Module - 1

Higher order linear differential equation with constant co-efficient.

Differential equation: An equation which contains differential coefficients of dependent variable w.r.t to independent variable is called differential equation.

Ex: $\frac{dy}{dx} + xy = 0$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x$$

Order of differential equation: The highest derivative in the differential equation is called order of differential equation.

Ex: $\frac{dy}{dx} + xy = 0$, $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x$

order = 1 order = 2

Degree of differential equation: The highest power of highest derivative is called degree of the differential equation.

Ex: $\frac{d^3y}{dx^3} + \left(\frac{d^2y}{dx^2}\right)^2 + \frac{dy}{dx} + y = x^2$

order = 3 degree = 1

Higher order linear differential equations with constant coefficients

The differential equation whose order is more than "1" and derivatives of dependent variable y w.r.t to x of any order must occurs only one is called higher order linear differential equation, and the

The general form

$$a_n\frac{d^ny}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1\frac{dy}{dx} + a_0y = Q(x)$$

classmate

Date
Page

A higher order linear differential equations with constant coefficient of n^{th} order

- where $a_n, a_{n-1}, \ldots a_1, a_0$ are relation real & $Q(x)$ is real cts function
- If $Q(x) = 0$ then the equation is called as Homogeneous equation of higher order linear differential equation.
- If $Q(x) \neq 0$ then it is called as Non-homogeneous equ higher order linear differential equation.

Solution of homogeneous higher order linear differential equ

The general form is

$(a_n D^n + a_{n-1} D^{n-1} + \ldots + a_1 D + a_0) y = 0$ where $D = \frac{dy}{dx}$

Let $P(D) = a_n D^n + a_{n-1} D^{n-1} + \ldots a_1 D + a_0$

So $P(D) y = 0$

general solution is $y = C.F = y_c$

Steps to find C.F (y_c)

1. Consider auxillary eq^n by substituting D as m in P(D) so we get $P(m) = 0$
2. Find the roots of $p(m) = 0$
3. If the roots of A.E is real & distinct i.e let $m_1, m_2, \ldots m_n$ are real & distinct

$$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots c_n e^{m_n x}$$

→ If the roots are real and equal i.e let $m_1 = m_2 = \ldots = m_n = a$ (real)

$$y_c = e^{ax} (c_1 + c_2 x + \ldots + c_n x^{n-1})$$

→ If the roots are imaginary (complex) like

$$m = \alpha \pm i\beta$$

$$y_c = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x)$$

If $m = \alpha \pm \beta$ repeats twice

$$y_c = e^{\alpha x} ((c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x)$$

1. Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$

$(D^2 + D - 2)y = 0$

$p(D) = D^2 + D - 2$

general sol^n $\Rightarrow$ y = complementary function $= y_c$

To find y_c

A.E = put $D = m$ in $P(D)$

$\Rightarrow m^2 + m - 2 = 0$

$\Rightarrow m^2 + 2m - m - 2 = 0$

$m(m+2) - 1(m+2) = 0$

$(m+2)(m-1) = 0$

$m = -2 \quad m = 1$

$y_c = c_1 e^{-2x} + c_2 e^{x}$

2. Solve $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$

$(D^2 + 6D + 9)y = 0$

$P(D) = D^2 + 6D + 9$

general sol^n $\Rightarrow$ y = complementary function $= y_c$

To find y_c

A.E = $p(D)$

$= m^2 + 6m + 9$

roots $\Rightarrow m^2 + 3m + 3m + 9 = 0$

$m(m+3) + 3(m+3) = 0$

$(m+3)(m+3) = 0$

$m = -3 \quad m = -3$

$y_c = e^{mx}(c_1 + c_2 x + \ldots)$

$y_c = e^{-2x}(c_1 + c_2 x)$

3. Solve $(D^3 - 2D^2 - 3D)y = 0$

$P(D) = D^3 - 2D^2 - 3D$

To find y_c

A.E $\Rightarrow m^3 - 2m^2 - 3m$

$\Rightarrow m(m^2 - 2m$

$\Rightarrow m(m^2 - 2m - 3) = 0$

$\Rightarrow m(m^2 - 3m + m - 3) = 0$

$\Rightarrow m = 0 \quad m(m-3)(1(m-3) = 0$

$m = 0, \; m = 3, \; m = -1$

$y_c = c_1 e^{0x} + c_2 e^{3x} + c_3 e^{-x}$

4. Solve $(D^2 + 1)y = 0$

A.E $= m^2 + 1 = 0$

$m^2 = -1$

$m = \pm\sqrt{-1}$

$m = \pm i$

$m = 0 \pm i$

$a = 0$

$y_c = e^{0x}(c_1 \cos bx + c_2 \sin bx) \; \sqrt{-1} = i$

$y_c = e^{0x}(c_1 \cos x + c_2 \sin x)$

$g.s = y = y_c = c_1 \cos x + c_2 \sin x$

5. Solve $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = 0$

$P(D) = D^2 + D + 1$

classmate

Date

Page

A.E $\Rightarrow$ $m^2+m+1=0$

$\Downarrow$

$am^2+bm+c=0$

$a=1 \quad b=1 \quad c=1$

$$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{1^2-4}}{2(1)}$$

$$= \frac{-1 \pm \sqrt{-3}}{2}$$

$$= \frac{-1}{2} \pm \frac{i\sqrt{3}}{2}$$

$$m = \frac{-1}{2} \pm \frac{i\sqrt{3}}{2}$$

$$y_c = e^{-1/2 x}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right)$$

6. Solve $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = 0$

A.E : $m^3 - 3m + 2 = 0$

$m=1$ is one root

$$\begin{array}{c|cccc} 1 & 1 & 0 & -3 & 2 \\ & 0 & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & 0 \end{array}$$

$m^2+m-2=0$ $\quad -2m^2$

$m^2-m+2m-2=0$ $\quad -1m+2m$

$m(m-1)+2(m-1)=0$

$(m+2)(m-1)=0$

$m=-2 \quad m=1 \quad m=1$

$$y_c = e^{mx}(c_1 + c_2 x) + c_3 e^{mx}$$

$$= e^{x}(c_1 + c_2 x) + c_3 e^{-2x}$$

7. Solve $(4D^2 - 4D + 1)y = 0$

A.E :

$$4m^2 - 4m + 1 = 0$$
$$4m^2 - 2m - 2m + 1 = 0$$
$$2m(2m-1) - 1(2m-1) = 0$$
$$(2m-1)(2m-1) = 0$$
$$m_1 = \frac{1}{2} \qquad m_2 = \frac{1}{2}$$

$$y_c = e^{1/2 x}(c_1 + c_2 x)$$

8. Solve $(D^3 + 3D^2 + 3D + 1)y = 0$

A.E : $m^3 + 3m^2 + 3m + 1 = 0$

$$(m+1)^3 = 0$$
$$m = -1, -1, -1$$
$$y = y_c = e^{-x}(c_1 + c_2 x + c_3 x^2)$$

Non-homogeneous higher order differential equation

The general form is $P(D)y = Q(x)$
where $Q(x) \neq 0$ then $y_p = \frac{Q(x)}{P(D)}$

Here $Q(x)$ can take the function of the form e^{ax}, $\sin ax$ or $\cos ax$, x^k and $e^{ax}V$

Case 1:

- If $Q(x) = e^{ax}$

Find a
Substitute $D=a$ in $P(D)$ so we get $P(a)$ if $P(a) \neq 0$
then $y_p = \frac{Q(x)}{P(a)}$

If $P(a) = 0$ write $P(D) = (D-a)^n f(D)$ then
$y_p = \frac{x^n Q(x)}{n!\, f(a)}$

Solve $(D^2 + 5D + 6)y = e^x$
Sol^n:

G.S $y = y_c + y_p$

To find y_c

A.E $\Rightarrow m^2 + 5m + 6 = 0$
$m^2 + 2m + 3m + 6 = 0$
$m(m+2) + 3(m+2) = 0$
$(m+3)(m+2) = 0$
$m = -3, \ m = -2$

$y_c = c_1 e^{-3x} + c_2 e^{-2x}$

To find y_p

$a=1$

Substituting $a=1$ in $P(D)$

$= D^2+5D+6$

$= 1^2+5+6$

$= 12 \neq 0$

$$y_p = \frac{Q(x)}{P(a)}$$

$$= \frac{e^x}{12}$$

G.S $= y_c + y_p$

$$y = c_1e^{-3x} + c_2e^{-2x} + \frac{e^x}{12}$$

Solve $(D+2)(D-1)^2y = e^{-2x}$

Soln:

G.S $y = y_c + y_p$

To find y_p

A.E $\Rightarrow$ $(m+2)(m-1)^2 = 0$

$\Rightarrow (m+2)(m-1)^2 = 0$

$m=-2,\ m=1,\ m=1$

$y_c = e^x(c_1+c_2x) + c_3e^{-2x}$

To find y_p

$a=-2$

Substituting $a=-2$ in $P(D)$

$= (-2+2)(-2-1)^2 = $

$\Rightarrow 0 = 0$

Since $P(a) = 0$

$P(D) = (D-a)^n f(D)$

compare with

$P(D) = (D+2)(D-1)^2$

$P(D) = (D-(-2))(D-1)^2$

$\Rightarrow n=1, \quad f(D) = (D-1)^2 \Rightarrow f(a) = (-2-1)^2 = 9$

$$y_p = \frac{x^n Q(x)}{n!\, f(a)}$$

$$= \frac{x e^{-2x}}{9}$$

G.S $= y_c + y_p$

$$= e^x(c_1 + c_2 x) + c_3 e^{-2x} + \frac{x e^{-2x}}{9}$$

3. Solve $(D^2 - 4D + 3)y = e^x$

Sol^n:

G.S $\Rightarrow y = y_c + y_p$

To find y_c:

A.E $\quad m^2 - 4m + 3 = 0$

$m^2 - 3m - m + 3 = 0$

$m(m-3) - 1(m-3) = 0$

$(m-1)(m-3) = 0 \quad \text{--- (1)}$

$m = 1, \; m = 3$

$y_c = c_1 e^x + c_2 e^{3x}$

To find y_p:

$$y_p = \frac{Q(x)}{P(D)} = \frac{e^x}{D^2 - 4D + 3}$$ here $a = 1$

$P(D) = D^2 - 4D + 3$

put $D = a = 1$ in $p(a)$

$p(a) = 1 - 4 + 3 = 4 - 4 = 0$

$\therefore p(D) = (D-1)(D-3)$ [from (1)]

compare $(D-a)^n f(D)$

$n = 1$, $f(D) = D - 3$

$f(1) = 1 - 3 = -2$

$$y_p = \frac{x^n}{n!\, f(a)} Q(x) = \frac{x^1}{1!\,(-2)} e^x = \frac{xe^x}{-2}$$

G.S, $y = c_1 e^x + c_2 e^{3x} + \frac{xe^x}{-2}$

4. Solve $(D^2 + 2D + 1)y = e^{2x}$

G.S, $y = y_c + y_p$

To find y_c:

A.E: $m^2 + 2m + 1 = 0$

$(m+1)^2 = 0$

$m = -1, -1$

$y_c = e^{-x}(c_1 + c_2 x)$

To find y_p:

$$y_p = \frac{Q(x)}{P(D)} = \frac{e^{2x}}{D^2 + 2D + 1} \quad \text{here } a = 2$$

$P(D) = D^2 + 2D + 1$

put $D = a = 2$;

$P(2) = 4 + 4 + 1 = 5 \neq 0$

$$y_p = \frac{Q(x)}{P(a)} = \frac{e^{2x}}{5}$$

$$y = e^{-x}(c_1 + c_2 x) + \frac{e^{2x}}{5}$$

Case 2: If $Q(x) = \cos ax$ or $\sin ax$

$$y_p = \frac{Q(x)}{P(D)}$$

Put $D^2 = -a^2$ in $P(D)$

Date

Page

If $P(-a^2) = 0$

If $Q(x) = \sin ax$, then $y_p = \frac{-x}{2a} \cos ax$

If $Q(x) = \cos ax$ then $y_p = \frac{x}{2a} \sin ax$

1. Solve $(D^2+4)y = \sin 2x$

g.s $y = y_c + y_p$

To find y_c

A.E $\Rightarrow m^2+4 = 0$

$m^2 = -4$

$m = \pm\sqrt{-4} = \pm 2i$

$y_c = e^{0x}(c_1\cos 2x + c_2 \sin 2x)$

To find y_p

$y_p = \frac{Q(x)}{P(D)} = \frac{\sin 2x}{D^2+4}$ here $a = 2$

$P(D) = D^2+4$

put $D^2 = -a^2 = -4$ in $P(D)$

$P(-a^2) = -4+4 = 0$

$Q(x) = \sin ax$

So $y_p = \frac{-x}{2a} \cos ax = \frac{-x}{4} \cos 2x$

g.s $y = (c_1\cos 2x + c_2 \sin 2x) - \frac{x}{4} \cos 2x$

2. Solve $(D^2+4)y = \cos 2x$

y_c is same as above

$y_p = \frac{\cos 2x}{D^2+4}$ here $a = 2$

$P(D) = D^2+4$

put $D^2 = -a^2 = -4$

$P(-a^2) = -4+4 = 0$

$Q(x) = \cos 2x$

So $y_p = \frac{x}{2a}\sin ax = \frac{x}{4}\sin 2x$

$y = (c_1\cos 2x + c_2\sin 2x) + \frac{x}{4}\sin 2x$

3. Solve $(D^2+1)y = 3\sin 2x$

y_c : A.E. $m^2+1=0$

$m^2 = -1$

$m = \pm\sqrt{-1} = \pm i$

$y_c = e^{0x}(c_1\cos x + c_2\sin x)$

To find y_p

$y_p = \frac{3\sin 2x}{D^2+1}$ $a=2$

put $D^2 = -a^2 = -4 \neq 0$

$y_p = \frac{3\sin 2x}{-3}$

G.S. $= c_1\cos x + c_2\sin x - \frac{\sin 2x}{3}$

4. Solve $(D^2+2D+1)y = \sin 2x$

$y = y_c + y_p$

To find y_c :-

A.E $= m^2+2m+1=0$

$(m+1)^2 = 0$

$m=-1,\ m=-1$

$y_c = e^{-x}(c_1 + c_2 x)$

To find y_p :-

$y_p = \frac{\sin 2x}{D^2+2D+1}$

$P(D) = D^2+2D+1$, $a=2$

put $D^2 = -2^2$

$= -4$

$P(-a^2) = -4+2D+1 = 2D-3 \neq 0$

~~put $D^2 = -2^2 = -4$~~

$$y_p = \frac{\sin 2x}{2D-3}$$

$$= \frac{1}{2D-3} \cdot \frac{2D+3}{2D+3} \sin 2x$$

$$= \frac{(2D+3)\sin 2x}{4D^2-9}$$

$$= \frac{2D\sin 2x + 3\sin 2x}{-16-9} \qquad \left(D = \frac{d}{dx}\right)$$

$$y_p = \frac{2\cos x + 3\sin 2x}{-25}$$

$$y = e^{-x}(c_1 + c_2 x) + \frac{2\cos x + 3\sin 2x}{-25}$$

case (iii): If $Q(x) = x^k$

$$y_p = \frac{Q(x)}{P(D)}$$

1. Solve $(D^2+2D+1)y = x^2$

$$y = y_c + y_p$$

To find y_c:

$$A.E = m^2+2m+1 = 0$$

$$(m+1)^2 = 0$$

$$m = -1,\ m = -1$$

$$y_c = e^{-x}(c_1 + c_2 x)$$

To find y_p:

$$y_p = \frac{x^3}{D^2+2D+1}$$

$$= (D^2+2D+1)^{-1} x^3$$

$$= [1 - (D^2+2D) + (D^2+2D)^2 - \dots] x^3$$

$$= x^3 - (D^2+2D)x^3 + (D^4 + 4D^2 + 4D^3)x^3 + \dots$$

$$= x^3 - 6x - 6x^2 + 0 + 24x + 24$$

$$y_p = x^3 - 6x^2 + 18x + 24$$

$$y = e^{-x}(c_1 + c_2 x) + x^3 - 6x^2 + 18x + 24$$

2. Solve $(D^2+1)y = x^2$

$y = y_c + y_p$

To find y_c

A.E $\Rightarrow m^2+1=0$

$m^2 = -1$

$m = \pm\sqrt{-1} = \pm i$

$y_c = e^{0x}(c_1\cos x + c_2\sin x)$

To find y_p:

$y_p = \frac{x^2}{1+D^2}$

$= x^2[1+D^2]^{-1} \Rightarrow x^2(1-D^2+D^4\dots)$

$= x^2 - D^2x^2 + D^4x^2\dots$

$y_p = x^2-2$

$y = (c_1\cos x + c_2\sin x) + x^2-2$

3. Solve $(D^2-1)y = 5$

$y = y_c + y_p$

To find y_c:

A.E $\Rightarrow m^2-1=0$

$m^2 = 1$

$m = \pm 1$

$y_c = c_1e^{x}+c_2e^{-x}$

To find y_p:

$y_p = \frac{5}{D^2-1}$

$= \frac{5}{-1(1-D^2)}$ $\quad (1-x)^{-1} = 1+x^2+x+x$

$D5 = 0$

$= -5(1-D^2)^{-1}$ $\quad D^25 = 0$

$= -5(1+D^2+D^4+\dots)$

$= -5 + D^2(-5) + D^4(-5) + \dots$

$y_p = -5$

$y = c_1e^{x} + c_2e^{-x} - 5$

4. Solve $(D^2-D)y = x$

$y = y_c + y_p$

To find y_c:-

A.E $\Rightarrow$ $m^2 - m = 0$

$m(m-1) = 0$

$m = 0 \quad m = 1$

$y_c = c_1 e^{0x} + c_2 e^{x}$

To find y_p:

$y_p = \frac{x}{D^2 - D}$

$= \frac{x}{D(D-1)}$

$= \frac{x}{-D(1-D)}$

$= \frac{-x}{D}(1-D)^{-1}$

$= \frac{-1}{D}(1 + D + D^2 \ldots)x$

$= \frac{-1}{D}(x + Dx + D^2x + \ldots)$

$y_p = \frac{-1}{D}(x+1)$

$= -\int x+1 \, dx$

$= \frac{-x^2}{2} + x$

$y = c_1 e^{0x} + c_2 e^{x} + \frac{x^2}{2} - x$

5. Solve $\frac{d^2y}{dx^2} + \frac{5dy}{dx} + 6y = e^{3x}$

G.S $\Rightarrow$ $y = y_c + y_p$

To find y_c

A.E $\Rightarrow$ $m^2 + 5m + 6 = 0$

$m^2 + 2m + 3m + 6 = 0$

$m(m+2) + 3(m+2) = 0$

$m=-3 \quad m=-2$

$y_c = c_1 e^{-3x} + c_2 e^{-2x}$

To find y_p

$y_p \Rightarrow \dfrac{2e^{3x}}{D^2+5D+6}$

where $p(D) = D^2+5D+6$, $a=3$

put $D = a = 3$ in $p(D)$

$p(a) = 9+15+6 = 30 \neq 0$

$y_p = \dfrac{Q(x)}{p(a)} = \dfrac{2e^{3x}}{30}$

$y = c_1 e^{-3x} + c_2 e^{-2x} + \dfrac{2e^{3x}}{30}$

6. Solve $\dfrac{d^2y}{dx^2} + 7\dfrac{dy}{dx} + 10y = 3\sin 3x$

g.s $\Rightarrow y = y_c + y_p$

To find y_c

A.E $\Rightarrow m^2+7m+10=0$

$m^2+5m+2m+10=0$

$m(m+5)+2(m+5)=0$

$(m+2)(m+5)=0$

$m=-2 \quad m=-5$

$y_c = c_1 e^{-2x} + c_2 e^{-5x}$

To find y_p:

$y_p = \dfrac{3\sin 3x}{D^2+7D+10}$

$p(D) = D^2+7D+10 \quad a=3$

put $D^2 = -a^2 = -9$ in $P(D)$

$p(-a^2) = -9+7D+10 = 7D+1 \neq 0$

$y_p = \dfrac{Q(x)}{P(-a^2)} = \dfrac{3\sin 3x}{7D+1} \times \dfrac{7D-1}{7D-1}$

$= \dfrac{3\sin 3x(7D-1)}{49D^2-1}$

$= \dfrac{21D\sin 3x - 3\sin 3x}{-441-1}$

$$= \frac{81(\cos 3x)3 - 8\sin 3x)}{21 - 442}$$

$$y_P = \frac{81}{-442}(21\cos 3x - 8\sin 3x)$$

$$y = c_1 e^{-2x} + c_2 e^{-5x} - \frac{3}{442}(21\cos 3x - 8\sin 3x)$$

To Solve $(D^2 + D - 1)y = x^2$

$G.S \Rightarrow y = y_c + y_P$

To find y_c

$A.E \Rightarrow m^2 + m - 1 = 0$

$a = 1,\ b = 1,\ c = -1$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{1+4}}{2}$$

$$= \frac{-1 \pm \sqrt{5}}{2}$$

$$m_1 = \frac{-1+\sqrt{5}}{2},\quad m_2 = \frac{-1-\sqrt{5}}{2}$$

$$y_c = c_1 e^{\frac{-1+\sqrt{5}}{2}x} + c_2 e^{\frac{-1-\sqrt{5}}{2}x}$$

To find y_P

$$y_P = \frac{x^2}{D^2+D-1} = \frac{x^2}{-1(1-(D^2+D))}$$

$$= -x^2(1-(D^2+D))^{-1}$$

$$= -x^2[1 + (D^2+D) + (D^2+D)^2 + \dots]$$

$$= -x^2[1 + D^2 + D + D^2 + \dots]$$

$$= [-x^2 - 2D^2x^2 + Dx^2]$$

$$= -[x^2 + 2 + 2 + 2x]$$

$$y_P = -[x^2 + 2x + 4]$$

$$G.S \Rightarrow y = c_1 e^{\left(\frac{-1+\sqrt{5}}{2}\right)x} + c_2 e^{\left(\frac{-1-\sqrt{5}}{2}\right)x} - x^2 - 2x - 4$$

Solution of system of linear differential equation

General solⁿ:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = x_1'$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = x_2'$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = x_n'$$

The matrix form is $AX = X'$ where A = coefficients

i.e. $A = \begin{bmatrix} a_{11}, & a_{12} & \cdots & a_{1n} \\ a_{21}, & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{n1}, & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $X' = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}$

G.S $\Rightarrow X = C_1V_1e^{\lambda_1 t} + C_2V_2e^{\lambda_2 t} + C_3V_3e^{\lambda_3 t} + \cdots + C_nV_ne^{\lambda_n t}$

where $\lambda_1, \lambda_2, \lambda_3 \cdots, \lambda_n$

are Eigen values of a matrix A & $v_1, v_2 \dots v_n$ are Eigen vectors corresponding to Eigen values

1. Solve $\frac{dx}{dt} = -3x-4y$, $\frac{dy}{dt} = 5x+6y$ by using diagonalization method

Given a System of linear differential equation as

$$\frac{dx}{dt} = -3x-4y$$

$$\frac{dy}{dt} = 5x+6y$$

matrix form $AX = X'$

where $A = \begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $X' = \begin{bmatrix} x' \\ y' \end{bmatrix}$

To find Eigen values of A

$|A - \lambda I| = 0$ (or) $\lambda^2 - S_1\lambda + S_2 = 0$

$S_1 = -3+6 = 3$

$S_2 = |A| = -18+20 = 2$

$\lambda^2 - 3\lambda + 2 = 0$

$\lambda^2 - 2\lambda - \lambda + 2 = 0$

$\lambda(\lambda-2) - 1(\lambda-2) = 0$

$(\lambda-1)(\lambda-2) = 0$

$\lambda = 1, 2$

$\therefore$ Eigen values are $\lambda_1 = 1$, $\lambda_2 = 2$

To find Eigen vector of $\lambda_1 = 1$

$(A - \lambda I)X = 0$ $\quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\left[\begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3-\lambda & -4 \\ 5 & 6-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{--- (1)}$$

put $\lambda = 1$

$$\begin{bmatrix} -4 & -4 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-4x_1 - 4x_2 = 0$

$x_1 + x_2 = 0$

$5x_1 + 5x_2 = 0$

$x_1 + x_2 = 0$

$x_1 = -x_2$

$\frac{x_1}{-1} = \frac{x_2}{1}$

$$V_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

To find Eigen vector for $\lambda_2 = 2$

$(A - \lambda I)X = 0$ $\quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\begin{bmatrix} -3-\lambda & -4 \\ 5 & -6-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

put $\lambda = 2$

$$\begin{bmatrix} -5 & -4 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-5x_1 - 4x_2 = 0$

$5x_1 - 4x_2 = 0$

$-5x_1 = +4x_2$

$5x_1 = -4x_2$

$\frac{x_1}{-4} = \frac{x_2}{5}$

$$V_2 = \begin{bmatrix} -4 \\ 5 \end{bmatrix}$$

$X = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} -4 \\ 5 \end{bmatrix} e^{2t}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -c_1 e^{t} \\ c_1 e^{t} \end{bmatrix} + \begin{bmatrix} -4c_2 e^{2t} \\ 5c_2 e^{2t} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -c_1 e^{t} - 4c_2 e^{2t} \\ c_1 e^{t} + 5c_2 e^{2t} \end{bmatrix}$$

$\therefore x = -c_1 e^{t} - 4c_2 e^{2t}$

$y = c_1 e^{t} + 5c_2 e^{2t}$

2. A particle moving in a planar force field has a position vector x that satisfies $X' = AX$. The matrix A has eigen values 4 and 2 with corresponding eigen vectors $v_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Find the position of the particle at time t assuming that $x(0) = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$

given: $x' = AX$ (system of D.E)

Eigen values of A as $\lambda_1 = 4, \lambda_2 = 2$

eigen vectors of A as $v_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$; $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

position of the particle is

$x(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$

$x(t) = c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2t}$

put $t = 0$ $x(0) = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$

$x(0) = c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{4(0)} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2(0)}$

$$\begin{bmatrix} -6 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$-3c_1 - c_2 = -6$ —— (1)

$c_1 + c_2 = 1$ —— (2) $\quad c_1 = \cancel{1} - c_2$

from ①

$$-3(1-C_2) - C_2 = -6$$

$$-3 + 3C_2 - C_2 = -6$$

$$-3 + 2C_2 = -6$$

$$+2C_2 = -6+3$$

$$+2C_2 = -3$$

$$C_2 = -\frac{3}{2}$$

$$C_1 = 1 - C_2$$

$$= 1 + \frac{3}{2}$$

$$= \frac{5}{2}$$

$$x(t) = \frac{5}{2}\begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{4t} - \frac{3}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2t}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{15}{2} e^{4t} + \frac{3}{2} e^{2t} \\ 2e^{4t} - 2e^{2t} \end{bmatrix}$$

$$x = \frac{-15}{2} e^{4t} + \frac{3}{2} e^{2t}$$

$$y = 2e^{4t} - 2e^{2t}$$