Mini-Project

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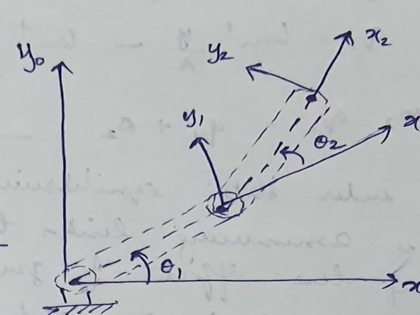
0. Derivation of Kinematic and Dynamic equations governing the motion of 2R Manipulator

Mini Project

a. Include a neat handwritten derivation of the 6 key equations derived with regard to the kinematics, dynamics and statics of the elbow manipulator.

Consider 2-link planar arm.
The joint axes z_0 and z_1 are normal to the plane.
Now, let $q_1 = \theta_1$ and $q_2 = q_1 + \theta_2$



Considering the geometry of the robot i.e. link 1 having length 'l_1' and 'm_1' mass & link 2 having length l_2 and m_2 mass, we can express the end effector position x, y as below –

$$x = l_1 \cos q_1 + l_2 \cos q_2 = l_1 c_1 + l_2 c_2$$
$$y = l_1 \sin q_1 + l_2 \sin q_2 = l_1 s_1 + l_2 s_2 \quad \text{— (1)}$$

Inorder to obtain the relation between ~~xy~~ end effector velocities and joint velocities, we may differentiate (1) w.r.t 't'

We get,
$$\dot{x} = -l_1 s_1 \dot{q}_1 - l_2 s_2 \dot{q}_2$$
$$\dot{y} = l_1 c_1 \dot{q}_1 + l_2 c_2 \dot{q}_2 \quad \text{— (2)a}$$

This can also be written as,

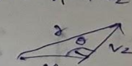
$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -l_1 s_1 & -l_2 s_2 \\ l_1 c_1 & l_2 c_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \quad \text{— (2)b}$$

where $s_1 = \sin q_1$, $s_2 = \sin q_2$, $c_1 = \cos q_1$ & $c_2 = \cos q_2$

Inorder to obtain a relation between joint angles and position of end effector, we can apply the cosine rule as described below –

$$x^2 + y^2 = l_1^2 + l_2^2 - 2 l_1 l_2 \cos(\pi - \theta_2)$$

$\because$ Cosine rule states $r^2 = v_1^2 + v_2^2 - 2 v_1 v_2 \cos\theta$



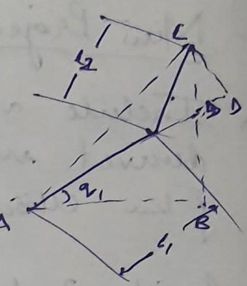
$$\Rightarrow x^2 + y^2 = l_1^2 + l_2^2 + 2 l_1 l_2 \cos\theta_2$$

$$\cos\theta_2 = \frac{-l_1^2 - l_2^2 + x^2 + y^2}{2 l_1 l_2}$$

$$\theta_2 = \cos^{-1}\left[\frac{x^2 + y^2 - l_1^2 - l_2^2}{2 l_1 l_2}\right] \quad \text{— (3)a}$$

We know that $q_1 = q_2 - \theta_2$

Also $q_1 = \tan^{-1}\frac{CB}{AB} - \tan^{-1}\frac{CD}{AD}$.

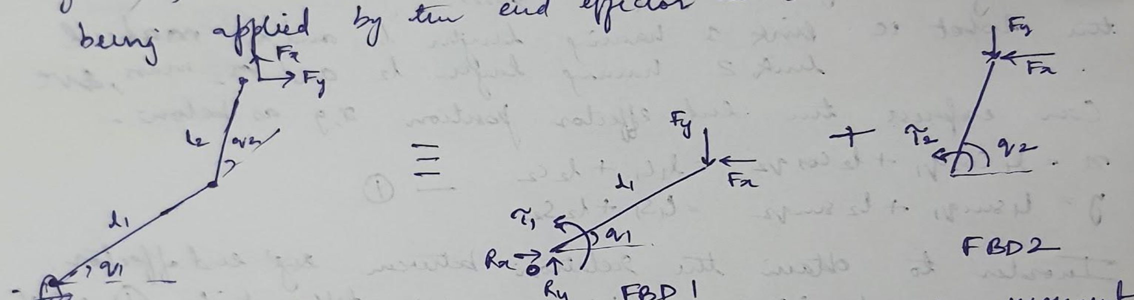


Expressing in terms of l_1 and l_2, x & y.

$$q_1 = \tan^{-1}\frac{y}{x} - \tan^{-1}\frac{l_2 \sin\theta}{l_1 + l_2\cos\theta} \quad \text{— (3) b}$$

Also, $q_2 = q_1 + \theta_2$ — (3) c.

Now, under static equilibrium for the given configuration, and assuming links to be of negligible mass and that is ignoring the effect of gravity, Consider force F_x and F_y being applied by the end effector as shown below.



Consider FBD1, the static equilibrium condition for moment balance at 'O' can be written as,

$\tau_1 = -F_x l_1 S_1 + F_y l_1 C_1$ — (4) a

IIIarly for link 2,

$\tau_2 = -F_x l_2 S_2 + F_y l_2 C_2$ — (4) b

These combined can be written as below;

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -l_1 S_1 & l_1 C_1 \\ -l_2 S_2 & l_2 C_2 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix} \quad \text{— (4) c}$$

The above equations give the relation between motor torques & end effector forces.

Now considering the masses of the link and additional torque would be required to counter the effect of gravity.

To compute the dynamics of the 2R link planar manipulator we will utilise the Langrange's equation.

Lagrangian L, is defined as below –

$L = K - V$ where K – Kinetic energy of the system.
V – Potential energy of the system.

where $V = V_1 + V_2$, V_1 – Potential energy of link 1.
V_2 – Potential energy of link 2

$V_1 = m_1 g \frac{l_1}{2} S_1$ & $V_2 = m_2 g \left(l_1 S_1 + \frac{l_2}{2} S_2\right)$.

Thus $V = m_1 g \frac{l_1}{2} S_1 + m_2 g \left(l_1 S_1 + \frac{l_2}{2} S_2\right)$ — (5)a

Now, $K = K_1 + K_2$ $\quad K_1 = \frac{1}{2} I_0 \dot{q}_1^2$ & $K_2 = \frac{1}{2} m_2 V_c^2 + \frac{1}{2} I_2 \dot{q}_2^2$

where

$$V_c^2 = \dot{x}_c^2 + \dot{y}_c^2 = l_1^2 \dot{q}_1^2 + \frac{l_2^2}{4}(\dot{q}_1 + \dot{q}_2)^2 + l_1 l_2 C_2 \dot{q}_1 (\dot{q}_1 + \dot{q}_2)$$

& $x_c = l_1 C_1 + \frac{l_2}{2} C_{12}$ $\quad y_c = l_1 S_1 + \frac{l_2}{2} S_{12}$ (from geometry)

Thus
$$K = \frac{1}{2}(I_0 + m_2 l_1^2)\dot{q}_1^2 + \frac{1}{2} m_2 l_1 l_2 C_2 \dot{q}_1 (\dot{q}_1 + \dot{q}_2) + \left(\frac{1}{8} m_2 l_2^2 + \frac{1}{2} I_2\right)(\dot{\theta}_1 + \dot{\theta}_2)^2$$ — (5)b

The general forces (Torque in our case) is given by.

$$\tau_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} \quad \text{— (5) c}$$

Computing the partial differentials in the above expression,

$$\frac{\partial L}{\partial \dot{q}_1} = (I_0 + m_2 l_1^2)\dot{q}_1 + m_2 l_1 l_2 C_2 \left(\dot{q}_1 + \frac{1}{2}\dot{q}_2\right) + \left(\frac{1}{4} m_2 l_2^2 + I_2\right)(\dot{q}_1 + \dot{q}_2).$$

$$\frac{\partial L}{\partial \dot{q}_2} = \frac{1}{2} m_2 l_1 l_2 C_2 \dot{q}_1 + \left(\frac{1}{4} m_2 l_2^2 + I_2\right)(\dot{q}_1 + \dot{q}_2)$$

$$\frac{\partial L}{\partial q_1} = -m_1 g \frac{l_1}{2} C_1 - m_2 g \left(l_1 C_1 + \frac{l_2}{2} C_{12}\right)$$

$$\frac{\partial L}{\partial q_2} = -m_2 g \frac{l_2}{2} C_{12} - \frac{1}{2} m_2 l_1 l_2 S_2 \dot{q}_1 (\dot{q}_1 + \dot{q}_2).$$

Substituting the above we get,

$$\tau_1 = \frac{1}{3} m_1 l_1^2 \ddot{q}_1 + m_2 l_1^2 \ddot{q}_1 + m_2 l_1 l_2 \ddot{q}_2 \cos(q_2 - q_1) - m_2 \frac{l_1 l_2}{2} \dot{q}_2 (\dot{q}_2 - \dot{q}_1) \sin(q_2 - q_1)$$
$$+ m_1 g \frac{l_1}{2} C_2 + m_2 g l_1 C_2$$

$$\tau_2 = \frac{1}{2} m_2 l_2^2 \ddot{q}_7 + \underset{\tau_1}{m_2 l_2^2 \ddot{q}_2} + m_2 \frac{l_1 l_2}{2} \ddot{q}_1 \cos(q_2 - q_1) - m_2 \frac{l_1 l_2}{2} \dot{q}_1 (\dot{q}_2 - \dot{q}_1) \sin(q_2 - q_1) \cancel{\sin(q_2 - q_1)}$$
$$+ m_2 g \frac{l_2}{2} \sin q_2$$

How to run the codes?

All the codes will have prompts taking user input with expected format mentioned there itself.

All dynamic analysis and its variation with end-effector speeds are visualised using graphs.

The equations used are the same as derived in class.