Programming task for the application of the post of Project Assistant

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1 Analytical Solution

The analytical solution for the PDE can be derived using the method of seperable variables. The governing PDE for the problem is.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{1}$$

Assuming the solution to be of the form u(x,t) = p(x)q(t), we subtitute in the differential equation.

$$\frac{dq(t)}{dt}p(x) = \frac{d^2p(x)}{dx^2}q(t) \tag{2}$$

$$\frac{p''}{p} = \frac{q'}{q} = k$$

Further analytical analysis requires 3 cases.

1.1 Case 1: $k = 0$

$$p''(x) = 0 \tag{3}$$

$$q'(t) = 0 \tag{4}$$

From equations (3) and (4)
$p(x) = ax + b$ and $q(t) = c$, where a,b,c are constants. Thus finally the general solution would be

$$u(x,t) = c(ax + b) \tag{5}$$

1.2 Case 2: $k > 0$

Let $k = m^2$

$$\frac{dq(t)}{dt} - m^2q(t) = 0 \tag{6}$$

$$\frac{d^2p(x)}{dx^2} - m^2p(x) = 0 \tag{7}$$

From eq (6) and (7) the solution for $u(x,t)$ is of the form:

$$u(x,t) = A_1 exp(m^2 t)[A_2 exp(mx) + A_3 exp(-mx)] \tag{8}$$

This equation does not satisfy the initial boundary condition $u(x.0) = sin(2\pi x)$. Thus it can be eliminated as the solution for the BVP.

1.3 Case 3: $k < 0$

Using the similar analysis method as in case 2. With a minor change where $k = -m^2$ since $k < 0$.
The governing equation remains the same, and as the characteristic polynomial for the differential equation generates complex values ($k < 0$). Euler's identity is applied and the solution takes the form:

$$u(x,t) = A_1 exp(-m^2 t)\,[A_2 cos(mx) + A_3 sin(mx)] \tag{9}$$

Substituting the initial conditions, $u(x,0) = sin(2\pi x)$, the value of $m = 2\pi$. Thus the solution boils down to:

$$u(x,t) = exp(-4\pi^2 t) sin(2\pi x) \tag{10}$$

2 Numerical implementation

The numerical schemes implemented in the code are, the central difference scheme for the spatial discritization and explicit Euler method for time.

$$u(x, t+\delta t) = u(x,t) + \delta t \frac{u(x+\delta x, t) - 2u(x,t)u(x-\delta x, t)}{2(\delta x)^2} \tag{11}$$

128 grid points were chosen along the x direction, to discritize space between [0,1]. And 1000 time steps of 10^{-5} seconds each.

2.1 Solutions

The numerical solution follows the as expected from the analytical results. With time, the amplitude of the sine wave decreases exponentiallt owing to the term $exp(-4\pi^2 t)$.

The error calculations are performed at every time step by taking the average of the difference between u calculated from the numerical simulation and the u obtained from the analytical solution.

$$Error = \frac{1}{N}\sum_{i=1}^{N}[u(x_i,t) - uexact(x,t)] \tag{12}$$

Where $uexact = exp(-4\pi^2 t) sin(2\pi x)$.

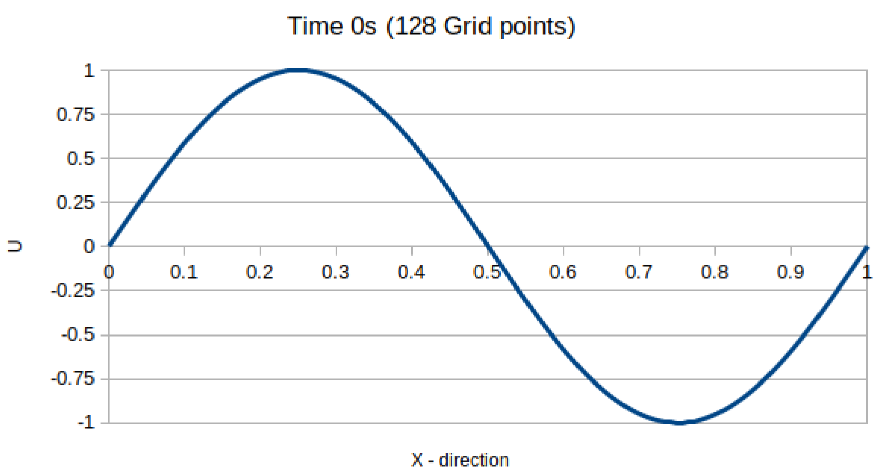

Figure 1: 0th Time step

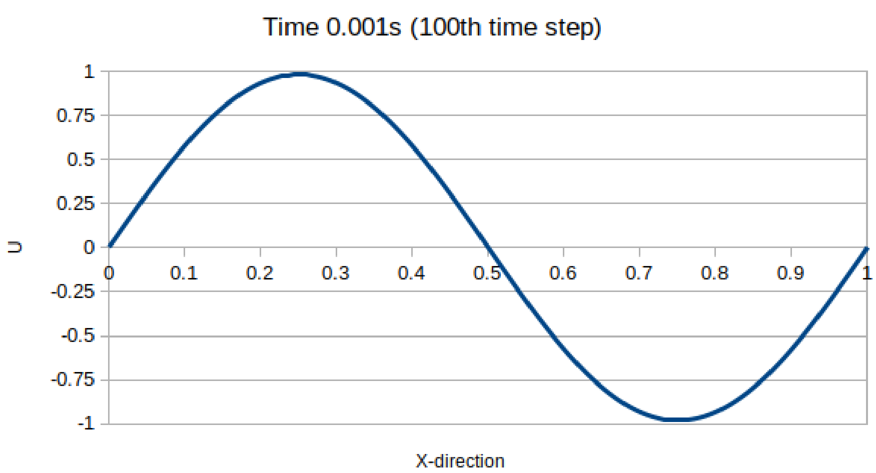

Figure 2: 100th Time step

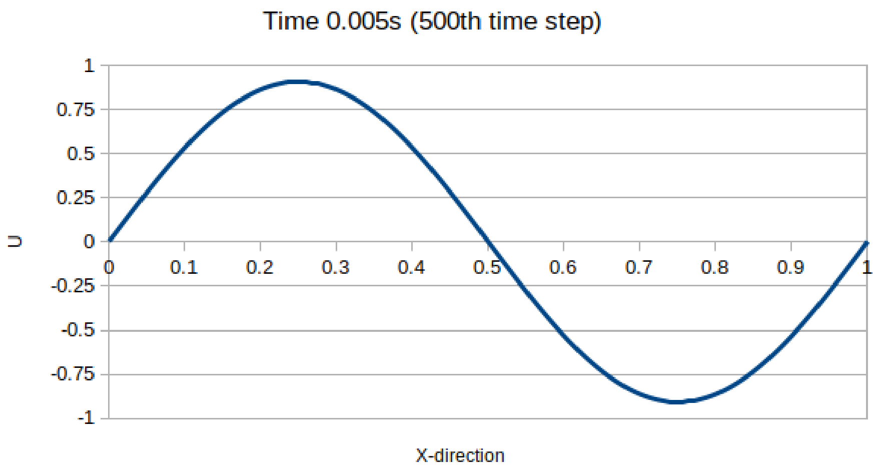

Figure 3: 500th Time step

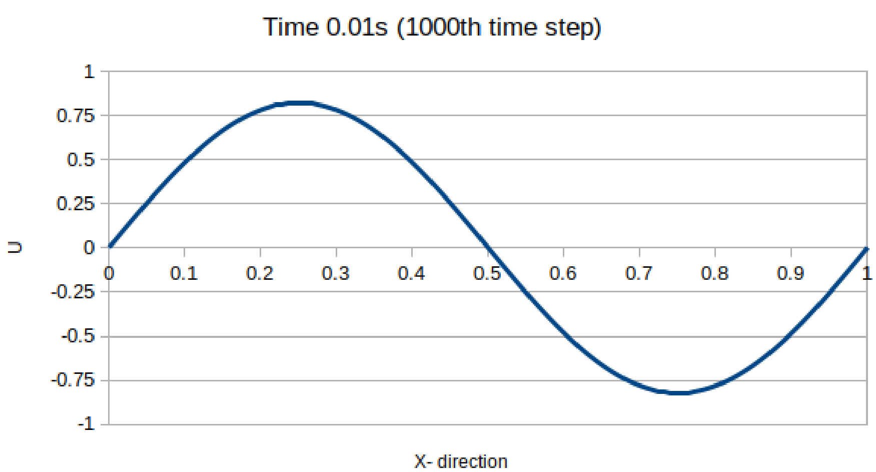

Figure 4: 1000th Time step