explain in 2 lines what you can say about the chromatic number of the graph from the solution of (2).

(c) [**10 marks**] **Application in your field.** Pick a real-world application from your discipline that maps naturally to graph coloring (e.g., conflict-free scheduling of maintenance tasks; frequency/channel assignment with interference constraints; exam timetable preparation). Explain the application and reduce it to the formulation in (b).

Note: You are not expected to solve the problem for this assignment, but are strongly encouraged to try it out for your own learning