CSC 611: Analysis of Algorithms

Lecture 7

Medians and Order Statistics

Medians and Order Statistics

*Def.: The i-th **order statistic** of a set of n elements is the i-th smallest element.*

- The minimum of a set of elements:
  - The first order statistic i = 1
- The maximum of a set of elements:
  - The n-th order statistic i = n
- The median is the "halfway point" of the set
  - i = (n+1)/2, is unique when n is odd
  - i = $\lfloor (n+1)/2 \rfloor$ = n/2 (lower median) and $\lceil (n+1)/2 \rceil$ = n/2+1 (upper median), when n is even

Finding Minimum or Maximum

Alg.: MINIMUM(A, n)

```
min ← A[1]
for i ← 2 to n
    do if min > A[i]
        then min ← A[i]
return min
```

- How many comparisons are needed?
  - n – 1: each element, except the minimum, must be compared to a smaller element at least once
  - The same number of comparisons are needed to find the maximum
  - The algorithm is optimal with respect to the number of comparisons performed

Simultaneous Min, Max

- Find min and max independently
  - Use n – 1 comparisons for each ⇒ total of **2n – 2**
- However, we can do better: at most **3n/2** comparisons
  - Process elements in pairs
  - Maintain the minimum and maximum of elements seen so far
  - Don't compare each element to the minimum and maximum separately
  - Compare the elements of a pair to each other
  - Compare the larger element to the maximum so far, and compare the smaller element to the minimum so far
  - This leads to only 3 comparisons for every 2 elements

Analysis of Simultaneous Min, Max

- Setting up initial values:
  - n is odd: set both **min** and **max** to the first element
  - n is even: compare the first two elements, assign the smallest one to **min** and the largest one to **max**
- Total number of comparisons:
  - n is odd: we do $3(n-1)/2$ comparisons
  - n is even: we do 1 initial comparison + $3(n-2)/2$ more comparisons = $3n/2 - 2$ comparisons

Example: Simultaneous Min, Max

- $n = 5$ (odd), array A = {2, 7, 1, 3, 4}
  1. Set **min** = **max** = 2
  2. Compare elements in pairs:
     - 1 < 7 ⇒ compare 1 with **min** and 7 with **max**
       ⇒ **min** = 1, **max** = 7
       3 comparisons
     - 3 < 4 ⇒ compare 3 with **min** and 4 with **max**
       ⇒ **min** = 1, **max** = 7
       3 comparisons

We performed: $3(n-1)/2 = 6$ comparisons

Example: Simultaneous Min, Max

- $n = 6$ (even), array $A = \{2, 5, 3, 7, 1, 4\}$

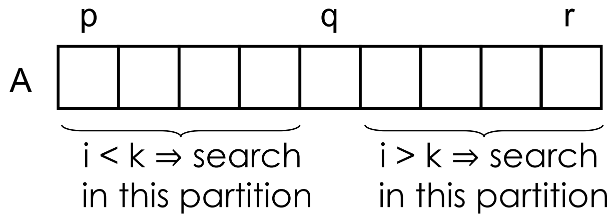
1. Compare 2 with 5: $2 < 5$ — 1 comparison
2. Set **min =** 2, **max** = 5
3. Compare elements in pairs:
   - $3 < 7 \Rightarrow$ compare 3 with **min** and 7 with **max** $\Rightarrow$ **min** = 2, **max** = 7 — 3 comparisons
   - $1 < 4 \Rightarrow$ compare 1 with **min** and 4 with **max** $\Rightarrow$ **min** = 1, **max** = 7 — 3 comparisons

We performed: $3n/2 - 2 = 7$ comparisons

General Selection Problem

- Select the i-th order statistic (i-th smallest element) form a set of n distinct numbers

$k = q – p + 1$



- Idea:
  - Partition the input array similarly with the approach used for Quicksort (use RANDOMIZED-PARTITION)
  - Recurse on one side of the partition to look for the i-th element depending on where i is with respect to the pivot

- We will show that selection of the **i**-th smallest element of the array *A* can be done in $\Theta(n)$ time

Randomized Select

Alg.: RANDOMIZED-SELECT(*A*, p, r, i)

if p = r

  then return *A*[p]

q ←RANDOMIZED-PARTITION(*A*, p, r)

k ← q - p + 1

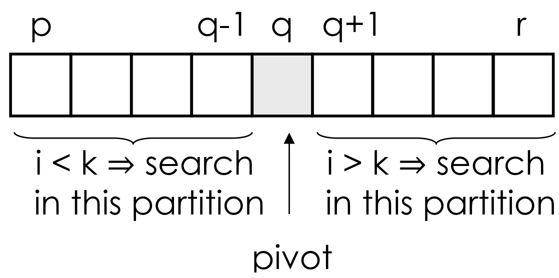
if i = k

  then return *A*[q]   ▷ pivot value is the answer

elseif i < k

    then return RANDOMIZED-SELECT(*A*, p, q-1, i)

else return RANDOMIZED-SELECT(*A*, q + 1, r, i-k)



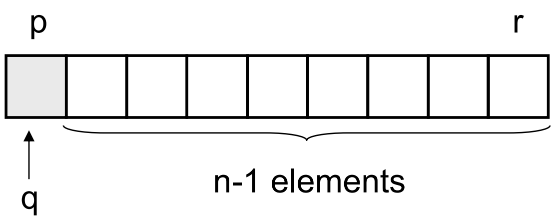
RANDOMIZED-PARTITION(*A*, *p*, *r*)

1 $i = \text{RANDOM}(p, r)$
2 exchange $A[r]$ with $A[i]$
3 **return** PARTITION(A, p, r)

Analysis of Running Time

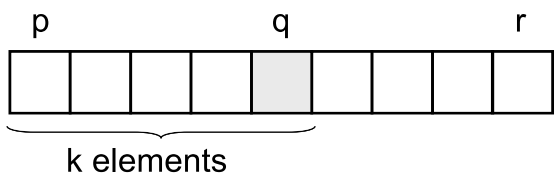
- **Worst case** running time: **$\Theta(n^2)$**
  - If we always partition around the largest/smallest remaining element
  - Partition takes $\Theta(n)$ time
  - $T(n) = \Theta(1)$ (compute k) + $\Theta(n)$ (partition) + $T(n-1)$

    $= 1 + n + T(n-1) = \Theta(n^2)$



Analysis of Running Time

- **Expected** running time (on **average**)
  - Let T(n) be a random variable denoting the running time of RANDOMIZED-SELECT



  - RANDOMIZED-PARTITION is equally likely to return any element of *A* as the pivot ⇒
  - For each k such that $1 \le k \le n$, the subarray $A[p \, . \, . \, q]$ has k elements (all ≤ pivot) with probability $1/n$

Analysis of Running Time

- When we call RANDOMIZED-SELECT we could have three situations:
  - The algorithm terminates with the answer ($i = k$), or
  - The algorithm recurses on the subarray $A[p..q-1]$, or
  - The algorithm recurses on the subarray $A[q+1..r]$
- The decision depends on where the i-th smallest element falls relative to $A[q]$
- To obtain an upper bound for the running time $T(n)$:
  - assume the i-th smallest element is always in the larger subarray

Analysis of Running Time (cont.)

$$E[T(n)] = \underbrace{\text{Probability that T(n) takes a value} \times \text{The value of the random variable T(n)}}_{\text{Summed over all possible values}}$$

$$E[T(n)] = \frac{1}{n}[T(\max(0,n-1))] + \frac{1}{n}[T(\max(1,n-2))] + \ldots + \frac{1}{n}[T(\max(n-1,0))] + O(n)$$

since select recurses only on the larger partition

PARTITION

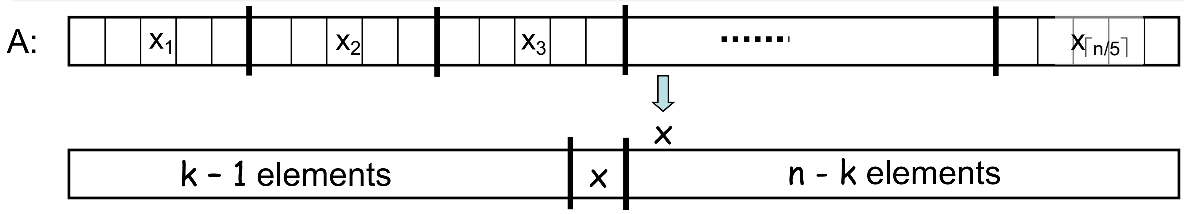
$$= \frac{1}{n}\left[T(n-1) + T(n-2) + T(n-3) \ldots + T(n/2) \ldots + T(n-3) + T(n-2) + T(n-1) \right] + O(n)$$

$$E[T(n)] \le \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} [T(k)] + O(n)$$ **T(n) = O(n) (prove by substitution)**

A Better Selection Algorithm

- Can perform Selection in ***O*(n)** Worst Case
- Idea: guarantee a good split on partitioning
  - Running time is influenced by how "balanced" are the resulting partitions
- Use a modified version of PARTITION
  - Takes as input the element around which to partition

Selection in $O(n)$ Worst Case



1. Divide the n elements into groups of 5 ⇒ $\lceil n/5 \rceil$ groups
2. Find the median of each of the $\lceil n/5 \rceil$ groups
   - Use insertion sort, then pick the median
3. Use SELECT recursively to find the median x of the $\lceil n/5 \rceil$ medians
4. Partition the input array around x, using the modified version of PARTITION
   - There are k-1 elements on the low side of the partition and n-k on the high side
5. If $i = k$ then return x. Otherwise, use SELECT recursively:
   - Find the i-th smallest element on the low side if $i < k$
   - Find the $(i-k)$-th smallest element on the high side if $i > k$

Example

- Find the 11th smallest element in the array:

  A = {12, 34, 0, 3, 22, 4, 17, 32, 3, 28, 43, 82, 25, 27, 34,
  2 ,19 ,12 ,5 ,18 ,20 ,33, 16, 33, 21, 30, 3, 47}

1. Divide the array into groups of 5 elements

12	4	43	2	20	30
34	17	82	19	33	3
0	32	25	12	16	47
3	3	27	5	33	
22	28	34	18	21	

Example (cont.)

2. Sort the groups and find their medians

0	4	25	2	20	3
3	3	27	5	16	30
12	17	34	12	21	47
34	32	43	19	33	
22	28	82	18	33	

3. Find the median of the medians

12, 12, 17, 21, 34, 30

Example (cont.)

4. Partition the array around the median of medians (17)

   First partition:
   {12, 0, 3, 4, 3, 2, 12, 5, 16, 3}

   Pivot:
   17 (position of the pivot is q = 11)

   Second partition:
   {34, 22, 32, 28, 43, 82, 25, 27, 34, 19, 18, 20, 33, 33, 21, 30, 47}

To find the 6-th smallest element we would have to recurse our search in the first partition.

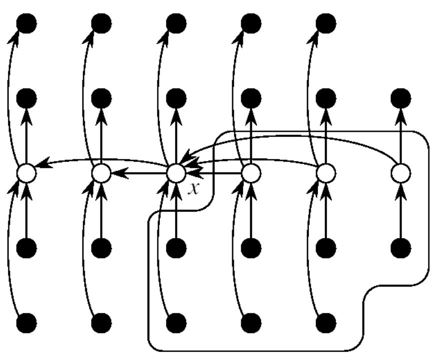
Analysis of Running Time

- Step 1: making groups of 5 elements takes $O(n)$
- Step 2: sorting $n/5$ groups in $O(1)$ time each takes $O(n)$
- Step 3: calling SELECT on $\lceil n/5 \rceil$ medians takes time $T(\lceil n/5 \rceil)$
- Step 4: partitioning the n-element array around x takes $O(n)$
- Step 5: recursion on one partition takes

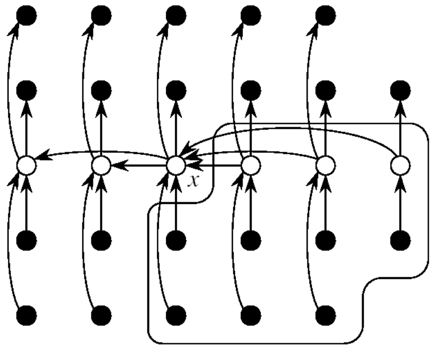
depends on the size of the partition!!

Analysis of Running Time

- First determine an upper bound for the sizes of the partitions
  - See how bad the split can be
- Consider the following representation
  - Each column represents one group of 5 (elements in columns are sorted)
  - Columns are sorted by their medians



Analysis of Running Time



- At least half of the medians found in step 2 are ≥ x: $\left\lceil \frac{1}{2} \left\lceil \frac{n}{5} \right\rceil \right\rceil$
- All but two of these groups contribute 3 elements > x

$\left\lceil \frac{1}{2} \left\lceil \frac{n}{5} \right\rceil \right\rceil - 2$ groups with 3 elements > x

- At least $3\left(\left\lceil \frac{1}{2} \left\lceil \frac{n}{5} \right\rceil \right\rceil - 2\right) \geq \frac{3n}{10} - 6$ elements greater than x
- SELECT is called on at most $n - \left(\frac{3n}{10} - 6\right) = \frac{7n}{10} + 6$ elements

Recurrence for the Running Time

- Step 1: making groups of 5 elements takes $O(n)$
- Step 2: sorting **n/5** groups in ***O*(1)** time each takes $O(n)$
- Step 3: calling SELECT on $\lceil n/5 \rceil$ medians takes time $T(\lceil n/5 \rceil)$
- Step 4: partitioning the n-element array around **x** takes $O(n)$
- Step 5: recursion on one partition takes time ≤ $T(7n/10 + 6)$
- $T(n) = T(\lceil n/5 \rceil) + T(7n/10 + 6) + O(n)$
- We can show that $T(n) = O(n)$

Substitution

- $T(n) = T(\lceil n/5 \rceil) + T(7n/10 + 6) + O(n)$
  Show that $T(n) \le cn$ for some constant $c > 0$ and all $n \ge n_0$

  $$\begin{aligned} T(n) &\le c\lceil n/5 \rceil + c(7n/10 + 6) + an \\ &\le cn/5 + c + 7cn/10 + 6c + an \\ &= 9cn/10 + 7c + an \\ &= cn + (-cn/10 + 7c + an) \\ &\le cn \qquad \text{if: } -cn/10 + 7c + an \le 0 \end{aligned}$$

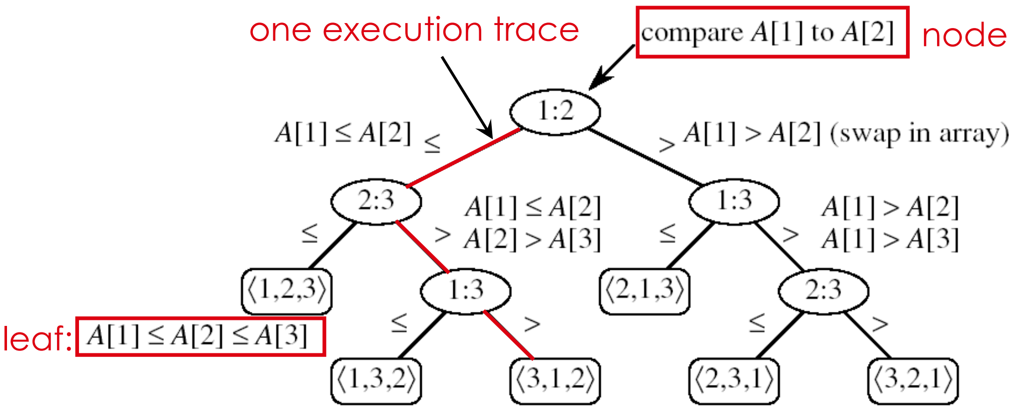
- $c \ge 10a(n/(n-70))$
  - choose $n_0 > 70$ and obtain the value of c

How Fast Can We Sort?

- Insertion sort, Bubble Sort, Selection Sort $\Theta(n^2)$
- Merge sort $\Theta(n\lg n)$
- Quicksort $\Theta(n\lg n)$
- What is common to all these algorithms?
  - These algorithms sort by making comparisons between the input elements
- To sort n elements, comparison sorts must make $\Omega(n\lg n)$ comparisons in the worst case

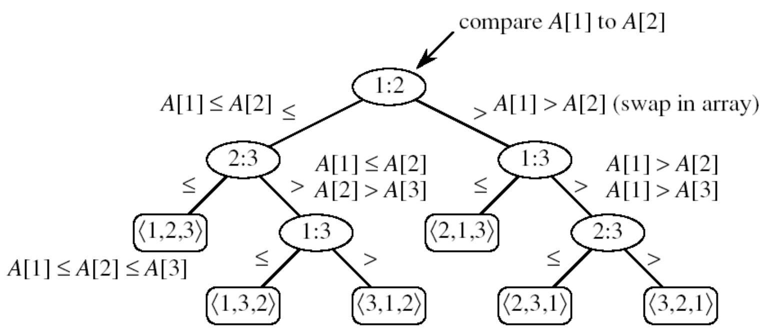
Decision Tree Model

- Represents the comparisons made by a sorting algorithm on an input of a given size: models all possible execution traces
- Control, data movement, other operations are ignored
- Count only the comparisons
- Decision tree for insertion sort on three elements:



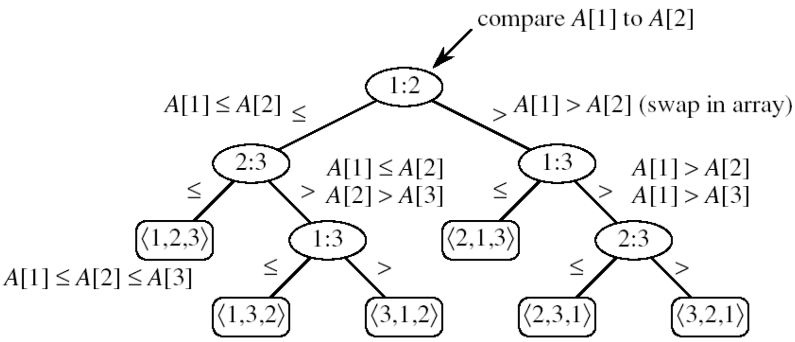
Decision Tree Model

- All permutations on n elements must appear as one of the leaves in the decision tree n! permutations
- Worst-case number of comparisons
  - the length of the longest path from the root to a leaf
  - the height of the decision tree



Decision Tree Model

- Goal: finding a lower bound on the running time on any comparison sort algorithm
  - find a lower bound on the heights of all decision trees for all algorithms



Lemma

- Any binary tree of height **h** has at most **2^h leaves**

Proof: induction on **h**

Basis: **h** = 0 ⇒ tree has one node, which is a leaf

$2^h = 1$

Inductive step: assume true for **h-1**

- Extend the height of the tree with one more level
- Each leaf becomes parent to two new leaves

No. of leaves for tree of height **h** =

= 2 × (no. of leaves for tree of height **h-1**)

$\le 2 \times 2^{h-1}$

$= 2^h$

Lower Bound for Comparison Sorts

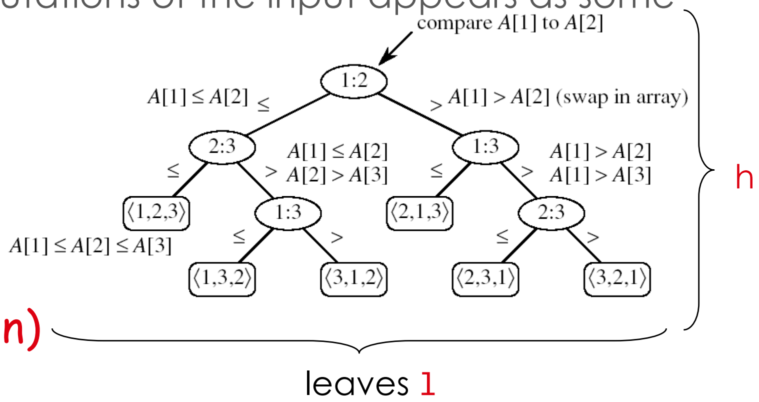
Theorem: Any comparison sort algorithm requires **Ω(nlgn)** comparisons in the worst case.

Proof: How many leaves does the tree have?

- ℓ reachable leaves
- Each of the **n!** permutations of the input appears as some leaf $\Rightarrow n! \le \ell$
- At most 2^h leaves

$\Rightarrow \quad n! \le \ell \le 2^h$

$\Rightarrow \quad h \ge \lg(n!) = \Omega(n\lg n)$



We can beat the **Ω**(nlgn) running time if we use other operations than comparisons!

Counting Sort

- Assumption:
  - The elements to be sorted are integers in the range **0** to **k**
- Idea:
  - Determine for each input element **x**, the number of elements smaller than **x**
  - Place element **x** into its correct position in the output array

A

1	2	3	4	5	6	7	8
2	5	3	0	2	3	0	3

C

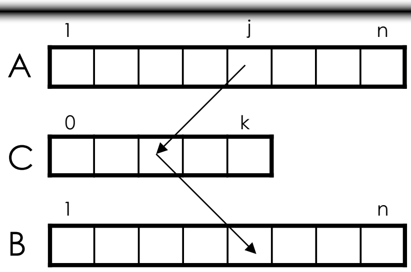
0	1	2	3	4	5
2	2	4	7	7	8

B

1	2	3	4	5	6	7	8
0	0	2	2	3	3	3	5

COUNTING-SORT

Alg.: COUNTING-SORT(A, B, n, k)




```
for i ← 0 to k
    do C[ i ] ← 0
for j ← 1 to n
    do C[A[ j ]] ← C[A[ j ]] + 1
▷C[i] contains the number of elements equal to i
for i ← 1 to k
    do C[ i ] ← C[ i ] + C[i -1]
▷C[i] contains the number of elements ≤ i
for j ← n downto 1
    do B[C[A[ j ]]] ← A[ j ]
       C[A[ j ]] ← C[A[ j ]] - 1
```


Example

	1	2	3	4	5	6	7	8
A	2	5	3	0	2	3	0	3

	0	1	2	3	4	5
C	2	0	2	3	0	1

	0	1	2	3	4	5
C	2	2	4	7	7	8

	1	2	3	4	5	6	7	8
B							3	

	0	1	2	3	4	5
C	2	2	4	6	7	8

	1	2	3	4	5	6	7	8
B		0					3	

	0	1	2	3	4	5
C	1	2	4	6	7	8

	1	2	3	4	5	6	7	8
B		0				3	3	

	0	1	2	3	4	5
C	1	2	4	5	7	8

	1	2	3	4	5	6	7	8
B		0		2		3	3	

	0	1	2	3	4	5
C	1	2	3	5	7	8

Example (cont.)

A

1	2	3	4	5	6	7	8
2	5	3	0	2	3	0	3

B

1	2	3	4	5	6	7	8
0	0		2		3	3	

C

0	1	2	3	4	5
0	2	3	5	7	8

B

1	2	3	4	5	6	7	8
0	0		2	3	3	3	5

C

0	1	2	3	4	5
0	2	3	4	7	7

B

1	2	3	4	5	6	7	8
0	0		2	3	3	3	

C

0	1	2	3	4	5
0	2	3	4	7	8

B

1	2	3	4	5	6	7	8
0	0	2	2	3	3	3	5

Analysis of Counting Sort

Alg.: COUNTING-SORT(A, B, n, k)

```
for i ← 0 to k                              } Θ(k)
    do C[ i ] ← 0
for j ← 1 to n                              } Θ(n)
    do C[A[ j ]] ← C[A[ j ]] + 1
▷C[i] contains the number of elements equal to i
for i ← 1 to k                              } Θ(k)
    do C[ i ] ← C[ i ] + C[i -1]
▷C[i] contains the number of elements ≤ i
for j ← n downto 1                          } Θ(n)
    do B[C[A[ j ]]] ← A[ j ]
       C[A[ j ]] ← C[A[ j ]] - 1
```

Overall time: $\Theta(n + k)$

Analysis of Counting Sort

- Overall time: $\Theta(n + k)$
- In practice we use COUNTING sort when $k = O(n)$

  $\Rightarrow$ running time is $\Theta(n)$
- Counting sort is **stable**
  - Numbers with the same value appear in the same order in the output array
  - Important when additional data is carried around with the sorted keys

Radix Sort

- Considers keys as numbers in a base-k number
  - A **d**-digit number will occupy a field of **d** columns
- Sorting looks at one column at a time
  - For a **d** digit number, sort the least significant digit first
  - Continue sorting on the next least significant digit, until all digits have been sorted
  - Requires only **d** passes through the list

326
453
608
835
751
435
704
690

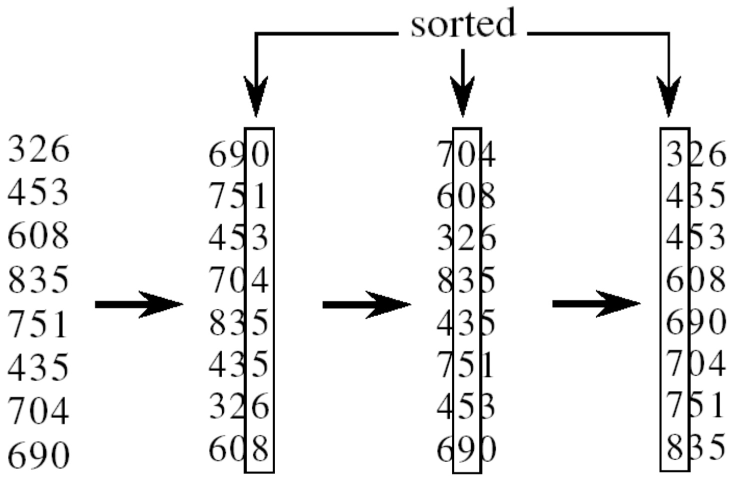
RADIX-SORT

Alg.: RADIX-SORT(*A*, d)

for *i* ← 1 **to** d

do use a stable sort to sort array ***A*** on digit ***i***

- 1 is the lowest order digit, d is the highest-order digit



Analysis of Radix Sort

- Given **n** numbers of **d** digits each, where each digit may take up to **k** possible values, RADIX-SORT correctly sorts the numbers in $\Theta(d(n+k))$
  - One pass of sorting per digit takes $\Theta(n+k)$ assuming that we use counting sort
  - There are **d** passes (for each digit)

Correctness of Radix sort

- We use induction on the number **d** of passes through the digits
- **Basis:** If **d** = 1, there's only one digit, trivial
- **Inductive step:** assume digits 1, 2, . . . , **d-1** are sorted
  - Now sort on the **d**-th digit
  - If $a_d < b_d$, sort will put **a** before **b**: correct **a < b** regardless of the low-order digits
  - If $a_d > b_d$, sort will put **a** after **b**: correct **a > b** regardless of the low-order digits
  - If $a_d = b_d$, sort will leave **a** and **b** in the same order and **a** and **b** are already sorted on the low-order **d-1** digits

	704		326
	608		435
	326		453
	835		608
→	435	→	690
	751		704
	453		751
	690		835

Bucket Sort

- Assumption:
  - the input is generated by a random process that distributes elements uniformly over [0, 1)
- Idea:
  - Divide [0, 1) into **n** equal-sized buckets
  - Distribute the **n** input values into the buckets
  - Sort each bucket
  - Go through the buckets in order, listing elements in each one

- **Input:** $A[1 \,..\, n]$, where $0 \le A[i] < 1$ for all i
- **Output:** elements in A sorted
- **Auxiliary array:** $B[0 \,..\, n - 1]$ of linked lists, each list initially empty

BUCKET-SORT

Alg.: BUCKET-SORT(A, n)

for i ← **1 to** n

do insert A[i] into list **B[**$\lfloor$**nA[i]**$\rfloor$**]**

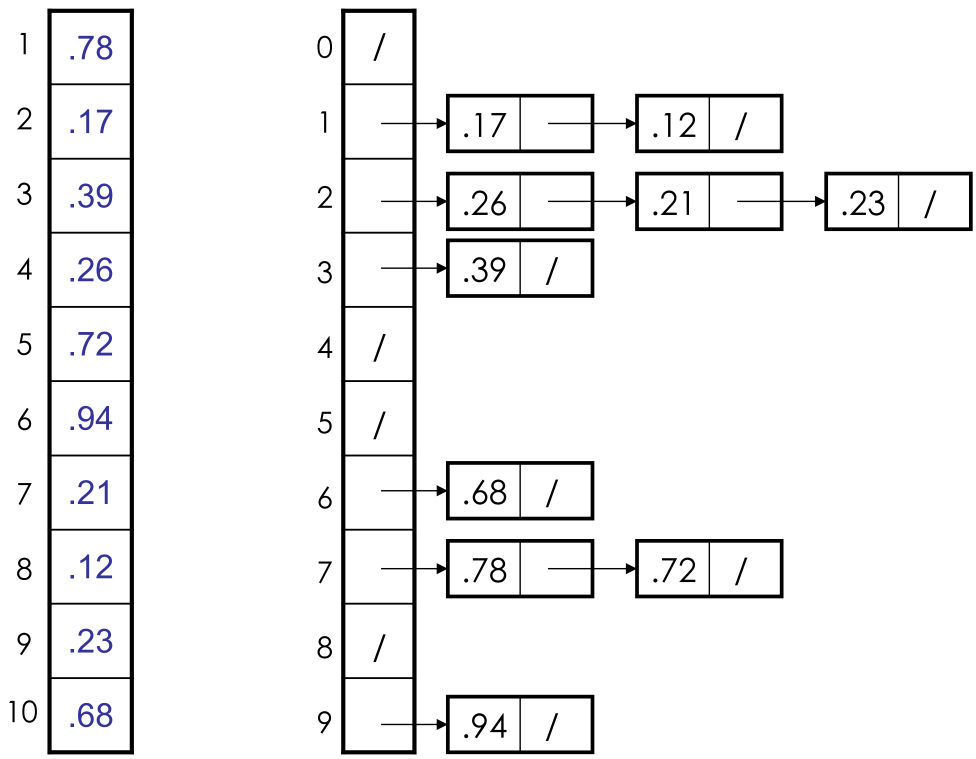
for i ← **0 to n - 1**

do sort list **B[i]** with insertion sort

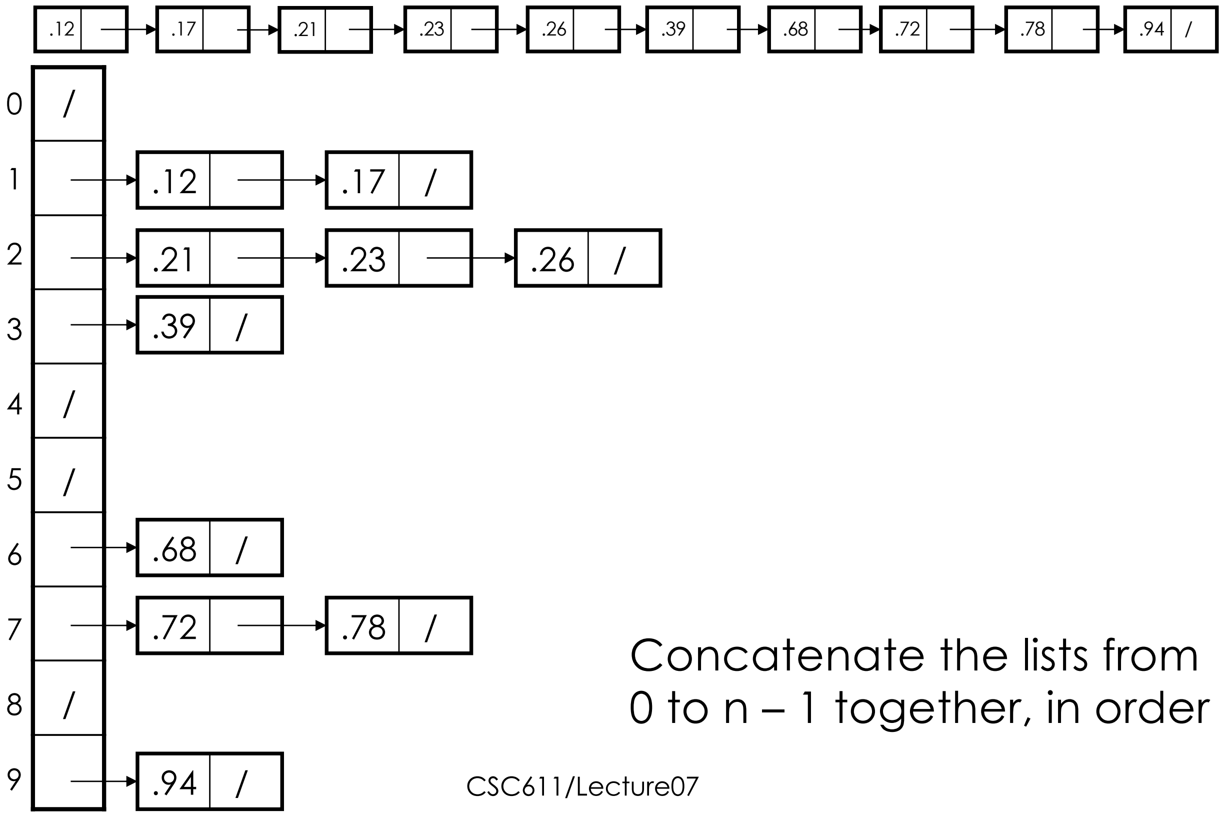
concatenate lists **B[0], B[1], . . . , B[n -1]** together in order

return the concatenated lists

Example - Bucket Sort



Example - Bucket Sort



Concatenate the lists from 0 to n – 1 together, in order

Correctness of Bucket Sort

- Consider two elements $A[i]$, $A[j]$
- Assume without loss of generality that $A[i] \le A[j]$
- Then $\lfloor nA[i] \rfloor \le \lfloor nA[j] \rfloor$
  - $A[i]$ belongs to the same group as $A[j]$ or to a group with a lower index than that of $A[j]$
- If $A[i]$, $A[j]$ belong to the same bucket:
  - insertion sort puts them in the proper order
- If $A[i]$, $A[j]$ are put in different buckets:
  - concatenation of the lists puts them in the proper order

Analysis of Bucket Sort

Alg.: BUCKET-SORT(A, n)

for i ← 1 **to** n

  do insert A[i] into list $B[\lfloor nA[i] \rfloor]$ — **O**(n)

for i ← 0 **to** n - 1

  do sort list B[i] with insertion sort — Θ(n)

concatenate lists B[0], B[1], . . . , B[n -1]
together in order — **O**(n)

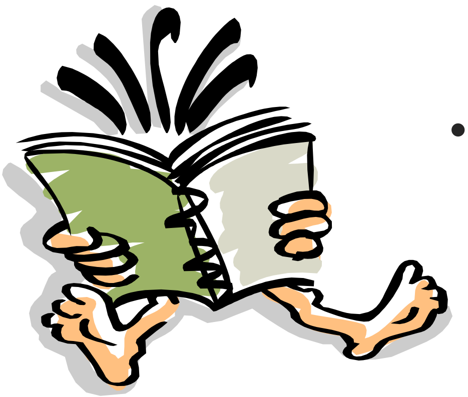
return the concatenated lists

Θ(n)

Conclusion

- Any comparison sort will take at least **nlgn** to sort an array of **n** numbers
- We can achieve a better running time for sorting if we can make certain assumptions on the input data:
  - **Counting sort:** each of the n input elements is an integer in the range **0** to **k**
  - **Radix sort:** the elements in the input are integers represented with **d** digits
  - **Bucket sort:** the numbers in the input are uniformly distributed over the interval [0, 1)

Readings



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