ME639
Assignment-3&4

Que :- Singularities

→ Specific conf. of manipulator where it losses its ability to generate velocities in particular direction/orientation (or All).
- losses some DoF, making it impossible to achieve certain end-effector orientation/motions.

→ In simple words, it's a point where the robot becomes mechanically "stuck" or unable to move in particular dir.

⇒ How to find singular configurations

→ when Jacobian $J(q)$ becomes rank defficient

$$\text{Singular conf} = \{ q \mid \det(J(q)) = 0 \}$$

Exam :- fully extended joint-link conf.

⇒ Detecting a particular conf. is close to a singular conf.

→ There are certain indices to show how close conf is to singularity
Manipulability index $w = \sqrt{\det(JJ^T)}$ shows how close the conf. is to become singular. If $w=0$, then it singular conf.

ME639
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Q.7: three different configurations of 2R manipulator

① Direct drive
→ both revolute joints directly actuated by motors
Advantages:- Simple & straightforward
- Precise control of E.E's position & orientation
- good stiffness & higher accuracy.
Disadvantages:- range of motion may be limited due to physical constraints
- Complex inverse Kinematics

② Remotely driven
→ manipulator has extra extension/links driven by actuators located away from the joints & End-effector.
Advantages:- extended reach than original configuration
- Simplified Inverse Kinematics
-

Disadvantages:- additional links & actuators increases complexity & weight of system.
- stiffness may be reduced, affecting precision & control

③ 5-bar parallelogram Arrangement:
→ an additional parallelogram linkage is added to 2R manipulator.
Advantages:- improve stability by keeping end-effector level in balance during motion
- can handle higher loads comparitively.
Disadvantages:- Complex Kinematics due to additional linkage.
- limited range of motion compared to direct drive.

ME-639

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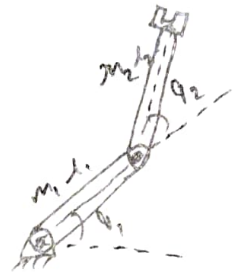
Q.8: Dynamic equation of 2R manipulator:

⇒ Lagrange's equation



$$\mathcal{L} = K - V$$

V → Total potential energy of robot

K → Total kinetic energy of robot.

→ In general for rigid body.

$$K = \frac{1}{2} m V_c^2 + \frac{1}{2} I_c \omega^2$$

{ ω - for rigid body (whole)
V_c - for point on body

⇒ For 2R manipulator.

$$K = \sum_{i=1}^{n=2} \left[\frac{1}{2} m_i V_{c_i}^2 + \frac{1}{2} I_{c_i} \omega_i^2 \right]$$

$$= \sum_{i=1}^{n=2} \left[\frac{1}{2} m V_{c_i}^T V_{c_i} + \frac{1}{2} \omega_i^T I_{c_i} \omega_i \right]$$

$$K = \frac{1}{2} \sum_{i,j}^{n} d_{ij}(q) \dot{q}_i \dot{q}_j = \frac{1}{2} \dot{q}^T D \dot{q}$$

{ using Jacobian

⇒ Lagrangian $L = K - V = \frac{1}{2} \sum_{i,j}^{n} d_{ij}(q) \dot{q}_i \dot{q}_j - V(q)$ ~ eq (1)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \tau_k$$

{ K = No of DoFs
$K = 2$ (for 2R manipulator) }

$$\Rightarrow \frac{\partial L}{\partial \dot{q}_k} = \frac{1}{2}\left(\sum_i d_{ik}(q)\,\dot{q}_i\right) + \sum_j d_{kj}(q)\,\dot{q}_j\Big)$$

$$= \sum_i d_{ik}(q)\,\dot{q}_i \qquad \left\{\begin{matrix}\text{as } i,j \text{ are}\\ \text{dummy Indices}\end{matrix}\right\}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \sum_i d_{ik}(q)\,\ddot{q}_i + \sum_i\left(\frac{d}{dt}\,d_{ik}(q)\right)\dot{q}_i$$

$$= \sum_i d_{ik}(q)\,\ddot{q}_i + \sum_i\left[\sum_j \frac{\partial}{\partial q_j}\,d_{ik}(q)\,\dot{q}_j\right]\dot{q}_i$$

$$= \underline{\sum_i d_{ik}(q)\,\ddot{q}_i + \sum_{i,j} \frac{\partial}{\partial q_j}\,d_{ik}(q)\,\dot{q}_i\,\dot{q}_j}$$

$$\Rightarrow \underline{\frac{\partial L}{\partial q_k} = \frac{1}{2}\sum_{ij} \frac{\partial d_{ij}(q)}{\partial q_k}\,\dot{q}_i\,\dot{q}_j - \frac{\partial V(q)}{\partial q_k}}$$

Therefore, Lagrange's Equations are...

$$\underline{\sum_i d_{ik}(q)\,\ddot{q}_i + \sum_{ij} \frac{\partial}{\partial q_j}\,d_{ik}(q)\,\dot{q}_i\,\dot{q}_j - \frac{1}{2}\sum_{ij}\frac{\partial d_{ij}(q)}{\partial q_k}\,\dot{q}_i\,\dot{q}_j + \frac{\partial V(q)}{\partial q_k}} = \tau_k$$

$$\boxed{\sum_i d_{ik}(q)\,\ddot{q}_i + \frac{1}{2}\sum_{ij}\left[\frac{\partial d_{ik}}{\partial q_i} + \frac{\partial d_{kj}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right]\dot{q}_i\,\dot{q}_j + \frac{\partial V(q)}{\partial q_k} = \tau_k}$$

→ The term

$$c_{ijk} = \frac{1}{2}\left[\frac{\partial d_{ik}(q)}{\partial q_j} + \frac{\partial d_{jk}(q)}{\partial q_i} - \frac{\partial d_{ij}(q)}{\partial q_k}\right]$$

are known as Christoffel Symbols (of the ~~first~~ 3rd kind)

→ Then Lagrange's Equation become

$$\sum_i d_{ik}(q)\ddot{q}_i + \sum c_{ijk}(q,\dot{q})\dot{q}_i\dot{q}_j + \phi_k(q) = \tau_k \qquad \Big\{ \phi_k = \frac{\partial V(q)}{\partial q_k}$$

more commonly,

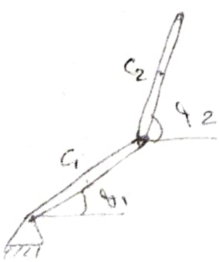
$$\boxed{D(q)\ddot{q} + C(q,\dot{q})\dot{q} + \phi(q) = \tau}$$

example

Mini Project Elbow Manipulator [MZT Checker

(but with q_2 relative angle)
remotely driven)

→



$$V_{c_1} = \begin{bmatrix} -l_1/2 \sin q_1 \\ l_1/2 \cos q_2 \\ 0 \end{bmatrix} \dot{q}_1$$

$$V_{c_2} = \begin{bmatrix} -l_1 s q_1 & -\frac{l_2}{2} s q_2 \\ l_1 c q_1 & \frac{l_2}{2} c q_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

$$w_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix} \;\&\; w_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{q}_2 \end{bmatrix}$$

$$K = \frac{1}{2}\sum_{i=1}^{n} m_i v_{c_i}^T v_{c_i} + \frac{1}{2}\sum_{i=1}^{n} w_i^T I_i w_i$$

$\Rightarrow$ In Jaccobians

$$V_{c_i} = J_{v_{c_i}} \dot{q}$$

$$w_i = R_i^T(q) J_{w_i}(q) \dot{q}$$

$$\boxed{\begin{aligned} K &= \frac{1}{2} \dot{q}^T \sum_{i=1}^{n} \Big[m_i J_{v_{c_i}}(q)^T J_{v_{c_i}}(q) + J_{w_i}(q)^T R_i(q) R_i^T(q) J_{w_i}(q) \Big] \dot{q} \\ K &= \frac{1}{2} \dot{q}^T D(q) \dot{q} \end{aligned}}$$

Always true for all examples.

$\Rightarrow$ In our example

$$D(q) = \begin{bmatrix} m_1 \frac{l_1^2}{4} + m_2 l_1^2 + I_1 & m_2 l_1 \frac{l_2}{2} \cos(q_2 - q_1) \\ m_2 l_1 \frac{l_2}{2} \cos(q_2 - q_1) & m_2 \frac{l_2^2}{4} + I_2 \end{bmatrix}$$

{Always a symmetric +ve definate matrix}

- Computing Cristoffel symbols.

$$C_{111} = \frac{1}{2} \left[\frac{\partial d_{11}}{\partial q_1} + \frac{\partial d_{11}}{\partial q_1} - \frac{\partial d_{11}}{\partial q_1} \right]$$

$$C_{121} = C_{211} = \frac{1}{2} \left[\frac{\partial d_{21}}{\partial q_1} + \frac{\partial d_{12}}{\partial q_2} - \frac{\partial d_{12}}{\partial q_1} \right]$$

$$C_{ijk} = \frac{1}{2}\left[\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ik}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right]$$

$$D(q) = \begin{bmatrix} m_1\frac{l_1^2}{4} + m_1 l_1^2 + I & m_2\frac{l_1 l_2}{2}\cos(q_2 - q_1) \\ m_2\frac{l_1 l_2}{2}\cos(q_2 - q_1) & m_2\frac{l_2^2}{4} + I_2 \end{bmatrix}$$

$$\Rightarrow D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$

$$\phi_1 = \frac{\partial V}{\partial q_1}$$

$$\phi_2 = \frac{\partial V}{\partial q_2}$$

$$C_{111} = \frac{1}{2}\left[\frac{\partial d_{11}}{\partial q_1} + \cancel{\frac{\partial d_{11}}{\partial q_1}} - \cancel{\frac{\partial d_{11}}{\partial q_1}}\right] = \frac{1}{2}\frac{\partial d_{11}}{\partial q_1} = 0$$

$$C_{121} = C_{211} = \frac{1}{2}\left[\frac{\partial d_{11}}{\partial q_2} + \cancel{\frac{\partial d_{21}}{\partial q_1}} - \cancel{\frac{\partial d_{12}}{\partial q_1}}\right] = \frac{1}{2}\frac{\partial d_{11}}{\partial q_2} = 0$$

$$C_{221} = \frac{1}{2}\left[\frac{\partial d_{21}}{\partial q_2} + \frac{\partial d_{21}}{\partial q_2} - \frac{\partial d_{22}}{\partial q_1}\right] = \frac{\partial d_{21}}{\partial q_2} - \frac{1}{2}\underbrace{\frac{\partial d_{22}}{\partial q_1}}_{0}$$

$$= -m_2\frac{l_1 l_2}{2}\sin(q_2 - q_1)$$

$$C_{112} = \frac{1}{2}\left[\frac{\partial d_{12}}{\partial q_1} + \frac{\partial d_{12}}{\partial q_1} - \frac{\partial d_{11}}{\partial q_2}\right] = \frac{\partial d_{12}}{\partial q_1} - \frac{1}{2}\underbrace{\frac{\partial d_{11}}{\partial q_2}}_{0}$$

$$= m_2\frac{l_1 l_2}{2}\sin(q_2 - q_1)$$

$$C_{122} = C_{212} = \frac{1}{2}\left[\frac{\partial d_{22}}{\partial q_1} + \cancel{\frac{\partial d_{21}}{\partial q_2}} - \cancel{\frac{\partial d_{12}}{\partial q_2}}\right] = 0$$

$$C_{222} = \frac{1}{2}\left[\frac{\partial d_{22}}{\partial q_2} + \cancel{\frac{\partial d_{22}}{\partial q_2}} - \cancel{\frac{\partial d_{22}}{\partial q_2}}\right] = 0$$

Q-10: Key steps to derive equations of motion when $D(q)$ & $V(q)$ already provided.

39

$\Rightarrow$ The Lagrange's equation of motion

$$\boxed{D(q)\ddot{q} + C(q,\dot{q})\dot{q} + \frac{\partial}{\partial q_k} V(q) = \tau}$$

$\tau = \tau_k$ joint torque

$k = 1, 2, \ldots, n$

(No of DoF)

- here $D(q)$ & $V(q)$ already given

Unknowns

- q current joint angles
- $\dot{q}, \ddot{q}$ - Vel. & Acc. for Joints (given by Trajectory planning algorithm)
- $C(q,\dot{q})$ ⟨ Christoffel Symbol

→ Christoffel symbols can be computed by.

$$C_{ijk} = \frac{1}{2}\left[\frac{\partial d_{ik}(q)}{\partial q_j} + \frac{\partial d_{kj}(q)}{\partial q_i} - \frac{\partial q_{ij}(q)}{\partial q_k}\right]$$

→ Formulating above unknowns & knows in Lagrange's Equation of motion, we can get E.O.M. for particular Manipulator