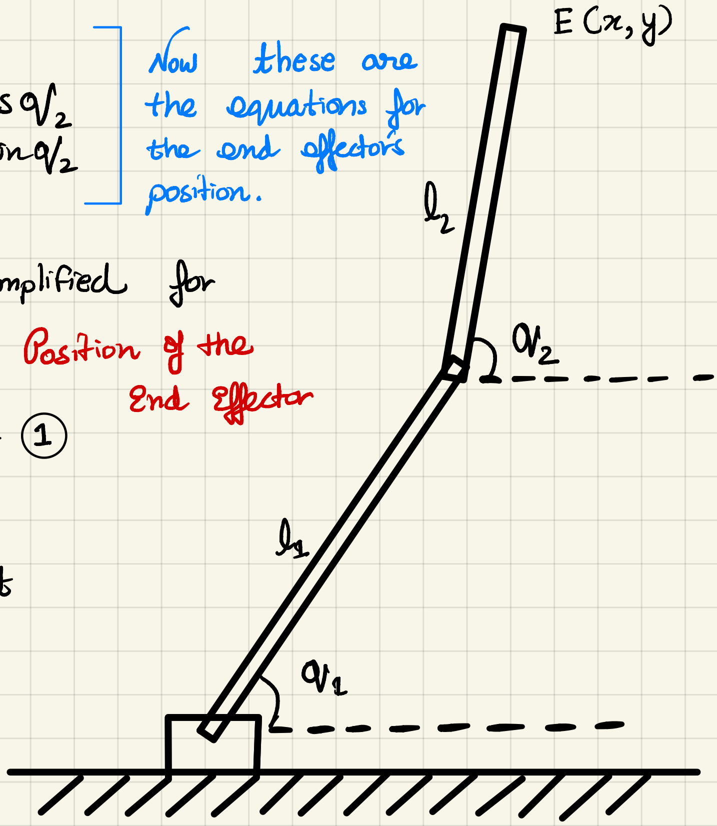
# FORWARD KINEMATICS

TASK 1: Given arbitory trajectory of end effector (given $(x,y)$) as function of time make the robot follow the trajectory.

Now,

$$x = l_1 \cos q_1 + l_2 \cos q_2$$
$$y = l_1 \sin q_1 + l_2 \sin q_2$$

Now these are the equations for the end effectors position.




The above equations are simplified for simpler notation purposes to

$$\boxed{\begin{aligned} x &= l_1 c q_1 + l_2 c q_2 \\ y &= l_1 s q_1 + l_2 s q_2 \end{aligned}} \quad \text{--- } (1)$$

Position of the End Effector

Now differentiating (1) we get

$$\therefore \quad \dot{x} = -l_1 s q_1 \dot{q}_1 - l_2 s q_2 \dot{q}_2$$
$$\dot{y} = l_1 c q_1 \dot{q}_1 + l_2 c q_2 \dot{q}_2$$

Remember that we are differentiating with respect to $x$ and with respect to $y$.

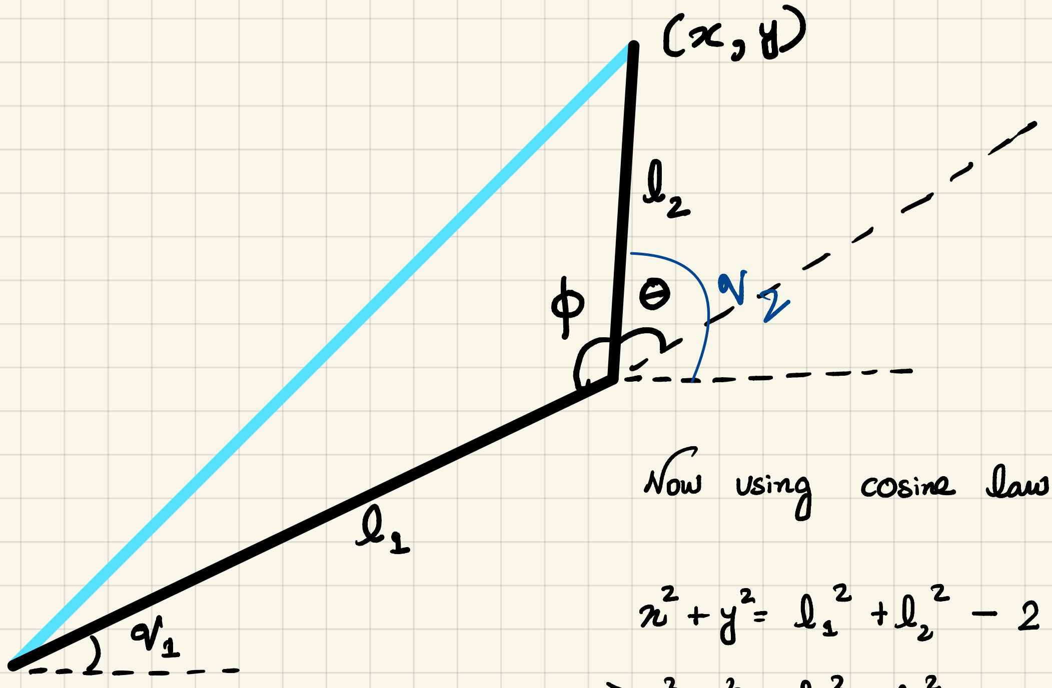
END EFFECTOR VELOCITY

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -l_1 s q_1 & -l_2 s q_2 \\ l_1 c q_1 & l_2 c q_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \quad \text{--- } (2)$$

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -l_1 s q_1 & -l_2 s q_2 \\ l_1 c q_1 & l_2 c q_2 \end{bmatrix}^{-1} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

Now we need to find the joint angles and joint velocities.

Hence for Equation 2 we need to find the inverse




$$\phi = 180 - \theta$$

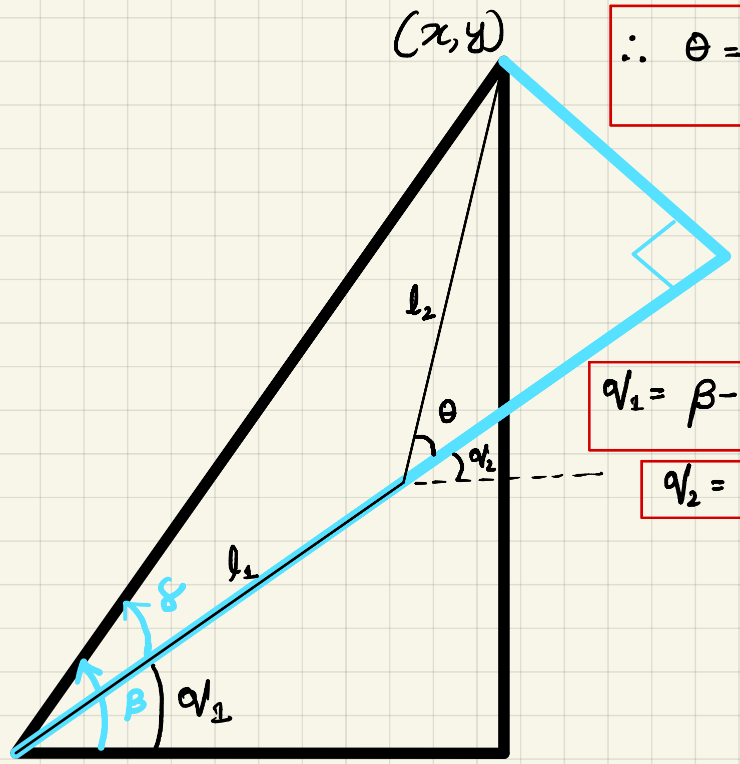
Now using cosine law,

$$x^2 + y^2 = l_1^2 + l_2^2 - 2\, l_1 l_2 \cos(\phi)$$

$$\Rightarrow x^2 + y^2 = l_1^2 + l_2^2 - 2\, l_1 l_2 \cos(180 - \theta)$$

$$\Rightarrow x^2 + y^2 = l_1^2 + l_2^2 + 2\, l_1 l_2 \cos\theta$$

$$\boxed{\therefore\ \theta = \cos^{-1}\left(\frac{x^2 + y^2 - l_1^2 - l_2^2}{2\, l_1 l_2}\right)} \quad \text{— ③}$$



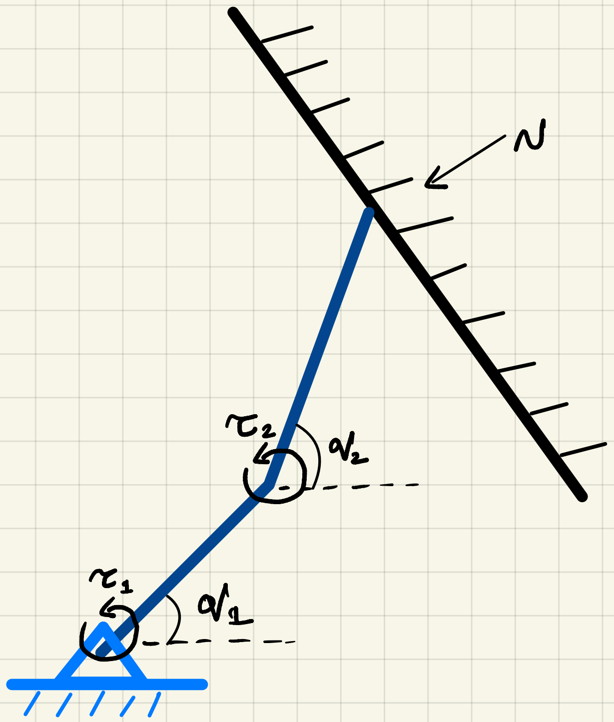

$$\boxed{q_1 = \beta - \gamma = \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{l_2 \sin\theta}{l_1 + l_2 \cos\theta}\right)}$$

$$\boxed{q_2 = q_1 + \theta} \quad \text{— ④}$$

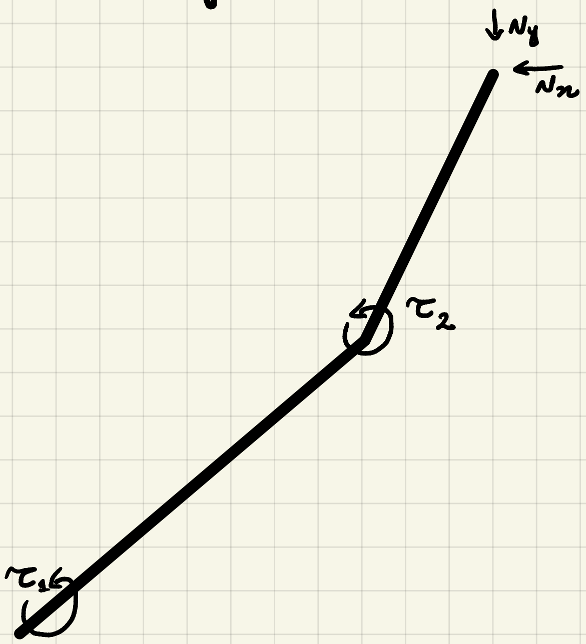
TASK 2 : Given a location (and orientation on a wall make the robot touch the wall and apply a prespecified force at that location

We will later start using the notion $x_d$ and $y_d$ (and $q_1 d$ and $q_2 d$) here for desired values.

(they are not necessarily the actual values)




## FBD of the entire robot




We are neglecting gravity here!

Forces applied by the manipulator

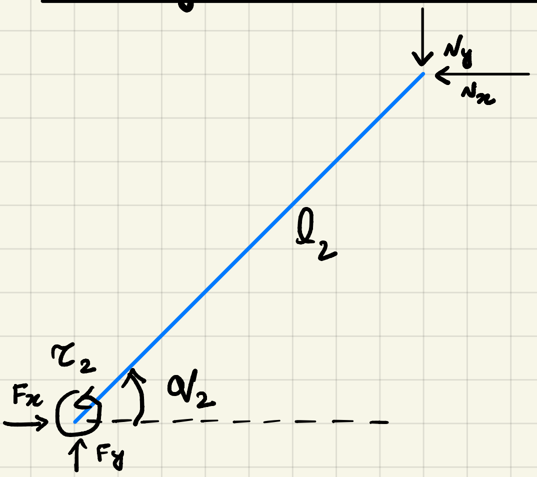
$$F_x = -N_x$$

$$F_y = -N_y$$

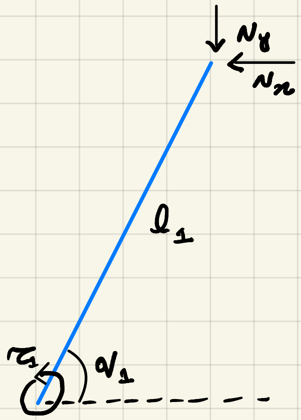
## STATIC EQUILIBRIUM

- Sum of moments equal to zero.
- Sum along $x$-directions equal to zero.
- Sum along $y$-directions equal to zero.

- ## FBD of Each Link Seperately




- Link 2




- Link 1

$$\Sigma M_{O_2} = 0$$

$$\Rightarrow -N_y l_2 \sin(90^\circ + \theta_2) + N_x l_2 \sin(180^\circ - \theta_2) + \tau_2 = 0$$

$$\Rightarrow -N_y l_2 \cos\theta_2 + N_x l_2 \sin\theta_2 + \tau_2 = 0$$

$$\Rightarrow \tau_2 = N_y l_2 \cos\theta_2 - N_x l_2 \sin\theta_2$$

$$\Rightarrow \tau_2 = N_y l_2 c\theta_2 - N_x l_2 s\theta_2$$

$$\Sigma M_{O_1} = 0$$

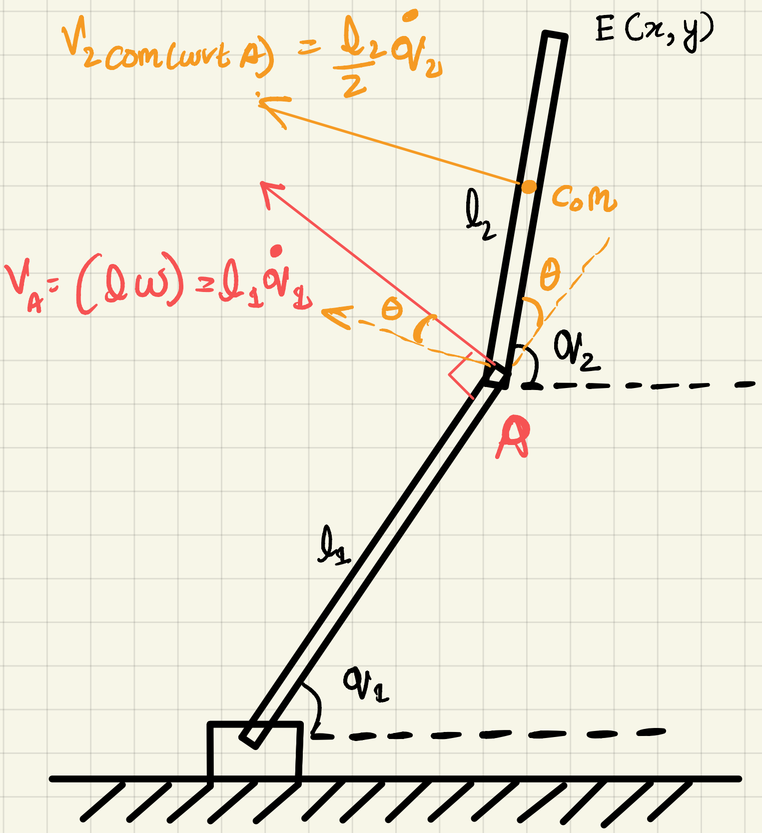
$$\Rightarrow -N_y l_1 \sin(90^\circ + \theta_1) + N_x l_1 (180^\circ - \theta_1) + \tau_1 = 0$$

$$\Rightarrow -N_y l_1 \cos(\theta_1) + N_x l_1 \sin\theta_1 + \tau_1 = 0$$

$$\therefore \tau_1 = N_y l_1 c\theta_1 - N_x l_1 s\theta_1$$

$$\boxed{\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -l_1 s\theta_1 & l_1 c\theta_1 \\ -l_2 s\theta_2 & l_2 c\theta_2 \end{bmatrix} \begin{bmatrix} N_x \\ N_y \end{bmatrix}} \quad — (4)$$

**TASK 3 : Make the robot behave like a virtual spring connected from E to a given point $(x_0, y_0)$**




## KINETIC ENERGY FOR LINK 1

$K = \frac{1}{2} I \omega^2$ — $\omega$: Angular Velocity $\omega = \frac{dq_1}{dt}$ ; $I$: Moment of Inertia

$I = \int_0^{l_1} dm\, r^2$ — $r$: distance of dm from the axis of rotation

$dm = \lambda_1 dr$ $\quad \boxed{\lambda_1 = \frac{m_1}{l_1}}$

$= \int_0^{l_1} \lambda_1 r^2 dr$

$= \lambda_1 \frac{r^3}{3}\Big|_0^{l_1} = \frac{1}{3}\lambda_1 l_1^3$

$= \frac{1}{3} m_1 l_1^2$

## KINETIC ENERGY FOR LINK 2

$K =$ Rotational Energy about CoM $+$ Translational K.E

$= \frac{1}{2} I\omega^2 + - - - - - -$

$I = \int_{-l_2/2}^{+l_2/2} dm\, r^2$ $\quad \lambda_2 = \frac{m_2}{l_2}$

$= \int_{-l_2/2}^{l_2/2} \lambda_2 r^2 dr$

$= \lambda_2 \frac{r^3}{3}\Big|_{-l_2/2}^{l_2/2}$

$= \lambda_2 \frac{1}{24}(l_2^3 + l_2^3)$

$= \lambda_2 \frac{1}{\cancel{24}_{12}}(\cancel{2} l_2^3)$

$= \lambda_2 \frac{1}{12} l_2^3$

$= \frac{1}{12}\lambda_2 l_2^3$

$= \frac{1}{12} m_2 l_2^2$

$$K = \frac{1}{2}\left(\frac{1}{3} m_1 l_1^2\right)\dot{q}_1^2$$

$$= \frac{1}{6} m_1 l_1^2 \dot{q}_1^2$$

$$K = \frac{1}{24} m_2 l_2^2 \dot{q}_2^2 + \text{Translational KE}$$

$$\text{Translatory KE} = \frac{1}{2} m_2 v_{c_2}^2$$

$$\vec{V}_{c2(\text{wrt }O)} = \vec{V}_{2\,com\,(\text{wrt }A)} + \vec{V}_A$$

$$V_{c_2}^2 = V_{2\,com(\text{wrt }A)}^2 + V_A^2 + 2 V_{2\,com\,(\text{wrt }A)} V_A \cos\theta$$

$$= l_1^2 \dot{q}_1^2 + \frac{l_2^2}{4}\dot{q}_2^2 + \cancel{2} l_1 \dot{q}_1 \times \frac{l_2}{\cancel{2}}\dot{q}_2 \cos(q_2 - q_1)$$

$$\frac{1}{2} m_2 l_1^2 \dot{q}_1^2 + \frac{1}{8} m_2 l_2^2 \dot{q}_2^2 + \frac{1}{2} m_2 l_1 l_2 \dot{q}_1 \dot{q}_2 \cos(q_2 - q_1)$$

$$K = \frac{1}{6} m_1 l_1^2 \dot{q}_1^2 + \frac{1}{24} m_2 l_2^2 \dot{q}_2^2 + \frac{1}{2} m_2 l_1^2 \dot{q}_1^2 + \frac{1}{8} m_2 l_2^2 \dot{q}_2^2 + \frac{1}{2} m_2 l_1 l_2 \dot{q}_1 \dot{q}_2 \cos(q_2 - q_1)$$

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Now PE for Link 1 & Link 2

$$V = \frac{1}{2} m_1 g l_1 \sin q_1 + m_2 g\left(l_1 \sin q_1 + \frac{l_2}{2} \sin q_2\right)$$

$$= \frac{1}{2} m_1 g l_1 \sin q_1 + m_2 g\, l_1 \sin q_1 + \frac{1}{2} m_2 l_2 g \sin q_2$$

$$\mathcal{L} = \frac{1}{6} m_1 l_1^2 \dot{q}_1^2 + \frac{1}{24} m_2 l_2^2 \dot{q}_2^2 + \frac{1}{2} m_2 l_1^2 \dot{q}_1^2 + \frac{1}{8} m_2 l_2^2 \dot{q}_2^2 + \frac{1}{2} m_2 l_1 l_2 \dot{q}_1 \dot{q}_2 \cos(q_2 - q_1) - \frac{1}{2} m_1 g l_1 \sin q_1 - m_2 g l_1 \sin q_1 - \frac{1}{2} m_2 l_2 g \sin q_2$$

$$\frac{\partial \mathcal{L}}{\partial q_1} = \frac{1}{2} m_2 l_1 l_2 \dot{q}_1 \dot{q}_2 \sin(q_2 - q_1) - \frac{1}{2} m_1 g l_1 \cos q_1 - m_2 g l_1 \cos q_1$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = Q_i'}$$

$$\tau_1 = \frac{1}{3} m_1 l_1^2 \ddot{q}_1 + m_2 l_1^2 \ddot{q}_1 + m_2 \frac{l_1 l_2}{2} \ddot{q}_2 \cos(q_2 - q_1) - \frac{m_2 l_1 l_2}{2} \dot{q}_2 (\dot{q}_2 - \dot{q}_1) \sin(q_2 - q_1) + m_1 g \frac{l_1}{2} c q_1 + m_2 g l_1 c q_1$$

$$\tau_2 = \frac{1}{3} m_2 l_2^2 \ddot{q}_1 + m_2 \frac{l_2^2}{4} \ddot{q}_2 + m_2 \frac{l_1 l_2}{2} \ddot{q}_1 \cos(q_2 - q_1) - m_2 \frac{l_1 l_2}{2} \dot{q}_1 (\dot{q}_2 - \dot{q}_1) \sin(q_2 - q_1) + m_2 g \frac{l_2}{2} s q_2 \qquad (6)$$

NEXT, we note that (4) is valid for any force $F_x, F_y$ (not just wall forces)

can be applied to other systems too.

Want,

$$F_x = Kx \qquad F_y = Ky$$

– more generally

$$F_x = K_x(x - x_s)$$
$$F_y = K_y(y - y_0$$

There is no dampening so it will keep vibrating

Real world scenario it will dampen.

From (1)

$$F_x = K(l_1 c\theta_1 + l_2 c\theta_2)$$

$$F_y = K(l_1 s\theta_1 + l_2 s\theta_2)$$

From (4)

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -l_1 s\theta_1 & l_1 c\theta_1 \\ -l_2 s\theta_2 & l_2 c\theta_2 \end{bmatrix} \begin{bmatrix} N_x \\ N_y \end{bmatrix}$$

$$\boxed{\begin{aligned} K(l_1 s\theta_1 + l_2 s\theta_2) l_2 c\theta_2 - K(l_1 c\theta_1 + l_2 c\theta_2) l_2 s\theta_2 &= \tau_{2_s} \\ K(l_1 s\theta_1 + l_2 s\theta_2) l_1 s\theta_2 - K(l_1 c\theta_1 + l_2 c\theta_2) l_1 s\theta_1 &= \tau_{1_s} \end{aligned}} \quad (7)$$