2/29/20

$$\sum_{u_1,u_2}\Big(\sum_\alpha A^{u_1}_\alpha B^{u_2}_\alpha - \sum_\beta C^{u_1}_\beta D^{u_2}_\beta\Big)^2$$

u: d phys. dim
α: D_1 bond dim
β: D_2 bond dim

$$= \sum_{u_1 u_2}\Big(\sum_{\alpha\alpha'} A^{u_1}_\alpha B^{u_2}_\alpha A^{u_1}_{\alpha'} B^{u_2}_{\alpha'} - 2\sum_{\alpha\beta} A^{u_1}_\alpha B^{u_2}_\alpha C^{u_1}_\beta D^{u_2}_\beta$$

$$+ \underbrace{\sum_{\beta\beta'} C^{u_1}_\beta D^{u_2}_\beta C^{u_1}_{\beta'} D^{u_2}_{\beta'}}\Big)$$

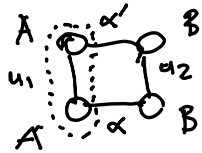
Dropping the last term because it is constant, and reordering the summations gives two terms:

$$\sum_{u_2\alpha\alpha'} B^{u_2}_\alpha\Big[\sum_{u_1} A^{u_1}_\alpha A^{u_1}_{\alpha'}\Big] B^{u_2}_{\alpha'} + \sum_{u_2\alpha\beta} B^{u_2}_\alpha\Big[\sum_{u_1} A^{u_1}_\alpha C^{u_1}_\beta\Big] D^{u_2}_\beta$$

The number of contractions required for each sum depends on the order in which the summations are taken.

The first term can be computed in 3! different ways:

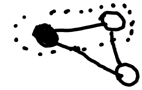
- $\displaystyle\sum_{u_2}\sum_{\alpha}\sum_{\alpha'} B^{u_2}_{\alpha}\left[\sum_{u_1} A^{u_1}_{\alpha} A^{u_1}_{\alpha'}\right] B^{u_2}_{\alpha'}$



$d \times D_1^2$

$+$

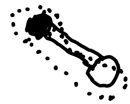
$\displaystyle = \sum_{u_2}\sum_{\alpha}\sum_{\alpha'} B^{u_2}_{\alpha} M_{\alpha\alpha'} B^{u_2}_{\alpha'}$



$D_1 \times (D_1 d)$

$+$

$\displaystyle = \sum_{u_2}\sum_{\alpha} B^{u_2}_{\alpha} M^{u_2}_{\alpha}$



$D_1 \times d$

$+$

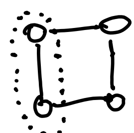
$\displaystyle = \sum_{u_2} M^{u_2 u_2}$

d

$= M$

$= 2dD_1^2 + dD_1 + d$

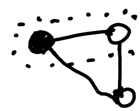
- $\displaystyle\sum_{\alpha}\sum_{u_2}\sum_{\alpha'} B^{u_2}_{\alpha}\left[\sum_{u_1} A^{u_1}_{\alpha} A^{u_1}_{\alpha'}\right] B^{u_2}_{\alpha'}$



$d \times D_1^2$

$+$

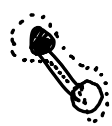
$\displaystyle = \sum_{\alpha}\sum_{u_2}\sum_{\alpha'} B^{u_2}_{\alpha} M_{\alpha\alpha'} B^{u_2}_{\alpha'}$



$D_1 \times (d D_1)$

$+$

$\displaystyle = \sum_{\alpha}\sum_{u_2} B^{u_2}_{\alpha} M^{u_2}_{\alpha}$



$d \times D_1$

$+$

$\displaystyle = \sum_{\alpha} M_{\alpha\alpha}$

D_1

$= M$

$= 2dD_1^2 + dD_1 + D_1$

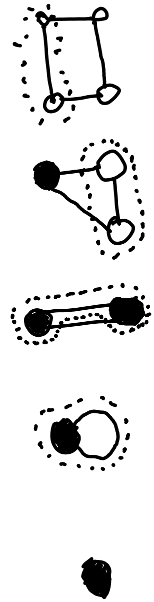
$$\sum_{\alpha}\sum_{\alpha'}\sum_{u_2} B^{u_2}_{\alpha}\left[\sum_{u_1} A^{u_1}_{\alpha} A^{u_1}_{\alpha'}\right] B^{u_2}_{\alpha'}$$

$$= \sum_{\alpha}\sum_{\alpha'}\sum_{u_2} B^{u_2}_{\alpha} M_{\alpha\alpha'} B^{u_2}_{\alpha'}$$

$$= \sum_{\alpha}\sum_{\alpha'} M^{(1)}_{\alpha\alpha'} M^{(2)}_{\alpha\alpha'}$$

$$= \sum_{\alpha} M_{\alpha\alpha}$$

$$= M$$

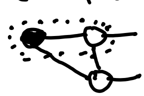
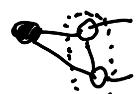


$$d\times D_1^2 + d\times (D_1^2) + D_1\times D_1 + D_1$$

$$= 2dD_1^2 + D_1^2 + D_1$$

The other rearrangements ($\sum_{\alpha'}\sum_{\alpha}\sum_{u_2}$, $\sum_{u_2}\sum_{\alpha'}\sum_{\alpha'}$, etc.) can be obtained by swapping $\alpha \leftrightarrow \alpha'$ in these calculations, so they lead to the same number of contractions. That might not be true for the second term, but.. probably is.

what's important to notice is that the leading order contribution from the second step onwards (which is the step which matters when generalizing to larger N) is always of order dD_1^2 no matter in which order the contractions are taken.

With more than 2 sites, the contraction scheme <u>does</u> matter:  $= dD_1^3$, but  $= dD_1^4$. If the bond dims aren't equal, it becomes $dD_1^2 D_2$ vs $dD_1^2 D_2^2$ instead.

The 3-site calculation has cost function

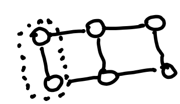
$$\sum_{u_1u_2u_3}\Big(\sum_{\alpha_1\alpha_2} A^{(1)u_1}_{1\alpha_1} A^{(2)u_2}_{\alpha_1\alpha_2} A^{(3)u_3}_{\alpha_2\alpha_3} - \sum_{\beta_1\beta_2} B^{(1)u_1}_{1\beta_1} B^{(2)u_2}_{\beta_1\beta_2} B^{(3)u_3}_{\beta_2 1}\Big)^2$$

$$= \sum_{u_i}\Big(\sum_{\alpha_i\alpha_i'} A^{(1)u_1}_{1\alpha_1} A^{(2)u_2}_{\alpha_1\alpha_2} A^{(3)u_3}_{\alpha_2 1} A^{(1)u_1}_{1\alpha_1'} A^{(2)u_2}_{\alpha_1'\alpha_2'} A^{(3)u_3}_{\alpha_2' 1}$$

$$- 2\sum_{\alpha_i\beta_i} A^{(1)u_1}_{1\alpha_1} A^{(2)u_2}_{\alpha_1\alpha_2} A^{(3)u_3}_{\alpha_2 1} B^{(1)u_1}_{1\beta_1} B^{(2)u_2}_{\beta_1\beta_2} B^{(3)u_3}_{\beta_2 1}\Big) + \text{const.}$$

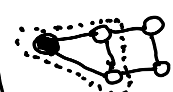
The second term can be computed efficiently as:

$$\sum_{u_3\alpha_2\beta_2} A^{(3)u_3}_{\alpha_2 1}\Big[\sum_{u_2\alpha_1\beta_1} A^{(2)u_2}_{\alpha_1\alpha_2}\Big[\underbrace{\sum_{u_1} A^{(1)u_1}_{1\alpha_1} B^{(1)u_1}_{1\beta_1}}_{M_{\alpha_1\beta_1},\ \mathcal{O}(dD_1D_2)}\Big] B^{(2)u_2}_{\beta_1\beta_2}\Big] B^{(3)u_3}_{\beta_3 1}$$



$+$

$M_{\alpha_2\beta_2}$, $\mathcal{O}(dD_1^2D_2)$ [or $\mathcal{O}(dD_1D_2^2)$]



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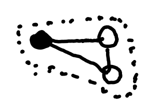
M, $\mathcal{O}(dD_1D_2)$



$$= \mathcal{O}(dD_1^2D_2)$$

In general, this term can be evaluated in $\mathcal{O}((N-2)dD_1^2D_2) = \mathcal{O}(NdD_1^2D_2)$ time [or $\mathcal{O}(NdD_1D_2^2)$].

3/1/20

- We can reduce the computational time by about half by combining the sums:

$$\sum_{u_i}\sum_{\alpha_i} A^{(1)u_1}_{1\alpha_1} A^{(2)u_2}_{\alpha_1\alpha_2} A^{(3)u_3}_{\alpha_2 1} \Big(\sum_{\alpha_i'} A^{(1)u_1}_{1\alpha_1'} A^{(2)u_2}_{\alpha_1'\alpha_2'} A^{(3)u_3}_{\alpha_2' 1}$$

$$- 2\sum_{\beta_i'} B^{(1)u_1}_{1\beta_1} B^{(2)u_2}_{\beta_1\beta_2} B^{(3)u_3}_{\beta_2 1} \Big)$$

Actually, this doesn't help ... we would have to distribute things out again to do the recursion trick.

- The recursion trick doesn't seem to work with the scheme of interpreting this as a quadratic:

$$\sum_{u_i\alpha_i} \Big(A^{(1)u_1}_{1\alpha_1} A^{(2)u_2}_{\alpha_1\alpha_2} X_{u_3\alpha_2\alpha_3} A^{(4)u_4}_{\alpha_3\alpha_4} A^{(5)}_{\alpha_4 1} \Big)$$

$$\times \sum_{\alpha_i'} \Big(A^{(1)u_1}_{1\alpha_1'} A^{(2)u_2}_{\alpha_1'\alpha_2'} X_{u_3\alpha_2'\alpha_3'} A^{(4)u_4}_{\alpha_3'\alpha_4'} A^{(5)u_5}_{\alpha_4' 1} \Big)$$

can be treated as a quadratic in X if we write it in the form

$$\sum_{\substack{u\ \alpha_i' \\ \alpha\ ..}} X_{u\alpha\tilde{\alpha}} M_{u\alpha\tilde{\alpha},\, u'\alpha'\tilde{\alpha}'} X_{u'\alpha'\tilde{\alpha}'}$$

which can be done with

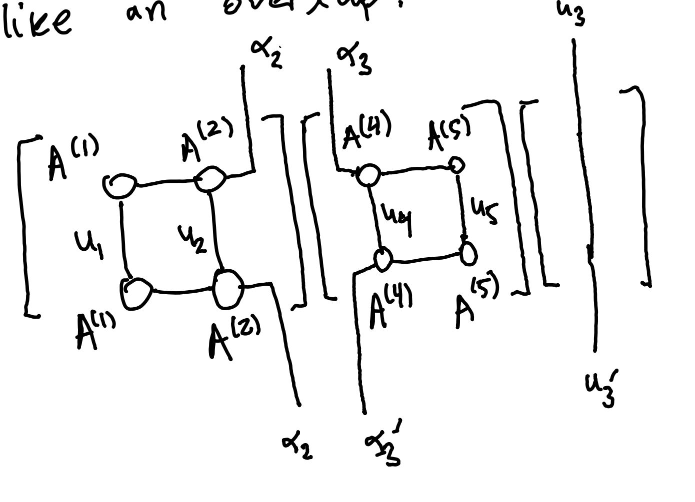
$$M_{u_3\alpha_2\alpha_3,u_3'\alpha_2'\alpha_3'} = \delta_{u_3u_3'}\sum_{u\alpha\alpha'} A^{(1)u_1}_{1\alpha_1} A^{(2)u_2}_{\alpha_1\alpha_2} A^{(4)u_4}_{\alpha_3\alpha_4} A^{(5)}_{\alpha_4 1}$$
$$\times A^{(1)u_1}_{1\alpha_1'} A^{(2)u_2}_{\alpha_1'\alpha_2'} A^{(4)u_4}_{\alpha_3'\alpha_4'} A^{(5)}_{\alpha_4' 1}$$

This seems daunting at first, but we actually <u>can</u> turn this into an efficient contraction scheme using the MPO structure by rearranging sums like so:

$$= \left[\sum_{u_5\alpha_4\alpha_4'} A^{(5)u_5}_{\alpha_4 1}\left[\sum_{u_4} A^{(4)u_4}_{\alpha_3\alpha_4}\left[\sum_{u_2\alpha_1\alpha_1'} A^{(2)u_2}_{\alpha_1\alpha_2}\left[\sum_{u_1} A^{(1)u_1}_{1\alpha_1} A^{(1)u_1}_{1\alpha_1'}\right] A^{(2)u_2}_{\alpha_1'\alpha_2'}\right] A^{(4)u_4}_{\alpha_3'\alpha_4'}\right] A^{(5)u_5}_{\alpha_4' 1}\right] \delta_{u_3u_3'}$$

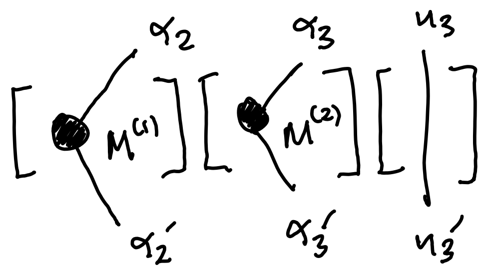
with the exception of this sum and this factor, this looks just like an overlap!

$$M_{u_3\alpha_2\alpha_3,u_3'\alpha_2'\alpha_3'} =$$

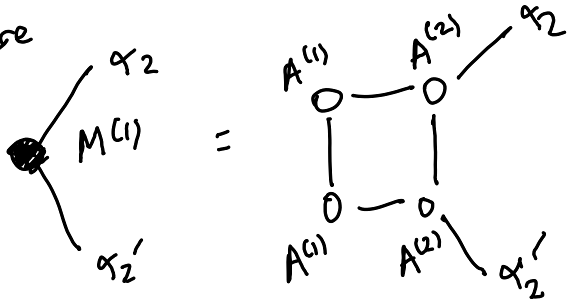


It seems a bit redundant to define the function this way due to the $\delta_{u_3 u_3'}$ term, but it might be necessary if I end up using a quadratic solver. Note that I can cache a significant part of the computation to avoid some of the redundancy:

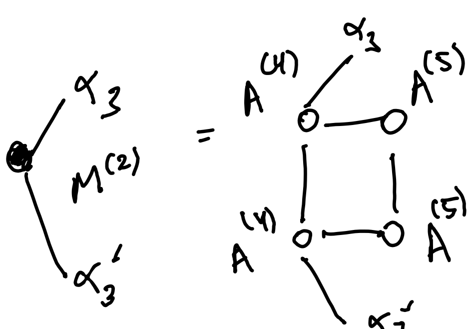
$M_{u_3\alpha_2\alpha_3,u_3'\alpha_2'\alpha_3'} =$



where



and



For larger N, this is the largest part of the calculation!

- Useful plot to make: $\sum_{u \in N} \mathrm{tr}(A_u \rho)^2$ vs $\sum_{u \in N} \mathrm{tr}(A_u \rho)$, color coded with an indicator for when any of the elements in the sum are larger than 1 (which would mean that $\left|\sum_{u \in N} \mathrm{tr}(A_u \rho)^2\right|$ is not a lower bound for $\left|\sum_{u \in N} \mathrm{tr}(A_u \rho)\right|$).

Which ρ to compute these for? Scott: Generate a random mixed state by generating N random states $|\phi_i\rangle$ and letting $\rho = \frac{\sum p_i |\phi_i\rangle\langle\phi_i|}{\sum p_i}$ for random $\{p_i\}_{i=1}^{N}$ positive.

As $N \to \infty$, $\rho \to \mathbb{1}$, so you should keep N fixed and not too large.

How close to the identity are we? The expectation value of a pauli string $X^{a_1} Z^{b_1} X^{a_2} Z^{b_2}$ in the identity is $\mathrm{tr}(X^{a_1} Z^{b_1} X^{a_2} Z^{b_2} \mathbb{1}) = 0$ I think...
if so then $\sum_{a_i b_i} \left(\mathrm{tr}(X^{a_1} Z^{b_1} X^{a_2} Z^{b_2})^2\right)$ is a measure..

gotta check this.

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Ok.. let's do the damn thing.

- MPO out of state:

$\rho \rightarrow W[i,j] = tr(\rho A[i] \times A[j])$ where $i \in \{0,..,8\}$, $j \in \{0,..,8\}$ phase space ops.

$$\rightarrow \sum_{j=1}^{D_1} U[i,j] S_j V[j,k] = W[i,k] \text{ by SVD}$$

$$\rightarrow \underset{(9\times D_1)}{A[0,i,j]} = U[i,j], \quad \underset{(D_1\times 9)}{A[1,i,j]} = S_i V[i,j]$$

- Generating random positive operator:

$$\underset{(9\times D_2)}{B[0,i,j]} = \text{randuniform}[0,1], \text{ same w/ } \underset{(D_2\times 9)}{B[1,i,j]}$$

- Defining cost function:

> First define a way to treat $B[k]$ as an object indexed by one index ranging from 0 to $8D_2-1$:

$$\begin{cases} I_0(n) = [\frac{n}{8}, n \% D_2] \\ I_1(n) = [n \% D_2, \frac{n}{8}] \end{cases}$$

Note: I could also use $\frac{n}{D_2}$ or $n \% 8$. not sure which convention is ~~better~~

> Then the cost functions are:

$$\begin{cases} f_0(x) = \| A[0]A[1] - B[0][I_0(x)]\,B[1] \|^2 \\ f_1(x) = \| A[0]A[1] - B[0]\,B[1]\,[I_1(x)] \|^2 \end{cases}$$

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- Reshaping phase space operators into one list:

$$\underbrace{\begin{pmatrix} A_{0,0} & A_{0,1} & A_{0,2} \\ A_{1,0} & A_{1,1} & A_{1,2} \\ A_{2,0} & A_{2,1} & A_{2,2} \end{pmatrix}}_{\text{"M"}} \Rightarrow \underbrace{\begin{pmatrix} A_{0,0} \\ A_{0,1} \\ A_{0,2} \\ A_{1,0} \\ \vdots \\ A_{2,2} \end{pmatrix}}_{\text{"V"}}$$

This means:

$$\{V_0 = M_{0,0},\ V_1 = M_{0,1},\ V_2 = M_{0,2},\ V_3 = M_{1,0}, \ldots, V_8 = M_{2,2}\}.$$

In general,

$$\boxed{V_k = M_{\lfloor \frac{k}{3} \rfloor, k\%3}}$$

Then the 2-qubit phase space operators are just

$$V^{(2)}_{k,l} = V_k \otimes V_l$$

- Schmidt decomposition of density matrices: Remember that ρ is a matrix in the computational basis and a vector in the phase space operator basis. The singular values of $(W_\rho)_{ij} = tr(\rho A_i \times A_j) = U_{ik} S_k V^\dagger_{kj}$ aren't related to the singular values $\rho_{ij} = \tilde{U}_{ik} \tilde{S}_k \tilde{V}^\dagger_{kj}$.

3/14/20

- Density matrices and MPOs for 2-sites:
  > cfnts_to_mpo(c):

    $(U, \Sigma, V^\dagger) = \mathrm{svd}(c)$

    return np.array([U, $\Sigma V^\dagger$])

  > mpo_to_cfts (M):

    return np.dot(M[0], M[1])

  > cfts_to_state (c, A):

    return $c_{ij} A_i \times A_j$

  > state_to_cfts (ρ, A): # assuming A_i satisfy $\mathrm{tr}(A_i \times A_j) = \delta_{ij}$

    return np.array([[tr($\rho A_i \times A_j$) for i in d] for j in d])

- Change of basis:

$\rho = c_{ij} A_i \times A_j = \tilde{c}_{ij} B_i \times B_j$. Given c_{ij}, want $\tilde{c}_{ij}$:

$\tilde{c}_{ij} = \mathrm{tr}(\rho B_i \times B_j)$ <u>if $\mathrm{tr}(B_i \times B_j) = \delta_{ij}$</u>

$= \mathrm{tr}(c_{kl} A_k \times A_l B_i \times B_j)$

$= c_{kl} \mathrm{tr}((A_k \times A_l)(B_i \times B_j))$

$= c_{kl} \mathrm{tr}(A_k B_i) \mathrm{tr}(A_l B_j)$

$\equiv U_{ki} c_{kl} U_{lj}$

so $\tilde{c} = U c U^T$

$U_{ij} \equiv \mathrm{tr}(A_i B_j)$ change-of-basis coefficients-

They satisfy:

$U_{ji} = \mathrm{tr}(A_j B_i) = \mathrm{tr}(B_i A_j) = U_{ij}$

$(U^{-1})_{ij} = \mathrm{tr}(B_i A_j) = \mathrm{tr}(A_j B_i) = U_{ij}$

$(U^\dagger)_{ij} = \mathrm{tr}(B_j^\dagger A_i^\dagger) \underset{?}{=} \mathrm{tr}(B_j A_i)$

If this is true, they are unitary.. not sure.

Changing basis is also easy to do at the MPO level:

$$\rho = M^{(1)}_{ik} M^{(2)}_{kj} A_i \times A_j = \tilde{M}^{(1)}_{ik} \tilde{M}^{(2)}_{kj} B_i \times B_j$$

Given M, get $\tilde{M}$ by representing $A_i \times A_j$ in $B_i \times B_j$ basis:

$$\rho = M^{(1)}_{ik} M^{(2)}_{kj} \operatorname{tr}\left((B_l \times B_m)(A_i \times A_j)\right) B_l \times B_m$$

$$= M^{(1)}_{ik} M^{(2)}_{kj} \operatorname{tr}(B_l A_i) \operatorname{tr}(B_m A_j) B_l \times B_m$$

$$\equiv \left(U_{li} M^{(1)}_{ik}\right)\left(M^{(2)}_{kj} U^T_{jm}\right) B_l \times B_m$$

$$\equiv \tilde{M}^{(1)}_{lk} \tilde{M}^{(2)}_{km} B_l \times B_m$$

so $\tilde{M}^{(1)} = UM^{(1)}$ and $\tilde{M}^{(2)} = M^{(2)} U^T$. This is also clear from our last result: Here, $c_{ij} = M^{(1)}_{ik} M^{(2)}_{kj}$, i.e. $c = M^{(1)} M^{(2)}$. So a change of basis means $\tilde{c} = UcU^T = UM^{(1)} M^{(2)} U^T$, which is accomplished by the above choice.

But note that we didn't <u>have</u> to do it this way. we could have set $\tilde{M}^{(1)} = UM^{(1)} M^{(2)}$ and $\tilde{M}^{(2)} = U^T$. In fact, this choice would move all the degrees of freedom into one matrix. Might be worth thinking about...

To summarize:

> mpo_basis_change(M, U):

    return np.array([UM[0], M[1]U^T])

> cfts_basis_change(c, U):

    return UcU^T

- optimization:

The cost functions for two site optimization are local restrictions of the frobenius norm $\| A^{(1)}A^{(2)} - B^{(1)}B^{(2)} \|^2$

$$= \langle A^{(1)}A^{(2)} - B^{(1)}B^{(2)}, A^{(1)}A^{(2)} - B^{(1)}B^{(2)} \rangle = \mathrm{tr}\left(\left(A^{(1)}A^{(2)} - B^{(1)}B^{(2)} \right)^2 \right)$$

Or... maybe it's $\| (A^{(1)}A^{(2)} - B^{(1)}B^{(2)})_{ij} A_i \times A_j \|^2$

$$= (A^{(1)}A^{(2)} - B^{(1)}B^{(2)})_{ij}^2 \, \mathrm{tr}\left((A_i \times A_j)^2 \right)$$

$$= (A^{(1)}A^{(2)} - B^{(1)}B^{(2)})_{ij}^2 \, \mathrm{tr}(A_i^2)\mathrm{tr}(A_j^2)$$

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- I'm not sure which of the above leads to the cost fct

$$\sum_u \left(\sum_\alpha A_\alpha^{(1)u} A_\alpha^{(2)u} - \sum_\beta B_\beta^{(1)u} B_\beta^{(2)u} \right)^2,$$

but that's the one I'm going to use.

(Actually, it turned out to be $A^{(1)u_1} A^{(2)u_2}$ not $A^{(1)u} A^{(2)u}$)

3/20/20

> renyi_entropy(M, α)

    (U, Σ, V†) = np.svd(np.dot(M[0], M[1]))
    return math.pow(1-α, -1) math.log(np.sum(np.square(Σ, 2)))

$$\left(\frac{1}{1-\alpha} \log \mathrm{tr}(\Sigma^{\alpha}) \right)$$

fixed bond dimension - best we can do?!

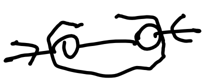
- If $\rho_1 = |\psi_1\rangle\langle\psi_1| r_1$ and $\rho_2 = |\psi_2\rangle\langle\psi_2| r_2$, then

$$\rho_1 + \rho_2 = \sum_i r_i |\psi_i\rangle\langle\psi_i|$$

has larger bond dimension <u>iff</u> $|\psi_i\rangle$ are lin. independent.

(In general, MPO bond dim doesn't add - for random mixed states there will generally be d^2 (maximal) lin. independent states in each mixture.)

- <u>operator space entanglement entropy</u> = treat it like an MPS and ask what



~~etc~~

$$C^{u_1 u_2} A_{u_1} \times A_{u_2} = \rho$$

"entropy of the system"

$d^n \times d^n$

$A^{u_1}_{\alpha\beta} A^{u_2}_{\beta}$

eigenvalues vs.

- mutual information

- need to normalize it like a state

• Adding two MPSs:

$$A^{u_1}_{\alpha\beta} A^{u_2}_{\beta\alpha}$$

$$v^{(1)}_i |\psi^{(1)}_i\rangle\langle\psi^{(1)}_k| + v^{(2)}_j |\psi^{(2)}_j\rangle\langle\psi^{(2)}_j|$$

$$= v^1_\alpha A^{u_1}_{\alpha\beta} A^{u_2}_{\beta\gamma} v'_\gamma \quad + \quad v^2_\alpha A^{u_1}_{\alpha\beta} A^{u_2}_{\beta\gamma} v'_\gamma$$

$$= \tilde{v}_\alpha \tilde{A}^{u_1}_{\alpha\beta} \tilde{A}^{u_2}_{\beta\gamma} \tilde{v}_\alpha$$

where $\tilde{v}, \tilde{A}$ are direct sums of v, A!

$$\left\{ \begin{matrix} [v_1][1][1'][v'_1] \\ [v_2][2][2'][v'_2] \end{matrix} \right\} \Rightarrow \left[\begin{array}{c|c} v_1 & v_2 \end{array}\right] \left[\begin{array}{c|c} 1 & \\ \hline & 2 \end{array}\right] \left[\begin{array}{c|c} 1' & \\ \hline & 2' \end{array}\right] \times \left[\begin{array}{c} v'_1 \\ \hline v'_2 \end{array}\right]$$

It's not clear how the positivity requirement affects convexity.

Seperable states ($\rho = \sum_k p_k \rho_1^k \otimes \rho_2^k$, $p_k \geq 0$) might have nice props but we don't necessarily have that..

SVDs might not give positive values

Main point:

If you relax the positivity (but keep fixed bond dim) criteria, you are not left with a convex space. So it's reasonable to guess that adding the positivity requirement ~~also~~ doesn't help.. in fact you could prove it's not convex by finding two ops. which leave the space.

$\left(\mathrm{tr}((\sigma^x)^2) = \mathrm{tr}(I) = 2\right)$

= physical states

= stabilizer states

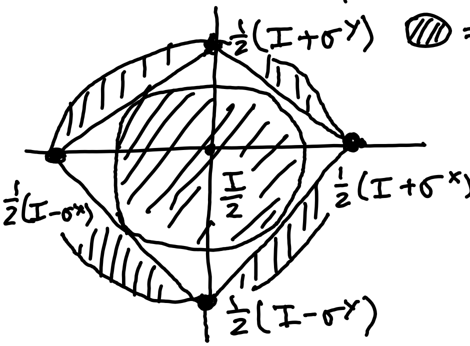
= ball of largest size inside stabilizer states

3/22/20

- Visualization of space:

(Qubits)

(projected onto σ^x/σ^y plane)

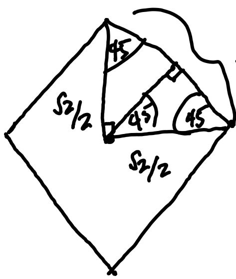


> Question: what are the radii of and ?

: radius $= \sqrt{\langle \frac{I}{2}, \frac{1}{2}(I+\sigma^x)\rangle} = \sqrt{\mathrm{tr}\left(\frac{1}{4}(I+\sigma^x)\right)}$

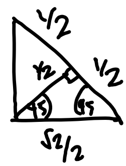
$= \sqrt{\frac{1}{4}\underbrace{\mathrm{tr}(I)}_{=2} + \frac{1}{4}\underbrace{\mathrm{tr}(\sigma^x)}_{=0}} = \sqrt{\frac{1}{2}} = \boxed{\frac{\sqrt{2}}{2}}$

:



$\rightarrow \sqrt{2\left(\frac{\sqrt{2}}{2}\right)^2} = \sqrt{2\left(\frac{2}{4}\right)} = 1$

so:



radius $= \boxed{\frac{1}{2}}$

- what is 2^{-1} modulo d? $(2)(2^{-1}) = 1$ is satisfied ~~by~~ if

$(2)(2^{-1}) = d+1 \rightarrow (2^{-1}) = \left(\frac{d+1}{2}\right)$. ex: $\begin{cases} d=3 \Rightarrow 2^{-1} = 2 \\ d=5 \Rightarrow 2^{-1} = 3 \end{cases}$.

If d is odd then $\frac{d+1}{2}$ is always an integer in $[0, d-1]$.

Not sure if 2^{-1} exists if d is even.. who cares.

3/25/20

- Generalizing to L sites:
  - > Change of basis:

$$\rho = M^{(1)u_1}_{\alpha_1\alpha_2} M^{(2)u_2}_{\alpha_2\alpha_3} M^{(3)u_3}_{\alpha_3\alpha_4} \; A_{u_1} \times A_{u_2} \times A_{u_3}$$

$$= \tilde{M}^{(1)}_{\alpha_1\alpha_2} \tilde{M}^{(2)u_2}_{\alpha_2\alpha_3} \tilde{M}^{(3)u_3}_{\alpha_3\alpha_4} \; B_{u_1} \times B_{u_2} \times B_{u_3}$$

Due to orthonormality of the $B_{u_1} \times B_{u_2} \times B_{u_3}$ basis, we have:

$$\tilde{M}^{(1)u_1}_{\alpha_1\alpha_2} \tilde{M}^{(2)u_2}_{\alpha_2\alpha_3} \tilde{M}^{(3)u_3}_{\alpha_3\alpha_4} = tr(B_{u_1} \times B_{u_2} \times B_{u_3}\rho)$$

$$= tr\left(B_{u_1} \times B_{u_2} \times B_{u_3}\left[M^{(1)v_1}_{\beta_1\beta_2} M^{(2)v_2}_{\beta_2\beta_3} M^{(3)v_3}_{\beta_3\beta_4} A_{v_1} \times A_{v_2} \times A_{v_3}\right]\right)$$

$$= M^{(1)v_1}_{\beta_1\beta_2} M^{(2)v_2}_{\beta_2\beta_3} M^{(3)v_3}_{\beta_3\beta_4} \underbrace{tr\left((B_{u_1} \times B_{u_2} \times B_{u_3})(A_{v_1} \times A_{v_2} A_{v_3})\right)}$$

$$tr(B_{u_1}A_{v_1} \times \cdots \times B_{u_3}A_{v_3})$$

$$= tr(B_{u_1}A_{v_1}) \cdots tr(B_{u_3}A_{v_3})$$

$$= U_{u_1v_1} U_{u_2v_2} U_{u_3v_3}$$

$$(U_{ij} = tr(B_iA_j))$$

$$= \left(U_{u_1v_1} M^{(1)v_1}_{\beta_1\beta_2}\right) \times \left(U_{u_2v_2} M^{(2)v_2}_{\beta_2\beta_3}\right) \times \left(U_{u_3v_3} M^{(3)v_3}_{\beta_3\beta_4}\right)$$

So: $$\tilde{M}^{(1)u_1}_{\alpha_1\alpha_2} = U_{u_1v_1} M^{(1)v_1}_{\alpha_1\alpha_2}$$

- Understanding this GOD DAMN cost function:

$D(\rho \| \sigma)$ needs to satisfy the following:

$$M_2(\rho) = \min_{\sigma} D(\rho \| \sigma)$$

3/28/20

- $\text{mana}(\rho) = \sum_{u, W_\rho(u)<0} |W_\rho(u)|$, and $\text{tr}\rho = 1 \Rightarrow \sum_u W_\rho(u) = 1$.

$$\sum_u W_\rho(u) = \sum_{u, W_\rho(u) \geq 0} W_\rho(u) + \sum_{u, W_\rho(u)<0} W_\rho(u) = 1$$

$$\Rightarrow \quad 1 = \sum_{u, W_\rho(u) \geq 0} |W_\rho(u)| - \sum_{u, W_\rho(u)<0} |W_\rho(u)| \ldots$$

that wasn't going anywhere. How about this:

$$\sum_u |W_\rho(u)| = \sum_{u, W_\rho(u) \geq 0} |W_\rho(u)| + \sum_{u, W_\rho(u)<0} |W_\rho(u)|$$

$$= \underbrace{\sum_{u, W_\rho(u) \geq 0} W_\rho(u)}_{1 - \sum_{u, W_\rho(u)<0} W_\rho(u)} + \underbrace{\sum_{u, W_\rho(u)<0} |W_\rho(u)|}_{\text{mana}(\rho)}$$

$$= 1 + \sum_{u, W_\rho(u)<0} |W_\rho(u)| = 1 + \text{mana}(\rho)$$

So $\sum_u |W_\rho(u)| = 1 + 2\,\text{mana}(\rho)$

$$\Rightarrow \boxed{\text{mana}(\rho) = \frac{1}{2}\left(\sum_u |W_\rho(u)| - 1\right)}$$

- $\|\rho\|_W \equiv \sum_u |W_\rho(u)|$, $W_\rho(u) = \frac{1}{d^n}\operatorname{tr}(A_u\rho)$

$$\Rightarrow \|\rho\otimes\sigma\|_W = \sum_{u,v} |W_{\rho\otimes\sigma}(u\oplus v)|$$

$$= \sum_{u,v} \left|\frac{1}{d^{2n}}\operatorname{tr}(A_{u\oplus v}\,\rho\otimes\sigma)\right|$$

$$= \sum_{u,v} \left|\frac{1}{d^{2n}}\operatorname{tr}((A_u\otimes A_v)(\rho\otimes\sigma))\right|$$

$$= \sum_{u,v} \left|\frac{1}{d^{2n}}\operatorname{tr}(A_u\rho)\operatorname{tr}(A_v\sigma)\right|$$

$$= \sum_{u,v} |W_\rho(u)\,W_\sigma(v)|$$

$$= \sum_{u,v} |W_\rho(u)|\,|W_\sigma(v)|$$

$$= \left(\sum_u |W_\rho(u)|\right)\left(\sum_v |W_\sigma(v)|\right)$$

$$= \|\rho\|_W\,\|\sigma\|_W$$

- $$\mathrm{sn}(\rho \otimes \sigma) = \frac{1}{2}\left(\|\rho \otimes \sigma\|_W - 1\right)$$
$$= \frac{1}{2}\left(\|\rho\|_W \|\sigma\|_W - 1\right)$$
$$= \frac{1}{2}\left((2\mathrm{sn}(\rho)+1)(2\mathrm{sn}(\sigma)+1) - 1\right)$$

$$\mathrm{sn}(\rho) = \frac{1}{2}(\|\rho\| - 1)$$
$$\Downarrow$$
$$\|\rho\| = 2\mathrm{sn}(\rho) + 1$$

so:
$$\mathrm{sn}(\rho \otimes \rho) = \frac{1}{2}\left((2\mathrm{sn}(\rho)+1)^2 - 1\right)$$
$$\mathrm{sn}(\rho^{\otimes 3}) = \mathrm{sn}(\rho^{\otimes 2} \otimes \rho)$$
$$= \frac{1}{2}\left((2\mathrm{sn}(\rho^{\otimes 2})+1)(2\mathrm{sn}(\rho)+1) - 1\right)$$
$$= \frac{1}{2}\left((2\mathrm{sn}(\rho)+1)^3 - 1\right)$$
$$\Rightarrow \boxed{\mathrm{sn}(\rho^{\otimes n}) = \frac{1}{2}\left((2\mathrm{sn}(\rho)+1)^n - 1\right)}$$

- Is it possible to write $\mathrm{sn}(\rho)$ in terms of $\sum_u W_\rho(u)$? Of course not... $\sum_u W_\rho(u) = 1$ is a constant if ρ is normalized.. if not it just equals $\mathrm{tr}\rho$.
Can't use $\mathrm{tr}\rho$ to find info about $\mathrm{sn}(\rho)$.. [illegible]

• Hermiticity of generalized pauli operators:

$$T_{(a_1, a_2)} = \omega^{-\frac{a_1 a_2}{2}} X^{a_1} Z^{a_2}$$

where

$$X|j\rangle = |(j+1) \bmod d\rangle, \quad Z|j\rangle = \omega^{j}|j\rangle.$$

The definitions of X and Z give:

$$\begin{aligned} \langle j|X^{\dagger}|i\rangle &= \langle (j+1) \bmod d|i\rangle \\ &= \langle i|(j+1) \bmod d\rangle^{*} \\ &= \langle i|X|j\rangle^{*} \end{aligned}$$

DuH.. this is always true! [I never did this but this.. weird]

$$X = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow X^{\dagger} = X^{T} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$Z = \begin{bmatrix} 1 & & \\ & \omega & \\ & & \omega^2 \end{bmatrix} \Rightarrow Z^{\dagger} = \begin{bmatrix} 1 & & \\ & \omega^{*} & \\ & & (\omega^{*})^2 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & \omega^2 & \\ & & \omega \end{bmatrix}$$

using the fact that $\omega = e^{\frac{2\pi i}{d}} \Rightarrow \omega^{*} = e^{-\frac{2\pi i}{d}} = (\omega)^{-1}$,

And $\omega^{d} = 1$ so $\omega^{-1} = \omega^{-1+d} = \omega^{d-1} \underset{(d=3)}{=} \omega^2$. And $\omega^{-2} = \omega^{d-2} \underset{(d=3)}{=} \omega$

So:

$$T_{(0,0)} = \omega^0 X^0 Z^0 = \mathbb{1}_3$$ is hermitian.

But

$$T_{(1,0)} = \omega^{-\frac{(1)(0)}{2}} X^1 Z^0 = X$$ is <u>not</u>!

But that's ok.. these aren't the hermitian basis elements we'll be expanding in.. those are the phase space operators. Note that:

$$\Rightarrow \langle i | Z^\dagger Z | j \rangle = (Z|i\rangle)^\dagger (Z|j\rangle) = (\omega^i |i\rangle)^\dagger (\omega^j |j\rangle)$$

$$= \langle i | \omega^{-i} \omega^j | j \rangle = \omega^{j-i} \langle i | j \rangle = \omega^{j-i} \delta_{ij}$$

$$= \begin{cases} 0 & i \neq j \\ \omega^0 = 1 & i = j \end{cases} = \delta_{ij}$$

and

$$\Rightarrow \langle i | X^\dagger X | j \rangle = \langle (i+1) \bmod d \,|\, (j+1) \bmod d \rangle$$

$$= \delta_{(i+1),(j+1)} = \delta_{ij}$$

$$\Rightarrow \langle i | (\omega^{-\frac{q_1 q_2}{2}} \mathbb{1})^\dagger (\omega^{-\frac{q_1 q_2}{2}} \mathbb{1}) | j \rangle = \delta_{ij} \text{ b/c } \omega^\dagger \omega = 1$$

So $T_{(a_1,a_2)}$ is a product of <u>unitary</u> operators, and is ~~therefore~~ therefore itself unitary.

Now the phase space operators

$$A_u = T_u^\dagger A_0 T_u = T_u^\dagger \left(\sum_v \frac{1}{d^n} T_v \right) T_u$$

satisfy

$$A_u^\dagger = \frac{1}{d^n}\left(\sum_v T_u^\dagger T_v T_u \right)^\dagger = \frac{1}{d^n} \left(\sum_v T_u^\dagger T_v^\dagger T_u \right)$$

$$= \frac{1}{d^n} T_u^\dagger \sum_v T_v^\dagger \; T_u$$

But it turns out that $\sum_v T_v^\dagger = \sum_v T_v$.

> <u>Proof</u>:

$$T_{(a,b)}^\dagger = (Z^b)^\dagger (X^a)^\dagger (\omega^{-\frac{ab}{2}})^\dagger$$

$$= \omega^{\frac{ab}{2}} (Z^{-1})^b (X^{-1})^a$$

$$= \omega^{\frac{ab}{2}} (Z^{-b})(X^{-a})$$

$$= \omega^{\frac{ab}{2}} Z^{d-b} X^{d-a}$$

Note: $X^d |j\rangle$
$= |(j+d) \bmod d\rangle$
$= |j\rangle$
and
$Z^d |j\rangle = \omega^{dj} |j\rangle$
$= |j\rangle$,
so $X^d = Z^d = \mathbb{1}_d$

To proceed, we need to move Z^{d-b} through X^{d-a}. Note that $[X, Z]$ can be found like so:

$$(XZ - ZX)|j\rangle = XZ|j\rangle - ZX|j\rangle = X\omega^j|j\rangle - Z|(j+1) \bmod d\rangle$$

$\Downarrow$ from now on all addition is mod d, ~~not that it matters~~ in the exponent of ω anyway.

$$= \omega^j|j+1\rangle - \omega^{j+1}|j+1\rangle = (\omega^j - \omega^{j+1})|j+1\rangle$$

$$= \omega^j(1-\omega)|j+1\rangle = \omega^j(1-\omega)X|j\rangle$$

$$= (1-\omega)X\omega^j|j\rangle = (1-\omega)XZ|j\rangle$$

$$\Rightarrow \boxed{[X, Z] = (1-\omega)XZ}$$

$\Downarrow$

$$XZ - ZX = (1-\omega)XZ$$

$$-ZX = (1-\omega-1)XZ$$

$$-ZX = -\omega XZ$$

$$\boxed{ZX = \omega XZ} \Rightarrow Z^a X^b = \omega^{\min(a,b)} X^b Z^a \ ???$$

so: $T_u^\dagger = \omega^{\frac{ab}{2}} \ldots ?$

• <u>3/29/20</u>

Lets see if the previous result makes sense:

$$ZX|k\rangle = Z|k\oplus 1\rangle = \omega^{k\oplus 1}|k\oplus 1\rangle$$

$$\omega XZ|k\rangle = \omega X\omega^{k}|k\rangle = \omega^{k+1}|k\rangle \left(= \omega^{k\oplus 1}|k\rangle \text{ b/c } \omega^{d}=1\text{, need to make this formal?}\right)$$

so:

$$T^{\dagger}_{(1,0)} = \left(\omega^{-\frac{(1)(0)}{2}} X^{1} Z^{0}\right)^{\dagger} = X^{\dagger} = X^{-1} = X^{d-1} = T_{(d-1,0)}$$

in general

$$T^{\dagger}_{(a,0)} = \left(\omega^{-\frac{(a)(0)}{2}} X^{a} Z^{0}\right)^{\dagger} = X^{-a} = X^{d-a} = T_{(d-a,0)}$$

$$T^{\dagger}_{(0,a)} = \left(Z^{a}\right)^{\dagger} = Z^{-a} = Z^{d-a} = T_{(0,d-a)}$$

Hm.. for the general case, I could swap X and Z first, before the hermitian conjugate, which will swap them back to the right order!

$$T^{\dagger}_{(a,b)} = \left(\omega^{-\frac{ab}{2}} X^{a} Z^{b}\right)^{\dagger} = \underbrace{\left(\omega^{-\frac{ab}{2}}\, \omega^{ab} Z^{b} X^{a}\right)}$$

also, its not that complicated: $X^{a}Z^{b} = X\cdot X\cdots X Z\cdots Z = \omega^{ab} Z\cdots Z X\cdots X$

so $T_{(a,b)}^{\dagger} = (\omega^{-\frac{ab}{2}} X^a Z^b)^{\dagger} = (\omega^{-\frac{ab}{2}} \omega^{ab} Z^b X^a)^{\dagger}$

$= (\omega^{\frac{ab}{2}} Z^b X^a)^{\dagger} = \omega^{-\frac{ab}{2}} X^{-a} Z^{-b}$

$= \omega^{-\frac{(-a)(-b)}{2}} X^{-a} Z^{-b} \overset{?}{=} T_{(-a,-b)}$

Not sure how to justify the last step.. I need to explicitly get $T_{(d-a,\,d-b)}$ to come out, not $T_{(-a,-b)}$, because I'm not convinced that $\omega^{(a \oplus b)(c \oplus d)} = \omega^{(a+b)(c+d)}$, or... even if that's the right property... Let's just see if $\omega^{-\frac{ab}{2}} = \omega^{-\frac{(d-a)(d-b)}{2}}$:

$$\omega^{-\frac{(d-a)(d-b)}{2}} = \omega^{-\frac{(d^2 - db - ad + ab)}{2}}$$

$$= \omega^{-\frac{ab}{2}} \omega^{-\frac{d^2}{2}} \omega^{-\frac{db}{2}} \omega^{-\frac{ad}{2}}$$

$$= \omega^{-\frac{ab}{2}} \underbrace{(\cancel{\omega^d})^{-\frac{d}{2}} (\cancel{\omega^d})^{-\frac{b}{2}} (\cancel{\omega^d})^{-\frac{a}{2}}}_{1 \quad 1 \quad 1}$$

$$= \omega^{-\frac{ab}{2}}$$

note that 2^{-1} is the multiplicative inverse of 2, <u>not</u> the square root operation!

So it works out! Maybe it should be obvious that the last step worked ~~out~~.. Anyway!

$$\sum_u T_u^\dagger = \sum_{a,b} T_{(a,b)}^\dagger = \sum_{a,b} T_{(d-a,d-b)}$$

$$= \sum_{a'b'} T_{(a',b')} = \sum_u T_u$$

And we have:

$$A_u^\dagger = \frac{1}{d^n}\left(T_u^\dagger \sum_v T_v T_u\right)^\dagger = \frac{1}{d^n} T_u^\dagger \left(\sum_v T_v\right)^\dagger (T_u)$$

$$= \frac{1}{d^n} T_u^\dagger \left(\sum_v T_v\right) T_u = A_u \quad \checkmark$$

Another way to see that A_u is hermitian is to note that it is the unitary conjugation of the hermitian operator $\sum_v T_v$:

$$\boxed{T_u \text{ unitary}, \sum_u T_u \text{ hermitian} \rightarrow A_u \text{ hermitian}}$$

Now, at long last, the beast of all beasts:

$$\mathrm{tr}\left[A_u A_v\right] = \frac{1}{d^n}\,\delta_{uv}$$

<u>Proof:</u>

$$\mathrm{tr}\left[A_u A_v\right] = \mathrm{tr}\left[T_u^\dagger \sum_{u'} T_{u'} T_u T_v^\dagger \sum_{v'} T_{v'} T_v\right]$$

$$= \sum_{u'v'} \mathrm{tr}\left[T_u^\dagger T_{u'} T_u T_v T_{v'} T_v\right]$$

we want to move these guys over to use the fact that $T_u^\dagger T_u = \mathbb{1}$.

So we need to look at:

$$T_{(a,b)} T_{(a',b')} = \omega^{-\frac{ab}{2}} X^a Z^b \omega^{-\frac{a'b'}{2}} X^{a'} Z^{b'}$$

$$= \left(\omega^{-\frac{a'b'}{2}} \omega^{-\frac{ab}{2}}\right)\left(X^a Z^b X^{a'} Z^{b'}\right)$$

picks up $\omega^{+a'b}$ (see note on next page)

$$= \omega^{-\frac{a'b'}{2}} \omega^{-ab/2} \left(\bar{\omega}^{a'b} X^a X^{a'} Z^b Z^{b'}\right)$$

all good   all good

$$= \omega^{-\frac{ab}{2}} \omega^{-\frac{a'b'}{2}} \left(\omega^{a'b} X^{a'} X^{a} Z^{b'} Z^{b} \right)$$

picks up $\omega^{-ab'}$ (see note below)

$$= \omega^{-\frac{ab}{2}} \omega^{-\frac{a'b'}{2}} \omega^{a'b} \omega^{-ab'} X^{a'} Z^{b'} X^{a} Z^{b}$$

$$= \omega^{a'b-ab'} \omega^{-\frac{a'b'}{2}} X^{a'} Z^{b'} \omega^{-\frac{ab}{2}} X^{a} Z^{b}$$

$$= \omega^{a'b-ab'} T_{(a',b')} T_{(a,b)}$$

(Note that:

$\omega X Z = Z X \Rightarrow X Z = \omega^{-1} Z X$)

So:

$$\mathrm{tr}[A_u^\dagger A_v] = \sum_{u'v'} \mathrm{tr}[T_u^\dagger T_{u'} T_u T_v^\dagger T_{v'} T_v]$$

(let $u=(a,b)$ and $v=(c,d)$)

picks up $\omega^{a'b-ab'}$

$$= \sum_{(a',b'),v'} \omega^{a'b-ab'} \mathrm{tr}[T_{(a',b')} \underbrace{\cancel{T^\dagger_{(a,b)} T_{(a,b)}}}_{\mathbb{1}} T_v^\dagger T_{v'} T_v]$$

$$= \sum_{(a',b'),(c',d')} \omega^{a'b-ab'} \omega^{c'd-cd'} \underbrace{\mathrm{tr}[T_{(a',b')} T_{(c',d')}]}_{\text{now we need to compute this!}}$$

$$\mathrm{tr}\left[T_{(a,b)}T_{(c,d)}\right] = \omega^{-\frac{ab}{2}}\omega^{-\frac{cd}{2}}\,\mathrm{tr}\left[X^a Z^b X^c Z^d\right]$$

$= \ldots$ not sure what to do here.

But I remember Chris mentioned that the property is useful!

$$\sum_{a=0}^{d} \omega^a = \frac{1-\omega^{d+1}}{1-\omega} = \frac{1-\omega}{1-\omega} = 1$$

Also:

$$\sum_{a=0}^{d} (\omega^b)^a = \frac{1-(\omega^b)^{d+1}}{1-\omega^b} = \frac{1-\cancel{(\omega^b)}^{d \to 1}(\omega^b)^1}{1-\omega^b} = 1$$

which means that:

$$\mathrm{tr}\left[A^\dagger_{(a,b)}A_{(c,d)}\right] = \sum_{a'b'c'd'} \omega^{a'b-ab'}\omega^{c'd-cd'}\,\mathrm{tr}\left[T_{(a',b')}T_{(c',d')}\right]$$

$$= \sum_{a'}(\omega^b)^{a'}\sum_{b'}(\omega^{-a})^{b'}\sum_{c'}(\omega^d)^{c'}\sum_{d'}(\omega^{-c})^{d'}\,\mathrm{tr}\left[T_{(a',b')}T_{(c',d')}\right]$$

Hm... not much we can do here..

I think I might have screwed up the

$$T_{a,b}T_{a',b'} = \omega^{a'b-ab'}T_{a'b'}T_{ab}$$

computation. Let's redo it:

$$T_{a,b}T_{a',b'} = \omega^{-\frac{ab}{2}}X^aZ^b\omega^{-\frac{a'b'}{2}}X^{a'}Z^{b'}$$

$$= \left(\omega^{-\frac{ab}{2}}\omega^{-\frac{a'b'}{2}}\right)\left(X^a\underbrace{Z^b\,X^{a'}}\,Z^{b'}\right)$$

...so we have to switch these.

we don't want to switch these b/c we'll just have to switch them back later...

$$Z^bX^{a'} = (ZZ\cdots Z)(XX\cdots X)$$

[and $ZX = \omega XZ$] $= \omega^b X(Z\cdots Z)(X\cdots X)$

$\cdots = (\omega^b)^{a'}(X\cdots X)(Z\cdots Z)$

$= \omega^{a'b}X^{a'}Z^b$

$$= \omega^{-\frac{ab}{2}}\omega^{-\frac{a'b'}{2}}\left(X^a\omega^{a'b}X^{a'}Z^bZ^{b'}\right)$$

$$= \omega^{-\frac{ab}{2}}\omega^{-\frac{a'b'}{2}}\omega^{a'b}\left(X^{a'}\underbrace{X^aZ^{b'}}Z^b\right)$$

$$X^aZ^{b'} = \omega^{-ab'}Z^{b'}X^a$$

$$= \omega^{-\frac{ab}{2}}\omega^{-\frac{a'b'}{2}}\omega^{a'b-ab'}\left(X^{a'}Z^{b'}X^aZ^b\right) = \text{above.}$$

If that's not the issue then I need to compute $\mathrm{tr}\{T_{ab}T_{a'b'}\}$. let's try some examples

$$\mathrm{tr}\{T_{00}T_{a0}\} = \mathrm{tr}[\mathbb{1}T_{a0}] = \mathrm{tr}[X^a] = \sum_j^a \langle j|X|j\rangle$$

$$= \left\lfloor \frac{a}{d} \right\rfloor \quad \text{b/c } X^d = \mathbb{1} \text{ but } \mathrm{tr}X^{a<d} = 0$$

$$= 0 \quad \text{since } a<d \quad (\text{unless } a=0)$$

$$\mathrm{tr}[T_{00}T_{0a}] = \mathrm{tr}[\mathbb{1}T_{0a}] = \mathrm{tr}[Z^a] = \sum \langle j|Z^a|j\rangle$$

$$= \sum_{j=0}^{d-1} \omega^{aj} = \frac{1-(\omega^a)^d}{1-\omega^a} = \frac{1-(\omega^d)^a}{1-\omega^a} = \frac{1-1}{1-\omega^a} = 0 \left(\begin{array}{c}\text{unless}\\ a=0\\ \text{or } d\end{array}\right)$$

$$\mathrm{tr}[XZ] = \sum_{j=0}^{d-1} \langle j|XZ|j\rangle = \sum_{j=0}^{d-1} \omega^j \underbrace{\langle j|X|j\rangle}_{=0} = 0$$

In fact, since X^a is traceless, if $a\neq 0$, $a<d$, and Z^b is diagonal:
$$\boxed{\mathrm{tr}[X^a Z^b] = 0 \quad \text{for } 0<a<d}$$

Now:

$$\mathrm{tr}\left[T_{ab}T_{a'b'}\right] = \omega^{-\frac{ab}{2}}\omega^{-\frac{a'b'}{2}}\,\mathrm{tr}\left[X^a Z^b X^{a'} Z^{b'}\right]$$

$$= \omega^{-\frac{ab}{2}}\omega^{-\frac{a'b'}{2}}\,\omega^{ba'}\,\underbrace{\mathrm{tr}\left[X^a X^{a'} Z^b Z^{b'}\right]}$$

$$= \mathrm{tr}\left[X^{a+a'} Z^{b+b'}\right]$$

New note that if $a+a'=d$, then:

$$\mathrm{tr}\left[X^{a+a'} Z^{b+b'}\right] = \mathrm{tr}\left[\mathbb{1} Z^{b+b'}\right] = \sum_j \left(\omega^{b+b'}\right)^j$$

$$= \begin{cases} \text{if } b+b'=d, & \sum_{j=0}^{d-1} (1)^j = d \\ \text{if } 0<b+b'<d, & \sum_{j=0}^{d-1} \left(\omega^{b+b'}\right)^j = \frac{1-1}{1-\omega^{b+b'}} = 0 \end{cases}$$

$$= d\,\delta_{b+b',d}$$

So: $\boxed{\mathrm{tr}\left[X^{a+a'} Z^{b+b'}\right] = d\,\delta_{a+a',d}\,\delta_{b+b',d}}$

and $\omega^{-\frac{ab}{2}}\omega^{-\frac{a'b'}{2}}\omega^{ba'} = \omega^{-\frac{ab}{2}+\frac{2ba'}{2}+\frac{a'b'}{2}} = \omega^{-\frac{1}{2}(ab+2ba'+a'b')}$

$$= \omega^{-\frac{1}{2}(ab + a'b + a'b + a'b')} = \omega^{-\frac{1}{2}(b(a+a') + a'(b'+b))}$$

$$= \omega^{-\frac{1}{2}(bd + a'd)} = -\frac{1}{2} d(b+a')$$

(if $a+a' = d$, $b+b' = d$)

$$= \left(\omega^{d}\right)^{-\frac{(b+a')}{2}} \Big\} \text{ just an integer}$$

$$= 1$$

So:

$$\boxed{\mathrm{tr}\left[T_{ab} T_{a'b'}\right] = d\, \delta_{a+a',d}\, \delta_{b+b',d}}$$

And:

$$\mathrm{tr}\left[A^{\dagger}_{a_1b_1} A_{a_2b_2}\right] = \sum_{a_1'b_1'a_2'b_2'} \omega^{a_1'b_1 - a_1b_1'} \omega^{a_2'b_2 - a_2b_2'} \underbrace{\mathrm{tr}\left[T_{a_1'b_1'} T_{a_2'b_2'}\right]}_{d\,\delta_{a_1'+a_2',d}\,\delta_{b_1'+b_2',d}}$$

$$= d \sum_{a_1'b_1'} \omega^{a_1'b_1 - a_1b_1'} \omega^{(d-a_1')(b_2) - (a_2)(d-b_1')}$$

all d's go away b/c $\omega^d = 1$

$$= d \sum_{a_1'b_1'} \omega^{a_1'b_1 - a_1b_1'} \omega^{-a_1'b_2 + a_2b_1'}$$

[might want to just let $\delta_{a_1+a_2,0}$ be it instead..]

$$= d \sum_{a_1' b_1'} \omega^{a_1'(b_1-b_2)} \omega^{b_1'(a_2-a_1)}$$

$$= d\left(\sum_{a_1'} \omega^{a_1'(b_1-b_2)}\right)\left(\sum_{b_1'} \omega^{b_1'(a_2-a_1)}\right)$$

$$= d\, \delta_{b_1,b_2}\, \delta_{a_1,a_2} = d\, \delta_{u,v}$$

Summary of results:

$$\mathrm{tr}[X^a Z^b] = d\, \delta_{a,0}\, \delta_{b,0}$$

[where by $\delta_{a,0}$ we mean $\delta_{a,0} + \delta_{a,d} + \delta_{a,2d} + \dots$ from now on]

$$\Downarrow$$

$$\mathrm{tr}[T_{ab} T_{a'b'}] = d\, \delta_{a_1-a'}\, \delta_{b_1-b'}$$

$$\Downarrow$$

$$\mathrm{tr}[A_u^\dagger A_v] = d\, \delta_{u,v} \quad (1 \text{ qudit})$$

$$\Downarrow$$

$$\mathrm{tr}[A_u^\dagger A_v] = \mathrm{tr}[(A_{u_1}^\dagger \otimes \cdots \otimes A_{u_n}) * (A_{v_1} \otimes \cdots \otimes A_{v_n})]$$

$$= \prod_i \mathrm{tr}[A_{u_i}^\dagger A_{v_i}] = \prod_i (d\, \delta_{u_i v_i}) = d^n \delta_{u,v}$$

And nowhere in <u>any</u> of this, did I end up using the $\frac{1}{2}$ in $\omega^{-\frac{ab}{2}}$!

In fact, I didn't even use the fact that $\omega^{-\frac{ab}{2}}$ was there at all! I think..

Actually, no. I did use it in

$$\mathrm{tr}(X^a Z^b) \longrightarrow \mathrm{tr}(T_{ab} T_{a'b'})$$

and in fact the $\frac{1}{2}$ was crucial in figuring out that! :)

So $W_\rho(u) = \frac{tr[\rho A_u]}{d^n} \rightarrow \sum_u W_\rho(u) = \frac{1}{d^n}\sum_u tr[\rho A_u]$

And $\rho = \sum_u C_u A_u \Rightarrow tr[\rho A_u] = tr[(\sum_v C_v A_v) A_u]$

$= \sum_v tr[C_v A_v A_u] = \sum_v C_v \, tr[A_v A_u]$

$= \sum_v C_v d^n \delta_{uv} = d^n C_u \Rightarrow C_u = \frac{tr[\rho A_u]}{d^n}$,

so $\rho = \sum_u \left(\frac{1}{d^n} tr[\rho A_u]\right) A_u$

$= \frac{1}{d^n}\sum_u tr\left(\left(\sum_v \left\{\frac{1}{d^n} tr[\rho A_v]\right\} A_v\right) A_u\right)$

$= \left(\frac{1}{d^n}\right)^2 \sum_{uv} tr\left[tr[\rho A_v] A_v A_u\right]$

wtf ... lets restart:

$\sum_u W_\rho(u) = \sum_u \frac{tr[\rho A_u]}{d^n} = \frac{1}{d^n} tr[\rho \underbrace{\sum_u A_u}_{}]$

maybe $\sum_u A_u = 1$?

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- Yes, $\sum_u A_u \propto \mathbb{1}$ because $\sum_u A_u = \frac{1}{d^n}\sum_u T_u^\dagger \left(\sum_v T_v\right) T_u = \frac{1}{d^n}\left(\sum_v T_v\right)\left(\sum_u T_u^{-1} T_u\right)$

$$= \frac{1}{d^n}\left(\sum_v T_v\right)\left(\sum_u \mathbb{1}\right) = \frac{1}{d^n}\mathbb{1}\left((d^n)^2 \mathbb{1}\right) = d^n$$

so $\sum_u W_u(\rho) = \sum_u \frac{tr[\rho A_u]}{d^n} = \frac{1}{d^n} tr\left[\rho \sum_u A_u\right] = \frac{1}{d^n} tr\left[d^n \rho \mathbb{1}\right]$

$$= tr[\rho] = 1$$

Also, from the PDF: $tr(A_u) = \frac{1}{d^n} tr\left(\cancel{T_u^\dagger}\left(\sum_v T_v\right)\cancel{T_u}\right)$

$= \frac{1}{d^n} tr(\mathbb{1}) = 1$, which means that

$$tr[\rho] = tr\left[\frac{\sum_u tr[\rho A_u] A_u}{d^n}\right] = \sum_u tr[A_u W_u(\rho)]$$

$$= \sum_u tr[A_u] W_u(\rho) = \sum_u W_u(\rho)$$

- mana vs. our qty:

> $sn(\rho) = \sum_{u \in N(\rho)} |W_u(\rho)| = \frac{1}{2}\left(\sum_u |W_u(\rho)| - 1\right) \to \sum_u |W_u(\rho)| = 2sn(\rho) + 1$

$\mathcal{M}(\rho) = \log\left(\sum_u |W_\rho(u)|\right) = \log(2sn(\rho)+1)$

> $\tilde{\mathcal{M}}(\rho) = \sqrt{\sum_{u \in N(\rho)} W_u(\rho)^2}$. Anything else to say here??

- Bounds on $\tilde{\mathcal{M}}(\rho)$ and $\mathcal{M}(\rho)$:
  > The following inequalities might help:

(sum of squares) $\frac{1}{n}\sum_{i=1}^{n} a_i \leq \sqrt{\frac{1}{n}\sum_{i=1}^{n} a_i^2} \Rightarrow \left(\sum a\right)^2 \leq n \sum a^2$

(Cauchy-Schwartz) $\sum_{i=1}^{n} a_i b_i \leq \sqrt{\sum_i a_i^2}\sqrt{\sum_i b_i^2}$

(Jensen's) $f\left(\sum \lambda_i a_i\right) \leq \sum \lambda_i f(a_i)$ if f concave up / convex

> we also know that for a single qutrit,

$$\mathrm{sn}(\rho) \leq \mathrm{sn}(|S\rangle\langle S|) = +\frac{1}{3}$$

$$\Rightarrow \mathcal{M}(\rho) \leq \log\left(2\,\mathrm{sn}(|S\rangle\langle S|)+1\right) = \log\left(\frac{2}{3}+1\right) = \log\left(\frac{5}{3}\right)$$

So for a product state of two qutrits:

$$\mathcal{M}(\rho\otimes\sigma) = \mathcal{M}(\rho) + \mathcal{M}(\sigma) \leq 2\log\left(\frac{5}{3}\right)$$

In general, $\mathcal{M}(\rho^{\otimes n}) \leq n\log\left(\frac{5}{3}\right)$

> I don't see how this bound can be extended for a general bound on even non-product states though...

> $\tilde{\mathcal{M}}(\rho) = \sum_{u \in N(\rho)} W_u(\rho)^2$ which, according to Chris,

looks like an "operator norm" / "second Renyi entropy"

which corresponds to $tr[O^\dagger O]$ / $-\log \sum_i p_i^2 = \log tr[\rho^2]$.

I don't see how either of these are related:

- Relationship to optimization problem:

$$\tilde{\mathcal{M}}(\rho) = \min_{\sigma \in \{X \in L(d^N, d^N) \text{ s.t. } \mathcal{M}(X)=0\}} D(\rho \| \sigma)^2$$

where

$$D(\rho\|\sigma)^2 = \sum_u \left(W_u(\rho) - W_u(\sigma)\right)^2$$

$$= \sum_u \left(tr[\rho A_u] - tr[\sigma A_u]\right)^2$$

$$= \sum_u tr[(\rho - \sigma) A_u]^2$$

Now note that the frobenius norm of $\rho - \sigma$ is

$$\sqrt{\langle \rho-\sigma, \rho-\sigma \rangle} = \sqrt{tr[(\rho-\sigma)^\dagger(\rho-\sigma)]}$$

$$= \sqrt{tr[(\rho-\sigma)^2]} = \sqrt{\sum_i (\rho-\sigma)^2_{ii}}$$

So, that's not the right quantity. What about the coefficient matrices of ρ and σ (vectors?) in the A_u basis?

$$W(\rho) \equiv \sum_u W_u(\rho) A_u$$
$$W(\sigma) = \sum_u W_u(\sigma) A_u$$
$$\Downarrow$$
$$\| W(\rho) - W(\sigma) \|_F^2 = \quad \ldots \text{ but it's not a matrix.}$$

> From looking @ code:

AHA! the frobenius norm comes from:

$$\rho = M^u_{1\alpha} M^v_{\alpha 1} A_u \otimes A_v$$

$$W_{u,v}(\rho) = \frac{1}{d^2} tr[\rho A_u \otimes A_v] = \frac{1}{d^2} M^{u'}_{1\alpha} M^{v'}_{\alpha 1} tr[A_u A_{u'}] \, tr[A_v A_{v'}]$$

$$= \frac{1}{\cancel{d^2}} M^{u'}_{1\alpha} M^{v'}_{\alpha 1} \delta_{uu'} \delta_{vv'} \cancel{d^2}$$

$$= M^u_{1\alpha} M^v_{\alpha 1}$$

So :

$$D(\rho\|\sigma)^2 = \|W_{u,v}(\rho) - W_{u,v}(\sigma)\|_F^2$$

$$= tr[$$

- Meeting with Chris: note that

$$tr[\rho^\dagger \rho] = \rho_{ij}^* \rho_{ji} = (U\rho)^\dagger (U\rho)$$

① Frobenius norm is invariant under superunitaries (which is why you can relate this to $H_2(\rho)$) → use Jensen's to get:

$$m(\rho) \leq \frac{1}{2}(\ln 3 - H_2(\rho)) \leq 0.549 - \frac{1}{2} S_2$$

② and $S_2(\rho) > 0$

~~[illegible]~~ $\ln(3) - H_2$ is kind of "how far from the identity matrix you are"

- working out what Chris said in the interview:

To find a bound on the mana of a general N-qutrit state, we can square the abs. val sum:

$$\left(\sum_u |W_u(\rho)|\right)^2 \leq \sum_u |W_u(\rho)|^2 \quad \text{(Jensen's [}x^2\text{ concave])}$$

$$= \sum_u W_u(\rho)^2$$

So

$$\mathcal{M}(\rho) = \log\left(\sum_u |W_u(\rho)|\right)$$

$$= \frac{1}{2} \cdot 2 \log\left(\sum_u |W_u(\rho)|\right)$$

$$= \frac{1}{2} \log\left(\left(\sum_u |W_u(\rho)|\right)^2\right)$$

$$\leq \frac{1}{2} \log\left(\sum_u (W_u(\rho)^2)\right) \quad \text{(Previous result + log increasing)}$$

weird... idk how to get to Chris' result. But I think I can use the fact that $tr[\rho^2]$ is invariant under a change of basis to bound $\tilde{\mathcal{M}}$ in terms of $H_2(\rho)$.

The second Renyi entropy is

$$S_2(\rho) = -\ln\left(\mathrm{tr}[\rho^2]\right)$$

Now, $W_u(\rho) = \frac{1}{d^n}\mathrm{tr}[\rho A_u]$ and $\rho = \sum_u W_u(\rho) A_u$:

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- what does Chris mean by "frobenius norm is invariant under superunitary change of basis"? At first, it actually seems like the statement is trivial:

$$\rho = \sum_u \mathrm{tr}[\rho \sigma_u]\sigma_u = \sum_u W_u(\rho) A_u$$

generalized paulis

of course these are just the same operator, so $\mathrm{tr}[\rho^2] = \mathrm{tr}[RHS^2]$. But what does this tell us about the coefficients $W_u(\rho)$ in terms of $S_2(\rho) = \mathrm{tr}[\rho^2]$?

$$\mathrm{tr}[\rho^2] = S_2(\rho) = \mathrm{tr}\left[\left(\sum_u W_u(\rho) A_u\right)^2\right] = \mathrm{tr}\left[\sum_{uv} W_u(\rho) W_v(\rho) A_u A_v\right]$$

$$= \sum_{uv} W_u(\rho) W_v(\rho)\,\mathrm{tr}[A_u A_v] = d^n \sum_u W_u(\rho)^2$$

so:

$$\boxed{S_2(\rho) = d^n \sum_u W_u(\rho)^2}$$

• Now: $\sum_{u\in N}|W_u| = \frac{1}{2}\left(\sum_u |W_u| - 1\right)$, $M = \log\left(\sum_u |W_u|\right)$

$$= \log\left(2\sum_{u\in N}|W_u| + 1\right)$$

How can we get a bound on M out of these? Let's just try squaring all of them:

$$\Rightarrow \left(\sum_{u\in N}|W_u|\right)^2 = \frac{1}{4}\left(\left(\sum_u |W_u|\right)^2 - 2\sum_u |W_u| + 1\right)$$

$$\Rightarrow \quad sn^2 \leq \frac{1}{4}\left(\sum_u W_u^2 - 2\sum_u |W_u| + 1\right)$$

$$\Rightarrow \quad sn^2 \leq \frac{1}{4}\left(\frac{S_2}{d^n} - 2\sum_u |W_u| + 1\right)$$

$$\Rightarrow \quad 4\,sn^2 - \frac{S_2}{d^n} - 1 \leq -2\sum_u |W_u|$$

$$\Rightarrow -2\,sn^2 + \frac{1}{2}\left(\frac{S_2}{d^n} + 1\right) \geq \sum_u |W_u|$$

$$\Rightarrow \log\left(-2sn^2 + \frac{1}{2}\left(\frac{S_2}{d^n} + 1\right)\right) \geq \log\left(\sum_u |W_u|\right)$$

$$\Rightarrow \quad M \leq \log\left(-2sn^2 + \frac{1}{2}\left(\frac{S_2}{d^n} + 1\right)\right)$$

Not sure how to continue with this because:

+ I know $sn \leq \frac{1}{3}$ for a <u>single qutrit</u>, but I don't know anything for multiple

+ All I know about S_2 is that it is ≥ 0 ... no way to use that in a $\leq$ inequality...

Maybe Chris was just talking about a bound on the mana of a single qutrit.. but can't you get that out of the definition of mana in terms of sn and the fact that $sn \leq \frac{1}{3}$?

$$\begin{aligned} M &= \log(2sn + 1) \\ &\leq \log\left(2\left(\tfrac{1}{3}\right) + 1\right) \\ &= \log\left(\tfrac{5}{3}\right). \quad \checkmark \end{aligned}$$

ok... so that's not it. somehow Chris gets a $-S_2$ instead of $+S_2$ to come out. How's that?

$$\mathcal{M} = \ln \sum_u |W_u|, \quad \sum_{u \in N} |W_u| = \frac{1}{2}\left(\sum_u |W_u| - 1\right)$$

$$\Rightarrow \left(\sum_{u\in N} |W_u|\right)^2 = \frac{1}{4}\left(\sum_u |W_u| - 1\right)^2$$

$$= \frac{1}{4}\left(\left(\sum_u |W_u|\right)^2 - 2\sum_u |W_u| + 1\right)$$

$$\leq \frac{1}{4}\left(\sum_u W_u^2 - 2\sum_u |W_u| + 1\right) \quad [\text{Jensen's}]$$

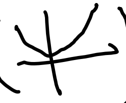
$$= \frac{1}{4}\left(\frac{S_2}{d^n} - 2\sum_u |W_u| + 1\right) \quad \left[S_2 = \operatorname{tr}\rho^2 = d^2 \sum_u W_u^2\right]$$

$$\Rightarrow 4\left(\sum_{u\in N} |W_u|\right)^2 - \left(\frac{S_2}{d^n} + 1\right) \leq -2\sum_u |W_u|$$

$$\Rightarrow -2\left(\sum_{u\in N} |W_u|\right)^2 + \frac{1}{2}\left(\frac{S_2}{d^n} + 1\right) \geq \sum_u |W_u|$$

$$\Rightarrow \mathcal{M} = \ln \sum_u |W_u| \leq \quad \ldots$$

The signs are all wrong.. maybe you did the inequality in reverse?

Review of Jensen's: for f convex ():

$$f\left(\frac{\sum_i \lambda_i a_i}{\sum_i \lambda_i}\right) \leq \frac{\sum \lambda_i f(a_i)}{\sum \lambda_i}$$

So you were using it incorrectly! You need weights to sum to 1.. and indeed $\sum_u W_u = 1$ so:

$$\left(\sum_u W_u X_u\right)^2 \leq \sum_u W_u X_u^2$$

for all X_u ... hm.. what if we let $X_u = W_u$?

$$\left(\sum_u W_u W_u\right)^2 \leq \sum_u W_u W_u^2 \Rightarrow \sum_u W_u^2 \leq \sum_u W_u^3$$

what about $X_u = 1$?

$$\left(\sum_u W_u\right)^2 \leq \sum_u W_u = 1$$

Aha.. I think this is it!

$$\left(\sum_{u\in N}|W_u|\right)^2 = \frac{1}{4}\left(\cancel{\sum_u W_u^2} - 2\sum_u |W_u| + 1\right)$$

WRONG! should be $\left(\sum_u |W_u|\right)^2$

$$\leq \frac{1}{4}\left(1 - 2\sum_u |W_u| + 1\right) \quad \text{[Jensen's]}$$

$$= \frac{1}{2}\left(1 - \sum_u |W_u|\right)$$

$$\Rightarrow \quad 2sn^2 \leq 1 - \sum_u |W_u|$$

$$\Rightarrow \quad 1 - 2sn^2 \geq \sum_u |W_u|$$

$$\Rightarrow \quad \log\left(\sum_u |W_u|\right) = \boxed{M \leq \log(1 - 2sn^2)}$$

I can't get a bound on M out of this, even for a single qutrit, because $sn \leq \frac{1}{3}$ doesn't give me any info on M

wait... what if I let $X_u = \begin{cases} +1 & W_u > 0 \\ -1 & W_u \leq 0 \end{cases}$?

Then:

$$f\left(\sum_u W_u X_u\right) = f\left(\sum_{u \in P} W_u - \sum_{u \in N} W_u\right) = f\left(\sum_u |W_u|\right) !$$

$$\leq \sum_u W_u f(X_u)$$

If we let $f(x) = |x|$ we get:

$$\left|\sum_u |W_u|\right| \leq \sum_u W_u |X_u| \Rightarrow \sum_u |W_u| \leq \sum_u W_u = 1$$

$$\Rightarrow \ln \sum_u |W_u| \leq \ln(1) = 0$$

I don't think that's right.. in fact, I know for a fact that $\sum_u |W_u| \geq 1$ because $\sum_u W_u = 1$, so if some W_u are negative then $\sum_u |W_u| > 1$. I guess $|x|$ isn't convex.

However, if $f(x) = x^2$:

$$\left(\sum_u |W_u|\right)^2 \leq \sum_u W_u (X_u)^2 = \sum_u W_u = 1$$

<u>This</u> is the real inequality I need, but unfortunately all it does is show that the unhelpful result $M \leq \log(1-2sn^2)$ which I found earlier is actually true, even though I had derived it by mistaking $(\sum_u |W_u|)^2$ for $(\sum_u W_u)^2$...

I could let $X_u = \begin{cases} -1 & W_u < 0 \\ 0 & W_u \geq 0 \end{cases}$. Then!

$$f\left(\sum_u W_u X_u\right) = f\left(\sum_{u \in N} |W_u|\right) \leq \sum_u W_u f(X_u).$$

For $f(x) = x^2$:

$$\left(\sum_{u \in N} |W_u|\right)^2 \leq \sum_u W_u (X_u)^2 = \sum_{u \in N} W_u = -\sum_{u \in N} |W_u|$$

$$\Rightarrow \sum_{u \in N} |W_u| \leq -1 \text{ or } \textstyle\sum_{u \in N} |W_u| = 0.$$

NOT TRUE! It's ≥ 0....

I just don't get what to do..

$$\sum_{u \in N} |W_u| = \frac{1}{2}\left(\sum_u |W_u| - 1\right)$$

$$\mathcal{M} = \ln\left(\sum_u |W_u|\right) = \ln\left(2\sum_{u\in N}|W_u| + 1\right) \dots$$

Hm.. I could let $X_u = \begin{cases} +1 & w_u < 0 \\ 0 & \text{else} \end{cases}$ and $f = -\ln(2x+1)$:

Then $f(\sum_u W_u X_u) \leq \sum_u W_u f(X_u)$ gives:

$$-\ln\left(\cancel{-}2\sum_{u\in N}|W_u| + 1\right) \leq \sum_{u\in N} W_u \ln(3)$$

but the negative sign ruins it..

If we let $f = -\ln(2x-1)$:

$$-\ln\left(\underbrace{(-1)\left(2\sum_{u\in N}|W_u| + 1\right)}_{\text{this is undefined..}}\right) \leq \sum_{u\in N} W_u \ln(3)$$

And it seems like letting X_u depend on W_u might not work anyway.. see 2 pages ago.