CSC 611: Analysis of Algorithms

Lecture 4

Recurrence Relations [Continued]

Recurrences - Intuition

- For a recurrence of the type:

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

- It takes f(n) to make the processing for the problem of size n
- The algorithm divides the problem into a subproblems, each of size n/b
- T(n) = number of subproblems * Running time(n/b) + processing of the problem of size n

Master's method

- "Cookbook" for solving recurrences of the form:

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

where, $a \geq 1$, $b > 1$, and $f(n) > 0$

Idea: compare $f(n)$ with $n^{\log_b a}$

- $f(n)$ is asymptotically smaller or larger than $n^{\log_b a}$ by a polynomial factor n^{ε}
- $f(n)$ is asymptotically equal with $n^{\log_b a}$

Master's method

- "Cookbook" for solving recurrences of the form:

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

where, $a \geq 1$, $b > 1$, and $f(n) > 0$

Case 1: if $f(n) = O(n^{\log_b a - \varepsilon})$ for some $\varepsilon > 0$, then: $T(n) = \Theta(n^{\log_b a})$

Case 2: if $f(n) = \Theta(n^{\log_b a})$, then: $T(n) = \Theta(n^{\log_b a} \lg n)$

Case 3: if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some $\varepsilon > 0$, and if

$af(n/b) \leq cf(n)$ for some $c < 1$ and all sufficiently large n, then:

$T(n) = \Theta(f(n))$

regularity condition

Why $n^{\log_b a}$?

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$$T(n) = a\underbrace{T\left(\frac{n}{b}\right)}_{}$$

$$a^2\underbrace{T\left(\frac{n}{b^2}\right)}_{}$$

$$a^3T\left(\frac{n}{b^3}\right)$$

$$\vdots$$

$$T(n) = a^iT\left(\frac{n}{b^i}\right) \quad \forall i$$

- Assume $n = b^k \Rightarrow k = \log_b n$
- At the end of iterations, i = k:

$$T(n) = a^{\log_b n}T\left(\frac{b^i}{b^i}\right) = a^{\log_b n}T(1) = \Theta\left(a^{\log_b n}\right) = \Theta\left(n^{\log_b a}\right)$$

- Case 1:
  - If $f(n)$ is dominated by $n^{\log_b a}$:
    - $T(n) = \Theta(n^{\log_b n})$
- Case 3:
  - If $f(n)$ dominates $n^{\log_b a}$:
    - $T(n) = \Theta(f(n))$
- Case 2:
  - If $f(n) = \Theta(n^{\log_b a})$:
    - $T(n) = \Theta(n^{\log_b a} \log n)$

Examples

$T(n) = 2T(n/2) + n$

$a = 2$, $b = 2$, $\log_2 2 = 1$

Compare $n^{\log_2 2}$ with $f(n) = n$

$\Rightarrow f(n) = \Theta(n) \Rightarrow$ Case 2

$\Rightarrow T(n) = \Theta(n\lg n)$

Examples

$$T(n) = 2T(n/2) + n^2$$

$a = 2$, $b = 2$, $\log_2 2 = 1$

Compare n with $f(n) = n^2$

$\Rightarrow f(n) = \Omega(n^{1+\varepsilon})$

Case 3 $\Rightarrow$ verify regularity cond.: $a\, f(n/b) \le c\, f(n)$

$\Rightarrow 2\, n^2/4 \le c\, n^2 \Rightarrow c = \frac{1}{2}$ is a solution $(c<1)$

$\Rightarrow T(n) = \Theta(n^2)$

Examples (cont.)

$$T(n) = 2T(n/2) + \sqrt{n}$$

$a = 2$, $b = 2$, $\log_2 2 = 1$

Compare n with $f(n) = n^{1/2}$

$\Rightarrow f(n) = O(n^{1-\varepsilon})$ Case 1

$\Rightarrow T(n) = \Theta(n)$

Examples

$$T(n) = 3T(n/4) + n\lg n$$

$a = 3$, $b = 4$, $\log_4 3 = 0.793$

Compare $n^{0.793}$ with $f(n) = n\lg n$

$f(n) = \Omega(n^{\log_4 3+\varepsilon})$

Case 3: check regularity condition:

$3(n/4)\lg(n/4) \le (3/4)n\lg n = c\,f(n)$, $c=3/4$

$\Rightarrow T(n) = \Theta(n\lg n)$

Examples

$$T(n) = 2T(n/2) + n\lg n$$

$a = 2$, $b = 2$, $\log_2 2 = 1$

- **Compare n with $f(n) = n\lg n$**
  - seems like case 3 should apply
- $f(n)$ must be polynomially larger by a factor of n^{ε}
- In this case it is only larger by a factor of $\lg n$

The Sorting Problem

- **Input:**
  - A sequence of n numbers $a_1, a_2, \ldots, a_n$
- **Output:**
  - A permutation (reordering) $a_1', a_2', \ldots, a_n'$ of the input sequence such that $a_1' \leq a_2' \leq \cdots \leq a_n'$

Why Study Sorting Algorithms?

- There are a variety of situations that we can encounter
  - Do we have randomly ordered keys?
  - Are all keys distinct?
  - How large is the set of keys to be ordered?
  - Need guaranteed performance?
  - Does the algorithm sort in place?
  - Is the algorithm stable?
- Various algorithms are better suited to some of these situations

Stability

- A STABLE sort preserves relative order of records with equal keys

Sort file on first key:

Aaron	4	A	664-480-0023	097 Little
Andrews	3	A	874-088-1212	121 Whitman
Battle	4	C	991-878-4944	308 Blair
Chen	2	A	884-232-5341	11 Dickinson
Fox	1	A	243-456-9091	101 Brown
Furia	3	A	766-093-9873	22 Brown
Gazsi	4	B	665-303-0266	113 Walker
Kanaga	3	B	898-122-9643	343 Forbes
Rohde	3	A	232-343-5555	115 Holder
Quilici	1	C	343-987-5642	32 McCosh

Sort file on second key:

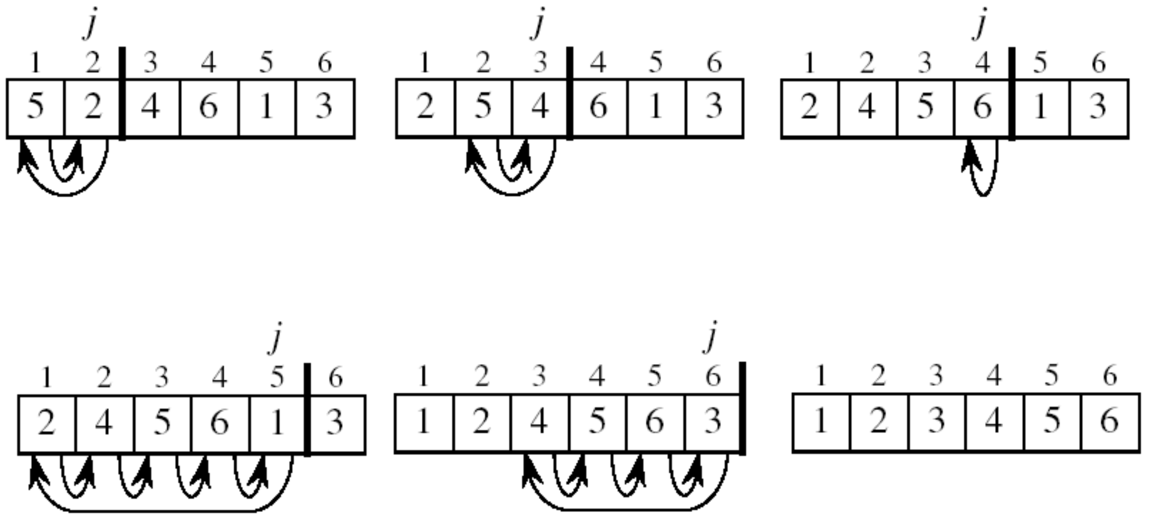
Fox	1	A	243-456-9091	101 Brown
Quilici	1	C	343-987-5642	32 McCosh
Chen	2	A	884-232-5341	11 Dickinson
Kanaga	3	B	898-122-9643	343 Forbes
Andrews	3	A	874-088-1212	121 Whitman
Furia	3	A	766-093-9873	22 Brown
Rohde	3	A	232-343-5555	115 Holder
Battle	4	C	991-878-4944	308 Blair
Gazsi	4	B	665-303-0266	113 Walker
Aaron	4	A	664-480-0023	097 Little

Records with key value 3 are not in order on first key!!

Insertion Sort

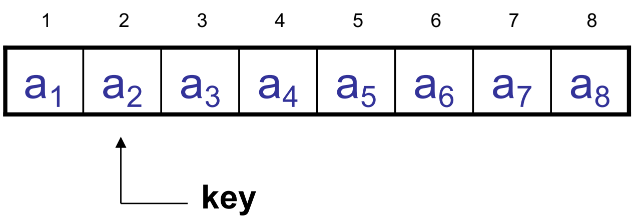
- Idea: like sorting a hand of playing cards
  - Start with an empty left hand and the cards facing down on the table
  - Remove one card at a time from the table, and insert it into the correct position in the left hand
    - compare it with each of the cards already in the hand, from right to left
  - The cards held in the left hand are sorted
    - these cards were originally the top cards of the pile on the table

Example



INSERTION-SORT

Alg.: INSERTION-SORT(*A*)

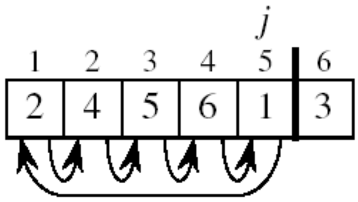


for j ← 2 **to** n
    do key ← A[j]
        ▷Insert A[j] into the sorted sequence A[1 . . j -1]
        i ← j - 1
        while i > 0 and A[i] > key
            do A[i + 1] ← A[i]
                i ← i - 1
        A[i + 1] ← key

- Insertion sort – sorts the elements in place

Loop Invariant for Insertion Sort

Alg.: INSERTION-SORT(*A*)



for j ← 2 **to** n
  do key ← A[j]
    Insert A[j] into the sorted sequence A[1 . . j -1]
    i ← j - 1
    while i > 0 and A[i] > key
      do A[i + 1] ← A[i]
        i ← i - 1
    A[i + 1] ← key

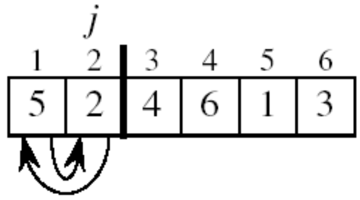
Invariant: at the start of each iteration of the for loop, the elements in A[1 . . j-1] are in sorted order

Proving Loop Invariants

- Proving loop invariants works like induction
- **Initialization (base case):**
  – It is true prior to the first iteration of the loop
- **Maintenance (inductive step):**
  – If it is true before an iteration of the loop, it remains true before the next iteration
- **Termination:**
  – When the loop terminates, the invariant gives us a useful property that helps show that the algorithm is correct

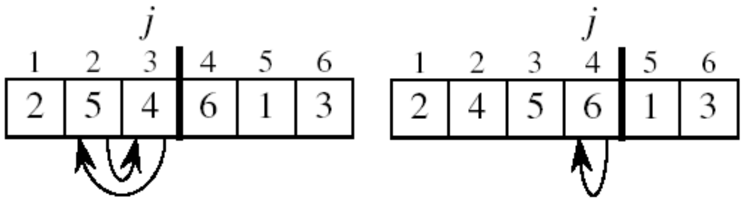
Loop Invariant for Insertion Sort

- **Initialization:**
  – Just before the first iteration, j = 2: the subarray A[1 . . j-1] = A[1], (the element originally in A[1]) – is sorted



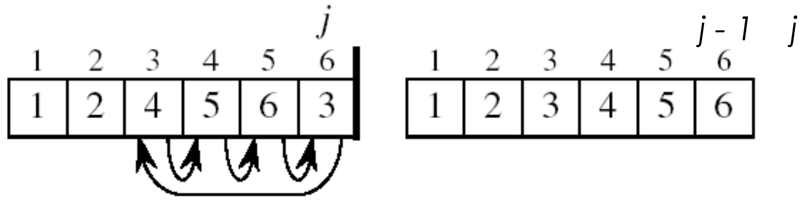
Loop Invariant for Insertion Sort

- **Maintenance:**
  – the **while** inner loop moves A[j -1], A[j -2], A[j -3], and so on, by one position to the right until the proper position for key (which has the value that started out in A[j]) is found
  – At that point, the value of key is placed into this position.



Loop Invariant for Insertion Sort

- **Termination:**
  - The outer **for** loop ends when $j = n + 1 \Rightarrow j-1 = n$
  - Replace n with $j-1$ in the loop invariant:
    - the subarray $A[1 .. n]$ consists of the elements originally in $A[1 .. n]$, but in sorted order



- The entire array is sorted!

Analysis of Insertion Sort

INSERTION-SORT(*A*)	cost	times
for j ← 2 **to** n	c_1	n
do key ← A[j]	c_2	n-1
▷Insert A[j] into the sorted seq. A[1 . . j -1]	0	n-1
i ← j - 1	c_4	n-1
while i > 0 and A[i] > key	c_5	$\sum_{j=2}^{n} t_j$
do A[i + 1] ← A[i]	c_6	$\sum_{j=2}^{n} (t_j - 1)$
i ← i – 1	c_7	$\sum_{j=2}^{n} (t_j - 1)$
A[i + 1] ← key	c_8	n-1

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5\sum_{j=2}^{n} t_j + c_6\sum_{j=2}^{n}(t_j - 1) + c_7\sum_{j=2}^{n}(t_j - 1) + c_8(n-1)$$

Best Case Analysis

- The array is already sorted "**while** i > 0 and A[i] > key"
  - A[i] ≤ **key** upon the first time the **while** loop test is run (when $i = j - 1$)
  - $t_j = 1$
- $T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5(n-1) + c_8(n-1)$

  $= (c_1 + c_2 + c_4 + c_5 + c_8)n - (c_2 + c_4 + c_5 + c_8)$

  $= an + b = \Theta(n)$

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5\sum_{j=2}^{n} t_j + c_6\sum_{j=2}^{n}\left(t_j - 1\right) + c_7\sum_{j=2}^{n}\left(t_j - 1\right) + c_8(n-1)$$

Worst Case Analysis

- The array is reversely sorted "while i > 0 and A[i] > key"
  - Always A[i] > **key** in **while** loop test
  - Have to compare **key** with all elements to the left of the **j**-th position ⇒ compare with **j-1** elements ⇒ $t_j = j$

$$\sum_{j=2}^{n} j = \frac{n(n+1)}{2} - 1 \quad and \quad \sum_{j=2}^{n}(j-1) = \frac{n(n-1)}{2}$$

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5\left(\frac{n(n+1)}{2} - 1\right) + c_6\frac{n(n-1)}{2} + c_7\frac{n(n-1)}{2} + c_8(n-1)$$

$$= an^2 + bn + c$$ a quadratic function of n

- $T(n) = \Theta(n^2)$ order of growth in n^2

Comparisons and Exchanges in Insertion Sort

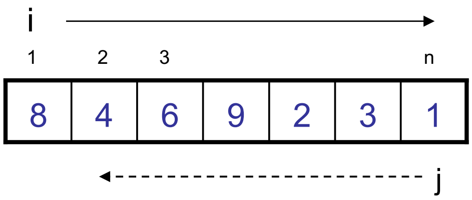
INSERTION-SORT(*A*)	cost	times
for j ← 2 **to** n	c_1	n
do key ← A[j]	c_2	n-1
▷Insert A[j] into the sorted sequence A[1 . . j -1]	0	n-1
i ← j - 1 ≈$n^2/2$ comparisons	c_4	n-1
while i > 0 and A[i] > key	c_5	$\sum_{j=2}^{n} t_j$
do A[i + 1] ← A[i]	c_6	$\sum_{j=2}^{n} (t_j - 1)$
i ← i – 1 ≈$n^2/2$ exchanges	c_7	$\sum_{j=2}^{n} (t_j - 1)$
A[i + 1] ← key	c_8	n-1

Insertion Sort - Summary

- Idea: like sorting a hand of playing cards
  - Start with an empty left hand and the cards facing down on the table.
  - Remove one card at a time from the table, and insert it into the correct position in the left hand
- Advantages
  - Good running time for "almost sorted" arrays $\Theta(n)$
- Disadvantages
  - $\Theta(n^2)$ running time in worst and average case
  - ≈$n^2/2$ comparisons and $n^2/2$ exchanges

Bubble Sort

- Idea:
  - Repeatedly pass through the array
  - Swaps adjacent elements that are out of order



- Easier to implement, but slower than Insertion sort

Example

8	4	6	9	2	3	1

i = 1 j

8	4	6	9	2	1	3

i = 1 j

8	4	6	9	1	2	3

i = 1 j

8	4	6	1	9	2	3

i = 1 j

8	4	1	6	9	2	3

i = 1 j

8	1	4	6	9	2	3

i = 1 j

1	8	4	6	9	2	3

i = 1 j

1	8	4	6	9	2	3

i = 2 j

1	2	8	4	6	9	3

i = 3 j

1	2	3	8	4	6	9

i = 4 j

1	2	3	4	8	6	9

i = 5 j

1	2	3	4	6	8	9

i = 6 j

1	2	3	4	6	8	9

i = 7
j

Bubble Sort

Alg.: BUBBLESORT(A)

for i ← 1 **to** length[A]
  do for j ← length[A] **downto** i + 1
    do if A[j] < A[j -1]
      then exchange A[j] ⟺A[j-1]

i →

8	4	6	9	2	3	1

i = 1 ← - - - - - - - - - - - - j

Bubble-Sort Running Time

Alg.: BUBBLESORT(A)

for i ←1 **to** length[A]
  do for j ←length[A] **downto** i + 1
    do if A[j] < A[j -1]
      then exchange A[j] ⟺A[j-1]

Comparisons: $\approx n^2/2$

Exchanges: $\approx n^2/2$

$$T(n) = c_1(n+1) + c_2\sum_{i=1}^{n}(n-i+1) + c_3\sum_{i=1}^{n}(n-i) + c_4\sum_{i=1}^{n}(n-i)$$

$$= \Theta(n) + (c_2 + c_3 + c_4)\sum_{i=1}^{n}(n-i)$$

$$\approx \sum_{i=1}^{n}(n-i) = \sum_{i=1}^{n}n - \sum_{i=1}^{n}i = n^2 - \frac{n(n+1)}{2} = \frac{n^2}{2} - \frac{n}{2}$$

$T(n) = \Theta(n^2)$

Selection Sort

8	4	6	9	2	3	1

- Idea:
  - Find the smallest element in the array
  - Exchange it with the element in the first position
  - Find the second smallest element and exchange it with the element in the second position
  - Continue until the array is sorted
- Invariant:
  - All elements to the left of the current index are in sorted order and never changed again
- Disadvantage:
  - Running time depends only slightly on the amount of order in the file

Example

8	4	6	9	2	3	1
1	4	6	9	2	3	8
1	2	6	9	4	3	8
1	2	3	9	4	6	8

1	2	3	4	9	6	8
1	2	3	4	6	9	8
1	2	3	4	6	8	9
1	2	3	4	6	8	9

Selection Sort

Alg.: SELECTION-SORT(*A*)

8	4	6	9	2	3	1

```
n ← length[A]
for j ← 1 to n - 1
    do smallest ← j
        for i ← j + 1 to n
            do if A[i] < A[smallest]
                then smallest ← i
        exchange A[j] ⇔ A[smallest]
```


Analysis of Selection Sort

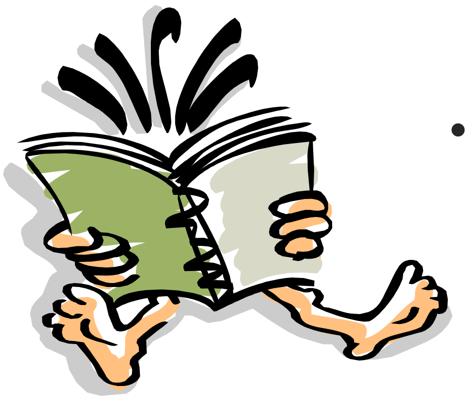
Alg.: SELECTION-SORT(*A*)	cost	times
n ← length[A]	c_1	1
for j ← 1 to n - 1	c_2	n
do smallest ← j	c_3	n-1
for i ← j + 1 to n	c_4	$\sum_{j=1}^{n-1}(n-j+1)$
do if A[i] < A[smallest]	c_5	$\sum_{j=1}^{n-1}(n-j)$
then smallest ← i	c_6	$\sum_{j=1}^{n-1}(n-j)$
exchange A[j] ⇔ A[smallest]	c_7	n-1

$\approx n^2/2$ comparisons

$\approx n$ exchanges

$T(n) = \Theta(n^2)$

Readings



- Chapter 4