# ASSIGNMENT 5

## HARITHA R AI20BTECH11010

### Download all python codes from

https://github.com/harithar1234/EE3900-Haritha/blob/main/Assignment5/assignment5.py

#### **QUESTION**

#### Quadratic Forms/Q 2.8

Find the area bounded by the curves

$$\left\|\mathbf{x} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\| = 1 \text{ and } \left\|\mathbf{x}\right\| = 1$$

#### **SOLUTION**

General equation of circle is

$$\mathbf{x}^{\mathsf{T}}\mathbf{x} - 2\mathbf{O}^{\mathsf{T}}\mathbf{x} + f = 0 \tag{0.0.1}$$

Taking equation of the first circle to be,

$$\left\|\mathbf{x} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\|^2 = 1^2 \tag{0.0.2}$$

$$\mathbf{x}^{\mathsf{T}}\mathbf{x} - 2(1 \quad 0)\mathbf{x} + 1 = 1$$
 (0.0.3)

$$\mathbf{x}^{\mathsf{T}}\mathbf{x} - 2(1 \quad 0)\mathbf{x} = 0 \tag{0.0.4}$$

$$\mathbf{x}^{\mathsf{T}}\mathbf{x} - 2\mathbf{O}_{\mathbf{1}}^{\mathsf{T}}\mathbf{x} + f_{1} = 0 \tag{0.0.5}$$

$$\mathbf{O_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{0.0.6}$$

$$f_1 = 0 (0.0.7)$$

Taking equation of the second circle to be,

$$\|\mathbf{x}\|^2 = 1^2 \tag{0.0.8}$$

$$\mathbf{x}^{\mathsf{T}}\mathbf{x} - 1 = 0 \tag{0.0.9}$$

$$\mathbf{x}^{\mathsf{T}}\mathbf{x} - 2\mathbf{O}_{2}^{\mathsf{T}}\mathbf{x} + f_{2} = 0 \tag{0.0.10}$$

$$\mathbf{O_2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{0.0.11}$$

$$f_2 = -1 \tag{0.0.12}$$

substituting  $O_2$  in equation of first circle,

$$\left\|\mathbf{O}_2 - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\| = \left\| \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| = 1 \qquad (0.0.13)$$

so  $O_2$  lies on first circle. substituting  $O_1$  in equation of second circle,

$$\|\mathbf{O_1}\| = \left\| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| = 1 \tag{0.0.14}$$

1

so  $O_1$  lies on second circle. So each circle passes through the centre of the other.

Subtracting (0.0.9) from (0.0.4) we get the chord of intersection as,

$$\mathbf{x}^{\mathsf{T}}\mathbf{x} - 2(1 \quad 0)\mathbf{x} - {\mathbf{x}^{\mathsf{T}}\mathbf{x} - 1} = 0 - 0 \quad (0.0.15)$$

$$\mathbf{x}^{\mathsf{T}}\mathbf{x} - 2(1 \quad 0)\mathbf{x} = \mathbf{x}^{\mathsf{T}}\mathbf{x} - 1 \quad (0.0.16)$$

$$-2(1 \ 0)\mathbf{x} = -1 \ (0.0.17)$$

$$(1 \quad 0)\mathbf{x} = \frac{1}{2} \quad (0.0.18)$$

This can be written as,

$$(1 \quad 0)\mathbf{x} = \frac{1}{2} = 0.5 \tag{0.0.19}$$

$$\mathbf{x} = \begin{pmatrix} 0.5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{0.0.20}$$

$$\mathbf{x} = \mathbf{q} + \lambda \mathbf{m} \tag{0.0.21}$$

$$\mathbf{q} = \begin{pmatrix} 0.5 \\ 0 \end{pmatrix} \tag{0.0.22}$$

$$\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{0.0.23}$$

Substituting (0.0.21) in (0.0.9) we get

$$(\mathbf{q} + \lambda \mathbf{m})^{\mathsf{T}} (\mathbf{q} + \lambda \mathbf{m}) - 1 = 0$$
(0.0.24)

$$\mathbf{q}^{\mathsf{T}}(\mathbf{q} + \lambda \mathbf{m}) + \lambda \mathbf{m}^{\mathsf{T}}(\mathbf{q} + \lambda \mathbf{m}) - 1 = 0$$
(0.0.25)

$$\mathbf{q}^{\mathsf{T}}\mathbf{q} + \lambda \mathbf{q}^{\mathsf{T}}\mathbf{m} + \lambda \mathbf{m}^{\mathsf{T}}\mathbf{q} + \lambda^{2}\mathbf{m}^{\mathsf{T}}\mathbf{m} - 1 = 0$$
(0.0.26)

$$\|\mathbf{q}\|^2 + 2\lambda \mathbf{q}^{\mathsf{T}} \mathbf{m} + \lambda^2 \|\mathbf{m}\|^2 - 1 = 0$$
(0.0.27)

$$\|\mathbf{q}\|^2 + \lambda^2 \|\mathbf{m}\|^2 - 1 = 0$$
(0.0.28)

since  $\mathbf{q}^{\mathsf{T}}\mathbf{m} = 0$ 

$$\lambda^2 = \frac{1 - ||\mathbf{q}||^2}{||\mathbf{m}||^2} \tag{0.0.29}$$

$$\lambda^2 = \frac{3}{4} \tag{0.0.30}$$

$$\lambda = +\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2} \tag{0.0.31}$$

Substituting the value of  $\lambda$  in (0.0.21) we get point of intersections of the circles as

$$\mathbf{A} = \begin{pmatrix} 0.5 \\ +\frac{\sqrt{3}}{2} \end{pmatrix} \tag{0.0.32}$$

$$\mathbf{B} = \begin{pmatrix} 0.5 \\ -\frac{\sqrt{3}}{2} \end{pmatrix} \tag{0.0.33}$$

Now finding the direction vectors

$$\mathbf{m}_{\mathbf{O_1A}} = \begin{pmatrix} 1\\0 \end{pmatrix} - \begin{pmatrix} 0.5\\\frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} 0.5\\-\frac{\sqrt{3}}{2} \end{pmatrix} \tag{0.0.34}$$

$$\mathbf{m_{O_1B}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0.5 \\ -\frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} 0.5 \\ \frac{\sqrt{3}}{2} \end{pmatrix}$$
 (0.0.35)

$$\mathbf{m}_{\mathbf{O_2A}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0.5 \\ \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} -0.5 \\ -\frac{\sqrt{3}}{2} \end{pmatrix}$$
 (0.0.36)

$$\mathbf{m_{O_2B}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0.5 \\ -\frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} -0.5 \\ \frac{\sqrt{3}}{2} \end{pmatrix}$$
 (0.0.37)

(0.0.38)

Finding the angle  $\angle AO_1B$ 

$$\cos \theta_1 = \frac{(\mathbf{m}_{\mathbf{O}_1 \mathbf{A}})^{\mathsf{T}} \mathbf{m}_{\mathbf{O}_1 \mathbf{B}}}{\|\mathbf{m}_{\mathbf{O}_1 \mathbf{A}}\| \|\mathbf{m}_{\mathbf{O}_1 \mathbf{B}}\|}$$
(0.0.39)

$$\cos \theta_1 = \frac{-2}{4} = \frac{-1}{2} \tag{0.0.40}$$

$$\theta_1 = 120^{\circ}$$
 (0.0.41)

Finding the angle  $\angle AO_2B$ 

$$\cos \theta_2 = \frac{(\mathbf{m}_{\mathbf{O}_2 \mathbf{A}})^{\mathsf{T}} \mathbf{m}_{\mathbf{O}_2 \mathbf{B}}}{\|\mathbf{m}_{\mathbf{O}_2 \mathbf{A}}\| \|\mathbf{m}_{\mathbf{O}_2 \mathbf{B}}\|}$$
(0.0.42)

$$\cos \theta_2 = \frac{-2}{4} = \frac{-1}{2} \tag{0.0.43}$$

$$\theta_2 = 120^{\circ}$$
 (0.0.44)

In general Area of a Segment in degrees is  $(\frac{\pi}{360})\theta r^2 - \frac{1}{2}r^2\sin\theta$ The radius of  $||\mathbf{x}||^2 = 1^2$  is 1.

Area of a Segment  $O_1AB$  is

$$Area_{O_1AB} = (\frac{\pi}{360})(120)(1^2) - \frac{1}{2}(1^2)\sin(120)$$

$$(0.0.45)$$

$$Area_{O_1AB} = \frac{\pi}{3} - \frac{1}{2}(\frac{\sqrt{3}}{2}) = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$$

$$(0.0.46)$$

The radius of  $\left\|\mathbf{x} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\|^2 = 1^2$  is 1. Area of a Segment  $O_2AB$  is

$$Area_{O_2AB} = (\frac{\pi}{360})(120)(1^2) - \frac{1}{2}(1^2)\sin(120)$$

$$(0.0.47)$$

$$Area_{O_2AB} = \frac{\pi}{3} - \frac{1}{2}(\frac{\sqrt{3}}{2}) = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$$

Area of region bounded by curves is area of the region  $O_1AO_2B$ 

$$Area_{O_1AO_2B} = Area_{O_1AB} + Area_{O_2AB} = 2(\frac{\pi}{3} - \frac{\sqrt{3}}{4})$$

$$(0.0.49)$$

$$Area_{O_1AO_2B} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

$$(0.0.50)$$

Area=
$$\frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

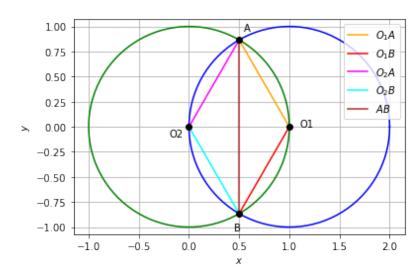


Fig. 1: plot