Module 5:
Integral calculus

Integration formulas (Indefinite)

1. $\int 1\,dx = x + c$, $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x$

2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$

3. $\int \frac{1}{x}\,dx = \log x + c$

4. $\int e^x\,dx = e^x + c$

5. $\int \sin x\,dx = -\cos x + c$

6. $\int \cos x\,dx = \sin x + c$

7. $\int \tan x\,dx = \log \sec x + c$

8. $\int \cot x\,dx = \log \sin x + c$

9. $\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + c$

10. $\int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c$

Eg: $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

$= -\int \frac{-\sin x}{\cos x}\,dx$

$= -\log \cos x + c$

$= \log \cos^{-1}x + c$

$= \log \sec x + c$

11. $\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)}$

Definite Integral

Definite Integral is used to calculate area under the curve lies between the limits a to b. It is represented by $\int_a^b f(x)dx$ where b is upper limit and a is lower limit

$$\int_a^b f(x)dx = \left(F(x)\right)_a^b = F(b) - F(a)$$

This depends on the limits but not on the Integration of the variable.

1. Evaluate $\int_{-1}^{1} x+1 \, dx$

$$I = \int_{-1}^{1} x+1 \, dx$$

$$= \left[\frac{x^2}{2} + x\right]_{-1}^{1}$$

$$= \left[\frac{1}{2} + 1\right] - \left[\frac{1}{2} - 1\right]$$

$$= 1 + 1$$

$$= 2$$

2. Evaluate $\int_2^3 \frac{1}{x}\,dx$

$$I = \int_2^3 \frac{1}{x}\,dx$$

$$= \left[\log x\right]_2^3$$

$$= \log 3 - \log 2$$

$$= \log \frac{3}{2}$$

3. Evaluate $\int_1^2 4x^3 - 5x^2 + 6x + 9\,dx$

$$= \left[\frac{\cancel{4}x^4}{\cancel{4}} - \frac{5x^3}{3} + \frac{\cancel{6}^{3}x^2}{\cancel{2}} + 9x\right]_1^2$$

$$= \left[x^4 - \frac{5x^3}{3} + 3x^2 + 9x\right]_1^2$$

$$= \left[16 - \frac{5}{3}\times 8 + 3(4) + 9(2)\right] - \left[1 - \frac{5}{3} + 3 + 9\right]$$

$$= \left[16 - \frac{40}{3} + 12 + 18\right] - \left[13 - \frac{5}{3}\right]$$

$$= \left[46 - \frac{40}{3}\right] - \left[13 - \frac{5}{3}\right]$$

$$= 46 - \frac{40}{3} - 13 + \frac{5}{3}$$

$$= 33 - \frac{35}{3}$$

$$= \frac{99 - 35}{3}$$

$$= \frac{64}{3}$$

4. Evaluate $\int_0^1 \frac{1}{\sqrt{1-x^2}} dx$

$I = \int_0^1 \frac{1}{\sqrt{1-x^2}} dx$

$= [\sin^{-1} x]_0^1$

$= \sin^{-1} 1 - \sin^{-1} 0$

$= \frac{\pi}{2} - 0$

$= \frac{\pi}{2}$

5. Evaluate $\int_0^1 \frac{1}{1+x^2} dx$

$I = \int_0^1 \frac{1}{1+x^2} dx$

$= [\tan^{-1} x]_0^1$

$= \tan^{-1} 1 - \tan^{-1} 0$

$= \frac{\pi}{4} - 0$

$= \frac{\pi}{4}$

Improper Integrals:

Improper Integral is a definite integral that has the limit as infinite.

1. Evaluate $\int_1^\infty \frac{1}{x^2} dx$

$= \lim_{t \to \infty} \int_1^t x^{-2} dx$

$\Rightarrow \lim_{t \to \infty} \left[\frac{x^{-1}}{-1}\right]_1^t$

6. $\displaystyle\int_1^2 \frac{2x}{\sqrt{x^2+1}}\,dx$

$= \left(2\sqrt{x^2+1}\right)_1^2$

$= 2\sqrt{3} - 2\sqrt{2}$

$= 2\left(\sqrt{3}-\sqrt{2}\right)$

7. $\displaystyle\int_2^3 \frac{2x}{x^2+1}\,dx$

Let $f(x) = x^2+1$

$f'(x) = 2x$

$\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \log f(x)$

$= \left[\log(x^2+1)\right]_2^3$

$= \log(9+1) - \log(4+1)$

$= \log \frac{10}{5}$

$= \log 2$

8. $\int_0^3 x e^{x^2} dx$

$\int_0^3 x e^{x^2} dx$

Let $t = e^{x^2}$

$dt = e^{x^2} 2x\, dx$

$\frac{1}{2} dt = x e^{x^2} dx$

$t = e^{x^2}$ $\Rightarrow$ If $x = 0 \Rightarrow t = 1$

$\Rightarrow$ If $x = 3 \Rightarrow t = e^9$

$= \int_0^3 \frac{1}{2} dt$

$= \frac{1}{2}\left[t\right]^{e^9}$

$= \frac{1}{2}\left[e^9 - 0\right]$

$= \frac{e^9}{2}$

9. $\int_0^{\pi/2} \cos x + \sin x \quad dx$

$= \left(-\sin x - \cos x\right)_0^{\pi/2}$

$= \left(\sin \pi/2 - \cos \pi/2\right) - \left(\sin 0 - \cos 0\right)$

$= 1 - 0 - 0 + 1$

$= 2$

Improper Integrals

Improper Integral is a definite integral that has the limit as infinite.

1. Evaluate $\int_1^{\infty} \frac{1}{x^2} dx$

$$= \lim_{t \to \infty} \int_1^t x^{-2} dx$$

$$\Rightarrow \lim_{t \to \infty} \left[\frac{x^{-1}}{-1} \right]_1^t$$

$= -\lim_{t \to \infty} \left(\frac{1}{t} - \frac{1}{1} \right)$

$= -\left(\frac{1}{\infty} - 1 \right)$

$= -(0-1)$

$= 1$

2. Evaluate $\int_1^\infty \frac{1}{x}\,dx$

$= \lim_{t \to \infty} \int_1^t \frac{1}{x}\,dx$

$= \lim_{t \to \infty} [\log x]_1^t$

$= \lim_{t \to \infty} \log t - \log 1$

$= \lim_{t \to \infty} \log t$

$= \log \infty$

$= \infty$

3. Evaluate $\int_{-2}^\infty \sin x\,dx$

$= \lim_{t \to \infty} \int_{-2}^t \sin x\,dx$

$= \lim_{t \to \infty} [-\cos x]_{-2}^t$

$= -\lim_{t \to \infty} [\cos t - \cos(-2)]$

$= -[\cos \infty - \cos 2]$

$= \infty$

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4. Evaluate $\int_{\sqrt{3}}^{\infty} \frac{1}{1+x^2} dx$

$= \lim_{t\to\infty} \int_{\sqrt{3}}^{t} \frac{1}{1+x^2} dx$

$= \lim_{t\to\infty} \left[\tan^{-1}x\right]_{\sqrt{3}}^{t}$

$= \lim_{t\to\infty} \left[\tan^{-1}t - \tan^{-1}\sqrt{3}\right]$

$= \tan^{-1}\infty - \tan^{-1}\sqrt{3}$

$= \cancel{\tan^{-1}}\left(\cancel{\tan}\frac{\pi}{2}\right) - \cancel{\tan^{-1}}\left(\cancel{\tan}\frac{\pi}{3}\right)$

$= \frac{\pi}{2} - \frac{\pi}{3}$

$= \frac{\pi}{6}$

5. Evaluate $\int_{-\infty}^{0} \frac{1}{\sqrt{3+x}} dx$

$= \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{\sqrt{3+x}} dx$

let $f = 3+x$

$f' = 1$

It is in the form of

$\int \frac{f'(x)}{\sqrt{f(x)}} = 2\sqrt{f(x)}$

$= \lim_{t\to\infty} \left[2\sqrt{3+x}\right]_{t}^{0}$

$= \lim_{t\to\infty} \left[2\left(\sqrt{3+0} - \sqrt{3+t}\right)\right]$

$= 2\left(\sqrt{3} - \sqrt{3-\infty}\right)$

$= \infty$

6. Evaluate $\int_{4}^{\infty} 2 - 4x + 6x^2 \, dx$

$$= \lim_{t \to \infty} \int_{4}^{t} 2 - 4x + 6x^2 \, dx$$

$$= \lim_{t \to \infty} \left[2x - 2x^2 + 2x^3 \right]_{4}^{t}$$

$$= \lim_{n \to \infty} \left(2t - 2t^2 + 2t^3 \right) - \left(8 - 32 + 128 \right)$$

$$= \lim_{n \to \infty} \left(2t - 2t^2 + 2t^3 \right) - (104)$$

Gamma function

The gamma function is an extended form of an factorial function and which is defined by

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x}\, dx$$

and $\boxed{\Gamma(n) = (n-1)!}$

1. Evaluate $\int_0^{\infty} x^{7} e^{-x}\, dx$ using gamma function

By gamma function

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x}\, dx$$

$$I = \int_0^{\infty} x^{7} e^{-x}\, dx$$

On comparing ① & ②

$7 = n-1$

$n = 8$

$\Gamma(n) = (n-1)!$

$\Gamma(8) = 7!$

$= 5040$

Note

- $\Gamma(n+1) = n!$
- $\Gamma(1) = 1$
- $\Gamma(0) = \infty$

- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

2. Evaluate $\int_0^\infty x^{\frac{7}{2}} e^{-x}\, dx$ using gamma function

By gamma function

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx \quad \text{--- (1)}$$

$$I = \int_0^\infty x^{\frac{7}{2}} e^{-x}\, dx \quad \text{--- (2)}$$

On comparing (1) & (2)

$$\frac{7}{2} = n-1$$

$$n = \frac{9}{2}$$

$$\Gamma\left(\frac{9}{2}\right) = \left(\frac{9}{2} - 1\right)!$$

$$= \frac{7}{2}!$$

$$=$$

Beta function

The beta function is defined by

$$\beta(m,n) = \int_0 x^{m-1} (1-x)^{n-1}\, dx$$

The formula to find

$$\beta(m,n) = \frac{\Gamma m \cdot \Gamma n}{\Gamma m+n} = \frac{(m-1)!\,(n-1)!}{(m+n-1)!}$$

1. Find $\beta(8,6)$

$m = 8$

$n = 6$

$$\beta(8,6) = \frac{\Gamma 8 \cdot \Gamma 6}{\Gamma 14}$$

$$= \frac{7!\ 5!}{13!}$$

$$= \frac{5040}{\cancel{7\times6\times5\times4\times3\times2}\times\cancel{5}\times\cancel{4}\times\cancel{3}\times\cancel{2}}{13\times12\times11\times10\times9\times8\times\cancel{7\times6\times5\times3\times4\times2}}$$

$$= \frac{1}{10296}$$

2. Evaluate $\int_0^1 x^7 (1-x)^4\,dx$ ② using β function

By β function

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad ①$$

$$I = \int_0^1 x^7 (1-x)^4\,dx$$

By comparing ① & ②

$m-1 = 7$, $m = 8$

$n-1 = 4$, $n = 5$

$$\therefore \beta(8,5) = \frac{\Gamma 8\ \Gamma 5}{\Gamma 13} = \frac{7!\ 4!}{12!}$$

$$= \frac{7!\,4!}{12!}$$

$$= \frac{\cancel{7!}\ \cancel{4}\times\cancel{3}\times\cancel{2}}{12\times 11\times 10\times\cancel{9}\times\cancel{8}\times\cancel{7!}}$$

$$= \frac{1}{}$$

3. Evaluate $\int_0^1 x^{7/2}(1-x)^{-3/2}\,dx$ using Γ function

By using Beta function

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$I = \int_0^1 x^{7/2}(1-x)^{-3/2}\,dx$$

On comparing ① & ②

$$m-1 = \frac{7}{2} \Rightarrow m = 9/2$$

$$n-1 = \frac{-3}{2} \Rightarrow n = -1/2$$

$$\beta\left(\frac{9}{2}, -1/2\right) = \frac{\Gamma\frac{9}{2}\ \Gamma{-1/2}}{\Gamma\left(\frac{9}{2}-\frac{1}{2}\right)}$$

$$= \frac{\Gamma\frac{9}{2}\ \Gamma{-1/2}}{\Gamma\frac{8}{2}}$$

$$= \frac{\left(\frac{9}{2}-1\right)!\left(\frac{-1}{2}-1\right)!}{3!}$$

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4. Evaluate $\int_0^1 x^9 (1-x)^{-1/2} dx$ ①

By using Beta function

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

$$I = \int_0^1 x^9 (1-x)^{-1/2} dx$$

On comparing ① & ②

$$m-1 = 9 \Rightarrow m = 10$$

$$n-1 = \frac{-1}{2} \Rightarrow n = \frac{1}{2}$$

$$\beta\left(10, \frac{1}{2}\right) = \frac{\Gamma 10 \; \Gamma \frac{1}{2}}{\Gamma\left(10 + \frac{1}{2}\right)}$$

$$= \frac{\Gamma 10 \; \Gamma \frac{1}{2}}{\Gamma \frac{21}{2}}$$

$$= \frac{9! \; \frac{1}{2}!}{\frac{21}{2}!}$$

5. Evaluate $\int_0^1 x^{10} (1-x)^5 dx$

By using Beta function

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

$$I = \int_0^1 x^{10} (1-x)^5 dx$$

On comparing ① & ②

$$m-1 = 10 \Rightarrow m = 11$$

$$n-1 = 5 \Rightarrow n = 6$$

$$\beta(11, 6) = \frac{\Gamma 11 \, \Gamma 6}{\Gamma 17}$$

$$= \frac{10! \; 5!}{16!}$$

$$= \frac{\cancel{10!} \, \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2}}{16 \times \cancel{15} \times \cancel{14} \times 13 \times \cancel{12} \times 11 \times \cancel{10!}}$$

$$= \frac{1}{48048}$$

6. Evaluate $\int_0^1 y^{4/2} (1-y)^{1/2} \frac{1}{2} y^{-1/2} dy$

$= \frac{1}{2} \int_0^1 y^{3/2} (1-y)^{1/2} dy$

comparing with

$\frac{1}{2} \int_0^1 x^{m-1} (1-x)^{n-1} dx$

$m-1 = \frac{3}{2} \quad , \quad n-1 = \frac{1}{2}$

$m = \frac{5}{2} \quad , \quad n = \frac{3}{2}$

By beta function

$\beta\left(\frac{5}{2}, \frac{3}{2}\right) = \frac{\left(\frac{5}{2}-1\right)! \left(\frac{3}{2}-1\right)!}{(4-1)!}$

$= \frac{\frac{3}{2}! \; \frac{1}{2}!}{3!}$

$= \frac{\cancel{3}^{1}\sqrt{\pi}/4 \;\; \frac{\sqrt{\pi}}{2}}{\cancel{6}_{2}}$

$$= \frac{\pi}{16}$$

7. Evaluate $\int_0^1 \frac{x^3}{3\sqrt{1-x}}\,dx$ using beta function

$$\frac{1}{3}\int_0^1 x^3(1-x)^{-1/2}\,dx \quad \text{--- (1)}$$

By beta function

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad \text{--- (2)}$$

By comparing (1) & (2)

$m-1 = 3, \qquad n-1 = -1/2$

$m = 4, \qquad n = (1/2)$

$$\beta(m,n) = \frac{(m-1)!\,(n-1)!}{(m+n-1)!}$$

$$\frac{1}{3}\beta(4, 1/2) = \frac{3!\;\frac{-1}{2}!}{\frac{7}{2}!}$$

$$= \frac{6 \times \sqrt{\pi}}{\frac{105\sqrt{\pi}}{16}}$$

$$= \frac{1}{3} \times \frac{32}{35}$$

$$= \frac{32}{105}$$

$$= \frac{6 \times 16}{105}$$

$$= \frac{96}{105}$$

$$= \frac{32}{35}$$

8. Evaluate $\int_0^3 \frac{x^3}{\sqrt{3-x}}\,dx$ using beta function

$$\int_0^3 x^3 (3-x)^{-1/2}\,dx$$

$$\int_0^3 x^3\, 3^{-1/2}\left(1-\frac{x}{3}\right)^{-1/2} dx$$

let $\frac{x}{3} = z$ $\Rightarrow$ If $x=0 \Rightarrow z=0$

If $x=3 \Rightarrow z=1$

$x = 3z$

$dx = 3dz$

$$I = \frac{1}{\sqrt{3}}\int_0^1 (3z)^3 (1-z)^{-1/2}\, 3\,dz$$

$$= \sqrt{3}\, 27 \int_0^1 z^3 (1-z)^{-1/2}\, dz$$

By beta function

$m-1=3$ $\quad$ $n-1=-1/2$

$m=4$ $\quad$ $n=1/2$

$$I = 27\sqrt{3}\, \beta(4, 1/2)$$

$$\beta(4,1/2) = \frac{3!\ \left(-\frac{1}{2}\right)!}{\frac{7}{2}!} = \frac{6\times\sqrt{\pi}}{\frac{105\sqrt{\pi}}{16}} = \frac{6\times 16}{105}$$

$$I = \frac{\overset{9}{\cancel{27}}\sqrt{3}\times 6\times 16}{\underset{35}{\cancel{105}}}$$

Reduction formula

1. $\int \sin^n x\, dx = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x\, dx$

2. $\int \cos^n x\, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x\, dx$

3. $\int \tan^n x\, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx$

4. $\int \cot^n x\, dx = \frac{-\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\, dx$

5. $\int \sec^n x\, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x\, dx$

6. $\int \operatorname{cosec}^n x\, dx = \frac{-\operatorname{cosec}^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} \int \operatorname{cosec}^{n-2} x\, dx$

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7. $\int x^n e^{ax} dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1} e^{ax} dx$

8. $\int \sin^m x \cos^n x dx = \frac{-\sin^{m-1}x \cos^{n+1}x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2}x \cos^{n-2}x dx$

9. $\int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \cos^n x \, dx$

If n is even $= \left[\frac{(n-1)(n-3)(n-5)\dots}{n(n-2)(n-4)(n-6)\dots}\right]\frac{\pi}{2}$

If n is odd $= \frac{(n-1)(n-3)(n-5)\dots 2}{n(n-2)(n-4)(n-6)\dots 3}$

10. $\int_0^{\pi/2} \sin^m x \cos^n x \, dx$

If m & n are even $= \left[\frac{(m-1)(m-3)\dots(n-1)(n-3)\dots}{m+n(m+n-2)(m+n-4)(m+n-6)\dots}\right]\pi/2$

If m & n are odd $= \frac{(m-1)(m-3)\dots(n-1)(n-3)\dots 2}{m+n(m+n-2)(m+n-4)\dots(n+3)(n+1)\dots}$

1. Evaluate $\int \sin^4 x \, dx$

It is form of $\int \sin^n x \, dx = \frac{-\sin^{n-1}x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2}x dx$

Here $n=4$

$\int \sin^4 x \, dx = \frac{-\sin^3 x \cos x}{4} + \frac{3}{4}\int \sin^2 x \, dx$

Here $n=2$

$= \frac{-\sin^3 x \cos x}{4} + \frac{3}{4}\left[\frac{-\sin x \cos x}{2} + \frac{1}{2}\int \sin^0 x \, dx\right]$ x

$$\int \sin^4 x\,dx = \frac{-\sin^3 x\cos x}{4} - \frac{3\sin x\cos x}{8} + \frac{3}{8}x$$

2. Evaluate $\int \cos^3 x\,dx$

$$\int \cos^n x\,dx = \frac{\cos^{n-1}x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

put $n=3$

$$= \frac{\cos^2 x\sin x}{3} + \frac{2}{3}\int \cos x\,dx$$

$$= \frac{\cos^2 x\sin x}{3} + \frac{2}{3}\sin x$$

3. Evaluate $\int_0^{\pi/2} \cos^6 x\,dx$

$$\int_0^{\pi/2} \cos^6 x\,dx = \left[\frac{(n-1)(n-3)\ldots}{n(n-2)(n-4)\ldots}\right]\frac{\pi}{2} \quad n=6$$

Put $n=6$

$$= \left[\frac{5\times 3\times 1}{6\times 4\times 2}\right]\frac{\pi}{2}$$

$$= \frac{5\pi}{32}$$

4. Evaluate $\int_0^{\pi/6} \sin^5 3x\,dx$

let $3x = t$

$$3 = \frac{dt}{dx}$$

$$3dx = dt$$

To find limits for t

$3x = t$

Sub $x = 0 \Rightarrow t = 0$ (lower limit)

Sub $x = \frac{\pi}{6} \Rightarrow t = \frac{\cancel{3}\pi}{\cancel{6}_2}$ (upper limit)

$$\int_0^{\pi/2} \sin^5 t \, \frac{dt}{3}$$

$$= \frac{1}{3}\int_0^{\pi/2} \sin^5 t \, dt$$

$$= \frac{1}{3}\left[\frac{(n-1)(n-3)(n-5)\ldots\ldots 2}{n(n-2)(n-4)\ldots\ldots 3}\right]$$

Here $n = 5$ (odd)

$$= \frac{1}{3}\left[\frac{4 \times 2}{5 \times 3}\right]$$

$$= \frac{8}{45}$$

5. Evaluate $\int \sin^4 x \cos^2 x \, dx$

$$\int \sin^m x \cos^n x = \frac{-\sin^{m-1} x \cos^{n+1}}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} x \cos^{n-2} x \, dx$$

put $m = 4, n = 2$

$$= -\frac{\sin^3 x \cos^3 x}{6} + \frac{\cancel{3}}{\cancel{6}_2}\int \sin^2 x \cancel{\cos^0 x} \, dx$$

$$= \frac{-\sin^3 x \cos^3 x}{6} + \frac{1}{2}\left[\frac{-\sin x \cos x}{2} + \frac{1}{2}\int \sin^0 x \, dx\right]$$

$$= \frac{-\sin^3 x \cos^3 x}{6} - \frac{\sin x \cos x}{4} + \frac{1}{4}$$

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6. $\int \cos^4 x\, dx$

$\int \cos^4 x\, dx = \frac{\cos^3 x \sin x}{4} + \frac{3}{4}\int \cos^2 x\, dx$

when $n=2$

$= \frac{\cos^3 x \sin x}{4} + \frac{3}{4}\left[\frac{\cos x \sin x}{2} + \frac{1}{2}\int \cos^0 x\, dx\right]$

$= \frac{\cos^3 x \sin x}{4} + \frac{3}{8}\cos x \sin x + \frac{3}{8}\int 1\, dx$

$\int \cos^4 dx = \frac{\cos^3 x \sin x}{4} + \frac{3}{8}\cos x \sin x + \frac{3}{8}x$

7. $\int \sin^3 x\, dx$

$\int \sin^3 x\, dx = -\frac{\sin^2 x \cos x}{3} + \frac{2}{3}\int \sin x\, dx$

$= -\frac{\sin^2 x \cos x}{3} + \frac{2}{3}(-\cos x)$

$\int \sin^3 x\, dx = -\frac{\sin^2 x \cos x}{3} - \frac{2\cos x}{3}$

8. $\int \sin^5 x\, dx$

$\int \sin^5 x\, dx = -\frac{\sin^4 x \cos x}{5} + \frac{4}{5}\int \sin^3 x\, dx$

$= -\frac{\sin^4 x \cos x}{5} + \frac{4}{5}\left[-\frac{\sin^2 x \cos x}{3} + \frac{2}{3}\int \sin x\, dx\right]$

$= -\frac{\sin^4 x \cos x}{5} - \frac{4}{15}\sin^2 x \cos x + \frac{8}{15}(-\cos x)$

9. $\int \cos^5 x\, dx$

$\int \cos^5 x\, dx = \frac{\cos^4 x \sin x}{5} + \frac{4}{5}\int \cos^3 x\, dx$

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$$= \frac{\cos^4 x \sin x}{5} + \frac{4}{5}\left[\frac{\cos^2 x \sin x}{3} + \frac{2}{3}\int \cos x\, dx\right]$$

$$\int \cos^5 x\, dx = \frac{\cos^4 x \sin x}{5} + \frac{4}{15}\cos^2 x \sin x + \frac{8}{15}\sin x$$

10. $\int \tan^5 x\, dx$

$$\int \tan^5 x\, dx = \frac{\tan^4 x}{4} - \int \tan^3 x\, dx$$

$$= \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \int \tan x\, dx$$

$$\int \tan^5 x\, dx = \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \log \sec x$$

11. $\int \cot^6 x\, dx$

$n = 6$

$$\int \cot^6 x\, dx = -\frac{\cot^5 x}{5} - \int \cot^4 x\, dx$$

$$= -\frac{\cot^5 x}{5} - \left[-\frac{\cot^3 x}{3} - \int \cot^2 x\, dx\right]$$

$$= -\frac{\cot^5 x}{5} + \frac{\cot^3 x}{3} + \int \cot^2 x\, dx$$

$$= -\frac{\cot^5 x}{5} + \frac{\cot^3 x}{3} - \frac{\cot x}{1} - \int \cot^0 x\, dx$$

$$= -\frac{\cot^5 x}{5} + \frac{\cot^3 x}{3} - \cot x - \int 1\, dx$$

$$= -\frac{\cot^5 x}{5} + \frac{\cot^3 x}{3} - \cot x - x$$

12. $\int_0^{\pi/4} \sec^4 x\, dx$

$n = 4$

$$\int_0^{\pi/4} \sec^4 x\, dx = \left[\frac{\sec^2 x \tan x}{3} + \frac{2}{3}\int \sec^2 x\, dx\right]_0^{\pi/4}$$

$$= \left[\frac{\sec^2 x \tan x}{3} + \frac{2}{3}\tan x\right]_0^{\pi/4}$$

$$= \frac{\sec^2 \pi/4 \tan \pi/4}{3} + \frac{2}{3}\tan \pi/4$$

$$= \frac{2}{3} + \frac{2}{3}$$

$$= \frac{4}{3}$$

13. $\int_0^{\pi/2} \cos^9 x \, dx$

$n = 9$ (odd)

$$= \frac{(n-1)(n-3)(n-5)(n-7)}{n(n-2)(n-4)\ldots}$$

$$= \frac{8\times 6\times 4\times 2}{9\times 7\times 5\times 3\times 1}$$

$$= \frac{128}{315}$$

14. $\int_0^{\pi/6} \cos^5 3\theta \, d\theta$

Let $3\theta = x$ If $\theta = 0 \Rightarrow x = 0$

$3d\theta = dx$ If $\theta = \frac{\pi}{6} \Rightarrow x = \pi/2$

$$= \frac{1}{3}\int_0^{\pi/2} \cos^5 x \, dx$$

$n = 5$ (odd)

$$= \frac{1}{3}\left[\frac{(n-1)(n-3)(n-5)\ldots 2}{n(n-2)(n-4)\ldots 3}\right]$$

$$= \frac{1}{3}\left[\frac{4\times 2}{5\times 3}\right]$$

$= \frac{8}{45}$

15. $\int_0^{\pi/2} \sin^9 x \, dx$

$n = 9$ (odd)

$$\int_0^{\pi/2} \sin^9 x \, dx = \left[\frac{(n-1)(n-3)(n-5)\cdots 2}{n(n-2)(n-4)\cdots 3}\right]$$

$$= \frac{8 \times \cancel{6}^{2} \times 4 \times 2}{9 \times 7 \times 5 \times \cancel{3}_{1}}$$

$$= \frac{128}{945}$$