CSC 611: Analysis of Algorithms

Lecture 2

Mathematical Background and Order Complexity

Algorithms Analysis

- The amount of resources used by the algorithm
  - Space
  - Computational time
- Running time:
  - The number of primitive operations (steps) executed before termination
- Order of growth
  - The leading term of a formula
  - Expresses the behavior of a function toward infinity
  - Ignores machine depending constants and looks at growth of T(n) as $n \rightarrow \propto$

Asymptotic Notations

- A way to describe behavior of functions in the limit
  - How we indicate running times of algorithms
  - Describe the running time of an algorithm as n grows to ∞
- O notation: asymptotic "less than": f(n) "≤" g(n)
- **Ω** notation: asymptotic "greater than": f(n) "≥" g(n)
- Θ notation: asymptotic "equality": f(n) "=" g(n)

Asymptotic Notations

- Theorem 1
  - Leading constants and lower order terms don't matter
  - Justification
    - Can choose constant big enough to make high-order term swamp other terms
- Theorem 2
  - (O and $\Omega \Leftrightarrow \Theta$)
  - How easy can you prove this?

Logarithms

- In algorithm analysis we often use the notation **"log n"** without specifying the base

Binary logarithm $\lg n = \log_2 n$

Natural logarithm $\ln n = \log_e n$

$$\log^k n = (\log n)^k$$
$$\log\log n = \log(\log n)$$
$$\log x^y = y \log x$$
$$\log xy = \log x + \log y$$
$$\log \frac{x}{y} = \log x - \log y$$
$$\log_a x = \log_a b \log_b x$$
$$a^{\log_b x} = x^{\log_b a}$$

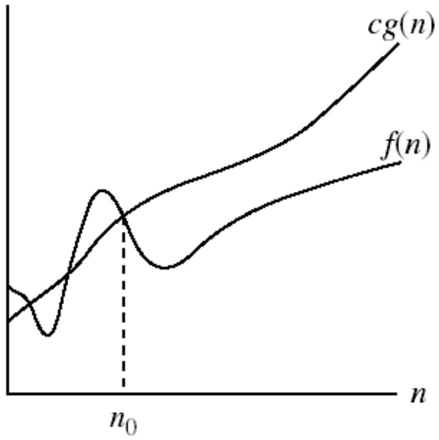
Asymptotic Notations - Examples

- For each of the following pairs of functions, either f(n) is O(g(n)), f(n) is Ω(g(n)), or f(n) is Θ(g(n)). Determine which relationship is correct.
  - f(n) = log n^2; g(n) = log n + 5 — **f(n) = Θ (g(n))**
  - f(n) = n; g(n) = log n^2 — **f(n) = Ω(g(n))**
  - f(n) = log log n; g(n) = log n — **f(n) = O(g(n))**
  - f(n) = n; g(n) = $\log^2$ n — **f(n) = Ω(g(n))**
  - f(n) = n log n + n; g(n) = log n — **f(n) = Ω(g(n))**
  - f(n) = 10; g(n) = log 10 — **f(n) = Θ(g(n))**
  - f(n) = 2^n; g(n) = $10n^2$ — **f(n) = Ω(g(n))**
  - f(n) = 2^n; g(n) = 3^n — **f(n) = O(g(n))**

Asymptotic notations

- *O-notation*

$$O(g(n)) = \{f(n) : \text{there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0\}.$$



- Intuitively: O(g(n)) = the set of functions with a smaller or same order of growth as g(n)

$g(n)$ is an ***asymptotic upper bound*** for $f(n)$.

Examples

- $2n^2 = O(n^3)$: $2n^2 \le cn^3 \Rightarrow 2 \le cn \Rightarrow c = 1$ and $n_0 = 2$
- $n^2 = O(n^2)$: $n^2 \le cn^2 \Rightarrow c \ge 1 \Rightarrow c = 1$ and $n_0 = 1$
- $1000n^2+1000n = O(n^2)$:

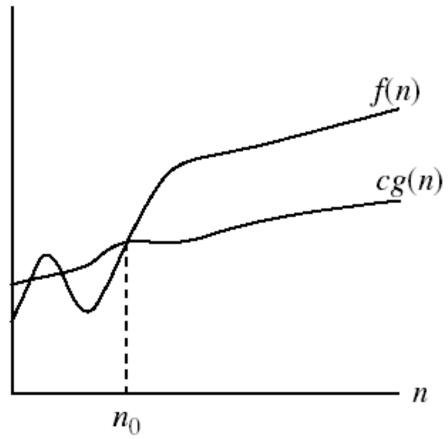
  $1000n^2+1000n \le 1000n^2+ 1000n^2 = 2000n^2$

  $\Rightarrow c=2000$ and $n_0 = 1$
- $n = O(n^2)$: $n \le cn^2 \Rightarrow cn \ge 1 \Rightarrow c = 1$ and $n_0 = 1$

Asymptotic notations (cont.)

- **Ω** - *notation*

$$\Omega(g(n)) = \{f(n) : \text{there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \le cg(n) \le f(n) \text{ for all } n \ge n_0\} .$$



- Intuitively: **Ω**(g(n)) = the set of functions with a larger or same order of growth as g(n)

$g(n)$ is an ***asymptotic lower bound*** for $f(n)$.

Examples

– $5n^2 = \Omega(n)$

$\exists\ c, n_0$ such that: $0 \le cn \le 5n^2 \Rightarrow cn \le 5n^2 \Rightarrow c = 1$ and $n_0 = 1$

– $100n + 5 \ne \Omega(n^2)$

$\exists\ c, n_0$ such that: $0 \le cn^2 \le 100n + 5$

$100n + 5 \le 100n + 5n\ (\forall\ n \ge 1) = 105n$

$cn^2 \le 105n \Rightarrow n(cn - 105) \le 0$

Since n is positive $\Rightarrow cn - 105 \le 0 \Rightarrow n \le 105/c$

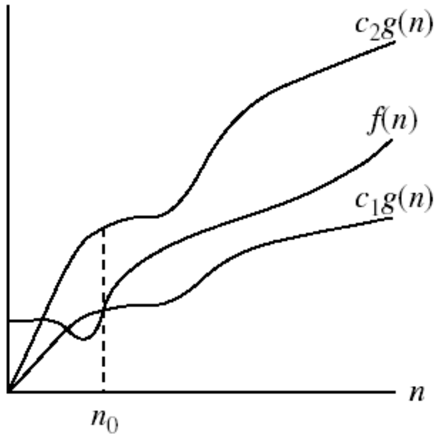
$\Rightarrow$ contradiction: n cannot be smaller than a constant

– $n = \Omega(2n)$, $n^3 = \Omega(n^2)$, $n = \Omega(\log n)$

Asymptotic notations (cont.)

- Θ*-notation*

$\Theta(g(n)) = \{f(n) :$ there exist positive constants c_1, c_2, and n_0 such that $0 \le c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \ge n_0\}$.



- Intuitively $\Theta(g(n))$ = the set of functions with the same order of growth as g(n)

$g(n)$ is an ***asymptotically tight bound*** for $f(n)$.

Θ Notation: Example 1

Prove that: $\frac{1}{2}n^2 - 3n \in \Theta(n^2)$

Well, $c_1 n^2 \le \frac{1}{2}n^2 - 3n \le c_2 n^2 \Rightarrow$

$c_1 \le \frac{1}{2} - \frac{3}{n} \le c_2$

c_1 has to be positive $\Rightarrow c_1 \le \frac{1}{2} - \frac{3}{7} = \frac{1}{14}$

c_2 has to be positive $c_2 \ge \frac{1}{2}$

Θ Notation: Example 2

Prove that: $\frac{1}{2}n^2 + 2n \in \Theta(n^2)$

Well, $c_1 n^2 \le \frac{1}{2}n^2 + 2n \le c_2 n^2 \Rightarrow$

$c_1 \le \frac{1}{2} + \frac{2}{n} \le c_2$

c_1 has to be positive $\Rightarrow$ $c_1 \le \frac{1}{2} + \frac{2}{n} = \frac{5}{2}$

c_2 has to be positive $c_2 \ge \frac{1}{2}$

$n_0 = 1$

Θ Notation: Example 3

- $n^2/2 - n/2 = \Theta(n^2)$
  - $\frac{1}{2} n^2 - \frac{1}{2} n \le \frac{1}{2} n^2 \ \forall n \ge 0 \quad \Rightarrow \quad c_2 = \frac{1}{2}$
  - $\frac{1}{2} n^2 - \frac{1}{2} n \ge \frac{1}{2} n^2 - \frac{1}{2} n * \frac{1}{2} n \ (\forall n \ge 2) = \frac{1}{4} n^2$

    $\Rightarrow \quad c_1 = \frac{1}{4}$

- $n \ne \Theta(n^2)$: $c_1 n^2 \le n \le c_2 n^2$

  $\Rightarrow$ only holds for: $n \le 1/c_1$

Θ Notation: Example 4

- $6n^3 \neq \Theta(n^2)$: $c_1 n^2 \leq 6n^3 \leq c_2 n^2$

  $\Rightarrow$ only holds for: $n \leq c_2 / 6$

- $n \neq \Theta(\log n)$: $c_1 \log n \leq n \leq c_2 \log n$

  $\Rightarrow c_2 \geq n/\log n, \forall n \geq n_0$ – impossible

More on Asymptotic Notations

- There is no unique set of values for n_0 and c in proving the asymptotic bounds
- Prove that $100n + 5 = O(n^2)$
  - $100n + 5 \leq 100n + n = 101n \leq 101n^2$

    for all $n \geq 5$

    $n_0 = 5$ and $c = 101$ is a solution
  - $100n + 5 \leq 100n + 5n = 105n \leq 105n^2$

    for all $n \geq 1$

    $n_0 = 1$ and $c = 105$ is also a solution

Must find **SOME** constants c and n_0 that satisfy the asymptotic notation relation

Comparisons of Functions

- *Theorem:*

  $f(n) = \Theta(g(n)) \Leftrightarrow f = O(g(n))$ and $f = \Omega(g(n))$
- Transitivity:
  - $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n)) \Rightarrow f(n) = \Theta(h(n))$
  - Same for O and Ω
- Reflexivity:
  - $f(n) = \Theta(f(n))$
  - Same for O and Ω
- Symmetry:
  - $f(n) = \Theta(g(n))$ if and only if $g(n) = \Theta(f(n))$
- Transpose symmetry:
  - $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$

Asymptotic Notations in Equations

- On the right-hand side
  - $\Theta(n^2)$ stands for some anonymous function in $\Theta(n^2)$

  $2n^2 + 3n + 1 = 2n^2 + \Theta(n)$ means:

  There exists a function $f(n) \in \Theta(n)$ such that

  $2n^2 + 3n + 1 = 2n^2 + f(n)$
- On the left-hand side

  $2n^2 + \Theta(n) = \Theta(n^2)$

  No matter how the anonymous function is chosen on the left-hand side, there is a way to choose the anonymous function on the right-hand side to make the equation valid.

Limits and Comparisons of Functions

- Using limits for comparing orders of growth:

$$\lim_{n\to\infty}\frac{t(n)}{g(n)} = \begin{cases} 0, t(n) \text{ has a smaller order of growth than } g(n): t(n)\in O(g(n)) \\ c, t(n) \text{ has the same order of growth as } g(n): t(n)\in \Theta(g(n)) \\ \infty, t(n) \text{ has a larger order of growth than } g(n): t(n)\in \Omega(g(n)) \end{cases}$$

- Compare ½ n (n-1) and n^2

$$\lim_{n\to\infty}\frac{\frac{1}{2}n(n-1)}{n^2} = \frac{1}{2}\lim_{n\to\infty}\frac{n^2-n}{n^2} = \frac{1}{2}\lim_{n\to\infty}\left(1-\frac{1}{n}\right) = \frac{1}{2}$$

Limits and Comparisons of Functions

- Any positive exponential function grows faster than any polynomial

$$\lim_{n\to\infty}\frac{n^b}{a^n} = 0 \Rightarrow n^b \in o(a^n)$$

Limits and Comparisons of Functions

- L'Hopital rule:

$$\lim_{n\to\infty}\frac{t(n)}{g(n)}=\lim_{n\to\infty}\frac{t'(n)}{g'(n)}$$

- Compare lgn and $\sqrt{n}$

$$\lim_{n\to\infty}\frac{(\lg n)'}{\left(\sqrt{n}\right)'}=\lim_{n\to\infty}\frac{\frac{1}{n}\lg e}{\frac{1}{2\sqrt{n}}}=2\lg e\lim_{n\to\infty}\frac{\sqrt{n}}{n}=0$$

Mathematical Review

- Changing the base of logarithm changes the value of the logarithm by only a constant factor

$$\log_b a=\frac{\log_c a}{\log_c b}$$

Mathematical Review: Common Functions

- Factorials

$$n! = \prod_{k=1}^{n} k$$

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\Theta\left(\frac{1}{n}\right)\right)$$ Stirling's approximation

$$\lg(n!) = \Theta(n \lg n)$$

- Floors and ceilings
  - $\lfloor x \rfloor$: the greatest integer less than or equal to x
  - $\lceil x \rceil$: the least integer greater than or equal to x
  - $x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$

Some Simple Summation Formulas

- Arithmetic series: $\sum_{k=1}^{n} k = 1 + 2 + ... + n = \frac{n(n+1)}{2}$
- Geometric series: $\sum_{k=0}^{n} x^k = 1 + x + x^2 + ... + x^n = \frac{x^{n+1}-1}{x-1} (x \neq 1)$
  - Special case: $x < 1$: $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$
- Harmonic series: $\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + ... + \frac{1}{n} \approx \ln n$
- Other important formulas: $\sum_{k=1}^{n} \lg k \approx n \lg n$

  $\sum_{k=1}^{n} k^p = 1^p + 2^p + ... + n^p \approx \frac{1}{p+1} n^{p+1}$

Mathematical Induction

- Used to prove a sequence of statements (**S(1)**, **S(2)**, … **S(n)**) indexed by positive integers
- Proof:
  - **Basis step**: prove that the statement is true for **n = 1**
  - **Inductive step:** assume that **S(n)** is true and prove that **S(n+1)** is true for all **n ≥ 1**
- Find case **n** "within" case **n+1**

Example 1

- Prove that: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ *for all* $n \geq 1$
- Basis step:
  - n = 1: $\sum_{i=1}^{1} i = \frac{1(1+1)}{2} = 1$
- Inductive step:
  - Assume inequality is true for n, and prove it is true for (n+1):
  - Assume $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and prove: $\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$

$$\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2}$$

Example 2

- Prove that: $2n + 1 \le 2^n$ for all $n \ge 3$
- **Basis step:**
  - $n = 3$: $2 \times 3 + 1 \le 2^3 \Leftrightarrow 7 \le 8$ TRUE
- **Inductive step:**
  - Assume inequality is true for n, and prove it for (n+1)

Assume: $2n + 1 \le 2^n$

Must prove: $2(n + 1) + 1 \le 2^{n+1}$

$2(n + 1) + 1 = (2n + 1) + 2 \le 2^n + 2 \le$

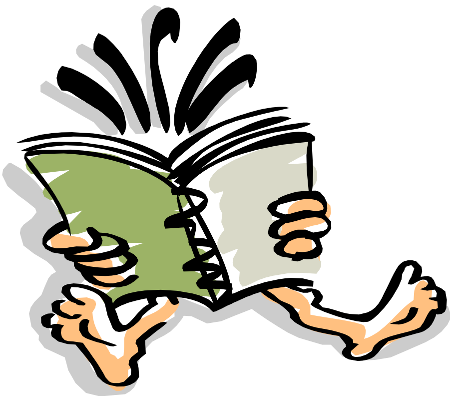
$\le 2^n + 2^n = 2^{n+1}$, since $2 \le 2^n$ for $n \ge 1$

More Examples

$$\sum_{i=1}^{n} (2i - 1) = n^2 \quad \forall n \ge 1$$

$$n! \ge 2^{n-1} \quad \forall n \ge 1$$

Readings



- Chapter 3
- Apendix A