Quantum Optics: Homework #2

Due on May 11, 2018

Professor Pei Zhang

Shihao Ru

Problem 1

Show that

$$a^\dagger |\alpha\rangle \langle\alpha| = \left(\alpha^* + \frac{\partial}{\partial\alpha}\right) |\alpha\rangle \langle\alpha| \,,$$

and

$$|\alpha\rangle \langle\alpha| \, a = \left(\alpha + \frac{\partial}{\partial\alpha^*}\right) |\alpha\rangle \langle\alpha| \,.$$

Solution

At first, we know that

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle = e^{-|\alpha|^2/2} e^{\alpha a^\dagger} |0\rangle \,,$$

so that

$$\begin{aligned} |\alpha\rangle \langle\alpha| &= e^{-|\alpha|^2/2} e^{\alpha a^\dagger} |0\rangle \langle 0| e^{\alpha^* a} e^{-|\alpha|^2/2} \\ &= e^{-\alpha^*\alpha} e^{\alpha a^\dagger} |0\rangle \langle 0| e^{\alpha^* a}. \end{aligned}$$

Then make the α (or α^*) derivation of the upper-formula,

$$\begin{aligned} \frac{\partial}{\partial\alpha} (|\alpha\rangle \langle\alpha|) &= \frac{\partial}{\partial\alpha} \left(e^{-\alpha^*\alpha} e^{\alpha a^\dagger} |0\rangle \langle 0| e^{\alpha^* a}\right) \\ &= \left(-\alpha^* + a^\dagger\right) \left(e^{-\alpha^*\alpha} e^{\alpha a^\dagger} |0\rangle \langle 0| e^{\alpha^* a}\right) \\ &= \left(-\alpha^* + a^\dagger\right) (|\alpha\rangle \langle\alpha|) \,, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial\alpha^*} (|\alpha\rangle \langle\alpha|) &= \frac{\partial}{\partial\alpha^*} \left(e^{-\alpha^*\alpha} e^{\alpha a^\dagger} |0\rangle \langle 0| e^{\alpha^* a}\right) \\ &= \left(-\alpha^* + a^\dagger\right) \left(e^{-\alpha^*\alpha} e^{\alpha a^\dagger} |0\rangle \langle 0| e^{\alpha^* a}\right) \\ &= (-\alpha + a) (|\alpha\rangle \langle\alpha|) \,. \end{aligned}$$

So we prove the

$$a^\dagger |\alpha\rangle \langle\alpha| = \left(\alpha^* + \frac{\partial}{\partial\alpha}\right) |\alpha\rangle \langle\alpha| \,,$$

and

$$|\alpha\rangle \langle\alpha| \, a = \left(\alpha + \frac{\partial}{\partial\alpha^*}\right) |\alpha\rangle \langle\alpha| \,.$$

Problem 2

Show that the following useful unitary transformation properties of the squeezed operator

$$S^\dagger(\xi)aS(\xi) = a\cosh r - a^\dagger e^{i\theta}\sinh r,$$
$$S^\dagger(\xi)a^\dagger S(\xi) = a^\dagger\cosh r - ae^{-i\theta}\sinh r,$$

where $\xi = r\exp(i\theta)$.

Proof. We have proved the Baket-Hausdorf lemma:

$$e^A Be^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \cdots$$

So that

$$\begin{aligned} S^\dagger(\xi)aS(\xi) &= S(-\xi)aS(\xi) \\ &= \exp\left(\frac{1}{2}\xi a^{\dagger 2} - \frac{1}{2}\xi^* a^2\right) a \exp\left(\frac{1}{2}\xi^* a^2 - \frac{1}{2}\xi a^{\dagger 2}\right) \\ where\ A &= \frac{1}{2}\xi a^{\dagger 2} - \frac{1}{2}\xi^* a^2, B = a. \end{aligned}$$

Now, I caculate the previous items:

$$\begin{aligned} [A, B] &= [\frac{1}{2}\left(\xi a^{\dagger 2} - \xi^* a^2\right), a] \\ &= \frac{1}{2}\xi[a^{\dagger 2}, a] = -a^\dagger\xi, \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2!}[A, [A, B]] &= \frac{1}{2}[A, -a^\dagger\xi] \\ &= \frac{1}{4}|\xi|^2[a^2, a^\dagger] \\ &= \frac{1}{2}|\xi|^2 a = \frac{1}{2}r^2 a. \end{aligned}$$

Then, we can get

$$\frac{1}{(2k-1)!}\underbrace{[A, \cdots [A}_{2k-1}, B]\cdots] = -a^\dagger e^{i\theta}\frac{(-1)^{k-1}}{(2k-1)!}r^{2k-1},$$

and

$$\frac{1}{2k!}\underbrace{[A, \cdots [A}_{2k}, B]\cdots] = a\frac{1}{(2k)!}r^{2k}.$$

So that

$$\begin{aligned} S^\dagger(\xi)aS(\xi) &= S(-\xi)aS(\xi) \\ &= a\sum_k a\frac{1}{(2k)!}r^{2k} - a^\dagger\sum_k e^{i\theta}\frac{(-1)^{k-1}}{(2k-1)!}r^{2k-1} \\ &= a\cosh r - a^\dagger e^{i\theta}\sinh r, \end{aligned}$$

and in the same way, it's easy to get

$$S^\dagger(\xi)a^\dagger S(\xi) = a^\dagger\cosh r - ae^{-i\theta}\sinh r.$$

□

Problem 3

A squeeze coherent state $|\alpha,\xi\rangle$ has two different definition, one is obtained by first acting with the displacement operator $D(\alpha)$ on the vacuum followed by the squeeze operactor $S(\xi)$, i.e.,

$$|\alpha,\xi\rangle = S(\xi)D(\alpha)\left|0\right\rangle,$$

another one is obtained by first acting with the squeeze operactor $S(\xi)$ on the vacuum followed by the displacement operator $D(\alpha)$, i.e.,

$$|\alpha,\xi\rangle = D(\alpha)S(\xi)\left|0\right\rangle,$$

calculate the $\langle X_1\rangle$ and $\langle X_2\rangle$ of them.

Solution

Part One

If the squeeze coherent is defined by $|\alpha,\xi\rangle = S(\xi)D(\alpha)\left|0\right\rangle$, we can know

$$\begin{aligned}
\langle X_1\rangle &= \langle\alpha,\xi|\, X_1 \,|\alpha,\xi\rangle = \langle 0|\, D^\dagger(\alpha)S^\dagger(\xi)X_1S(\xi)D(\alpha)\,|0\rangle \\
&= \langle 0|\, D^\dagger(\alpha)S^\dagger(\xi)\frac{1}{2}\left(a+a^\dagger\right)S(\xi)D(\alpha)\,|0\rangle \\
&= \frac{1}{2}\langle\alpha|\, a\cosh r - a^\dagger e^{i\theta}\sinh r + a^\dagger\cosh r - ae^{-i\theta}\sinh r\,|\alpha\rangle \\
&= \frac{1}{2}\left((\alpha+\alpha^*)\cosh r - \left(\alpha^* e^{i\theta} + \alpha e^{-i\theta}\right)\sinh r\right),
\end{aligned}$$

and

$$\begin{aligned}
\langle X_2\rangle &= \langle\alpha,\xi|\, X_1 \,|\alpha,\xi\rangle = \langle 0|\, D^\dagger(\alpha)S^\dagger(\xi)X_2S(\xi)D(\alpha)\,|0\rangle \\
&= \langle 0|\, D^\dagger(\alpha)S^\dagger(\xi)\frac{1}{2i}\left(a-a^\dagger\right)S(\xi)D(\alpha)\,|0\rangle \\
&= \frac{1}{2i}\langle\alpha|\, a\cosh r - a^\dagger e^{i\theta}\sinh r - a^\dagger\cosh r + ae^{-i\theta}\sinh r\,|\alpha\rangle \\
&= \frac{1}{2i}\left((\alpha-\alpha^*)\cosh r - \left(\alpha^* e^{i\theta} - \alpha e^{-i\theta}\right)\sinh r\right).
\end{aligned}$$

Part Two

But if the squeeze coherent is defined by $|\alpha,\xi\rangle = D(\alpha)S(\xi)\left|0\right\rangle$, we can know

$$\begin{aligned}
\langle X_1\rangle &= \langle\alpha,\xi|\, X_1 \,|\alpha,\xi\rangle = \langle 0|\, S^\dagger(\xi)D^\dagger(\alpha)X_1D(\alpha)S(\xi)\,|0\rangle \\
&= \langle 0|\, S^\dagger(\xi)D^\dagger(\alpha)\frac{1}{2}\left(a+a^\dagger\right)D(\alpha)S(\xi)\,|0\rangle \\
&= \frac{1}{2}\langle 0|\, S^\dagger(\xi)(a+\alpha+a^\dagger+\alpha^*)S(\xi)\,|0\rangle \\
&= \frac{1}{2}\langle 0|\,(a\cosh r - a^\dagger e^{i\theta}\sinh r + a^\dagger\cosh r - ae^{-i\theta}\sinh r + \alpha + \alpha^*)\,|0\rangle \\
&= \frac{1}{2}(\alpha+\alpha^*),
\end{aligned}$$

and

$$\begin{aligned}
\langle X_2\rangle &= \langle\alpha,\xi|\, X_2 \,|\alpha,\xi\rangle = \langle 0|\, S^\dagger(\xi)D^\dagger(\alpha)X_2D(\alpha)S(\xi)\,|0\rangle \\
&= \langle 0|\, S^\dagger(\xi)D^\dagger(\alpha)\frac{1}{2i}\left(a-a^\dagger\right)D(\alpha)S(\xi)\,|0\rangle \\
&= \frac{1}{2i}\langle 0|\, S^\dagger(\xi)(a+\alpha-a^\dagger-\alpha^*)S(\xi)\,|0\rangle \\
&= \frac{1}{2i}\langle 0|\,(a\cosh r - a^\dagger e^{i\theta}\sinh r - a^\dagger\cosh r + ae^{-i\theta}\sinh r + \alpha - \alpha^*)\,|0\rangle \\
&= \frac{1}{2i}(\alpha-\alpha^*).
\end{aligned}$$

Problem 4

Consider a two-mode squeezed state defined by

$$|\alpha_1, \alpha_2, \xi\rangle = D_1(\alpha_1)D_2(\alpha_2)S_{12}(\xi)\,|0\rangle\,,$$

where

$$D_i(\alpha_i) = \exp(\alpha_i a_i^\dagger - \alpha_i^* a_i) \quad (i = 1, 2),$$

is the coherent displacement operator for the two modes described by destruction and creation operators a_i and $a_i^\dagger$, respectively,

$$S_{12}(\xi) = \exp(\xi^* a_1 a_2 - \xi a_1^\dagger a_2^\dagger)$$

is the two-mode squeeze operator, and $|0\rangle$ is the two-mode vacuum state. Show that there is no squeezing in the two individual modes.

Proof. It is easy to get

$$S_{12}(-\xi) = S_{12}^\dagger(\xi) = S_{12}^{-1}(\xi),$$

so that

$$S_{12}^\dagger a_i S_{12} = a_i \cosh r - a_j^\dagger e^{i\theta} \sinh r, \quad i, j \in \{\, 1, 2 \,\} \ \textit{and}\ i \neq j.$$

Then need to calculate $\langle a_i \rangle$, $\langle a_i^2 \rangle$, and $\langle a_i^\dagger a_i \rangle$.

$$\begin{aligned}
\langle a_i \rangle &= \langle \alpha_1, \alpha_2, \xi |\, a_i \,| \alpha_1, \alpha_2, \xi \rangle \\
&= \langle 0 |\, S_{12}^\dagger D_2^\dagger(\alpha_1) D_1^\dagger(\alpha_1) a_i D_1(\alpha_1) D_2(\alpha_2) S_{12}(\xi)\, |0\rangle \\
&= \langle 0 |\, S_{12}^\dagger(\xi)(a_i + \alpha_i) S_{12}(\xi)\, |0\rangle \\
&= \alpha_i + \langle 0 |\, S_{12}^\dagger(\xi) a_i S_{12}(\xi)\, |0\rangle \\
&= \alpha_i, \\
\langle a_i^2 \rangle &= \langle \alpha_1, \alpha_2, \xi |\, a_i^2 \,| \alpha_1, \alpha_2, \xi \rangle \\
&= \langle 0 |\, S_{12}^\dagger D_2^\dagger(\alpha_1) D_1^\dagger(\alpha_1) a_i^2 D_1(\alpha_1) D_2(\alpha_2) S_{12}(\xi)\, |0\rangle \\
&= \langle 0 |\, S_{12}^\dagger \left(a_i + \alpha_i\right)^2 S_{12}\, |0\rangle \\
&= \alpha_i^2, \\
\left\langle a_i a_i^\dagger \right\rangle &= \cosh^2 r + |\alpha_i|^2, \\
\left\langle a_i a_i^\dagger \right\rangle &= \sinh^2 r + |\alpha_i|^2.
\end{aligned}$$

So that

$$\begin{aligned}
\langle \Delta Y_1^{(i)} \rangle^2 &= \left\langle \left(\frac{a_i \exp(-i\theta/2) + a_i^\dagger \exp(i\theta/2)}{2} \right)^2 \right\rangle - \left\langle \frac{a_i \exp(-i\theta/2) + a_i^\dagger \exp(i\theta/2)}{2} \right\rangle^2 \\
&= \frac{1}{4}(2\sinh^2 r + 1) \geq \frac{1}{4}, \\
\langle \Delta Y_2^{(i)} \rangle^2 &= \frac{1}{4}(2\sinh^2 r + 1) \geq \frac{1}{4},
\end{aligned}$$

which means there is no squeezing in the two individual modes.

□

Problem 5

Consider the Hermitian operators corresponding to the real and imaginary parts of the square of the complex amplitude of the field

$$X_1 = \frac{1}{2}\left(a^2 + a^{\dagger 2}\right),$$
$$X_2 = \frac{1}{2i}\left(a^2 - a^{\dagger 2}\right).$$

Show that the squeezing condition is

$$\left\langle \Delta X_i^2 \right\rangle < \left\langle a^\dagger a \right\rangle + \frac{1}{2} \quad (i = 1 \text{ or } 2).$$

This type of squeezing is called amplitude-squared squeezing. Show that the amplitude-squared squeezing is a nonclassical effect.

Proof.

$$\begin{aligned}
[X_1, X_2] &= \frac{1}{4i}[a^2 + a^{\dagger 2}, a^2 - a^{\dagger 2}] \\
&= \frac{1}{4i}\left([a^2, -a^{\dagger 2}] + [a^{\dagger 2}, a^2]\right) \\
&= \frac{1}{2i}[a^{\dagger 2}, a^2] \\
&= \frac{1}{2i}\left(a^\dagger [a^\dagger, a^2] + [a^\dagger, a^2] a^\dagger\right) \\
&= \frac{1}{2i}\left(-2a^\dagger a - 2aa^\dagger\right) \\
&= i(a^\dagger a + aa^\dagger) \\
&= 2i\left(a^\dagger a + \frac{1}{2}\right),
\end{aligned}$$

So the squeezing condition is

$$\left\langle \Delta X_i^2 \right\rangle < \left\langle a^\dagger a \right\rangle + \frac{1}{2} \quad (i = 1 \text{ or } 2).$$

Since it appears the average photon number in the above formula, which is a nonclassical effect. □