ME 639

Intro to Robotics

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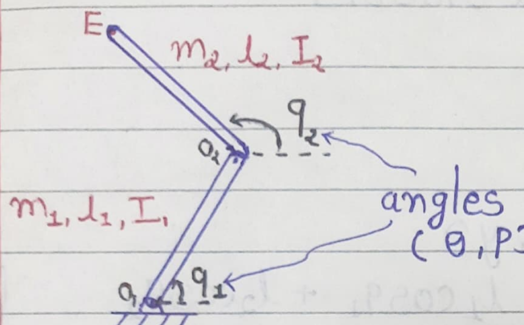
* 2R Manipulator (Elbow Manipulator)

2R → Revolute Joint (other → Presmatic Joint)



[Open Serial Chain]

E → End Effector
E(x,y) - position
↳ Assume there are motors connected to θ_1 & θ_2 that are providing torques τ_1, τ_2 or controlling the angles q_1 & q_2 as desired.

* Mini Project :-

Task-1 Given an arbitary trajectory of end effector (given E(x,y) function of time), make the robot to follow the trajectory.

Task-2 Given a location of a wall, make the robot touch the wall and apply an constant normal force against the wall.

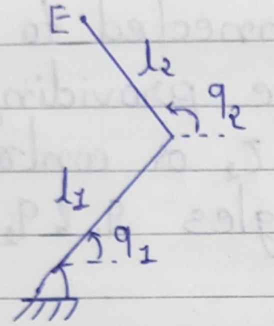
Task-3 Make the robot behave like a virtual spring connected from E to given virtual point (x_0, y_0).

* Robot :-

- Mechanical moving parts
- Actuated (usually electic)
- Autonomy (sense + decide)

* Manipulators
  - Serial
  - Parallel
- → Mobile Robots → ground robots
- → Aeriel Robots → drones

* 2R Manipulator



$E(x, y)$

$x = l_1 \cos q_1 + l_2 \cos q_2$

$y = l_1 \sin q_1 + l_2 \sin q_2$

(position kinematic

$$\left.\begin{aligned} x &= l_1 c q_1 + l_2 c q_2 \\ y &= l_1 s q_1 + l_2 s q_2 \end{aligned}\right\} \quad -(1)$$

$\dot{x} = -l_1 s q_1 \dot{q}_1 - l_2 s q_2 \dot{q}_2$

$\dot{y} = l_1 c q_1 \dot{q}_1 + l_2 c q_2 \dot{q}_2$

(velocity Kinematics)

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -l_1 s q_1 & -l_2 s q_2 \\ l_1 c q_1 & l_2 c q_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \quad -(2)$$

velocity of End effector — Task space — Cartesian space

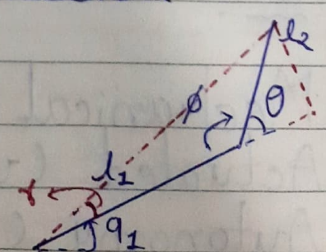
joint space — Angular velocity of joints

For task-1 ⇒ This is not enough.

→ Inverse kinematics
  - ↳ Numerically
  - ↳ Derive closed form expression

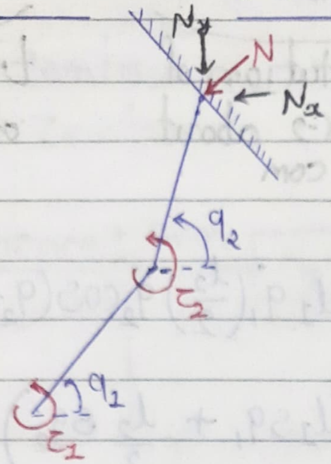
Inverse Kinematics

$$\boxed{\begin{aligned} * \quad & \theta = \cos^{-1}\left(\frac{x^2 + y^2 - l_1^2 - l_2^2}{2 l_1 l_2}\right) \\ * \quad & q_2 - q_1 = \theta \end{aligned}}$$



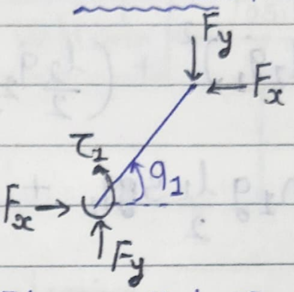
$$\boxed{q_1 = \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{l_2 \sin\theta}{l_1 + l_2\cos\theta}\right)} \quad -(3)$$

No gravity



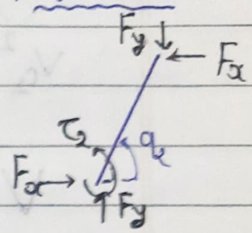
→ Static Equillibrium

FBD of Link-1



$\tau_1 = -F_x l_1 s q_1 + F_y l_1 c q_1$

FBD of Link-2



$\tau_2 = -F_x l_2 s q_2 + F_y l_2 c q_2$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -l_1 s q_1 & +l_1 c q_1 \\ -l_2 s q_2 & +l_2 c q_2 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix} \quad -(4)$$

(Simple version) For Task-3

(x_0, y_0)

$F_x = k(x - x_0)$

$F_y = k(y - y_0)$

$k \rightarrow$ User defined stiffness

* Next level of Task-1 & Task-3 :-

→ Lagrange's Equations

Lagrangian $\boxed{\mathcal{L} = K - V}$

K → kinetic Energy, V → Potential Energy

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = Q_i} \quad i = 1, 2, \ldots n$$

$Q_i \rightarrow$ generliged force

$$\boxed{\tau_i = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i}} \quad i = 1, 2, \ldots n$$

$$K = \underbrace{\frac{1}{2}\left(\frac{1}{3}m_1 l_1^2\right)\dot{q}_1^2}_{\text{pure rotation of link-1}} + \underbrace{\frac{1}{2}\left(\frac{1}{12}m_2 l_2^2\right)\dot{q}_2^2}_{\text{rotation of link-2 about com}} + \underbrace{\frac{1}{2}m_2 v_{c_2}^2}_{\text{translation of com of link-2}}$$

$$v_{c_2}^2 = (l_1\dot{q}_1)^2 + \left(\frac{l_2}{2}\dot{q}_2\right)^2 + 2l_1\dot{q}_1\left(\frac{l_2}{2}\right)\dot{q}_2\cos(q_2-q_1)$$

$$V = m_1 g\frac{l_1}{2}sq_1 + m_2 g\left(l_1 sq_1 + \frac{l_2}{2}sq_2\right)$$

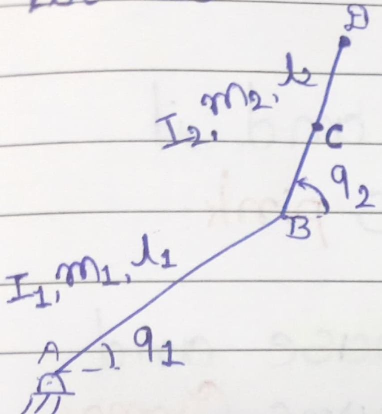
$$\tau_1 = \frac{1}{3}m_1 l_1^2\ddot{q}_1 + m_2 l_1^2\ddot{q}_1 + m_2\frac{l_1 l_2}{2}\ddot{q}_2\cos(q_2-q_1) - m_2\frac{l_1 l_2}{2}\dot{q}_2(\dot{q}_2-\dot{q}_1)\sin(q_2-q_1)$$
$$+ m_1 g\frac{l_1}{2}cq_2 + m_2 g l_1 cq_2$$

$$\tau_2 = \frac{1}{3}m_2 l_2^2\ddot{q}_2 + m_2\frac{l_2^2}{4}\ddot{q}_2 + m_2\frac{l_1 l_2}{2}\ddot{q}_1\cos(q_2-q_1) - \frac{m_2 l_1 l_2}{2}\dot{q}_1(\dot{q}_2-\dot{q}_1)\sin(q_2-q_1)$$
$$+ m_2 g\frac{l_2}{2}sq_2$$

$$F_x = k(x-x_0) = k(l_1cq_1 + l_2cq_2 - x_0)$$
$$F_y = k(y-y_0) = k(l_1sq_1 + l_2sq_2 - y_0)$$

* What goes wrong if you don't have feedback control?
* What goes wrong with no dynamics and only statics.
* What goes wrong with trying to acheive force and trajectory control together?

KEA



$$x = l_1 \cos q_1 + l_2 \cos q_2$$
$$y = l_1 \sin q_1 + l_2 \sin q_2$$

$$\dot{x} = -l_1 \sin q_1 \cdot \dot{q}_1 - l_2 \sin q_2 \dot{q}_2$$
$$\dot{y} = l_1 \cos q_1 \dot{q}_1 + l_2 \cos q_2 \dot{q}_2$$

Total velocity $V_t^2 = \dot{x}^2 + \dot{y}^2$

$$= l_1^2 \dot{q}_1^2 \sin^2 q_1 + l_2^2 \dot{q}_2^2 \sin^2 q_2 + 2 l_1 l_2 \dot{q}_1 \dot{q}_2 \sin q_1 \sin q_2 + l_1^2 \dot{q}_1^2 \cos^2 q_1 + l_2^2 \dot{q}_2^2 \cos^2 q_2 + 2 l_1 l_2 \dot{q}_1 \dot{q}_2 \cos q_1 \cdot \cos q_2$$

$$V_t^2 = l_1^2 \dot{q}_1^2 + l_2^2 \dot{q}_2^2 + 2 l_1 l_2 \dot{q}_1 \dot{q}_2 \cos(q_1 - q_2)$$

Total Kinetic Energy $K = K_1 + K_2$

Teacher's Signature ____________

$$K = \left(\frac{1}{2} I_1 \dot{q}_1^2\right) + \left(\frac{1}{2} I_2 \dot{q}_2^2 + \frac{1}{2} m_2 v_c^2\right)$$

velocity of com of link-2

→ position of com of link-2

$$x_c = l_1 \cos q_1 + \frac{l_2}{2} \cos q_2$$

$$y_c = l_1 \sin q_1 + \frac{l_2}{2} \sin q_2$$

→ velocity of com of link-2

$$\dot{x}_c = -l_1 \sin q_1 \dot{q}_1 - \frac{l_2}{2} \sin q_2 \dot{q}_2$$

$$\dot{y}_c = l_1 \cos q_1 \dot{q}_1 + \frac{l_2}{2} \cos q_2 \dot{q}_2$$

Total velocity of

$$v_c^2 = \dot{x}_c^2 + \dot{y}_c^2$$

$$= l_1^2 \dot{q}_1^2 \sin^2 q_1 + \frac{l_2^2}{4} \dot{q}_2^2 \sin^2 q_2 + l_1 l_2 \dot{q}_1 \dot{q}_2 \sin q_1 \sin q_2$$

$$+ l_1^2 \dot{q}_1^2 \cos^2 q_1 + \frac{l_2^2}{4} \dot{q}_2^2 \cos^2 q_2 + l_1 l_2 \dot{q}_1 \dot{q}_2 \cos q_1 \cdot \cos q_2$$

$$v_c^2 = l_1^2 \dot{q}_1^2 + \frac{l_2^2 \dot{q}_2^2}{4} + l_1 l_2 \dot{q}_1 \dot{q}_2 \cos(q_1 - q_2)$$

$$K = \frac{1}{2}\left(\frac{1}{3} m_1 l_1^2\right) \dot{q}_1^2 + \frac{1}{2}\left(\frac{1}{12} m_2 l_2^2\right) \dot{q}_2^2 + \frac{1}{2} m_2 l_1^2 \dot{q}_1^2 + \frac{m_2 l_2^2 \dot{q}_2^2}{8}$$

$$+ \frac{m_2 l_1 l_2 \dot{q}_1 \dot{q}_2 \cos(q_1 - q_2)}{2}$$

$$K = \dot{q}_1^2 \left(\frac{m_1 l_1^2}{6} + \frac{m_2 l_1^2}{2}\right) + \dot{q}_2^2 \left(\frac{m_2 l_2^2}{6}\right) + \dot{q}_1 \dot{q}_2 \left(\frac{m_2 l_1 l_2 \cos(q_1 - q_2)}{2}\right)$$

$$V = \frac{m_1 l_1 g \sin(q_1)}{2} + m_2 g \left(l_1 \sin(q_1) + \frac{l_2}{2} \sin q_2\right)$$

→ Lagrangian for 2R Manipulator

$$L = K - V$$

$$= \dot{q}_1^2 \left(\frac{m_1 l_1^2}{6} + \frac{m_2 l_1^2}{2} \right) + \dot{q}_2^2 \left(\frac{m_2 l_2^2}{6} \right) + \dot{q}_1 \dot{q}_2 \left(\frac{m_2 l_1 l_2 \cos(q_1 - q_2)}{2} \right)$$

$$- \frac{m_1 l_1 g \sin(q_1)}{2} - m_2 g \left(l_1 \sin q_1 + \frac{l_2 \sin q_2}{2} \right)$$

→ Derivatives of Lagrangian to find torque

$$\tau_1 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1} \right) - \frac{\partial L}{\partial q_1}$$

$$\frac{\partial L}{\partial \dot{q}_1} = \dot{q}_1 \left(\frac{m_1 l_1^2}{3} + m_2 l_1^2 \right) + \dot{q}_2 \left(\frac{m_2 l_1 l_2 \cos(q_1 - q_2)}{2} \right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1} \right) = \ddot{q}_1 \left(\frac{m_1 l_1^2}{3} + m_2 l_1^2 \right) + \ddot{q}_2 \left(\frac{m_2 l_1 l_2 \cos(q_1 - q_2)}{2} \right)$$

$$- \dot{q}_2 \cdot (\dot{q}_1 - \dot{q}_2) \frac{m_2 l_1 l_2 \sin(q_1 - q_2)}{2}$$

$$\frac{\partial L}{\partial q_1} = - \dot{q}_1 \dot{q}_2 \frac{m_2 l_1 l_2 \sin(q_1 - q_2)}{2} - \frac{m_1 l_1 g \cos q_1}{2} - m_2 g l_1 \cos q_1$$

$$\therefore \tau_1 = \ddot{q}_1 \left(\frac{m_1 l_1^2}{3} + m_2 l_1^2 \right) + \ddot{q}_2 \left(\frac{m_2 l_1 l_2 \cos(q_1 - q_2)}{2} \right)$$

$$- \dot{q}_2 (\dot{q}_1 - \dot{q}_2) \frac{m_2 l_1 l_2 \cancel{\sin(q_1 - q_2)}}{2} + \dot{q}_1 \dot{q}_2 \frac{m_2 l_1 l_2 \cancel{\sin(q_1 - q_2)}}{2}$$

$$+ \frac{m_1 l_1 g \cos q_1}{2} + m_2 g l_1 \cos q_1$$

$$\boxed{\begin{aligned}\tau_1 = {} & \ddot{q}_1 \overset{a}{\left(\frac{m_1 l_1^2}{3} + m_2 l_1^2\right)} + \ddot{q}_2 \overset{b}{\left(\frac{m_2 l_1 l_2}{2} \cos(q_1 - q_2)\right)} \\ & + \underset{c}{\dot{q}_2^2 \frac{m_2 l_1 l_2}{2} \sin(q_1 - q_2)} + \overset{d}{\frac{m_1 l_1 g}{2} \cos q_1} + \overset{e}{m_2 g l_1 \cos q_1}\end{aligned}}$$

$$\tau_2 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2}$$

$$\frac{\partial L}{\partial \dot{q}_2} = \dot{q}_2\left(\frac{m_2 l_2^2}{3}\right) + \dot{q}_1\left(\frac{m_2 l_1 l_2}{2}\cos(q_1 - q_2)\right)$$

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) = {} & \ddot{q}_2\left(\frac{m_2 l_2^2}{3}\right) + \ddot{q}_1\left(\frac{m_2 l_1 l_2}{2}\cos(q_1 - q_2)\right) \\ & - \dot{q}_1(\dot{q}_1 - \dot{q}_2)\frac{m_2 l_1 l_2}{2} \overset{\sin}{\cancel{\cos}}(q_1 - q_2) \\ = {} & \ddot{q}_2\left(\frac{m_2 l_2^2}{3}\right) + \ddot{q}_1\left(\frac{m_2 l_1 l_2}{2}\cos(q_1 - q_2)\right) \\ & - \dot{q}_1^2 \cdot \frac{m_2 l_1 l_2}{2}\sin(q_1 - q_2) + \dot{q}_1 \dot{q}_2 \frac{m_2 l_1 l_2}{2}\sin(q_1 - q_2)\end{aligned}$$

$$\frac{\partial L}{\partial q_2} = \dot{q}_1 \dot{q}_2\left(\frac{m_2 l_1 l_2}{2}\sin(q_1 - q_2)\right) - \frac{m_2 g l_2}{2}\cos q_2$$

$$\boxed{\begin{aligned}\tau_2 = {} & \ddot{q}_2 \underset{f}{\left(\frac{m_2 l_2^2}{3}\right)} + \ddot{q}_1 \overset{h}{\left(\frac{m_2 l_1 l_2}{2}\cos(q_1 - q_2)\right)} \\ & - \overset{j}{\dot{q}_1^2 \cdot \frac{m_2 l_1 l_2}{2}\sin(q_1 - q_2)} + \overset{K}{\frac{m_2 g l_2}{2}\cos q_2}\end{aligned}}$$

$$\tau_1 = a\ddot{q}_1 + b\ddot{q}_2\cos(q_1-q_2) + c\dot{q}_2^2\sin(q_1-q_2) + d\cos q_1 + e\cos q_1$$

$$\tau_2 = f\ddot{q}_2 + h\ddot{q}_1\cos(q_1-q_2) - j\dot{q}_1^2\sin(q_1-q_2) + K\cos q_2$$

$$\tau_1 = a\ddot{q}_1 + c\dot{q}_2^2\sin(q_1-q_2) + d\cos q_1 + e\cos q_1 + \frac{b\cos(q_1-q_2)}{f}\left[(\tau_2 - h\ddot{q}_1\cos(q_1-q_2) + j\dot{q}_1^2\sin(q_1-q_2) - K\cos q_2)\right]$$

$$\tau_1 = a\ddot{q}_1 + c\dot{q}_2^2\sin(q_1-q_2) + d\cos q_1 + e\cos q_1 + \frac{b\cos(q_1-q_2)\tau_2}{f} - \frac{bh\ddot{q}_1\cos^2(q_1-q_2)}{f} + \frac{bj\dot{q}_1^2\sin(q_1-q_2)\cos(q_1-q_2)}{f} - \frac{bK\cos(q_1-q_2)\cos q_2}{f}$$

$$\ddot{q}_1 = \left(\frac{1}{a - \frac{bh\cos^2(q_1-q_2)}{f}}\right)\left[\tau_1 - c\dot{q}_2^2\sin(q_1-q_2) - d\cos q_1 - e\cos q_1 - \frac{b\cos(q_1-q_2)\tau_2}{f} - \frac{bj\dot{q}_1^2\sin(q_1-q_2)\cos(q_1-q_2)}{f} + \frac{bK}{f}\cos(q_1-q_2)\cdot\cos q_2\right]$$

$$\boxed{\ddot{q}_1 = \left(\frac{1}{af - bh\cos^2(q_1-q_2)}\right)\left[f\tau_1 - cf\dot{q}_2^2\sin(q_1-q_2) - fd\cos q_1 - fe\cos q_1 - b\tau_2\cos(q_1-q_2) - bj\dot{q}_1^2\sin(q_1-q_2)\cos(q_1-q_2) + bK\cos(q_1-q_2)\cos q_2\right]}$$

$$\boxed{\ddot{q}_2 = \frac{1}{f}\left[\tau_2 - h\ddot{q}_1\cos(q_1-q_2) + j\dot{q}_1^2\sin(q_1-q_2) - K\cos q_2\right]}$$

$a = \frac{m_1 l_1^2}{3} + m_2 l_1^2$

$b = m_2\frac{l_1 l_2}{2}$

$c = m_2\frac{l_1 l_2}{2}$

$d = m_1\frac{l_1 g}{2}$

$e = m_2 g l_1$

$f = \frac{m_2 l_2^2}{3}$

$h = m_2\frac{l_1 l_2}{2}$

$j = m_2\frac{l_1 l_2}{2}$

$K = \frac{m_2 g l_2}{2}$