Iterative Improvement

- Start with some solution.
- Iteratively improve the solution.
- Start with some partial solution.
- Iteratively add/improve the solution.

Stable Matching/Marriage

- Matching in a graph $G = (V, E)$ is a subset E' of edges such that for no two edges have same endpoints: that is, if $(u, v), (u', v') \in E'$ then $\{u, v\} \cap \{u', v'\} = \emptyset$ or $\{u, v\} = \{u', v'\}$.
- Maximum Matching: a matching E' of maximum size.
- Bipartite graph: $V = V_1 \cup V_2$, where $V_1 \cap V_2 = \emptyset$, and $E \subseteq V_1 \times V_2$.

Stable Marriage

- n men and n women.
- Each man has an order of preference among the women (no ties). $M(i, j)$ denotes the preference order for i-th man for the j-woman.
- Each woman has an order of preference among the men (no ties). $W(i, j)$ denotes the preference order for i-th woman for the j-man.
- Thus, for all i, j, $M(i, j), W(i, j) \in \{1, 2, \ldots, n\}$ and for all i, for all j, j' such that $j \neq j'$, $M(i, j) \neq M(i, j')$ and $W(i, j) \neq W(i, j')$.

- Find a stable-matching: i.e., a bijective function $S : \{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, n\}$ such that
If $S(i) = j$ and $S(i') = j'$, then $\neg[M(i', j) < M(i', j')$ and $W(j, i') < W(j, i)]$
That is, if one matches i-th man to j-th woman and i'-th man to j'-woman, then it is not the case that i'-th man prefers j-th woman more than j'-th woman and j-th woman prefers the i'-th man more than i-th man: in which case there is motivation for both i'-th man and j-th woman to switch.

Example

	A	B	C
X	A	B	C
Y	B	A	C
Z	A	B	C

	X	Y	Z
A	Y	X	Z
B	X	Y	Z
C	X	Y	Z

- Is $X - C$, $Y - B$, $Z - A$ stable?
- No. X prefers B over C and B prefers X over Y.
- Is $X - A$, $Y - B$, $Z - C$ stable?
- Yes.

Algorithm

StableMarriage
Initally every one is unmatched;
While there is some free man
Arbitrarily select a man m who is free.
m proposes to the next woman w in his preference list to whom he has not proposed before.
If w is free, then m and w get engaged
Else If w is already engaged to some m', but prefers m to m' then w and m get engaged, and m' becomes free.
Else w rejects m.
Endwhile

Accuracy

- Claim: Every man gets engaged:
- Note that a woman, once engaged, never gets un-engaged (though may switch).
- Each woman/man is engaged to at most one man/woman at any particular time.
- Thus, if m is not engaged at some point, then there must also be a woman w who is also not engaged at the end. Thus, noone has proposed to w, and in particular m has not proposed to w. As m will eventually propose to w (if not already engaged), eventually m and w will get engaged.
- There are at most n^2 proposals: at most n by each man. Thus, the algorithm runs for at most n^2 loops.

- Why is the final assignment stable?
- Suppose m–w is an unstable pair: that is m is assigned to w' and m' is assigned to w, but m prefers w over w' and w prefers m over m'.
- Thus, m must have proposed to w before proposing to w'.
- At that time w must have rejected m: but that means at that time w was assigned to some m'' whom w prefers more than m.
- By the upward assignment property, m' must be prefered more by w than m (and even m'' if m' is not same as m'').
- Contradiction.