Unravelling Bayesian Inference
A Journey Through MCMC, Rejection Sampling, and ABC Methods

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Introduction

Parameter of interest : $\boldsymbol{p}$, Prior : $\boldsymbol{\pi}(\boldsymbol{p}) \sim Beta(1,1)$ $(i.e. Unif(0,1))$,
Likelihood : $\boldsymbol{f}(\boldsymbol{x}|\boldsymbol{p}) \sim Bin(n, p)$, $\boldsymbol{n} = \mathbf{10}$ i.e. 10 coin-tosses , each coin toss has probability p of a head , $\boldsymbol{x_{obs}} = \mathbf{5}$ (observed 5 heads when tossing 10 times)

Posterior : $\boldsymbol{\pi}(\boldsymbol{p}|\boldsymbol{x_{obs}}) = \frac{\pi(p) f(x_{obs}|p)}{\int \pi(p) f(x_{obs}|p) dp} \sim Beta(6, 6)$ [3]

Implemented MCMC, Rejection Sampling, ABC Rejection Sampling and ABC-MCMC for this example

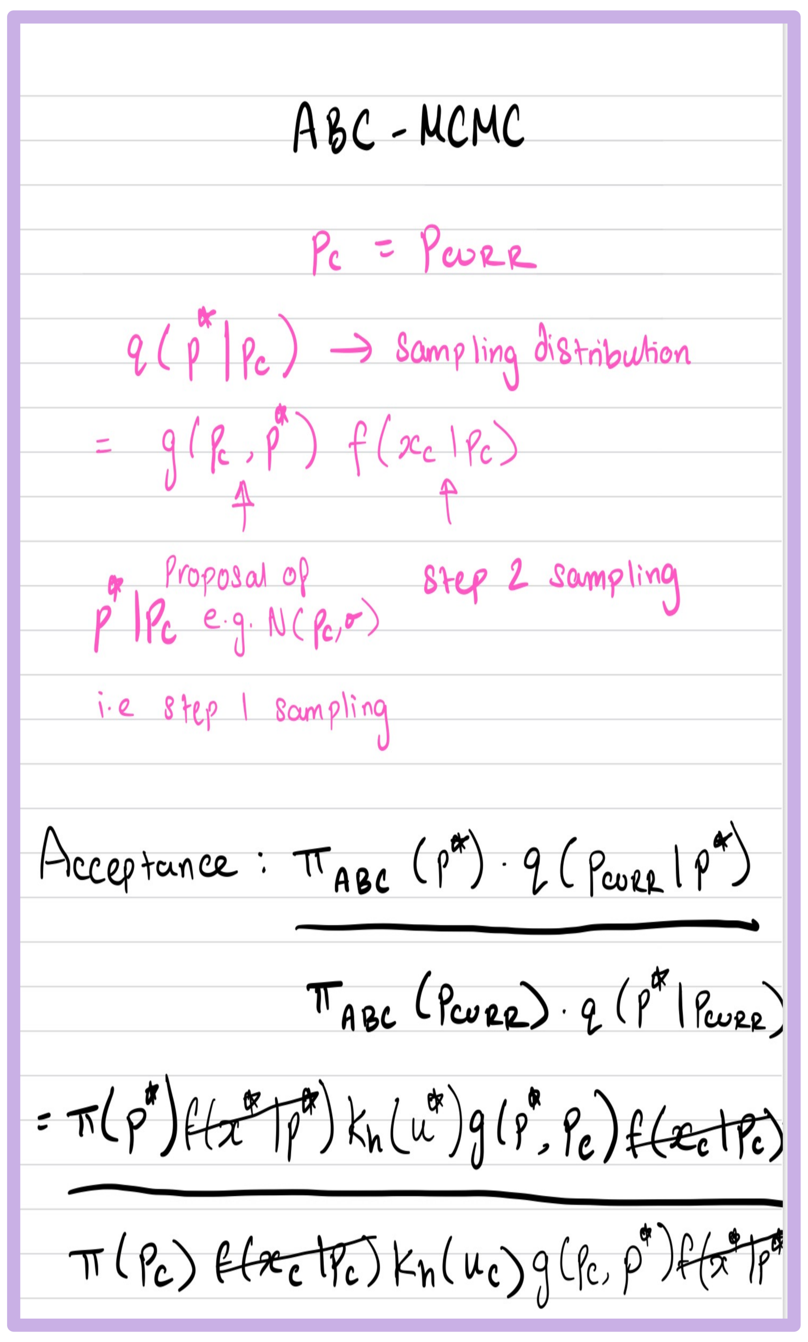
AIM: samples from the posterior , or close enough to the posterior (e.g., to estimate expectation of posterior).

ABC Rejection Sampling [1]

ABC Rejection Sampling [1], accept values 'close enough' to x_{obs} giving approximation to posterior :

1. $p \sim g(p)$
2. $x \sim f(.|p)$
3. Accept p with probability $\frac{\pi(p) K_h(||x - x_{obs}||)}{K g(p)}$, $K \geq K_h(0) \max_p \frac{\pi(p)}{g(p)}$
4. Repeat 1-3 until enough samples obtained

Rejection sampling is same, but step 3 is replaced by:
If $x = x_{obs}$, accept p with probability $\frac{\pi(p)}{K g(p)}$, $K \geq \max_p \frac{\pi(p)}{g(p)}$



ABC-MCMC [2]

Initialise:

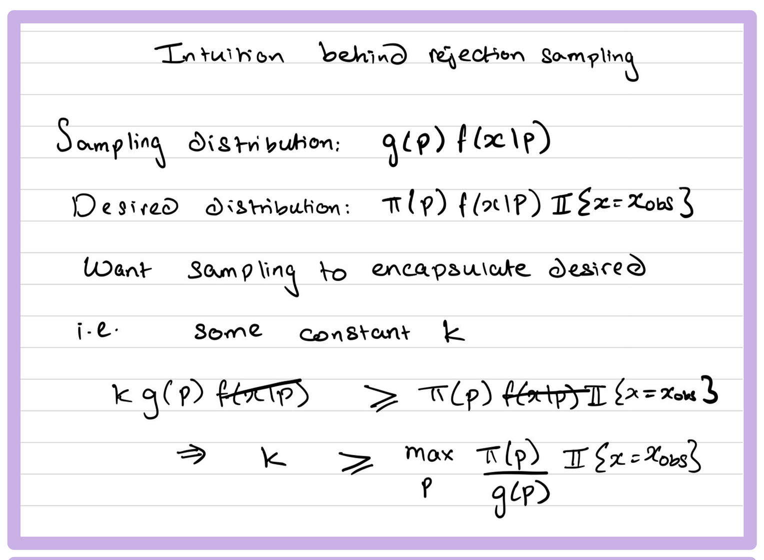
1. $p_{CURR} \sim \pi(p)$
2. $x_{CURR} \sim f(.|p_{CURR})$
3. $K_h(||x_{CURR} - x_{obs}||) > 0$, move to **sampling** else repeat 1-2

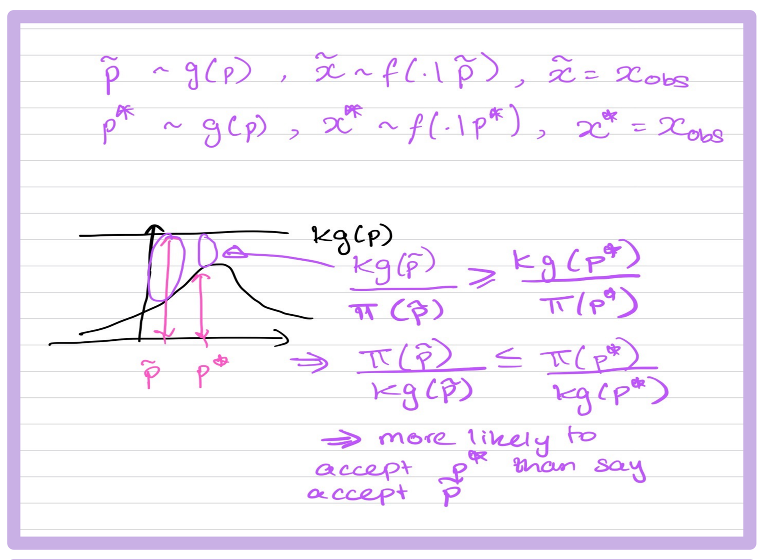
Sampling:

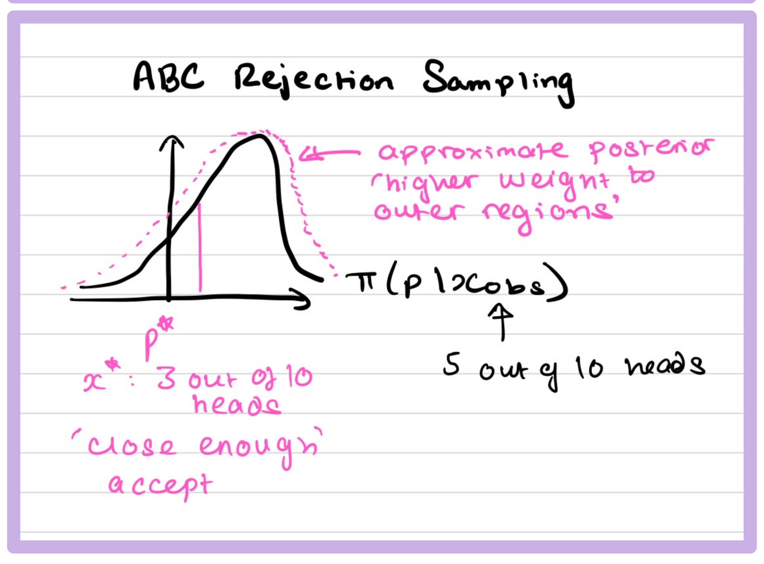
For $i = 1, \ldots, N$:

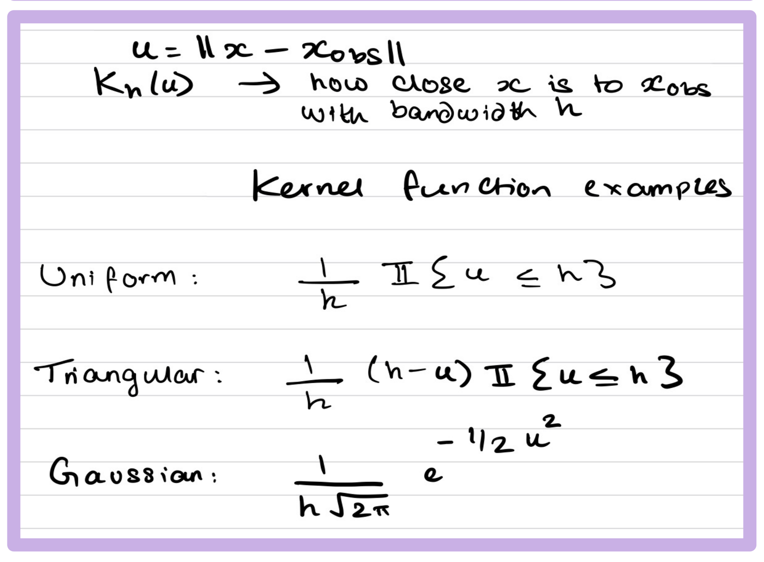
1. $p^* \sim g(p, p^*)$
2. $x^* \sim f(.|p^*)$
3. Accept p^* with probability

$\min\{1, \frac{\pi(p^*) K_h(||x^* - x_{obs}||) g(p^*, p_{CURR})}{\pi(p_{CURR}) K_h(||x_{CURR} - x_{obs}||) g(p_{CURR}, p^*)}\}$









ABC-MCMC - Findings

$g(p_{CURR}, p^*)$ is a normal truncated at 0 and 1, mean p_{CURR} with fixed variation 0.01

For smaller bandwidth: if end up in region of low posterior density, stuck there for a long time (sampled values nearby p_{CURR} more likely to be rejected)

Higher weight assigned to p_{CURR} , chain hence takes longer to converge

Larger bandwidth: easier to escape regions of low posterior density , however less accurate posterior achieves

Trade-off between convergence time and accuracy

ABC Rejection Sampling - Findings

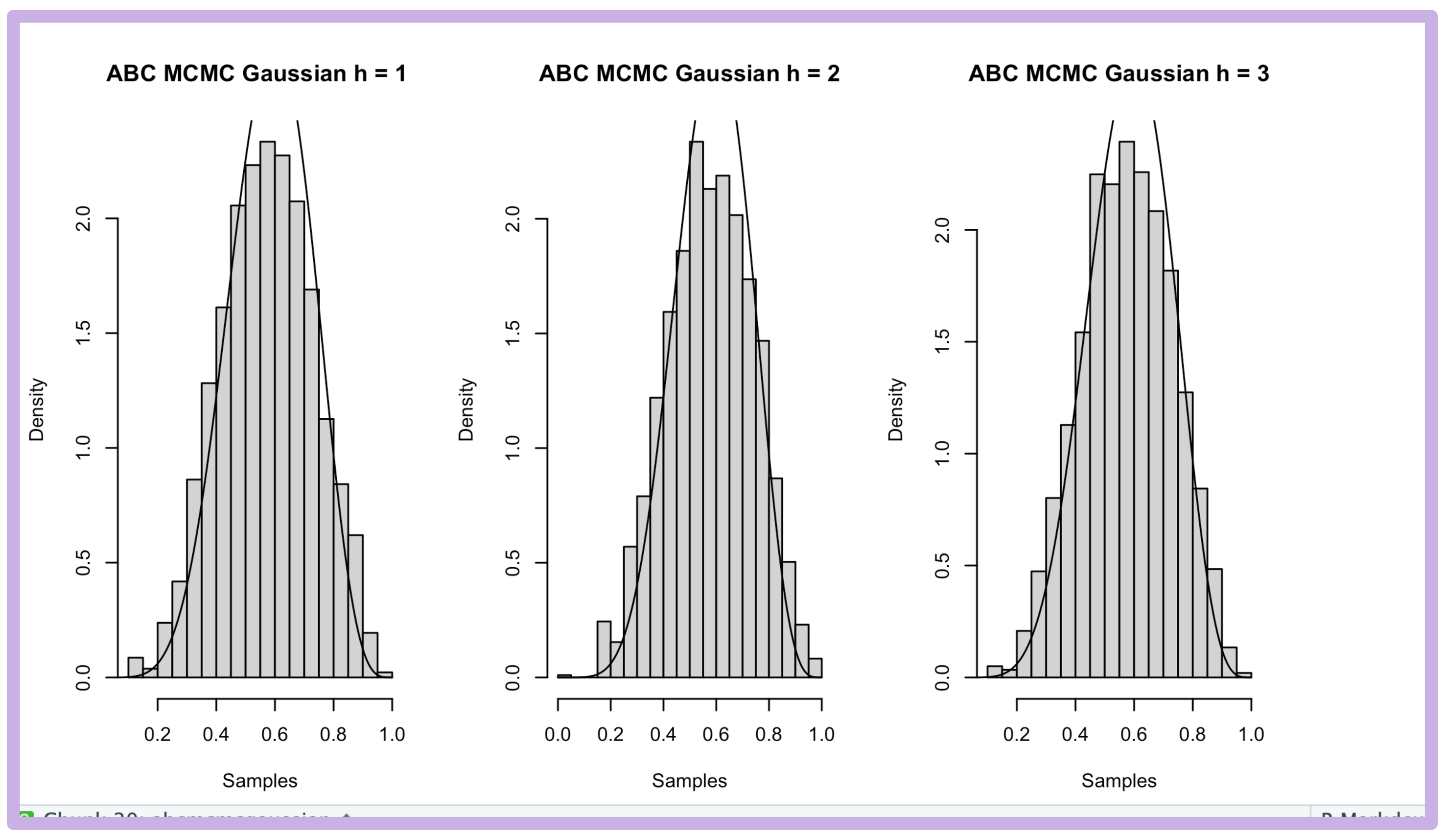
For $\pi(p) \sim Unif(0,1), g(p) \sim Unif(0,1)$, $K = K_h(0)$ in theory:

h = 1: uniform and triangular give samples from posterior , gaussian gives approximation to the posterior

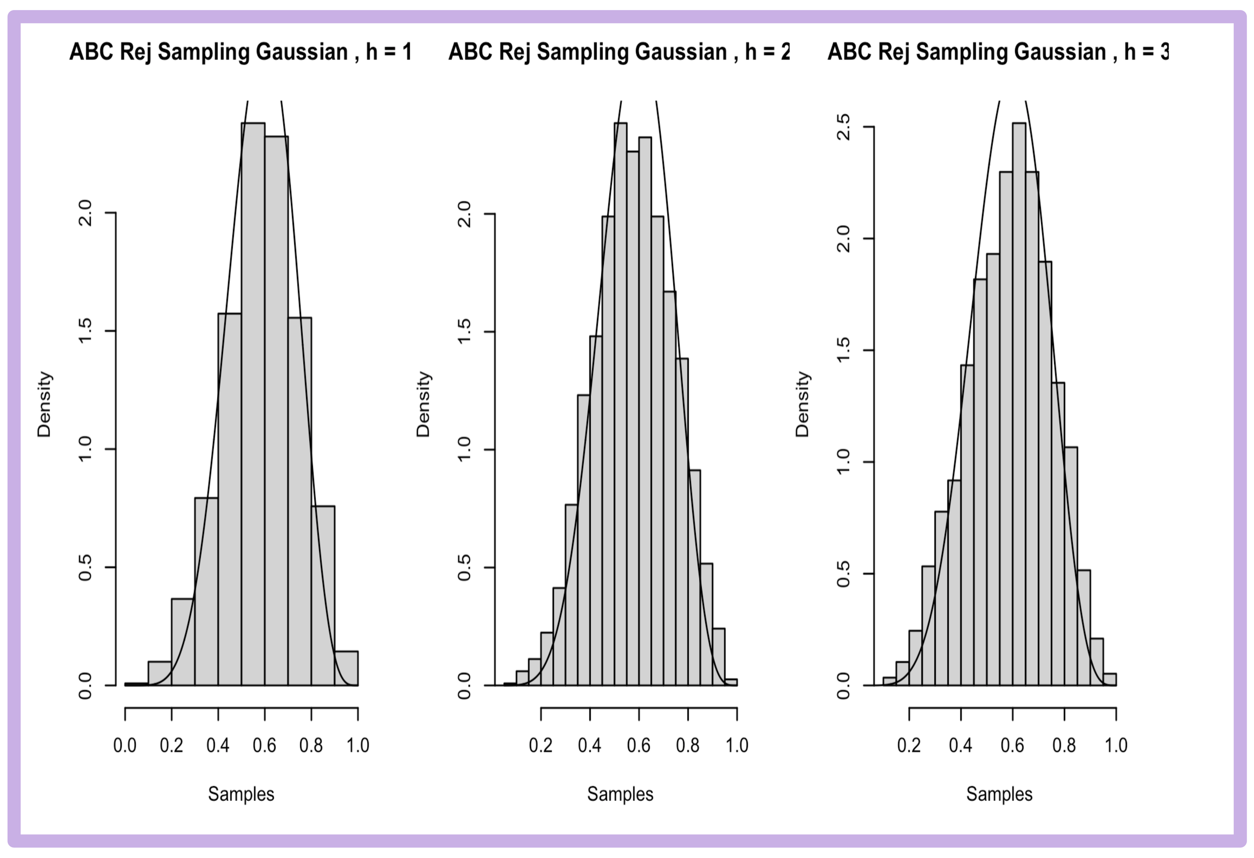
h = 2: triangular gives closest approximation to posterior , then gaussian then uniform

h = 3: gaussian gives closest approximation to posterior. , then triangular then uniform

For $K = K_h(0)$, gaussian kernel independent of bandwidth (so h = 1, 2 and 3 should give similar approximations to the posterior)



	mean	std
True	0.5833333	0.1367354
MCMC MH RW	0.5813443	0.1363355
ABC rejection sampling	0.5868384	0.1384273
ABC Uniform Kernel h=1	0.5835204	0.1497778
ABC Uniform Kernel h=2	0.5865621	0.1783927
ABC Uniform Kernel h=3	0.5783527	0.2125274
ABC Triangular Kernel h=1	0.5827805	0.1403692
ABC Triangular Kernel h=2	0.5813172	0.1526662
ABC Triangular Kernel h=3	0.5818682	0.1755546
ABC Gaussian Kernel h=1	0.5824817	0.1614007
ABC Guassian Kernel h=2	0.5797562	0.1612592
ABC Gaussian Kernel h=3	0.5848723	0.1562577
ABC MCMC Uniform Kernel h = 1	0.5771836	0.1518173
ABC MCMC Uniform Kernel h = 2	0.5870190	0.1451889
ABC MCMC Uniform Kernel h = 3	0.5987897	0.2131671
ABC MCMC Triangular Kernel h = 1	0.5821289	0.1221116
ABC MCMC Triangular Kernel h = 2	0.5794697	0.1429829
ABC MCMC Triangular Kernel h = 3	0.5987897	0.1704211
ABC MCMC Gaussian Kernel h = 1	0.5914076	0.1567081
ABC MCMC Gaussian Kernel h = 2	0.5876735	0.1624184
ABC MCMC Gaussian Kernel h = 3	0.5782485	0.1568744



[1] Sisson, S.A., Fan, Y. and Beaumont, M.A., 2018. Overview of ABC. In Handbook of approximate Bayesian computation (pp. 3-54). Chapman and Hall/CRC.
[2] Fan, Y. and Sisson, S.A., 2018. ABC samplers. arXiv preprint arXiv:1802.09650.
[3] Jeremy Orloff and Jonathan Bloom. Conjugate priors: Beta and normal. https://math.mit.edu/~dav/05.dir/class15-prep.pdf