Quantum Optics: Homework #4

(Date: May 28, 2018)

Professor Pei Zhang

Shihao Ru

Problem 1

As for the thermal state, proof

$$\left\langle n^2\right\rangle = 2\left\langle n\right\rangle^2 + \left\langle n\right\rangle,$$

where

$$\left\langle n\right\rangle = \frac{1}{e^{\hbar\nu/k_BT}-1}.$$

Proof. In the chapter 3, we have finished to the proof about

$$\begin{aligned}\rho &= \sum_m \frac{\left\langle n\right\rangle^m}{(1+\left\langle n\right\rangle)^{m+1}}\left|m\right\rangle\left\langle m\right| \\ &= \sum_m \frac{\exp(\hbar\nu/k_BT)-1}{\exp((m+1)\hbar\nu/k_BT)}\left|m\right\rangle\left\langle m\right|\end{aligned}$$

so that

$$\begin{aligned}2\left\langle n\right\rangle^2 + \left\langle n\right\rangle &= 2\left(\frac{1}{\exp(\hbar\nu/k_BT)-1}\right)^2 + \frac{1}{\exp(\hbar\nu/k_BT)-1} \\ &= \frac{\exp(\hbar\nu/k_BT)+1}{\left[\exp(\hbar\nu/k_BT)-1\right]^2},\end{aligned}$$

and

$$\begin{aligned}\left\langle n^2\right\rangle &= \mathrm{Tr}\big(n^2\rho\big) = \mathrm{Tr}\big(a^\dagger a\rho a^\dagger a\big) \\ &= \mathrm{Tr}\left(\sum_m a^\dagger a\frac{\exp(\hbar\nu/k_BT)-1}{\exp\left((m+1)\hbar\nu/k_BT\right)}\left|m\right\rangle\left\langle m\right| a^\dagger a\right) \\ &= \mathrm{Tr}\left(\sum_m \frac{\exp(\hbar\nu/k_BT)-1}{\exp\left((m+1)\hbar\nu/k_BT\right)}m^2\left|m\right\rangle\left\langle m\right|\right) \\ &= \sum_m \left\langle m\right| m^2\frac{\exp(\hbar\nu/k_BT)-1}{\exp\left((m+1)\hbar\nu/k_BT\right)}\left|m\right\rangle \\ &= \sum_m m^2\frac{\exp(\hbar\nu/k_BT)-1}{\exp\left((m+1)\hbar\nu/k_BT\right)},\end{aligned}$$

here we using the

$$\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA) = \mathrm{Tr}(CAB)$$

Noting $x = \hbar\nu/k_BT$, we have

$$\left\langle n^2\right\rangle = \frac{e^x-1}{e^x}\sum_m m^2 e^{-mx},$$

which is easy calculational by Mathemctica

```
Sum[m^2 × e^(-m×x), {m, 0, Infinity}]
 求和                      无穷大
```

$$\frac{e^x\,(1+e^x)}{(-1+e^x)^3}$$

.

So that

$$\left\langle n^2\right\rangle = \frac{\exp(\hbar\nu/k_BT)+1}{\left[\exp(\hbar\nu/k_BT)-1\right]^2},$$

just meaning

$$\left\langle n^2\right\rangle = 2\left\langle n\right\rangle^2 + \left\langle n\right\rangle.$$

□

Problem 2

Show that the radition field state which is a linear superposition of the vacuum state and a single photon state, i.e.,

$$|\psi\rangle = a_0 |0\rangle + a_1 |1\rangle ,$$

where a_0 and a_1 are complex coefficients, is a nonclassical state.

Proof. From the eqs. (4.2.21), we know that

$$g^{(2)}(\tau) = \frac{\left\langle a^\dagger(t) a^\dagger(t+\tau) a(t+\tau) a(t) \right\rangle}{\left\langle a^\dagger a \right\rangle^2},$$

so that

$$\begin{aligned} \left\langle a^\dagger a^\dagger a a \right\rangle &= \left(\langle 0| a_0^* + \langle 1| a_1^* \right) a^\dagger a^\dagger a a \left(a_0 |0\rangle + a_1 |1\rangle \right) \\ &= 0, \end{aligned}$$

just meaning that

$$g^{(2)}(0) = 0 < 1,$$

so it is a nonclassical state. $\square$