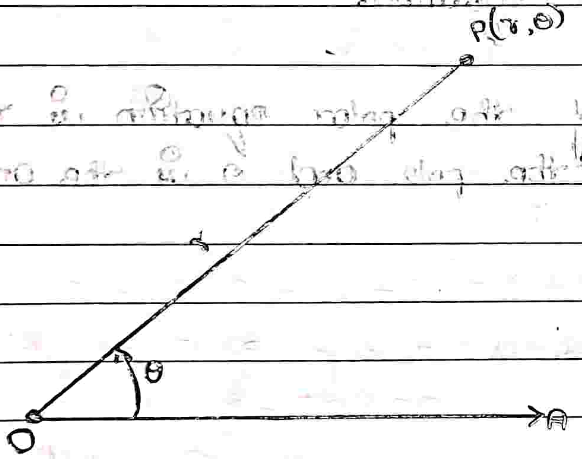
Module-2

Polar curves

Polar Co-ordinate System

The polar Co-ordinate System

- The polar co-ordinates system is a coordinate system in which the co-ordinates of a point in a plane are its distances from a fixed point and its direction from a fixed line. The fixed point is called the origin or the pole and the fixed line is called the polar axis. The co-ordinates given in this way are called polar co-ordinates.

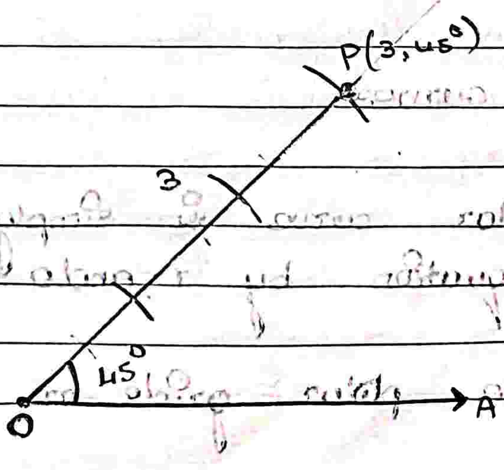


where:

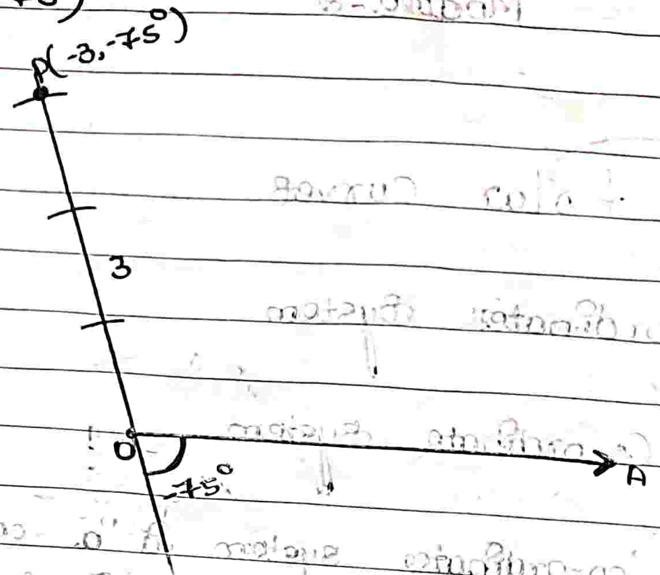
O - origin or pole
OA - polar Axis
r :- Radius vector
θ - vectorial angle
P(r, θ) - polar coordinate

Examples:
Plot the given points on given polar co-ordinate system.

1. $P(3, 45^\circ)$



2. P(-3, -75°)



Polar equation

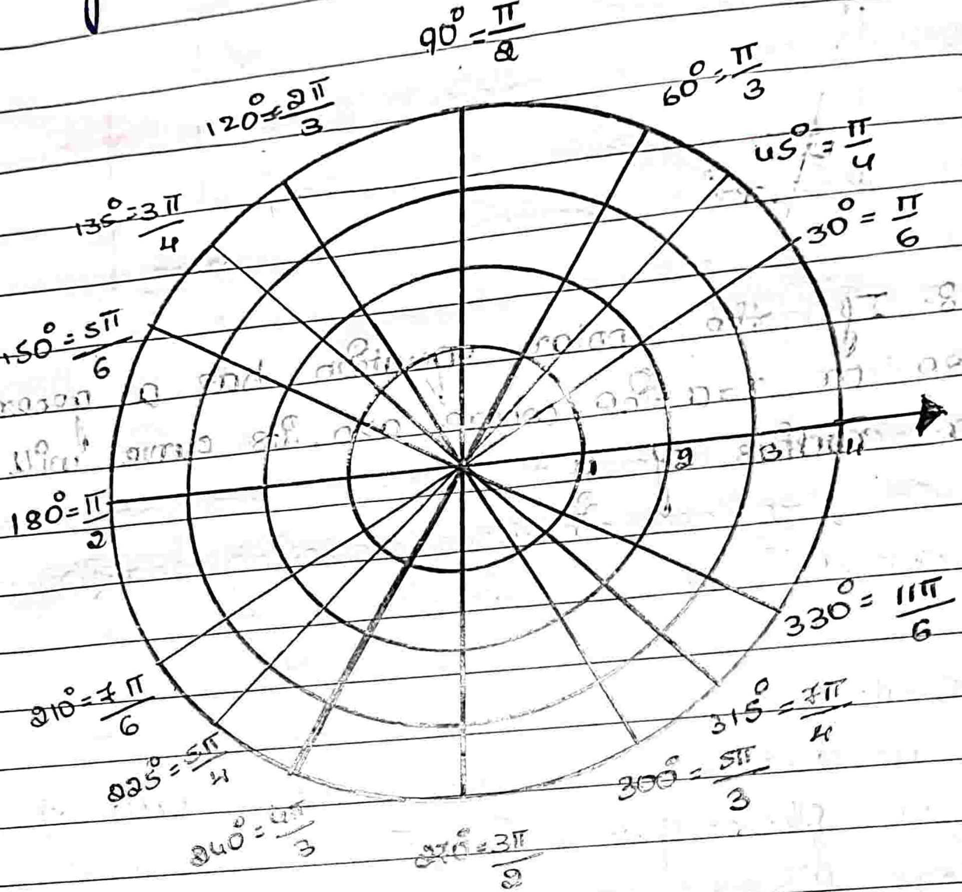
- An equation in the form of polar co-ordinates (r, is known as "polar equation".
- The general form of the polar equation is $r = f(\theta)$, wh r is distance from the pole and θ is the angle.
- Examples :-
- $r = a + a\sin(\theta)$
- $r = a + b\cos(\theta)$
- $r = a e^{b\theta}$

Polar curves

- A polar curve is simply the resulting graph of polar equation by r and θ
- We use polar grids or polar planes to plot the

polar curve.

- This graph is defined by all sets of (r,θ), that satisfy the given polar equation, $r=f(\theta)$.

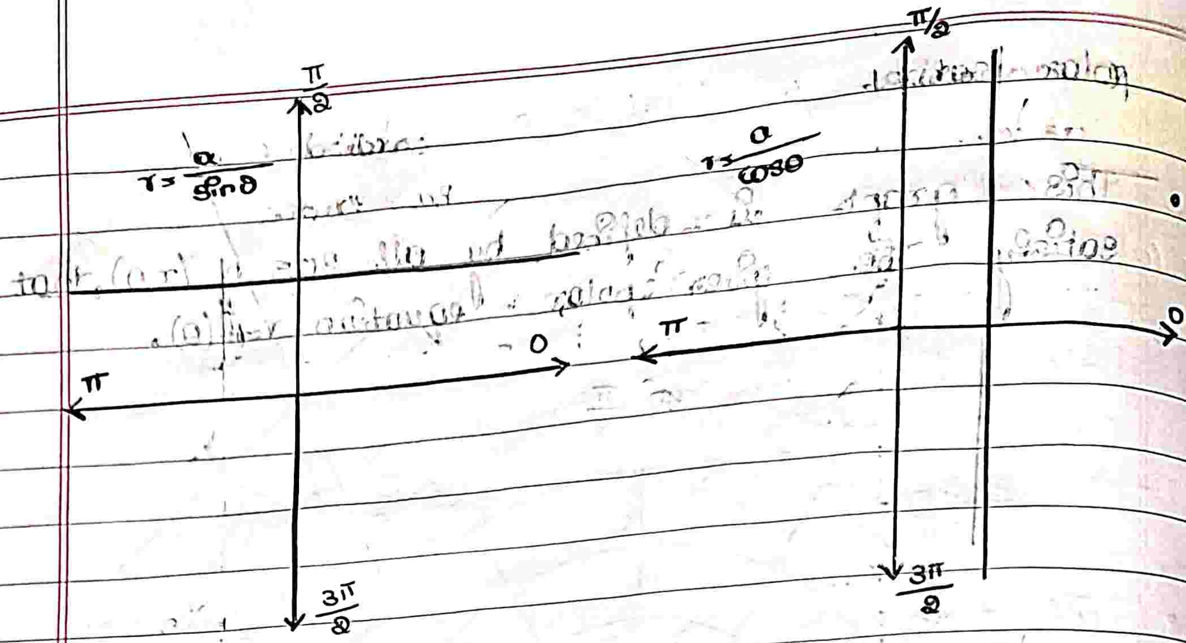


Types of polar curves

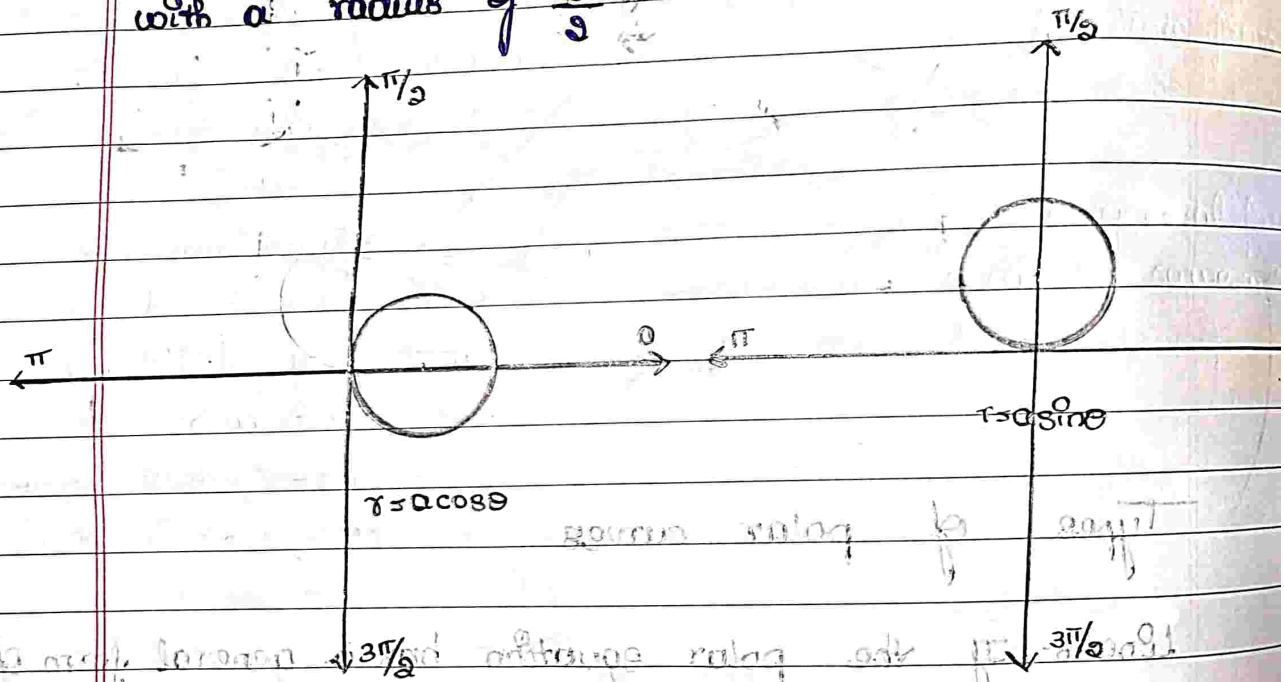
Lines- If the polar equation has a general form of $r = \frac{a}{\sin\theta}$ or $r = \frac{a}{\cos\theta}$ where $a \neq 0$.

- Its curve will be horizontal line or a vertical line respectively.

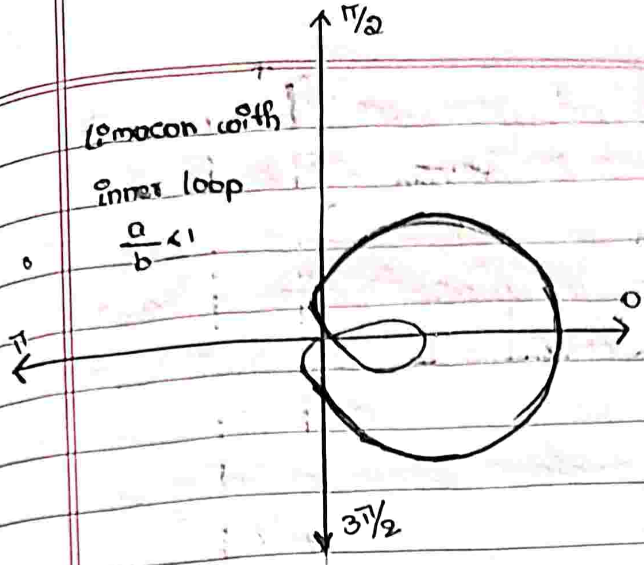
- The line passes through either $(0,a)$ or $(a,0)$.

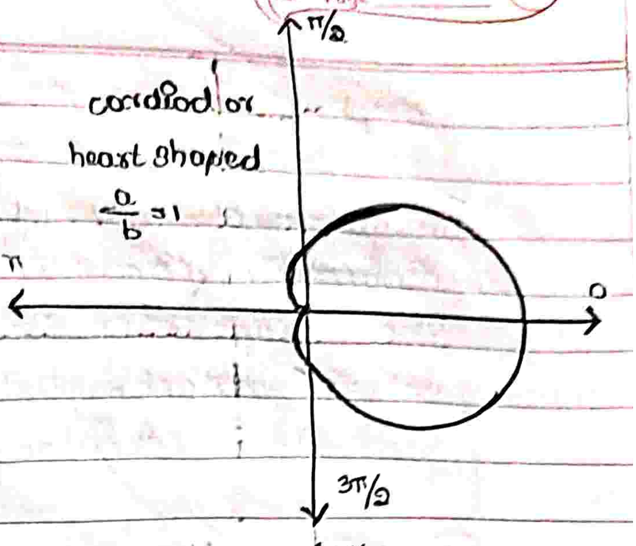


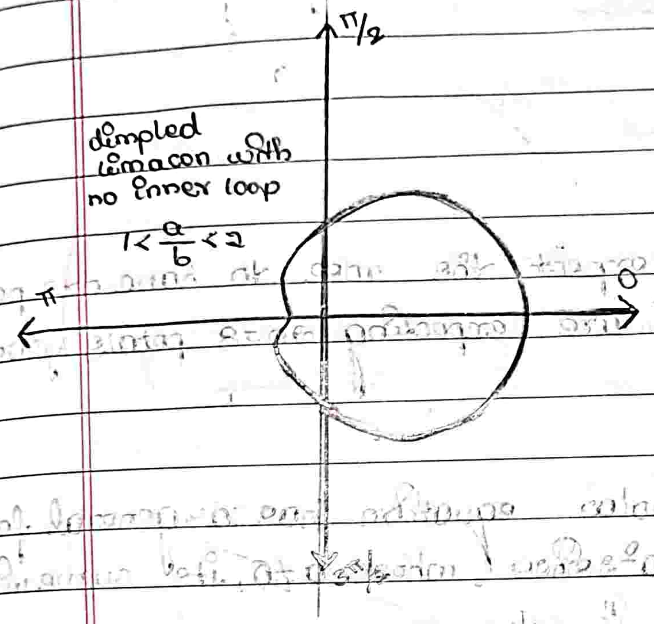
Circles:- If the polar equation has a general form $r = a\cos\theta$ or $r = a\sin\theta$, where $a>0$. Its curve will be a circle with a radius of $\frac{a}{2}$

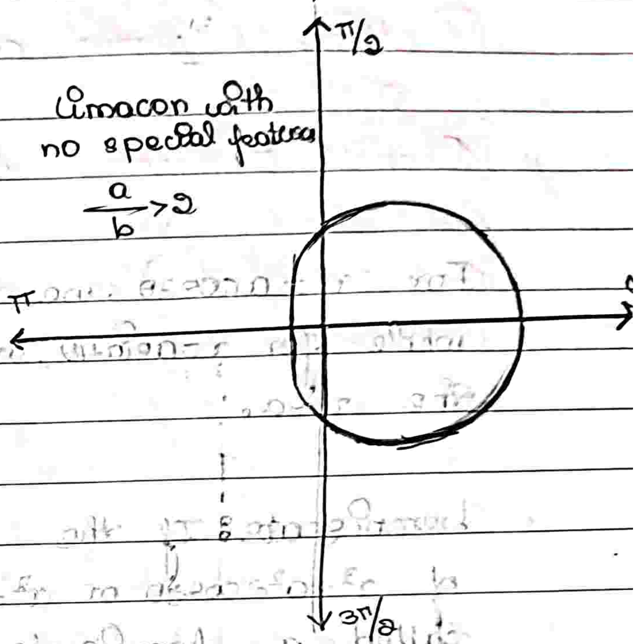


Limacons:- If the polar equation has a general form $r = a \pm b\cos\theta$ or $r = a \pm b\sin\theta$, where $a, b > 0$, its curve will be a curve called the limacon (its French for snail). You'll encounter four variations of limacons and their shape depends on the value of $\frac{a}{b}$.



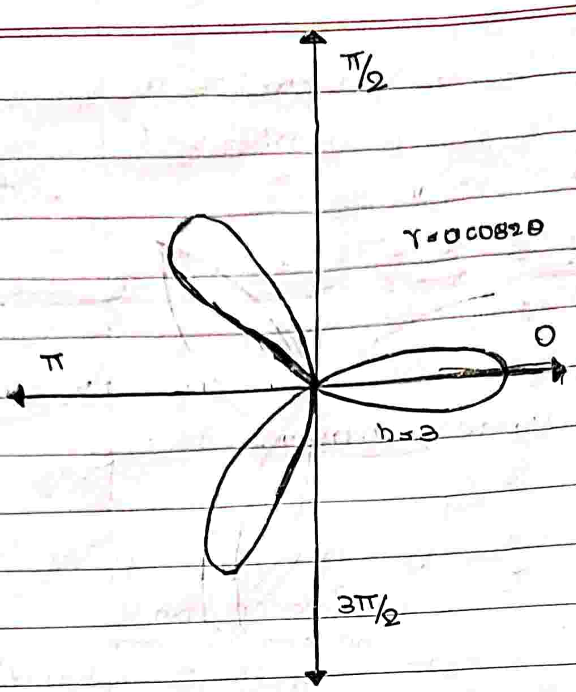


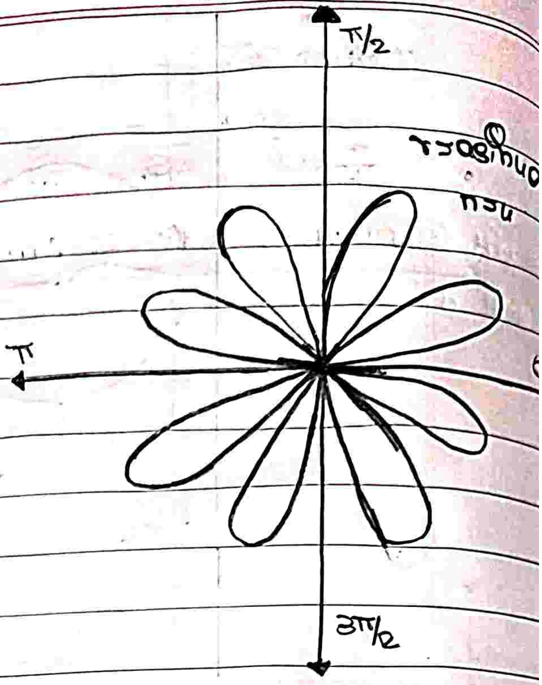




Roses:- If the polar equation has a general form of $r = a\cos n\theta$ or $r = a\sin n\theta$, where $a \neq 0$, its graph will be a polar curve called the rose. We call each segment of the rose petals and the number of petals will depend on whether n is even or odd.

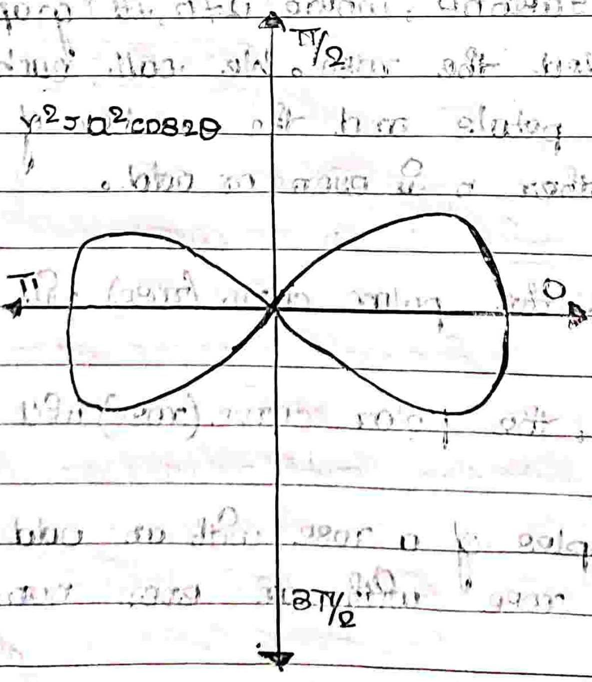
- If n is even, the polar curve (rose) will show 2n petals
- If n is odd, the polar curve (rose) will show n petals
- Here are examples of a rose with an odd number of petals and a rose with an even number of petals.

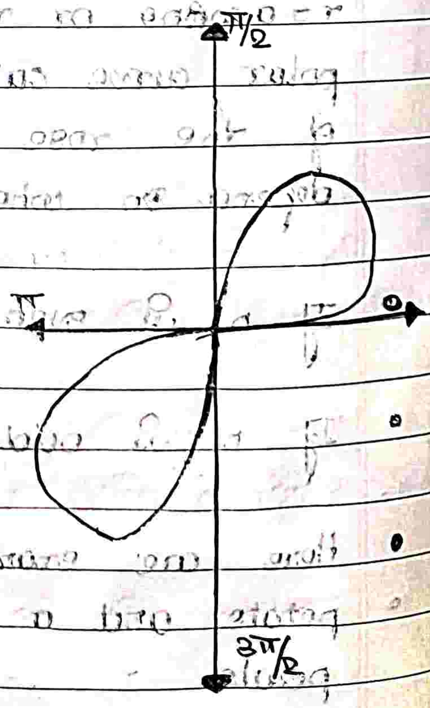




For $r = a\cos 3\theta$, we expect the rose to have $n=3$ peta[ls] while for $r = a\sin 4\theta$, we are expecting $2n = 8$ petals from the rose.

Lemniscate: If the polar equation has a general form of $r^2 = a^2 \cos 2\theta$ or $r^2 = a^2 \sin 2\theta$, where $a \neq 0$, its curve will [be] called a lemniscate.

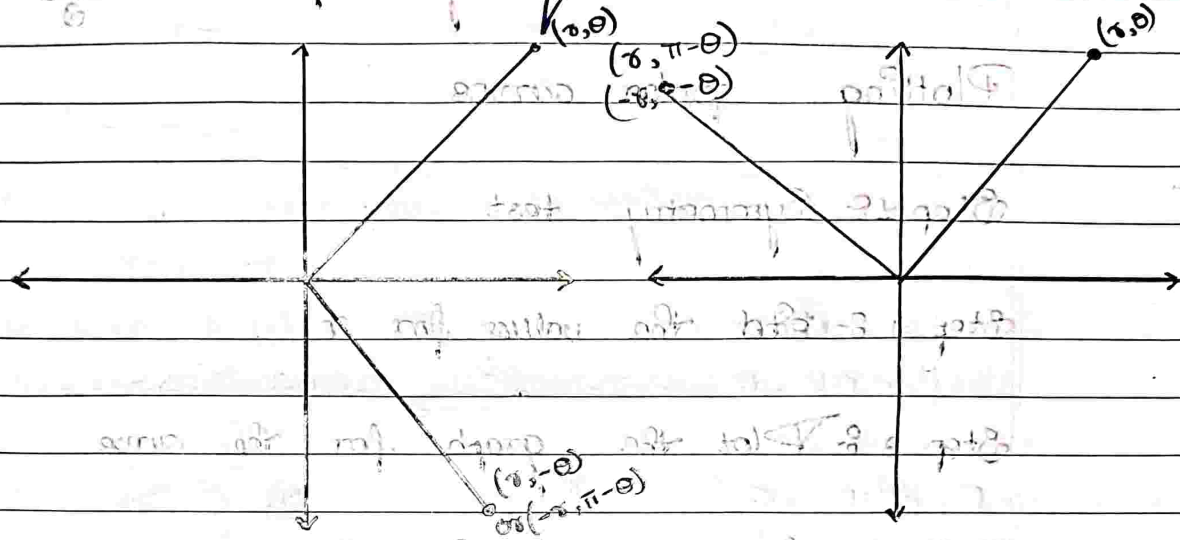


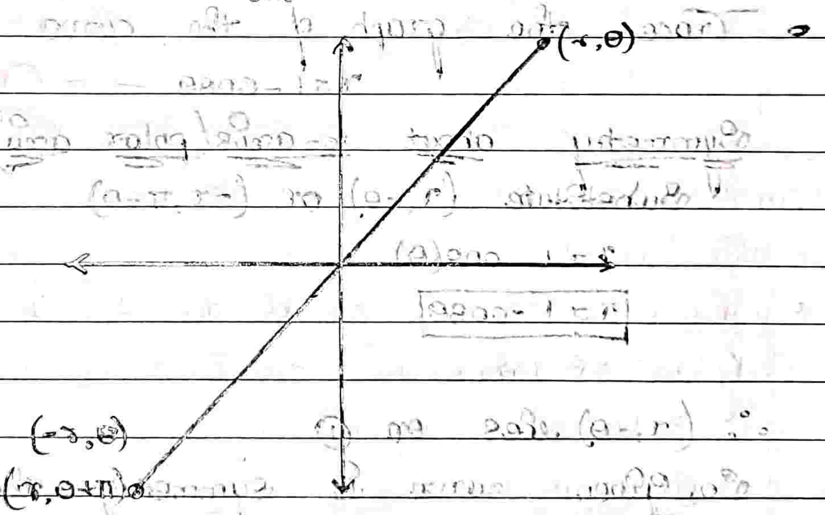


Symmetry test for polar curves

- How do we test for symmetry of an equation in polar coordinates?

- Let us look as the following diagrams to determine the answer to this question.





- Symmetry test for polar graphs.

1. Symmetry about the polar axis (x-axis): If the point (r,θ) lies on the graph, then the point (r,-θ) or (-r, π-θ) also lies on the graph.
   We consider angle θ from 0 to π.

2. Symmetry about the y-axis: If the point (r,θ) lies on the graph, then the point (r, π-θ) or (-r,-θ) also lies on the

graph.

We consider angle θ from $\frac{-\pi}{2}$ to $\frac{\pi}{2}$.

3. Symmetry about the origin: If the point (r,θ) lies the graph, then the point $(-r,\theta)$ or $(r,\pi+\theta)$ also lies on the graph.

We consider angle θ from 0 to $\frac{\pi}{2}$.

Plotting polar curves

Step 1: Symmetry test

Step 2: Find the values for r

Step 3: Plot the graph for the curve

- Trace the graph of the curve $r=1-\cos\theta$

$r=1-\cos\theta$ —— (1)

Symmetry about x-axis (polar axis)

Substitute $(r,-\theta)$ or $(-r,\pi-\theta)$

$r=1-\cos(-\theta)$

$\boxed{r=1-\cos\theta}$

$\therefore$ $(r,-\theta)$ lies on (1)

So, given curve is symmetry about x-axis.

Symmetry about y-axis

Substitute $(-r,\theta)$ or $(r,\pi-\theta)$ in (1)

$-r=1-\cos(-\theta)$

$-r=1-\cos\theta$

The point $(-r,-\theta)$ not lies on the curve.

Sub $(r,\pi-\theta)$ in (1)

$r=1-\cos(\pi-\theta)$

$r=1+\cos\theta$

The point $(r, \pi-\theta)$ not lies on the curve.

∴ Given curve not symmetry about y-axis.

∴ Curve is symmetry about only x-axis. So, it is not symmetry about origin.

So, we consider $\theta : 0$ to π

Let $\theta : 0^\circ, 30^\circ, 45^\circ, 60^\circ, 90^\circ, 120^\circ, 135^\circ, 150^\circ, 180^\circ$

To find values for r

$r = 1 - \cos\theta$

Let $\theta = 0$ in (1)

$r = 1 - \cos 0^\circ = 1 - 1 = 0$

$\boxed{r = 0}$

Let $\theta = 30^\circ$ in (1)

$r = 1 - \cos 30^\circ = 1 - \sqrt{3}/2 = 0.13$

$\boxed{r = 0.13}$

Let $\theta = 45^\circ$ in (1)

$r = 1 - \cos 45^\circ = 1 - \frac{1}{\sqrt{2}} = 0.29$

$\boxed{r = 0.29}$

Let $\theta = 60^\circ$ in (1)

$r = 1 - \cos 60^\circ = 1 - \frac{1}{2} = 0.5$

$\boxed{r = 0.5}$

Let $\theta = 90^\circ$ in (1)

$r = 1 - \cos 90^\circ = 1 - 0 = 1$

$\boxed{r = 1}$

Let $\theta = 120^\circ$

$r = 1 - \cos(120^\circ)$

$= 1 - \cos(90 + 30)$

$= 1 + \sin(30^\circ) = 1 + \frac{1}{2} = 1.5$

$\boxed{r = 1.5}$

Let $\theta = 135^\circ$

$r = 1 - \cos(135^\circ)$

$= 1 - \cos(90 + 45)$

$= 1 + \sin(45^\circ) = 1 + \frac{1}{\sqrt{2}} = 1.7$

$\boxed{r = 1.7}$

Let $\theta = 150^\circ$

$r = 1 - \cos(150^\circ)$

$= 1 - \cos(90 + 60)$

$= 1 + \sin(60) = 1 + \frac{\sqrt{3}}{2} = 1.86$

$\boxed{r = 1.86}$

Let $\theta = 180^\circ$

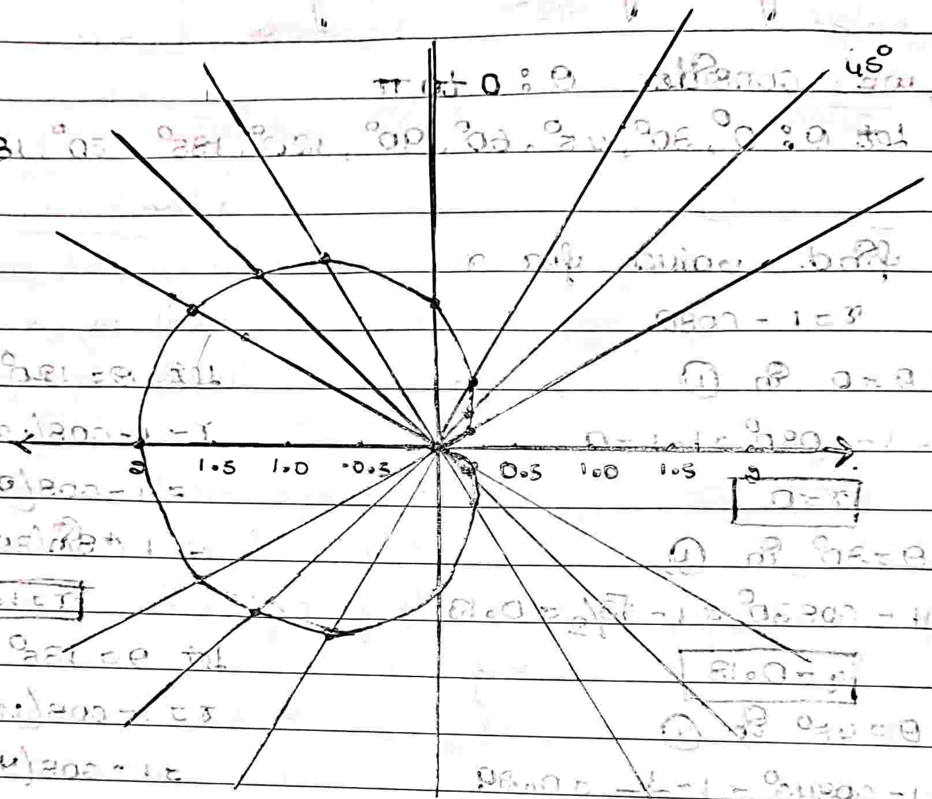
$r = 1 - \cos(180^\circ)$

$r = 1 - (-1)$

$\boxed{r = 2}$

θ	0°	30°	45°	60°	90°
r	0	0.13	0.29	0.5	1

θ	120°	135°	150°	180°
r	1.5	1.7	1.89	2



- Trace the graph of $r = 1 - \sin\theta$

given : $r = 1 - \sin\theta$ — ①

Symmetry about x-axis

Substitute $(r, -\theta)$ or $(-r, \pi - \theta)$ in ①

$r = 1 - \sin(-\theta)$

$r = 1 + \sin\theta$

So, the point $(r, -\theta)$ not lies on the curve

Sub $(-r, \pi - \theta)$ in ①

$-r = 1 - \sin(\pi - \theta)$

$-r = 1 - \sin\theta$

So, the point $(-r, \pi - \theta)$ not lie on the curve

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$\therefore$ The curve is not symmetry about x-axis.

Symmetry about y-axis

Sub $(-r, -\theta)$ (or) $(r, \pi-\theta)$ in ①

$-r = 1 - \sin(-\theta)$

$-r = 1 + \sin\theta$

So, the point $(-r, -\theta)$ not lies the curve.

Sub $(r, \pi-\theta)$ in ①

$r = 1 - \sin(\pi-\theta)$

$r = 1 - \sin\theta$

$\therefore (r, \pi-\theta)$ lies on ①

So, the given curve is symmetry about y-axis.

$\therefore$ Curve is symmetry about only y-axis. So, it is not symmetry about origin.

So, we consider : $\frac{-\pi}{2}$ to $\frac{\pi}{2}$

Let θ : $-\frac{\pi}{2}(-90^\circ)$, -60°, -45°, -30°, 0, 30°, 45°, 60°, 90°.

$0 \quad 30 \quad 45 \quad 60 \quad 90$

$0, \frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}, 1$

To find the values of r

$r = 1 - \sin\theta$

Let $\theta = -90^\circ$ in ①

$r = 1 - \sin(-90^\circ)$

$= 1 + 1$

$\boxed{r = 2}$

Let $\theta = -60^\circ$

$r = 1 - \sin(-60^\circ)$

$= 1 + \frac{\sqrt{3}}{2}$

$\boxed{r = 1.86}$

Let $\theta = -45^\circ$ in ①

$r = 1 - \sin(-45^\circ)$

$= 1 + \frac{1}{\sqrt{2}}$

$\boxed{r = 1.7}$

Let $\theta = -30^\circ$

$r = 1 - \sin(-30^\circ)$

$= 1 + \frac{1}{2}$

$\boxed{r = 1.5}$

Let $\theta = 0$ in ①

$r = 1 - \sin 0$

$= 1 - 0$

$\boxed{r = 1}$

Let $\theta = 30^\circ$ in ①

$r = 1 - \sin 30^\circ$

$= 1 - \frac{1}{2}$

$\boxed{r = 0.5}$

Let $\theta = 45^\circ$

$r = 1 - \sin 45^\circ$

$= 1 - \frac{1}{\sqrt{2}}$

$\boxed{r = 0.29}$

Let $\theta = 60^\circ$

$r = 1 - \sin 60^\circ$

$= 1 - \frac{\sqrt{3}}{2}$

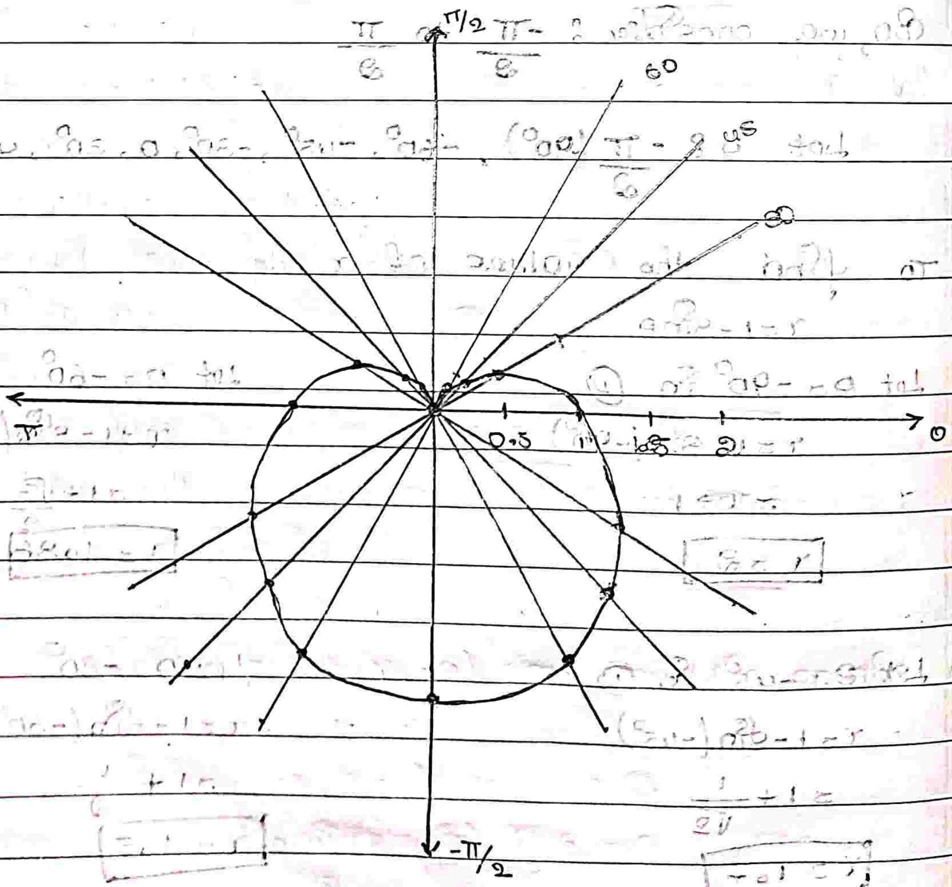
$\boxed{r = 0.13}$

Let $\theta = 90^\circ$

$r = 1 - \sin 90^\circ$

$\boxed{r = 0}$

θ	-90°	-60°	-45°	-30°	0	30°	45°	60°
r	2	1.86	1.7	1.5	1	0.5	0.29	0.13



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- Plot the graph of $r^2 = \cos\theta$

given: $r^2 = \cos\theta$ — ①

Symmetry about x-axis

Sub $(+r, \theta)$ or $(-r, \pi-\theta)$ in ①

$r^2 = \cos(-\theta)$

$r^2 = \cos\theta$

∴ So, the point $(r, -\theta)$ lies on ①

So, given curve is symmetry about x-axis

Symmetry about y-axis

Sub $(-r, -\theta)$ or $(r, \pi-\theta)$ in ①

$(-r)^2 = \cos(-\theta)$

$r^2 = \cos\theta$

∴ the point $(r, -\theta)$ lies on ①

So, given curve is symmetry about y-axis

∴ The given curve is symmetry both x-axis and y-axis. So, it also symmetry about origin.

We consider: 0 to $90^\circ (\pi/2)$

The θ is $0, 30^\circ, 45^\circ, 60^\circ, 90^\circ$.

To find the values of r.

$r^2 = \cos\theta$

Let $\theta = 0$ in ①

$r^2 = \cos 0$

$r = \sqrt{1}$

$r = 1$

Let $\theta = 30^\circ$ in ①

$r^2 = \cos 30$

$= \frac{\sqrt{3}}{2}$

$r = \pm 0.92$

Let $\theta = 45^\circ$

$r^2 = \cos 45$

$r = \pm 0.84$

Let $\theta = 60^\circ$

$r^2 = \cos 60^\circ$

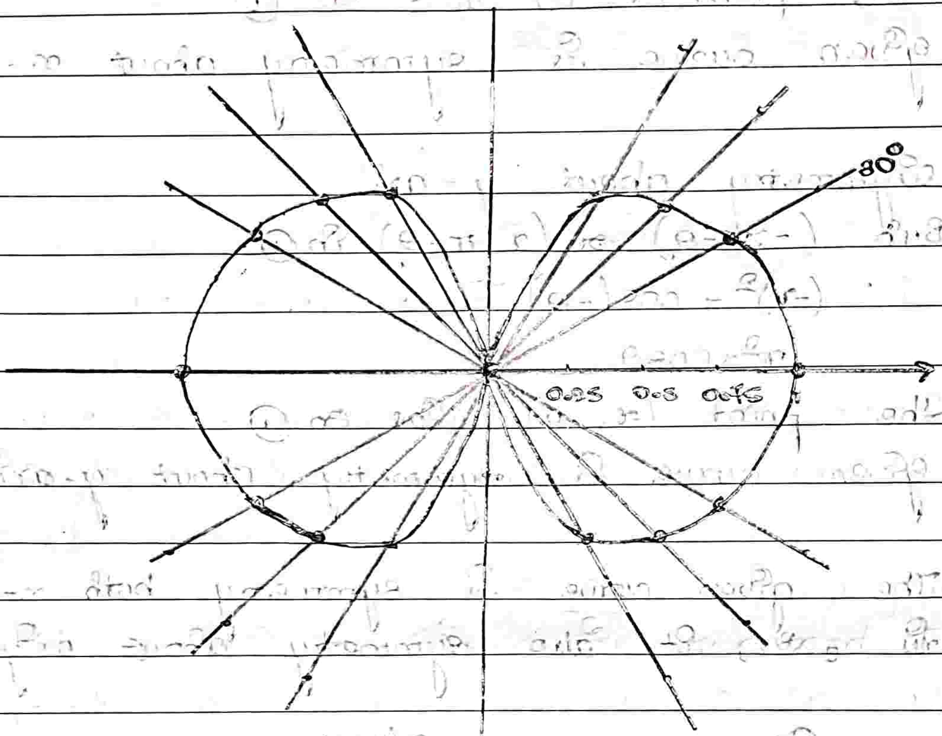
$r = 0.7$

Let $\theta = 90^\circ$

$r^2 = \cos 90^\circ$

$r = 0$

θ	0	30°	45°	60°	90°
r	± 1	± 0.9	± 0.8	± 0.7	0



- Plot the graph of $r^2 = \sin\theta$

given : $r^2 = \sin\theta$ —— (1)

Symmetry about x-axis

Sub $(r, -\theta)$ or $(-r, \pi-\theta)$ in eqn (1)

$r^2 = \sin(-\theta)$

$r^2 = -\sin\theta$

$\therefore$ the point $(r, -\theta)$ does not lie on the curve

Sub $(-r, \pi-\theta)$ in eqn (1)

$(-r)^2 = \sin(\pi-\theta)$

$r^2 = \sin\theta$

$\therefore$ the point $(-r, \pi-\theta)$ lies on the curve

So, the given curve is symmetry about x-axis

Symmetry about y-axis

Sub $(-r, -\theta)$ or $(r, \pi-\theta)$ in eq^n (1)

$(-r)^2 = \sin(-\theta)$

$r^2 = -\sin\theta$

$\therefore$ the point $(-r, -\theta)$ does not lie on the curve

Sub $(r, \pi-\theta)$ in eq^n (1)

$r^2 = \sin(\pi-\theta)$

$r^2 = \sin\theta$

$\therefore$ the point $(r, \pi-\theta)$ lies on the curve

So the given curve is symmetry about y-axis

$\therefore$ The given curve is symmetry about both x-axis and y-axis so, the given curve is symmetry about origin

We consider θ 0 to 90°

Let θ :- 0 30° 45° 60° 90°

To find r

$r^2 = \sin\theta$ — (1)

Let $\theta = 0$ in (1)

$r^2 = \sin 0$

$\boxed{r = 0}$

$r^2 = \sin\theta$ — (1)

Let $\theta = 30°$ in (1)

$r^2 = \sin 30°$

$\boxed{r = \pm 0.7}$

Let $\theta = 45°$ in (1)

$r^2 = \sin 45°$

$\boxed{r = \pm 0.84}$

Let $\theta = 60°$ in (1)

$r^2 = \sin 60°$

$\boxed{r = \pm 0.93}$

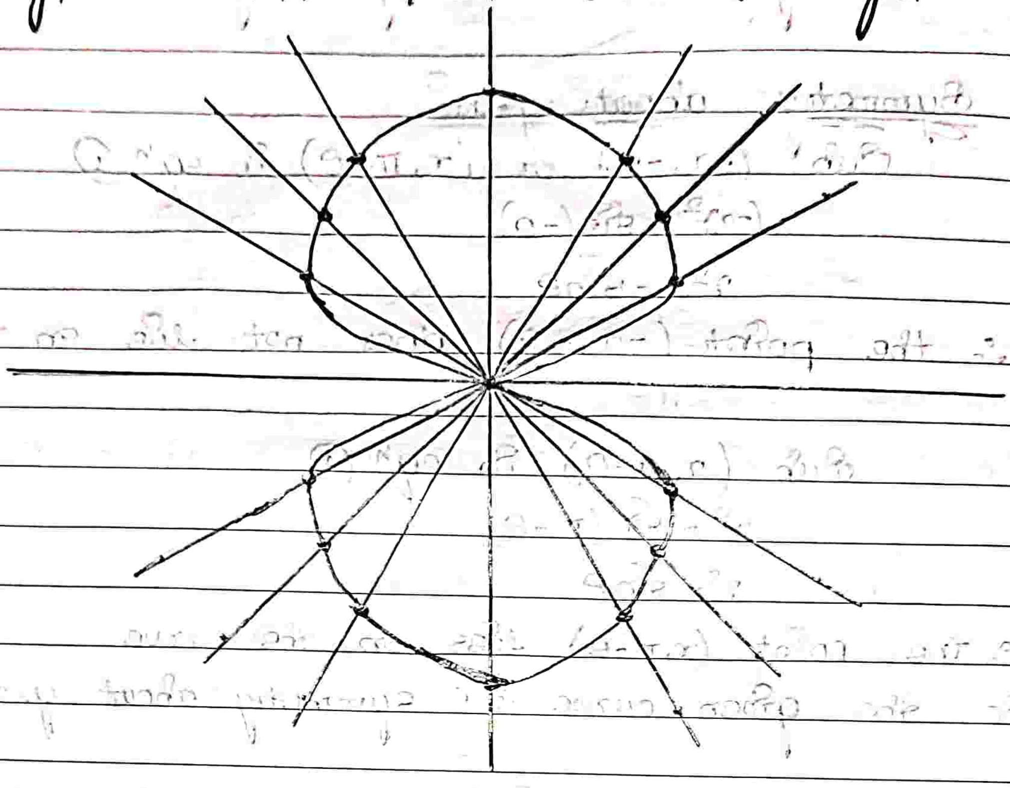
Let $\theta = 90°$ in (1)

$r^2 = \sin 90°$

$\boxed{r = \pm 1}$

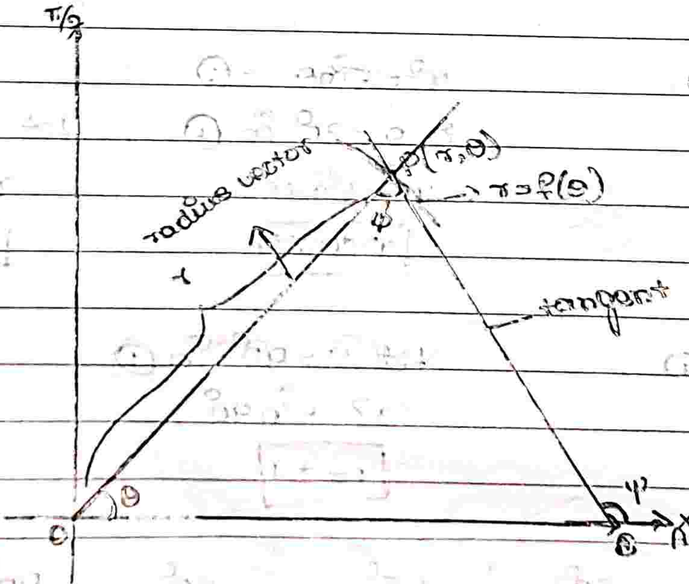
θ	0	30°	45°	60°	90°
r	0	± 0.7	± 0.84	± 0.9	± 1

Angle between radius vector & tangent



Angle between radius vector and tangent to the polar curve

Let $r = f(\theta)$ be the polar curve and $P(r,\theta)$ be the point on the polar curve



From fig, let $\angle OPB = \phi$, $\angle POB = \theta$, $\angle PBA = \psi$.

From $\triangle OPB$, we have

$$\psi = \phi + \theta$$

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apply tan on B.S

$\tan\psi = \tan(\theta+\phi)$

$\tan\psi = \dfrac{\tan\theta + \tan\phi}{1-\tan\theta\tan\phi}$ ——— ①

$\tan\psi$ is slope of tangent

$\tan\psi = \dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

$\because p(r,\theta),\ r = f(\theta)$

Cartesian form of $P(r,\theta)$ is

$x = r\cos\theta,\ y = r\sin\theta$

$x = f(\theta)\cos\theta,\ y = f(\theta)\sin\theta$

$\dfrac{dy}{d\theta} = \dfrac{d}{d\theta}(\underset{u}{f(\theta)}\cdot\underset{v}{\sin\theta})$

$\boxed{\dfrac{dy}{d\theta} = f(\theta)\cos\theta + \sin\theta f'(\theta)}$

$\dfrac{dx}{d\theta} = \dfrac{d}{d\theta}(\underset{u}{f(\theta)}\underset{v}{\cos\theta})$

$\dfrac{dx}{d\theta} = f(\theta)(-\sin\theta) + \cos\theta f'(\theta)$

$\tan\psi = \dfrac{f(\theta)\cos\theta + \sin\theta f'(\theta)}{-f(\theta)\sin\theta + \cos\theta f'(\theta)}$

divide by $\cos\theta f'(\theta)$

$$\tan\psi = \dfrac{\dfrac{f(\theta)\cos\theta + \sin\theta f'(\theta)}{\cos\theta f'(\theta)}}{\dfrac{-f(\theta)\sin\theta + \cos\theta f'(\theta)}{\cos\theta f'(\theta)}}$$

$$\tan\psi = \dfrac{\dfrac{f(\theta)}{f'(\theta)} + \tan\theta}{-\dfrac{f(\theta)}{f'(\theta)}\tan\theta + 1}$$

On comparing eqn ①

$$\tan\psi = \frac{\tan\theta + \frac{f(\theta)}{f'(\theta)}}{1 - \tan\theta \frac{f(\theta)}{f'(\theta)}}$$

$$\frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} = \frac{\tan\theta + \frac{f(\theta)}{f'(\theta)}}{1 - \tan\theta \frac{f(\theta)}{f'(\theta)}}$$

$$\tan\phi = \frac{f(\theta)}{f'(\theta)}$$

$$= \frac{r}{r'}$$

$$= \frac{r}{\frac{dr}{d\theta}}$$

$$\boxed{\tan\phi = r \cdot \frac{d\theta}{dr}}$$

- For the cardiod $r = a(1-\cos\theta)$ prove that $\phi = \theta/2$

Soln : $r = a(1-\cos\theta)$ —— ①

Given : P.T $\phi = \theta/2$

W.K.T $\tan\phi = r \cdot \frac{d\theta}{dr}$

$$\frac{dr}{d\theta} = \frac{d}{d\theta}(a(1-\cos\theta))$$

$$= a(0+\sin\theta)$$

$$\frac{dr}{d\theta} = a\sin\theta$$

$$\frac{d\theta}{dr} = \frac{1}{a\sin\theta}$$

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$$\tan\phi = \cancel{a}(1-\cos\theta) \cdot \frac{1}{\cancel{a}\sin\theta}$$

$$= \frac{1-\cos\theta}{\sin\theta}$$

$\cos 2\theta = 1 - 2\sin^2\theta$
$= 2\cos^2\theta - 1$

$$= \frac{\cancel{1}-\cancel{1}+\cancel{2}\sin^{\cancel{2}}\theta/2}{\cancel{2}\sin\theta/2 \cdot \cos\theta/2}$$

$\sin 2\theta = 2\sin\theta \cdot \cos\theta$

$$\tan\phi = \frac{\sin\theta/2}{\cos\theta/2}$$

$$\tan\phi = \tan\theta/2$$

$$\therefore \boxed{\phi = \theta/2}$$

- Prove that $\phi = \pi/2 + \theta/2$ for the polar curve $r = a(1+\cos\theta)$

Sol^n: $r = a(1+\cos\theta)$ —— (1)

given: P.T $\phi = \pi/2 + \theta/2$

W.K.T $\tan\phi = r \cdot \frac{d\theta}{dr}$

$$\frac{dr}{d\theta} = \frac{d}{d\theta}(a(1+\cos\theta))$$

$$= a(0-\sin\theta)$$

$$\frac{dr}{d\theta} = -a\sin\theta$$

$$\frac{d\theta}{dr} = \frac{-1}{a\sin\theta}$$

$$\tan\phi = \cancel{a}(1+\cos\theta) \cdot \frac{-1}{\cancel{a}\sin\theta}$$

$$= \frac{-(1+\cos\theta)}{\sin\theta}$$

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$$= \frac{\cancel{1} + \cancel{2}\cos^2\theta/2 - \cancel{1}}{\cancel{2}\sin\theta/2 \cos\theta/2}$$

$\cos 2\theta = 1 - 2\sin^2\theta$
$= 2\cos^2\theta - 1$

$\tan\phi = \cot \theta/2$

$\cancel{\tan}\phi = \cancel{\tan}(\pi/2 + \theta/2)$

$$\boxed{\phi = \pi/2 + \theta/2}$$

- Prove that for parabola $\frac{2a}{r} = 1 - \cos\theta$, $\phi = \pi - \theta/2$

Sol^n:
given: $\frac{2a}{r} = 1 - \cos\theta$ —— (1)
$\phi = \pi - \theta/2$

W.K.T $\tan\phi = r \cdot \frac{d\theta}{dr}$

applying log on B.S of eq^n (1)

$$\log\left(\frac{2a}{r}\right) = \log(1 - \cos\theta)$$

$$\log(2a) - \log r = \log(1 - \cos\theta)$$

diff w.r.t θ

$$\frac{d}{d\theta}(\log(2a) - \log r) = \frac{d}{d\theta}(\log(1 - \cos\theta))$$

$$0 - \frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{1}{1 - \cos\theta} \cdot \sin\theta$$

$$-\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1 - \cos\theta}$$

$$-\frac{1}{r}\frac{dr}{d\theta} = \frac{\cancel{2}\sin\theta/2 \cos\theta/2}{\cancel{1} - \cancel{1} + \cancel{2}\sin^2\theta/2}$$

$\cos 2\theta = 1 - 2\sin^2\theta/2$
$= 2\cos^2\theta/2 - 1$
$\sin 2\theta = 2\sin\theta/2 \cos\theta/2$

$$= \frac{\cos\theta/2}{\sin\theta/2}$$

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$$-\frac{1}{r}\frac{dr}{d\theta} = \cot\theta/2$$

$$\frac{1}{r}\frac{dr}{d\theta} = -\cot\theta/2$$

$$r\frac{d\theta}{dr} = -\tan\theta/2$$

$$\tan\phi = -\tan\theta/2$$

$$\tan\phi = \tan(\pi - \theta/2)$$

$$\boxed{\phi = \pi - \theta/2}$$

Note:-

- If $f(\theta) = \log g(\theta)$, then $f'(\theta) = \frac{1}{g(\theta)} g'(\theta)$
- If $f(\theta) = \sin g(\theta)$, then $f'(\theta) = \cos g(\theta) \cdot g'(\theta)$
- If $f(\theta) = \cos g(\theta)$, then $f'(\theta) = -\sin g(\theta) \cdot g'(\theta)$
- If $f(\theta) = \sin(\log\theta)$, then $f'(\theta) = \cos(\log\theta) \cdot \frac{1}{\theta}$

- Show that in the equiangular spiral $r = a(e^{\theta\cot\alpha})$, the tangent is inclined at constant angle to the radius vector.

Given:- $r = ae^{\theta\cot\alpha}$ — ①

applying

P.T:- ϕ = constant

applying log on both s in eqn ①

$$\log r = \log(a e^{\theta\cot\alpha})$$

$$\log r = \log a + \log_a e^{\theta\cot\alpha}$$

$$\log r = \log a + \theta\cot\alpha \log_e e$$

$$\log r = \log a + \theta\cot\alpha \cdot 1$$

apply $\frac{d}{d\theta}$ on b.s

$$\frac{d}{d\theta}\log r = \frac{d}{d\theta}(\log a + \theta\cot\alpha \cdot 1)$$

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \cot\alpha(1)$$

$$\frac{1}{r}\frac{dr}{d\theta} = \cot\alpha$$

$$r\frac{d\theta}{dr} = \tan\alpha$$

$$\tan\phi = \tan\alpha$$

$$\boxed{\phi = \alpha}$$

$\therefore\ \alpha = \text{constant}$

- Show that the tangent to the cardioid $r = a(1+\cos\theta)$ at the point $\theta = \pi/3$ and $\theta = 2\pi/3$ are parallel and perpendicular to the $\underbrace{\text{initial line}}_{\text{polar axis (x-axis)}}$ respectively.

Given : $r = a(1+\cos\theta)$ —— ①

P.T : $\phi = 0$ & $\phi = 90^\circ$ or $\pi/2$

W.K.T. $\tan\phi = r\cdot\frac{d\theta}{dr}$

$$\frac{dr}{d\theta} = \frac{d}{d\theta}(a(1+\cos\theta))$$

$$\frac{dr}{d\theta} = a(0 - \sin\theta)$$

$$\frac{dr}{d\theta} = -a\sin\theta$$

$$\frac{d\theta}{dr} = \frac{1}{-a\sin\theta}$$

$$r\cdot\frac{d\theta}{dr} = \frac{\cancel{a}(1+\cos\theta)}{-\cancel{a}\sin\theta}$$

$$= \frac{1+2\cos^2\theta/2 - 1}{2\sin\theta/2\,\cos\theta/2}$$

$$\tan\phi = \tan(\pi/2 + \theta/2)$$

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$\phi = \frac{\pi}{2} + \theta/2 \text{ ---- } ③$

Sub $\theta = \pi/3$ in ϕ

$\phi = \pi/2 + \pi/6$

$= \frac{3\pi + \pi}{6}$

$= \frac{\cancel{4}^{2}\pi}{\cancel{6}_{3}}$

$\phi = \frac{2\pi}{3}$

<u>Slope :</u> $\tan\psi = \tan(\theta+\phi)$

$= \tan(\pi/3 + \frac{2\pi}{3})$

$= \tan\pi$

$= \tan(90+90)$

$\tan\psi = -\cot 90^\circ$

$\boxed{\tan\psi = 0}$

$\cot 90^\circ = 0$

$\therefore$ If the slope of tangent is 0 then it is parallel to initial line

Sub $\theta = 2\pi/3$ in ②

$\phi = \pi/2 + \frac{2\pi}{6}$

$= \frac{3\pi + 2\pi}{6}$

$= 5\pi/6$

Slope : $\tan\psi = \tan(\theta+\phi)$

$= \tan(2\pi/3 + 5\pi/6)$

$= \tan\left(\frac{4\pi + 5\pi}{6}\right)$

$= \tan\left(\frac{\cancel{9}^{3}\pi}{\cancel{6}_{2}}\right)$

$= \tan\left(\frac{3\pi}{2}\right)$

$= \tan(\pi + \pi/2)$

$= \cancel{-\cot}$

$= \tan(\pi/2)$

$\psi = \pi/2$

The slope of curve is 90° so it is $\perp^r$ to the initial line

- find angle between tangent at any point P and the line joining P to the origin of the curve $\log(x^2+y^2) = k\tan^{-1}\left(\frac{y}{x}\right)$

Sol^n:

given curve:

$$\log(x^2+y^2) = k\tan^{-1}\left(\frac{y}{x}\right) \quad \text{—— (1)}$$

Sub $x = r\cos\theta$, $y = r\sin\theta$ in (1)

$$\log(r^2\cos^2\theta + r^2\sin^2\theta) = k\tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta}\right)$$

$$\log(r^2 \cdot 1) = k\tan^{-1}(\tan\theta)$$

$$\log r^2 = k\theta \quad \text{—— (2)}$$

To find angle b/w tangent and radius vector

$$\tan\phi = r \cdot \frac{d\theta}{dr}$$

from (2)

$\log r^2 = k\theta$

diff w.r.t θ

$$\frac{d}{d\theta}\log r^2 = \frac{d}{d\theta}k\theta$$

$$\frac{2\cancel{r}}{r^{\cancel{2}}} \cdot \frac{dr}{d\theta} = k \cdot 1$$

$$\frac{2}{r} \cdot \frac{dr}{d\theta} = k$$

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{k}{a}$$

$$r\cdot\frac{d\theta}{dr} = \frac{a}{k}$$

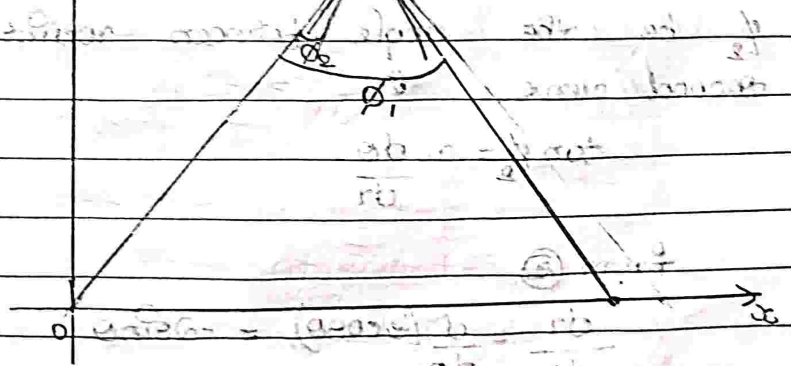
$$\tan\phi = \frac{a}{k}$$

$$\boxed{\phi = \tan^{-1}\left(\frac{a}{k}\right)}$$

- $\tan\psi = \tan(\theta+\phi)$
- $\tan\phi = r\cdot\frac{d\theta}{dr}$
- $x = r\cos\theta$, $y = r\sin\theta$

Angle between intersection of two curves

Let $C_1 : r = f_1(\theta)$ and $C_2 : r = f_2(\theta)$ be the two curves intersecting at a point P. Let PT_1 be the tangent to curve C_1 at a point P and also PT_2 be the tangent to curve C_2 at a point P.



From dia. , angle between inte. of two

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~~It equal the angle between~~ point of intersection is equal to angle between their tangents at point of intersection to the angle between intersections of two curves is

$$|\phi_2 - \phi_1|.$$

It is an acute angle.
If this angle is 90, then the two curves are orthogonal.

- Find the angle of intersection of the curves $r = 2\sin\theta$, $r = 2\cos\theta$

Solⁿ:
Given:

$r = 2\sin\theta$ —— ①

$r = 2\cos\theta$ —— ②

Let ϕ_1 be the angle between radius and tangent to the first curve.

$$\tan\phi_1 = r\frac{d\theta}{dr}$$

$$\frac{dr}{d\theta} = \frac{d}{d\theta}(2\sin\theta) = 2\cos\theta$$

$$\tan\phi_1 = \cancel{2}\sin\theta \cdot \frac{1}{\cancel{2}\cos\theta}$$

$$\cancel{\tan}\phi_1 = \cancel{\tan}\theta$$

$$\phi_1 = \theta$$

Let ϕ_2 be the angle between radius and tangent to the second curve.

$$\tan\phi_2 = r.\frac{d\theta}{dr}$$

from ②

$$\frac{dr}{d\theta} = \frac{d}{d\theta}(2\cos\theta) = -2\sin\theta$$

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$$\tan\phi_2 = \cancel{a}\cos\theta \cdot \frac{-1}{\cancel{a}\sin\theta}$$

$$\tan\phi_2 = -\cot\theta$$

$$\tan\phi_2 = \tan\left(\frac{\pi}{2}+\theta\right)$$

$$\phi_2 = \frac{\pi}{2}+\theta$$

Angle between intersection of two curves

$$|\phi_2-\phi_1| = \left|\pi/_2+\cancel{\theta}-\cancel{\theta}\right| = \frac{\pi}{2}$$

- Prove that the curves $r = a(1+\cos\theta)$ and $r=b(1-\cos\theta)$ intersect at right angles.

Solⁿ:

Given:

$$r = a(1+\cos\theta)$$

$$r = b(1-\cos\theta)$$

Let ϕ_1 be the angle between radius and tangent to the first curve.

$$\tan\phi_1 = r\cdot\frac{d\theta}{dr}$$

$$\frac{dr}{d\theta} = \frac{d}{d\theta}\left(a(1+\cos\theta)\right)$$

$$= a(-\sin\theta)$$

$$\boxed{= -a\sin\theta}$$

$$\tan\phi_1 = -a\sin\theta$$

$$\tan\phi_1 = \cancel{a}(1+\cos\theta)\cdot\frac{1}{-\cancel{a}\sin\theta}$$

$$= \frac{1+\cos\theta}{-\sin\theta}$$

$$= \frac{\cancel{2}\cos^2\theta/_2 \cancel{-1} \cancel{+1}}{-\cancel{2}\sin\theta/_2\cos\theta/_2}$$

$$= \frac{\cos\theta}{-\sin\theta}$$

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$\tan\phi_1 = -\cot\theta/2$

$\cancel{\tan}\phi_1 = \cancel{\tan}(\pi/2 + \theta/2)$

$\phi_1 = (\pi/2 + \theta/2)$

$(-\theta/2 - \pi/2$

$= \pi/2 + \theta/2$

Let ϕ_2 be the angle between radius and tang to the second curve

$\tan\phi_2 = r \cdot \frac{d\theta}{dr}$

$\frac{dr}{d\theta} = \frac{d}{d\theta}(b(1-\cos\theta))$

$= b\sin\theta$

$\tan\phi_2 = \cancel{b}(1-\cos\theta) \cdot \frac{1}{\cancel{b}\sin\theta}$

$= \frac{1-\cos\theta}{\sin\theta}$

$= \frac{\cancel{1}-(\cancel{1}-2\sin^2\theta/2)}{2\sin\theta/2\cos\theta/2}$

$= \frac{\sin\theta/2}{\cos\theta/2}$

$\cancel{\tan}\phi_2 = \cancel{\tan}\theta/2$

$\boxed{\phi_2 = \theta/2}$

Angle between intersection of two curves

$|\phi_2 - \phi_1| = |\cancel{\theta/2} - (\pi/2 + \cancel{\theta/2})|$

$= |-\pi/2|$

$= \pi/2$

The given two curves intersect at right angle

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- Show that the two curves $r = \frac{a}{1+\cos\theta}$ & $r = \frac{b}{1-\cos\theta}$ intersect at right angles.

Soln:

Given: $r = \frac{a}{1+\cos\theta}$

$r = \frac{b}{1-\cos\theta}$

Let ϕ_1 be the angle between radius and tangent to the first curve

$$r = \frac{a}{1+\cos\theta}$$

apply log on B.S

$$\log r = \log\left(\frac{a}{1+\cos\theta}\right)$$

$$\log r = \log a - \log(1+\cos\theta)$$

$$\tan\phi_1 = r\cdot\frac{d\theta}{dr}$$

diff w.r.t θ

$$\frac{d}{d\theta}\log r = \frac{d}{d\theta}\log a - \frac{d}{d\theta}\log(1+\cos\theta)$$

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 - \frac{1}{1+\cos\theta}\cdot -\sin\theta$$

$$r\cdot\frac{d\theta}{dr} = \boxed{\frac{\sin\theta}{1+\cos\theta}} = \frac{1+\cos\theta}{\sin\theta}$$

$$\tan\phi_1 = \frac{2\sin\theta/2\cos\theta/2}{(1+\cos^2\theta/2)/1} = \frac{\sin\theta/2}{\cos\theta/2} = \frac{\cos\theta/2}{\sin\theta/2}$$

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$$\tan \phi_1 = \cancel{\tan\theta/2} \cot \theta/2$$
$$\phi_1 = \pi/2 - \theta/2$$

Let ϕ_2 be the angle between the radius and tangent

$$\log r = \log b - \log(1-\cos\theta)$$
diff
$$\frac{1}{r}\frac{dr}{d\theta} = 0 - \frac{1}{1-\cos\theta}\sin\theta$$
$$\left(r \cdot \frac{d\theta}{dr}\right) = -\frac{\sin\theta}{1-\cos\theta}$$
$$= \left(\frac{-2\sin\theta/2 \cos\theta/2}{1-1-2\sin^2\theta/2}\right) \times$$

$$r \cdot \frac{d\theta}{dr} = \frac{1-\cos\theta}{-\sin\theta}$$

$$= \frac{\cancel{1}-\cancel{1}+\cancel{2}\sin^{\cancel{2}}\theta/2}{-\cancel{2}\cancel{\sin\theta/2}\cos\theta/2}$$

$$= -\tan\theta/2 \Rightarrow \tan(\pi - \theta/2)$$
$$\phi_2 = \pi - \theta/2$$

Angle between intersection of two curves
$$|\phi_2 - \phi_1| = |\pi - \cancel{\theta/2} - \pi/2 + \cancel{\theta/2}|$$
$$= \frac{2\pi - \pi}{2}$$
$$\boxed{|\phi_2 - \phi_1| = \pi/2}$$

Show that two curves $r^n = a^n \cos n\theta$ and $r^n = b^n \sin n\theta$ cut each other orthogonally

Solⁿ:
Given:

$$r^n = a^n \cos n\theta$$
$$r^n = b^n \sin n\theta$$

Let ϕ_1 be the angle between the radius and tangent

$$r^n = a^n \cos n\theta$$

apply log on B.S

$$\log r^n = \log(a^n \cos n\theta)$$

$$n \log r = \log a^n + \log \cos n\theta$$

$$n \log r = n \log a + \log \cos n\theta$$

diff w.r.t θ

$$\frac{d}{d\theta} n \log r = \frac{d}{d\theta} n \log a + \frac{d}{d\theta} \log \cos n\theta$$

$$\frac{\cancel{n}}{r} \frac{dr}{d\theta} = 0 + \frac{1}{\cos n\theta} (-\sin n\theta) \cdot \cancel{n}$$

$$r \cdot \frac{d\theta}{dr} = \frac{\cos n\theta}{-\sin n\theta}$$

$$= -\cot n\theta$$

$$= \tan\left(\frac{\pi}{2} + n\theta\right)$$

$$\phi_1 = \frac{\pi}{2} + n\theta$$

Let ϕ_2 be the angle between the radius and tangent

$$n \log r = n \log b + \log \sin n\theta$$

diff w.r.t θ

$$\frac{\cancel{n}}{r} \frac{dr}{d\theta} = 0 + \frac{1}{\sin n\theta} \cos n\theta \cdot \cancel{n}$$

$$r \cdot \frac{d\theta}{dr} = \tan n\theta$$

$$\cancel{\tan}\,\phi_2 = \cancel{\tan}\,n\theta$$
$$\phi_2 = n\theta$$

Angle between intersection of two curves

$$|\phi_2 - \phi_1| = |\pi/2 + \cancel{n\theta} - \cancel{n\theta}|$$

$$|\phi_2 - \phi_1| = \pi/2$$

- Find the angle of intersection of the curves $r^2 = 16\sin 2\theta$ and $r^2\sin 2\theta = 4$

Solⁿ:

Given:

$$r^2 = 16\sin 2\theta$$
$$r^2\sin 2\theta = 4$$

To Let ϕ_1 be the angle between radius and tangent

$$r^2 = 16\sin 2\theta$$
$$\log r^2 = \log(16\sin 2\theta)$$

$$2\log r = \log 16 + \log\sin 2\theta$$

diff w.r.t θ

$$\frac{\cancel{2}}{r}\frac{dr}{d\theta} = 0 + \frac{\cancel{2}\cos 2\theta}{\sin 2\theta}$$

$$r\cdot\frac{d\theta}{dr} = \frac{\sin 2\theta}{\cos 2\theta}$$

$$\tan\phi_1 = \tan 2\theta$$
$$\phi_1 = 2\theta$$

Let ϕ_2 be the angle between radius and tangent

$$r^2\sin 2\theta = 4$$

$$\log r^2\sin 2\theta = \log 4$$

$$2\log r + \log\sin 2\theta = \log 4$$

diff w.r.t θ

$$\frac{2}{r}\frac{dr}{d\theta} + \frac{2}{\sin 2\theta}\cos 2\theta = 0$$

$$\frac{\cancel{2}}{r}\frac{dr}{d\theta} = -\boxed{\frac{\cancel{2}\cos 2\theta}{\sin 2\theta}}$$

$$r\frac{d\theta}{dr} = -\frac{\sin 2\theta}{\cos 2\theta}$$

$$\boxed{\cancel{\tan}\phi_2 = -\cancel{\tan}2\theta}$$

$$\boxed{\phi_2 = -2\theta}$$

Angle between intersection of two curves

$$|\phi_1 - \phi_2| = |2\theta + 2\theta|$$
$$= 4\theta$$

To find θ

Solving ① & ②

$$r^2 = 16\sin 2\theta$$
$$r^2\sin 2\theta = 4$$

Sub eq^n ① in ②

$$(\overset{4}{\cancel{16}}\sin 2\theta)(\sin 2\theta) = \cancel{4}$$
$$4\sin^2 2\theta = 1$$

$$\sin^2 2\theta = \frac{1}{4}$$

Sqrt on B.S

$$\sin 2\theta = \frac{1}{2}$$

$$\sin 2\theta = \sin \pi/6$$

$$2\theta = \pi/6$$

$$\boxed{\theta = \frac{\pi}{12}}$$

$$\therefore |\phi_1 - \phi_2| = 4\theta$$
$$= \cancel{4}\left(\frac{\pi}{\cancel{12}\,3}\right)$$

$$\boxed{|\phi_1 - \phi_2| = \pi/3}$$

- Find angle of intersection of the curves $r = \sin\theta + \cos\theta$ and $r = 2\sin\theta$

Solⁿ:

given:

$r = \sin\theta + \cos\theta$

$r = 2\sin\theta$

Let ϕ_1 be the angle between radius and tangent.

$$\tan\phi_1 = r\frac{d\theta}{dr}$$

$$r = \sin\theta + \cos\theta$$

diff w.r.t θ

$$\frac{dr}{d\theta} = \cos\theta - \sin\theta$$

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$$\tan\phi_1 = r \cdot \frac{d\theta}{dr}$$

$$\tan\phi_1 = \frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta}$$

Let ϕ_2 be the angle between radius and tangent

$$\tan\phi_2 = r \cdot \frac{d\theta}{dr}$$

$$r = 2\sin\theta$$

$$\frac{dr}{d\theta} = 2\cos\theta$$

$$\tan\phi_2 = \frac{\cancel{2}\sin\theta}{\cancel{2}\cos\theta}$$

$$\cancel{\tan}\phi_2 = \cancel{\tan}\theta$$

$$\boxed{\phi_2 = \theta}$$

To find θ

Solve (1) & (2)

$$r = \sin\theta + \cos\theta$$

$$r = 2\sin\theta$$

Sub (2) in (1)

$$2\sin\theta = \sin\theta + \cos\theta$$

$$2\sin\theta - \sin\theta = \cos\theta$$

$$\sin\theta = \cos\theta$$

$$\frac{\sin\theta}{\cos\theta} = 1$$

$$\tan\theta = 1$$

$$\theta = \tan^{-1}(1)$$

$$\theta = \pi/4$$

Sub $\theta = \pi/4$ in $\tan\phi_1$

$$\tan\phi_1 = \frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta} = \frac{\sin\pi/4 + \cos\pi/4}{\cos\pi/4 - \sin\pi/4}$$

$$= \frac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}}$$

$$= \frac{2}{\sqrt{2}(0)}$$

= UNDEFINED

$$\tan\phi_1 = \tan \pi/2$$

$$\boxed{\phi_1 = \pi/2}$$

$$\boxed{\phi_2 = \pi/4}$$

Angle between intersection of two curves

$$|\phi_1 - \phi_2| = |\pi/2 - \pi/4|$$

$$= \frac{2\pi - \pi}{4}$$

$$\boxed{|\phi_1 - \phi_2| = \pi/4}$$

Pedal Equation

Pedal Equation is a relation b/w r and P where r be the radius vector of any point on the curve and P be the length of the perpendicular from the pole on the tangent at that point. i.e,

$$\boxed{P = r\sin\phi}$$

or

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

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Steps to find pedale equation of the given curve.

- **Step 1:** Write the given curve
- **Step 2:** Find θ from the given curve
- **Step 3:** Find ϕ $\left(\tan\phi = r\dfrac{d\theta}{dr}\right)$
- **Step 4:** Use formula $P = r\sin\phi$
- **Step 5:** Substitute θ in the above equation.

- Find the pedale equation of the following curves.

(i) $r = a(1+\cos\theta)$

$r = a(1+\cos\theta)$ — (1)

Find θ

$a(1+\cos\theta) = r$

$a(1+2\cos^2\theta/_2-1) = r$ $\quad\because 2\cos^2\theta/_2 - 1 = \cos\theta$

$2a\cos^2\theta/_2 = r$

$\cos^2\theta/_2 = \dfrac{r}{2a}$

$$\boxed{\cos\theta/_2 = \sqrt{\frac{r}{2a}}}$$

Find ϕ:

$\tan\phi = r\dfrac{d\theta}{dr}$

$\dfrac{dr}{d\theta} = \dfrac{d}{d\theta}\left(a(1+\cos\theta)\right)$

$\dfrac{dr}{d\theta} = a(0-\sin\theta)$

$\dfrac{dr}{d\theta} = -a\sin\theta$

$\tan\phi = r\cdot\dfrac{d\theta}{dr}$

$= \cancel{a}(1+\cos\theta)\cdot\dfrac{1}{-\cancel{a}\sin\theta}$

$$= \frac{(\cancel{1} + \cancel{2}\cos^2\theta/2 - \cancel{1})}{-\cancel{2}\sin\theta/2 \cos\theta/2}$$

$$= -\cot\theta/2$$

$$= \tan(\theta/2 + \pi/2)$$

or

$$\tan\phi = \tan(\pi/2 + \theta/2)$$

$$\boxed{\phi = \pi/2 + \theta/2}$$

Use formula

$$P = r\sin\phi$$

$$= r\sin(\pi/2 + \theta/2)$$

$$= r\cos\theta/2$$

$$P = r\sqrt{\frac{r}{2a}}$$

$$p^2 = r^2 \cdot \frac{r}{2a}$$

$$\boxed{p^2 2a = r^3}$$

$\therefore$ This is the pedale equation.

(iii) $r^2 = a^2\sin^2\theta$

$$r^2 = a^2\sin 2\theta \quad \text{--- (1)}$$

Find θ

$$a^2\sin^2\theta = r^2$$

$$\sin^2\theta = \frac{r^2}{a^2}$$

$$\sin\theta = \frac{r}{a}$$

$$\tan\phi = r \cdot \frac{d\theta}{dr}$$

$$\log r^2 = \log(a^2 \sin^2\theta)$$

$$2\log r = 2\log a + 2\log \sin\theta$$

$$\log r = \log a + \log \sin\theta$$

diff w.r.t θ

$$\frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{1}{\sin\theta} \cdot \cos\theta$$

$$r \cdot \frac{d\theta}{dr} = \frac{\sin\theta}{\cos\theta}$$

$$\tan\phi = \tan\theta$$

$$\phi = \theta$$

$$p = r\sin\phi$$

$$p = r\sin\theta$$

$$p = r \cdot \frac{r}{a}$$

$$\boxed{pa = r^2}$$

Find the pedale equation for the curve $r^m \cos m\theta = a^m$

$$r^m \cos m\theta = a^m$$

Find θ, $\cos m\theta = \frac{a^m}{r^m}$

Find ϕ $\log r^m \cos m\theta = \log a^m$

$$m \log r + \log \cos m\theta = m \log a$$

diff

$$\frac{m}{r} \frac{dr}{d\theta} + \frac{(-\sin m\theta)(m}{\cos m\theta} = 0$$

$$\frac{r}{m}\frac{d\theta}{dr} + \frac{(-\sin m\theta)m}{\cos m\theta} = 0$$

$$\frac{r}{m}\frac{d\theta}{dr} = \frac{\sin m\theta}{\cos m\theta}$$

$$r\frac{d\theta}{dr} = \frac{\sin m\theta}{\cos m\theta}$$

$$\tan\phi = \frac{1}{\tan m\theta}$$

$$\tan\phi = \cot m\theta$$

$$\tan\phi = \tan\left(\pi/2 - m\theta\right)$$

$$\phi = \pi/2 - m\theta$$

$$p = r\sin\phi$$

$$p = r\sin\left(\pi/2 - m\theta\right)$$

$$p = r\cos m\theta$$

$$p = r\frac{a^m}{r^m}$$

$$\boxed{r^{m-1}p = a^m}$$

or

$$\boxed{p = r^{1-m}a^m}$$

(iii) $r^n = a^n\cos n\theta$

$$r^n = a^n\cos n\theta$$

$$a^n\cos n\theta = r^n$$

$$\cos n\theta = \frac{r^n}{a^n}$$

$$\tan\phi = r\frac{d\theta}{dr}$$

$$\log r^n = \log a^n\cos n\theta$$

$$n\log r = n\log a + \log\cos n\theta$$

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$$\cancel{r}\cdot\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{(-\sin n\theta)}{\cos n\theta}\cancel{n}$$

$$r\cdot\frac{d\theta}{dr} = -\cot n\theta$$

$$\tan\phi = \tan\left(\frac{\pi}{2}+n\theta\right)$$

$$\phi = \frac{\pi}{2}+n\theta$$

$$p = r\sin\phi$$

$$p = r\sin\left(\frac{\pi}{2}+n\theta\right)$$

$$p = r\cos n\theta$$

$$p = r\cdot\frac{r^n}{a^n}$$

$$\boxed{a^n p = r^{n+1}}$$

(iii) $\frac{2a}{r} = 1-\cos\theta$

$$\frac{2a}{r} = 1-\cos\theta$$

$$\frac{2a}{r} = \cancel{1}-\cancel{1}+2\sin^2\theta/2$$

$$\frac{\cancel{2}a}{r} = \cancel{2}\sin^2\theta/2$$

$$\frac{a}{r} = \sin^2\theta/2$$

$$\tan\phi = r\cdot\frac{d\theta}{dr}$$

$$\log\frac{2a}{r} = \log(1-\cos\theta)$$

$$\log 2a - \log r = \log(1-\cancel{\log}\cos\theta)$$

diff w.r.t θ

$$0 - \frac{1}{r}\frac{dr}{d\theta} = \frac{1}{1-\cos\theta}(\sin\theta)$$

$$-r\frac{d\theta}{dr} = \frac{1-\cos\theta}{\sin\theta}$$

$$= \frac{\cancel{1}-\cancel{1}+\cancel{2}\sin^2\theta/2}{\cancel{2}\sin\theta/2\cos\theta/2}$$

$$= \frac{\sin\theta/2}{\cos\theta/2}$$

$$-\tan\phi = \tan\theta/2$$

$$-\phi = \theta/2$$

$$\phi = -\theta/2$$

$$p = r\sin\phi$$

$$p = r\sin(-\theta/2)$$

$$p = -r\sin\theta/2$$

$$p = -r\sqrt{\frac{a}{r}}$$

$$p^2 = r^2\frac{a}{r}$$

$$p^2 = ra$$

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Radius of curvature

Curvature :- The length of bending of curve is called the curvature.

Radius of curvature :- The reciprocal of curvature is called radius of curvature.

Cartesian form of the curve is $y=f(x)$
The radius of curvature for cartesian form of the curve is

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

where,

$y_1 = \frac{dy}{dx}$

$y_2 = \frac{d^2y}{dx^2}$

If the curvature is in the form $x=f(y)$ then

$$\rho = \frac{(1+x_1^2)^{3/2}}{x_2}$$

where,

$x_1 = \frac{dx}{dy}$

$x_2 = \frac{d^2x}{dy^2}$

1. Find the radius of curvature of the curve
(i) $y = x^3(x-a)$ at the point $(a,0)$
given point $(a,0)$
given curve $\Rightarrow y = x^3(x-a)$
$= x^4 - ax^3$

Radius of curvature :

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$y_1 = \frac{dy}{dx}$

$= \frac{d}{dx}(x^4 - ax^3)$

$= 4x^3 - a^3x^2$

$y_2 = \frac{d^2y}{dx^2}$

$= \frac{d}{dx}(4x^3 - a^3x^2)$

$= 12x^2 - 6ax$

Substitute $(a,0)$ in y_1 and y_2

$y_1(a,0) = 4a^3 - a^2a^2$

$= 4a^3 - 3a^3$

$= a^3$

$y_2(a,0) = 12a^2 - 6a^2$

$= 6a^2$

$\rho = \frac{(1+(a^3)^2)^{3/2}}{6a^2}$

$= \frac{(1+a^6)^{3/2}}{6a^2}$

(Q) $y = 4\sin x - \sin 2x$ at $x = \pi/2$

Given

$y = 4\sin x - \sin 2x$

$x = \pi/2$

$$y_1 = \frac{dy}{dx}$$

$$= \frac{d}{dx}(4\sin x - \sin 2x)$$

$$= 4\cos x - 2\cos 2x$$

$$y_2 = \frac{d^2y}{dx^2}$$

$$= \frac{d}{dx}(4\cos x - 2\cos 2x)$$

$$= 4\sin 2x - 4\sin x$$

Radius of curvature

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

Substitute $x = \pi/2$ in y_1 and y_2

$$y_1(\pi/2) = 4\cos \pi/2 - 2\cos \cancel{2}\pi/\cancel{2}$$

$$= 0 - 2(-1)$$

$$= 2$$

$$y_2(\pi/2) = 4\sin \cancel{2}\pi/\cancel{2} - 4\sin \pi/2$$

$$= 0 - 4$$

$$= -4$$

$$\rho = \frac{(1+(2)^2)^{3/2}}{-4}$$

$$= \frac{(5)^{3/2}}{-4}$$

$$\boxed{\rho = \frac{5\sqrt{5}}{4}}$$

(iii) $y = c\left[\cosh\left(\frac{x}{c}\right)\right]$ at the point $(0,c)$

given :- $y = c\cosh\left(\frac{x}{c}\right)$

point $(0,c)$

Radius of curvature

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$y_1 = \frac{dy}{dx}$

$= \frac{d}{dx}\left[c\cosh\left(\frac{x}{c}\right)\right]$

$= \cancel{c}\sinh\left(\frac{x}{c}\right)\cdot\frac{1}{\cancel{c}}$

$= \sinh\left(\frac{x}{c}\right)$

$y_2 = \frac{d^2y}{dx^2}$

$= \frac{d}{dx}\left[\sinh\left(\frac{x}{c}\right)\right]$

$= \cosh\left(\frac{x}{c}\right)\cdot\frac{1}{c}$

Substitute $(0,c)$ in $(y_1$ and $y_2)$

$y_{1(0,c)} = \sinh\left(\frac{0}{c}\right)$

$= \sin 0$

$= 0$

$y_{2(0,c)} = \cosh\left(\frac{0}{c}\right)\cdot\frac{1}{c}$

$= 1\cdot\frac{1}{c}$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$= \frac{(1+0)^{3/2}}{1}$

$$\rho = c(1^{3/2})$$

$$\boxed{\rho = c}$$

(iv) $y^2 = 4ax$ at the point $(at^2, 2at)$

given $y^2 = 4ax$

$$y = \sqrt{4ax}$$

$$y = 2(ax)^{1/2}$$

Radius of curvature

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$y_1 = \frac{dy}{dx}$$

$$= \frac{d}{dx}(2\sqrt{a}\sqrt{x})$$

$$= \cancel{2}\sqrt{a} \cdot \frac{1}{\cancel{2}\sqrt{x}}$$

$$= \frac{\sqrt{a}}{\sqrt{x}}$$

$$y_2 = \frac{d^2y}{dx^2}$$

$$= \frac{d}{dx}\left(\frac{\sqrt{a}}{\sqrt{x}}\right)$$

$$= \sqrt{a} \cdot \frac{-1}{2(x)^{3/2}}$$

Substitute $(at^2, 2at)$ in y_1 and y_2

$$y_1(at^2) = \frac{\sqrt{a}}{\sqrt{at^2}} = \frac{\sqrt{\cancel{a}}}{\sqrt{\cancel{a}} \cdot t}$$

$$= \frac{1}{t}$$

$$y_2(at^2) = \sqrt{a} \cdot \frac{-1}{2at^2 \cdot \sqrt{at^2}} = \frac{-1}{2at^3}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{\left(1 + \left(\frac{1}{t}\right)^2\right)^{3/2}}{\frac{-1}{2at^3}}$$

$$= \frac{\left(\frac{t+1}{t^2}\right)^{3/2}}{\frac{-1}{2at^3}}$$

$$= \frac{(t^2+1)^{3/2}}{(t)^{3/2}} x$$

$$= \frac{(t^2+1)^{3/2}}{\frac{-t^3}{2at^3}}$$

$$\boxed{\rho = 2a\,(t^2+1)^{3/2}}$$

(v) $\sqrt{x} + \sqrt{y} = \sqrt{\alpha}$ at the point $\left(\frac{a}{4}, \frac{a}{4}\right)$

Given:- curve $\Rightarrow \sqrt{x} + \sqrt{y} = \sqrt{\alpha}$

$$\sqrt{y} = \sqrt{\alpha} + \sqrt{x}$$

$$y = (\sqrt{\alpha} - \sqrt{x})^2$$

point $\Rightarrow \left(\frac{a}{4}, \frac{a}{4}\right)$

Radius of curvature

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$y_1 = 2(\sqrt{\alpha} - \sqrt{x}) \cdot \frac{d}{dx}(\sqrt{\alpha} - \sqrt{x})$$

$$= \cancel{2}(\sqrt{\alpha} - \sqrt{x}) \cdot \left(\cancel{\sqrt{\alpha}}^{\,0} - \frac{1}{\cancel{2}\sqrt{x}}\right)$$

$$= -\left(\frac{\sqrt{a} - \sqrt{x}}{\sqrt{x}}\right)$$

$$= -\frac{\sqrt{y}}{\sqrt{x}}$$

$$y_2 = \frac{d^2y}{dx^2}$$

$$= \frac{d}{dx}\left(\frac{\sqrt{y}}{\sqrt{x}}\right)$$

$$= -\left[\frac{\sqrt{x}\,\frac{d\sqrt{y}}{dx} - \sqrt{y}\,\frac{d\sqrt{x}}{dx}}{x}\right]$$

$$= -\left[\frac{\sqrt{x}\,\frac{1}{2\sqrt{y}}\cdot\frac{dy}{dx} - \sqrt{y}\cdot\frac{1}{2\sqrt{x}}}{x}\right]$$

Substitute $\left(\frac{a}{4}, \frac{a}{4}\right)$ in y_1 and y_2

$$y_1 = -\frac{\sqrt{\cancel{\frac{a}{4}}}}{\sqrt{\cancel{\frac{a}{4}}}}$$

$$= -1$$

$$y_2 = \frac{\sqrt{\cancel{\frac{a}{4}}}\cdot\frac{1}{2\sqrt{\cancel{\frac{a}{4}}}}\cdot\frac{dy}{dx} - \sqrt{\frac{a}{4}}}{\frac{a}{4}}$$

$$= \frac{\frac{1}{2}\frac{dy}{dx} - \frac{1}{2}}{\frac{a}{4}}$$

$$= \frac{\frac{1}{2}\,\cancel{\frac{d\left(\frac{a}{4}\right)}{dx}}^{-1} - \frac{1}{2}}{\frac{a}{4}}$$

$$= \frac{1}{2} + \frac{1}{2} \times \frac{4}{a}$$

$$y_2 = \frac{4}{2a}$$

$$\rho = \frac{(1+(-1)^2)^{3/2}}{\frac{4}{a}}$$

$$= \frac{(1+1)^{3/2}}{\frac{4}{a}}$$

$$= 2^{3/2} \times \frac{a}{4}$$

$$= \cancel{2}\sqrt{\cancel{2}} \cdot \frac{a}{\cancel{4}\,\cancel{2}\sqrt{2}}$$

$$\boxed{\rho = \frac{a}{\sqrt{2}}}$$

Show that the radius of curvature for the rectangular hyperbola $xy = c^2$ is $\rho = \frac{(x^2+y^2)^{3/2}}{2c^2}$

$$xy = c^2$$

$$y = \frac{c^2}{x}$$

$$y_1 = \frac{c^2}{x}$$

$$y_1 = \frac{d}{dx}\left(\frac{c^2}{x}\right) \qquad y_2 = \frac{d}{dx}\left(\frac{-c^2}{x^2}\right)$$

$$= \frac{-c^2}{x^2} \qquad = -c^2 \cdot \frac{-2}{x^3}$$

$$= \frac{2c^2}{x^3}$$

$$\rho = \frac{(x^2+y_1^2)^{3/2}}{2c^2}$$

$$\cancel{\rho} \cancel{=} \cancel{1+y_1^2} \qquad \rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$\rho = \frac{\left(1+\frac{c^4}{x^4}\right)^{3/2}}{\frac{2c^2}{x^3}}$$

$$= \frac{(x^4+c^2y^2)^{3/2}}{\cancel{(x^4)^{3/2}}} \times \frac{\cancel{x^3}}{2c^2}$$

$$= \frac{\cancel{(x^2)^{3/2}}(x^2+y^2)^{3/2}\,\cancel{x^3}}{\cancel{x^6}\,2c^2}$$

$$\rho = \frac{(x^2+y^2)^{3/2}}{2c^2}$$