CSC 611: Analysis of Algorithms

Lecture 2

Algorithms Analysis

Algorithms Analysis

- The amount of resources used by the algorithm
  - Space
  - Computational time
- Running time:
  - The number of primitive operations (steps) executed before termination
- Order of growth
  - The leading term of a formula
  - Expresses the behavior of a function toward infinity

Asymptotic Notations

- A way to describe behavior of functions in the limit
  - How we indicate running times of algorithms
  - Describe the running time of an algorithm as n grows to ∞
- O notation: asymptotic "less than": f(n) "≤" g(n)
- **Ω** notation: asymptotic "greater than": f(n) "≥" g(n)
- Θ notation: asymptotic "equality": f(n) "=" g(n)

Logarithms

- In algorithm analysis we often use the notation **"log n"** without specifying the base

Binary logarithm $\lg n = \log_2 n$

Natural logarithm $\ln n = \log_e n$

$$\log^k n = (\log n)^k$$

$$\log \log n = \log(\log n)$$

$$\log x^y = y \log x$$

$$\log xy = \log x + \log y$$

$$\log \frac{x}{y} = \log x - \log y$$

$$\log_a x = \log_a b \log_b x$$

$$a^{\log_b x} = x^{\log_b a}$$

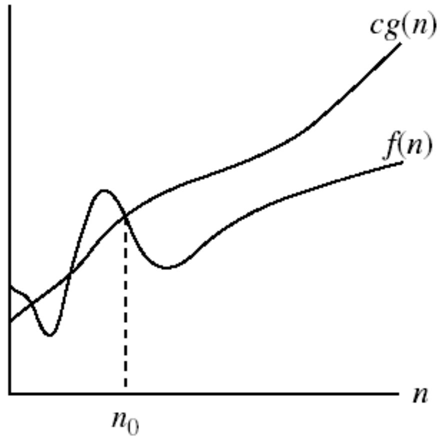
Asymptotic Notations - Examples

- For each of the following pairs of functions, either f(n) is O(g(n)), f(n) is Ω(g(n)), or f(n) is Θ(g(n)). Determine which relationship is correct.
  - $f(n) = \log n^2$; $g(n) = \log n + 5$ — $f(n) = \Theta(g(n))$
  - $f(n) = n$; $g(n) = \log n^2$ — $f(n) = \Omega(g(n))$
  - $f(n) = \log \log n$; $g(n) = \log n$ — $f(n) = O(g(n))$
  - $f(n) = n$; $g(n) = \log^2 n$ — $f(n) = \Omega(g(n))$
  - $f(n) = n \log n + n$; $g(n) = \log n$ — $f(n) = \Omega(g(n))$
  - $f(n) = 10$; $g(n) = \log 10$ — $f(n) = \Theta(g(n))$
  - $f(n) = 2^n$; $g(n) = 10n^2$ — $f(n) = \Omega(g(n))$
  - $f(n) = 2^n$; $g(n) = 3^n$ — $f(n) = O(g(n))$

Asymptotic notations

- *O-notation*

$$O(g(n)) = \{f(n) : \text{there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0\}.$$



- Intuitively: O(g(n)) = the set of functions with a smaller or same order of growth as g(n)

$g(n)$ is an ***asymptotic upper bound*** for $f(n)$.

Examples

- $2n^2 = O(n^3)$: $2n^2 \le cn^3 \Rightarrow 2 \le cn \Rightarrow c = 1$ and $n_0 = 2$

- $n^2 = O(n^2)$: $n^2 \le cn^2 \Rightarrow c \ge 1 \Rightarrow c = 1$ and $n_0 = 1$

- $1000n^2+1000n = O(n^2)$:

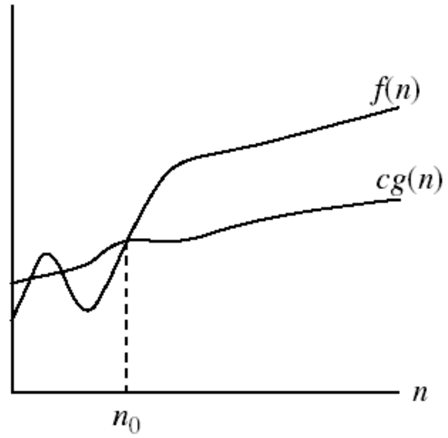
  $1000n^2+1000n \le 1000n^2+ 1000n^2 = 2000n^2$
  $\Rightarrow c=2000$ and $n_0 = 1$

- $n = O(n^2)$: $n \le cn^2 \Rightarrow cn \ge 1 \Rightarrow c = 1$ and $n_0 = 1$

Asymptotic notations (cont.)

- **Ω** - *notation*

$$\Omega(g(n)) = \{f(n) : \text{there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \le cg(n) \le f(n) \text{ for all } n \ge n_0\}.$$



- Intuitively: **Ω**(g(n)) = the set of functions with a larger or same order of growth as g(n)

$g(n)$ is an ***asymptotic lower bound*** for $f(n)$.

Examples

– $5n^2 = \Omega(n)$

$\exists$ c, n_0 such that: $0 \le cn \le 5n^2$ $\Rightarrow$ $cn \le 5n^2$ $\Rightarrow$ $c = 1$ and $n_0 = 1$

– $100n + 5 \ne \Omega(n^2)$

$\exists$ c, n_0 such that: $0 \le cn^2 \le 100n + 5$

$100n + 5 \le 100n + 5n$ ($\forall$ $n \ge 1$) $= 105n$

$cn^2 \le 105n \Rightarrow n(cn - 105) \le 0$

Since n is positive $\Rightarrow cn - 105 \le 0$ $\Rightarrow n \le 105/c$

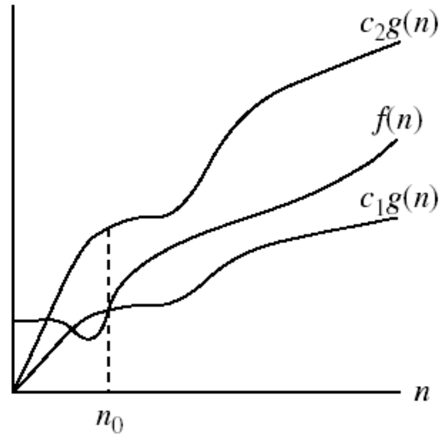
$\Rightarrow$ contradiction: n cannot be smaller than a constant

– $n = \Omega(2n)$, $n^3 = \Omega(n^2)$, $n = \Omega(\log n)$

Asymptotic notations (cont.)

- Θ*-notation*

$\Theta(g(n)) = \{f(n) :$ there exist positive constants c_1, c_2, and n_0 such that $0 \le c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \ge n_0\}$.



- Intuitively $\Theta(g(n))$ = the set of functions with the same order of growth as g(n)

$g(n)$ is an ***asymptotically tight bound*** for $f(n)$.

Examples

- $n^2/2 - n/2 = \Theta(n^2)$
  - $\frac{1}{2} n^2 - \frac{1}{2} n \le \frac{1}{2} n^2 \ \forall n \ge 0 \quad \Rightarrow \quad c_2 = \frac{1}{2}$
  - $\frac{1}{2} n^2 - \frac{1}{2} n \ge \frac{1}{2} n^2 - \frac{1}{2} n * \frac{1}{2} n \ (\forall n \ge 2) = \frac{1}{4} n^2$

    $\Rightarrow \quad c_1 = \frac{1}{4}$

- $n \ne \Theta(n^2)$: $c_1 n^2 \le n \le c_2 n^2$

  $\Rightarrow$ only holds for: $n \le 1/c_1$

Examples

- $6n^3 \ne \Theta(n^2)$: $c_1 n^2 \le 6n^3 \le c_2 n^2$

  $\Rightarrow$ only holds for: $n \le c_2 /6$

- $n \ne \Theta(\log n)$: $c_1 \log n \le n \le c_2 \log n$

  $\Rightarrow c_2 \ge n/\log n, \forall n \ge n_0$ – impossible

More on Asymptotic Notations

- There is no unique set of values for n_0 and c in proving the asymptotic bounds
- Prove that $100n + 5 = O(n^2)$
  - $100n + 5 \leq 100n + n = 101n \leq 101n^2$

    for all $n \geq 5$

    $n_0 = 5$ and $c = 101$ is a solution
  - $100n + 5 \leq 100n + 5n = 105n \leq 105n^2$

    for all $n \geq 1$

    $n_0 = 1$ and $c = 105$ is also a solution

  Must find **SOME** constants c and n_0 that satisfy the asymptotic notation relation

Comparisons of Functions

- *Theorem:*

  $$f(n) = \Theta(g(n)) \Leftrightarrow f = O(g(n)) \text{ and } f = \Omega(g(n))$$
- Transitivity:
  - $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n)) \Rightarrow f(n) = \Theta(h(n))$
  - Same for O and Ω
- Reflexivity:
  - $f(n) = \Theta(f(n))$
  - Same for O and Ω
- Symmetry:
  - $f(n) = \Theta(g(n))$ if and only if $g(n) = \Theta(f(n))$
- Transpose symmetry:
  - $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$

Asymptotic Notations in Equations

- On the right-hand side
  - $\Theta(n^2)$ stands for some anonymous function in $\Theta(n^2)$

  $2n^2 + 3n + 1 = 2n^2 + \Theta(n)$ means:

  There exists a function $f(n) \in \Theta(n)$ such that

  $2n^2 + 3n + 1 = 2n^2 + f(n)$
- On the left-hand side

  $2n^2 + \Theta(n) = \Theta(n^2)$

  No matter how the anonymous function is chosen on the left-hand side, there is a way to choose the anonymous function on the right-hand side to make the equation valid.

Some Simple Summation Formulas

- Arithmetic series: $$\sum_{k=1}^{n} k = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$
- Geometric series: $$\sum_{k=0}^{n} x^k = 1 + x + x^2 + \ldots + x^n = \frac{x^{n+1}-1}{x-1}\ (x \neq 1)$$
  - Special case: $x < 1$: $$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$
- Harmonic series: $$\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \ldots + \frac{1}{n} \approx \ln n$$
- Other important formulas: $$\sum_{k=1}^{n} \lg k \approx n \lg n$$ $$\sum_{k=1}^{n} k^p = 1^p + 2^p + \ldots + n^p \approx \frac{1}{p+1} n^{p+1}$$

Mathematical Induction

- Used to prove a sequence of statements ($S(1)$, $S(2)$, … $S(n)$) indexed by positive integers
- Proof:
  - **Basis step**: prove that the statement is true for $n = 1$
  - **Inductive step:** assume that $S(n)$ is true and prove that $S(n+1)$ is true for all $n \geq 1$
- Find case n "within" case $n+1$

Example

- Prove that: $2n + 1 \leq 2^n$ for all $n \geq 3$
- **Basis step:**
  - $n = 3$: $2 \times 3 + 1 \leq 2^3 \Leftrightarrow 7 \leq 8$ TRUE
- **Inductive step:**
  - Assume inequality is true for n, and prove it for (n+1)

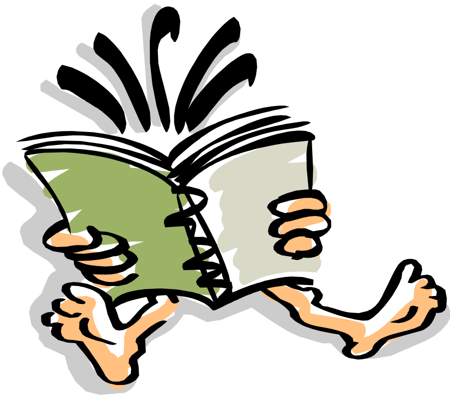
Assume: $2n + 1 \leq 2^n$

Must prove: $2(n + 1) + 1 \leq 2^{n+1}$

$2(n + 1) + 1 = (2n + 1) + 2 \leq 2^n + 2 \leq$

$\leq 2^n + 2^n = 2^{n+1}$, since $2 \leq 2^n$ for $n \geq 1$

Readings



- Chapter 3
- Apendix A