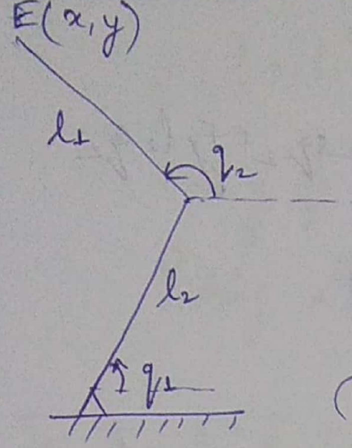
1.

Forward kinematics.



$$x = l_1 \cos q_1 + l_2 \cos q_2$$
$$y = l_1 \sin q_1 + l_2 \sin q_2$$

$$\Rightarrow \left.\begin{aligned} x &= l_1 cq_1 + l_2 cq_2 \\ y &= l_1 sq_1 + l_2 sq_2 \end{aligned}\right\} \text{— (I)}$$

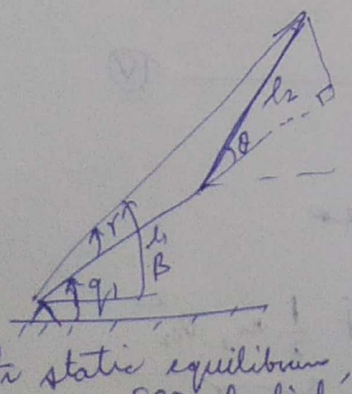
Differentiating w.r.t. time

$$\dot{x} = -l_1 sq_1 \dot{q}_1 - l_2 cq_2 \dot{q}_2$$
$$\dot{y} = l_1 cq_1 \dot{q}_1 + l_2 cq_2 \dot{q}_2$$

$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}}_{\text{Cartesian space}} = \begin{bmatrix} -l_1 sq_1 & -l_2 sq_2 \\ l_1 cq_1 & l_2 cq_2 \end{bmatrix} \underbrace{\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}}_{\text{Joint space}} \text{— (II)}$$

Velocity kinematics

Inverse kinematics



Cosine rule + switch to acute angle

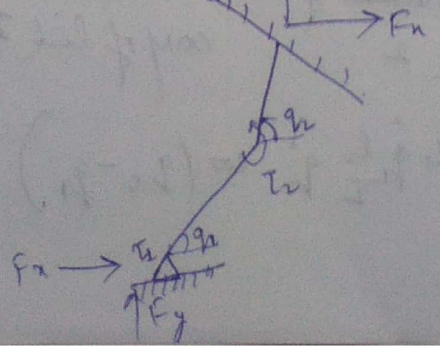
$$\theta = \cos^{-1}\left(\frac{x^2 + y^2 - l_1^2 - l_2^2}{2 l_1 l_2}\right)$$

From right angled Δ, $q_1 = \beta - \gamma$

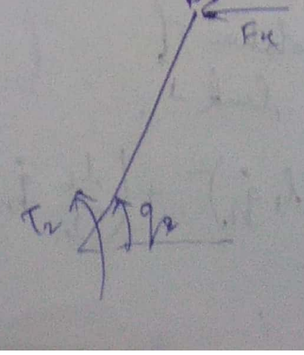
$$\left.\begin{aligned} q_1 &= \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{l_2 \sin\theta}{l_1 + l_2\cos\theta}\right) \\ q_2 &= q_1 + \theta \end{aligned}\right\} \text{— (III)}$$

In static equilibrium,

FBD of link 1



FBD of link 2



Gravity is ignored here,

$\tau_1 = -F_x l_1 sq_1 + F_y l_2 cq_1$, $\tau_2 = -F_x l_2 sq_2 + F_y l_2 q_2$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -l_1 sq_1 & l_2 cq_1 \\ -l_2 sq_2 & l_2 cq_2 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix} \quad \text{—— (IV)}$$

For virtual spring, force needs to be a restoring force

Let the desired location be (x_0, y_0),

$F_x = k(x - x_0)$

$F_y = k(y - y)$, where k is user-defined stiffness.

When we need to account for dynamics, we use

Lagrange's eq^n: $F = ma$

Lagrangian: $\mathcal{L} = \underset{\substack{\downarrow \\ \text{Kinetic} \\ \text{energy}}}{K} - \underset{\substack{\searrow \\ \text{Potential} \\ \text{energy}}}{V}$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = Q_i' \quad \text{—— (V)}$$

where Q_i' is the generalised forces

$$K = \underbrace{\frac{1}{2}\left(\frac{1}{3} m_1 l_1^2\right) \dot{q}_1^2}_{\text{pure rotation of link 1}} + \underbrace{\frac{1}{2}\left(\frac{1}{12} m_2 l_2^2\right) \dot{q}_2^2}_{\text{pure rotation of link 2}} + \underbrace{\frac{1}{2} m_2 v_{c_2}^2}_{\text{translation of COM of link 2}}$$

$$v_c^2 = (l_1 \dot{q}_1)^2 + \left(\frac{l_2}{2} \dot{q}_2\right)^2 + 2 l_1 \dot{q}_1 \frac{l_2}{2} \dot{q}_2 \cos(q_2 - q_1)$$

Scanned with CamScanner

$P.E. = V = m_1 g \frac{l_1}{2} sq_1 + m_2 g \left(l_1 sq_1 + \frac{l_2}{2} sq_2\right)$

$\mathcal{L} = k - V$

$= \frac{1}{2}\left(\frac{1}{3} m_1 l_1^2\right)\dot{q}_1^2 + \frac{1}{2}\left(\frac{1}{12} m_2 l_2^2 \dot{q}_2^2\right) + \frac{1}{2} m_2 v_c^2$

$+ m_1 g \frac{l_1}{2} sq_1 + m_2 g \left(l_1 sq_1 + \frac{l_2}{2} sq_2\right)$

Putting $\mathcal{L}$ in eq^n (1)

$\tau_1 = \frac{1}{3} m_1 l_1^2 \ddot{q}_1 + m_2 l_1^2 \ddot{q}_1 + \frac{m_2 l_1 l_2}{2} \ddot{q}_2 \cos(q_2 - q_1)$

$- m_2 \frac{l_1 l_2}{2} \dot{q}_2 (\dot{q}_2 - \dot{q}_1) \sin(q_2 - q_1) + m_1 g \frac{l_1}{2} cq_1 + m_2 g l_1 cq_1$

$\tau_2 = \frac{1}{3} m_2 l_2^2 \ddot{q}_2 + \frac{m_2 l_2^2}{4} \ddot{q}_2 + m_2 \frac{l_1 l_2}{2} \ddot{q}_1 \cos(q_2 - q_1) -$

$m_2 \frac{l_1 l_2}{2} \dot{q}_1 (\dot{q}_2 - \dot{q}_1) \sin(q_2 - q_1) + \frac{m_2 g l_2}{2} sq_2$