18.438 Advanced Combinatorial Optimization	Updated February 18, 2012.

Lecture 5

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In this lecture, we establish the connection between nowhere-zero k-flows and nowhere-zero $\mathbb{Z}_k$-flows. Then, we present several theorems of relations between edge-connectivity of a graph and the existence of nowhere-zero flows.

1 Nowhere-zero k-flow

Let us first recall some definitions from the previous lecture.

Definition 1 *Let $G = (V, E)$ be a directed graph, and Γ be an abelian group. A nowhere-zero* Γ-flow *is $\phi : E \to \Gamma \setminus \{0\}$ such that*

$$\sum_{e\in\delta^-(v)} \phi(e) = \sum_{e\in\delta^+(v)} \phi(e) \qquad (\textit{flow conservation}\ ).$$

If G is undirected, then we say that it has a nowhere-zero Γ flow if the graph admits a nowhere-zero Γ flow after giving an orientation to all the edges.

As we saw, if one orientation works then any does, since inverses exist in abelian groups.

Definition 2 *Let G be an undirected graph. For integer $k \geq 2$, a* nowhere-zero k-flow *ϕ is an assignment $\phi : E \to \{1, \ldots, k-1\}$ such that for some orientation of G flow conservation is achieved, i.e.,*

$$\sum_{e\in\delta^-(v)} \phi(e) = \sum_{e\in\delta^+(v)} \phi(e)$$

for all $v \in V$.

It is often convenient to fix an orientation and let ϕ take values in $\{\pm 1, \cdots, \pm(k-1)\}$.

Theorem 1 (Tutte 1950) *Let G be an undirected graph. Then G has a nowhere-zero k-flow $\iff$ G has a nowhere-zero $\mathbb{Z}_k$-flow.*

Proof: ($\Rightarrow$): By definition of nowhere-zero flows.
($\Leftarrow$): Let ϕ be a nowhere-zero $\mathbb{Z}_k$-flow, define $e(v) = \sum_{e\in\delta^+(v)} \phi(e) - \sum_{e\in\delta^-(v)} \phi(e)$ for all $v \in V$, under group operation in $\mathbb{Z}$. Observe that all $e(v)$'s are multiples of k. Without loss of generality, we can assume ϕ is the nowhere-zero $\mathbb{Z}_k$-flow such that $\sum_{v\in V} |e(v)|$ is minimized where we minimize over all ϕ and all orientations of G..

Suppose $\sum_{v\in V} |e(v)| = 0$, then we have obtained a nowhere-zero k-flow. Otherwise, let $S = \{v : e(v) > 0\}$ and $T = \{v : e(v) < 0\}$. Since $\sum_{v\in V} |e(v)| > 0$ and $\sum_{v\in V} e(v) = 0$, we have that both S and T are nonempty. Let U be the set of vertices reachable from S. If $U \cap T = \emptyset$, then $0 < \sum_{v\in U} e(v) = \sum_{e\in\delta^+(U)} \phi(e) - \sum_{e\in\delta^-(U)} \phi(e)$. But $\delta^+(U) = \emptyset$, and thus implies $\sum_{e\in\delta^+(U)}\phi(e)-\sum_{e\in\delta^-(U)}\phi(e)\le 0$, which is a contradiction. Thus, we must have $U\cap T\neq\emptyset$, which implies we can find a directed path P from some $s\in S$ to $t\in T$.

Now, we "revert" path P to create another nowhere-zero $\mathbb{Z}_k$-flow. More formally, for each arc $e\in P$, reverse the direction of e and define

$$\begin{aligned}\phi'(e)&=k-\phi(e) && \forall e\in P\\ \phi'(e)&=\phi(e) && \forall e\notin P\end{aligned}$$

Then ϕ' is also a nowhere-zero $\mathbb{Z}_k$-flow (for the new orientation). Let $e'(v)=\sum_{e\in\delta^+(v)}\phi'(e)-\sum_{e\in\delta^-(v)}\phi'(e)$ for all $v\in V$. Observe that for any $v\in V\setminus\{s,t\}$, $e'(v)=e(v)$. And $e'(s)=e(s)-k$, $e'(t)=e(t)+k$. This implies $\sum_{v\in V}|e'(v)|<\sum_{v\in V}|e(v)|$, contradicting the minimality of ϕ. Therefore, we have $\sum_{v\in V}|e(v)|=0$, which implies we also have a nowhere-zero k-flow. □

Theorem 1 implies that if G has a nowhere-zero k-flow then G has a nowhere-zero k'-flow, for any integer $k'\ge k$. Combine with the theorem from previous lecture which states that the existence of a nowhere-zero Γ-flow depends not on the group structure of Γ, but only on the size of Γ, we have obtained the following corollary:

Corollary 2 *G has a nowhere-zero Γ-flow $\implies$ G has a nowhere-zero Γ' flow for any $|\Gamma'|\ge|\Gamma|$.*

2 Nowhere-zero Flow and Edge Connectivity

Now, we discuss some open problems and known results of the relation between a graph's edge connectivity and the existence of its nowhere-zero flows. We begin with a famous conjecture of Tutte.

Conjecture 1 *Every 4-edge-connected graph has a nowhere-zero 3-flow.*

And here is a weaker version of this conjecture.

Conjecture 2 *There exists a positive integer k such that every k-edge-connected graph has a nowhere-zero 3-flow.*

This was open for many years and was very recently settled. Carsten Thomasen [2] just showed that every 8-edge-connected graph has a nowhere-zero 3-flow, and this was improved [3] to show that the same is true for 6-edge-connected graphs.

Now, we present some of the results that is known about nowhere-zero flow and edge connectivity.

Theorem 3 (Jaeger) *If an undirected graph $G=(V,E)$ is 4-edge-connected, then G has a nowhere-zero 4-flow (or nowhere-zero $\mathbb{Z}_2\times\mathbb{Z}_2$-flow).*

It is also shown that there is an efficient algorithm for finding a nowhere-zero 4-flow of any given graph G.

Theorem 4 (Jaeger) *If an undirected graph $G = (V, E)$ is 2-edge-connected, then G has a nowhere-zero 8-flow (or nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$-flow).*

Theorem 5 (Seymour) *If G is 2-edge-connected, then G has a nowhere-zero 6-flow.*

The proofs for all three theorems stated are constructive. Seymour also conjectured that if G is 2-edge-connected, then G has a nowhere-zero 5-flow.

Here, we will present the proofs of the two theorems by Jaeger. Before we present the proofs, we first state and prove a very useful proposition.

Proposition 6 *Given an undirected graph $G = (V, E)$, G has a nowhere-zero 2^p-flow $\Longleftrightarrow$ there exists $F_1, F_2, ..., F_p \subset E$, such that $E = \cup_{i=1}^{p} F_i$, and for any $1 \leq i \leq p$, F_i is an even graph (every vertex of F_i has even degree).*

Proof: ($\Leftarrow$): Given $F_1, ..., F_p$ define a $\mathbb{Z}_2 \times \mathbb{Z}_2 \times ... \times \mathbb{Z}_2$ circulation ϕ as:

$$[\phi(e)]_i = \begin{cases} 1 & \text{if } e \in F_i \\ 0 & \text{otherwise} \end{cases}$$

Since $E = \cup_{i=1}^{p} F_i$, ϕ is a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2 \times ... \times \mathbb{Z}_2$-flow.
($\Rightarrow$): Let ϕ be a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2 \times ... \times \mathbb{Z}_2$-flow of G (which exists since G has a nowhere-zero 2^p-flow). For $1 \leq i \leq p$, define

$$F_i = \{e | e \in E \text{ and } [\phi(e)]_i = 1\}$$

This means F_i is an even graph for any i and $E = \cup_{i=1}^{p} F_i$. $\square$

We also make use of a theorem by Nash-Williams, which we state without proof; we will prove it later in the class.

Theorem 7 (Nash-Williams) *If an undirected graph $G = (V, E)$ is $2k$-edge-connected, then G has k-edge-disjoint spanning trees.*

Now, we have enough tools to present the proofs of Jaeger.
Proof: [Proof of Theorem 3] By the theorem of Nash-Williams, we can find two edge disjoint spanning trees of G, say T_1, T_2.

Claim 8 *$\exists A_1 \subset T_1$, such that $(E \setminus T_1) \cup A_1$ is even (again, by even, we mean for all $v \in V$, v has even degree in $(E \setminus T_1) \cup A_1$).*

Proof: [Proof of the Claim]: We prove the claim by describe an algorithm to find A_1. Start the algorithm with $U = E \setminus T_1$, $A_1 = \emptyset$ and $T = T_1$. At every iteration, find a leaf vertex v of T. If the degree of v in U is even, leave U as it is, and if degree of v in U is odd, add the edge of T incident to v into U and A_1. Next, delete v, update T and start another iteration. At the end of the algorithm (when T has exactly one vertex left), we have that U has all even degrees except possibly at that last vertex. But the sum of degrees (in U) of all vertices must be even, so all vertices in U must have even degrees. So we have $U = (E \setminus T_1) \cup A_1$ and U is even. $\square$

By similar argument, there also exists $A_2 \subset T_2$, such that $(E \setminus T_2) \cup A_2$ is even. Now let

$F_1 = (E \setminus T_1) \cup A_1$, $F_2 = (E \setminus T_2) \cup A_2$. Then $T_1 \cap T_2 = \emptyset$ implies that $F_1 \cup F_2 = E$. By proposition 6, we have that G has a nowhere-zero 4-flow. $\square$

Now, we prove Theorem 4, which is similar to the previous proof.

Proof: [Proof of Theorem 4] Suppose G is 2-edge-connected but not 3-edge-connected. Then, there exists a cut $\{e_1, e_2\}$. Without loss of generality, assume that, in our orientation of G, e_1 and e_2 disagree in orientation within the cut $\{e_1, e_2\}$. Let $G' = G/e_1$, and by induction (on the number of vertices), G' has a nowhere-zero 8-flow ϕ'. Define

$$[\phi(e)] = \begin{cases} \phi'(e) & \text{if } e \neq e_1 \\ \phi'(e_2) & \text{if } e = e_1 \end{cases}$$

Then $\phi(e)$ is a nowhere-zero 8-flow of G (by flow conservation across the cut $\{e_1, e_2\}$).

Now, suppose G is 3-edge-connected, then we duplicate every edge of G to create $G' = (V, E')$. Clearly, G' is 6-edge-connected. Again, by Nash-Williams, there exists pairwise disjoint spanning trees $T'_1, T'_2, T'_3 \subset E'$, such that $T'_1 \cup T'_2 \cup T'_3 \subseteq E'$. Define $T_i = \{e \in E | e$ is in T'_i or the duplicate of e is in $T'_i\}$. Then T_1, T_2, T_3 are spanning trees in (V, E), such that for any $e \in E$, there is i such that $e \notin T_i$. By similar arguments as in the last proof, we can find A_1, A_2, A_3 such that $(E \setminus T_i) \cup A_i$ is even. Let $F_i = (E \setminus T_i) \cup A_i$. Since $\forall e \in E$, there is some i such that $e \notin T_i$ (and thus $e \in F_i$), we have $E = F_1 \cup F_2 \cup F_3$. Now, the proof is completed by applying Proposition 6. $\square$

3 Transforming nowhere-zero flows

The proof we gave of Tutte's theorem stating that the existence of a nowhere-zero Γ-flow in G depends only on $|\Gamma|$ is non-constructive. In this section, we prove it algorithmically by going from any nowhere-zero Γ-flow to a nowhere-zero Γ'-flow where $|\Gamma| = |\Gamma'|$. This does not seem to be known (or at least widely known); Jensen and Toft [1, p. 210] write:

> No constructive proof has been published so far as we know. However, the arguments given by Minty [1967] for the case $k = 4$ seem to provide a key to a constructive proof also in general.

For this purpose, we use the fundamental theorem of finite abelian groups which says that any finite abelian group is isomorphic to $\mathbb{Z}_{q_1} \times \cdots \times \mathbb{Z}_{q_\ell}$ where the q_i's are prime powers (the constructions below will work even for general q_i's). Throughout this section, we consider a directed graph $G = (V, E)$, so when we talk about a nowhere-zero k-flow ϕ, we assume that ϕ takes values in $\{\pm 1, \cdots, \pm(k-1)\}$.

As a warm-up, we show algorithmically how to find a nowhere-zero $\mathbb{Z}_4$-flow from a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow. Let ϕ be a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow. Then, let $F_i = \{e | [\phi(e)]_i \neq 0\}$, for $i = 1, 2$. This implies (V, F_i) has a nowhere-zero $\mathbb{Z}_2$-flow for $i = 1, 2$. Then, we can find $\phi_i : F_i \to \{-1, 1\}$, where ϕ_i is a 2-flow (under operations in $\mathbb{Z}$) for $i = \{1, 2\}$, and we can extend it with 0 values to E. Finally, define

$$\phi' = 2\phi_1 + \phi_2.$$

Then ϕ' satisfies the flow conservation and $\forall e \in E$, $-3 \leq \phi'(e) \leq 3$, and $\phi'(e) \neq 0$. This shows ϕ' is a nowhere-zero 4-flow, and taking values modulo 4, we get a $\mathbb{Z}_4$-flow of G.

We can extend this procedure to all finite abelian groups, that is, for any group $\Gamma = \mathbb{Z}_{q_1} \times ... \times \mathbb{Z}_{q_\ell}$, where q_i's are prime powers for $1 \leq i \leq \ell$. Let $k = (q_1 \cdot q_2 \cdot ... \cdot q_\ell)$, given a nowhere-zero Γ-flow ϕ, we can find a nowhere-zero $\mathbb{Z}_k$-flow ϕ' algorithmically in the following way. The ith coordinate of ϕ is a $\mathbb{Z}_{q_i}$-flow and thus, we can transform it into a q_i-flow (over $\mathbb{Z}$) (instead of changing the orientation as we did in the proof of Theorem 1, we can simply decrease the flow value along P by q_i):

$$\phi_i : E \to \{-(q_i - 1), ..., -1, 0, 1, ..., (q_i - 1)\}$$

for each $i = 1, 2, ..., \ell$. Observe that, by construction, for every e, there exists i such that $\phi_i(e) \neq 0$. Then define

$$\phi'(e) = \sum_{i=1}^{\ell} (\prod_{j=1}^{i-1} q_j) \phi_i(e)$$

for each $e \in E$. Then, we claim that ϕ' is a nowhere-zero k-flow in G. Indeed, by induction on h, we have that

$$\left| \sum_{i=1}^{h} \left(\prod_{j=1}^{i-1} q_j \right) \phi_i(e) \right| \leq \left(\prod_{j=1}^{h} q_j \right) - 1,$$

implying that (i) $\phi'(e) \neq 0$ (by considering the largest i with $\phi_i(e) \neq 0$) and (ii) $|\phi'(e)| < k$.

The converse can also be done. From a nowhere-zero $\mathbb{Z}_k$-flow, one can construct a nowhere-zero Γ-flow for $\Gamma = \mathbb{Z}_{q_1} \times ... \times \mathbb{Z}_{q_\ell}$ where $k = (q_1 \cdot q_2 \cdot ... \cdot q_\ell)$. For this, we show how to get a nowhere-zero $\mathbb{Z}_{q_1} \times \mathbb{Z}_{q_2}$ flow ϕ' from a nowhere-zero $\mathbb{Z}_{q_1 q_2}$-flow ϕ for any q_1, q_2. Remember we have the choice of the orientation; if we flip edge e, we can maintain that ϕ is a $\mathbb{Z}_{q_1 q_2}$-flow by replacing $\phi(e)$ by $q_1 q_2 - \phi(e)$. Choose the orientation in such a way that if we define λ by $\phi(e) \equiv \lambda(e) \pmod{q_1}$ for all $e \in E$ then not only is λ a $\mathbb{Z}_{q_1}$-flow but also a q_1-flow. This is possible since as we flip edge e, $\lambda(e)$ gets replaced by $q_1 - \lambda(e)$ which is what we need. This flow λ takes value 0 on the edges of $E_1 = \{e : \phi(e) \equiv 0 \pmod{q_1}\}$. Observe that every value in $\phi - \lambda$ is a multiple of q_1, and thus we can define

$$\mu(e) = \frac{1}{q_1} \left(\phi(e) - \lambda(e) \right).$$

Observe that μ satisfies the flow conservation constraints modulo q_2 (as they were satisfied modulo $q_1 q_2$ prior to dividing by q_1), i.e. μ defines a $\mathbb{Z}_{q_2}$ flow. Furthermore, $\mu(e) \not\equiv 0 \pmod{q_2}$ on E_1 (since otherwise $\phi(e)$ would have been 0). Thus, (λ, μ) constitutes a nowhere-zero $\mathbb{Z}_{q_1} \times \mathbb{Z}_{q_2}$-flow on G.

References

[1] T.R. Jensen and B. Toft, "Graph Coloring Problems", Wiley, 1995.

[2] C. Thomassen, "The weak 3-flow conjecture and the weak circular conjecture", Journal of Combinatorial Theory, Series B, 102, 521–529, 2012.

[3] C. Thomassen, Y. Wu and C.-Q. Zhang, "3-flows for 6-edge-connected graphs", AMS 2012 Spring Eastern Sectional Meeting, George Washington University, March 2012.