

ASSIGNMENT 5

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Download all python codes from

<https://github.com/harithar1234/EE3900-Haritha/blob/main/Assignment5/assignment5.py>

QUESTION

Quadratic Forms/Q 2.8

Find the area bounded by the curves

$$\left\| \mathbf{x} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| = 1 \text{ and } \|\mathbf{x}\| = 1$$

SOLUTION

General equation of circle is

$$\mathbf{x}^T \mathbf{x} - 2\mathbf{O}^T \mathbf{x} + f = 0 \quad (0.0.1)$$

Taking equation of the first circle to be,

$$\left\| \mathbf{x} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|^2 = 1^2 \quad (0.0.2)$$

$$\mathbf{x}^T \mathbf{x} - 2 \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} + 1 = 1 \quad (0.0.3)$$

$$\mathbf{x}^T \mathbf{x} - 2 \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \quad (0.0.4)$$

$$\mathbf{x}^T \mathbf{x} - 2\mathbf{O}_1^T \mathbf{x} + f_1 = 0 \quad (0.0.5)$$

$$\mathbf{O}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (0.0.6)$$

$$f_1 = 0 \quad (0.0.7)$$

Taking equation of the second circle to be,

$$\|\mathbf{x}\|^2 = 1^2 \quad (0.0.8)$$

$$\mathbf{x}^T \mathbf{x} - 1 = 0 \quad (0.0.9)$$

$$\mathbf{x}^T \mathbf{x} - 2\mathbf{O}_2^T \mathbf{x} + f_2 = 0 \quad (0.0.10)$$

$$\mathbf{O}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (0.0.11)$$

$$f_2 = -1 \quad (0.0.12)$$

substituting \mathbf{O}_2 in equation of first circle,

$$\left\| \mathbf{O}_2 - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| = 1 \quad (0.0.13)$$

so \mathbf{O}_2 lies on first circle.

substituting \mathbf{O}_1 in equation of second circle,

$$\|\mathbf{O}_1\| = \left\| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| = 1 \quad (0.0.14)$$

so \mathbf{O}_1 lies on second circle. So each circle passes through the centre of the other.

Subtracting (0.0.9) from (0.0.4) we get the chord of intersection as,

$$\mathbf{x}^T \mathbf{x} - 2 \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} - \{\mathbf{x}^T \mathbf{x} - 1\} = 0 - 0 \quad (0.0.15)$$

$$\mathbf{x}^T \mathbf{x} - 2 \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = \mathbf{x}^T \mathbf{x} - 1 \quad (0.0.16)$$

$$-2 \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = -1 \quad (0.0.17)$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = \frac{1}{2} \quad (0.0.18)$$

This can be written as,

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = \frac{1}{2} = 0.5 \quad (0.0.19)$$

$$\mathbf{x} = \begin{pmatrix} 0.5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (0.0.20)$$

$$\mathbf{x} = \mathbf{q} + \lambda \mathbf{m} \quad (0.0.21)$$

$$\mathbf{q} = \begin{pmatrix} 0.5 \\ 0 \end{pmatrix} \quad (0.0.22)$$

$$\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (0.0.23)$$

Substituting (0.0.21) in (0.0.9) we get

$$(\mathbf{q} + \lambda \mathbf{m})^T (\mathbf{q} + \lambda \mathbf{m}) - 1 = 0 \quad (0.0.24)$$

$$\mathbf{q}^T (\mathbf{q} + \lambda \mathbf{m}) + \lambda \mathbf{m}^T (\mathbf{q} + \lambda \mathbf{m}) - 1 = 0 \quad (0.0.25)$$

$$\mathbf{q}^T \mathbf{q} + \lambda \mathbf{q}^T \mathbf{m} + \lambda \mathbf{m}^T \mathbf{q} + \lambda^2 \mathbf{m}^T \mathbf{m} - 1 = 0 \quad (0.0.26)$$

$$\|\mathbf{q}\|^2 + 2\lambda \mathbf{q}^T \mathbf{m} + \lambda^2 \|\mathbf{m}\|^2 - 1 = 0 \quad (0.0.27)$$

$$\|\mathbf{q}\|^2 + \lambda^2 \|\mathbf{m}\|^2 - 1 = 0 \quad (0.0.28)$$

since $\mathbf{q}^\top \mathbf{m} = 0$

$$\lambda^2 = \frac{1 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \quad (0.0.29)$$

$$\lambda^2 = \frac{3}{4} \quad (0.0.30)$$

$$\lambda = +\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2} \quad (0.0.31)$$

Substituting the value of λ in (0.0.21) we get point of intersections of the circles as

$$\mathbf{A} = \begin{pmatrix} 0.5 \\ +\frac{\sqrt{3}}{2} \end{pmatrix} \quad (0.0.32)$$

$$\mathbf{B} = \begin{pmatrix} 0.5 \\ -\frac{\sqrt{3}}{2} \end{pmatrix} \quad (0.0.33)$$

Now finding the direction vectors

$$\mathbf{m}_{O_1A} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0.5 \\ \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} 0.5 \\ -\frac{\sqrt{3}}{2} \end{pmatrix} \quad (0.0.34)$$

$$\mathbf{m}_{O_1B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0.5 \\ -\frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} 0.5 \\ \frac{\sqrt{3}}{2} \end{pmatrix} \quad (0.0.35)$$

$$\mathbf{m}_{O_2A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0.5 \\ \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} -0.5 \\ -\frac{\sqrt{3}}{2} \end{pmatrix} \quad (0.0.36)$$

$$\mathbf{m}_{O_2B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0.5 \\ -\frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} -0.5 \\ \frac{\sqrt{3}}{2} \end{pmatrix} \quad (0.0.37)$$

$$(0.0.38)$$

Finding the angle $\angle AO_1B$

$$\cos \theta_1 = \frac{(\mathbf{m}_{O_1A})^\top \mathbf{m}_{O_1B}}{\|\mathbf{m}_{O_1A}\| \|\mathbf{m}_{O_1B}\|} \quad (0.0.39)$$

$$\cos \theta_1 = \frac{-2}{4} = \frac{-1}{2} \quad (0.0.40)$$

$$\theta_1 = 120^\circ \quad (0.0.41)$$

Finding the angle $\angle AO_2B$

$$\cos \theta_2 = \frac{(\mathbf{m}_{O_2A})^\top \mathbf{m}_{O_2B}}{\|\mathbf{m}_{O_2A}\| \|\mathbf{m}_{O_2B}\|} \quad (0.0.42)$$

$$\cos \theta_2 = \frac{-2}{4} = \frac{-1}{2} \quad (0.0.43)$$

$$\theta_2 = 120^\circ \quad (0.0.44)$$

In general Area of a Segment in degrees is $(\frac{\pi}{360})\theta r^2 - \frac{1}{2}r^2 \sin \theta$

The radius of $\|\mathbf{x}\|^2 = 1^2$ is 1.

Area of a Segment O_1AB is

$$Area_{O_1AB} = (\frac{\pi}{360})(120)(1^2) - \frac{1}{2}(1^2) \sin(120) \quad (0.0.45)$$

$$Area_{O_1AB} = \frac{\pi}{3} - \frac{1}{2}(\frac{\sqrt{3}}{2}) = \frac{\pi}{3} - \frac{\sqrt{3}}{4} \quad (0.0.46)$$

The radius of $\left\| \mathbf{x} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|^2 = 1^2$ is 1.

Area of a Segment O_2AB is

$$Area_{O_2AB} = (\frac{\pi}{360})(120)(1^2) - \frac{1}{2}(1^2) \sin(120) \quad (0.0.47)$$

$$Area_{O_2AB} = \frac{\pi}{3} - \frac{1}{2}(\frac{\sqrt{3}}{2}) = \frac{\pi}{3} - \frac{\sqrt{3}}{4} \quad (0.0.48)$$

Area of region bounded by curves is area of the region O_1AO_2B

$$Area_{O_1AO_2B} = Area_{O_1AB} + Area_{O_2AB} = 2(\frac{\pi}{3} - \frac{\sqrt{3}}{4}) \quad (0.0.49)$$

$$Area_{O_1AO_2B} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2} \quad (0.0.50)$$

$$Area = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

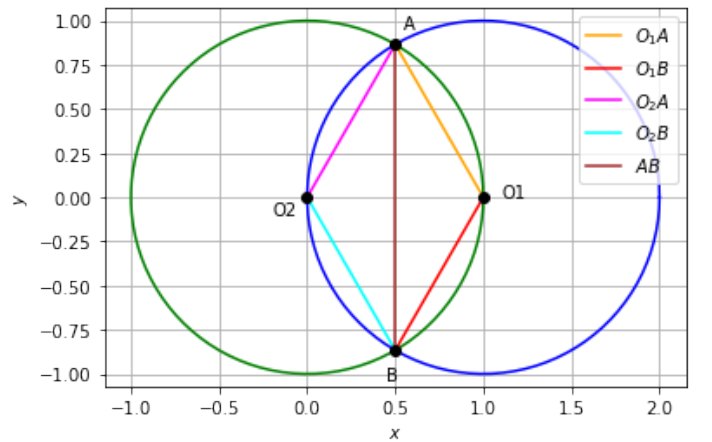


Fig. 1: plot