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Einstein and Jordan frame correspondence in quantum cosmology: expansion-collapse duality

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Abstract The conformal correspondence between FLRW universes in the Einstein and Jordan frames allows for an expansion-collapse duality – an always expanding Einstein frame universe can have a dual Jordan frame description that is contracting forever. The scenario eventually runs into an apparent paradox. When the contracting Jordan frame universe becomes sufficiently small, the classical description becomes inadequate and the universe is expected to develop quantum characteristics. However, at this time, the corresponding Einstein frame universe is expected to behave classically, due to the arbitrarily large size it has grown to. The conformal map here appears to be providing a duality between a quantum effect-dominated universe and a classical universe. We investigate the status of the conformal map at the quantum level in such a scenario, focusing on addressing this paradox. The Einstein and Jordan frame universes are quantized using the Wheeler-DeWitt prescription. We show that the classical conformal map holds at the quantum level when compared through expectation values of scale factors. The relative quantum fluctuation in the scale factor becomes conformally invariant, it increases in both the past and future directions according to the internal clock. Expectedly, the quantum fluctuations in the collapsing Jordan frame increase as it shrinks towards singularity. More surprisingly, the quantum fluctuations in the expanding Einstein frame increase as well, even as its classical scale factor becomes larger. Despite having drastically different cosmological evolutions, the rise in quantum characteristics in a collapsing frame implies the same in its expanding counterpart, thereby resolving the apparent paradox.

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1 Introduction

It is well-known that scalar-tensor theories of gravity can be put in the form of Einstein's gravity with a minimally coupled scalar-field in a conformally connected spacetime, where the former and latter representations are referred to as the *Jordan* and *Einstein frames*, respectively [1,2]. The two conformally connected frames are mathematically equivalent by construction; however, whether the frames can be considered as 'physically equivalent' is highly debated [3–17]. While the conformal transformation removes the non-minimal coupling between the scalar field and curvature in the Jordan frame action, this comes at a cost. An ordinary matter component in the Jordan frame becomes non-trivially coupled to the scalar field in the Einstein frame, resulting in an extra 'fifth force' term in the geodesic equation of a material particle. The modification in the geodesic equation can be seen as a violation of the *Einstein's equivalence principle*. This is often used as the main argument for the Einstein frame not being the physical frame. The counter arguments to this is largely based the notion of 'running units' in the Einstein frame [8,18–20]. As originally suggested by Dicke [19], the conformal transformation of the metric naturally leads to spacetime dependent scaling of the units of fundamental physical quantities, such as length, time, and mass. It is argued that, if one consistently adopts this running units in the Einstein frame, then the two conformally connected frames can be regarded as physically equivalent (see [8] for a review).

Regardless of Dicke's argument, the Einstein and Jordan frame descriptions generally differ in their cosmological evolutions [21–29]. Although different conformal frames are mathematically equivalent, the equations of motion in these frames may lead to quite different properties of the scale factors, Hubble parameters, accelerations, and any higher-order derivatives of the scale factors in the two frames. One may use this duality to study standard cosmological models of the physical universe through conformally connected universes with contrasting features. Such conformal maps have interesting implications. For example, in [24], it is shown that an accelerated expansion of space can alternatively be realized as a decelerating phase in a conformally connected universe, demonstrating that cosmic acceleration is a frame dependent effect. In [27], the author introduces a 'slow freeze' model of the early universe as a dual description of the hot big-bang model, where the universe is contracting instead of expanding in the radiation and matter-dominated eras. Consequently, the universe is free from a big-bang-like singularity in the slow-freeze description. In [26], the authors propose an early-time 'anamorphic phase' of the universe, during which the cosmological evolution shows features of both the inflationary model and the ekpyrotic model depending on the choice of different conformally invariant criteria. By combining features of both the inflationary and ekpyrotic scenarios, the anamorphic universe model attempts to bypass some shortcomings associated with inflationary and bouncing models when considered individually. The 'conflation model', introduced in [25], also explores the duality between the inflationary and the ekpyrotic model. During the 'conflation'-phase, the accelerated expansion in one frame maps to a contracting phase in another frame.

It is also possible to realize the dark energy-driven standard late-time cosmological models in conformal frames with contrasting cosmological evolutions. In [28], a class of quintessence models is shown to be conformally dual to $f(R)$ gravity driven Jordan frames, where the Jordan frame may undergo a collapse. In [29], it is shown that the cosmological evolutions of the ΛCDM and quintessence models in the Einstein frames always correspond to bouncing or collapsing Jordan frames driven by Brans–Dicke theories. With suitable choice for the Brans–Dicke parameter, the current epoch of the standard cosmological evolution can be aligned with a bounce in the Jordan frame, during which the Jordan frame scale factor becomes arbitrarily small.

In general, the expansion-collapse duality between the Einstein and Jordan frames has interesting consequences. Given the choice of the theories in the two frames, an ever-expanding universe in the Einstein frame may correspond to a Jordan frame universe which is contracting indefinitely. As the collapsing Jordan frame universe becomes sufficiently small, one may expect it to develop quantum characteristics; for example, the quantum uncertainties in different cosmological observables may increase as the scale factor gets smaller. At the same time, the scale factor in the dual Einstein frame universe keeps on increasing. This raises the questions – what is the status of the classical conformal map when one considers the quantum descriptions of the Einstein and Jordan frame universes in such a scenario? Subsequently, how do the increasing quantum uncertainties in the collapsing Jordan frame universe correspond to the uncertainties in the expanding Einstein frame? One can speculate that the quantum variances in the Einstein frame may get suppressed due to the expansion of space. If this is true, then the conformal correspondence seems to provide a map between a quantum-effect dominated universe and a universe with negligible quantum corrections. Alternatively, if the quantum fluctuations in the Einstein and Jordan frames evolve similarly despite their contrasting cosmological evolutions, then this would indicate that the increase in quantum characteristics is a frame-independent effect.

Previous studies have explored different aspects of the Einstein–Jordan frame duality in quantum descriptions [6,8,9,13,14,30–37]. For example, in [6], the Einstein and Jordan frame representations of the Brans–Dicke theory are studied in the loop quantum cosmology framework. The loop quantizations in different frames are shown to be leading to non-equivalent results. In [13], a Brans–Dicke model with a perfect fluid is quantized in the Wheeler–DeWitt prescription in both frames. In this case, the Einstein and Jordan frame wave packet solutions are found to be incompatible when compared via the conformal map. However, it is shown in a subsequent study [38] that the Einstein and Jordan frame wave packets do become conformally connected in the absence of additional matter components.

In this paper, we investigate the expansion-collapse duality between the conformal frames in a fully quantum mechanical framework according to the Wheeler–DeWitt quantization scheme [39,40]. We consider the simple example of the Brans–Dicke model without a potential in the Jordan frame that corresponds to a massless canonical scalar field in the Einstein frame. In the classical description, one may choose a Brans–Dicke parameter for which the Jordan frame contracts indefinitely, corresponding to an ever-expanding Einstein frame. To describe the expansion-collapse duality at the quantum level, the Einstein and Jordan frame universes are individually quantized using the Wheeler–DeWitt prescription, following [38]. We derive the expectation values for relevant cosmological quantities in both frames. To see how the classical conformal correspondence translates into the quantum description, we find the relations between different Einstein and Jordan frame quantities through their expectation values and compare these relations with their classical counterparts. In this regard, our approach differs from that of [38], where the status of the conformal map is investigated at the level of wave packets in the two frames. The maps between the Einstein and Jordan frame expectation values also determine whether an expansion-collapse duality between the frames is possible in the quantum description. Finally, we see how quantum fluctuations of different quantities are related via the conformal map and address whether the increasing nature of quantum characteristics in one frame implies the same in the other frame.

The paper is organized as follows. In Sect. 2, we discuss the classical expansion-collapse duality between a Jordan frame with the Brans–Dicke model, and its corresponding Einstein frame. In Sect. 3, we set up the classical Hamiltonian formalism in the Einstein and Jordan frames. The universes in the two frames are then quantized in Sect. 4, and we obtain expectation values and quantum fluctuations of different cosmological quantities. In Sect. 5, we investigate the status of the conformal map at the quantum level and discuss how the quantum fluctuations in the two frames are related. We end with a summary and discussion in Sect. 6.

2 Classical expansion-collapse duality

Depending on the scalar field model, the cosmological evolution in the Einstein and Jordan frame descriptions may be drastically different. In this paper, we are interested in a scenario where an ever-expanding Einstein frame universe corresponds to a Jordan frame that is contracting indefinitely. In fact, one can realize such an expansion-collapse duality even through a simple Brans–Dicke model in the Jordan frame. In this section, we consider the original Brans–Dicke model with zero potential in the Jordan frame and show that the expansion-collapse duality can be achieved for certain choice of the Brans–Dicke parameter.

Let us consider the Brans–Dicke theory action in the Jordan frame, governed by the Brans–Dicke scalar field λ and the metric g_{ab},

$$S_J^{BD} = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{\lambda}{16\pi}R - \frac{w_{\rm BD}}{16\pi\lambda}g^{ab}\partial_a\lambda\partial_b\lambda\right), \quad (2.1)$$

where R is the Ricci scalar in the Jordan frame and $w_{\rm BD}$ is the constant Brans–Dicke parameter [1,2,41]. The conformal transformation

$$\tilde{g}_{ab} = G\lambda g_{ab} \quad (2.2)$$

leads to the Einstein frame action

$$S_E = \int \mathrm{d}^4x\sqrt{-\tilde{g}}\left[\frac{1}{2\kappa^2}\tilde{R} - \frac{1}{2}\tilde{g}^{ab}\partial_a\varphi\partial_b\varphi - V(\varphi)\right], \quad (2.3)$$

where G is the Newtonian constant of gravity, $\kappa^2 = 8\pi G$, $\tilde{R}$ is the Ricci scalar for the metric $\tilde{g}_{ab}$, and the Einstein frame scalar field φ is identified as [1]

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\lambda} = \sqrt{\frac{\varpi}{2}}\frac{1}{\kappa\lambda}, \quad (2.4)$$

where we define the parameter

$$\varpi \equiv 2w_{\rm BD} + 3. \quad (2.5)$$

Note that $\varpi > 0$ is required in order for the field φ to be real. We consider that both the Jordan and Einstein frames have spatially flat FLRW metrics,

$$g_{ab} \equiv \mathrm{diag}\left[-1, a^2(t), a^2(t), a^2(t)\right], \quad (2.6a)$$

$$\tilde{g}_{ab} \equiv \mathrm{diag}\left[-1, \tilde{a}^2(\tilde{t}), \tilde{a}^2(\tilde{t}), \tilde{a}^2(\tilde{t})\right], \quad (2.6b)$$

where the Einstein and Jordan frame scale factors $(\tilde{a}, a)$ and coordinate times $(\tilde{t}, t)$ are then related via the conformal transformation as

$$\tilde{a} = \sqrt{G\lambda}a, \quad (2.7a)$$

$$\mathrm{d}\tilde{t} = \sqrt{G\lambda}\mathrm{d}t. \quad (2.7b)$$

The energy density, pressure and equation of state parameter of the homogeneous Einstein frame scalar field φ are given by the usual definitions,

$$\rho_\varphi = \frac{1}{2}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\tilde{t}}\right)^2, \quad P_\varphi = \frac{1}{2}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\tilde{t}}\right)^2, \quad w_\varphi = \frac{P_\varphi}{\rho_\varphi} = 1. \quad (2.8)$$

The possibility of an expansion-collapse duality between the Einstein and Jordan frames can easily be observed by expressing the Jordan frame scale factor (a) in terms of the Einstein frame scale factor $(\tilde{a})$ from Eq. (2.7a). To obtain the relation, we rewrite Eq. (2.8) as

$$\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\tilde{t}}\right)^2 = \rho_\varphi + P_\varphi = \rho_\varphi(1 + w_\varphi), \quad (2.9a)$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tilde{a}} = \frac{1}{\tilde{a}\tilde{H}}\sqrt{\rho_\varphi(1 + w_\varphi)}, \quad (2.9b)$$

where the Einstein frame Hubble parameter $\tilde{H}$ is constrained via the Friedmann equation

$$\tilde{H}^2 \equiv \left(\frac{1}{\tilde{a}}\frac{\mathrm{d}\tilde{a}}{\mathrm{d}\tilde{t}}\right)^2 = \frac{\kappa^2}{3}\rho_\varphi. \quad (2.10)$$

Choosing an expanding Einstein frame ($\tilde{H} > 0$) and replacing $\tilde{H}$ in Eq. (2.9b), the Einstein frame scalar field can be solved as

$$\kappa(\varphi - \varphi_0) = \sqrt{6}\ln\tilde{a}, \quad (2.11)$$

where $\varphi_0 = \varphi(\tilde{a} = 1)$, and we have put $w_\varphi = 1$ from Eq. (2.8). For simplicity, we set $\varphi_0 = 0$, therefore fixing the origin of the Einstein frame scalar field as $\varphi(\tilde{a} = 1) = 0$. The Brans–Dicke scalar field λ can be written in terms of φ, from Eq. (2.4),

$$\lambda(\varphi) = \lambda_0 \exp\left(\sqrt{\frac{2}{\varpi}}\kappa\varphi\right), \tag{2.12}$$

where $\lambda_0 = \lambda(\varphi = 0)$. Finally, from Eq. (2.7a), the Jordan frame scale factor can be written in terms of the Einstein frame scale factor as,

$$a = \sqrt{\frac{8\pi}{\kappa^2\lambda_0}}\tilde{a}^{1-\sqrt{\frac{3}{\varpi}}}, \tag{2.13}$$

where we have used Eqs. (2.12) and (2.11). It is evident that the condition required for an increasing Einstein frame scale factor to map to a decreasing Jordan frame scale factor is given by

$$\frac{\mathrm{d}a}{\mathrm{d}\tilde{a}} < 0 \iff \varpi < 3. \tag{2.14}$$

Noting that the Einstein frame scale factor $\tilde{a}$ monotonically increases with the Einstein frame coordinate time $\tilde{t}$, and the Jordan frame coordinate time t is monotonically changing with $\tilde{t}$ (see Eq. (2.7b)), the Jordan frame scale factor $a(t)$ is always decreasing if $\varpi < 3$. Therefore, for an expanding Einstein frame universe, the conformally dual Jordan frame universe contracts (expands) indefinitely if it is driven by a Brans–Dicke model with $\varpi < 3$ ($\varpi > 3$) (see Fig. 1).

The expansion-collapse duality also becomes evident if one considers the relation between the Jordan and Einstein frame Hubble parameters, H and $\tilde{H}$. One can obtain from Eqs. (2.7a) and (2.7b)

$$H \equiv \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} = \sqrt{\frac{\kappa^2\lambda}{8\pi}}\left(1 - \sqrt{\frac{3}{\varpi}}\right)\tilde{H}, \tag{2.15}$$

therefore, given an expanding Einstein frame ($\tilde{H} > 0$), the Jordan frame Hubble parameter H becomes negative for $\varpi < 3$, depicting a collapsing Jordan frame.

From Eq. (2.13) we see that depending on the value of the integration constant λ_0, a can become arbitrarily small for a large $\tilde{a}$. After a time when the Jordan frame scale factor becomes sufficiently small, the quantum effects in the Jordan frame can no longer be ignored. One may assume that the quantum effects become more robust as the Jordan frame contracts further. At the same time, the conformally dual Einstein frame scale factor keeps on increasing and one can anticipate that the quantum effects are no longer relevant. We are interested in how the quantum effects in the contracting Jordan frame maps to that of the expanding Einstein frame universe. In particular, we seek whether the quantum features in the Einstein frame are suppressed as the space becomes arbitrarily large, or whether the Einstein frame quantum effects

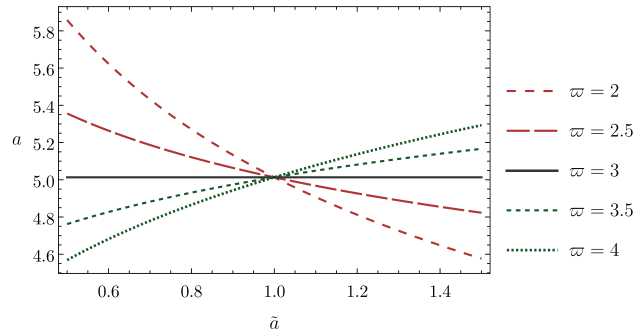

Fig. 1 Expansion-collapse duality in the classical description. The plots show evolution of the Jordan frame scale factors (a) corresponding to different Brans–Dicke models specified by the parameter ϖ, with respect to the Einstein frame scale factor ($\tilde{a}$) (where $\lambda_0 = 1$). For the models with $\varpi > 3$, the Jordan frame universes expand with the Einstein frame universe; while for $\varpi < 3$, the Jordan frames are always collapsing. $\varpi = 3$ leads to a static Jordan frame universe

increase similarly to its Jordan frame counterpart, despite the expansion of space.

In order to address this question, we seek whether there exists a conformal map between the Einstein and Jordan frames at the quantum level. Then we find how the quantum fluctuations of different cosmological quantities in the two frames are related via the conformal map. We begin with treating both the Einstein and Jordan frame universes in the minisuperspace formalism of quantum cosmology. The following section introduces Hamiltonian formalism in the Einstein and Jordan frame descriptions. Later on, this is used the quantize the conformally connected universes.

3 Canonical descriptions of the conformal frames

In this section, we introduce Hamiltonian formalism in both Einstein and Jordan frame universes, where the Jordan frame is driven by a Brans–Dicke model with zero potential, and the corresponding Einstein frame consists of a minimally coupled canonical massless scalar field.

3.1 Hamiltonian formalism in the Einstein frame

Let us consider the line element in the Einstein frame to be

$$\mathrm{d}\tilde{s}^2 = -\tilde{n}^2(\tilde{t})\mathrm{d}\tilde{t}^2 + \tilde{a}^2(\tilde{t})\delta_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta, \tag{3.1}$$

where $\tilde{n}$ is the lapse function in the Einstein frame. Using this in the Lagrangian of the Einstein frame

$$\tilde{L} = \sqrt{-\tilde{g}}\left[\frac{1}{2\kappa^2}\tilde{R} - \frac{1}{2}\tilde{g}^{ab}\partial_a\varphi\partial_b\varphi\right], \tag{3.2}$$

and adding appropriate GHY (Gibbons–Hawking–York) boundary term to make the variational principle well-posed, we obtain

$$\tilde{L} = -\frac{3\tilde{a}\dot{\tilde{a}}^2}{\kappa^2 \tilde{n}} + \frac{\tilde{a}^3 \dot{\varphi}^2}{2\tilde{n}}. \tag{3.3}$$

Note that in this subsection, the overdots denote derivatives with respect to $\tilde{t}$. Under the variable transformation

$$\tilde{\chi} = \ln \tilde{a}, \tag{3.4}$$

the Lagrangian takes the form

$$\tilde{L} = \frac{1}{\tilde{n}} \exp(3\tilde{\chi}) \left(-\frac{3}{\kappa^2} \dot{\tilde{\chi}}^2 + \frac{1}{2} \dot{\varphi}^2 \right). \tag{3.5}$$

The primary constraint of the system is $P_{\tilde{n}} \approx 0$, where P_j are the conjugate momenta associated with the variable j. Legendre transformation of this Lagrangian system leads to the Einstein frame Hamiltonian

$$\tilde{\mathbb{H}} = \tilde{n} \exp(-3\tilde{\chi}) \left(-\frac{\kappa^2}{12} P_{\tilde{\chi}}^2 + \frac{1}{2} P_{\varphi}^2 \right). \tag{3.6}$$

Following [38], we do a canonical transformation in the scalar field sector

$$\tilde{T} = \frac{\varphi}{P_{\varphi}}, \tag{3.7}$$

$$P_{\tilde{T}} = \frac{1}{2} P_{\varphi}^2, \tag{3.8}$$

leading to the Hamiltonian that is linear in the momentum conjugate to the variable $\tilde{T}$,

$$\tilde{\mathbb{H}} \left(\tilde{\chi}, \tilde{n}, \tilde{T}, P_{\tilde{\chi}}, P_{\tilde{n}}, P_{\tilde{T}} \right) = \frac{\tilde{n}}{12} \mathrm{e}^{-3\tilde{\chi}} \left(-P_{\tilde{\chi}}^2 + 12 P_{\tilde{T}} \right), \tag{3.9}$$

here we have set $\kappa^2 = 1$. The consistency equation for the primary constraint, $\dot{P}_{\tilde{n}} \approx 0$ leads to the secondary (Hamiltonian) constraint,

$$\tilde{\mathcal{H}} \equiv -P_{\tilde{\chi}}^2 + 12 P_{\tilde{T}} \approx 0. \tag{3.10}$$

The Hamilton's equations of motion in the Einstein frame become

$$\dot{\tilde{T}} = \tilde{n} \exp(-3\tilde{\chi}), \tag{3.11a}$$

$$P_{\tilde{T}} \equiv P_{\tilde{T}} (\text{constant}), \tag{3.11b}$$

$$\dot{\tilde{\chi}} = -\frac{\tilde{n}}{6} \exp(-3\tilde{\chi}) P_{\tilde{\chi}}, \tag{3.11c}$$

$$P_{\tilde{\chi}} \equiv P_{\tilde{\chi}} (\text{constant}). \tag{3.11d}$$

Using Eqs. (3.11a) and (3.11c), one can solve for the classical trajectory $\tilde{\chi}(\tilde{T})$ as

$$\tilde{\chi}(\tilde{T}) = -\frac{1}{6} P_{\tilde{\chi}} \tilde{T} + c_1, \tag{3.12}$$

$$\tilde{\chi}(\tilde{T}) = \tilde{H}_0 \tilde{T}, \tag{3.13}$$

where in the second line we have identified $\tilde{H}_0 = \tilde{H}(\tilde{\chi} = 0) = -P_{\tilde{\chi}}/6$ from Eq. (3.14) and set $c_1 = 0$, thus choosing the origin of $\tilde{T}$ as $\tilde{\chi}(\tilde{T} = 0) = 0$.

For the choice of $\tilde{n} = 1$, that is, taking $\tilde{t}$ to be the comoving time coordinate in the Einstein frame, the Hubble parameter in this frame can be written in terms of the phase-space variables as

$$\tilde{H} = \frac{1}{\tilde{a}} \frac{\mathrm{d}\tilde{a}}{\mathrm{d}\tilde{t}} = \dot{\tilde{\chi}}, \tag{3.14a}$$

$$\tilde{H} = -\frac{1}{6} P_{\tilde{\chi}} \exp(-3\tilde{\chi}). \tag{3.14b}$$

3.2 Hamiltonian formalism in the Jordan frame

Let us consider the Jordan frame line element

$$\mathrm{d}s^2 = -n^2(t)\mathrm{d}t^2 + a^2(t)\delta_{\alpha\beta}\mathrm{d}x^{\alpha}\mathrm{d}x^{\beta}, \tag{3.15}$$

where n is the Jordan frame lapse function. Using this in the Lagrangian of the Jordan frame,

$$L = \frac{\sqrt{-g}}{16\pi} \left(\lambda R - \frac{w_{\mathrm{BD}}}{\lambda} g^{ab} \partial_a \lambda \partial_b \lambda \right), \tag{3.16}$$

and adding appropriate boundary term we get

$$L = \frac{1}{16\pi n} \left(w_{\mathrm{BD}} a^3 \frac{\dot{\lambda}^2}{\lambda} - 6\lambda a \dot{a}^2 - 6a^2 \dot{\lambda} \dot{a} \right). \tag{3.17}$$

Here the overdots denote derivatives with respect to t. Following [38], we choose the variable transformation

$$a = \exp\left(-\frac{\alpha}{2} + \beta \right), \tag{3.18a}$$

$$\lambda = \exp(\alpha), \tag{3.18b}$$

leading to the Lagrangian

$$L = \frac{1}{16\pi n} \exp\left(-\frac{\alpha}{2} + 3\beta \right) \left[\frac{1}{2} \varpi \dot{\alpha}^2 - 6\dot{\beta}^2 \right]. \tag{3.19}$$

Again, $P_n \approx 0$ is the primary constraint on the phase-space of this system, and the Jordan frame Hamiltonian can be obtained as

$$\mathbb{H} = \frac{16\pi n}{24} \exp\left(\frac{\alpha}{2} - 3\beta \right) \left(-P_{\beta}^2 + \frac{12}{\varpi} P_{\alpha}^2 \right). \tag{3.20a}$$

The canonical transformation

$$T = \frac{\alpha}{P_\alpha}, \tag{3.21a}$$

$$P_T = \frac{P_\alpha^2}{2}, \tag{3.21b}$$

is used to then replace the phase-space variables $(\alpha, P_\alpha) \to (T, P_T)$, leading to the Hamiltonian

$$\mathbb{H}\left(n, \beta, T, P_n, P_\beta, P_T\right) = \frac{16\pi n}{24} \exp\left(\frac{T\sqrt{P_T}}{\sqrt{2}} - 3\beta\right)\left(-P_\beta^2 + \frac{24}{\varpi} P_T\right). \tag{3.22}$$

The consistency equation of the primary constraint leads to the Hamiltonian constraint

$$\mathcal{H} \equiv -P_\beta^2 + \frac{24}{\varpi} P_T \approx 0. \tag{3.23}$$

The Hamilton equations of motion in the Jordan frame take the forms

$$P_\beta \equiv P_\beta(\text{constant}), \tag{3.24a}$$

$$\dot{\beta} = \frac{16\pi n}{24} \exp\left(\frac{T\sqrt{P_T}}{\sqrt{2}} - 3\beta\right)(-2P_\beta) \tag{3.24b}$$

$$P_T \equiv P_T(\text{constant}), \tag{3.24c}$$

$$\dot{T} \approx \frac{16\pi n}{24} \exp\left(\frac{T\sqrt{P_T}}{\sqrt{2}} - 3\beta\right)\left(\frac{24}{\varpi}\right). \tag{3.24d}$$

One can solve for the classical trajectory in the Jordan frame from Eqs. (3.24b) and (3.24d) as

$$\beta = -\frac{\varpi P_\beta}{12} T + c_2, \tag{3.25a}$$

where c_2 is the integration constant. Using this and the variable transformations Eqs. (3.18), (3.21) and (3.23) we find

$$\chi = \ln a = -\frac{\alpha}{2} + \beta = \sqrt{\frac{P_T}{6}}\left(\sqrt{\varpi} - \sqrt{3}\right) T - \sqrt{\frac{\varpi}{12}} \ln \lambda_0 - \ln\sqrt{\frac{\kappa^2}{8\pi}}, \tag{3.26}$$

where we have set $c_2 = -\sqrt{\varpi/12}\ln\lambda_0 - \ln(\kappa^2/(8\pi))$ in order to be consistent with the conformal transformation in Eq. (2.13).

The Hubble parameter in the Jordan frame can be written as a function of the phase-space variables as (see Eqs. (3.21a) and (3.21b))

$$H = \dot{\chi} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\beta - \sqrt{\frac{P_T}{2}} T\right), \tag{3.27a}$$

$$H = 16\pi \exp\left(T\sqrt{\frac{P_T}{2}} - 3\beta\right)\left(-\frac{P_\beta}{12} - \frac{1}{\varpi}\sqrt{\frac{P_T}{2}}\right); \tag{3.27b}$$

here we have chosen $n = 1$, therefore setting t to be the comoving time in the Jordan frame.

4 Quantum descriptions of the conformal frames

With the phase-space structure at hand, one can proceed to quantize the systems in both frames. According to Dirac's prescription, the primary constraints imply that the wave functions are independent of the lapse choices; while, the secondary constraints lead to the Wheeler–DeWitt equations in both frames. In this analysis, we have deparameterized the systems such that the scalar field φ serves as a clock in the Einstein frame, whereas, the clock in the Jordan frame is provided by α, which is a combination of both the Brans–Dicke field and the scale factor.

4.1 Quantum description in the Einstein frame

In the coordinate representation, the Hamiltonian constraint, Eq. (3.10), leads to the Wheeler–DeWitt equation $\widehat{\tilde{\mathcal{H}}}\tilde{\psi} = 0$,

$$\left(\frac{\partial^2}{\partial\tilde{\chi}^2} - 12i\frac{\partial}{\partial\tilde{T}}\right)\tilde{\psi} = 0. \tag{4.1}$$

The operator is self-adjoint with the Hilbert space $L^2(\mathbb{R}, \mathrm{d}\tilde{\chi})$. The Wheeler–DeWitt equation has the form of a free particle Schrödinger equation and it admits a plane wave solution,

$$\tilde{\psi}_m\left(\tilde{\chi}, \tilde{T}\right) = \exp\left(i\left(m^2\tilde{T} - \sqrt{12}m\tilde{\chi}\right)\right). \tag{4.2}$$

We construct a normalized wave packet from the stationary states using a Gaussian distribution

$$\tilde{\phi}(m) = \exp\left(-\tilde{\gamma}(m - \tilde{m}_0)^2\right), \tag{4.3}$$

$$\tilde{\Psi}\left(\tilde{\chi}, \tilde{T}\right) = \int_{-\infty}^{\infty} \mathrm{d}m\, \tilde{\phi}(m)\tilde{\psi}_m\left(\tilde{\chi}, \tilde{T}\right) = \left(\frac{\pi^{3/2}}{\sqrt{6\tilde{\gamma}}}\right)^{-1/2}\sqrt{\frac{\pi}{\tilde{\gamma} - i\tilde{T}}} \times \exp\left(-\frac{3\tilde{\chi}^2 - i\tilde{m}_0\gamma\left(\tilde{m}_0\tilde{T} - 2\sqrt{3}\tilde{\chi}\right)}{\tilde{\gamma} - i\tilde{T}}\right). \tag{4.4}$$

The wave packet constructed here satisfies DeWitt's criteria as $\lim_{\tilde{\chi}\to-\infty} \tilde{\Psi}(\tilde{\chi}, \tilde{T}) = 0$ and hints at singularity resolution [40,42]. The expectation of $\tilde{\chi}$ can be obtained as

$$\langle\tilde{\chi}\rangle_E = \int_{-\infty}^{\infty} \mathrm{d}\chi\ \tilde{\Psi}^* \tilde{\chi} \tilde{\Psi} = \frac{\tilde{m}_0}{\sqrt{3}} \tilde{T}, \tag{4.5}$$

where the subscript E denotes that the expectation is evaluated with respect to the Einstein frame wave packet. If one chooses the wave packet parameter as

$$\tilde{m}_0 = \sqrt{3}\tilde{H}_0 = \sqrt{\frac{2}{\varpi}}\sqrt{P_T}, \tag{4.6}$$

where the second equality follows from the conformal transformation, then

$$\langle\tilde{\chi}\rangle_E = \tilde{H}_0 \tilde{T}. \tag{4.7}$$

Thus, $\langle\tilde{\chi}\rangle_E$ always follows the classical trajectory, Eq. (3.13), and thereby suggesting singularity non-resolution for the state that satisfy DeWitt's criteria.

The variance of $\tilde{\chi}$ is found to be

$$\Delta_E \tilde{\chi} = \langle\tilde{\chi}^2\rangle_E - \langle\tilde{\chi}\rangle_E^2 = \frac{\tilde{T}^2 + \tilde{\gamma}^2}{12\tilde{\gamma}}. \tag{4.8}$$

We see that the variance becomes minimum as $|\tilde{T}| \to 0$, and diverges as $|\tilde{T}| \to \infty$. The expectation value of the Einstein frame scale factor is obtained as

$$\langle\tilde{a}\rangle_E = \langle\exp(\tilde{\chi})\rangle_E = \exp\left(\frac{\tilde{m}_0}{\sqrt{3}}\tilde{T} + \frac{1}{24}\left(\frac{\tilde{T}^2}{\tilde{\gamma}} + \tilde{\gamma}\right)\right), \tag{4.9a}$$

which can be put in the form

$$\langle\tilde{a}\rangle_E = \exp\left(\langle\tilde{\chi}\rangle_E + \frac{\Delta_E\tilde{\chi}}{2}\right) = \tilde{a}(\tilde{T}) \exp\left(\frac{\Delta_E\tilde{\chi}}{2}\right), \tag{4.9b}$$

where $\tilde{a}(\tilde{T})$ is the classical trajectory (from Eq. (3.13)),

$$\tilde{a}(\tilde{T}) = \exp\left(\tilde{H}_0\tilde{T}\right), \tag{4.10}$$

and we have used the parameter choice in Eq. (4.6). As we can see, $\langle\tilde{a}\rangle_E$ does not follow the classical trajectory for all $\tilde{T}$, in fact, the deviation from the classical trajectory is dictated by the fluctuation in $\tilde{\chi}$ and it increases as $|\tilde{T}|$ becomes larger. The expectation value of the scale factor suggests a regular universe with a non-zero minimum at $\tilde{T} = -12\tilde{H}_0\tilde{\gamma}$ and therefore singularity resolution.

The variance and the relative fluctuation in the scale factor can be obtained as

$$\begin{aligned}\Delta_E\tilde{a} &= \langle\tilde{a}^2\rangle_E - \langle\tilde{a}\rangle_E^2 \\ &= \langle\exp(2\tilde{\chi})\rangle_E - \langle\exp(\tilde{\chi})\rangle_E^2,\end{aligned} \tag{4.11a}$$

$$\begin{aligned}\Delta_E\tilde{a} = &\left(\exp\left(\frac{\tilde{T}^2 + \tilde{\gamma}^2}{12\tilde{\gamma}}\right) - 1\right) \\ &\times \exp\left(\frac{2\tilde{m}_0}{\sqrt{3}}\tilde{T} + \frac{\tilde{T}^2}{12\tilde{\gamma}} + \frac{\tilde{\gamma}}{12}\right),\end{aligned} \tag{4.11b}$$

$$\frac{\sqrt{\Delta_E\tilde{a}}}{\langle\tilde{a}\rangle_E} = \sqrt{\exp\left(\frac{\tilde{T}^2 + \tilde{\gamma}^2}{12\tilde{\gamma}}\right) - 1} = \sqrt{\exp(\Delta_E\tilde{\chi}) - 1}. \tag{4.11c}$$

The relative fluctuation $\sqrt{\Delta_E\tilde{a}}/\langle\tilde{a}\rangle_E$ is at its minimum when $|\tilde{T}| \to 0$ and it diverges as $|\tilde{T}| \to \infty$. One can construct the operator for the Einstein frame Hubble parameter, Eq. (3.14b), as

$$\hat{\tilde{H}} = -\frac{1}{12}\left(\exp\left(-3\hat{\tilde{\chi}}\right)\hat{P}_{\tilde{\chi}} + \hat{P}_{\tilde{\chi}}\exp\left(-3\hat{\tilde{\chi}}\right)\right). \tag{4.12}$$

With the parameter choice from Eq. (4.6), the expectation value of $\hat{\tilde{H}}$ can be written as

$$\begin{aligned}\langle\tilde{H}\rangle_E &= \tilde{H}_0\exp\left(-3\tilde{H}_0\tilde{T}\right) \\ &\quad\times \exp\left(\frac{3\left(\tilde{T}^2 + \tilde{\gamma}^2\right)}{8\tilde{\gamma}}\right)\left(1 - \frac{\tilde{T}}{4\tilde{\gamma}\tilde{H}_0}\right),\end{aligned} \tag{4.13a}$$

$$= \tilde{H}(\tilde{T})\exp\left(\frac{9}{2}\Delta_E\tilde{\chi}\right)\left(1 - \frac{\tilde{T}}{4\tilde{\gamma}\tilde{H}_0}\right) \tag{4.13b}$$

where $\tilde{H}(\tilde{T})$ is the classical expression,

$$\tilde{H} = \tilde{H}_0\exp\left(-3\tilde{H}_0\tilde{T}\right). \tag{4.14}$$

The deviation of $\langle\tilde{H}\rangle$ from its classical trajectory increases as $|T|$ gets larger. The expectation value of the Hubble parameter also suggests a bouncing universe with the bounce occurring at $\tilde{T} = 4\tilde{H}_0\tilde{\gamma}$.

4.2 Quantum description in the Jordan frame

The Hamiltonian constraint in the Jordan frame, Eq. (3.23), leads to the Wheeler-DeWitt equation $\widehat{\mathcal{H}}\psi = 0$,

$$\left(\frac{\partial^2}{\partial\beta^2} - \frac{24}{\varpi}i\frac{\partial}{\partial T}\right)\psi = 0 \tag{4.15}$$

The differential operator is self-adjoint on the Hilbert space $L^2(\mathbb{R}, \mathrm{d}\beta)$. As in the Einstein frame, the Jordan frame Wheeler–DeWitt equation has the form of a free particle Schrödinger equation and it admits a plane wave solution,

$$\psi_m(\beta, T) = \exp\left(i\left(m^2 T - \sqrt{\frac{24}{\varpi}} m\beta\right)\right). \tag{4.16}$$

We construct a normalized wave packet from the stationary state using a Gaussian distribution

$$\phi(m) = \exp\left(-\gamma(m - m_0)^2 + i\beta_0 m\right), \tag{4.17}$$

$$\begin{aligned}\Psi(\beta, T) &= \int_{-\infty}^{\infty} \mathrm{d}m\, \phi(m)\psi_m(\beta, T)\\ &= \left(\frac{\pi^{3/2}}{2\sqrt{3}}\sqrt{\frac{\varpi}{\gamma}}\right)^{-1/2}\sqrt{\frac{\pi}{\gamma - iT}}\\ &\quad\times \exp\left(-\frac{\left(\sqrt{\frac{6}{\varpi}}\beta - \frac{\beta_0}{2}\right)^2 - im_0\gamma\left(m_0 T - 2\sqrt{\frac{6}{\varpi}}\beta + \beta_0\right)}{\gamma - iT}\right).\end{aligned} \tag{4.18}$$

In this case as well, the DeWitt's criteria is satisfied by the wave packet, $\lim_{\beta\to-\infty}\Psi(\chi, T) = 0$, hinting at singularity resolution. The expectation value of χ with respect to the Jordan frame wave packet can be obtained as

$$\langle\chi\rangle_J = \int_{-\infty}^{\infty}\mathrm{d}\beta\, \Psi^*\left(\beta - \sqrt{\frac{P_T}{2}}T\right)\Psi \tag{4.19a}$$

$$= \left(\frac{m_0\sqrt{\varpi}}{\sqrt{6}} - \frac{\sqrt{P_T}}{\sqrt{2}}\right)T + \sqrt{\frac{\varpi}{6}}\frac{\beta_0}{2}. \tag{4.19b}$$

For the choice of parameters

$$m_0 = \sqrt{P_T}, \tag{4.20a}$$

$$\beta_0 = -2\sqrt{\frac{6}{\varpi}}\ln\left(\frac{1}{\sqrt{8\pi}}\lambda_0^{\sqrt{\frac{\varpi}{12}}}\right), \tag{4.20b}$$

$\langle\chi\rangle_J$ follows the classical trajectory, Eq. (3.26), for all T,

$$\langle\chi\rangle_J = \sqrt{\frac{P_T}{6}}\left(\sqrt{\varpi} - \sqrt{3}\right)T + \ln\left(\sqrt{8\pi}\right) - \sqrt{\frac{\varpi}{12}}\ln(\lambda_0), \tag{4.21}$$

and again represents singularity non-resolution. The variance of χ can be calculated as

$$\Delta_J\chi = \langle\chi^2\rangle_J - \langle\chi\rangle_J^2 = \frac{\varpi\left(T^2 + \gamma^2\right)}{24\gamma}. \tag{4.22}$$

$\Delta_J\chi$ is minimum when $|T| \to 0$ and it diverges as $|T| \to \infty$.

With the parameter choices in Eq. (4.20), the expectation value of the Jordan frame scale factor a takes the form

$$\begin{aligned}\langle a\rangle_J &= \langle\exp(\chi)\rangle_J\\ &= \exp\left(\sqrt{\frac{P_T}{6}}\left(\sqrt{\varpi} - \sqrt{3}\right)T + \ln\left(\sqrt{8\pi}\right)\right.\\ &\quad\left. - \sqrt{\frac{\varpi}{12}}\ln(\lambda_0)\right)\exp\left(\frac{\varpi}{48}\left(\frac{T^2}{\gamma} + \gamma\right)\right)\end{aligned} \tag{4.23}$$

which can be put in the form

$$\langle a\rangle_J = \exp\left(\langle\chi\rangle_J + \frac{\Delta_J\chi}{2}\right) = a(T)\exp\left(\frac{\Delta_J\chi}{2}\right), \tag{4.24}$$

where $a(T)$ is the classical trajectory (from Eq. (3.26))

$$a(T) = \exp\left(\sqrt{\frac{P_T}{6}}\left(\sqrt{\varpi} - \sqrt{3}\right)T - \sqrt{\frac{\varpi}{12}}\ln\lambda_0 - \ln\sqrt{\frac{\kappa^2}{8\pi}}\right). \tag{4.25}$$

The relation between the expectation values of the scale factor and χ has a similar structure to its Einstein frame counterpart Eq. (4.9b). We see that $\langle a\rangle_J$ deviates from its classical trajectory, where the deviation is parameterized by the fluctuations in χ and it increases as $|T|$ becomes larger. Furthermore, the expectation value of scale factor represents a singularity free universe with its non-zero minimum being at $T = -24\sqrt{(P_T/6)}\gamma(\varpi - 3)/\varpi$. The variance and relative fluctuation in the scale factor are found to be

$$\Delta_J a = \langle a^2\rangle_J - \langle a\rangle_J^2 = \langle\exp(2\chi)\rangle_J - \langle\exp\chi\rangle_J^2 \tag{4.26a}$$

$$\begin{aligned}&= \frac{8\pi}{\lambda_0^{\sqrt{\frac{\varpi}{3}}}}\exp\left(2\sqrt{\frac{P_T}{6}}\left(\sqrt{\varpi} - \sqrt{3}\right)T\right.\\ &\quad\left. + 2\frac{\varpi}{48}\left(\frac{T^2}{\gamma} + \gamma\right)\right)\\ &\quad\times\left(\exp\left(\frac{\varpi\left(T^2 + \gamma^2\right)}{24\gamma}\right) - 1\right),\end{aligned} \tag{4.26b}$$

$$\frac{\sqrt{\Delta_J a}}{\langle a\rangle_J} = \sqrt{\exp\left(\frac{\varpi\left(T^2 + \gamma^2\right)}{24\gamma}\right) - 1} = \sqrt{\exp(\Delta_J\chi) - 1}. \tag{4.26c}$$

The relative fluctuation in the scale factor becomes minimum as $|T| \to 0$ and it diverges in the limit $|T| \to \infty$.

Starting with Eq. (3.27b), we construct the operator for the Jordan frame Hubble parameter as,

$$\hat{H} = 16\pi \exp\left(T\sqrt{\frac{P_T}{2}}\right) \times \left(-e^{-3\hat{\beta}}\frac{1}{\varpi}\sqrt{\frac{P_T}{2}} - \frac{1}{2\times 12}\left(e^{-3\hat{\beta}}\hat{P}_\beta + \hat{P}_\beta e^{-3\hat{\beta}}\right)\right). \tag{4.27}$$

The expectation of the Hubble parameter, with the parameter choice from Eq. (4.20), becomes

$$\langle H\rangle_J = \frac{\lambda_0^{\frac{\sqrt{3\varpi}}{2}}}{2\sqrt{3\pi}}\frac{\sqrt{P_T}}{\varpi}\exp\left(T\sqrt{\frac{P_T}{2}}\left(1-\sqrt{3\varpi}\right)\right) \times \exp\left(\frac{3\varpi}{16}\left(\frac{T^2}{\gamma}+\gamma\right)\right) \times \left(\sqrt{\varpi}-\sqrt{3}-\sqrt{\frac{6}{P_T}}\frac{\varpi}{8\gamma}T\right) \tag{4.28a}$$

which can be put in the form

$$\langle H\rangle_J = H(T)\exp\left(\frac{9}{2}\Delta_J\chi\right)\left(1-\sqrt{\frac{6}{P_T}}\frac{\varpi}{8\gamma}\frac{T}{\sqrt{\varpi}-\sqrt{3}}\right), \tag{4.28b}$$

while, one can show that the classical expression for the Hubble parameter is

$$H(T) = \frac{\lambda_0^{\frac{\sqrt{3\varpi}}{2}}}{2\sqrt{3\pi}}\frac{\sqrt{P_T}}{\varpi}\exp\left(T\sqrt{\frac{P_T}{2}}\left(1-\sqrt{3\varpi}\right)\right) \times \left(\sqrt{\varpi}-\sqrt{3}\right). \tag{4.29}$$

Clearly, the expectation of the Hubble parameter differs from its classical trajectory; the deviation increases as $|T|$ becomes larger. In this case as well, the expectation value of the Hubble parameter represents a bouncing universe with bounce occurring at $T = 8\sqrt{(P_T/6)}\gamma(\varpi-3)/\varpi$.

5 Conformal map and expansion-collapse duality in the quantum description

We will now see how the classical conformal map translates into the quantum descriptions of the Einstein and Jordan frame universes. In a previous study [38], the authors investigate the conformal map between the Brans–Dicke Jordan frame and its corresponding Einstein frame in the quantum description. It is argued that the classical conformal map holds true when compared at the level of the Wheeler–DeWitt equations and the wave packets in the two frames. However, the comparison at the level of wave packets may not be sufficient, as the expectation values of different cosmological operators in the two frames still may deviate from their classical conformal relations. In this paper, we investigate the status of the conformal map in the quantum description at the level of expectation values of relevant cosmological quantities.

In the previous section, the expectation values of different Einstein and Jordan frame operators are obtained in terms of the clock variables in these frames, $\tilde{T}$ and T, respectively. In order to compare the expectation values of an operator in different frames, we need to consider the relation between $\tilde{T}$ and T. In fact, one can use the conformal correspondence to obtain the relation between the Einstein and Jordan frame clock variables as

$$\tilde{T} = \frac{\varpi}{2}\left(T - \frac{\ln\lambda_0}{\sqrt{2P_T}}\right). \tag{5.1}$$

This shows that the clock variables are monotonically changing functions of each other.

We have seen that classically, a Jordan frame with $\varpi < 3$ contracts indefinitely, corresponding to an expanding Einstein frame. In the quantum description, we first find the relations between the expectation values of the logarithmic scale factors, scale factors, and Hubble parameters in the two frames, and compare these relations with their classical counterparts, set up by the conformal transformation. Then we seek whether the quantum expectation values of different operators suggest an expansion-collapse duality between the frames, similar to the classical description. Finally, we find the relations between quantum fluctuations in the two frames for different quantities. In particular, we seek whether the rise in fluctuation in one frame implies the same in the other, regardless of classical cosmological evolutions therein.

5.1 Logarithmic scale factors

The classical conformal map between the Einstein and Jordan frame logarithmic scale factors can be obtained from Eq. (2.13) as

$$\tilde{\chi} - \chi = \sqrt{\frac{P_T}{2}}T + \ln\frac{1}{\sqrt{8\pi}}, \tag{5.2}$$

where we have used the relations from Eqs. (3.18b) and (3.21). On the other hand, from Eqs. (4.5) and (4.19a) we get

$$\langle\tilde{\chi}\rangle_E - \langle\chi\rangle_J = \sqrt{\frac{P_T}{2}}T + \ln\frac{1}{\sqrt{8\pi}}, \tag{5.3}$$

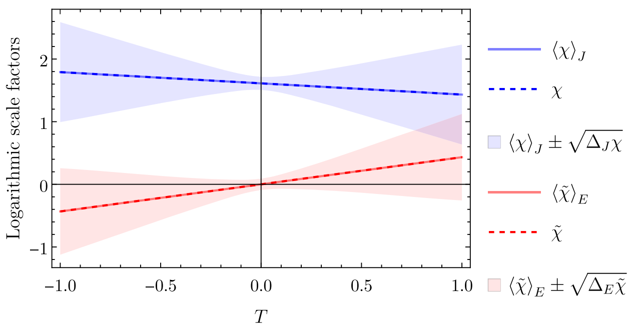

Fig. 2 Evolution of the expectation values of the Jordan and Einstein frame logarithmic scale factors ($\langle\chi\rangle_J, \langle\tilde{\chi}\rangle_E$) (solid plots), along with their classical trajectories ($\chi, \tilde{\chi}$) (dashed plots), with respect to the Jordan frame clock T. The shaded regions depict fluctuations in the observables. We have used $\varpi = 1.5$, $\gamma_E = 0.1$, $\tilde{m}_0 = 1$, and other parameters are chosen according to Eqs. (4.6), (4.20) and (5.4). Both $\langle\chi\rangle_J, \langle\tilde{\chi}\rangle_E$ follow their corresponding classical trajectories throughout. The fluctuations in both the cases are minimum near $|T| \to 0$ and get larger as $|T| \to \infty$. Since $\varpi < 3$, $\langle\chi\rangle_J$ is always decreasing while $\langle\chi\rangle_E$ expands forever, consistent with the classical expansion-collapse duality scenario

where we have replaced $\tilde{T}$ with T using the conformal relation Eq. (5.1), and the wave packet parameters are chosen according to Eqs. (4.6) and (4.20). For these parameter choices, the RHSs of the Eqs. (5.2) and (5.3) are equal for all T (see Fig. 2). Therefore, *the expectation values of the Einstein and Jordan frame logarithmic scale factors follow the classical conformal relation for all time for a given choice of parameters*.

It is interesting to note that, given the parameter choices from Eqs. (4.6) and (4.20), if we further impose

$$\tilde{\gamma} = \frac{\varpi}{2}\gamma, \tag{5.4a}$$

and the integration constant

$$\beta_0 = 0 \implies \lambda_0 = 1, \tag{5.4b}$$

then the variances of $\tilde{\chi}$ and χ (from Eqs. (4.8) and (4.22)) become equal,

$$\Delta_E\tilde{\chi} = \Delta_J\chi. \tag{5.5}$$

Thus, for these parameter choices, the variance of logarithmic scale factor is conformally invariant.

In order to see how the expectation values of the logarithmic scale factors impose the condition for an expansion-collapse duality, let us consider the following quantity

$$\frac{\mathrm{d}\langle\chi\rangle_J}{\mathrm{d}\langle\tilde{\chi}\rangle_E} = \frac{1}{\sqrt{\varpi}}(\sqrt{\varpi} - \sqrt{3}), \tag{5.6}$$

where we have used the parameter choices mentioned above. Given $\langle\tilde{\chi}\rangle_E$ is increasing with T, a Jordan frame with $\varpi < 3$ will always have a decreasing $\langle\chi\rangle_J$. The expansion-collapse duality condition, in this case, is the same as the classical condition, Eq. (2.14), for all T. Therefore, in terms of the logarithmic scale factor operators, the Jordan frame (with $\varpi < 3$) is contracting indefinitely, corresponding to an expanding Einstein frame, similar to the classical description. However, both the expanding and contracting frames develop the same amount of quantum fluctuation as T increases, despite of their contrasting evolutions.

5.2 Scale factors

Classically, the conformal relation between the Einstein and Jordan frame scale factors can be written from Eq. (2.13) as

$$\frac{\tilde{a}}{a} = \frac{1}{\sqrt{8\pi}}\exp\left(\sqrt{\frac{P_T}{2}}T\right). \tag{5.7}$$

The ratio of the expectation values of the Einstein and Jordan frame scale factors is obtain from Eqs. (4.9a) and (4.23) as

$$\frac{\langle\tilde{a}\rangle_E}{\langle a\rangle_J} = \frac{1}{\sqrt{8\pi}}\lambda_0^{\frac{\varpi}{96P_T\gamma}\left(\ln\lambda_0 - 2\sqrt{2}\sqrt{P_T}T\right)} \times \exp\left(\sqrt{\frac{P_T}{2}}T + \frac{\varpi}{96}\left(\frac{\varpi}{\tilde{\gamma}} - \frac{2}{\gamma}\right)T^2 + \frac{1}{24}\left(\tilde{\gamma} - \frac{\varpi}{2}\gamma\right)\right), \tag{5.8}$$

where we have imposed the parameters as in Eqs. (4.6) and (4.20), and replaced $\tilde{T}$ with T from Eq. (5.1). If we further consider the parameter choices that ensure conformally invariant variance of χ, from Eq. (5.4), it takes the form

$$\frac{\langle\tilde{a}\rangle_E}{\langle a\rangle_J} = \frac{1}{\sqrt{8\pi}}\exp\left(\sqrt{\frac{P_T}{2}}T\right), \tag{5.9}$$

the RHS of which is the same as that of Eq. (5.7). Therefore, though the expectation values of the Einstein and Jordan frame scale factors do not follow their corresponding classical trajectories for all T, the ratio of them is the same as the ratio of their classical counterparts, provided the choice of parameters. The expectations of the scale factors of Einstein and Jordan frames satisfy the classical conformal transformation relation for a certain choice of wave packet parameters and boundary condition.

For the same parameter choice from Eqs. (4.6), (4.20) and (5.4), the quantum fluctuations of the Einstein and Jordan frame scale factors are related as

$$\sqrt{\frac{\Delta_E\tilde{a}}{\Delta_J a}} = \frac{1}{\sqrt{8\pi}}\exp\left(\sqrt{\frac{P_T}{2}}T\right) = \frac{\langle\tilde{a}\rangle_E}{\langle a\rangle_J}. \tag{5.10}$$

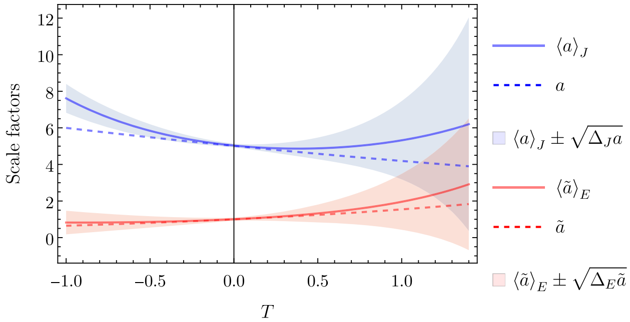

Fig. 3 Evolution of the expectation values of the Jordan and Einstein frame scale factors ($\langle a\rangle_J$, $\langle \tilde{a}\rangle_E$) (solid plots), along with their classical trajectories (a, $\tilde{a}$) (dashed plots), with respect to the Jordan frame clock T. The shaded regions depict fluctuations in the observables. We have used $\varpi = 1.5$, $\gamma_E = 0.1$, $\tilde{m}_0 = 1$, and other parameters according to Eqs. (4.6), (4.20) and (5.4). Both $\langle a\rangle_J$, $\langle \tilde{a}\rangle_E$ follow their corresponding classical trajectories near $|T| \to 0$, and deviate as $|T|$ gets larger. The fluctuations in both the cases also get larger as $|T| \to \infty$. Although $\langle a\rangle_J$ is decreasing near $|T| \to 0$, it eventually starts to increase at a larger T, while the classical trajectory $a(T)$ decreases forever, since $\varpi < 3$. In terms of $\langle a\rangle_J$, $\langle \tilde{a}\rangle_E$, the classical expansion-collapse scenario for $\varpi < 3$ is only observed near $|T| \to 0$

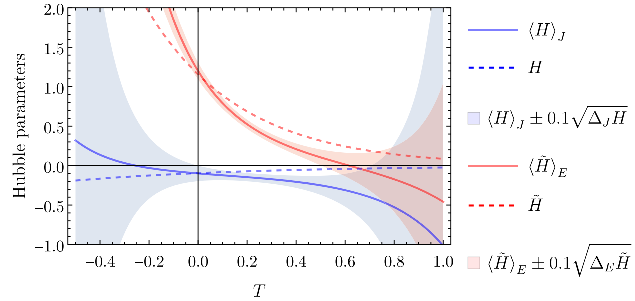

Fig. 4 Evolution of the expectation values of the Jordan and Einstein frame Hubble parameters ($\langle H\rangle_J$, $\langle \tilde{H}\rangle_E$) (solid plots), along with their classical trajectories (H, $\tilde{H}$) (dashed plots), with respect to the Jordan frame clock T. The shaded regions depict fluctuations in the observables. We have used $\varpi = 1.5$, $\gamma_E = 0.1$, $\tilde{m}_0 = 2$, and other parameters according to Eqs. (4.6), (4.20) and (5.4). Both $\langle H\rangle_J$, $\langle \tilde{H}\rangle_E$ align with their corresponding classical trajectories only near $|T| \to 0$, and deviate as $|T|$ gets larger. The fluctuations in both the cases also get larger as $|T| \to \infty$. Near $|T| \to 0$, $\langle H\rangle_J < 0$ and $\langle \tilde{H}\rangle_E > 0$, therefore indicating an expansion-collapse duality similar to their classical counterparts. However, at a larger T, both $\langle H\rangle_J$, $\langle \tilde{H}\rangle_E < 0$, indicating both the frames are contracting; while at a smaller negative T $\langle H\rangle_J$, $\langle \tilde{H}\rangle_E > 0$, indicating both the frames are expanding. In terms of $\langle H\rangle_J$, $\langle \tilde{H}\rangle_E$, the classical expansion-collapse scenario for $\varpi < 3$ is observed near $|T| \to 0$

That is, the ratio of the roots of variances of Einstein and Jordan frame scale factors is the same as the ratio of the expectations of the scale factors, which is according to the classical conformal map. Hence, *the relative fluctuation of scale factors is invariant under the conformal transformation,*

$$\frac{\sqrt{\Delta_E \tilde{a}}}{\langle \tilde{a}\rangle_E} = \frac{\sqrt{\Delta_J a}}{\langle a\rangle_J}. \tag{5.11}$$

The condition for the expansion-collapse duality, as imposed by the scale factor operators, can be written as

$$\frac{\mathrm{d}\langle a\rangle_J}{\mathrm{d}\langle \tilde{a}\rangle_E} < 0, \tag{5.12a}$$

where

$$\frac{\mathrm{d}\langle a\rangle_J}{\mathrm{d}\langle \tilde{a}\rangle_E} = \frac{\mathrm{d}\langle a\rangle_J}{\mathrm{d}T}\frac{\mathrm{d}T}{\mathrm{d}\langle \tilde{a}\rangle_E} = \frac{\mathrm{d}\langle a\rangle_J}{\mathrm{d}T}\frac{\mathrm{d}\tilde{T}}{\mathrm{d}\langle \tilde{a}\rangle_E}\frac{\mathrm{d}T}{\mathrm{d}\tilde{T}}, \tag{5.12b}$$

$$\frac{\mathrm{d}\langle a\rangle_J}{\mathrm{d}\langle \tilde{a}\rangle_E} = \frac{2\sqrt{\pi}}{\sqrt{\varpi}}\frac{24\sqrt{P_T}\gamma\left(\sqrt{\frac{\varpi}{3}}-1\right)+\sqrt{2}\varpi T}{4\sqrt{6P_T}\gamma+\sqrt{\varpi}T}. \tag{5.12c}$$

In the limit $|T| \to \infty$, $\frac{\mathrm{d}\langle a\rangle_J}{\mathrm{d}\langle \tilde{a}\rangle_E} > 0$, the scale factors' expectation values in the two frames change monotonically with each other regardless of ϖ. Therefore, according to the scale factor operators, there is no possibility of an expansion-collapse duality as $|T| \to \infty$ for all ϖ. This is in contrast with the classical condition as well as the condition set by the logarithmic scale factor operators. However, the quantum fluctuations in both the scale factors are large in this limit. In the $|T| \to 0$ limit, when the quantum fluctuations are small, $\frac{\mathrm{d}\langle a\rangle_J}{\mathrm{d}\langle \tilde{a}\rangle_E} < 0$ requires $\varpi < 3$, indicating that the classical condition for the expansion-collapse duality is recovered in this limit (see Fig. 3).

5.3 Hubble parameters

Let us now consider the relation between the expectation values of the Hubble parameters in the two frames. Classically, the ratio of the Jordan and Einstein frame Hubble parameters can be obtained using the conformal correspondence as

$$\frac{H}{\tilde{H}} = \frac{1}{\sqrt{8\pi}}\exp\left(\sqrt{\frac{P_T}{2}}\right)\left(1-\sqrt{\frac{3}{\varpi}}\right). \tag{5.13}$$

As it is expected, given $\tilde{H} > 0$, the Jordan frame Hubble parameter H becomes negative for $\varpi < 3$, depicting a contracting Jordan frame. With the parameter choices from Eqs. (4.6) and (4.20) and (5.4), and using the relation in Eq. (5.1), one can write the ratio of the expectations of Jordan and Einstein frame Hubble parameters (Eqs. (4.13a) and (4.28a)) as

$$\frac{\langle H\rangle_J}{\langle \tilde{H}\rangle_E} = \exp\left(\sqrt{\frac{P_T}{2}}T\right)\frac{24\sqrt{P_T}\gamma\left(1-\frac{\sqrt{\varpi}}{\sqrt{3}}\right)+3\sqrt{2}\varpi T}{4\sqrt{\pi}\left(3\varpi T-4\sqrt{6}\gamma\sqrt{\varpi P_T}\right)}. \tag{5.14}$$

Clearly, the ratio deviates from its classical counterpart Eq. (5.13), which is due to the classical conformal map. In particular, in the limit $|T| \to \infty$, $\frac{\langle H\rangle_J}{\langle \tilde{H}\rangle_E} > 0$ regardless of the choice of ϖ; hence there is no expansion-collapse duality in this limit, as determined by the expectation values of the Hubble parameters. However, the quantum fluctuations in both the Einstein and Jordan frame Hubble operators are large at this regime (see Fig. 4). The expansion-collapse requirement set by the Hubble operators in this regard is similar to that of the scale factor operators, which is in contrast with the classical condition as well as the condition set by the logarithmic scale factor operators. The quantum fluctuations in both the Hubble parameters become small as $|T| \to 0$. In this limit, $\frac{\langle H\rangle_J}{\langle \tilde{H}\rangle_E}$ becomes negative for $\varpi < 3$, thus the classical condition for the expansion-collapse duality is recovered in this limit (see Fig. 4).

6 Summary and conclusion

We study the conformal correspondence between Jordan and Einstein frames in a quantum mechanical framework. In particular, we consider a Brans–Dicke model without a potential in the Jordan frame, which corresponds to a massless scalar field in the Einstein frame. According to the classical conformal map, the Jordan frame universe may contract indefinitely to an arbitrarily small scale factor, while the Einstein frame continually expands. When the scale factor in the contracting Jordan frame becomes sufficiently small, and the universe approaches the singularity, the classical description of the system becomes inadequate. One expects the Jordan frame universe to develop significant quantum characteristics in this regime, for example, rise in quantum fluctuations in different cosmological quantities. However, the corresponding Einstein frame universe is arbitrarily large at this point, therefore it is expected to behave classically. This leads to the apparent paradox – does the conformal correspondence in this scenario provide a map between a quantum effect-dominated universe and a universe behaving classically?

To consistently address this question, we quantize the Einstein and Jordan frame universes following the Wheeler–DeWitt prescription. We first investigate whether the classical conformal map is valid at the quantum level by comparing the expectation values of different cosmological quantities in the two frames. We then find how the quantum fluctuations in different quantities are related in the two frames. The main results in this work can be summarized as follows:

- For a convenient choice of phase-space variables, the Wheeler–DeWitt equations in both frames take the simple form of the Schrödinger equation for a free particle. The wave packets constructed in both the Einstein and Jordan frames satisfy DeWitt's criteria, and therefore hint at singularity avoidance. However, the expectation values of both Einstein and Jordan frame logarithmic scale factors follow their corresponding classical trajectories for all values of the clock parameter (T). In this case, following their classical trajectories, the expectation values do not lead to singularity avoidance. On the other hand, the expectation values of the scale factors and Hubble parameters in both frames deviate from their classical trajectories as $|T|$ becomes larger. Eventually, the expectation values of these operators represent singularity-free bouncing universes.
- *The expectation values of the logarithmic scale factors in the two frames always satisfy the classical conformal correspondence for a given set of parameter choices*. Moreover, for the same set of parameters, *the variance in the logarithmic scale factor operator is found to be conformally invariant*, it takes the same value in both Einstein and Jordan frames at a given clock parameter (T). The variance is minimum as $|T| \to 0$, and it diverges as $|T| \to \infty$.
- Similarly, *the expectation values of the scale factors in the two frames also satisfy the classical conformal map*, even though the expectation values in this case do not always follow the classical trajectories. *The relative fluctuation in the scale factor operator is also conformally invariant*. In both the frames, the variances in the scale factors are minimum near $|T| \to 0$, when the expectation values of the scale factors follow the classical behavior. They keep on increasing as $|T|$ increases, and the expectation values keep on deviating from the classical trajectories.
- Unlike the logarithmic scale factors and the scale factors, the expectation values of the Hubble parameters do not always follow the classical conformal relation. However, similar to the other two operators, the quantum variances in both the Einstein and Jordan frame Hubble parameters are minimum in the $|T| \to 0$ limit, and they keep on increasing as $|T|$ increases. Similar to the scale factor operators, the expectation values of the Hubble parameters in both frames deviate from the classical trajectories as $|T|$ becomes larger.
- In terms of the logarithmic scale factor operators, the Einstein and Jordan frames undergo indefinite expansion and contraction, respectively, set by the classical condition of the expansion-collapse duality. However, according to both the scale factors and Hubble operators, there is no eternal expansion-collapse duality between the frames, but only in the limit when $|T| \to 0$, and the quantum fluctuations in these operators are small. This contrasts with the classical duality, as well as the expansion-collapse duality dictated by the logarithmic scale factors.

It is evident that different cosmological operators may disagree on whether one frame is expanding or contracting as $|T|$ becomes large and the fluctuations in the operators increase. As $|T| \rightarrow 0$, the fluctuations in the operators decrease, and their expectation values resemble their corresponding classical trajectories. In this limit, the expectation values of all the three operators depict an expansion-collapse duality between the conformal frames, similar to the classical description.

With regard to the initial question, we find that the classical conformal map holds true at the quantum level when compared through the expectation values of the logarithmic scale factors and the scale factors in the two frames. As the Jordan frame contracts and its classical scale factor approaches singularity in the limit $T \rightarrow \infty$, the quantum fluctuations in different cosmological operators expectedly become large, indicating the rise in the quantum characteristics in the Jordan frame universe. More interestingly, at the same time, the quantum fluctuations also increase in the expanding Einstein frame, despite its arbitrarily large scale factor. Therefore, the rise in quantum fluctuations in one frame indicates the same in the other, even though the cosmological evolutions in the frames are drastically different.

The present results indicate that the rise in quantum features is a frame independent effect, even the expanding conformal universe with arbitrarily large size can harbor large quantum fluctuations. The presence of significant quantum effects in a large expanding universe is previously argued in the literature. For example, [43] predicts non-trivial quantum effects when the physical universe transitions from the decelerating phase to an accelerating one. In [44], it is argued that quantum fluctuations may remain non-vanishing in the late-time universe and can become dominant over large scales. The revival of quantum correlations at the late-time matter-dominated universe is explored in [45].

This resolution of the apparent paradox at hand leads to another paradoxical notion of large quantum fluctuations in a macroscopic universe. One can naively expect the decoherence mechanism to suppress large quantum effects in the background geometry of the macroscopic universe. However, the universe is an isolated system, and there is no notion of an external environment to decohere the system. Still, in canonical quantum gravity, decoherence can be achieved by considering the inhomogeneous degrees of freedom, such as density fluctuations and quasi-normal modes, as the environment, while the background degrees of freedom as the system [40,46,47]. For example, inhomogeneous modes are found to decohere the quantized background geometry for WKB-like solutions of Wheeler–DeWitt equation [48–51]. However, a similar notion of a decohering environment does not exist for the model under consideration; therefore, it is not surprising to find large quantum fluctuations for the macroscopic universe in this case.

In this paper, the quantum analysis is done with convenient choices of clock variables, phase-space variables, and operator ordering of the Hamiltonians. It is worth exploring whether different choices of clock variables influence the results [52,53]. The present work can be extended by introducing ordinary matter components in the two conformal frames. It is argued in [13] that in the presence of matter, the conformal correspondence breaks down at the level of the wave packets. It would be interesting to investigate the conformal map in such a scenario at the level of expectation values of relevant cosmological operators. This will be pursued in a future work.

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