Homework 2

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1. Prove that T and regular matrix multiplication satisfy the properties listed above.

(a) Closure:

For set T, we have $T_1 = \begin{bmatrix} R_1 & p_1 \\ 0 & 1 \end{bmatrix}$ and $T_2 = \begin{bmatrix} R_2 & p_2 \\ 0 & 1 \end{bmatrix}$

$$T_1 \cdot T_2 = \begin{bmatrix} R_1 R_2 & R_1 p_2 + p_1 \\ 0 & 1 \end{bmatrix}$$

We know that $R_1 R_2$ is rotation matrices so that it from a group, also $R_1 p_2 + p_1$ is a rotated translation vector and plus another vector, by the fact that translation vectors and vector addition from a group we know that $R_1 p_2 + p_1$ is from a group. Therefore, we have $T_1 \cdot T_2$ from a group. Since all set T satisfy the closure property, the multiplication of set T and regular matrix also satisfy the closure property.

(b) Associativity:

Suppose we have: $T_1 = \begin{bmatrix} R_1 & p_1 \\ 0 & 1 \end{bmatrix}$, $T_2 = \begin{bmatrix} R_2 & p_2 \\ 0 & 1 \end{bmatrix}$ and $T_3 = \begin{bmatrix} R_3 & p_3 \\ 0 & 1 \end{bmatrix}$

$$(T_1 \cdot T_2) \cdot T_3 = \begin{bmatrix} R_1 R_2 & R_1 p_2 + p_1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} R_3 & p_3 \\ 0 & 1 \end{bmatrix}$$

$$(T_1 \cdot T_2) \cdot T_3 = \begin{bmatrix} R_1 R_2 R_3 & R_1 R_2 p_3 + R_1 p_2 + p_1 \\ 0 & 1 \end{bmatrix}$$

On the other hand:

$$T_1 \cdot (T_2 \cdot T_3) = \begin{bmatrix} R_1 & p_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2 R_3 & R_2 p_3 + p_2 \\ 0 & 1 \end{bmatrix}$$

$$T_1 \cdot (T_2 \cdot T_3) = \begin{bmatrix} R_1 R_2 R_3 & R_1 R_2 p_3 + R_1 p_2 + p_1 \\ 0 & 1 \end{bmatrix}$$

As we can see that $(T_1 \cdot T_2) \cdot T_3 = T_1 \cdot (T_2 \cdot T_3)$. So that the set T satisfies the associativity property.

(c) Identity:

I is an identity matrix in T.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore

$$T_1 I = \begin{bmatrix} R_1 & p_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1 & p_1 \\ 0 & 1 \end{bmatrix}$$

and

$$I T_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_1 & p_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1 & p_1 \\ 0 & 1 \end{bmatrix}$$

Thus we have

$$T_1 I = I T_1 = T_1$$

So the set T has the identity property.

(d) Inverse:

For $T_1 \cdot T_2 = \begin{bmatrix} R_1R_2 & R_1p_2 + p_1 \\ 0 & 1 \end{bmatrix}$, and $T_2 \cdot T_1 = \begin{bmatrix} R_2R_1 & R_2p_1 + p_2 \\ 0 & 1 \end{bmatrix}$

In order to have $T_1 \cdot T_2 = T_2 \cdot T_1 = I$, we need to show that

$$R_1R_2 = R_2R_1 = I$$

and

$$R_1p_2 + p_1 = R_2p_1 + p_2 = 0$$

For $R_1R_2 = R_2R_1$:

$$R_1R_2 = \begin{bmatrix} cos\theta_1 & -sin\theta_1 \\ sin\theta_1 & cos\theta_1 \end{bmatrix} \begin{bmatrix} cos\theta_2 & -sin\theta_2 \\ sin\theta_2 & cos\theta_2 \end{bmatrix}$$

$$= \begin{bmatrix} cos\theta_1 cos\theta_2 - sin\theta_1 sin\theta_2 & -sin\theta_2 cos\theta_1 - sin\theta_1 cos\theta_2 \\ sin\theta_1 cos\theta_2 + cos\theta_1 sin\theta_2 & cos\theta_1 cos\theta_2 - sin\theta_2 sin\theta_1 \end{bmatrix}$$

$$R_2R_1 = \begin{bmatrix} cos\theta_2 & -sin\theta_2 \\ sin\theta_2 & cos\theta_2 \end{bmatrix} \begin{bmatrix} cos\theta_1 & -sin\theta_1 \\ sin\theta_1 & cos\theta_1 \end{bmatrix}$$

$$= \begin{bmatrix} cos\theta_1 cos\theta_2 - sin\theta_1 sin\theta_2 & -sin\theta_2 cos\theta_1 - sin\theta_1 cos\theta_2 \\ sin\theta_1 cos\theta_2 + cos\theta_1 sin\theta_2 & cos\theta_1 cos\theta_2 - sin\theta_2 sin\theta_1 \end{bmatrix}$$

Thus we have $R_1R_2 = R_2R_1 = \begin{bmatrix} cos(\theta_1 + \theta_2) & -sin(\theta_1 + \theta_2) \\ sin(\theta_1 + \theta_2) & cos(\theta_1 + \theta_2) \end{bmatrix}$

So that if $\theta_1 + \theta_2 = 0$, we have $R_1R_2 = R_2R_1 = \begin{bmatrix} cos0 & -sin0 \\ sin0 & cos0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Also if we could let $R_1p_2 = -p_1$, then we have $R_1p_2 + p_1 = 0$

Then $R_2p_1 + p_2 = -(R_1R_2)p_1 + p_2 = -p_1I + p_2 = -p_1 + p_2 = 0$

Thus, we can find a inverse matrix T_2 for T_1 in Set T

2.

(a) What is a rotation of $\frac{\pi}{2}$ radians about the axis [0 0 1]^T as a unit quaternion, *q1* ?

Suppose we have a unit quaternion, *q1* = (cos(θ), sin(θ)N).

We have quaternion rotation equation: $q = (cos\frac{\theta}{2}, sin\frac{\theta}{2}N)$

Since the rotation is about axis [0 0 1]^T, we could write *q1* as:

$$q1 = \cos 45^{\circ} + sin45^{\circ}(0i + 0j + 1k)$$

$$q1 = \frac{\sqrt{2}}{2} + 0i + 0j + \frac{\sqrt{2}}{2}k$$

(b) Given the quaternion $q2 = 0 + 1i + 0j + 0k$, what is $q1 \cdot q2$?

We have $q1 = \frac{\sqrt{2}}{2} + 0i + 0j + \frac{\sqrt{2}}{2}k$ and $q2 = 0 + 1i + 0j + 0k$

So that:

$$q1 \cdot q2 = \left(\frac{\sqrt{2}}{2} \times 0\right) - \left(0i + 0j + \frac{\sqrt{2}}{2}k\right) \cdot (1i + 0j + 0k) + \frac{\sqrt{2}}{2}(1i + 0j + 0k)$$
$$+ 0\left(0i + 0j + \frac{\sqrt{2}}{2}k\right) + \left(0i + 0j + \frac{\sqrt{2}}{2}k\right) \times (1i + 0j + 0k)$$

Therefore, $q1 \cdot q2$ is:

$$q1 \cdot q2 = 0 + \frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}j + 0k$$

3. Determine the topology and dimension of the manipulator's configuration space.

(a) A manipulator with two prismatic points.

Two prismatic joints connect two rigid bodies. The movement of two bodies is defined by the linear sliding movement of two prismatic points. Thus, we could use the two prismatic joints to describe the movement of the manipulator. So that the DoF of this manipulator is 2, and the topology of the manipulator is R^2.

(b) A manipulator with three revolute joints.

The manipulator has three revolute joints. The configuration space should be the semi-circle of that three links can cover. Three angles, θ_1, θ_2 and θ_3 can specify the configuration of this manipulator. Thus, the dimension of configuration space is 3. The topology should be $S^1 \times S^1 \times S^1 = T^3$.

(c) A manipulator with two revolute joints and a prismatic joint.

The manipulator has 2 revolute joints and 1 prismatic joint. The configuration space is the semi-circle that all links could cover. The revolute joint and prismatic joint is considered as 1 dimension each. Thus, we have 3 dimension of configuration space for this manipulator. The topology should be $R^1 \times T^2$.

4. Figure shows a three-link kinematic chain in 2D. The lengths of link A_1, A_2 and A_3 are l_1, l_2 and l_3, respectively. The joint angles of the chain are θ_2 and θ_3.

(a) Determine the topology and dimension of the configuration space for this manipulator.

Since the joint angles of the chain are θ_2 and θ_3, there only 2 parameters to specify the configuration of the object. The dimension of the configuration space is 2, and the topology is T^2.

(b) Determine the Homogeneous coordinates v_3 of the point 2 in the local frame of link A_3.

In the local frame of link A_3, the homogeneous coordinate v_3 should be:

$$v_3 = [l_3, 0, 1]$$

(c) Determine the forward kinematic of this three-link chain. That is, calculate the homogenous coordinates v_1 of the point 2 in the local frame of link A_1.

We know that the homogenous transformation for 2D chains is:

$$x = a_1 cos\theta_1 + a_2 \, cos(\theta_1 + \theta_2) + a_3 cos(\theta_1 + \theta_2 + \theta_3)$$

$$y = a_1 sin\theta_1 + a_2\, sin(\theta_1 + \theta_2) + a_3 sin(\theta_1 + \theta_2 + \theta_3)$$

So that we could have the homogenous coordinates v_1:

$$v_1 = [\, l_1 + l_2 cos(\theta_2) + l_3 cos(\theta_2 + \theta_3),\ l_2 sin(\theta_2) + l_3 sin(\theta_2 + \theta_3), 1]$$

(d) Determine the homogeneous transformation from the local frame of A_3 to the local frame of A_1.

We could have the transformation matrix T_2:

$$T_2 = \begin{bmatrix} \cos\theta_2 & -sin\theta_2 & l_1 \\ sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We use the same way to get the transformation matrix T_3:

$$T_3 = \begin{bmatrix} \cos\theta_3 & -sin\theta_3 & l_2 \\ sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then $T_2 \cdot T_3$ moves A_3 from its local frame to the local frame of A_1.

$$T_2 \cdot T_3 = \begin{bmatrix} \cos\theta_2 & -sin\theta_2 & l_1 \\ sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_3 & -sin\theta_3 & l_2 \\ sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_2 \cdot T_3 = \begin{bmatrix} \cos\theta_2\cos\theta_3 - sin\theta_2 sin\theta_3 & -sin\theta_3\cos\theta_2 - sin\theta_2\cos\theta_3 & l_2\cos\theta_2 + l_1 \\ sin\theta_2\cos\theta_3 + \cos\theta_2 sin\theta_3 & -sin\theta_3 sin\theta_2 + \cos\theta_2\cos\theta_3 & l_2 sin\theta_2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_2 \cdot T_3 = \begin{bmatrix} cos(\theta_2 + \theta_3) & -sin(\theta_3 + \theta_2) & l_2 cos\theta_2 + l_1 \\ sin(\theta_2 + \theta_3) & cos(\theta_2 + \theta_3) & l_2 sin\theta_2 \\ 0 & 0 & 1 \end{bmatrix}$$

(e) Show that $\boldsymbol{v_1 = T_2 \cdot T_3 \cdot v_3}$.

So far we know that:

$$T_2 \cdot T_3 = \begin{bmatrix} cos(\theta_2 + \theta_3) & -sin(\theta_3 + \theta_2) & l_2 cos\theta_2 + l_1 \\ sin(\theta_2 + \theta_3) & cos(\theta_2 + \theta_3) & l_2 sin\theta_2 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$v_3 = [l_3, 0, 1]$$

Therefore, $T_2 \cdot T_3 \cdot v_3$ equals:

$$T_2 \cdot T_3 \cdot v_3 = \begin{bmatrix} \cos(\theta_2 + \theta_3) & -\sin(\theta_3 + \theta_2) & l_2 cos\theta_2 + l_1 \\ \sin(\theta_2 + \theta_3) & \cos(\theta_2 + \theta_3) & l_2 sin\theta_2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} l_3 \\ 0 \\ 1 \end{bmatrix}$$

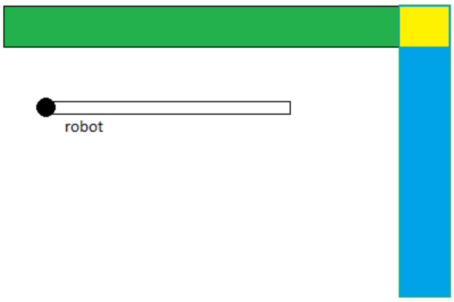
$$T_2 \cdot T_3 \cdot v_3 = \begin{bmatrix} l_3 \cos(\theta_2 + \theta_3) + 0 + (l_2 cos\theta_2 + l_1) \\ l_3 \sin(\theta_2 + \theta_3) + 0 + l_2 sin\theta_2 \\ 0 + 0 + 1 \end{bmatrix}$$

$$T_2 \cdot T_3 \cdot v_3 = \begin{bmatrix} l_1 + l_2 cos\theta_2 + l_3\, cos(\theta_2 + \theta_3) \\ l_2 sin\theta_2 + l_3\, sin(\theta_2 + \theta_3) \\ 1 \end{bmatrix} = v_1$$

5.

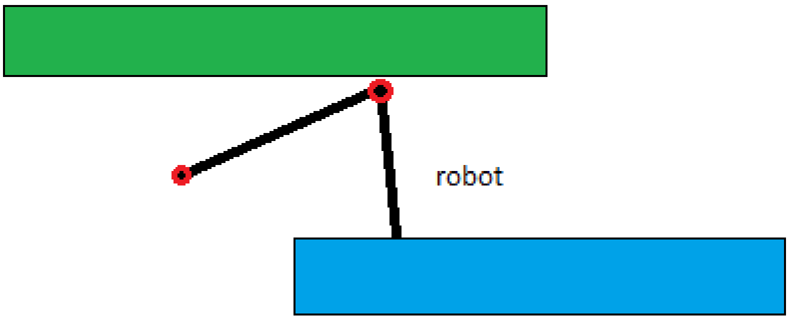
(a) Consider workspace obstacles A and B. If $A \cap B \neq \emptyset$, do the configuration space obstacles QA and QB always overlap?

No, we know that if two configuration space obstacles are overlap, then the robots can reach to both obstacles at the same time.



Imagine a workspace like above image. We have obstacle A in green and obstacle B in blue. The yellow part is the overlap of two obstacles. For the robot with one revolute joint, it will reach obstacle A in some angle. But it will never reach obstacle B and obstacle A at the same time. So that the configuration space obstacles QA and QB are not overlap under this situation.

(b) If $A \cap B = \emptyset$, is it possible for the configuration space obstacles QA and QB overlap?



Now two obstacles are not overlap. Consider the above image, obstacle A in green and obstacle B in blue. We have a 2 dimension robot with two red revolute joints. Under some certain angle, we could have this robot to reach both obstacles even if two obstacles are not overlap. Thus, the configuration space obstacles QA and QB are overlap.

6. Suppose five polyhedral bodies float freely in 3D world. They are each capable of rotating and translating. If these are treated as "one" composite robot, what is the topology of the resulting configuration space (assume that the bodies are not attached to each other)? What is the dimension of the composite configuration space?

Since each polyhedral body floats freely in 3D world and able to rotating and translating. So we need three dimension to describe the rotation of the polyhedral body, $(\theta_1, \theta_2, \theta_3)$, and three dimension to describe the translating, (x, y, z). Therefore, the dimension of configuration space for each polyhedral body should be 6. Since bodies are not attached to each other, then there is no constrains between polyhedral bodies. Thus, the dimension of composite configuration space for total 5 polyhedral bodies should be $6 \times 5 = 30$. The topology of the resulting configuration space should be $R^{15} \times T^{15}$.