Module - 3

Taylor series

The formula of taylor series is

$$f(x) = f(a) + (x-a) f'a + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \dots\dots$$

Problem

1. Expand $\log x$ in powers of $(x-1)$ and hence evaluate $\log 1.01$

Given function is $f(x) = \log x$

compare $x-1$ with $x-a$

So $a=1$

Taylor series formula

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \dots$$

$f(x) = \log x$ $\Rightarrow$

$f'(x) = \frac{1}{x}$ $\Rightarrow$ Substitute $a=1$ in $f'(x), f''(x), f'''(x), f$

$\Rightarrow f(a) = \log 1 = 0$

$f''(x) = -x^{-2}$ $\Rightarrow f'(a) = \frac{1}{1} = 1$

$\Rightarrow f''(a) = -(1)^{-2} = -1$

$f'''(x) = 2x^{-3}$ $\Rightarrow f'''(a) = 2(1)^{-3} = 2$

Sub

$$\therefore f(x) = 0 + (x-1)(1) + \frac{(x-1)^2}{2!}(-1) + \frac{(x-1)^3}{3!}(2)$$

$$f(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \dots$$

Substitute $x = 1.01$

$$f(1.1) = (1.1-1) - \frac{(1.1-1)^2}{2} + \frac{(1.1-1)^3}{3} + \ldots$$

$$\log 1.1 = (0.1) - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} + \ldots$$

2. Expand e^x in powers of x

$f(x) = e^x$

$\Rightarrow$ compare x with $x-a$

$a = 0$

Taylors series

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots$$

$f(x) = e^x \Rightarrow f(0) = e^0 = 1$

$f'(x) = e^x \Rightarrow f'(0) = e^0 = 1$

$f''(x) = e^x \Rightarrow f''(0) = e^0 = 1$

$f'''(x) = e^x \Rightarrow f'''(0) = e^0 = 1$

$$f(x) = 1 + (x-0)f'(0) + \frac{(x-0)^2}{2!}f''(0) + \frac{(x-0)^3}{3!}f'''(0) + \ldots$$

$$e^x = 1 + x(1) + \frac{x^2}{2}(1) + \frac{x^3}{6}(1) + \ldots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

3. Expand $\sin x$ in powers of $x - \pi/2$

$f(x) = \sin x$

$\Rightarrow$ Compare $x - \pi/2$ with $x-a$

$a = \pi/2$

Taylor Series

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a)$$

$f(x) = \sin x \Rightarrow f(\pi/2) = \sin \pi/2 = 1$

$f'(x) = \cos x \quad f'(\pi/2) = \cos \pi/2 = 0$

$f''(x) = -\sin x \quad f''(\pi/2) = -\sin \pi/2 = -1$

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$f'''(x) = -\cos x \Rightarrow f'''(\pi/2) = -\cos \pi/2 = 0$

$f(x) = f(\pi/2) + (x-\pi/2) f'(\pi/2) + \frac{(x-\pi/2)^2}{2!} f''(\pi/2) + \frac{(x-\pi/2)^3}{3!} f'''(\pi/2) + \dots$

$\sin x = 1 + (x-\pi/2)\, 0 + \frac{(x-\pi/2)^2}{2!}(-1) + \frac{(x-\pi/2)^3}{3!}(0) + \dots$

$\sin x = 1 - \frac{(x-\pi/2)^2}{2!} + \dots$

4. Expand $\cos x$ in powers of $x-\pi/2$

$f(x) = \cos x$

$\Rightarrow$ compare $x-\pi/2$ with $x-a$

$a = \pi/2$

Taylor series

$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \dots$

$f(x) = \cos x \Rightarrow f(\pi/2) = 0$

$f'(x) = -\sin x \Rightarrow f'(\pi/2) = -1$

$f''(x) = -\cos x \Rightarrow f''(\pi/2) = 0$

$f'''(x) = \sin x \Rightarrow f'''(\pi/2) = 1$

$f(x) = f(\pi/2) + (x-\pi/2) f'(\pi/2) + \frac{(x-\pi/2)^2}{2!} f''(\pi/2) + \frac{(x-\pi/2)^3}{3!} f'''(\pi/2) + \dots$

$\cos x = 0 + (x-\pi/2)(-1) + \frac{(x-\pi/2)^2}{2!}(0) + \frac{(x-\pi/2)^3}{3!}(1) + \dots$

$= -(x-\pi/2) + \frac{(x-\pi/2)^3}{3!}(1) + \dots$

$\cos x = \frac{(x-\pi/2)^3}{3!} - (x-\pi/2) + \dots$

5. Expand $f(x) = 2x^3 + 7x^2 + x - 6$ in powers of $x-2$

$f(x) = 2x^3 + 7x^2 + x - 6$

$a = 2$

Taylor series

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \dots$$

$f(x) = 2x^3 + 7x^2 + x - 6 \Rightarrow f(2) = 2(2)^3 + 7(2)^2 + 2 - 6 = 40$

$f'(x) = 6x^2 + 7x + 1 \Rightarrow f'(2) = 6(2)^2 + 7(2) + 1 = 39$

$f''(x) = 12x + 7 \Rightarrow f''(2) = 12(2) + 7 = 31$

$f'''(x) = 12 \Rightarrow f'''(2) = 12$

$f^{iv}(x) = 0 \Rightarrow f^{iv}(2) = 0$

$$f(x) = 40 + (x-2)39 + \frac{(x-2)^2}{2!}31 + \frac{(x-2)^3}{3!}12 + \frac{(x-2)^4}{4!}(0) + \dots$$

$$= 40 + (x-2)39 + \frac{(x-2)^2}{2}31 + \frac{(x-2)^3}{\cancel{6}}\cancel{12}^{2} + \frac{(x-2)^4}{4!}(0) + \dots$$

$$2x^3 + 7x^2 + x - 6 = 40 + 39(x-2) + 31\frac{(x-2)^2}{2} + 2(x-2)^3 + 0 + \dots$$

Expand $\log(1+x)$ in terms of x

$f(x) = \log(1+x)$

$a = 0$

Taylor series

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \dots$$

$f(x) = \log(1+x) \Rightarrow f(0) = \log 1 = 0$

$f'(x) = \frac{1}{1+x}$ $\quad f'(0) = \frac{1}{1+0} = 1$

$f''(x) = \frac{-1}{(1+x)^2}$ $\quad f''(0) = \frac{-1}{1} = -1$

$f'''(x) = \frac{2}{(1+x)^3}$ $\quad f'''(0) = \frac{2}{(1+0)^3} = 2$

$$f(x) = 0 + (x-0)1 + \frac{(x-0)^2}{2!}(-1) + \frac{(x-0)^3}{3!}(2) + \dots$$

$$\log(1+x) = x + \frac{x^2}{2!}(-1) + \frac{(x)^3}{3!}(2) + \dots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots$$

Expand $\cancel{\frac{1}{1}+x}$ $\frac{1}{1-x}$ in powers of x

$f(x) = \frac{1}{1-x}$

$a = 0$

Taylor series

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \dots$$

$f(x) = \frac{1}{(1-x)} \Rightarrow f(0) = \frac{1}{1} = 1$

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$$f'(x) = \frac{-1}{(1-x)^2} \cdot \frac{d}{dx}(1-x) = \frac{1}{(1-x)^2} \Rightarrow f'(0) = \frac{1}{(1-0)^2} = 1$$

$$f''(x) = \frac{-2}{(1-x)^3} \cdot \frac{d}{dx}(1-x) = \frac{2}{(1-x)^3} \Rightarrow f''(0) = \frac{2}{1} = 2$$

$$f'''(x) = \frac{-6}{(1-x)^4} \cdot \frac{d}{dx}(-1) = \frac{6}{(1-x)^4} \Rightarrow f'''(0) = \frac{6}{1} = 6$$

$$f^{iv}(x) = \frac{-24}{(1-x)^5} \cdot \frac{d}{dx}(-1) = \frac{24}{(1-x)^5} \Rightarrow f^{iv}(0) = \frac{24}{1} = 24$$

$$f(x) = 1 + (x-0)1 + \frac{(x-0)^2 2}{2!} + \frac{(x-0)^3 6}{3!} + \frac{(x-0)^4 24}{4!} + \ldots$$

$$= 1 + x + \frac{x^2(2)}{2!} + \frac{x^3}{6} \cdot 6 + \frac{x^4}{24} \cdot 24 + \ldots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots$$

Maclaurin's Series

The formula of maclaurin series is given by $f(x)$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots$$

Problem

1. Expand $\tan x$ by using maclaurin series

Sol^n $f(x) = \tan x$

$f(x) = \tan x \Rightarrow f(0) = \tan 0 = 0$

$f'(x) = \sec^2 x \Rightarrow f'(0) = \sec^2 0 = 1$

$f''(x) = 2\sec x \cdot \sec x \cdot \tan x \Rightarrow f''(0) = 2\sec^2 0 \cdot \tan 0 = 0 \cdot 1 = 0$

$f'''(x) = [2(1+\tan^2 x)\tan x]$

$\frac{d}{dx}(2\tan x + 2\tan^3 x) = 2\sec^2 x + 6\tan^2 x \sec^2 x$

$\Rightarrow f'''(0) = 2\sec^2 0 + 6\tan^2 0 \sec^2 0$

$= 2(1) = 2$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

$$\tan x = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \dots$$

$$\boxed{\tan x = 0 + x + \frac{x^3}{3} + \dots}$$

2. Expand $e^{\sin x}$ by using maclaurin series

$f(x) = e^{\sin x}$

$f(x) = e^{\sin x} \quad \Rightarrow f(0) = e^{\sin 0} = e^0 = 1$

$f'(x) = e^{\sin x} \cdot \cos x \quad \Rightarrow f'(0) = e^{\sin 0} \cdot \cos 0 = e^0 \cdot 1 = 1$

$f''(x) = e^{\sin x}(-\sin x) + \cos x \cdot e^{\sin x} \cdot \cos x$

$= -e^{\sin x} \cdot \sin x + e^{\sin x} \cdot \cos^2 x$

$= f(x) \cdot \cos^2 x - f(x) \cdot \sin x$

(or) $= f'(x) \cdot \cos x - f(x) \sin x \Rightarrow f''(0) = f'(0) \cdot \cos 0 - f(0) \sin x$

$= 1 \cdot 1 - 1 \cdot 0$

$= 1$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

$$e^{\sin x} = 1 + x(1) + \frac{x^2}{2!}(1) + \dots$$

$$\boxed{e^{\sin x} = 1 + x + \frac{x^2}{2!} + \dots}$$

3. Expand $\tan^{-1}x$ by using maclaurin series

$f(x) = \tan^{-1}x \quad \Rightarrow f(0) = \tan^{-1}(0) = \tan^{-1}(\tan 0) = 0$

$f'(x) = \frac{1}{1+x^2} \quad \Rightarrow f'(0) = \frac{1}{1+0} = \frac{1}{1} = 1$

$f''(x) = \frac{-1}{(1+x^2)^{+2}} = 0 + 2x \qquad \frac{1}{x} \qquad x^{-1-1} \qquad x^{-2}$

$\Rightarrow f''(0) = -1 \cdot 0 = 0$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots$$

$$f'''(x) = -2\left[x(-2(1+x^2)^{-3}\cdot 2x) + (1+x^2)^{-2}\cdot 1\right]$$

$$\Rightarrow f'''(0) = -2[0+1]$$
$$= -2$$

$$\tan^{-1}x = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-2) + \dots$$

$$\boxed{\tan^{-1}x = x - \frac{x^3}{3} + \dots}$$

4. Expand $\log(1+x)$ by using maclaurin's series

$f(x) = \log(1+x) \quad \Rightarrow f(0) = \log(1) = 0$

$f'(x) = \frac{1}{1+x} \quad \Rightarrow f'(0) = \frac{1}{1} = 1$

$f''(x) = \frac{-1}{(1+x)^2} \quad \Rightarrow f''(0) = \frac{-1}{1} = -1$

$f'''(x) = \frac{2}{(1+x)^3} \quad \Rightarrow f'''(0) = \frac{2}{1} = 2$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots$$

$$\log(1+x) = 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \dots$$

$$\boxed{\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots}$$

5. Expand $\log(\sec x)$ by using maclaurin series

$f(x) = \log(\sec x) \quad \Rightarrow f(0) = \log(\sec 0) = 0$

$f'(x) = \frac{1}{\sec x}\cdot \sec x \cdot \tan x$

$= \tan x \qquad \Rightarrow f'(0) = \tan 0 = 0$

$f''(x) = \sec^2 x \qquad \Rightarrow f''(0) = \sec^2(0) = 1$

$f'''(x) = 2\sec x . \sec x . \tan x$

$= 2\sec^2 x . \tan x \qquad \Rightarrow f'''(0) = 2\sec^2(0)\tan(0) = 0$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

$$\log(\sec x) = 0 + x(0) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(0) + \dots$$

$\log(\sec x) = \frac{x^2}{2} + \dots$ or $f^{iv}(0) = (1)$ last page

$$\boxed{\log(\sec x) = \frac{x^2}{2} + \frac{x^4}{4!} + \dots}$$

6. Expand $e^x \sin x$ by using maclaurin series

$f(x) = e^x \sin x \qquad \Rightarrow f(0) = e^0 \sin x = 1 . 0 = 0$

$f'(x) = e^x \cos x + \sin x \, e^x \qquad \Rightarrow f'(0) = e^0 \cos x + e^0 \sin 0 = 1$

$f''(x) = e^x \cancel{(-\sin x)} + \cos x \, e^x + \cancel{\sin x \, e^x} + e^x \cos e^x$

$= e^x \cos x + e^x \cos x$

$= 2(e^x \cos x) \qquad \Rightarrow f''(0) = 2(e^0 \cos 0) = 2$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

$$e^x \sin x = 0 + x(1) + \frac{x^2}{2!}(2) + \dots$$

$$\boxed{e^x \sin x = x + x^2 + \dots}$$

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Partial derivative

- If f is a function of 2 independent variable x & y then the first partial derivates are given by

$$f_x = \frac{\partial f}{\partial x}$$

$$f_y = \frac{\partial f}{\partial y}$$

- The second partial derivates are given by

$$f_{x^2} = \frac{\partial^2 f}{\partial x^2}$$

$$f_{y^2} = \frac{\partial^2 f}{\partial y^2}$$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$$

$$f_{yx} = \frac{\partial^2 f}{\partial y \partial x}$$

1. Find $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$ where $z = x^3 + y^3$

$$z = x^3 + y^3$$

$$\frac{\partial z}{\partial x} = \frac{d}{dx}(x^3 + y^3)$$
$$= 3x^2 + 0$$
$$= 3x^2$$

$$\boxed{\frac{\partial^2 z}{\partial x^2} = 6x}$$

$$\frac{\partial z}{\partial y} = \frac{d}{dx}(x^3 + y^3)$$
$$= 0 + 3y^2$$
$$= 3y^2$$

$$\boxed{\frac{\partial^2 z}{\partial x^2} = 6y}$$

2. Find $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$, where $z = x^2 + e^y + \sin x$

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$$z = x^2 + e^y + \sin x$$

$$\frac{\partial z}{\partial x} = 2x + e^y \cdot 0 + \cos x$$

$$= 2x + \cos x$$

$$\frac{\partial z}{\partial y} = 0 + e^y + 0$$

$$= e^y$$

$$\boxed{\frac{\partial^2 z}{\partial x^2} = 2 - \sin x}$$

$$\boxed{\frac{\partial^2 z}{\partial x^2} = e^y}$$

3. Find $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial z^2}$ where $f = xy + yz + 3x$

$$f = xy + yz + 3x$$

$$\frac{\partial f}{\partial x} = y + 0 + 3$$

$$= y + 3$$

$$\frac{\partial f}{\partial y} = x + z + 0$$

$$= x + z$$

$$\frac{\partial f}{\partial z} = 0 + y + x$$

$$= y + x$$

$$\boxed{\frac{\partial^2 f}{\partial x^2} = 0}$$

$$\boxed{\frac{\partial^2 f}{\partial x^2} = 0}$$

$$\boxed{\frac{\partial^2 f}{\partial z^2} = 0}$$

4. Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, where $z = \log(x^2 + y^2)$

$$z = \log(x^2 + y^2)$$

$$\frac{\partial z}{\partial x} = \frac{1}{x^2 + y^2} \cdot 2x$$

$$\frac{\partial z}{\partial y} = \frac{1}{x^2 + y^2} \cdot 2y$$

$$\frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2}$$

$$\frac{\partial z}{\partial y} = \frac{2y}{x^2 + y^2}$$

5. Find first & second partial derivatives for $z = x^3 + y^3 - 3axy$

$$z = x^3 + y^3 - 3axy$$

To find & $\frac{\partial z}{dx}$, $\frac{\partial z}{\partial y}$, $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$, $\frac{\partial^2 z}{\partial x.\partial y}$

$$\boxed{\frac{\partial z}{\partial x} = 3x^2 + 0 - 3ay}$$

$$\boxed{\frac{\partial z}{\partial y} = 0 + 3y^2 - 3ax}$$

$$\boxed{\frac{\partial^2 z}{\partial x^2} = 6x}$$

$$\boxed{\frac{\partial^2 z}{\partial xy^2} = 3y^2 \Rightarrow 6y}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}(3y^2 - 3ax$$

$$\boxed{\frac{\partial^2 z}{\partial x.\partial y} = -3a}$$

$$\boxed{\frac{\partial^2 z}{\partial x \partial y} = -3a}$$

6. Find $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}$, $z = x^2 + xy + y^2$

$$\frac{\partial z}{\partial x} = 2x + y$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(2x+y)$$

$$= 2$$

$$\frac{\partial z}{\partial y} = x + 2y$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial x}(x+2y)$$

$$= 2$$

7. Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ for $z = x^2 y - x\sin xy$

$$z = x^2 y - x\sin xy$$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^2 y) - \frac{\partial}{\partial x}(\underset{u}{x}\, \underset{v}{\sin xy})$$

$$= 2yx - (x\cos xy \cdot y + \sin xy)$$

$$= 2xy - x\cos xy\, y - \sin xy$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^2y - x\sin xy)$$

$$= x^2 - x\cos xy \cdot x$$

$$= x^2 - x^2\cos xy$$

$$\frac{\partial z}{\partial y} = x^2(1-\cos xy)$$

8. Prove that $\frac{\partial^2 z}{\partial t^2} = c^2\frac{\partial^2 z}{\partial x^2}$ for $z = f(x+ct) + \phi(x-ct)$

Given $z = f(x+ct) + \phi(x-ct)$

$$\frac{\partial z}{\partial t} = \frac{\partial}{\partial t} f(x+ct) + \frac{\partial}{\partial t}\phi(x-ct)$$

$$= f'(x+ct)\cancel{(+c)} + \phi'(x-ct)\cancel{(-c)}$$

$$= f'(x+ct)(+c) + \phi'(x-ct)(-c)$$

$$\frac{\partial z}{\partial t} = c\left[f'(x+ct) - \phi'(x-ct)\right]$$

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial t}\left[c(f'(x+ct) - \phi'(x-ct))\right]$$

$$= c\left[f''(x+ct)(+c) - \phi''(x-ct)(-c)\right]$$

$$\frac{\partial^2 z}{\partial t^2} = c^2\left[f''(x+ct) + \phi''(x-ct)\right] \quad \text{--- (1)}$$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left[f(x+ct) + \phi(x-ct)\right]$$

$$= f'(x+ct)(1) + \phi'(x-ct)(1)$$

$$= f'(x+ct) + \phi'(x-ct)$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left[f'(x+ct) + \phi'(x-ct)\right]$$

$$= f''(x+ct)(1) + \phi''(x-ct)(1)$$

$$= f''(x+ct) + \phi''(x-ct)$$

Sub $\frac{\partial^2 z}{\partial x^2}$ in (1)

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$$

Q. $u = \frac{y}{z} + \frac{z}{x}$ P.T $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{y}{z} + \frac{z}{x}\right)$$

$$= 0 + z\left(\frac{-1}{x^2}\right)$$

$$\frac{\partial u}{\partial x} = \frac{-z}{x^2} \quad \text{— (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y}{z}\right) + \frac{\partial}{\partial y}\left(\frac{z}{x}\right)$$

$$= \frac{1}{z} \quad \text{— (2)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}\left(\frac{y}{z}\right) + \frac{\partial}{\partial z}\left(\frac{z}{x}\right)$$

$$= y\left(\frac{-1}{z^2}\right) + \frac{1}{x}$$

$$\frac{\partial u}{\partial z} = \frac{-y}{z^2} + \frac{1}{x} \quad \text{— (3)}$$

$$x \times (1) + y \times (2) + z \times (3)$$

$$\Rightarrow x\left[\frac{-z}{x^2}\right] + y\left[\frac{1}{z}\right] + z\left[\frac{-y}{z^2}\right] + z\left[\frac{1}{x}\right]$$

$$\Rightarrow \cancel{\frac{-z}{x}} + \cancel{\frac{y}{z}} - \cancel{\frac{y}{z}} + \cancel{\frac{z}{x}} = 0$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

10. If $u = \log x^2+y^2+z^2$ P.T $(x^2+y^2+z^2)\sum \frac{\partial^2 u}{\partial x^2} = 2$

$u = \log(x^2+y^2+z^2)$

$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2+z^2}(2x)$

$= \frac{2x}{x^2+y^2+z^2}$

Illy

$\frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2+z^2}$ $\frac{\partial u}{\partial z} = \frac{2z}{x^2+y^2+z^2}$

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{2x}{x^2+y^2+z^2}\right)$

$= \frac{(x^2+y^2+z^2)2 - 2x(2x)}{(x^2+y^2+z^2)^2}$

$= \frac{2x^2+2y^2+2z^2-4x^2}{(x^2+y^2+z^2)^2}$

$= \frac{2y^2+2z^2-2x^2}{(x^2+y^2+z^2)^2}$

Illy

$\frac{\partial^2 u}{\partial y^2} = \frac{2x^2+2z^2-2y^2}{(x^2+y^2+z^2)^2}$ $\frac{\partial^2 u}{\partial z^2} = \frac{2x^2+2y^2-2z^2}{(x^2+y^2+z^2)^2}$

$\Rightarrow (x^2+y^2+z^2)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$

$\Rightarrow \cancel{(x^2+y^2+z^2)}\left(\frac{\cancel{2y^2}+2z^2-\cancel{2x^2}+\cancel{2x^2}+2z^2-\cancel{2y^2}+2x^2+2y^2-\cancel{2z^2}}{(x^2+y^2+z^2)^{\cancel{2}}}\right)$

$\Rightarrow \frac{2\cancel{(x^2+y^2+z^2)}}{\cancel{x^2+y^2+z^2}}$

$= 2$

Total Derivatives

If $u = f(x, y)$, $x = \phi(t)$, $y = \psi(t)$ then

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

If $u = f(x, y, z)$, $x = \phi(t)$, $y = \psi(t)$, $z = g(t)$ then

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$

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5. If $u = \sin\left(\frac{x}{y}\right)$, $x = e^t$, $y = t^2$, find $\frac{du}{dt}$

$u = \sin\left(\frac{x}{y}\right)$, $x = e^t$, $y = t^2$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \quad \text{---} \ (1)$$

$$\frac{\partial u}{\partial x} = \cos\frac{x}{y} \cdot \frac{1}{y}$$

$$\frac{\partial u}{\partial y} = x\cos\frac{x}{y} \cdot \left(\frac{-1}{y^2}\right)$$

$$= \frac{-x}{y^2}\cos\frac{x}{y}$$

$$\frac{dx}{dt} = \frac{d}{dt}(e^t) \qquad \frac{dy}{dt} = \frac{d}{dt}t^2$$

$$= e^t \qquad\qquad = 2t$$

from (1)

$$\frac{du}{dt} = \frac{\cos\left(\frac{x}{y}\right)}{y}e^t + \left[\frac{-x}{y^2}\cos\left(\frac{x}{y}\right)\right]2t$$

$$= \frac{\cos\left(\frac{e^t}{t^2}\right)e^t}{t^2} - \frac{e^t}{t^{\cancel{4}}}\cos\left(\frac{e^t}{t^2}\right)2\cancel{t}$$

$$= \frac{\cos\left(\frac{e^t}{t^2}\right)e^t}{t^2}\left[1 - \frac{2}{t}\right]$$

$$= \frac{\cos\left(\frac{e^t}{t^2}\right)e^t}{t^2}\left[\frac{t-2}{t}\right]$$

$$\frac{du}{dt} = \frac{(t-2)e^t\cos\left(\frac{e^t}{t^2}\right)}{t^3} = \frac{e^t(t-2)}{t^3}\cos\left(\frac{e^t}{t^2}\right)$$

1. $z = u^2+v^2$, $u = at^2$, $v = 2at$ find $\frac{dz}{dt}$

$$z = u^2+v^2$$
$$u = at^2 \quad v = 2at$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial u} \cdot \frac{du}{dt} + \frac{\partial z}{\partial v} \cdot \frac{dv}{dt}$$

$\frac{\partial z}{\partial u} = \frac{d(u^2+v^2)}{dx}$ $\quad \frac{\partial z}{\partial v} = 2v$

$\frac{\partial z}{\partial u} = 2u$

$\frac{du}{dt} = 2at$ $\quad \frac{dv}{dt} = 2a$

$$\frac{dz}{dt} = \frac{\partial z}{\partial u} \cdot \frac{du}{dt} + \frac{\partial z}{\partial v} \cdot \frac{dv}{dt}$$

$$\frac{dz}{dt} = 2u \cdot 2at + 2v \cdot 2a$$
$$= 4at(at^2) + 4a(2at)$$
$$= 4a^2t^3 + 8a^2t$$
$$\boxed{\frac{dz}{dt} = 4a^2t(t^2+2)}$$

2. $u = x^2-y^2$, $x = 2r+4$, $y = -r-5$ find $\frac{du}{dr}$

$$u = x^2-y^2$$
$$x = 2r+4 \quad y = r-5$$

$$\frac{du}{dr} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dr} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dr}$$

$\frac{\partial u}{\partial x} = 2x$ $\qquad$ $\frac{\partial u}{\partial y} = -2y$

$\frac{dx}{dr} = 2$ $\qquad$ $\frac{dy}{dr} = -1$

from (2)

$$\frac{du}{dr} = 2x \cdot 2 + (-2y)(-1)$$
$$= 4x + 2y$$
$$= 4(2r+4) + 2(-r-5)$$
$$= 8r+16 + (-2)(-10)$$
$$= 8r+16 + (-2r-10)$$
$$= 6r + 6$$
$$= 6(r+1)$$

~~If $u = x^2 - 2y$~~

3. If $u = y^2 - 4ax$, $x = at^2$, $y = 2at$ find $\frac{du}{dt}$

$$u = y^2 - 4ax$$
$$x = at^2 \quad y = 2at$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$\frac{\partial u}{\partial x} = 0 - 4a$ $\qquad$ $\frac{\partial u}{\partial y} = 2y$

$= -4a$

$\frac{dx}{dt} = 2at$ $\qquad$ $\frac{dy}{dt} = 2a$

$$= -4a(2at) + 2y(2a)$$

$$= -8a^2t + 4ay$$

$$= -8a^2t + 4a(2at)$$

$$= \cancel{-8a^2t + 8a^2t}$$

$$= 0$$

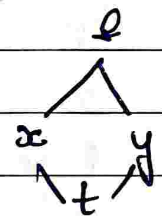
4. If x increases at the rate of change $2cm/s$ when $x = 3cm$ $y = 1cm$. At what rate y should changing in order that the function $f = 2xy - 3x^2y$ shall be neither increasing nor decreasing.

$$\frac{dx}{dt} = 2\,cms^{-1}$$

$x = 3\,cm$

$y = 1\,cm$

$\frac{dy}{dt} = ?$



$$\frac{df}{dt} = 0$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

$$0 = \frac{\partial}{\partial x}(2xy - 3x^2y) \cdot 2 + \frac{\partial}{\partial y}(2xy - 3x^2y) \cdot \frac{dy}{dt}$$

$$0 = 2y - 6xy \cdot 2 + 2x - 3x^2 \cdot \frac{dy}{dt}$$

$$0 = 2(1) - 6(3)(1) \cdot 2 + 2(3) - 3(3)^2 \cdot \frac{dy}{dt}$$

$$0 = 2 - 18 \cdot 2 + 6 - 27 \cdot \frac{dy}{dt}$$

$$0 = (-16) \cdot 2 - 21 \cdot \frac{dy}{dt}$$

$$0 = -32 - 21$$

$$\boxed{\frac{dy}{dt} = -\frac{32}{21}}$$

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Jacobian (u,v)

- If u and v are the functions of two independent variables x and y, then

$$\frac{\partial(u,v)}{\partial(x,y)} = J(u,v) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

- If u, v and w are the functions of three independent variables x, y and z, then

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

- Chain rule

- If u and v are functions of two independent variable x and y, x and y is again function of r & s, then

$$\frac{\partial(u,v)}{\partial(r,s)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,s)}$$

1. If $u = x^2 - y^2$, $v = x+y$, then find $\frac{\partial(u,v)}{\partial(x,y)}$

$u = x^2 - y^2$

$v = x+y$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$\frac{\partial y}{\partial x} = \begin{vmatrix} 2x & -2y \\ 1 & 1 \end{vmatrix}$$

$$= 2x - (-2y)$$

$$= 2(x+y)$$

2. If $u = x - y$, $v = xy$, then find $\frac{\partial(u,v)}{\partial(x,y)}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -1 \\ y & x \end{vmatrix}$$

$$= x - (-y)$$

$$= x + y$$

3. If $u = x^3 + y^3$, $v = x^2 + y^2$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 3x^2 & 3y^2 \\ 2x & 2y \end{vmatrix}$$

$$= 3x^2(2y) - [3y^2(2x)]$$

$$= 6x^2y - 6xy^2$$

$= 6xy(x-y)$

4. If $u = x^2 - y^2$, $v = x+y$ then find $\frac{\partial(u,v)}{\partial(x,y)}$

$u = x^2 - y^2$, $v = x+y$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$\frac{\partial u}{\partial x} = 2x$ $\qquad$ $\frac{\partial u}{\partial y} = -2y$

$\frac{\partial v}{\partial x} = 1$ $\qquad$ $\frac{\partial v}{\partial x} = 1$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & 2y \\ 1 & 1 \end{vmatrix} = 2x+2y$$

$$= 2(x+y)$$

5. If $u = x+3y^2-z^3$, $v = 4x^2yz$, $w = 2z^2 - xy$, find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ at $(1,-1,0)$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x+3y^2-z^3) = 1$ $\qquad$ $\frac{\partial v}{\partial x} = 8xyz$

$\frac{\partial u}{\partial y} = 6y$ $\qquad$ $\frac{\partial v}{\partial y} = 4x^2z$

$\frac{\partial u}{\partial z} = -3z^2$ $\qquad$ $\frac{\partial v}{\partial z} = 4x^2y$

$$\frac{\partial w}{\partial x} = -y$$

$$\frac{\partial w}{\partial y} = -x$$

$$\frac{\partial w}{\partial z} = 4z$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 6y & -3z^2 \\ 8xyz & 4x^2z & 4x^2y \\ -y & -x & 4z \end{vmatrix}$$

Put the point $(1,-1,0)$

i.e., $x=1$, $y=-1$, $z=0$

$$= \begin{vmatrix} 1 & -6 & 0 \\ 0 & 0 & -4 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= 4(-1+6)$$

$$= 4(5)$$

$$= 20$$

Maximum and minimum of function of two variables

Steps to find maximum and minimum of a function with two variable

Step 1: Write the given function f

Step 2: Find $\frac{\partial f}{\partial x}$ $\frac{\partial f}{\partial y}$.

$$r = \frac{\partial^2 f}{\partial x^2}$$

$$t = \frac{\partial^2 f}{\partial y^2}$$

$$s = \frac{\partial^2 f}{\partial x \partial y}$$

Step 3: Consider $\frac{\partial f}{\partial x} = 0$ — (1) and $\frac{\partial f}{\partial y} = 0$ — (2)

Solving (1) & (2) we get critical point

Step 4: Substitute critical point in r, t & s and also find $rt - s^2$

- If $rt - s^2 > 0$ and $r > 0$ then the function has minimum at that critical point

- If $rt - s^2 > 0$ and $r < 0$ then the function has maximum at that critical point

- If $rt - s^2 < 0$ then the function neither maximum nor minimum

- If $rt - s^2 = 0$ then method fail (Doubtfull case

1. Discuss maximum and minimum of the function
$f = x^2 + y^2 - 4x + 6y - 5$

Let $f = x^2 + y^2 - 4x + 6y - 5$

$\frac{\partial f}{\partial x} = 2x + 0 - 4 + 0 - 0$

$= 2x - 4$

$\frac{\partial f}{\partial y} = 0 + 2y - 0 + 6 - 0$

$= 2y + 6$

$r = \frac{\partial^2 f}{\partial x^2}$

$= 2$

$t = \frac{\partial^2 f}{\partial y^2}$

$= 2$

$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

$= \frac{\partial}{\partial x}(2y + 6)$

$= 0$

Let $f_x = 0$ & $f_y = 0$

$2x - 4 = 0$ $\quad$ $2y + 6 = 0$

$\boxed{x = 2}$ $\quad$ $\boxed{y = -3}$

∴ Critical point $(2, -3)$

Substitute $(2, -3)$ in r, t, s

$r(2, -3) = 2$

$t(2, -3) = 2$

$s(2, -3) = 0$

$rt - s^2 > 0$

$4 > 0$

$r = 2 > 0$

here $rt - s^2 > 0$ and $r > 0$, the function has minimum at $(2, -3)$

2. Discuss maximum and minimum of the function

$f = x^2 - 3xy + y^2 + 2x$

Let $f = x^2 - 3xy + y^2 + 2x$

$\frac{\partial f}{\partial x} = 2x - 3y + 2$ —— ①

$\frac{\partial f}{\partial y} = -3x + 2y$ —— ②

$r = \frac{\partial^2 f}{\partial x^2}$

$= 2$

$t = \frac{\partial^2 f}{\partial y^2}$

$= 2$

$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

$= \frac{\partial}{\partial x}(2y - 3x)$

$= -3$

Let $f_x = 0$ and $f_y = 0$

$2x - 3y + 2 = 0$ $\quad$ $2y + 3x = 0$

Solve ① & ②

$$\begin{matrix} -3 & 2 & 2 & -3 \\ 2 & 0 & 3 & 2 \end{matrix}$$

$$\frac{x}{-4} = \frac{y}{-6} = \frac{1}{-5}$$

$$\frac{x}{-4} = \frac{1}{-5} \Rightarrow x = \frac{4}{5}$$

$$\frac{y}{-6} = \frac{1}{-5} \Rightarrow y = \frac{6}{5}$$

Substitute $\left(\frac{4}{5}, \frac{6}{5}\right)$

$r\left(\frac{4}{5}, \frac{6}{5}\right) = 2$

$t\left(\frac{4}{5}, \frac{6}{5}\right) = 2$

$s\left(\frac{4}{5}, \frac{6}{5}\right) = -3$

$rt - s^2 = 0$

$4 - 9 = 0$

$-5 < 0$

∴ If here $rt - s^2 < 0$ so the function has neither maximum nor minimum

3. Discuss maximum and minimum of the function $f = x^3 + y^3 - 3x - 12y + 20$

Let $f = x^3 + y^3 - 3x - 12y + 20$

$\frac{\partial f}{\partial x} = 3x^2 - 3$, $\frac{\partial f}{\partial y} = 3y^2 - 12$

$$r = \frac{\partial^2 f}{\partial x^2}$$
$$= 6x$$

$$t = \frac{\partial^2 f}{\partial y^2}$$
$$= 6y$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}(3y^2 - 12)$$

$$= 0$$

Let

$3x^2 - 3 = 0$ | $3y^2 - 12 = 0$
$3x^2 = 3$ | $3y^2 = 12$
$x^2 = 1$ | $y^2 = 4$
$x = \pm 1$ | $y = \pm 2$

points $(1,2)$, $(1,-2)$, $(-1,2)$, $(-1,-2)$

points	$r = 6x$	$t = 6y$	$s = 0$	$rt - s^2$	cond
$(1,2)$	6	12	0	72	min
$(1,-2)$	6	-12	0	-72	No
$(-1,2)$	-6	12	0	-72	No
$(-1,-2)$	-6	-12	0	72	Max

4. Discuss the maximum and minimum of $f(x,y) = x^3y^2(1-x-y)$

Let $f(x,y) = x^3y^2(1-x-y)$
$$= x^3y^2 - x^4y^2 - x^3y^3$$

$$\frac{\partial f}{\partial x} = 3y^2x^2 - 4y^2x^3 - 3y^2x^3 \qquad \frac{\partial f}{\partial y} = 2x^3y - 2x^4y - 3x^3y^2$$

$$r = \frac{\partial^2 f}{\partial x^2}$$

$$= 6y^2x - 12x^2y^2 - 6y^3x$$

$$t = \frac{\partial^2 f}{\partial y^2}$$

$$= 2x^3 - 2x^4 - 6x^3y$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(2x^3y - 2x^4y - 3x^3y^2\right)$$

$$s = 6x^2y - 8x^3y - 9x^2y^2$$

let $\frac{\partial f}{\partial x} = 0$ $\quad \frac{\partial f}{\partial y} = 0$

$$\frac{\partial f}{\partial x} = 0 \qquad\qquad \frac{\partial f}{\partial y} = 0$$

$$3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0 \qquad\qquad 2x^3y - 2x^4y - 3x^3y^2 = 0$$

$$x^2y^2(3 - 4x - 3y) = 0 \qquad\qquad x^3y(2 - 2x - 3y) = 0$$

$$3 - 4x - 3y = 0 \text{ — (1)} \qquad\qquad 2 - 2x - 3y = 0 \text{ — (2)}$$

Solving (1) & (2) by 2312 -

$$\frac{x}{} \quad \frac{y}{} \quad \frac{1}{}$$

3	-3	4	3
3	-2	2	3

$$\frac{x}{-6+9} = \frac{y}{-6+8} = \frac{1}{12-6}$$

$$\frac{x}{\cancel{3}_1} = \frac{1}{\cancel{6}_2} \qquad\qquad \frac{y}{\cancel{2}_1} = \frac{1}{\cancel{6}_3}$$

$$x = \frac{1}{2} \qquad\qquad y = \frac{1}{3}$$

Stationary points $\left(\frac{1}{2},\frac{1}{3}\right)$ $(0,0)$

we find $rt-s^2$

$$r\left(\frac{1}{2},\frac{1}{3}\right) = 6xy^2 - 12x^2y^2 - 6xy^3$$

$$= 6\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^2 - 12\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right)^2 - 6\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^3$$

$$= \frac{1}{3} - \frac{1}{3} - \frac{1}{9}$$

$$= -\frac{1}{9}$$

$$s\left(\frac{1}{2},\frac{1}{3}\right) = 6x^2y - 8x^3y - 9x^2y^2$$

$$= 6\left(\frac{1}{4}\right)\left(\frac{1}{3}\right) - 8\left(\frac{1}{8}\right)\left(\frac{1}{3}\right) - 9\left(\frac{1}{4}\right)\left(\frac{1}{9}\right)$$

$$= \frac{1}{2} - \frac{1}{3} - \frac{1}{4}$$

$$= \frac{6-4-3}{12}$$

$$= -\frac{1}{12}$$

$$t\left(\frac{1}{2},\frac{1}{3}\right) = 2x^3 - 2x^4 - 6x^2y$$

$$= 2\left(\frac{1}{8}\right) - 2\left(\frac{1}{16}\right) - 6\left(\frac{1}{4}\right)\left(\frac{1}{3}\right)$$

$$= \frac{1}{4} - \frac{1}{8} - \frac{1}{2}$$

$$t = \frac{2-1-4}{8} = \frac{2-5}{8} = -\frac{3}{8}$$

$$rt - s^2 = -\frac{1}{9}\left(-\frac{3}{8}\right) - \frac{1}{144}$$

$$= \frac{1}{24} - \frac{1}{144}$$

$$= \frac{1}{24}\left(1-\frac{1}{6}\right) = \frac{1}{24}\left(\frac{6-1}{6}\right) = \frac{1}{24}\left(\frac{5}{6}\right)$$

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$rt - s^2 = \frac{5}{144} > 0$

$r = -\frac{1}{9} < 0$

f has maximum at $\left(\frac{1}{12}, \frac{1}{3}\right)$

$\Rightarrow r(0,0) = 0$, $s(0,0) = 0$ $t(0,0) = 0$
$rt - s^2 = 0$ fails

5. Find the maximum and minimum value of $x^3 + y^3 - 3axy$

Let $f = x^3 + y^3 - 3axy$

$\frac{\partial f}{\partial x} = 3x^2 - 3ay$

$\frac{\partial f}{\partial y} = 3y^2 - 3ax$

$\frac{\partial^2 f}{\partial x^2} = 6x = r$

$\frac{\partial^2 f}{\partial y^2} = 6y = t$

$s = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

$= \frac{\partial}{\partial x}(3y^2 - 3ax)$

$= -3a$

Let $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$

$\frac{\partial f}{\partial x} = 0$

$\frac{\partial f}{\partial y} = 0$

$3x^2 - 3ay = 0$

$x^2 = ay$

$3y^2 - 3ax = 0$

$y^2 = ax$

$x = \frac{y^2}{a}$

Sub (2) in (1)

$\frac{y^4}{a^2} = ay$

$y^4 = a^3 y$

$y^4 - a^3 y = 0$

$y(y^3 - a^3) = 0$

$y^3 - a^3 = 0$

$y = 0 \quad y = a$

Sub $y = 0$ in ② Sub $y = a$ in ②

$x = 0$ $x = \frac{a^2}{a} = a$

point $(0,0)$ point (a,a)

Now we find

$r(a,a) = 6(a) = 6a$

$s(a,a) = -3a$

$t(a,a) = 6a$

$rt - s^2$ at (a,a)

$= (6a)(6a) - 9a^2$

$= 36a^2 - 9a^2$

$= 27a^2 > 0$

$rt - s^2 > 0, \quad r = 6a > 0$ if $a > 0$

$r = 6a < 0$ if $a < 0$

If $a > 0$ then the function f has minimum value at (a,a)

If $a < 0$ then the function has maximum value.

at $(0,0)$, $rt - s^2 = 0 \times 0 - 9a^2 < 0$ then the function has no maximum, no minimum

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6. Find the maximum and minimum of
$f = 2(x^2 - y^2) - x^4 + y^4$

let $f = 2(x^2 - y^2) - x^4 + y^4$

$f = 2x^2 - 2y^2 - x^4 + y^4$

$\frac{\partial f}{\partial x} = 4x - 4x^3$ $\qquad$ $\frac{\partial f}{\partial y} = 4y^3 - 4y$

$r = \frac{\partial^2 f}{\partial x^2}$ $\qquad$ $t = \frac{\partial^2 f}{\partial y^2}$

$= 4 - 12x^2$ $\qquad$ $= 12y^2 - 4$

$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

$= \frac{\partial}{\partial x}(4y^3 - 4y)$

$s = 0$

let $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$4x - 4x^3 = 0$ $\qquad$ $4y^3 - 4y = 0$

$4x(1 - x^2) = 0$ $\qquad$ $+4y(y^2 - 1) = 0$

$x = 0, x^2 = 1$ $\qquad$ $y = 0, y^2 = 1$

$x = \pm 1$ $\qquad$ $y = \pm 1$

The points are $(0,0), (0,1), (0,-1), (1,0), (-1,0)$
$(1,1), (-1,-1)$

points	r	t	s	$rt-s^2$	conclusion
A(0,0)	4	-4	0	-16	no max no min
B(0,1)	-4	8	0	32	min
C(0,-1)	4	8	0	32	min
D(1,0)	-8	-4	0	32	max
E(-1,0)	-8	-4	0	32	max
F(1,1)	-8	8	0	-64	no max no min
G(1,-1)	-8	8	0	-64	no max no min