

Option pricing during post-crash relaxation times

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Received 15 September 2006; received in revised form 22 January 2007

Available online 3 March 2007

Abstract

This paper presents a model for option pricing in markets that experience financial crashes. The stochastic differential equation (SDE) of stock price dynamics is coupled to a post-crash market index. The resultant SDE is shown to have stock price and time dependent volatility. The partial differential equation (PDE) for call prices is derived using risk-neutral pricing. European call prices are then estimated using *Monte Carlo* and *finite difference methods*. Results of the model show that call option prices after the crash are systematically less than those predicted by the Black–Scholes model. This is a result of the effect of non-constant volatility of the model that causes a volatility skew.

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Keywords: Asset dynamics; Market crashes; Option pricing

1. Introduction

The dynamics of financial markets after the occurrence of a financial crash has been studied empirically in Refs. [1,2]. Sornette [1] shows that the post-crash dynamics follow a converging oscillatory motion. Lillo and Mantenga [2], on the other hand, show that financial markets follow power-law relaxation decay. Both empirical explanations do not agree with the traditional simple and stochastic volatility models used in the finance literature to explain the dynamics of stock markets. These empirical studies pave the way for the development of option pricing models that explain market option prices in the aftermath of financial market crashes. For example, in the post-1987 period the well-known volatility smile became more noticeable and hence the Black–Scholes model of option pricing does not model well option prices in such environments [3–5]. The Black–Scholes option pricing model is based on the assumption that the underlying stock price follows geometric Brownian motion [6]. This assumption on asset price dynamics has come lately under increased scrutiny as variations on the Brownian motion price dynamics have been introduced in the literature both at the theoretical and empirical levels. New models have been developed that produce asset price dynamics that include limit cycles or periodic orbits [7,8], chaotic attractors [9,10], and complex dynamics that generate both efficient solutions and speculative ones for different regimes of agents behavior [11]. Many option pricing models were developed based on empirical observations in financial markets, including cycles and chaotic

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motion, which deviate from the assumptions of the Black–Scholes model such as chaos [12], interest rate cycles [13], freight cycles [14], cycles and log-periodic cycles [15]. In Ref. [16], on the other hand, McCauley starts with the premise that option pricing models must reflect the empirical distribution of stock prices. He derives from market data an empirical distribution function and then derives a closed-form option pricing model.

In this paper an extension of the Black–Scholes is proposed. The extension takes into accounts the post-crash dynamics as posited by Sornette [1]. The stochastic differential equation (SDE) representing the underlying stock price dynamics is coupled to a post-crash index that exhibits exponentially decaying oscillatory motion. *Monte Carlo* and *finite difference* methods are then used to estimate European option prices for stocks governed by such an SDE. The remainder of the paper is divided as follows. In Section 2 the SDE that couples the post-crash market index to individual stock prices is derived. Section 3 presents the numerical estimation of European call option prices using the Monte Carlo and finite difference methods. We conclude with remarks in Section 4.

2. Option pricing in post-crash times

In Ref. [1], the post-crash time series for the S&P 500 index is represented by an exponentially decaying oscillating function. The results show that the market in the post-crash period still behaved in a cooperative manner finding the equilibrium price in a rapid fashion. The closed-form characteristic of the periodic motion allows for a possibility of incorporating such dynamics into the equations of motion of an individual stock price through an interaction term with the market index following a specific crash dynamics as follows. If we start with a deterministic ordinary differential equation for the stock price S .

$$\frac{dS}{dt} = \alpha S, \quad (1)$$

where S is the stock price and α is the growth rate and assuming that the coefficient is noise excited such that $\alpha = (a + \sigma\zeta)$ where ζ is white noise and σ is its intensity, the Langevin stock price growth equation can be written as

$$\frac{dS}{dt} = \alpha S + \sigma S \zeta \quad (2)$$

or

$$dS = aS dt + \sigma S dW, \quad (3)$$

where S is the asset price at time t , W is a Wiener process, and σ is the volatility of the stock price. Eq. (3) is the SDE for geometric Brownian process used in Black–Scholes option pricing. Assuming that the S&P 500 index exerts a forcing function, $g(t)$, on the dynamics of the stock price in Eq. (2) then the differential equation of the stock price can be written as

$$\frac{dS}{dt} = aS + \theta g(t). \quad (4)$$

If we noise excite the first and second terms of the RHS by $\alpha = (a + \sigma\zeta)$ and $\theta = b + \gamma\zeta$ respectively, then Eq. (4) becomes

$$dS = aS dt + \sigma S \zeta dt + bg(t) dt + \gamma g(t) \zeta dt \quad (5)$$

or

$$dS = (aS + bg(t)) dt + \sigma S dW + \gamma g(t) dW. \quad (6)$$

We can rewrite Eq. (6) in the following fashion:

$$dS = (aS + bg(t)) dt + \left(\sigma + \frac{\gamma g(t)}{S} \right) S dW, \quad (7)$$

where the SDE becomes a local volatility model with $\sigma(S, t) = \sigma + \gamma g(t)/S$.

The function $g(t)$ can represent Sornette's [1] empirical fitting of the stock market index in the aftermath of a market crash as a dissipative harmonic oscillator represented by an exponentially decaying sinusoidal function:

$$g(t) = A + Be^{\alpha t} \sin(\omega t). \quad (8)$$

Fig. 1 plots a representation of $g(t)$ between $t = 0$ and 5. Fig. 2 illustrates the local volatility generated by the $g(t)$ coupling plotted vs. the stock price S and the time to expiration t . The volatility surface demonstrates the typical volatility skew observed in financial markets.

In the case where the σ is a function of S and t , the usual risk-neutral pricing still applies. In order to derive the partial differential equation (PDE) for the European option price we can construct a portfolio of the call option and the underlying in the following manner. Let P be dollars invested in a combination of a call option $C(S, t)$ and the stock S in the following proportion:

$$P = C(S, t) - \frac{\partial C}{\partial S} S \quad (9)$$

or

$$dP = dC(S, t) - \frac{\partial C}{\partial S} dS. \quad (10)$$

Using Ito's lemma and the stochastic process Eq. (7) we get

$$dC = \frac{\partial C}{\partial S} (aS + bg(t)) dt + \sigma S dW + \gamma g(t) dW + \frac{\partial C}{\partial S} dt + \frac{1}{2} (\sigma S + \gamma g(t))^2 \frac{\partial^2 C}{\partial S^2} dt. \quad (11)$$

Substituting Eq. (11) in Eq. (10), we get the following risk-free portfolio:

$$dP = \frac{\partial C}{\partial t} dt + \frac{1}{2} (\sigma S + \gamma g(t))^2 \frac{\partial^2 C}{\partial S^2} dt. \quad (12)$$

Equating Eq. (12) to $rP dt$, the growth of a risk-free portfolio, we get the following PDE for the option price:

$$\frac{\partial C}{\partial t} = -r \frac{\partial C}{\partial S} S + rC - \frac{1}{2} (\sigma S + \gamma g(t))^2 \frac{\partial^2 C}{\partial S^2}. \quad (13)$$

The boundary condition for the European call are $C(S, T) = \max(S - X, 0)$ where T is the time to expiry and X is the strike price. The PDE in Eq. (13) reduces to the Black–Scholes PDE for $\gamma = 0$ as expected for an uncoupled stock price.

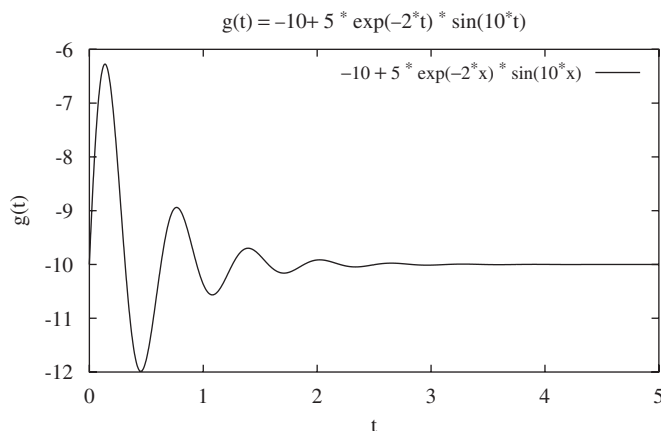


Fig. 1. The exponentially decaying sinusoidal function, $g(t) = -10 + 5 * e^{-2t} * \sin 10t$, that represents the market crash as a dissipative harmonic oscillator.

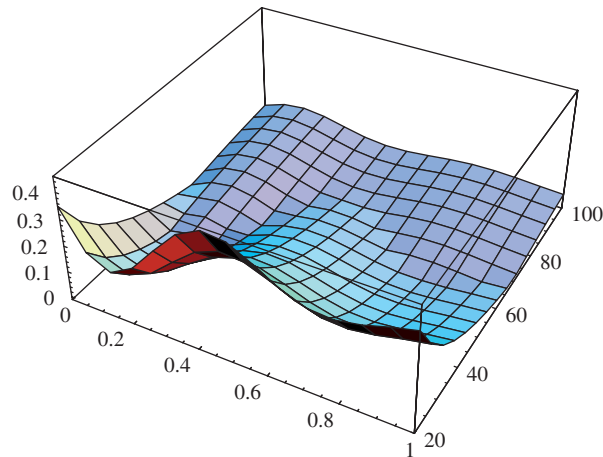


Fig. 2. $|\sigma|$ vs. S and t produced by $\sigma + \gamma g(t)/S$ where $\sigma = 0.15, \gamma = 1$, and $g(t)$ as shown in Fig. 1.

3. Numerical calculation of option prices

The PDE for the call price in Eq. (13) with time dependent parameters has no known closed-form solution. In this section, *Monte Carlo* and *finite difference* methods are used to solve numerically option prices for coupled calls.

3.1. Monte Carlo solution

The general *Monte Carlo* method was initially proposed by Stan Ulam (see [17]) in order to approximate combinatorial computations. The method was given the name “Monte Carlo” by Metropolis in 1949 when it was formally proposed as a general stochastic technique based on the observation that random sampling in a function may linearly add points to an accumulated sum that eventually becomes the function.

The use of Monte Carlo techniques in option pricing is straightforward and natural as derivatives pricing are inherently probabilistic problems. The use of Monte Carlo simulation in pricing options was first published by Boyle [18] and used later by various researchers in finance [19]. Typically, the price of a European option is estimated using M independent sample values of the payoff from the option in a risk-neutral world. The average of the sample payoffs is calculated in order to obtain an estimate of the expected payoff, $C(S, T) = \max(S - X, 0)$. Finally, the expected payoff is discounted using the risk-free interest rate in order to get an estimate of the value of the option.

The *Crash model* described in Eq. (13) implies that for a given stock price S at time t , simulated changes at time $t + \delta t$ can be generated using the SDE for the dynamics of the stock price rewritten here

$$dS = (aS + bg(t))dt + \sigma S dW + \gamma g(t) dW, \quad (14)$$

where function $g(t)$ is given in Eq. (8). Since risk-neutral pricing is assumed, the drift of the SDE (Eq. (14)) is replaced by the risk-free interest rate, r . The above SDE becomes

$$dS = rS dt + \sigma S dW + \gamma g(t) dW. \quad (15)$$

Call prices can be next generated from Eq. (15) using the Monte Carlo method.

In order to estimate the value of an option using Monte Carlo, the method proceeds as follows:

for ($i \leftarrow 0; i < M; i++$)

$\psi_i \leftarrow (\sum_{j=1}^{12} \psi_j) - 6;$

$S_i \leftarrow S_{i-1} + rS\sqrt{\delta t} + \sigma S_i \psi_i + \gamma g(t) \psi_i$

$$\begin{aligned}
C_i &\leftarrow \max(S_i - X, 0) \\
\text{Average} &\leftarrow 1/M \sum_{i=1}^M C_i \\
\text{Estimate of Value of Option} &\leftarrow e^{-rT} * \text{Average}
\end{aligned}$$

The method initially starts by approximating a standard normal variable from uniform random numbers that are generated using a pseudo-data encryption standard (pseudo-DES) method. The approximation generates and adds 12 uniformly random numbers, ψ_j , between 0 and 1. The sum has a mean of six and a standard deviation of 1.0. We next subtract 6 in order to adjust the mean to zero without changing the standard deviation. What we obtain is technically not normally distributed but is symmetric with mean zero and standard deviation of 1, which are three properties associated with the normal distribution [20].

The numerical integration for the SDE is next carried out using Euler's method, which is known to converge in expectation to the solution with an error that is equal approximately to the size of the step [21,22]. Thus, we compute the price change over the life of the option and add it to the current price in order to obtain the price of the asset at expiration. The price of the option at expiration is then computed using the formula $\max(0, S - X)$. This produces one possible option value of the call at expiration. The above procedure is repeated M times and the average value of all estimated calls at expiration is computed and then discounted at the risk-free rate. Fig. 3 shows this procedure for $M = 300$ for one stock price only.

Naturally, every one simulation run generates a different option price since the random numbers are different. However, since the option price is a sample average, it has been shown by Hull [23] that the error in this estimate is ω/\sqrt{M} where ω is the standard deviation of the sample payoffs. Thus, choosing a sufficiently large M will substantially reduce the error in the value.

The algorithm was run for $M = 50,000$ where in each run, the initial variables were randomized using uniform deviates. In order to guarantee random sampling, the random deviates were generated using a pseudo-DES that acts on 64 bits of input by iteratively applying a highly non-linear bit-mixing cipher function. The algorithm was very fast and all results were reported in a maximum of 2 CPU minutes. As the error is in the order of $O(1/\sqrt{M})$, the standard deviation varied between 1.65 and 9.48 for all cases. Thus, the error varied between 0.0074 and 0.042. The implementation of the algorithm goes thorough the following steps:

- Step 1: Accept the values of the following model's parameters: r, σ, γ, S and X .
- Step 2: Initialize the seed and set $\delta_t = 0.01$ and set $T = 100$ time steps.
- Step 3: Use Euler's method to compute the next S based on the following:

$$S_{t+1} = rS_t\delta_t + W_t\sqrt{\delta_t}[s_t\sigma + \gamma * g(t)] + S_t, \quad (16)$$

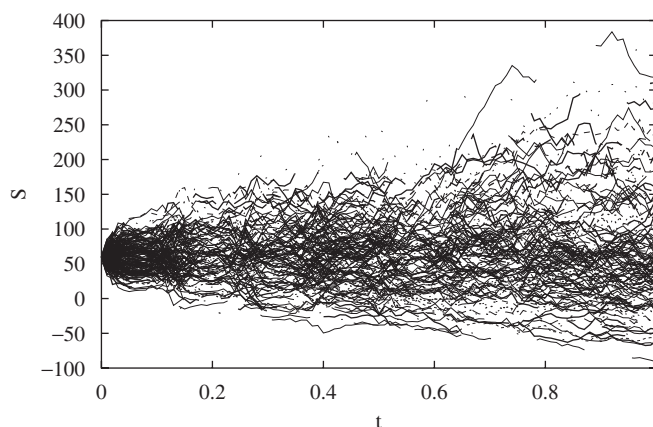


Fig. 3. Monte Carlo simulations to realize an asset price. The graph corresponds to 300 realization of an asset price random walk. Values are averaged in order to compute the final option price.

where

$$W_t = \left(\sum_{i=1}^{12} \psi_i \right) - 6 \quad \text{and} \quad g(t) = -10 + 5 * e^{-2t} * \sin 10t$$

and ψ_i 's are independent random variables.

- *Step 4:* Repeat *step 3* for the duration of time steps.
- *Step 5:* Repeat *steps 2* and *3* for $N = 50,000$ times.
- *Step 6:* Compute the *call*, C , where

$$C = \frac{1}{50,000} * \sum_{i=1}^{50,000} \max[(S_T - X), 0] * e^{-rT}, \quad (17)$$

where $C = 0$ if $S_T - X < 0$.

3.2. Finite difference solution

In order to evaluate the effectiveness and the stability of the results that were obtained by the Monte Carlo method, we solve the second-order linear, parabolic PDE in (13) using the explicit finite difference method. The method estimates the derivatives in the PDE using the differences in values over intervals of finite size.

Eq. (13) reduces to the following discretized equation:

$$\frac{C_j^{n+1} - C_j^n}{\delta t} + rS \frac{C_{j+1}^n - C_j^n}{\delta S} - rC + \frac{1}{2}(\sigma S + \gamma g(t))^2 \left[\frac{C_{j+1}^n - 2C_j^n + C_{j-1}^n}{\delta S^2} \right] = 0. \quad (18)$$

The above equation leads to the following equation that finds C_j^{n+1} knowing C_j^n :

$$C_j^{n+1} = -rS\delta t \frac{C_{j+1}^n - C_j^n}{\delta S} + rC\delta t - \delta t \frac{1}{2}(\sigma S + \gamma g(t))^2 \left[\frac{C_{j+1}^n - 2C_j^n + C_{j-1}^n}{\delta S^2} \right] + C_j^n, \quad (19)$$

with following boundary conditions:

$$\begin{aligned} C_0^n &= 0, \\ C_j^N &= \max(S_j^N - X, 0) \quad \text{where } j = 0, 1, 2, \dots, J-1, \\ C_j^n &= S_j^n - X e^{-r(T-t)} \quad \text{where } n = 0, 1, 2, \dots, N-1. \end{aligned}$$

The above difference equation is stable if $\delta t < \frac{1}{2}\delta S^2$. We chose $\delta t = 0.00001$ and $\delta S = 0.015$. We solved our proposed *Crash model* using the same parameters as for the Monte Carlo method.

3.3. Results

The Monte Carlo and the finite difference methods simulation were implemented using the C language on a Pentium Centrino 2.18 GHz. The source code has around 550 lines of code and is available from the authors.

The numerical results of the Monte Carlo and the finite difference methods presented in Figs. 4–7 show that the crash model systematically produces lower call prices than the Black–Scholes model. Note that the Monte Carlo results include error bars; however, they are not visible as the estimated error in the Monte Carlo method varied between ± 0.003203 for $X = 80$ and ± 0.060311 for $X = 40$ in all cases. The results of the numerical solution of the finite difference method agree with our Monte Carlo results. Furthermore, the results agree with the prevalence of a volatility skew in the market where the market participants expect the market to converge to equilibrium in a decaying fashion and hence the calls (which are bets on stock prices rising) would have a lower value than expected by the Black–Scholes model. In general, in the post-crash period, the model agrees with the market fact that in-the-money calls are less expensive than those predicted by Black–Scholes theory. The volatility skew makes the events on the left tail of the distribution more likely to

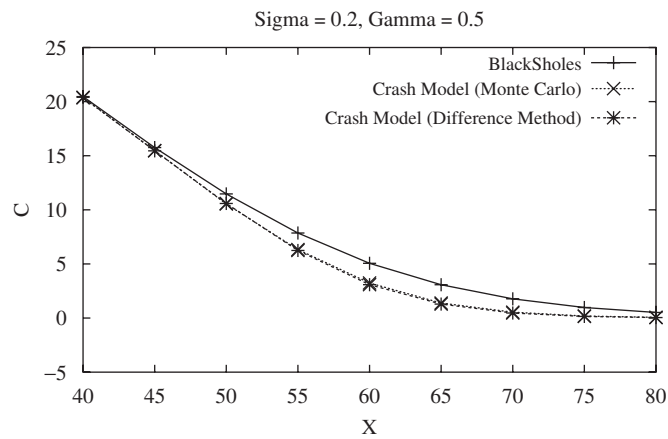


Fig. 4. Call option prices vs. exercise prices for Crash & Black–Scholes models with $\sigma = 0.2$, $\gamma = 0.5$, $r = 0.01$, $T = 1$, and $S = 60$.

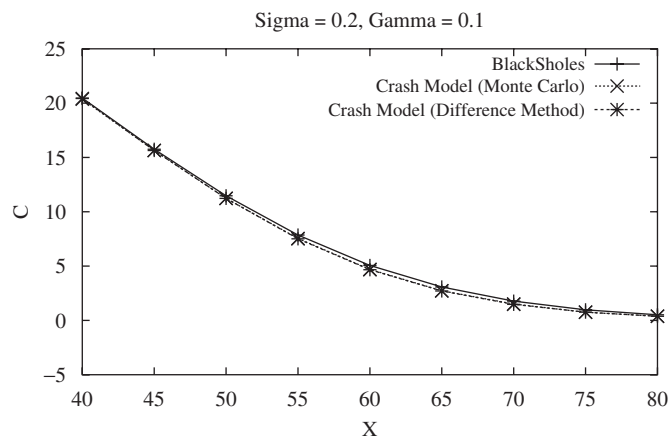


Fig. 5. Call option prices vs. exercise prices for Crash & Black–Scholes models with $\sigma = 0.2$, $\gamma = 0.1$, $r = 0.01$, $T = 1$ and $S = 60$.

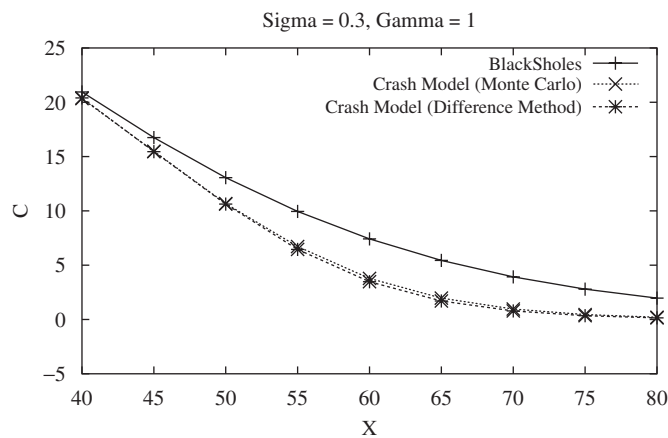


Fig. 6. Call option prices vs. exercise prices for Crash & Black–Scholes models with $\sigma = 0.3$, $\gamma = 1$, $r = 0.01$, $T = 1$ and $S = 60$.

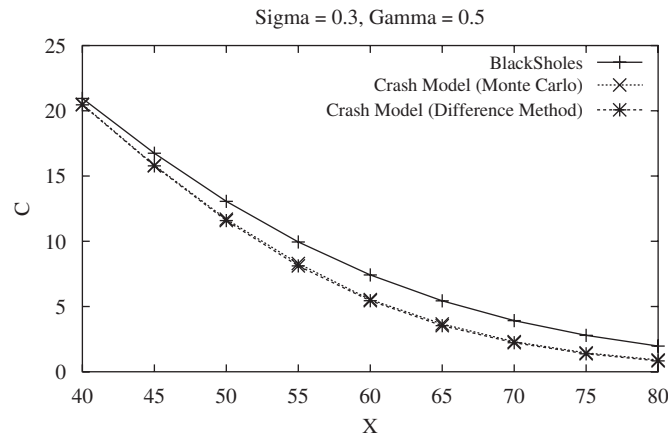


Fig. 7. Call option prices vs. exercise prices for Crash & Black–Scholes models with $\sigma = 0.3$, $\gamma = 0.5$, $r = 0.01$, $T = 1$ and $S = 60$.

occur causing the call price to be lower. Figs. 4–7 also show that the stronger the coupling of the stock to the market index as represented by γ , the larger the effect of the crashing index on the call price. For example, in Fig. 5 with $\gamma = 0.1$, the difference between the crash model price and the Black–Scholes theoretical price is almost negligible compared to the difference seen in Fig. 4 where $\gamma = 0.5$.

4. Conclusion

In this paper a simple model of valuation of European options written on stocks that are coupled to a market index that is experiencing post-crash equilibrating dynamics is developed. The model showed that individual stock options in a post-crash environment can be valued by Monte Carlo simulation of the stock price SDE generated by direct coupling to an exponentially decaying sinusoidal model of post-crash dynamics. The call price PDE is also solved using the explicit finite difference method. The coupled SDE with a local volatility that is a function of stock price and time generates a volatility skew. This causes the call option prices to be lower than those predicted by the standard Black–Scholes model. The model concentrated on the period immediately after a stock market crash where all individual stocks are more likely to be crashing also and hence can be modeled as coupled to a market index such as the S&P 500. This dynamic naturally makes stocks more likely to jump down than jump up which generates the volatility skew. Further work on the model will attempt at extending the “time window” of the applicability of the model to more normal times where the probability of up movements of individual stocks are higher than the current model permits. Such a more normal behavior agrees more with the empirical fact that individual stocks volatility surface exhibits normally the so-called volatility smile reflecting two-sided fat tails.

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