Quantum Optics: Homework #3

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Problem 1

Show that

$$\frac{1}{2}\left\langle aa^\dagger + a^\dagger a\right\rangle = \int W(a,a^*)|\alpha|^2 \mathrm{d}^2\alpha$$

where $W(a,a^*)$ is the Wigner-Weyl distribution.
Solution

Proof. From the Eq. (3.B.7), we can easily get

$$\begin{aligned} O_S(\alpha,\alpha^*) &= \alpha^*\alpha - \frac{1}{2} \qquad \text{while} \;\; O(a,a^\dagger) = a^\dagger a, \\ O_S(\alpha,\alpha^*) &= \alpha^*\alpha + \frac{1}{2} \qquad \text{while} \;\; O(a,a^\dagger) = aa^\dagger, \end{aligned}$$

so that

$$O_S(\alpha,\alpha^*) = 2\alpha^*\alpha \qquad \text{while} \;\; O(a,a^\dagger) = aa^\dagger + a^\dagger a,$$

just meaning

$$\frac{1}{2}\left\langle aa^\dagger + a^\dagger a\right\rangle = \int W(a,a^*)|\alpha|^2 \mathrm{d}^2\alpha.$$

□

Problem 2

Show that

$$\begin{aligned}\mathrm{Tr}[D(\alpha)] &= \pi\delta^{(2)}(\alpha),\\ \mathrm{Tr}[D(\alpha)D^\dagger(\alpha')] &= \pi\delta^{(2)}(\alpha-\alpha'),\end{aligned}$$

where $D(\alpha)$ is the displacement operator. Using these results, show that

$$\begin{aligned}&\mathrm{Tr}\left[\Delta^{(\Omega)}(\alpha-a,\alpha^*-a^\dagger)\bar{\Delta}^{(\Omega)}(\alpha'-a,\alpha^{*\prime}-a^\dagger)\right]\\ &=\frac{1}{\pi}\delta^{(2)}(\alpha-\alpha'),\end{aligned}$$

the operators $\Delta^{(\Omega)}$ and $\bar{\Delta}^{(\Omega)}$ are defined by

$$\begin{aligned}\Delta^{(\Omega)}(\alpha-a,\alpha^*-a^\dagger) =&\frac{1}{\pi^2}\int\exp[\Omega(\beta,\beta^*)]\\ &\times\exp[-\beta(\alpha^*-a^\dagger)+\beta^*(\alpha-a)]\mathrm{d}^2\beta,\\ \bar{\Delta}^{(\Omega)}(\alpha-a,\alpha^*-a^\dagger) =&\frac{1}{\pi^2}\int\exp[-\Omega(\beta,\beta^*)]\\ &\times\exp[\beta(\alpha^*-a^\dagger)-\beta^*(\alpha-a)]\mathrm{d}^2\beta.\end{aligned}$$

Solution
Part One

Proof.

$$\begin{aligned}\mathrm{Tr}[D(\alpha)] &= \frac{1}{\pi}\int\mathrm{d}^2\beta\,\langle\beta|\,e^{-|\alpha|^2/2}e^{\alpha a^\dagger}e^{-\alpha^* a}\,|\beta\rangle\\ &=\frac{1}{\pi}e^{-|\alpha|^2/2}\int\mathrm{d}^2\beta e^{\alpha\beta^*-\alpha^*\beta}\\ &=\frac{1}{\pi}e^{-|\alpha|^2/2}\pi^2\delta^2(\alpha)\\ &=\pi\delta^2(\alpha).\end{aligned}$$

□

Part Two

Proof. At first, it is easy to get

$$\begin{aligned}D^\dagger(\alpha') &= e^{\alpha a^\dagger-\alpha^* a}e^{-\alpha' a^\dagger+\alpha^{*\prime} a}\\ &=e^{\alpha a^\dagger-\alpha^* a-\alpha' a^\dagger+\alpha^{*\prime} a}e^{(\alpha^*\alpha'-\alpha\alpha^{*\prime})/2}\\ &=D(\alpha-\alpha')e^{(\alpha^*\alpha'-\alpha\alpha^{*\prime})/2},\end{aligned}$$

so that

$$\begin{aligned}\mathrm{Tr}[D(\alpha)D^\dagger(a')] &= \pi\delta^2(\alpha-\alpha')e^{(\alpha^*\alpha'-\alpha\alpha^{*\prime})/2}\\ &=\pi\delta^2(\alpha-\alpha')\end{aligned}$$

□

Part Three

Proof.

$$
\begin{aligned}
&\mathrm{Tr}\left[\Delta^{(\Omega)}(\alpha-a,\alpha^*-a^\dagger)\bar{\Delta}^{(\Omega)}(\alpha'-a,\alpha^{*\prime}-a^\dagger)\right]\\
&=\mathrm{Tr}\left[\frac{1}{\pi^2}\int e^{\Omega(\beta_1,\beta_1^*)}e^{-\beta_1(\alpha^*-a^\dagger)+\beta_1^*(\alpha-a)}\mathrm{d}^2\beta_1\times\frac{1}{\pi^2}\int e^{\Omega(\beta_2,\beta_2^*)}e^{-\beta_2(\alpha^{*\prime}-a^\dagger)+\beta_2^*(\alpha'-a)}\mathrm{d}^2\beta_1\right]\\
&=\mathrm{Tr}\left[\frac{1}{\pi^4}\iint \mathrm{d}^2\beta_1\mathrm{d}^2\beta_2 e^{\Omega(\beta_1,\beta_1^*)-\Omega(\beta_2,\beta_2^*)}e^{\beta_1\alpha^*+\beta_1^*\alpha+\beta_2\alpha^{*\prime}-\beta_2^*\alpha'}e^{\beta_1a^\dagger-\beta_1^*a}e^{-\beta_2a^\dagger+\beta_2^*a}\right]\\
&=\frac{1}{\pi^4}\iint \mathrm{d}^2\beta_1\mathrm{d}^2\beta_2 e^{\Omega(\beta_1,\beta_1^*)-\Omega(\beta_2,\beta_2^*)}e^{\beta_1\alpha^*+\beta_1^*\alpha+\beta_2\alpha^{*\prime}-\beta_2^*\alpha'}\mathrm{Tr}\left[D(\beta_1)D^\dagger(\beta_2)\right]\\
&=\frac{1}{\pi^4}\iint \mathrm{d}^2\beta_1\mathrm{d}^2\beta_2 e^{\Omega(\beta_1,\beta_1^*)-\Omega(\beta_2,\beta_2^*)}e^{\beta_1\alpha^*+\beta_1^*\alpha+\beta_2\alpha^{*\prime}-\beta_2^*\alpha'}\pi\delta^2(\beta_1-\beta_2)\\
&=\frac{1}{\pi^3}\int \mathrm{d}^2\beta_1 e^{-\beta_1(\alpha^*-\alpha^{*\prime})+\beta_1^*(\alpha-\alpha')}\\
&=\frac{1}{\pi}\delta^2(\alpha-\alpha').
\end{aligned}
$$

□

Problem 3

Show that the expectation value of the number operator n and the ρ for a thermal field is given by

$$\langle n\rangle = \mathrm{Tr}(a^\dagger a\rho) = \left[\exp\left(\frac{\hbar\nu}{k_BT}\right)-1\right]^{-1}$$

$$\rho = \sum_m \frac{\langle n\rangle^m}{(1+\langle n\rangle)^{m+1}} \left|m\right\rangle\left\langle m\right|.$$

Solution

Proof. The density operator for the thermal field is

$$\rho = \frac{\exp(-\mathscr{H}/k_BT)}{\mathrm{Tr}\left[\exp(-\mathscr{H}/k_BT)\right]},$$

where $\mathscr{H} = \hbar\nu(a^\dagger a + 1/2)$ and where

$$\begin{aligned}\mathrm{Tr}\left[\exp(-\mathscr{H}/k_BT)\right] &= \sum_{n=0}^{\infty}\left\langle n\right|\exp(\mathscr{H}/k_BT)\left|n\right\rangle \\ &= \sum_{n=0}^{\infty}\exp(-E_n/k_BT) \equiv Z\end{aligned}$$

is the partition function. With $E_n = \hbar\nu(n+1/2)$

$$Z = \exp(-\hbar\nu/2k_BT)\sum_{n=0}^{\infty}\exp(-\hbar\nu n/k_BT).$$

Since $\exp(-\hbar\nu/k_BT) < 1$,the sum is a geometric series and thus

$$Z = \frac{\exp(-\hbar\nu/2k_BT)}{1-\exp(-\hbar\nu/k_BT)},$$

so note the density operator itself can be written as

$$\begin{aligned}\rho &= \sum_{nm}\left|n\right\rangle\left\langle n\right|\rho\left|m\right\rangle\left\langle m\right| \\ &= \frac{1}{Z}\sum_m \exp(-E_n/k_BT)\left|m\right\rangle\left\langle m\right| \\ &= \sum_m P_m\left|m\right\rangle\left\langle m\right|.\end{aligned}$$

The average photon number of the thermal field is calcultated as

$$\begin{aligned}\langle n\rangle = \mathrm{Tr}(n\rho) &= \sum_m\left\langle m\right|n\rho\left|m\right\rangle \\ &= \sum_m mP_m = \exp(-\hbar\nu/2k_BT)\frac{1}{Z}\sum_m m\exp(-\hbar\nu m/k_BT).\end{aligned}$$

Noting that with $x = \hbar\nu/k_BT$,we have

$$\begin{aligned}\sum_{m=0}^{\infty} me^{-mx} &= -\frac{\mathrm{d}}{\mathrm{d}x}\sum_{m=0}^{\infty}e^{-mx} \\ &= -\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{1-e^{-x}}\right) \\ &= \frac{e^{-x}}{(1-e^{-x})^2}.\end{aligned}$$

Problem 3 continued on next page. . .

Thus

$$\begin{aligned} \langle n \rangle &= \frac{\exp(-\hbar\nu/k_B T)}{1-\exp(-\hbar\nu/k_B T)} \\ &= \frac{1}{\exp(\hbar\nu/k_B T)-1}. \end{aligned} \tag{1}$$

So that

$$\exp(-\hbar\nu/k_B T) = \frac{\langle n \rangle}{1+\langle n \rangle}$$

Then the density operators can be written as

$$\rho = \sum_m \frac{\langle n \rangle^m}{(1+\langle n \rangle)^{m+1}} \left| m \right\rangle \left\langle m \right|. \tag{2}$$

□