CSC 611: Analysis of Algorithms

Lecture 3

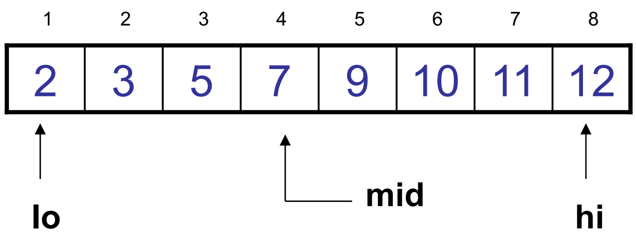
Recurrence Relations

Recurrent Algorithms: BINARY – SEARCH

- for an ordered array A, finds if **x** is in the array A[lo…hi]

Alg.: BINARY-SEARCH (A, lo, hi, x)

```
if (lo > hi)
    return FALSE
mid ← ⌊(lo+hi)/2⌋
if x = A[mid]
    return TRUE
if ( x < A[mid] )
    BINARY-SEARCH (A, lo, mid-1, x)
if ( x > A[mid] )
    BINARY-SEARCH (A, mid+1, hi, x)
```



Example

- A[8] = {1, 2, 3, 4, 5, 7, 9, 11}
  - lo = 1 hi = 8 x = 7

1	2	3	4	5	6	7	8
1	2	3	4	5	7	9	11

mid = 4, lo = 5, hi = 8

1	2	3	4	5	7	9	11

mid = 6, A[mid] = x
Found!

Example

- A[8] = {1, 2, 3, 4, 5, 7, 9, 11}
  - lo = 1 hi = 8 x = 6

1	2	3	4	5	6	7	8
1	2	3	4	5	7	9	11

mid = 4, lo = 5, hi = 8

1	2	3	4	5	7	9	11

mid = 6, A[6] = 7, lo = 5, hi = 5

1	2	3	4	5	7	9	11

mid = 5, A[5] = 5, lo = 6, hi = 5
NOT FOUND!

Analysis of BINARY-SEARCH

Alg.: BINARY-SEARCH (A, lo, hi, x)

```
if (lo > hi)                                  ← constant time: c1
    return FALSE
mid ← ⌊(lo+hi)/2⌋                             ← constant time: c2
if x = A[mid]                                 ← constant time: c3
    return TRUE
if ( x < A[mid] )
    BINARY-SEARCH (A, lo, mid-1, x)   ← same problem of size n/2
if ( x > A[mid] )
    BINARY-SEARCH (A, mid+1, hi, x)   ← same problem of size n/2
```

- $T(n) = c + T(n/2)$
  - $T(n)$ – running time for an array of size n

Recurrences and Running Time

Recurrences arise when an algorithm contains recursive calls to itself

- What is the actual running time of the algorithm?
- Need to solve the recurrence
  – Find an explicit formula of the expression (the generic term of the sequence)

Example Recurrences

- $T(n) = T(n-1) + n$ $\Theta(n^2)$
  - Recursive algorithm that loops through the input to eliminate one item
- $T(n) = T(n/2) + c$ $\Theta(\lg n)$
  - Recursive algorithm that halves the input in one step
- $T(n) = T(n/2) + n$ $\Theta(n)$
  - Recursive algorithm that halves the input but must examine every item in the input
- $T(n) = 2T(n/2) + 1$ $\Theta(n)$
  - Recursive algorithm that splits the input into 2 halves and does a constant amount of other work

Methods for Solving Recurrences

- Iteration method
- Substitution method
- Recursion tree method
- Master method

Iteration Method

Calls for expanding a recurrence as a summation of terms dependent on n and the initial conditions

The Iteration Method

$$T(n) = c + T(n/2)$$

$T(n) = c + T(n/2)$ $\qquad$ $T(n/2) = c + T(n/4)$

$= c + c + T(n/4)$ $\qquad$ $T(n/4) = c + T(n/8)$

$= c + c + c + T(n/8)$

Assume $n = 2^k$

$$T(n) = \underbrace{c + c + \ldots + c}_{k \text{ times}} + T(1)$$

$= c\lg n + T(1)$

$= \Theta(\lg n)$

Iteration Method – Example 1

$$T(n) = n + 2T(n/2)$$ Assume: $n = 2^k$

$T(n/2) = n/2 + 2T(n/4)$

$$
\begin{aligned}
T(n) &= n + 2T(n/2) \\
&= n + 2(n/2 + 2T(n/4)) \\
&= n + n + 4T(n/4) \\
&= n + n + 4(n/4 + 2T(n/8)) \\
&= n + n + n + 8T(n/8) \\
\ldots &= in + 2^i T(n/2^i) \\
&= kn + 2^k T(1) \\
&= n\lg n + nT(1) = \Theta(n\lg n)
\end{aligned}
$$

Iteration Method – Example 2

$$T(n) = n + T(n-1)$$

$$
\begin{aligned}
T(n) &= n + T(n-1) \\
&= n + (n-1) + T(n-2) \\
&= n + (n-1) + (n-2) + T(n-3) \\
\ldots &= n + (n-1) + (n-2) + \ldots + 2 + T(1) \\
&= n(n+1)/2 - 1 + T(1) \\
&= n^2 + T(1) = \Theta(n^2)
\end{aligned}
$$

Iteration Method – Example 3

- $T(n)=3T\left(\left\lfloor \frac{n}{4}\right\rfloor\right)+n$ Can be iterated as follows:

$$
\begin{aligned}
T(n) &= n+3T\left(\left\lfloor \frac{n}{4}\right\rfloor\right)\\
&= n+3\left(\left\lfloor \frac{n}{4}\right\rfloor\right)+3T\left(\left\lfloor \frac{n}{16}\right\rfloor\right)\\
&= n+3T\left(\left\lfloor \frac{n}{4}\right\rfloor\right)+3\left(\left\lfloor \frac{n}{16}\right\rfloor\right)+3T\left(\left\lfloor \frac{n}{64}\right\rfloor\right)\\
&= n+3T\left(\left\lfloor \frac{n}{4}\right\rfloor\right)+9\left\lfloor \frac{n}{16}\right\rfloor+27T\left(\frac{n}{64}\right)
\end{aligned}
$$

- i^{th} term is $3^i\left\lfloor \frac{n}{4^i}\right\rfloor$

Iteration stops when

$$\left\lfloor \frac{n}{4^i}\right\rfloor=1 \Rightarrow i \geq \log_4 n$$

Iteration Method – Example 3

- Thus,
  - $T(n) \leq n + 3n/4 + 9n/16 + 27n/64 + \ldots + 3^{\log_4 n}\,\Theta(1)$

$$
\begin{aligned}
&\leq n\sum_{i=0}^{\infty}\frac{3}{4}i+\Theta(n\log_4 3)\\
&= 4n+o(n)\\
&= O(n)
\end{aligned}
$$

$a^{\log_b n} = n^{\log_b a}$

The substitution method

1. *Guess a solution*
2. *Use induction to prove that the solution works*

Substitution method

- Guess a solution
  - $T(n) = O(g(n))$
  - Induction goal: apply the definition of the asymptotic notation
    - $T(n) \le d\, g(n)$, for some $d > 0$ and $n \ge n_0$
  - Induction hypothesis: $T(k) \le d\, g(k)$ for all $k < n$
- Prove the induction goal
  - Use the **induction hypothesis** to find some values of the constants d and n_0 for which the **induction goal** holds

Example: Binary Search

$$T(n) = c + T(n/2)$$

- Guess: $T(n) = O(\lg n)$
  - Induction goal: $T(n) \le d \lg n$, for some d and $n \ge n_0$
  - Induction hypothesis: $T(n/2) \le d \lg(n/2)$
- Proof of induction goal:

$$T(n) = T(n/2) + c \le d \lg(n/2) + c$$
$$= d \lg n - d + c \le d \lg n$$
if: $-d + c \le 0,\ d \ge c$

Example: Binary Search

$$T(n) = c + T(n/2)$$

- Boundary conditions:
  - Base case: $n_0 = 1 \Rightarrow T(1) = c$ has to verify condition:
    $T(1) \le d \lg 1 \;\Rightarrow\; c \le d \lg 1 = 0$ – contradiction
  - Choose $n_0 = 2 \Rightarrow T(2) = 2c$ has to verify condition:
    $T(2) \le d \lg 2 \Rightarrow\; 2c \le d \lg 2 = d \Rightarrow\;$ choose $d \ge 2c$
- We can similarly prove that $T(n) = \Omega(\lg n)$ and therefore: $T(n) = \Theta(\lg n)$

Example 2

$$T(n) = T(n-1) + n$$

- Guess: $T(n) = O(n^2)$
  - Induction goal: $T(n) \le c\, n^2$, for some c and $n \ge n_0$
  - Induction hypothesis: $T(n-1) \le c(n-1)^2$
- Proof of induction goal:

  $T(n) = T(n-1) + n \le c\,(n-1)^2 + n$

  $= cn^2 - (2cn - c - n) \le cn^2$

  if: $2cn - c - n \ge 0 \Rightarrow c \ge n/(2n-1) \Rightarrow c \ge 1/(2 - 1/n)$
  - For $n \ge 1 \Rightarrow 2 - 1/n \ge 1 \Rightarrow$ any $c \ge 1$ will work

Example 2

$$T(n) = T(n-1) + n$$

- Boundary conditions:
  - Base case: $n_0 = 1 \Rightarrow T(1) = 1$ has to verify condition:

    $T(1) \le c\,(1)^2 \Rightarrow 1 \le c \Rightarrow$ OK!
- We can similarly prove that $T(n) = \Omega(n^2)$ and therefore: $T(n) = \Theta(n^2)$

Example 3

$T(n) = 2T(n/2) + n$

- Guess: $T(n) = O(n\lg n)$
  - Induction goal: $T(n) \le cn \lg n$, for some c and $n \ge n_0$
  - Induction hypothesis: $T(n/2) \le cn/2 \lg(n/2)$
- Proof of induction goal:

$$T(n) = 2T(n/2) + n \le 2c\,(n/2)\lg(n/2) + n$$

$$= cn \lg n - cn + n \le cn \lg n$$

$$\text{if: } -cn + n \le 0 \Rightarrow c \ge 1$$

Example 3

$T(n) = 2T(n/2) + n$

- Boundary conditions:
  - Base case: $n_0 = 1 \Rightarrow T(1) = 1$ has to verify condition:

    $T(1) \le cn_0\lg n_0 \Rightarrow 1 \le c * 1 * \lg 1 = 0$ – contradiction
  - Choose $n_0 = 2 \Rightarrow T(2) = 4$ has to verify condition:

    $T(2) \le c * 2 * \lg 2 \Rightarrow 4 \le 2c \Rightarrow$ choose $c = 2$
- We can similarly prove that $T(n) = \Omega(n\lg n)$ and therefore: $T(n) = \Omega(n\lg n)$

Changing variables

$$T(n) = 2T(\sqrt{n}) + \lg n$$

– Rename: $m = \lg n \Rightarrow n = 2^m$

$T(2^m) = 2T(2^{m/2}) + m$

– Rename: $S(m) = T(2^m)$

$S(m) = 2S(m/2) + m \Rightarrow S(m) = O(m\lg m)$
(demonstrated before)

$T(n) = T(2^m) = S(m) = O(m\lg m) = O(\lg n \lg\lg n)$

Idea: transform the recurrence to one that you have seen before

Changing variables (cont.)

$$T(n) = T(n/2) + 1$$

– Rename: $n = 2^m$ $(n/2 = 2^{m-1})$

$T(2^m) = T(2^{m-1}) + 1$

– Rename: $S(m) = T(2^m)$

$S(m) = S(m-1) + 1$

$= 1 + S(m-1) = 1 + 1 + S(m-2) = \underbrace{1 + 1 + \ldots + 1}_{m\,-1\text{ times}} + S(1)$

$S(m) = O(m) \Rightarrow T(n) = T(2^m) = S(m) = O(m) = O(\lg n)$

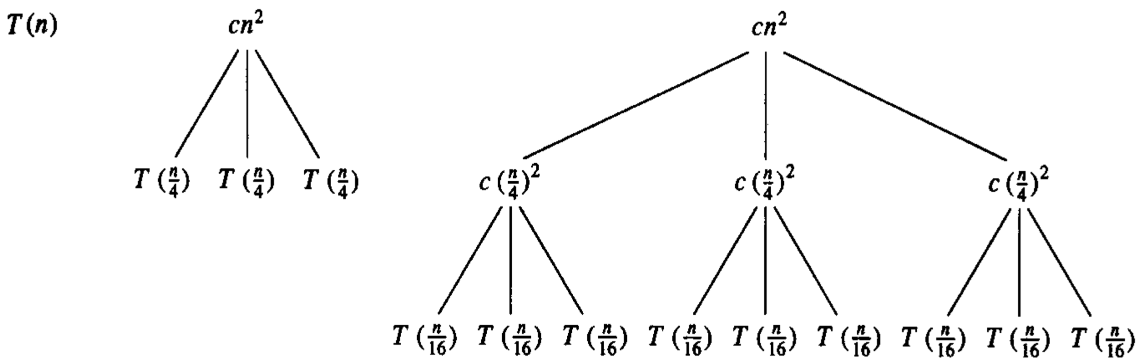
The recursion-tree method

Convert the recurrence into a tree:

- Each node represents the cost incurred at that level of recursion
- Sum up the costs of all levels

Used to “guess” a solution for the recurrence

E.g.: $T(n) = 3T(n/4) + cn^2$



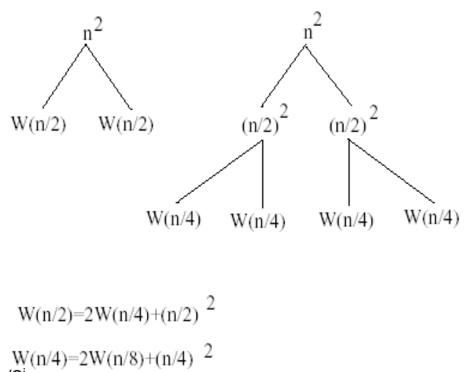
E.g.: T(n) = 3T(n/4) + cn² (contd.)

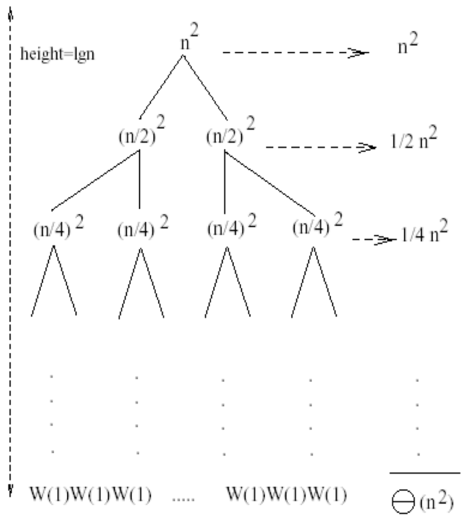
- Subproblem size at level i is: $n/4^i$
- Subproblem size hits 1 when $1 = n/4^i \Rightarrow i = \log_4 n$
- Cost of a node at level i = $c(n/4^i)^2$
- Number of nodes at level i = $3^i \Rightarrow$ last level has $3^{\log_4 n} = n^{\log_4 3}$ nodes
- Total cost:

$$T(n) = \sum_{i=0}^{\log_4 n-1}\left(\frac{3}{16}\right)^i cn^2 + \Theta\left(n^{\log_4 3}\right) \le \sum_{i=0}^{\infty}\left(\frac{3}{16}\right)^i cn^2 + \Theta\left(n^{\log_4 3}\right) = \frac{1}{1-\frac{3}{16}}cn^2 + \Theta\left(n^{\log_4 3}\right) = O(n^2)$$

$\Rightarrow T(n) \in O(n^2)$

$W(n) = 2W(n/2) + n^2$

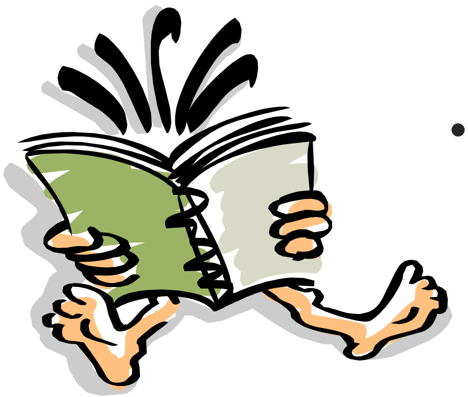




- Subproblem size at level i is: $n/2^i$
- Subproblem size hits 1 when $1 = n/2^i \Rightarrow i = \lg n$
- Cost of the problem at level i = $n^2/2^i$ No. of nodes at level i = 2^i
- h = Height of the tree ➔ $n/2^h=1$ ➔ h = lgn
- Total cost:
- $\Rightarrow W(n) = O(n^2)$

$$W(n) = \sum_{i=0}^{\lg n-1}\frac{n^2}{2^i} + 2^{\lg n}W(1) = n^2\sum_{i=0}^{\lg n-1}\left(\frac{1}{2}\right)^i + n \le n^2\sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^i + O(n) = n^2\frac{1}{1-\frac{1}{2}} + O(n) = 2n^2$$

Readings



- Chapter 4