Quantum Optics: Homework #1

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Problem 1

Show that

$$
\begin{aligned}
[a, e^{-\alpha a^\dagger a}] &= \left(e^{-\alpha} - 1\right) e^{-\alpha a^\dagger a} a, \\
[a^\dagger, e^{-\alpha a^\dagger a}] &= \left(e^{\alpha} - 1\right) e^{-\alpha a^\dagger a} a^\dagger,
\end{aligned}
$$

where α is a parameter.

Solution

Part One

$$
\begin{aligned}
[a, e^{-\alpha a^\dagger a}] &= [a, \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!} (a^\dagger a)^n] \\
&= \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!} \left[a, (a^\dagger a)^n\right] \\
&= \sum_{n=1}^{\infty} \frac{(-\alpha)^n}{n!} \left[a, (a^\dagger a)^n\right] \\
&= \sum_{n=1}^{\infty} \frac{(-\alpha)^n}{n!} \left[a (a^\dagger a)^{n-1} a^\dagger a - (a^\dagger a)^n a\right] \\
&= \sum_{n=1}^{\infty} \frac{(-\alpha)^n}{n!} \left[a (a^\dagger a)^{n-1} a^\dagger - (a^\dagger a)^n\right] a \\
&= \sum_{n=1}^{\infty} \frac{(-\alpha)^n}{n!} \left[(a a^\dagger)^n - (a^\dagger a)^n\right] a \\
&= (e^{-\alpha a a^\dagger} - e^{-\alpha a^\dagger a}) a \\
&= (e^{-\alpha(1 + a^\dagger a)} - e^{-\alpha a^\dagger a}) a \\
&= \left(e^{-\alpha} - 1\right) e^{-\alpha a^\dagger a} a;
\end{aligned}
$$

Part Two

$$
\begin{aligned}
[a^\dagger, e^{-\alpha a^\dagger a}] &= [a^\dagger, e^{-\alpha(a a^\dagger - 1)}] \\
&= e^{\alpha} [a^\dagger, e^{-\alpha a a^\dagger}] \\
&= e^{\alpha} \sum_{n=1}^{\infty} [a^\dagger (a a^\dagger)^n - (a a^\dagger)^n a^\dagger] \\
&= e^{\alpha} \sum_{n=1}^{\infty} [(a^\dagger a)^n - (a a^\dagger)^n] a^\dagger \\
&= e^{\alpha} \left(e^{-\alpha a^\dagger a} - e^{-\alpha a a^\dagger}\right) a^\dagger \\
&= e^{\alpha} \left(e^{-\alpha a^\dagger a} - e^{-\alpha(1 + a^\dagger a)}\right) a^\dagger \\
&= \left(e^{\alpha} - 1\right) e^{-\alpha a^\dagger a} a^\dagger.
\end{aligned}
$$

Problem 2

Show that the free-field Hamiltonian

$$\mathscr{H} = \hbar\nu\left(a^\dagger a + \frac{1}{2}\right)$$

can be written in terms of the number states as

$$\mathscr{H} = \sum_n E_n \left|n\right\rangle\left\langle n\right|,$$

and hence

$$e^{i\mathscr{H}t/\hbar} = \sum_n e^{iE_n t/\hbar}\left|n\right\rangle\left\langle n\right|.$$

Proof. In the number representation,
the matrix elements of $\mathscr{H} = \hbar\nu\left(a^\dagger a + 1/2\right)$ is

$$\mathscr{H}_{mn} = \left\langle m\right|\mathscr{H}\left|n\right\rangle = \hbar\nu\left(n + \frac{1}{2}\right)\delta_{mn} = E_n\delta_{mn},$$

and the matrix elements of $\mathscr{H} = \sum_n E_n \left|n\right\rangle\left\langle n\right|$ is

$$\mathscr{H}_{mk} = \sum_n E_n \left\langle m|n\right\rangle\left\langle n|k\right\rangle = E_k\delta_{km},$$

so that the free-field Hamiltonian $\mathscr{H} = \hbar\nu\left(a^\dagger a + 1/2\right)$ can be written in terms of the number states as $\mathscr{H} = \sum_n E_n \left|n\right\rangle\left\langle n\right|$.
As for $\exp\left(i\mathscr{H}t/\hbar\right)$, we can see that

$$\mathscr{H}^m = (\sum_n E_n\left|n\right\rangle\left\langle n\right|)^m = \sum_n E_n^m\left|n\right\rangle\left\langle n\right|.$$

Therefore,

$$\begin{aligned} e^{i\mathscr{H}t/\hbar} &= \sum_{m=0}^{\infty}\frac{(it/\hbar)^m}{m!}\mathscr{H}^m \\ &= \sum_{m=0}^{\infty}\sum_n\frac{(it/\hbar)^m}{m!}E_n^m\left|n\right\rangle\left\langle n\right| \\ &= \sum_n e^{iE_n t/\hbar}\left|n\right\rangle\left\langle n\right|. \end{aligned}$$

□