ASSIGNMENT 5

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Download all python codes from

https://github.com/harithar1234/EE3900−Haritha/blob/main/Assignment5/assignment5.py

QUESTION

Quadratic Forms/Q 2.8

Find the area bounded by the curves $\left\|\mathbf{x} - \begin{pmatrix}1\\0\end{pmatrix}\right\| = 1$ and $\|\mathbf{x}\| = 1$

SOLUTION

General equation of circle is

$$\mathbf{x}^\top\mathbf{x} - 2\mathbf{O}^\top\mathbf{x} + f = 0 \tag{0.0.1}$$

Taking equation of the first circle to be,

$$\left\|\mathbf{x} - \begin{pmatrix}1\\0\end{pmatrix}\right\|^2 = 1^2 \tag{0.0.2}$$

$$\mathbf{x}^\top\mathbf{x} - 2\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} + 1 = 1 \tag{0.0.3}$$

$$\mathbf{x}^\top\mathbf{x} - 2\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 0 \tag{0.0.4}$$

$$\mathbf{x}^\top\mathbf{x} - 2\mathbf{O_1}^\top\mathbf{x} + f_1 = 0 \tag{0.0.5}$$

$$\mathbf{O_1} = \begin{pmatrix}1\\0\end{pmatrix} \tag{0.0.6}$$

$$f_1 = 0 \tag{0.0.7}$$

Taking equation of the second circle to be,

$$\|\mathbf{x}\|^2 = 1^2 \tag{0.0.8}$$

$$\mathbf{x}^\top\mathbf{x} - 1 = 0 \tag{0.0.9}$$

$$\mathbf{x}^\top\mathbf{x} - 2\mathbf{O_2}^\top\mathbf{x} + f_2 = 0 \tag{0.0.10}$$

$$\mathbf{O_2} = \begin{pmatrix}0\\0\end{pmatrix} \tag{0.0.11}$$

$$f_2 = -1 \tag{0.0.12}$$

Subtracting (0.0.9) from (0.0.4) we get the chord of intersection as,

$$\mathbf{x}^\top\mathbf{x} - 2\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} - \{\mathbf{x}^\top\mathbf{x} - 1\} = 0 - 0 \tag{0.0.13}$$

$$\mathbf{x}^\top\mathbf{x} - 2\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = \mathbf{x}^\top\mathbf{x} - 1 \tag{0.0.14}$$

$$-2\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = -1 \tag{0.0.15}$$

$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = \frac{1}{2} \tag{0.0.16}$$

$$\tag{0.0.17}$$

This can be written as,

$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = \frac{1}{2} = 0.5 \tag{0.0.18}$$

$$\mathbf{x} = \begin{pmatrix}0.5\\0\end{pmatrix} + \lambda\begin{pmatrix}0\\1\end{pmatrix} \tag{0.0.19}$$

$$\mathbf{x} = \mathbf{q} + \lambda\mathbf{m} \tag{0.0.20}$$

$$\mathbf{q} = \begin{pmatrix}0.5\\0\end{pmatrix} \tag{0.0.21}$$

$$\mathbf{m} = \begin{pmatrix}0\\1\end{pmatrix} \tag{0.0.22}$$

Substituting (0.0.20) in (0.0.9) we get

$$(\mathbf{q} + \lambda\mathbf{m})^\top(\mathbf{q} + \lambda\mathbf{m}) - 1 = 0 \tag{0.0.23}$$

$$\mathbf{q}^\top(\mathbf{q} + \lambda\mathbf{m}) + \lambda\mathbf{m}^\top(\mathbf{q} + \lambda\mathbf{m}) - 1 = 0 \tag{0.0.24}$$

$$\mathbf{q}^\top\mathbf{q} + \lambda\mathbf{q}^\top\mathbf{m} + \lambda\mathbf{m}^\top\mathbf{q} + \lambda^2\mathbf{m}^\top\mathbf{m} - 1 = 0 \tag{0.0.25}$$

$$\|\mathbf{q}\|^2 + 2\lambda\mathbf{q}^\top\mathbf{m} + \lambda^2\|\mathbf{m}\|^2 - 1 = 0 \tag{0.0.26}$$

$$\|\mathbf{q}\|^2 + \lambda^2\|\mathbf{m}\|^2 - 1 = 0 \tag{0.0.27}$$

since $\mathbf{q}^\top \mathbf{m} = 0$

$$\lambda^2 = \frac{1 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \tag{0.0.28}$$

$$\lambda^2 = \frac{3}{4} \tag{0.0.29}$$

$$\lambda = +\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2} \tag{0.0.30}$$

Substituting the value of λ in (0.0.20) we get point of intersections of the circles as

$$\mathbf{A} = \begin{pmatrix} 0.5 \\ +\frac{\sqrt{3}}{2} \end{pmatrix} \tag{0.0.31}$$

$$\mathbf{B} = \begin{pmatrix} 0.5 \\ -\frac{\sqrt{3}}{2} \end{pmatrix} \tag{0.0.32}$$

Now finding the direction vectors

$$\mathbf{m_{O_1A}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0.5 \\ \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} 0.5 \\ -\frac{\sqrt{3}}{2} \end{pmatrix} \tag{0.0.33}$$

$$\mathbf{m_{O_1B}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0.5 \\ -\frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} 0.5 \\ \frac{\sqrt{3}}{2} \end{pmatrix} \tag{0.0.34}$$

$$\mathbf{m_{O_2A}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0.5 \\ \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} -0.5 \\ -\frac{\sqrt{3}}{2} \end{pmatrix} \tag{0.0.35}$$

$$\mathbf{m_{O_2B}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0.5 \\ -\frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} -0.5 \\ \frac{\sqrt{3}}{2} \end{pmatrix} \tag{0.0.36}$$

$$\tag{0.0.37}$$

Finding the angle $\angle AO_1B$

$$\cos\theta_1 = \frac{(\mathbf{m_{O_1A}})^\top \mathbf{m_{O_1B}}}{\|\mathbf{m_{O_1A}}\|\|\mathbf{m_{O_1B}}\|} \tag{0.0.38}$$

$$\cos\theta_1 = \frac{-2}{4} = \frac{-1}{2} \tag{0.0.39}$$

$$\theta_1 = 120^\circ \tag{0.0.40}$$

Finding the angle $\angle AO_2B$

$$\cos\theta_2 = \frac{(\mathbf{m_{O_2A}})^\top \mathbf{m_{O_2B}}}{\|\mathbf{m_{O_2A}}\|\|\mathbf{m_{O_2B}}\|} \tag{0.0.41}$$

$$\cos\theta_2 = \frac{-2}{4} = \frac{-1}{2} \tag{0.0.42}$$

$$\theta_2 = 120^\circ \tag{0.0.43}$$

In general Area of a Segment in degrees is $(\frac{\pi}{360})\theta r^2 - \frac{1}{2} r^2 \sin\theta$
The radius of $\|\mathbf{x}\|^2 = 1^2$ is 1.

Area of a Segment O_1AB is

$$Area_{O_1AB} = (\frac{\pi}{360})(120)(1^2) - \frac{1}{2}(1^2)\sin(120) \tag{0.0.44}$$

$$Area_{O_1AB} = \frac{\pi}{3} - \frac{1}{2}(\frac{\sqrt{3}}{2}) = \frac{\pi}{3} - \frac{\sqrt{3}}{4} \tag{0.0.45}$$

The radius of $\left\|\mathbf{x} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\|^2 = 1^2$ is 1.
Area of a Segment O_2AB is

$$Area_{O_2AB} = (\frac{\pi}{360})(120)(1^2) - \frac{1}{2}(1^2)\sin(120) \tag{0.0.46}$$

$$Area_{O_2AB} = \frac{\pi}{3} - \frac{1}{2}(\frac{\sqrt{3}}{2}) = \frac{\pi}{3} - \frac{\sqrt{3}}{4} \tag{0.0.47}$$

Area of region bounded by curves is area of the region O_1AO_2B

$$Area_{O_1AO_2B} = Area_{O_1AB} + Area_{O_2AB} = 2(\frac{\pi}{3} - \frac{\sqrt{3}}{4}) \tag{0.0.48}$$

$$Area_{O_1AO_2B} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2} \tag{0.0.49}$$

$$\text{Area}=\frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

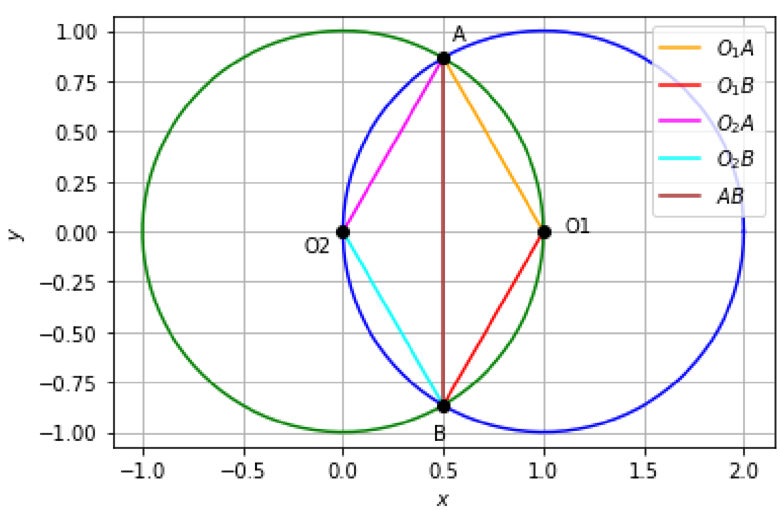

Fig. 1: plot