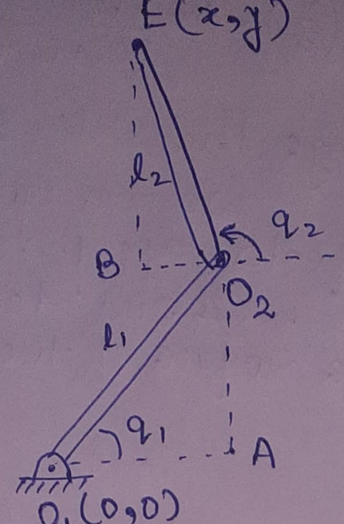
Task 0.



Assumption:
The motors are connected to O_1 and O_2 that are providing torques τ_1 and τ_2 or controlling the angles q_1 and q_2 as desired.
q_1, q_2 are the angle formed by link 1 and link 2.
E be the end effector.

$x = O_1A - O_2B$; $y = AO_2 + BE$

We can write the coordinate point of E(x,y).

$$\left.\begin{aligned} x &= l_1 \cos q_1 + l_2 \cos q_2 \\ y &= l_1 \sin q_1 + l_2 \sin q_2 \end{aligned}\right\} \quad \text{——— (1)}$$

differentiating equation (1), we get.

$$\dot{x} = -l_1 \sin q_1 \, \dot{q}_1 - l_2 \sin q_2 \, \dot{q}_2$$

$$\dot{y} = l_1 \cos q_1 \, \dot{q}_1 + l_2 \cos q_2 \, \dot{q}_2$$

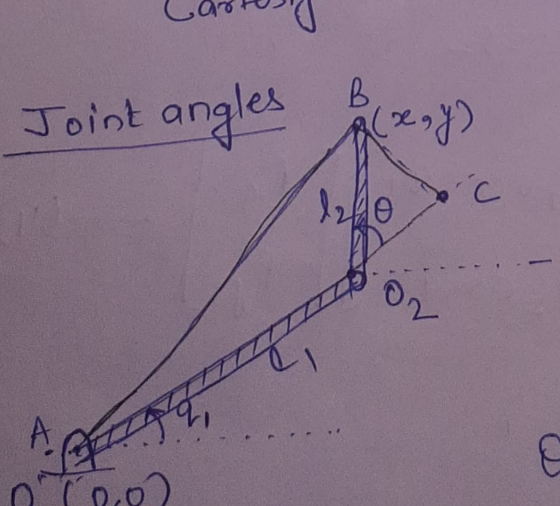
Velocity of End effector.

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -l_1 \sin q_1 & -l_2 \sin q_2 \\ l_1 \cos q_1 & l_2 \cos q_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ q_2 \end{bmatrix} \quad \text{— (2)}$$

Cartesign Space/Task Space

Joint Space.

Joint angles



We can write $q_2 = q_1 + \theta$

In Δ ABO_2

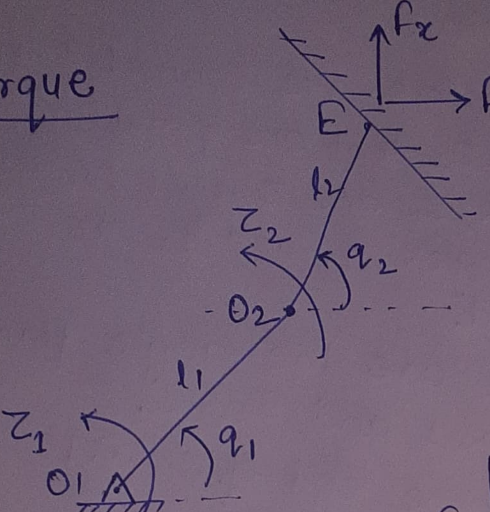
using Cosine Rule → $(AB)^2 = (O_1O_2)^2 + (O_2E)^2 - 2(O_1O_2)(O_2E)\cos(180-\theta)$

$$\theta = \cos^{-1}\left(\frac{x^2 + y^2 - l_1^2 - l_2^2}{2 l_1 l_2}\right)$$

$$q_1 = \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{l_2 \sin\theta}{l_1 + l_2\cos\theta}\right) \quad \text{———} \quad (3)$$

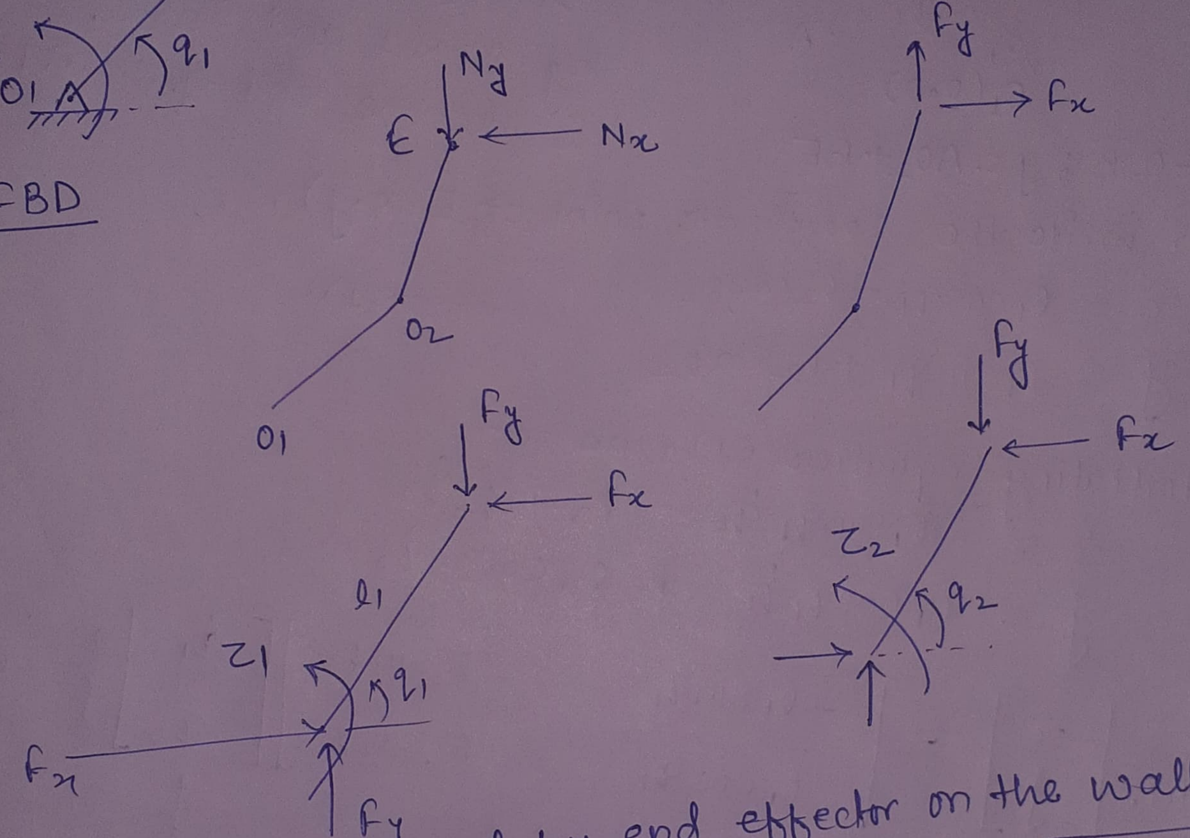
$$q_2 = \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{l_2 \sin\theta}{l_1 + l_2\cos\theta}\right) + \cos^{-1}\left(\frac{x^2+y^2-l_1^2-l_2^2}{2 l_1 l_2}\right)$$

<u>Torque</u>



Assuming: The Static Equblirium and we are ignoring gravity. End effector apply normal force on the wall and does not slip on the surface of the wall.

<u>FBD</u>



f_x and f_y is applied by end effector on the wall.

We can write the equilibrium condition $\Rightarrow$ $\boxed{\Sigma M_{O_2} = 0}$

$$z_1 = -f_x l_1 \sin q_1 + f_y l_1 \cos q_1$$

$$z_2 = -f_x l_2 \sin q_2 + f_y l_2 \cos q_2$$

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -l_1 \sin q_1 & l_1 \cos q_1 \\ -l_2 \sin q_2 & l_2 \cos q_2 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \end{bmatrix} \quad \text{———} \quad (4)$$

Torque

Lagrangian (L) = kinetic Energy (K) − Potential Energy (V)

Lagrangian Equation:

$$\left.\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \theta'_i \quad \right\} \quad \text{—(5)}$$

θ' : General Force drived using principle of virtual work.

$$K = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} m_2 V_{c_2}^2 + \frac{1}{2} I_2 \omega_2^2$$

Pure Rotation about O_1.

kinetic Energy by translation.

Rotation of L_2 about center of gravity.

V_{c_2} : velocity of center of mass.

$$K = \frac{1}{2}\left(\frac{1}{3} m_1 l_1^2\right)\dot{q}_1^2 + \frac{1}{2} m_2 V_{c_2}^2 + \frac{1}{2}\left(\frac{1}{12} m_2 l_2^2\right)\dot{q}_2^2$$

$$V_{c_2}^2 = (l\dot{q}_1) + \left(\frac{l_2}{2}\dot{q}_2\right)^2 + 2 l_1 \dot{q}_1 \frac{l_2}{2}\dot{q}_2 \cos(q_2 - q_1)$$

On consedering gravitation force;

$$V = m_1 g \frac{l_1}{2} \sin q_1 + m_2 g \left(l_1 \sin q_1 + \frac{l_2}{2} \sin q_2\right)$$

from equation (5), Solving further equation (5) to get desired result, we have -

$$\frac{1}{3} m_1 l_1^2 \ddot{q}_1 + m_2 l_1^2 \ddot{q}_1 + m_2 \frac{l_1 l_2}{2} \ddot{q}_2 \cos(q_2 - q_1) - m_2 \frac{l_1 l_2}{2} \dot{q}_2 (\dot{q}_2 - \dot{q}_1) \sin(q_2 - q_1)$$
$$+ m_1 g \frac{l_1}{2} \cos(q_2) + m_2 g l_1 \cos(q_1) = \tau_1$$

$$\frac{1}{3} m_2 l_2^2 \ddot{q}_2 + m_2 \frac{l_2^2}{4} \ddot{q}_2 + m_2 \frac{l_1 l_2}{2} \ddot{q}_1 \cos(q_2 - q_1) - m_2 \frac{l_1 l_2}{2} \dot{q}_1 (\dot{q}_2 - \dot{q}_1) \sin(q_2 - q_1)$$
$$+ m_2 g \frac{l_2}{2} \sin(q_2) = \tau_2 \qquad \text{—— (6)}$$

Spring Torque (virtual Spring)

2R Elbow manipulator should behave like a virtual Spring and whenever a displacement is given in any direction if given it should come to the Same point.

$$f_x = k(x - x_0)$$

$$f_y = k(y - y_0)$$

Where, K = Stiffness constant

using equation (1), we know that

$$x = l_1 \cos q_1 + l_2 \cos q_2$$

$$y = l_1 \sin q_1 + l_2 \sin q_2$$

~~from equation (4)~~

Therefore, the Spring force can be written as —

$$f_x = k[(l_1 \cos q_1 + l_2 \cos q_2) - x_0]$$

$$f_y = k[l_1 \sin q_1 + l_2 \sin q_2 - y_0]$$

using equation 4, we can calculate the torque

$$\therefore \quad \tau_{1s} = -k(l_1 \cos q_1 + l_2 \cos q_2 - x_0) l_1 \sin q_1 + k(l_1 \sin q_1 + l_2 \sin q_2 - y_0) l_1 \cos q_1 \quad (7)$$

$$\tau_{2s} = -k(l_1 \cos q_1 + l_2 \cos q_2) - x_0) l_2 \sin q_2 + k(l_1 \sin q_1 + l_2 \sin q_2 - y_0) l_2 \cos q_2$$

τ_{1s} and τ_{2s} are torque on 2R Elbow manipulator, also τ_1 and τ_2 are the torque by motor, combining torques i.e $\tau_1 + \tau_{1s}$ and $\tau_2 + \tau_{2s}$ will result the manipulator to behave like a virtual Spring.