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Semigroup

In mathematics, a **semigroup** is an algebraic structure consisting of a set together with an associative binary operation.

The binary operation of a semigroup is most often denoted multiplicatively: $x \cdot y$, or simply xy, denotes the result of applying the semigroup operation to the ordered pair (x, y). Associativity is formally expressed as that $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all x, y and z in the semigroup.

The name "semigroup" originates in the fact that a semigroup generalizes a group by preserving only associativity and closure under the binary operation from the axioms defining a group while omitting the requirement for an identity element and inverses.[note 1] From the opposite point of view (of adding rather than removing axioms), a semigroup is an associative magma. As in the case of groups or magmas, the semigroup operation need not be commutative, so $x \cdot y$ is not necessarily equal to $y \cdot x$; a typical example of associative but non-commutative operation is matrix multiplication. If the semigroup operation is commutative, then the semigroup is called a *commutative semigroup* or (less often than in the analogous case of groups) it may be called an *abelian semigroup*.

A monoid is an algebraic structure intermediate between groups and semigroups, and is a semigroup having an identity element, thus obeying all but one of the axioms of a group; existence of inverses is not required of a monoid. A natural example is strings with concatenation as the binary operation, and the empty string as the identity element. Restricting to non-empty strings gives an example of a semigroup that is not a monoid. Positive integers with addition form a commutative semigroup that is not a monoid, whereas the non-negative integers do form a monoid. A semigroup without an identity element can be easily turned into a monoid by just adding an identity element. Consequently, monoids are studied in the theory of semigroups rather than in group theory. Semigroups should not be confused with quasigroups, which are a generalization of groups in a different direction; the operation in a quasigroup need not be associative but quasigroups preserve from groups a notion of division. Division in semigroups (or in monoids) is not possible in general.

The formal study of semigroups began in the early 20th century. Early results include a Cayley theorem for semigroups realizing any semigroup as transformation semigroup, in which arbitrary functions replace the role of bijections from group theory. Other fundamental techniques of studying semigroups like Green's relations do not imitate anything in group theory though. A deep result in the classification of finite semigroups is Krohn–Rhodes theory. The theory of finite semigroups has been of particular importance in theoretical computer science since the 1950s because of the natural link between finite semigroups and finite automata via the syntactic monoid. In probability theory, semigroups are associated with Markov processes.[1][2] In other areas of applied mathematics, semigroups are fundamental models for linear time-invariant systems. In partial differential equations, a semigroup is associated to any equation whose spatial evolution is independent of time. There are numerous special classes of semigroups, semigroups with additional properties, which appear in particular applications. Some of these classes are even closer to groups by exhibiting some additional but not all properties of a group. Of these we mention: regular semigroups, orthodox semigroups, semigroups with involution, inverse semigroups and cancellative semigroups. There also interesting classes of semigroups that do not contain any groups except the trivial group; examples of the latter kind are bands and their commutative subclass—semilattices, which are also ordered algebraic structures.

Contents

Definition

A semigroup is a set S together with a binary operation "$\cdot$" (that is, a function $\cdot : S \times S \rightarrow S$) that satisfies the associative property:

> For all $a, b, c \in S$, the equation $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ holds.

More succinctly, a semigroup is an associative magma.

Examples of semigroups

- Empty semigroup: the empty set forms a semigroup with the empty function as the binary operation.
- Semigroup with one element: there is essentially only one (specifically, only one up to isomorphism), the singleton {*a*} with operation *a* · *a* = *a*.
- Semigroup with two elements: there are five which are essentially different.
- The set of positive integers with addition. (With 0 included, this becomes a monoid.)
- The set of integers with minimum or maximum. (With positive/negative infinity included, this becomes a monoid.)
- Square nonnegative matrices of a given size with matrix multiplication.
- Any ideal of a ring with the multiplication of the ring.
- The set of all finite strings over a fixed alphabet Σ with concatenation of strings as the semigroup operation — the so-called "free semigroup over Σ". With the empty string included, this semigroup becomes the free monoid over Σ.
- A probability distribution F together with all convolution powers of F, with convolution as the operation. This is called a convolution semigroup.
- A monoid is a semigroup with an identity element.
- A group is a monoid in which every element has an inverse element.
- Transformation semigroups and monoids.
- The set of continuous functions from a topological space to itself with composition of functions forms a monoid with the identity function acting as the identity. More generally, the endomorphisms of any object of a category form a monoid under composition.

Basic concepts

Identity and zero

A **left identity** of a semigroup S (or more generally, magma) is an element e such that for all x in S, $ex = x$. Similarly, a **right identity** is an element f such that for all x in S, $xf = x$. Left and right identities are both called **one-sided identities**. A semigroup may have one or more left identities but no right identity, and vice versa.

A **two-sided identity** (or just **identity**) is an element that is both a left and right identity. Semigroups with a two-sided identity are called **monoids**. A semigroup may have at most one two-sided identity. If a semigroup has a two-sided identity, then the two-sided identity is the only one-sided identity in the semigroup. If a semigroup has both a left identity and a right identity, then it has a two-sided identity (which is therefore the unique one-sided identity).

A semigroup S without identity may be embedded in a monoid formed by adjoining an element $e \notin S$ to S and defining $e \cdot s = s \cdot e = s$ for all $s \in S \cup \{e\}$.[3][4] The notation S^1 denotes a monoid obtained from S by adjoining an identity *if necessary* ($S^1 = S$ for a monoid).[4]

Similarly, every magma has at most one absorbing element, which in semigroup theory is called a **zero**. Analogous to the above construction, for every semigroup S, one can define S^0, a semigroup with 0 that embeds S.

Subsemigroups and ideals

The semigroup operation induces an operation on the collection of its subsets: given subsets *A* and *B* of a semigroup *S*, their product *A* · *B*, written commonly as *AB*, is the set { *ab* | *a* in *A* and *b* in *B* }. (This notion is defined identically as it is for groups.) In terms of this operation, a subset *A* is called

- a **subsemigroup** if *AA* is a subset of *A*,
- a **right ideal** if *AS* is a subset of *A*, and
- a **left ideal** if *SA* is a subset of *A*.

If *A* is both a left ideal and a right ideal then it is called an **ideal** (or a **two-sided ideal**).

If *S* is a semigroup, then the intersection of any collection of subsemigroups of *S* is also a subsemigroup of *S*. So the subsemigroups of *S* form a complete lattice.

An example of semigroup with no minimal ideal is the set of positive integers under addition. The minimal ideal of a commutative semigroup, when it exists, is a group.

Green's relations, a set of five equivalence relations that characterise the elements in terms of the principal ideals they generate, are important tools for analysing the ideals of a semigroup and related notions of structure.

The subset with the property that its every element commutes with any other element of the semigroup is called the **center** of the semigroup.[5] The center of a semigroup is actually a subsemigroup.[6]

Homomorphisms and congruences

A **semigroup homomorphism** is a function that preserves semigroup structure. A function *f*: *S* → *T* between two semigroups is a homomorphism if the equation

$$f(ab) = f(a)f(b).$$

holds for all elements *a*, *b* in *S*, i.e. the result is the same when performing the semigroup operation after or before applying the map *f*.

A semigroup homomorphism between monoids preserves identity if it is a monoid homomorphism. But there are semigroup homomorphisms which are not monoid homomorphisms, e.g. the canonical embedding of a semigroup S without identity into S^1. Conditions characterizing monoid homomorphisms are discussed further. Let $f : S_0 \to S_1$ be a semigroup homomorphism. The image of f is also a semigroup. If S_0 is a monoid with an identity element e_0, then $f(e_0)$ is the identity element in the image of f. If S_1 is also a monoid with an identity element e_1 and e_1 belongs to the image of f, then $f(e_0) = e_1$, i.e. f is a monoid homomorphism. Particularly, if f is surjective, then it is a monoid homomorphism.

Two semigroups *S* and *T* are said to be isomorphic if there is a bijection *f* : *S* ↔ *T* with the property that, for any elements *a*, *b* in *S*, *f*(*ab*) = *f*(*a*)*f*(*b*). Isomorphic semigroups have the same structure.

A **semigroup congruence** $\sim$ is an equivalence relation that is compatible with the semigroup operation. That is, a subset $\sim \subseteq S \times S$ that is an equivalence relation and $x \sim y$ and $u \sim v$ implies $xu \sim yv$ for every x, y, u, v in *S*. Like any equivalence relation, a semigroup congruence $\sim$ induces congruence classes

$$[a]_\sim = \{x \in S |\ x \sim a\}$$

and the semigroup operation induces a binary operation ∘ on the congruence classes:

$$[u]_\sim \circ [v]_\sim = [uv]_\sim$$

Because $\sim$ is a congruence, the set of all congruence classes of $\sim$ forms a semigroup with ∘, called the **quotient semigroup** or **factor semigroup**, and denoted $S/\sim$. The mapping $x \mapsto [x]_\sim$ is a semigroup homomorphism, called the **quotient map**, **canonical surjection** or **projection**; if S is a monoid then quotient semigroup is a monoid with identity $[1]_\sim$. Conversely, the kernel of any semigroup homomorphism is a semigroup congruence. These results are nothing more than a particularization of the first isomorphism theorem in universal algebra. Congruence classes and factor monoids are the objects of study in string rewriting systems.

A **nuclear congruence** on *S* is one which is the kernel of an endomorphism of *S*.[7]

A semigroup *S* satisfies the **maximal condition on congruences** if any family of congruences on *S*, ordered by inclusion, has a maximal element. By Zorn's lemma, this is equivalent to saying that the ascending chain condition holds: there is no infinite strictly ascending chain of congruences on *S*.[8]

Every ideal *I* of a semigroup induces a subsemigroup, the Rees factor semigroup via the congruence *x* ρ *y* ⇔ either *x* = *y* or both *x* and *y* are in *I*.

Quotients and divisions

The following notions[9] introduce the idea that a semigroup is contained in another one.

A semigroup **T** is a quotient of a semigroup **S** if there is a surjective semigroup morphism from **T** to **S**. For example, $(\mathbb{Z}/2\mathbb{Z}, +)$ is a quotient of $(\mathbb{Z}/4\mathbb{Z}, +)$, using the morphism consisting of taking the remainder modulo **p** of an integer.

A semigroup **T** divides a semigroup **S**, noted $T \preceq S$ if **T** is a quotient of a subsemigroup **S**. In particular, subsemigroups of **S** divides **T**, while it is not necessarily the case that there are a quotient of **S**.

Both of those relation are transitive.

Structure of semigroups

For any subset *A* of *S* there is a smallest subsemigroup *T* of *S* which contains *A*, and we say that *A* **generates** *T*. A single element *x* of *S* generates the subsemigroup { x^n | *n* is a positive integer }. If this is finite, then *x* is said to be of **finite order**, otherwise it is of **infinite order**. A semigroup is said to be **periodic** if all of its elements are of finite order. A semigroup generated by a single element is said to be monogenic (or cyclic). If a monogenic semigroup is infinite then it is isomorphic to the semigroup of positive integers with the operation of addition. If it is finite and nonempty, then it must contain at least one idempotent. It follows that every nonempty periodic semigroup has at least one idempotent.

A subsemigroup which is also a group is called a **subgroup**. There is a close relationship between the subgroups of a semigroup and its idempotents. Each subgroup contains exactly one idempotent, namely the identity element of the subgroup. For each idempotent *e* of the semigroup there is a unique maximal subgroup containing *e*. Each maximal subgroup arises in this way, so there is a one-to-one correspondence between idempotents and maximal subgroups. Here the term *maximal subgroup* differs from its standard use in group theory.

More can often be said when the order is finite. For example, every nonempty finite semigroup is periodic, and has a minimal ideal and at least one idempotent. The number of finite semigroups of a given size (greater than 1) is (obviously) larger than the number of groups of the same size. For example, of the sixteen possible "multiplication tables" for a set of two elements {a, b}, eight form semigroups[note 2] whereas only four of these are monoids and only two form groups. For more on the structure of finite semigroups, see Krohn–Rhodes theory.

Special classes of semigroups

- A monoid is a semigroup with identity.
- A subsemigroup is a subset of a semigroup that is closed under the semigroup operation.
- A band is a semigroup the operation of which is idempotent.
- A cancellative semigroup is one having the cancellation property:[10] $a \cdot b = a \cdot c$ implies $b = c$ and similarly for $b \cdot a = c \cdot a$.
- A semilattice is a semigroup whose operation is idempotent and commutative.
- 0-simple semigroups.
- Transformation semigroups: any finite semigroup *S* can be represented by transformations of a (state-) set *Q* of at most $|S| + 1$ states. Each element *x* of *S* then maps *Q* into itself $x: Q \to Q$ and sequence *xy* is defined by $q(xy) = (qx)y$ for each *q* in *Q*. Sequencing clearly is an associative operation, here equivalent to function composition. This representation is basic for any automaton or finite state machine (FSM).
- The bicyclic semigroup is in fact a monoid, which can be described as the free semigroup on two generators *p* and *q*, under the relation $pq = 1$.
- C_0-semigroups.
- Regular semigroups. Every element *x* has at least one inverse *y* satisfying $xyx=x$ and $yxy=y$; the elements *x* and *y* are sometimes called "mutually inverse".
- Inverse semigroups are regular semigroups where every element has exactly one inverse. Alternatively, a regular semigroup is inverse if and only if any two idempotents commute.
- Affine semigroup: a semigroup that is isomorphic to a finitely-generated subsemigroup of Z^d. These semigroups have applications to commutative algebra.

Structure theorem for commutative semigroups

There is a structure theorem for commutative semigroups in terms of semilattices.[11] A semilattice (or more precisely a meet-semilattice) $(L, \leq)$ is a partially ordered set where every pair of elements $a, b \in L$ has a greatest lower bound, denoted $a \wedge b$. The operation $\wedge$ makes L into a semigroup satisfying the additional idempotence law $a \wedge a = a$.

Given a homomorphism $f : S \to L$ from an arbitrary semigroup to a semilattice, each inverse image $S_a = f^{-1}\{a\}$ is a (possibly empty) semigroup. Moreover, S becomes **graded** by L, in the sense that

$S_a S_b \subseteq S_{a \wedge b}$

If f is onto, the semilattice L is isomorphic to the quotient of S by the equivalence relation $\sim$ such that $x \sim y$ iff $f(x) = f(y)$. This equivalence relation is a semigroup congruence, as defined above.

Whenever we take the quotient of a commutative semigroup by a congruence, we get another commutative semigroup. The structure theorem says that for any commutative semigroup S, there is a finest congruence $\sim$ such that the quotient of S by this equivalence relation is a semilattice. Denoting this semilattice by L, we get a homomorphism f from S onto L. As mentioned, S becomes graded by this semilattice.

Furthermore, the components S_a are all Archimedean semigroups. An Archimedean semigroup is one where given any pair of elements x, y, there exists an element z and $n > 0$ such that $x^n = yz$.

The Archimedean property follows immediately from the ordering in the semilattice L, since with this ordering we have $f(x) \leq f(y)$ if and only if $x^n = yz$ for some z and $n > 0$.

Group of fractions

The **group of fractions** or **group completion** of a semigroup *S* is the group *G* = *G*(*S*) generated by the elements of *S* as generators and all equations *xy* = *z* which hold true in *S* as relations.[12] There is an obvious semigroup homomorphism *j* : *S* → *G*(*S*) which sends each element of *S* to the corresponding generator. This has a universal property for morphisms from *S* to a group:[13] given any group *H* and any semigroup homomorphism *k* : *S* → *H*, there exists a unique group homomorphism *f* : *G* → *H* with *k*=*fj*. We may think of *G* as the "most general" group that contains a homomorphic image of *S*.

An important question is to characterize those semigroups for which this map is an embedding. This need not always be the case: for example, take *S* to be the semigroup of subsets of some set *X* with set-theoretic intersection as the binary operation (this is an example of a semilattice). Since *A*.*A* = *A* holds for all elements of *S*, this must be true for all generators of *G*(*S*) as well: which is therefore the trivial group. It is clearly necessary for embeddability that *S* have the cancellation property. When *S* is commutative this condition is also sufficient[14] and the Grothendieck group of the semigroup provides a construction of the group of fractions. The problem for non-commutative semigroups can be traced to the first substantial paper on semigroups.[15][16] Anatoly Maltsev gave necessary and sufficient conditions for embeddability in 1937.[17]

Semigroup methods in partial differential equations

Semigroup theory can be used to study some problems in the field of partial differential equations. Roughly speaking, the semigroup approach is to regard a time-dependent partial differential equation as an ordinary differential equation on a function space. For example, consider the following initial/boundary value problem for the heat equation on the spatial interval (0, 1) ⊂ **R** and times $t \geq 0$:

$$\begin{cases} \partial_t u(t,x) = \partial_x^2 u(t,x), & x \in (0,1), t > 0; \\ u(t,x) = 0, & x \in \{0,1\}, t > 0; \\ u(t,x) = u_0(x), & x \in (0,1), t = 0. \end{cases}$$

Let $X = L^2((0, 1)\ \mathbf{R})$ be the L^p space of square-integrable real-valued functions with domain the interval (0, 1) and let *A* be the second-derivative operator with domain

$$D(A) = \left\{ u \in H^2((0,1); \mathbf{R}) \middle| u(0) = u(1) = 0 \right\},$$

where H^2 is a Sobolev space. Then the above initial/boundary value problem can be interpreted as an initial value problem for an ordinary differential equation on the space *X*:

$$\begin{cases} \dot{u}(t) = Au(t); \\ u(0) = u_0. \end{cases}$$

On an heuristic level, the solution to this problem "ought" to be $u(t) = \exp(tA)u_0$. However, for a rigorous treatment, a meaning must be given to the exponential of tA. As a function of t, $\exp(tA)$ is a semigroup of operators from X to itself, taking the initial state u_0 at time $t = 0$ to the state $u(t) = \exp(tA)u_0$ at time t. The operator A is said to be the infinitesimal generator of the semigroup.

History

The study of semigroups trailed behind that of other algebraic structures with more complex axioms such as groups or rings. A number of sources[18][19] attribute the first use of the term (in French) to J.-A. de Séguier in *Élements de la Théorie des Groupes Abstraits* (Elements of the Theory of Abstract Groups) in 1904. The term is used in English in 1908 in Harold Hinton's *Theory of Groups of Finite Order*.

Anton Suschkewitsch obtained the first non-trivial results about semigroups. His 1928 paper "Über die endlichen Gruppen ohne das Gesetz der eindeutigen Umkehrbarkeit" ("On finite groups without the rule of unique invertibility") determined the structure of finite simple semigroups and showed that the minimal ideal (or Green's relations J-class) of a finite semigroup is simple.[19] From that point on, the foundations of semigroup theory were further laid by David Rees, James Alexander Green, Evgenii Sergeevich Lyapin, Alfred H. Clifford and Gordon Preston. The latter two published a two-volume monograph on semigroup theory in 1961 and 1967 respectively. In 1970, a new periodical called *Semigroup Forum* (currently edited by Springer Verlag) became one of the few mathematical journals devoted entirely to semigroup theory.

In recent years researchers in the field have become more specialized with dedicated monographs appearing on important classes of semigroups, like inverse semigroups, as well as monographs focusing on applications in algebraic automata theory, particularly for finite automata, and also in functional analysis.

Generalizations

Group-like structures

	Totality[α]	Associativity	Identity	Invertibility	Commutativity
Semigroupoid	Unneeded	Required	Unneeded	Unneeded	Unneeded
Category	Unneeded	Required	Required	Unneeded	Unneeded
Groupoid	Unneeded	Required	Required	Required	Unneeded
Magma	Required	Unneeded	Unneeded	Unneeded	Unneeded
Quasigroup	Required	Unneeded	Unneeded	Required	Unneeded
Loop	Required	Unneeded	Required	Required	Unneeded
Semigroup	Required	Required	Unneeded	Unneeded	Unneeded
Inverse Semigroup	Required	Required	Unneeded	Required	Unneeded
Monoid	Required	Required	Required	Unneeded	Unneeded
Group	Required	Required	Required	Required	Unneeded
Abelian group	Required	Required	Required	Required	Required

^α Closure, which is used in many sources, is an equivalent axiom to totality, though defined differently.

If the associativity axiom of a semigroup is dropped, the result is a magma, which is nothing more than a set *M* equipped with a binary operation $M \times M \to M$.

Generalizing in a different direction, an ***n*-ary semigroup** (also ***n*-semigroup**, **polyadic semigroup** or **multiary semigroup**) is a generalization of a semigroup to a set *G* with a *n*-ary operation instead of a binary operation.[20] The associative law is generalized as follows: ternary associativity is $(abc)de = a(bcd)e = ab(cde)$, i.e. the string *abcde* with any three adjacent elements bracketed. *N*-ary associativity is a string of length $n + (n - 1)$ with any *n* adjacent elements bracketed. A 2-ary semigroup is just a semigroup. Further axioms lead to an *n*-ary group.

A third generalization is the semigroupoid, in which the requirement that the binary relation be total is lifted. As categories generalize monoids in the same way, a semigroupoid behaves much like a category but lacks identities.

Infinitary generalizations of commutative semigroups have sometimes been considered by various authors.[21]

See also

- Absorbing element
- Biordered set
- Empty semigroup
- Identity element
- Light's associativity test
- Semigroup ring
- Weak inverse
- Quantum dynamical semigroup

Notes

1. The closure axiom is implied by the definition of a binary operation on a set. Some authors thus omit it and specify three *laws* for a group and only one law (associativity) for semigroup.
2. Namely: the trivial semigroup in which (for all *x* and *y*) *xy* = a and its counterpart in which *xy* = b, the semigroups based on multiplication modulo 2 (choosing a or b as the identity element 1), the groups equivalent to addition modulo 2 (choosing a or b to be the identity element 0), and the semigroups in which the elements are either both left identities or both right identities.

Citations

1. (Feller 1971)
2. Martino, Ivan; Martino, Luca (2013-11-14). "On the variety of linear recurrences and numerical semigroups" (https://link.springer.com/article/10.1007/s00233-013-9551-2). *Semigroup Forum*. **88** (3): 569–574. arXiv:1207.0111 (https://arxiv.org/abs/1207.0111). doi:10.1007/s00233-013-9551-2 (https://doi.org/10.1007%2Fs00233-013-9551-2). ISSN 0037-1912 (https://www.worldcat.org/issn/0037-1912).
3. Jacobson 2009, p. 30, ex. 5
4. Lawson 1998, p. 20 (https://books.google.com/books?id=_F78nQEACAAJ&pg=PA20&dq=%22adjoining+an+identity%22)

5. Kilp, Mati; Knauer, U.; Mikhalev, Aleksandr V. (2000). *Monoids, Acts, and Categories: With Applications to Wreath Products and Graphs : a Handbook for Students and Researchers* (https://books.google.com/books?id=4gPhmmW-EGcC&pg=PA25). Walter de Gruyter. p. 25. ISBN 978-3-11-015248-7. Zbl 0945.20036 (https://zbmath.org/?format=complete&q=an:0945.20036).
6. Li͡apin, E. S. (1968). *Semigroups* (https://books.google.com/books?id=G8pWKPp4tKwC&pg=PA96). American Mathematical Soc. p. 96. ISBN 978-0-8218-8641-0.
7. Lothaire 2011, p. 463
8. Lothaire 2011, p. 465
9. Pin, Jean-Éric (November 30, 2016). *Mathematical Foundations of Automata Theory* (http://www.liafa.jussieu.fr/~jep/PDF/MPRI/MPRI.pdf) (PDF). p. 19.
10. Clifford & Preston 1967, p. 3
11. Grillet 2001
12. Farb, B. (2006), *Problems on mapping class groups and related topics*, Amer. Math. Soc., p. 357, ISBN 0-8218-3838-5
13. Auslander, M.; Buchsbaum, D. A. (1974). *Groups, rings, modules*. Harper & Row. p. 50. ISBN 0-06-040387-X.
14. Clifford & Preston 1961, p. 34
15. (Suschkewitsch 1928)
16. Preston, G. B. (1990), *Personal reminiscences of the early history of semigroups* (https://web.archive.org/web/20090109045100/http://www.gap-system.org/~history/Extras/Preston_semigroups.html), archived from the original (http://www.gap-system.org/~history/Extras/Preston_semigroups.html) on 2009-01-09, retrieved 2009-05-12
17. Maltsev, A. (1937), "On the immersion of an algebraic ring into a field", *Math. Annalen*, **113**: 686–691, doi:10.1007/BF01571659 (https://doi.org/10.1007%2FBF01571659).
18. Earliest Known Uses of Some of the Words of Mathematics (http://jeff560.tripod.com/s.html)
19. An account of Suschkewitsch's paper by Christopher Hollings (https://www.webcitation.org/query?url=http://uk.geocities.com/cdhollings/suschkewitsch3.pdf&date=2009-10-25+04:13:15)
20. Dudek, W.A. (2001), "On some old problems in *n*-ary groups" (https://web.archive.org/web/20090714003319/http://www.quasigroups.eu/contents/contents8.php?m=trzeci), *Quasigroups and Related Systems*, **8**: 15–36, archived from the original (http://www.quasigroups.eu/contents/contents8.php?m=trzeci) on 2009-07-14
21. See references in Udo Hebisch and Hanns Joachim Weinert, *Semirings and Semifields*, in particular, Section 10, *Semirings with infinite sums*, in M. Hazewinkel, Handbook of Algebra, Vol. 1, Elsevier, 1996. Notice that in this context the authors use the term *semimodule* in place of *semigroup*.

References

General references

- Howie, John M. (1995), *Fundamentals of Semigroup Theory*, Clarendon Press, ISBN 0-19-851194-9, Zbl 0835.20077 (https://zbmath.org/?format=complete&q=an:0835.20077).
- Clifford, A. H.; Preston, G. B. (1961), *The Algebraic Theory of Semigroups* (https://books.google.com/books?id=4vXG2rjCUmUC), **1**, American Mathematical Society, ISBN 978-0-8218-0271-7, Zbl 0111.03403 (https://zbmath.org/?format=complete&q=an:0111.03403).
- Clifford, A. H.; Preston, G. B. (1967), *The Algebraic Theory of Semigroups* (https://books.google.com/books?id=756KAwAAQBAJ), **2**, American Mathematical Society, ISBN 978-0-8218-0272-4, Zbl 0178.01203 (https://zbmath.org/?format=complete&q=an:0178.01203).
- Grillet, Pierre A. (1995), *Semigroups: An Introduction to the Structure Theory* (https://books.google.com/books?id=yM544W1N2UUC), Marcel Dekker, ISBN 978-0-8247-9662-4, Zbl 0830.20079 (https://zbmath.org/?format=complete&q=an:0830.20079).
- Grillet, Pierre A. (2001), *Commutative Semigroups* (https://books.google.com/books?id=7QOOxYbCpggC), Springer Verlag, ISBN 978-0-7923-7067-3, Zbl 1040.20048 (https://zbmath.org/?format=complete&q=an:1040.20048).

- Hollings, Christopher (2014), *Mathematics across the Iron Curtain: A History of the Algebraic Theory of Semigroups*, American Mathematical Society, ISBN 978-1-4704-1493-1, Zbl 1317.20001 (https://zbmath.org/?format=complete&q=an:1317.20001).
- Petrich, Mario (1973), *Introduction to Semigroups*, Charles E. Merrill, ISBN 0-675-09062-8, Zbl 0321.20037 (https://zbmath.org/?format=complete&q=an:0321.20037).

Specific references

- Feller, William (1971), *An introduction to probability theory and its applications*, **II** (2nd ed.), Wiley, MR 0270403 (https://www.ams.org/mathscinet-getitem?mr=0270403).
- Hille, Einar; Phillips, Ralph S. (1974), *Functional analysis and semi-groups* (https://books.google.com/books?id=gf2VAwAAQBAJ), American Mathematical Society, ISBN 0821874640, MR 0423094 (https://www.ams.org/mathscinet-getitem?mr=0423094).
- Suschkewitsch, Anton (1928), "Über die endlichen Gruppen ohne das Gesetz der eindeutigen Umkehrbarkeit", *Mathematische Annalen*, **99** (1): 30–50, doi:10.1007/BF01459084 (https://doi.org/10.1007%2FBF01459084), ISSN 0025-5831 (https://www.worldcat.org/issn/0025-5831), MR 1512437 (https://www.ams.org/mathscinet-getitem?mr=1512437).
- Kantorovitz, Shmuel (2009), *Topics in Operator Semigroups* (https://books.google.com/books?id=bQdLAAAAQBAJ&pg=PP5), Springer, ISBN 978-0-8176-4932-6, Zbl 1187.47003 (https://zbmath.org/?format=complete&q=an:1187.47003).
- Jacobson, Nathan (2009), *Basic algebra*, **1** (2nd ed.), Dover, ISBN 978-0-486-47189-1
- Lawson, M.V. (1998), *Inverse semigroups: the theory of partial symmetries*, World Scientific, ISBN 978-981-02-3316-7, Zbl 1079.20505 (https://zbmath.org/?format=complete&q=an:1079.20505)
- Lothaire, M. (2011) [2002], *Algebraic combinatorics on words*, Encyclopedia of Mathematics and Its Applications, **90**, Cambridge University Press, ISBN 978-0-521-18071-9, Zbl 1221.68183 (https://zbmath.org/?format=complete&q=an:1221.68183)