

Programming task

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1 Analytical Solution

The analytical solution for the PDE can be derived using the method of separable variables. The governing PDE for the problem is.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad (1)$$

Assuming the solution to be of the form $u(x,t) = p(x)q(t)$, we substitute in the differential equation.

$$\frac{dq(t)}{dt}p(x) = \frac{d^2p(x)}{dx^2}q(t) \quad (2)$$

$$\frac{p''}{p} = \frac{q'}{q} = k$$

Further analytical analysis requires 3 cases.

1.1 Case 1: $k = 0$

$$p''(x) = 0 \quad (3)$$

$$q'(t) = 0 \quad (4)$$

From equations (3) and (4)

$p(x) = ax + b$ and $q(t) = c$, where a,b,c are constants. Thus finally the general solution would be

$$u(x,t) = c(ax + b) \quad (5)$$

1.2 Case 2: $k > 0$

Let $k = m^2$

$$\frac{dq(t)}{dt} - m^2q(t) = 0 \quad (6)$$

$$\frac{d^2p(x)}{dx^2} - m^2p(x) = 0 \quad (7)$$

From eq (6) and (7) the solution for $u(x,t)$ is of the form:

$$u(x,t) = A_1 \exp(m^2t)[A_2 \exp(mx) + A_3 \exp(-mx)] \quad (8)$$

This equation does not satisfy the initial boundary condition $u(x,0) = \sin(2\pi x)$. Thus it can be eliminated as the solution for the BVP.

1.3 Case 3: $k < 0$

Using the similar analysis method as in case 2. With a minor change where $k = -m^2$ since $k < 0$.

The governing equation remains the same, and as the characteristic polynomial for the differential equation generates complex values ($k < 0$). Euler's identity is applied and the solution takes the form:

$$u(x, t) = A_1 \exp(-m^2 t) [A_2 \cos(mx) + A_3 \sin(mx)] \quad (9)$$

Substituting the initial conditions, $u(x, 0) = \sin(2\pi x)$, the value of $m = 2\pi$. Thus the solution boils down to:

$$u(x, t) = \exp(-4\pi^2 t) \sin(2\pi x) \quad (10)$$

2 Numerical implementation

The numerical schemes implemented in the code are, the central difference scheme for the spatial discretization and explicit Euler method for time.

$$u(x, t + \delta t) = u(x, t) + \delta t \frac{u(x + \delta x, t) - 2u(x, t) + u(x - \delta x, t)}{2(\delta x)^2} \quad (11)$$

128 grid points were chosen along the x direction, to discretize space between [0,1]. And 1000 time steps of 10^{-5} seconds each.

2.1 Solutions

The numerical solution follows the as expected from the analytical results. With time, the amplitude of the sine wave decreases exponentially owing to the term $\exp(-4\pi^2 t)$.

The error calculations are performed at every time step by taking the average of the difference between u calculated from the numerical simulation and the u obtained from the analytical solution.

$$Error = \frac{1}{N} \sum_{i=1}^N [u(x_i, t) - u_{exact}(x, t)] \quad (12)$$

Where $u_{exact} = \exp(-4\pi^2 t) \sin(2\pi x)$.

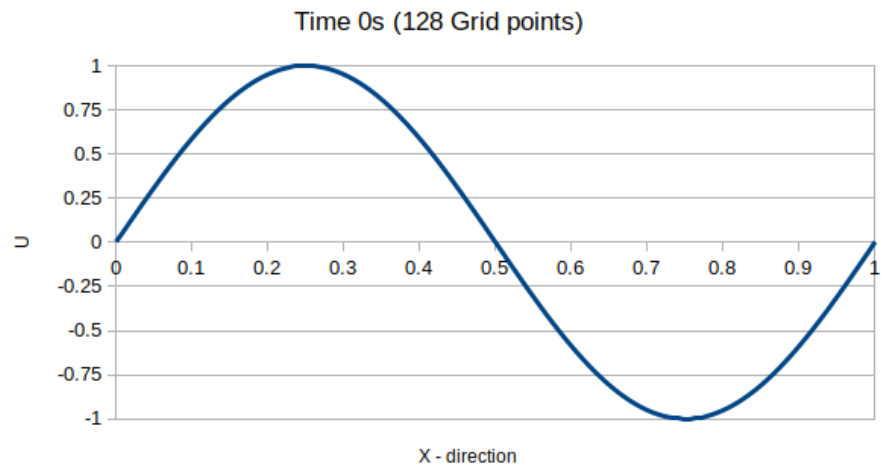


Figure 1: 0th Time step

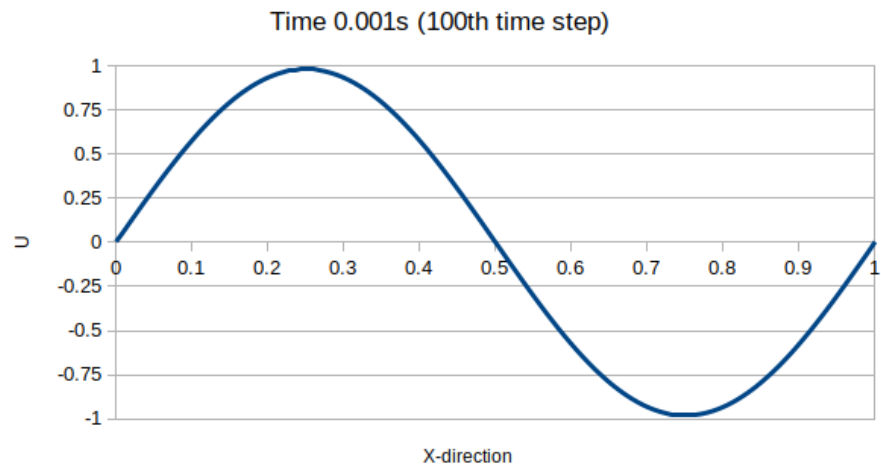


Figure 2: 100th Time step

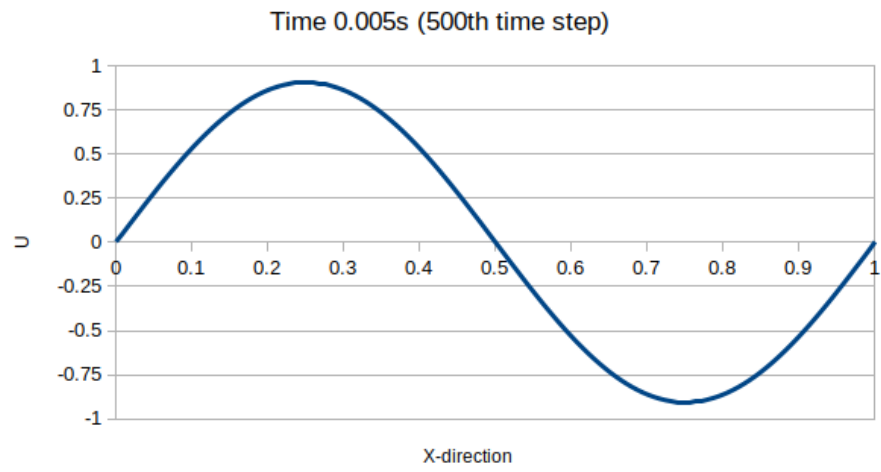


Figure 3: 500th Time step

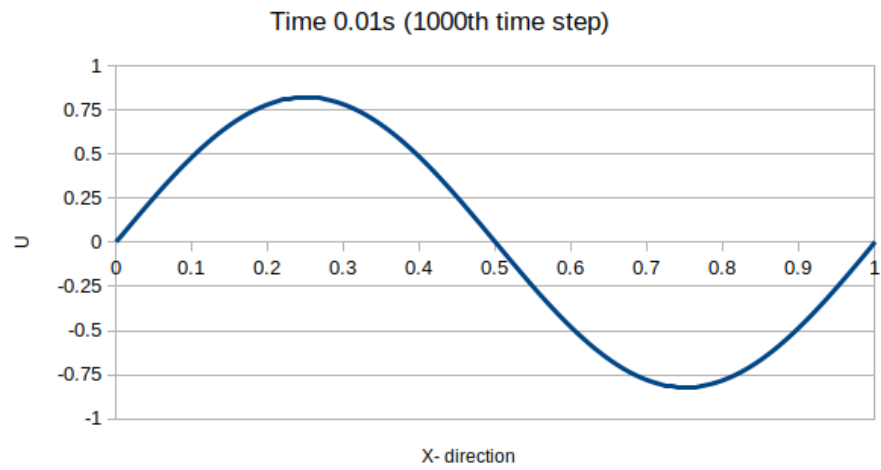


Figure 4: 1000th Time step