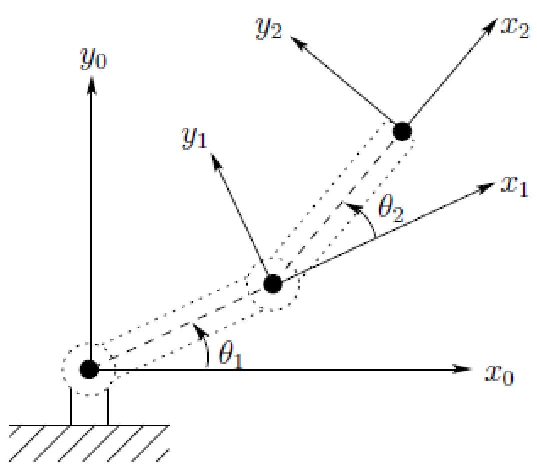
2R- Manipulator (Planar)



DH Parameter:

Link	d	θ	a	α
1	0	θ_1	l_1	0
2	0	θ_2	l_2	0

d = depth along the previous joint's Z-axis
θ = Rotation about the Z-axis to align the X-axis
a = length of common normal for both Z-axis
α = Rotation about the new X-axis to align the previous Z-axis.

$l_1 = 91.88\, mm,\ l_2 = 104.54mm$

Link	d	θ	a	α
1	0	θ_1	91.88	0
2	0	θ_2	104.54	0

Let $P_0 = (x_p\,,\ y_p)$ be end effector position w.r.t. Base frame.
Let's first calculate homogeneous transformation using DH parameters,

$$H_{i-1}^{i} = \begin{bmatrix} \cos\theta & -\sin\theta\cdot\cos\alpha & \sin\theta\cdot\sin\alpha & a.\cos\theta \\ \sin\theta & \cos\theta\cdot\cos\alpha & -\cos\theta\cdot\sin\alpha & a\cdot\sin\theta \\ 0 & \sin\alpha & \cos\alpha & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_{0}^{1} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & l_1\cdot\cos\theta_1 \\ \sin\theta_1 & \cos\theta_1 & 0 & l_1.\sin\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ -------------- (1)

$$H_1^2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & l_2 \cdot \cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & l_2 \cdot \sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_0^2 \ = \ H_0^1 \cdot H_1^2$$

$$H_0^2 = \begin{bmatrix} \cos(\theta_1+\theta_2) & -sin(\theta_1+\theta_2) & 0 & l_1.\cos\theta_1 + l_2 \cdot \cos(\theta_1+\theta_2) \\ sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 & l_1.sin\theta_1 + l_2 \cdot sin(\theta_1+\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ -------------- (2)

P_2 is a position of end effector w.r.t. frame 2

$$P_2 \ = \ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} P_0 \\ 1 \end{bmatrix} \ = \ H_0^2 \cdot \begin{bmatrix} P_2 \\ 1 \end{bmatrix}$$

Using the above values gives

$$\begin{bmatrix} P_0 \\ 1 \end{bmatrix} \ = \ \begin{bmatrix} l_1 \cdot \cos\theta_1 + l_2 \cdot \cos(\theta_1+\theta_2) \\ l_1 \cdot sin\theta_1 + l_2 \cdot sin(\theta_1+\theta_2) \\ 0 \\ 1 \end{bmatrix}$$

$$P_0 = \begin{bmatrix} l_1 \cdot \cos\theta_1 + l_2 \cdot \cos(\theta_1+\theta_2) \\ l_1 \cdot sin\theta_1 + l_2 \cdot sin(\theta_1+\theta_2) \\ 0 \end{bmatrix}$$

$$x_p \ = \ l_1 \cdot \cos\theta_1 \ + \ l_2 \cdot \cos(\theta_1+\theta_2)$$
$$y_p \ = \ l_1 \cdot \sin\theta_1 \ + \ l_2 \cdot \sin(\theta_1+\theta_2)$$

The velocity of the end-effector can be calculated using the Jacobian matrix.

$$\begin{bmatrix} v_0^n \\ w_0^n \end{bmatrix} = J \cdot \dot{q} = \begin{bmatrix} J_1 & J_2 \end{bmatrix} \cdot \begin{bmatrix} \dot{q_1} \\ \dot{q_2} \end{bmatrix}$$

Both joints are revolute joints, hence

$$J = \begin{bmatrix} Z_0 \times (o_2 - o_0) & Z_1 \times (o_2 - o_1) \\ Z_0 & Z_1 \end{bmatrix}$$

Where, $Z_i = R_0^i . k = R_0^i . \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$Z_0 = Z_1 = Z_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad o_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From (1) and (2),

$$H_0^1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & l_1 \cdot \cos\theta_1 \\ \sin\theta_1 & \cos\theta_1 & 0 & l_1 . \sin\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_0^1 \qquad o_1$$

$$H_0^2 = \begin{bmatrix} \cos(\theta_1 + \theta_2) & -sin(\theta_1 + \theta_2) & 0 & l_1 . \cos\theta_1 + l_2 \cdot \cos(\theta_1 + \theta_2) \\ sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) & 0 & l_1 . sin\theta_1 + l_2 \cdot sin(\theta_1 + \theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_0^2 \qquad o_2$$

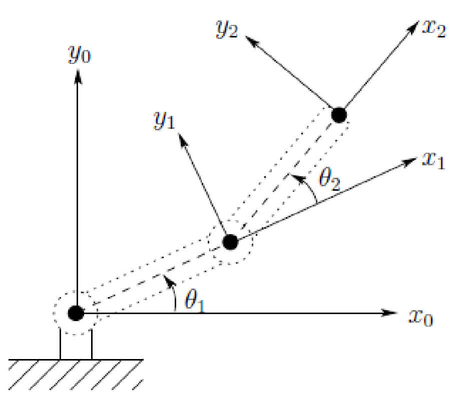
So, the Jacobian matrix becomes,

$$J = \begin{bmatrix} -l_1 \cdot \sin\theta_1 - l_2 \cdot \sin(\theta_1+\theta_2) & -l_2 \cdot \sin(\theta_1+\theta_2) \\ l_1 \cdot cos\theta_1 + l_2 \cdot cos(\theta_1+\theta_2) & l_2 \cdot cos(\theta_1+\theta_2) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

End effector velocity,

$$\begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ w_x \\ w_y \\ w_z \end{bmatrix} = \begin{bmatrix} -l_1 \cdot \sin\theta_1 - l_2 \cdot \sin(\theta_1+\theta_2) & -l_2 \cdot \sin(\theta_1+\theta_2) \\ l_1 \cdot cos\theta_1 + l_2 \cdot cos(\theta_1+\theta_2) & l_2 \cdot cos(\theta_1+\theta_2) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \dot{q_1} \\ \dot{q_2} \end{bmatrix}$$

2R- Manipulator (Planar) - Inverse Kinematics



Let the length of Link 1 be l_1

Let the length of Link 2 be l_2

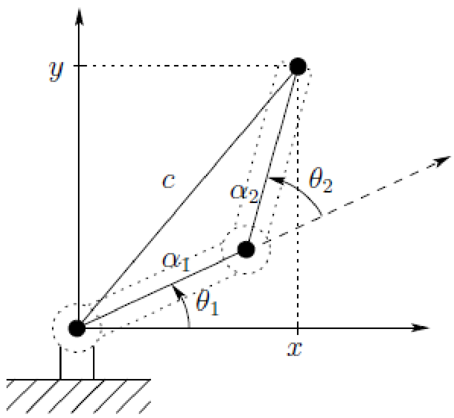
(x,y) ----------- Co-ordinates of an end effector with 0^{th} frame.
Using inverse kinematics, we can find the joint angles for the given x and y positions of an end effector.

Using the law of cosine,

$$\cos\theta_2 \;=\; \frac{x^2 + y^2 - l_1^2 \;-\; l_2^2}{2 \cdot l_1 \cdot l_2} \;=\; D$$

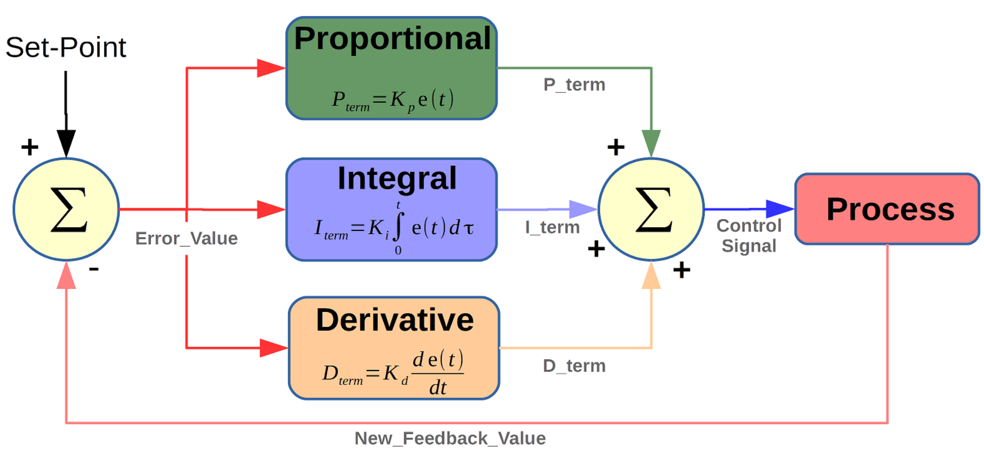
$$\sin\theta_2 \;=\; \pm\,\sqrt{1 \;-\; D^2}$$

$$\theta_2 \;=\; \cos^{-1}(D)$$



$$\theta_1 \;=\; \tan^{-1}\left(\frac{y}{x}\right) \;-\; \tan^{-1}\left(\frac{l_2 \cdot \sin\theta_2}{l_1 + l_2 \cdot \cos\theta_2}\right)$$

In this equation, we can find joint angles for any coordinates and can use PID control for position control.



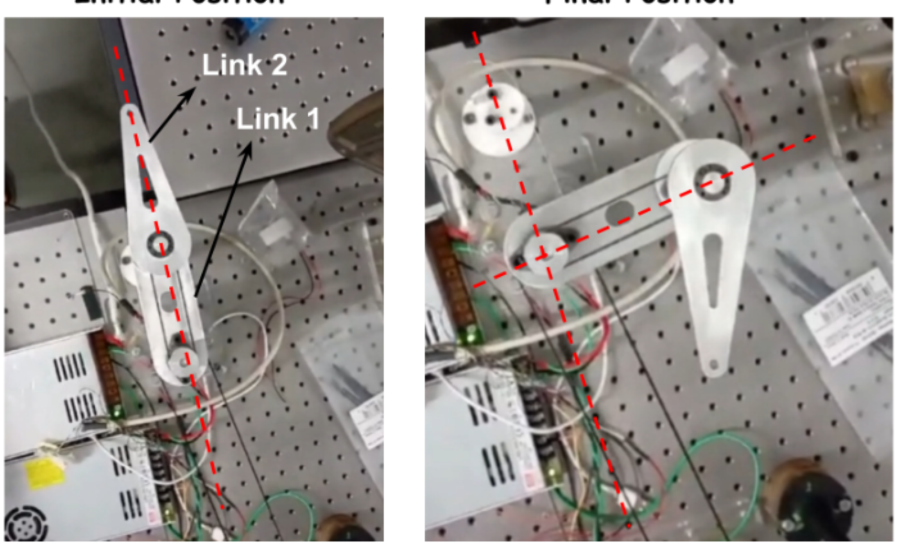
https://www.theengineeringconcepts.com/wp-content/uploads/2018/11/PID-CONTROLLER-BLOCK-DIAGRAM.png

```
int  target1 = 90 ;   // set target angle
// pid constants
int  target2 = 180;

E1 =  ( target1-degrees1) ;
E2 =  ( target2-degrees2) ;

Delta_E1 = ( E1 - prevE1)    ;  //derivative of the error
prevE1 = E1 ;                   // updating the error
E_integral1 = E_integral1 + E1 ;

Delta_E2 = ( E2 - prevE2)    ;  //derivative of the error
prevE2 = E2 ;                   // updating the error
E_integral2 = E_integral2 + E2 ;

u1 = kp1*E1 + kd1* Delta_E1 + ki1 * E_integral1 ;
u2 = kp2*E2 + kd2* Delta_E2 + ki2 * E_integral2 ;
```

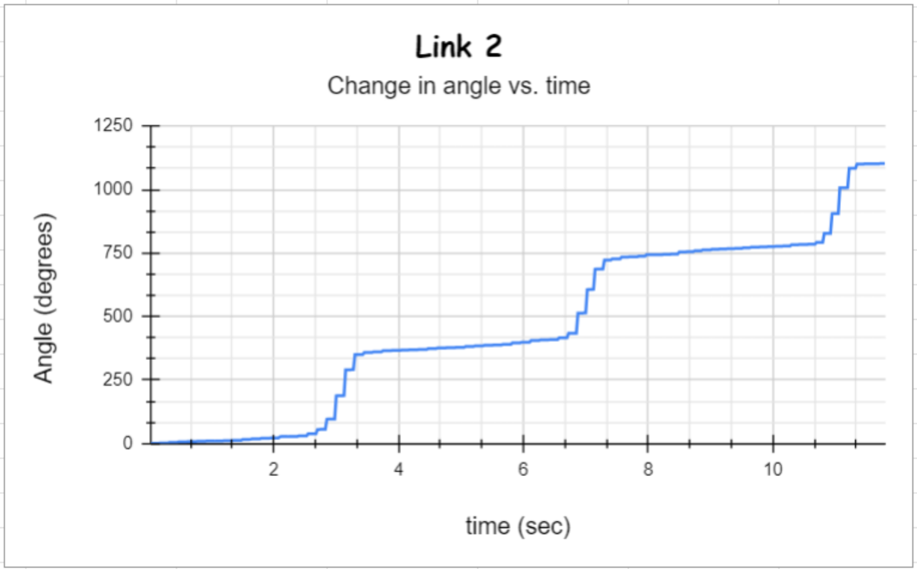
The above code will run in a loop until the error tends to zero.
Here, we are calculating the error between the target value and the feedback value given by the encoder. Then we multiply the error with constants Kp, Kd and Ki to tune control, and the summation of it is given as PWM value to the motor.

In the figure below, Link 1 is rotated through an angle of 90 degrees, and Link 2 is Rotated through an angle of 180 degrees.

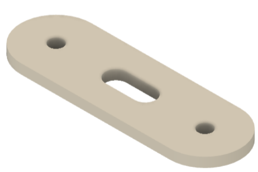
Initial Position **Final Position**



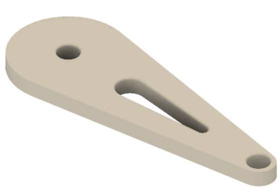
There are slight errors in the position of Link 2 because of some Hardware issues. The Link 2 is rotating slowly for some angles. The encoder value with respect to time is shown below.



Here, Link 2 is rotating slowly from angle 0 to 60 degrees. Which affects the precise trajectory following the manipulator.



Part	Link1
Material	Aluminum
Mass	127.804 g



Part	Link2
Material	Aluminum
Mass	75.235 g