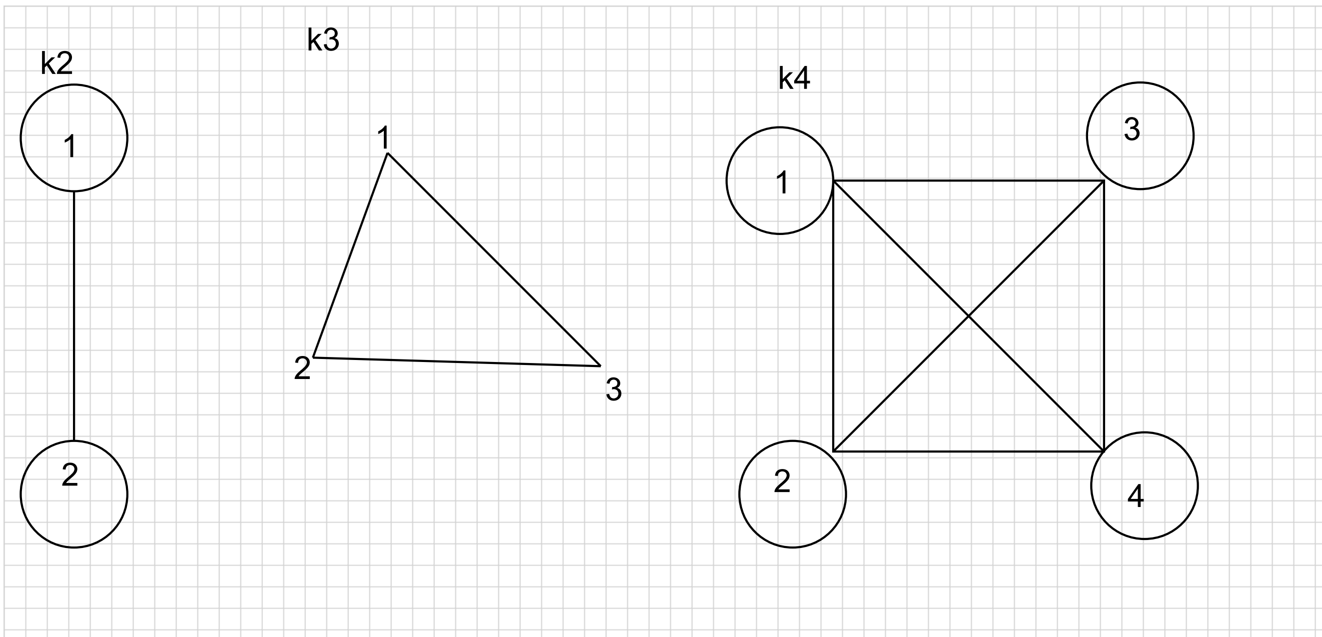
1) טורון

a)



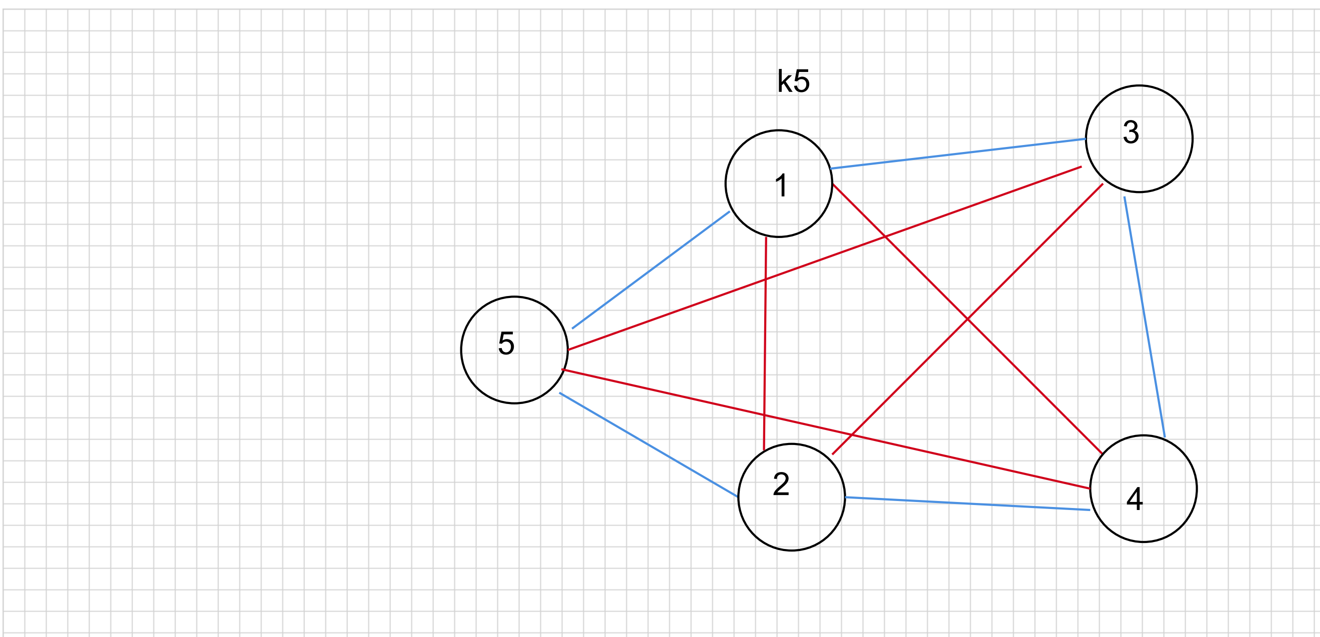
b)

This is similair to the problem of having a group of 6 friends where there is either
3 friends that know each other or 3 friends that don't know each other.
K_6 is a complete graph with 6 vertices (friends). Every single vetex is connected to each other by an edge. Each edge is either blue or white. There are $\binom{6}{2} = 15$ edges in total.
For each Vertez (person) there are 5 edges leaving them degree(v)=5
if we use the pingenhole principle putting 5 (edges) pigenons into two colours (wholes).
$\left\lceil \frac{5}{2} \right\rceil = 3$ We can assume 3 edges are either blue or white let's assume there are blue connecting V to v_1,v_2,v_3
now considering a subgraph of v_1,v_2,v_3 they are connect by the edges
$\{v_1,v_2\},\{v_2,v_3\},\{v_3,v_1\}$
if one of the edges are blue then it will form a blue triangle with v
if none of them are blue then they are all white forming a white triangle with itself
v_1,v_2,v_3

c)
This is not true, using the same pigenwhole expermint we will end up with

$\left\lceil \frac{4}{2} \right\rceil = 2$ these are edges that must be the same colour.

We will show a counter example as you can see there is no triangle of one colour.



d) To prove this it is similair to the proof in b) but we know have three colours and 16 people so our

pigenhole formula will be $\left\lceil \frac{16}{3} \right\rceil = 6$

meaning know each vertex has 6 different edges that are the same colour
let's assume the colour is blue and it is connected from V to $v_1, v_2, v_3, v_4, v_5, v_6$ repsiectivly
if one of the edges between themselves is blue we have a triangle
if none of them are they must all be either white or black.
for this sub graph there are 15 edges in total.
and know we are with the excat same problem from before with
k_6 and two different colours in which there must be a monchromatic triangle.

e) **not true.** $\left\lceil \frac{15}{3} \right\rceil = 5$

We can construc a counter example the following way
Divde the 16 vertices into 4 groups
the edges of the vertices in the group between each other will be blue
the edges between groups will be either white or black alternating

2)

a) For n=1 or n=2 there is no such way to form a triangle so it holds true trivally

for example if n=2 m > $\frac{2^2}{4} = 1$ m=1 this forms a triangle trivially.

inductive step:

lets assume this holds for n=k vertices m>$\frac{k^2}{4}$ this graph contains a triangle.

Step:

We know want to prove that for n=k+1 if m> $\frac{(k+1)^2}{4}$ then the grpah holds a triangle.

We find a vertex u in our graph that has a degree of $\frac{k+1}{2}$<d

if there exist two vertices v and w that are connected to u by an edge (u,v) and (u,w) then we are done

if not we will remove u from our grpah and focus on the subgraph that is without u and all of u's edges

we will call the subraph f. f has k vertices and m'=m-d edges.

m'=m-d< $\frac{(k+1)^2}{4} - \frac{k+1}{2} = \frac{k^2}{4}$

by induction this graph contains a clique.

if it is larger the amount then there exists a triangle becuase there is a vertex connected to more than half of the other vertex's

got it

b) We will prove this by taking the vertices and diving them into two seprate groups a and b with an equal number of vertices in each group becuase we are taking only even number of vertices.
every edge will connect a vertes in A to a vertex in B
to construc will will connect every vertex in A to every vertex in B
will will be excatly |A|*|B|= $\frac{n}{2} * \frac{n}{2} = \frac{n^2}{4}$ *total edges*
no clique exists becuase there are no edges within A or within B

3)

a) We will build a path from
$\sigma_1\sigma_2\sigma_3 \ldots \sigma_n$ *to* $\tau_1\tau_2\tau_3 .. \tau_n$ *each time we will for every edge we will add* τ *to the end in orde*
the new word the next vertex will remove the first letter in the previous vertex and add the first letter of the vertex we want to go to in the end to the end of this vertex.
We repeat this process until we get the vertex that we wanted to go to .
This path will be excatly $n-1$ *becuase we will perform the action of removing the first lett and adding the letter to the end until we get the new word and the word is of length* $n-1$ *so we will perform that action* $n-1$ *times.*
It is unique becuase our grpah only has 0 *and* 1*'s. therefore each step of the way we will on choose* 0 *or* 1, *if we choose wrongly we will end up with the wrong word.*

b)
We will to k turns and then keep going until we get to n-1 we we don't get to n-1 then f^k(a)=a
k is the smallest rotation
f is defined as $f(a_1a_2a_3a_4..a_n)=(a_2a_3a_4....a_{n-1}a_1)$
n-1=ak+b b<k stera that k is minimum

c) |V|=2^p = $\sum_{i=1}^{m} = |C_i| \quad 2^p = \sum xi \ = 1 + 1 + sigma\ p\ \ xi|p$
sigma= {0,1}
n=p+1

the words that are only 0's or only 1's then it has a self circle then it divees by 1 so 2^p-1

4)

a) We will start from one side.
Suppose G is not connected. V_g can be divided into two disconnected subsets V_{g1} V_{g2}
Now in $G \times H$ there will be nothing connecting vertices of the form (g,h) g $\in V_{g1}$ to vertices of the form (g^1,h^1) $g^1 \in V_{g2}$
which contraditcts the fact that $G \times H$ is connected. The same is true for H

Now assuming G and H are connected $G \times H$ must be connected

reason being for any g_1,g
$_2 \in V_g$ *there exist a path in G and the same is true for h for any two vertices*
From any (g,h) and (g1,h1) in G × H we can find a path between them
first we can fix h and move along G, since G is connected there is a path from G-G1
so we can traverse from (g,h)-(g1,h)
then we can fix g1 and move along H, H is connected so a path exist from h-h1
so we can traverse from (g1,h)-(g1,h1)
now we have shown that we can find a path between any two vertices.

b)This is true= $d((g,h))=d_G(g)+d_H(h)$
even +even = even

c)

this is false

here G × H is not an euler cycle but g is not not an euler cycle.

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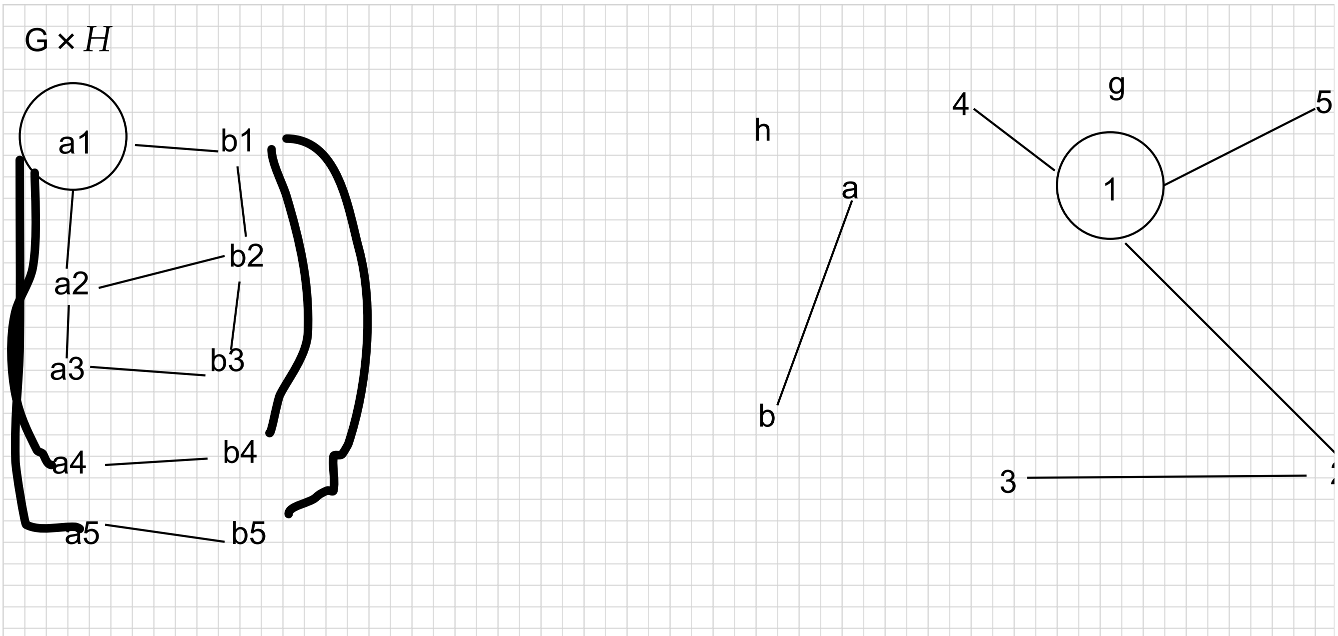
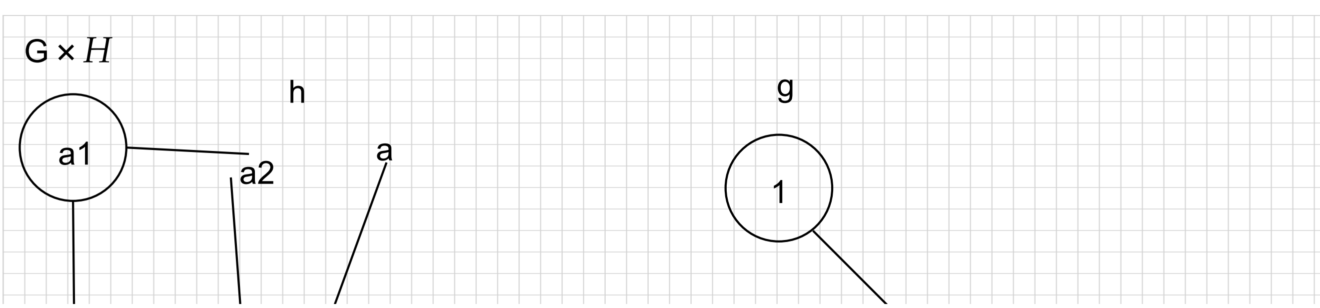
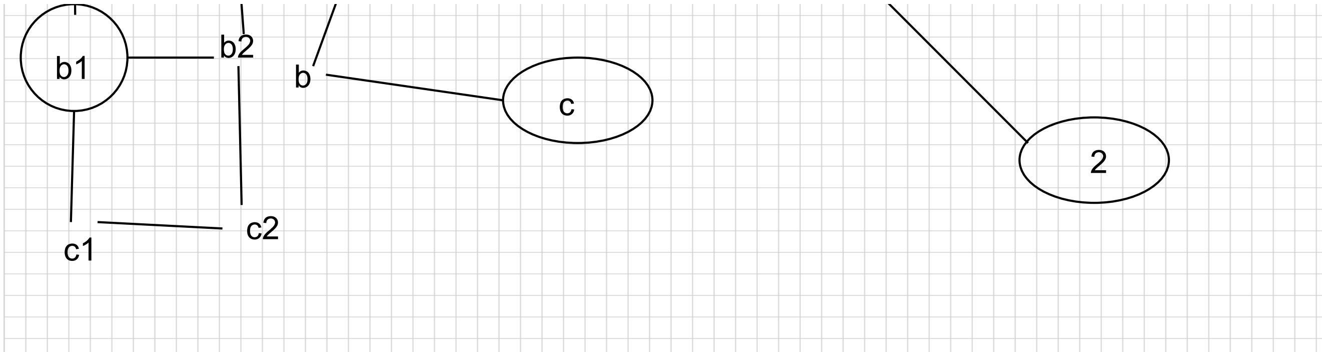
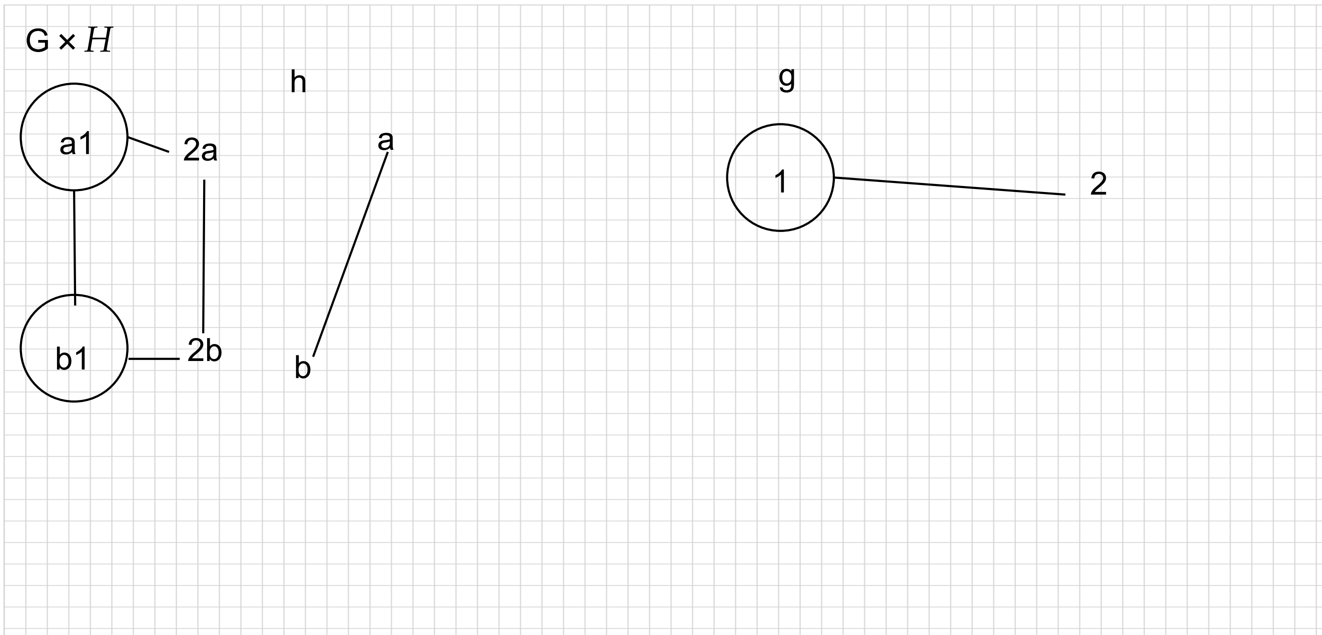


d)





e)
false

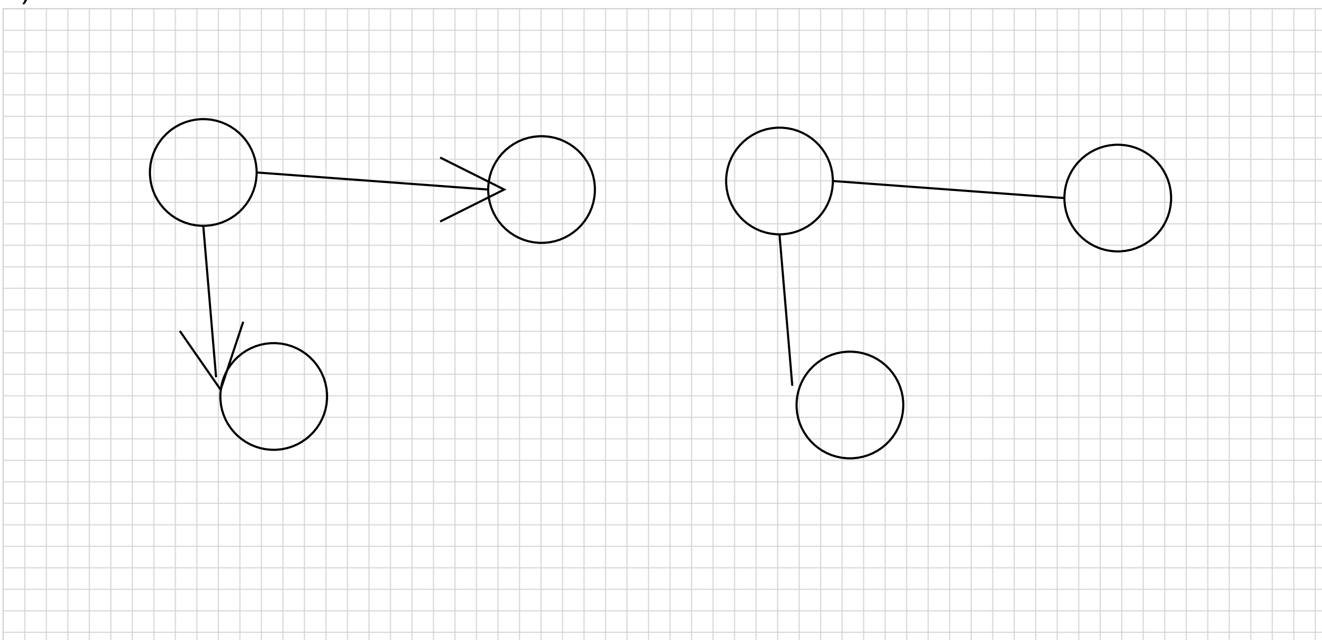


5)

a)true

If the graph of G is connected strongly then there is a directed path from u to v for every pair of vertices in the graph. We if remove the directions there will still be a path from u to v for every pair of vertices in the graph. We can assume that if there was not a path from u to v in the underlying graph then there would be no way to connect u to v in a directed manner which is a contradiction.

b) false



c)This claim is false becuase if G contains any self loops then when turning the graph into directed there is only one direction it can go. From itself to itself.

d) We want to show that there exist a simple cycle of length k in our graph.
We will take a vertex $a_1a_2a_3.... a_{n-1}$we know there exist an edge from there to $a_2a_3.... a_n$
We will choose a sequence of $a_1, a_2, a_{3,}... a_k$
We will know construct k binary strings $b_1, b_2, b_3.... b_k$ $of\ length\ n - 1$
and define $b_i = a_ia_{i+1}a_{i+2}.... a_{i+n-2}$
here the strings form a cycle $b_1 \succ b_2 \succ ... \succ b_k \succ b_1$
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e) We will prove this step by induction
if the graph has 1 vertex then this holds true becuase a graph with one vertex is hamiltonian.
now we will assume the claim is true for n vertices and we will prove it for n+1 vertices.

Now we we remove v from the graph the remaining n vertices must have a hamiltonian path which we will wrtite as

v1, v2, . . . vn.

Now we will look at all of the edges coming from v into the rest of the graph

case 1) if there is an edge from v-v_1 we now have a hamiltion path from V,V_1,V_2,...V_n

case2) if there is an edge from V_n-V we have a hamiltonion path from V_1,V_2,...V_n,V

now if case 1 and 2 do not hold.

we have case 3)

In this case there must be a edge from V_1-V and an edge V-V_n now when looking at the vertices in between if the edges are pointing towards V there must become a point where they are pointing away from v. There is at least one number 1 ≤ i ≤ n−1 for which vi-v is an edge and v-vi+1 is an edge and now we found a new hamiltonian path v1,v2,...,vi,v,vi+1,...,vn.