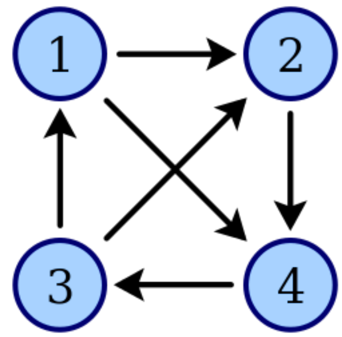
Tournament (graph theory)

A **tournament** is a directed graph (digraph) obtained by assigning a direction for each edge in an undirected complete graph. That is, it is an orientation of a complete graph, or equivalently a directed graph in which every pair of distinct vertices is connected by a single directed edge.

Tournament	
 A tournament on 4 vertices	
Vertices	n
Edges	$\binom{n}{2}$
Table of graphs and parameters	

Many of the important properties of tournaments were first investigated by Landau (1953) in order to model dominance relations in flocks of chickens. Current applications of tournaments include the study of voting theory and social choice theory among other things.

The name *tournament* originates from such a graph's interpretation as the outcome of a round-robin tournament in which every player encounters every other player exactly once, and in which no draws occur. In the tournament digraph, the vertices correspond to the players. The edge between each pair of players is oriented from the winner to the loser. If player a beats player b, then it is said that a *dominates* b. If every player beats the same number of other players (*indegree* = *outdegree*), the tournament is called *regular*.

Contents

Paths and cycles

Any tournament on a finite number n of vertices contains a Hamiltonian path, i.e., directed path on all n vertices (Rédei 1934). This is easily shown by induction on n: suppose that the statement holds for n, and consider any tournament T on $n+1$ vertices. Choose a vertex v_0 of T and consider a directed path $v_1, v_2, \ldots, v_n$ in $T \setminus \{v_0\}$. Now let $i \in \{0, \ldots, n\}$ be maximal such that for every $j \leq i$ there is a directed edge from v_j to v_0.

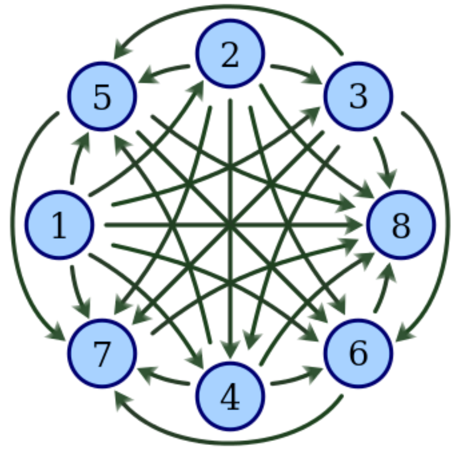
$$v_1, \ldots, v_i, v_0, v_{i+1}, \ldots, v_n$$

is a directed path as desired. This argument also gives an algorithm for finding the Hamiltonian path. More efficient algorithms, that require examining only $O(n \log n)$ of the edges, are known.[1]

This implies that a strongly connected tournament has a Hamiltonian cycle (Camion 1959). More strongly, every strongly connected tournament is vertex pancyclic: for each vertex v, and each k in the range from three to the number of vertices in the tournament, there is a cycle of length k containing v.[2] Moreover, if the tournament is 4-connected, each pair of vertices can be connected with a Hamiltonian path (Thomassen 1980).

Transitivity

A tournament in which $((a \to b)$ and $(b \to c)) \Rightarrow (a \to c)$ is called **transitive**. In other words, in a transitive tournament, the vertices may be (strictly) totally ordered by the edge relation, and the edge relation is the same as reachability.



A transitive tournament on 8 vertices.

Equivalent conditions

The following statements are equivalent for a tournament T on n vertices:

1. T is transitive.
2. T is a strict total ordering.
3. T is acyclic.
4. T does not contain a cycle of length 3.
5. The score sequence (set of outdegrees) of T is $\{0, 1, 2, \ldots, n-1\}$.
6. T has exactly one Hamiltonian path.

Ramsey theory

Transitive tournaments play a role in Ramsey theory analogous to that of cliques in undirected graphs. In particular, every tournament on n vertices contains a transitive subtournament on $1 + \lfloor \log_2 n \rfloor$ vertices.[3] The proof is simple: choose any one vertex v to be part of this subtournament, and form the rest of the subtournament recursively on either the set of incoming neighbors of v or the set of outgoing neighbors of v, whichever is larger. For instance, every tournament on seven vertices contains a three-vertex transitive subtournament; the Paley tournament on seven vertices shows that this is the most that can be guaranteed (Erdős & Moser 1964). However, Reid & Parker (1970) showed that this bound is not tight for some larger values of n.

Erdős & Moser (1964) proved that there are tournaments on n vertices without a transitive subtournament of size $2 + 2\lfloor \log_2 n \rfloor$ Their proof uses a counting argument: the number of ways that a k-element transitive tournament can occur as a subtournament of a larger tournament on n labeled vertices is

$$\binom{n}{k} k! 2^{\binom{n}{2} - \binom{k}{2}},$$

and when k is larger than $2 + 2\lfloor \log_2 n \rfloor$, this number is too small to allow for an occurrence of a transitive tournament within each of the $2^{\binom{n}{2}}$ different tournaments on the same set of n labeled vertices.

Paradoxical tournaments

A player who wins all games would naturally be the tournament's winner. However, as the existence of non-transitive tournaments shows, there may not be such a player. A tournament for which every player loses at least one game is called a 1-paradoxical tournament. More generally, a tournament $T = (V, E)$ is called k-paradoxical if for every k-element subset S of V there is a vertex v_0 in $V \setminus S$ such that $v_0 \to v$ for all $v \in S$. By means of the probabilistic method, Paul Erdős showed that for any fixed value of k, if $|V| \geq k^2 2^k \ln(2 + o(1))$, then almost every tournament on V is k-paradoxical.[4] On the other hand, an easy argument shows that any k-paradoxical tournament must have at least $2^{k+1} - 1$ players, which was improved to $(k+2)2^{k-1} - 1$ by Esther and George Szekeres (1965).[5] There is an explicit construction of k-paradoxical tournaments with $k^2 4^{k-1}(1 + o(1))$ players by Graham and Spencer (1971) namely the Paley tournament.

Condensation

The condensation of any tournament is itself a transitive tournament. Thus, even for tournaments that are not transitive, the strongly connected components of the tournament may be totally ordered.[6]

Score sequences and score sets

The score sequence of a tournament is the nondecreasing sequence of outdegrees of the vertices of a tournament. The score set of a tournament is the set of integers that are the outdegrees of vertices in that tournament.

Landau's Theorem (1953) A nondecreasing sequence of integers $(s_1, s_2, \cdots, s_n)$ is a score sequence if and only if :

1. $0 \leq s_1 \leq s_2 \leq \cdots \leq s_n$
2. $s_1 + s_2 + \cdots + s_i \geq \binom{i}{2}$, for $i = 1, 2, \cdots, n-1$
3. $s_1 + s_2 + \cdots + s_n = \binom{n}{2}$.

Let $s(n)$ be the number of different score sequences of size n. The sequence $s(n)$ (sequence A000571 in the OEIS) starts as:

1, 1, 1, 2, 4, 9, 22, 59, 167, 490, 1486, 4639, 14805, 48107, ...

Winston and Kleitman proved that for sufficiently large n:

$$s(n) > c_1 4^n n^{-\frac{5}{2}},$$

where $c_1 = 0.049.$ Takács later showed, using some reasonable but unproven assumptions, that

$$s(n) < c_2 4^n n^{-\frac{5}{2}},$$

where $c_2 < 4.858.$

Together these provide evidence that:

$$s(n) \in \Theta(4^n n^{-\frac{5}{2}}).$$

Here Θ signifies an asymptotically tight bound.

Yao showed that every nonempty set of nonnegative integers is the score set for some tournament.

Majority relations

In social choice theory, tournaments naturally arise as majority relations of preference profiles.[7] Let A be a finite set of alternatives, and consider a list $P = (\succ_1, \dots, \succ_n)$ of linear orders over A. We interpret each order $\succ_i$ as the preference ranking of a voter i. The (strict) majority relation $\succ_{\mathrm{maj}}$ of P over A is then defined so that $a \succ_{\mathrm{maj}} b$ if and only if a majority of the voters prefer a to b, that is $|\{i \in [n] : a \succ_i b\}| > |\{i \in [n] : b \succ_i a\}|$. If the number n of voters is odd, then the majority relation forms the dominance relation of a tournament on vertex set A.

By a lemma of McGarvey, every tournament on m vertices can be obtained as the majority relation of at most $m(m-1)$ voters.[7][8] Results by Stearns and Erdős & Moser later established that $\Theta(m/\log m)$ voters are needed to induce every tournament on m vertices.[9]

Laslier (1997) studies in what sense a set of vertices can be called the set of "winners" of a tournament. This revealed to be useful in Political Science to study, in formal models of political economy, what can be the outcome of a democratic process.[10]

See also

- Oriented graph
- Paley tournament
- Sumner's conjecture
- Tournament solution

Notes

1. Bar-Noy & Naor (1990).
2. Moon (1966), Theorem 1.
3. Erdős & Moser (1964).
4. Erdős (1963)
5. Szekeres, E.; Szekeres, G. (1965). "On a problem of Schütte and Erdős". *Math. Gaz*. **49**: 290–293.
6. Harary & Moser (1966), Corollary 5b.
7. Brandt, Felix and Brill, Markus and Harrenstein, Paul, "Tournament Solutions". Chapter 3 in: Brandt, Felix; Conitzer, Vincent; Endriss, Ulle; Lang, Jérôme; Procaccia, Ariel D. (2016). *Handbook of Computational Social Choice* (https://books.google.com/books?id=nMHgCwAAQBAJ). Cambridge University Press. ISBN 9781107060432. (free online version (http://www.cambridge.org/download_file/898428))
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10. Austen-Smith, D. and J. Banks, Positive Political theory, 1999, The University of Michigan Press.

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